

HYPERBOLIC CONSERVATION LAWS WITH DISCONTINUOUS FLUXES AND HYDRODYNAMIC LIMIT FOR PARTICLE SYSTEMS

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ABSTRACT. We study the following class of scalar hyperbolic conservation laws with discontinuous fluxes:

$$\partial_t \rho + \partial_x F(x, \rho) = 0. \quad (0.1)$$

The main feature of such a conservation law is the discontinuity of the flux function in the space variable x . Kruzkov's approach for the L^1 -contraction does not apply since it requires the Lipschitz continuity of the flux function; and entropy solutions even for the Riemann problem are not unique under the classical entropy conditions. On the other hand, it is known that, in statistical mechanics, some microscopic interacting particle systems with discontinuous speed parameter $\lambda(x)$, in the hydrodynamic limit, formally lead to scalar hyperbolic conservation laws with discontinuous fluxes of the form:

$$\partial_t \rho + \partial_x (\lambda(x)h(\rho)) = 0. \quad (0.2)$$

The natural question arises which entropy solutions the hydrodynamic limit selects, thereby leading to a suitable, physical relevant notion of entropy solutions of this class of conservation laws. This paper is a first step and provides an answer to this question for a family of discontinuous flux functions. In particular, we identify the entropy condition for (0.1) and proceed to show the well-posedness by combining our existence result with a uniqueness result of Audusse-Perthame (2005) for the family of flux functions; we establish a compactness framework for the hydrodynamic limit of large particle systems and the convergence of other approximate solutions to (0.1), which is based on the notion and reduction of measure-valued entropy solutions; and we finally establish the hydrodynamic limit for a ZRP with discontinuous speed-parameter governed by an entropy solution to (0.2).

1. INTRODUCTION

We are concerned with the following class of scalar hyperbolic conservation laws with discontinuous fluxes:

$$\partial_t \rho + \partial_x F(x, \rho(t, x)) = 0 \quad (1.1)$$

and with initial data:

$$\rho|_{t=0} = \rho_0(x), \quad (1.2)$$

where $F(\cdot, \rho)$ is continuous except on a set of measure zero.

The main feature of (1.1) is the discontinuity of the flux function in the space variable x . This feature causes new important difficulties in conservation laws. Kruzkov's approach in [18] for the L^1 -contraction does not apply; entropy solutions even for the Riemann problem of (1.1) are not unique under the classical entropy conditions; several admissibility criteria have been proposed in [1, 3, 8, 15, 17] and the references cited therein. In particular, a uniqueness theorem was established in Baiti-Jenssen [3] when $F(x, \cdot)$ is monotone and Audusse-Perthame [1] for more general flux functions that especially include non-monotone functions $F(x, \cdot)$ in (1.1) under their notion. However, the existence of entropy solutions for the non-monotone case under the notion

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of Audusse-Perthame [1] has not been established, and the entropy conditions proposed in the literature in general are not equivalent.

On the other hand, in statistical mechanics, some microscopic interacting particle systems with discontinuous speed parameter $\lambda(x)$, in the hydrodynamic limit, formally lead to scalar hyperbolic conservation laws with discontinuous flux of the form

$$\partial_t \rho + \partial_x (\lambda(x)h(\rho)) = 0 \quad (1.3)$$

and with initial data (1.2), where $\lambda(x)$ is continuous except on a set of measure zero and $h(\rho)$ is Lipschitz continuous. Equation (1.3) is equivalent to the following 2×2 hyperbolic system of conservation laws:

$$\begin{cases} \partial_t \rho + \partial_x (\lambda h(\rho)) = 0, \\ \partial_t \lambda = 0. \end{cases} \quad (1.4)$$

In particular, when $h(\rho)$ is not strictly monotone, system (1.4) is nonstrictly hyperbolic, one of the main difficulties in conservation laws (cf. [5, 7]). The natural question is which entropy solution the hydrodynamic limit selects, thereby leading to a suitable, physical relevant notion of entropy solutions of this class of conservation laws. This paper is a first step and provides an answer to this question for a family of discontinuous flux functions via an interacting particle system, namely, the attractive zero range process (ZRP). This ZRP leads to a conservation law of the form (1.3) with $\lambda(x) > 0$ and $h(\rho)$ being monotone in ρ , and its hydrodynamic limit naturally gives rise to an entropy condition of the type described in [1, 3] in the formal level.

Motivated by the hydrodynamic limit of the ZRP, in this paper, we adopt the notion of entropy solutions for a class of conservation laws with discontinuous flux functions, including the non-monotone case in the sense of Audusse-Perthame [1], and establish the existence of such an entropy solution via the method of compensated compactness in Section 3. This completes the well-posedness by combining a uniqueness result established in [1] for this class of conservation laws under the notion of entropy solutions.

In order to establish the hydrodynamic limit of large particle systems and the convergence of other approximate solutions to (1.1) rigorously, we establish another compactness framework for (1.1)–(1.2) in Section 2. This mathematical framework is based on the notion and reduction of measure-valued entropy solutions developed in Section 2, which is also applied for another proof of the existence of entropy solutions for the monotone case in Section 3.

In Section 4, we establish the hydrodynamic limit for a ZRP with discontinuous speed-parameter $\lambda(x)$ governed by the unique entropy solution of the Cauchy problem (1.2)–(1.3).

2. NOTION AND REDUCTION OF MEASURE-VALUED ENTROPY SOLUTIONS

In this section, we first develop the notion of measure-valued entropy solutions and establish their reduction to entropy solutions in L^∞ (provided that they exist) of the Cauchy problem (1.1)–(1.2) satisfying

- (H1) $F(x, \rho)$ is continuous at all points of $(\mathbb{R} \setminus \mathcal{N}) \times \mathbb{R}$ with \mathcal{N} a closed set of measure zero;
- (H2) \exists continuous functions f, g such that, for any $x \in \mathbb{R}$ and large ρ , $f(\rho) \leq |F(x, \rho)| \leq g(\rho)$ with $f(\rho) \geq 0$ and $f(\pm\infty) = \infty$;
- (H3) There exists a function $\rho_m(x)$ from \mathbb{R} to \mathbb{R} and a constant M_0 such that, for $x \in \mathbb{R} \setminus \mathcal{N}$, $F(x, \rho)$ is a locally Lipschitz, one to one function from $(-\infty, \rho_m]$ and $[\rho_m, \infty)$ to $[M_0, \infty)$ (or $(-\infty, M_0]$) with $F(x, \rho_m(x)) = M_0$;

or

- (H3') For $x \in \mathbb{R} \setminus \mathcal{N}$, $F(x, \cdot)$ is a locally Lipschitz, one to one function from \mathbb{R} to \mathbb{R} .

One example of the flux function satisfying (H1)–(H2) and (H3) or (H3') is

$$F(x, \rho) = \lambda(x)h(\rho), \quad (2.1)$$

where $\lambda(x)$ is continuous in $x \in \mathbb{R}$ with $0 < \lambda_1 \leq \lambda(x) \leq \lambda_2 < \infty$ for some constants λ_1 and λ_2 , except on a closed set \mathcal{N} of measure zero, $h(\rho)$ is locally Lipschitz and is either monotone or convex (or concave) with $h(\rho_m) = 0$ for some ρ_m in which case $M_0 = 0$.

It is easy to check that, if the flux function $F(x, \rho)$ satisfies (H1)–(H3), then, for any constant $\alpha \in [M_0, \infty)$ (or $\alpha \in (-\infty, M_0]$), there are two steady-state solutions m_α^+ from \mathbb{R} to $[\rho_m(x), \infty)$ and m_α^- from \mathbb{R} to $(-\infty, \rho_m(x)]$ of (1.1) such that

$$F(x, m_\alpha^\pm(x)) = \alpha. \quad (2.2)$$

In the case (H1)–(H2) and (H3'), $m_\alpha^+(x) = m_\alpha^-(x)$ which is even simpler.

2.1. Notion of measure-valued entropy solutions. First, the notion of entropy solutions in L^∞ introduced in Audusse-Perthame [1] and Baiti-Jenssen [3] can be further formulated into the following.

Definition 2.1 (Notion of entropy solutions in L^∞). *We say that an L^∞ function $\rho : \mathbb{R}_+^2 := \mathbb{R}_+ \times \mathbb{R} \mapsto \mathbb{R}$ is an entropy solution of (1.1)–(1.2) provided that, for each $\alpha \in [M_0, \infty)$ (or $\alpha \in (-\infty, M_0]$) and the corresponding two steady-state solutions $m_\alpha^\pm(x)$ of (1.1),*

$$\begin{aligned} & \int \left(|\rho(t, x) - m_\alpha^\pm(x)| \partial_t J + \text{sgn}(\rho(t, x) - m_\alpha^\pm(x)) (F(x, \rho(t, x)) - \alpha) \partial_x J \right) dt dx \\ & + \int |\rho_0(x) - m_\alpha^\pm(x)| J(0, x) dx \geq 0 \end{aligned} \quad (2.3)$$

for any test function $J : \mathbb{R}_+^2 \mapsto \mathbb{R}_+$.

It is easy to see that any entropy solution is a weak solution of (1.1)–(1.2) by choosing α such that $m_\alpha^+(x) \geq \|\rho\|_{L^\infty}$ and $m_\alpha^-(x) \leq -\|\rho\|_{L^\infty}$, respectively, for a.e. $x \in \mathbb{R}$.

From the uniqueness argument in Audusse-Perthame [1] (also see [6]), one can deduce that, for any $L > 0$,

$$\lim_{t \rightarrow 0} \int_{|x| \leq L} |\rho(t, x) - \rho_0(x)| dx = 0. \quad (2.4)$$

Following the notion of entropy solutions, we introduce the corresponding notion of measure-valued entropy solutions. We denote by $\mathcal{P}(\mathbb{R})$ the set of probability measures on \mathbb{R} .

Definition 2.2 (Notion of measure-valued entropy solutions). *We say that a measurable map*

$$\pi : \mathbb{R}_+^2 \rightarrow \mathcal{P}(\mathbb{R})$$

is a measure-valued entropy solution of (1.1)–(1.2) provided that $\langle \pi_{0,x}; k \rangle = \rho_0(x)$ for a.e. $x \in \mathbb{R}$ and, for each $\alpha \in [M_0, \infty)$ (or $\alpha \in (-\infty, M_0]$) and the corresponding two steady-state solutions $m_\alpha^\pm(x)$ of (1.1),

$$\begin{aligned} & \int \left(\langle \pi_{t,x}; |k - m_\alpha^\pm(x)| \rangle \partial_t J + \langle \pi_{t,x}; \text{sgn}(k - m_\alpha^\pm(x)) (F(x, k) - \alpha) \rangle \partial_x J \right) dx dt \\ & + \int |\rho_0(x) - m_\alpha^\pm(x)| J(0, x) dx \geq 0 \end{aligned} \quad (2.5)$$

for any test function $J : \mathbb{R}_+^2 \mapsto \mathbb{R}_+$.

If a measure-valued entropy solution $\pi_{t,x}(k)$ is a Dirac mass with the associated profile $\rho(t, x)$, i.e. $\pi_{t,x}(k) = \delta_{\rho(t,x)}(k)$, then $\rho(t, x)$ is an entropy solution of (1.1)–(1.2), which is unique as shown in [1].

Note that, when the flux function $F(x, \rho)$ is locally Lipschitz in both variables (x, ρ) , one can use the Kruzkov entropy inequality, instead of (2.5), to formulate the following notion of measure-valued solutions:

$$\partial_t \langle \pi_{t,x}; |k - c| \rangle + \partial_x \langle \pi_{t,x}; \text{sgn}(k - c) (F(x, k) - F(x, c)) \rangle + \langle \pi_{t,x}; \text{sgn}(k - c) \partial_x F(x, c) \rangle \leq 0 \quad (2.6)$$

in the sense of distributions and to establish their reduction as in DiPerna [12]. One of the new features in our formulation (2.5) in Definition 2.2 is that the constant c in (2.6) is replaced by the steady-state solutions $m_\alpha^\pm(x)$ so that the additional third term in (2.6) vanishes, as in [1, 3], and thereby allows the discontinuity of the flux functions on a closed set of measure zero for measure-valued entropy solutions.

2.2. Reduction of measure-valued entropy solutions. In this section we first establish the reduction of measure-valued entropy solutions of (1.1)–(1.2) and prove that any measure-valued entropy solution $\pi_{t,x}(k)$ in the sense of Definition 2.2 is the Dirac solution such that the associated profile $\rho(t, x)$ is an entropy solution in the sense of Definition 2.1. That is, our goal is to establish that, when $\pi_{0,x}(k) = \delta_{\rho_0(x)}(k)$,

$$\pi_{t,x}(k) = \delta_{\rho(t,x)}(k), \quad (2.7)$$

where $\rho : \mathbb{R}_+^2 \rightarrow \mathbb{R}$ is the unique entropy solution determined by (2.3). The reduction proof is achieved by two theorems. We start with the first theorem.

Theorem 2.1. *Assume $\rho : \mathbb{R}_+^2 \rightarrow \mathbb{R}$ is the unique entropy solution of (1.1)–(1.2) with initial data $\rho_0 \in L^\infty(\mathbb{R})$. Assume that there exists a measure-valued entropy solution $\pi : \mathbb{R}_+^2 \rightarrow \mathcal{P}(\mathbb{R})$ of (1.1) in the sense of Definition 2.2 with $\pi_{t,x}$ having a fixed compact support for a.e. (t, x) and $\pi_{0,x}(k) = \delta_{\rho_0(x)}(k)$ for a.e. $x \in \mathbb{R}$. Then*

$$\int (\langle \pi_{t,x}; |k - \rho(t, x)| \rangle \partial_t J + \langle \pi_{t,x}; \text{sgn}(k - \rho(t, x))(F(x, k) - F(x, \rho(t, x))) \rangle \partial_x J) dx dt \geq 0 \quad (2.8)$$

for any test function $J : \mathbb{R}_+^2 \mapsto \mathbb{R}_+$.

Proof. The proof is divided into six steps.

Step 1. We first rewrite

$$E := \partial_t \langle \pi_{t,x}; |k - \rho(t, x)| \rangle + \partial_x \langle \pi_{t,x}; \text{sgn}(k - \rho(t, x))(F(x, k) - F(x, \rho(t, x))) \rangle$$

in the entropy inequality (2.8). We notice the following:

- Under the assumption (H3'), $F(x, \rho(t, x))$ is continuous in x a.e. Then we can define a function $\tilde{\rho}(s, y, x)$ for a.e. $(s, y, x) \in \mathbb{R}_+ \times \mathbb{R}^2$ such that, for fixed (s, y) ,

$$F(x, \tilde{\rho}(s, y, x)) := F(x, m_{F(y, \rho(s, y))}(x)) = F(y, \rho(s, y)), \quad (2.9)$$

where the last equality follows from (2.2). Thus, we define

$$\beta(s, y) := F(y, \rho(s, y)) \quad \text{so that} \quad \tilde{\rho}(s, y, x) = m_{\beta(s, y)}(x).$$

- For the case (H3), we define the sign of the difference between the tilda function and $\rho_m(y)$ to be the same as the sign of the corresponding solution. Since $\rho_m(y)$ is the minimum (or maximum) point of the flux function with $F(y, \rho_m(y)) = M_0$, then, for

$$\tilde{\rho}(s, y, x) := m_{\beta(s, y)}^+(x) \text{sgn}_+(\rho(s, y) - \rho_m(y)) + m_{\beta(s, y)}^-(x) \text{sgn}_-(\rho(s, y) - \rho_m(y)), \quad (2.10)$$

we have as in (2.9)

$$\begin{aligned} F(x, \tilde{\rho}(s, y, x)) &:= F(x, m_{\beta(s, y)}^+(x) \text{sgn}_+(\rho(s, y) - \rho_m(y)) + m_{\beta(s, y)}^-(x) \text{sgn}_-(\rho(s, y) - \rho_m(y))) \\ &= F(y, \rho(s, y)) = \beta(s, y). \end{aligned}$$

With these notations, we set

$$\tilde{E} := \partial_t \langle \pi_{t,x}; |k - \tilde{\rho}(s, y, x)| \rangle + \partial_x \langle \pi_{t,x}; \text{sgn}(k - \tilde{\rho}(s, y, x))(F(x, k) - \beta(s, y)) \rangle. \quad (2.11)$$

Then, to obtain the inequality $E \leq 0$, it suffices to show that $\lim_{x \rightarrow y} \tilde{E} = E$.

Step 2. We now show that

$$\tilde{\rho}(s, y, x) \xrightarrow{x \rightarrow y} \tilde{\rho}(s, y, y) = \rho(s, y) \quad \text{for a.e. } (s, y) \in \mathbb{R}_+^2. \quad (2.12)$$

For the case (H3'), since the flux function is continuous outside a negligible set \mathcal{N} , then, for $x \in \mathbb{R} \setminus \mathcal{N}$,

$$F(x, \tilde{\rho}(s, y, y)) \xrightarrow{x \rightarrow y} F(y, \tilde{\rho}(s, y, y)).$$

On the other hand, we have $F(y, \tilde{\rho}(s, y, y)) = F(x, \tilde{\rho}(s, y, x))$. Therefore, we have

$$F(x, \tilde{\rho}(s, y, x)) - F(x, \tilde{\rho}(s, y, y)) \xrightarrow{x \rightarrow y} 0,$$

and (2.12) is a consequence of the fact that $F(x, \cdot)$ is a one to one function. The case (H3) is clear from the definition of $\tilde{\rho}(s, y, x)$ in (2.10).

Step 3. With Steps 1–2, to achieve inequality (2.8), it suffices by choosing $\alpha = \beta(s, y)$ in (2.5) to show the following inequality:

$$\begin{aligned} & \lim_{\tau, \omega \rightarrow 0} \int \partial_t J(t, x) \bar{H}_\tau(t-s) H_\omega(x-y) \langle \pi_{t,x}; |k - \tilde{\rho}(s, y, x)| \rangle dt dx ds dy \\ & + \lim_{\tau, \omega \rightarrow 0} \int \partial_x J(t, x) \bar{H}_\tau(t-s) H_\omega(x-y) \langle \pi_{t,x}; \text{sgn}(k - \tilde{\rho}(s, y, x)) (F(x, k) - \beta(s, y)) \rangle dt dx ds dy \\ & + \lim_{\tau, \omega \rightarrow 0} \int J(0, x) \bar{H}_\tau(-s) H_\omega(x-y) |\rho_0(x) - \tilde{\rho}(s, y, x)| dx ds dy \geq 0 \end{aligned} \quad (2.13)$$

for any test function $J \in C_0^\infty(\mathbb{R}_+^2)$ and verify that $\tilde{\rho}(s, y, x)$ can be replaced by $\rho(t, x)$ in the limit as $\tau, \omega \rightarrow 0$. Here the two families of functions $\bar{H}_\tau, H_\omega \in C_0^\infty(\mathbb{R})$ are defined as

$$\bar{H}_\tau(z) = \frac{1}{\tau} \bar{H}\left(\frac{z}{\tau}\right) \quad \text{and} \quad H_\omega(z) = \frac{1}{\omega} H\left(\frac{z}{\omega}\right) \quad \text{for } \tau, \omega > 0,$$

for a positive, compactly supported function $H \in C_0^\infty(\mathbb{R})$ and a positive function $\bar{H} \in C_0^\infty(\mathbb{R})$ with compact support in $(-2, -1)$ such that $\int_{\mathbb{R}} H(z) dz = 1$ and $\int_{(-2, -1)} \bar{H}(z) dz = 1$.

This can be easily seen by first choosing the test function in (2.5) as $J(t, x) \bar{H}_\tau(t-s) H_\omega(x-y) \geq 0$ for fixed (s, y) and then integrating with respect to (s, y) . We now estimate the three terms of (2.13) in Steps 4–6, respectively.

Step 4. We show that, as $\tau, \omega \rightarrow 0$, the first term converges to

$$\int \partial_t J(t, x) \langle \pi_{t,x}; |k - \rho(t, x)| \rangle dt dx.$$

Observe that

$$\begin{aligned} & \left| \int \partial_t J(t, x) \bar{H}_\tau(t-s) H_\omega(x-y) \langle \pi_{t,x}; |k - \tilde{\rho}(s, y, x)| \rangle dt dx ds dy \right. \\ & \quad \left. - \int \partial_t J(t, x) \bar{H}_\tau(t-s) H_\omega(x-y) \langle \pi_{t,x}; |k - \tilde{\rho}(s, y, y)| \rangle dt dx ds dy \right| \\ & \leq \int |\partial_t J(t, x)| \bar{H}_\tau(t-s) \left(\int H_\omega(x-y) |\tilde{\rho}(s, y, x) - \tilde{\rho}(s, y, y)| dx \right) dt ds dy \rightarrow 0 \quad \text{as } \omega \rightarrow 0, \end{aligned} \quad (2.14)$$

by the dominated convergence theorem and the fact that $\int H_\omega(x-y) |\tilde{\rho}(s, y, x) - \tilde{\rho}(s, y, y)| dx \rightarrow 0$ as $\omega \rightarrow 0$ for a.e. $(s, y) \in \mathbb{R}_+^2$ since $\tilde{\rho}(s, y, x) \xrightarrow{x \rightarrow y} \tilde{\rho}(s, y, y) = \rho(s, y)$ by (2.12). Then, to find the limit of the first term of (2.13), it suffices to compute the limit of

$$\int \partial_t J(t, x) \bar{H}_\tau(t-s) H_\omega(x-y) \langle \pi_{t,x}; |k - \rho(s, y)| \rangle dt dx ds dy. \quad (2.15)$$

Thus, it suffices to show that $\rho(s, y)$ can be replaced by $\rho(t, x)$ in (2.15), i.e., as $\tau, \omega \rightarrow 0$,

$$\begin{aligned} & \int \partial_t J(t, x) \bar{H}_\tau(t-s) H_\omega(x-y) |\rho(t, x) - \rho(s, y)| dt dx ds dy \\ & = \int \partial_t J(t, x) \bar{H}(-r) H(-z) |\rho(t, x) - \rho(t + \tau r, x + \omega z)| dt dx dr dz \rightarrow 0. \end{aligned} \quad (2.16)$$

This is guaranteed by the fact that

$$\lim_{\tau, \omega \rightarrow 0} \int |\rho(t, x) - \rho(t + \tau r, x + \omega z)| dt dx = 0,$$

and the dominated convergence theorem since all the functions involved are bounded. This implies that, in (2.15), we can indeed replace $\rho(s, y)$ by $\rho(t, x)$ to arrive at the result.

Step 5. We show that the second term of (2.13) converges to

$$\int \partial_x J(t, x) \langle \pi_{t,x}; \operatorname{sgn}(k - \rho(t, x)) (F(x, k) - \beta(t, x)) \rangle dt dx \quad \text{as } \tau, \omega \rightarrow 0.$$

The hypothesis (H2) on $F(x, \rho)$ implies

$$\begin{aligned} & \left| \operatorname{sgn}(k - \tilde{\rho}(s, y, x)) (F(x, k) - \beta(s, y)) - \operatorname{sgn}(k - \tilde{\rho}(s, y, y)) (F(x, k) - F(x, \tilde{\rho}(s, y, y))) \right| \\ &= \left| \operatorname{sgn}(k - \tilde{\rho}(s, y, x)) (F(x, k) - F(x, \tilde{\rho}(s, y, x))) - \operatorname{sgn}(k - \rho(s, y)) (F(x, k) - F(x, \rho(s, y))) \right| \\ &\leq C |\tilde{\rho}(s, y, x) - \rho(s, y)|. \end{aligned}$$

Integrating the last expression with respect to x against the function $H_\omega(x - y)$ yields its convergence to zero by Step 2 as $\omega \rightarrow 0$. Therefore, the limit of the second term of (2.13) is the same as the limit of

$$\int \partial_x J(t, x) \bar{H}_\tau(t - s) H_\omega(x - y) \langle \pi_{t,x}; \operatorname{sgn}(k - \rho(s, y)) (F(x, k) - F(x, \rho(s, y))) \rangle dt dx ds dy,$$

and it suffices to prove that, as $\tau, \omega \rightarrow 0$,

$$\int \partial_x J(t, x) \bar{H}_\tau(t - s) H_\omega(x - y) \langle \pi_{t,x}; \left| \operatorname{sgn}(k - \rho(s, y)) (F(x, k) - F(x, \rho(s, y))) - \operatorname{sgn}(k - \rho(t, x)) (F(x, k) - F(x, \rho(t, x))) \right| \rangle dt dx ds dy \rightarrow 0.$$

Using the Lipschitz property and fact (2.16), we achieve the result for the second term of (2.13).

Step 6. We now show that the third term of (2.13) converges to zero if $\tau, \omega \rightarrow 0$. Note that

$$\begin{aligned} & \int J(0, x) \bar{H}_\tau(-s) H_\omega(x - y) \left| |\rho_0(x) - \tilde{\rho}(s, y, x)| - |\rho_0(x) - \tilde{\rho}(s, y, y)| \right| dx ds dy \\ &\leq \int J(0, x) \bar{H}_\tau(-s) H_\omega(x - y) |\tilde{\rho}(s, y, x) - \tilde{\rho}(s, y, y)| dx ds dy. \end{aligned}$$

For the same reason as in the first and the second term of (2.13), the right hand side converges to zero if $\tau, \omega \rightarrow 0$. We therefore next compute the limit as $\tau, \omega \rightarrow 0$ of

$$\int J(0, x) \bar{H}_\tau(-s) H_\omega(x - y) |\rho_0(x) - \rho(s, y)| dx ds dy.$$

As before, $\lim_{\tau, \omega \rightarrow 0} \int J(0, x) H_\tau(-s) H_\omega(x - y) |\rho(s, x) - \rho(s, y)| dx ds dy = 0$. Therefore, the next goal is to compute the limit of

$$\int J(0, x) \bar{H}_\tau(-s) H_\omega(x - y) |\rho_0(x) - \rho(s, x)| dx ds dy = \int J(0, x) \bar{H}(-r) |\rho_0(x) - \rho(\tau r, x)| dx dr. \quad (2.17)$$

Since all the functions are bounded and $\operatorname{supp} H \subset (-2, -1)$, by the dominated convergence theorem and property (2.4) of the unique entropy solution $\rho(t, x)$, this converges to 0 as $\tau \rightarrow 0$ and thereby (2.17) converges to 0.

With Steps 3–6 and by (2.13), we complete the proof. \square

Then Theorem 2.1 yields the L^1 -contraction between the measure-valued entropy solution $\pi_{t,x}$ and the unique entropy solution $\rho(t, x)$ of (1.1)–(1.2).

Theorem 2.2 (L^1 -contraction). *The function $\int \langle \pi_{t,x}; |k - \rho(t,x)| \rangle dx$ is non-increasing in $t > 0$, which implies $\pi_{t,x}(k) = \delta_{\rho(t,x)}(k)$ when $\pi_{0,x}(k) = \delta_{\rho_0(x)}(k)$ for a.e. $x \in \mathbb{R}$.*

Proof. In expression (2.8), we choose the test function as the product test function $J_j(t)H(x)$, with $J_j(t)$ converging to the indicator function $\mathbb{1}_{[t_1,t_2]}(t)$ as $j \rightarrow \infty$ for $t_2 > t_1 \geq 0$. Then (2.8) is equal to

$$\begin{aligned} & \int H(x) \langle \pi_{t_1,x}(k); |k - \rho(t_1,x)| \rangle dx - \int H(x) \langle \pi_{t_2,x}(k); |k - \rho(t_2,x)| \rangle dx \\ & + \int_{t_1}^{t_2} \int H'(x) \langle \pi_{t,x}(k); \operatorname{sgn}(k - \rho(t,x)) (F(x,k) - \beta(t,x)) \rangle dx dt \geq 0. \end{aligned} \quad (2.18)$$

In (2.18), we choose

$$H(x) = e^{-\gamma\sqrt{1+|x|^2}} \chi\left(\frac{x}{N}\right), \quad \gamma, N > 0,$$

for $\chi \in C_0^\infty(-2, 2)$ with $\chi(x) = 1$ when $x \in [-1, 1]$ and $\chi(x) \geq 0$. Letting $N \rightarrow \infty$ first and $\gamma \rightarrow 0$ then yields that, for any $t_2 > t_1 \geq 0$,

$$\int \langle \pi_{t_2,x}; |k - \rho(t_2,x)| \rangle dx - \int \langle \pi_{t_1,x}; |k - \rho(t_1,x)| \rangle dx \leq 0.$$

In particular, when $t_2 = t > 0, t_1 \rightarrow 0$, then $\pi_{0,x}(k) = \delta_{\rho_0(x)}(k)$ implies

$$\int \langle \pi_{t,x}; |k - \rho(t,x)| \rangle dx \leq 0$$

so that $\pi_{t,x}(k) = \delta_{\rho(t,x)}(k)$ for any $t > 0$. □

3. EXISTENCE OF ENTROPY SOLUTIONS

In this section, we establish the existence of entropy solutions (1.1)–(1.2) in the sense of Definition 2.1, as required for the reduction of measure-valued entropy solutions. More precisely, for each fixed $\varepsilon > 0$, ρ^ε denotes the unique Kruzkov solution of (1.1)–(1.2) in the sense (3.3), where the flux function depends smoothly on the space variable x ; then it is shown that the sequence ρ^ε converges to an entropy solution of (1.1)–(1.2).

3.1. Existence of entropy solutions when F is smooth. Define $F_\varepsilon(x, \rho)$ the standard mollification of $F(x, \rho)$ in $x \in \mathbb{R}$:

$$F_\varepsilon(x, \rho) := (F(\cdot, \rho) * \theta^\varepsilon)(x) \rightarrow F(x, \rho) \quad a.e. \quad \text{as } \varepsilon \rightarrow 0, \quad (3.1)$$

with $\theta^\varepsilon(x) := \theta(\frac{x}{\varepsilon})$, $\theta(x) \geq 0$, $\operatorname{supp} \theta(x) \subset [-1, 1]$, and $\int_{-1}^1 \theta(x) dx = 1$. For fixed $\varepsilon > 0$, consider the following Cauchy problem:

$$\begin{cases} \partial_t \rho + \partial_x F_\varepsilon(x, \rho) = 0, \\ \rho|_{t=0} = \rho_0(x) \geq 0. \end{cases} \quad (3.2)$$

Kruzkov's result in [18] indicates that there exists a unique solution ρ^ε of (3.2) satisfying the Kruzkov entropy inequality:

$$\partial_t |\rho^\varepsilon(t, x) - c| + \partial_x (\operatorname{sgn}(\rho^\varepsilon(t, x) - c)(F_\varepsilon(x, \rho^\varepsilon(t, x)) - F_\varepsilon(x, c))) + (\operatorname{sgn}(\rho^\varepsilon(t, x) - c) \partial_x F_\varepsilon(x, c)) \leq 0 \quad (3.3)$$

in the sense of distributions. We now show that the entropy solution ρ^ε also satisfies (2.3).

Proposition 3.1. *Let $\rho^\varepsilon(t, x)$ be a solution of the Cauchy problem (3.2) satisfying the Kruzkov entropy inequality (3.3). Then $\rho^\varepsilon(t, x)$ also satisfies the entropy inequality (2.3) with steady-state solutions $m_\alpha^\pm = m_\alpha^{\varepsilon, \pm}$.*

Proof. In (3.3), we choose the constant $c = m_\alpha^{\varepsilon, \pm}(y)$ for any $\alpha \in [M_0, \infty)$ (or $\alpha \in (-\infty, M_0]$), integrate against a test function $J_\omega(t, x, y) := J(t, \frac{x+y}{2})H_\omega(x-y)$ with H_ω as in the proof of Theorem 2.1, integrate by parts in the term involving $H'_\omega(x-y)$ with respect to dy , and observe that $(\partial_x + \partial_y)J_\omega(t, x, y) = \partial_x J(t, \frac{x+y}{2})H_\omega(x-y)$ to obtain from (3.3) that

$$\begin{aligned}
& \int |\rho^\varepsilon(t, x) - m_\alpha^{\varepsilon, \pm}(y)| H_\omega(x-y) \partial_t J dt dx dy \\
& + \int \operatorname{sgn}(\rho^\varepsilon(t, x) - m_\alpha^{\varepsilon, \pm}(y)) (F_\varepsilon(x, \rho^\varepsilon(t, x)) - F_\varepsilon(x, m_\alpha^{\varepsilon, \pm}(y))) H_\omega(x-y) \partial_y J dt dx dy \\
& - \int \operatorname{sgn}(\rho^\varepsilon(t, x) - m_\alpha^{\varepsilon, \pm}(y)) \partial_y F_\varepsilon(x, m_\alpha^{\varepsilon, \pm}(y)) H_\omega(x-y) J dt dx dy \\
& - \int \operatorname{sgn}(\rho^\varepsilon(t, x) - m_\alpha^{\varepsilon, \pm}(y)) \partial_x F_\varepsilon(x, m_\alpha^{\varepsilon, \pm}(y)) H_\omega(x-y) J dt dx dy \\
& + \int |\rho^\varepsilon(0, x) - m_\alpha^{\varepsilon, \pm}(y)| H_\omega(x-y) J(0, \frac{x+y}{2}) dx dy \geq 0, \tag{3.4}
\end{aligned}$$

where we have used that

$$\int (F_\varepsilon(x, \rho^\varepsilon(t, x)) - F_\varepsilon(x, m_\alpha^{\varepsilon, \pm}(y))) H_\omega(x-y) J dy (\operatorname{sgn}(\rho^\varepsilon(t, x) - m_\alpha^{\varepsilon, \pm}(y))) dx dt = 0.$$

As in the proof of Theorem 2.1, we can replace $\rho^\varepsilon(t, x)$ by $\rho^\varepsilon(t, y)$ in the first term as $\omega \rightarrow 0$ and replace $\rho^\varepsilon(0, x)$ by $\rho^\varepsilon(0, y)$ in the last term as $\omega \rightarrow 0$.

The second term is equal to

$$\int \operatorname{sgn}(\rho^\varepsilon(t, x) - m_\alpha^{\varepsilon, \pm}(y)) (F_\varepsilon(x, \rho^\varepsilon(t, x)) - F_\varepsilon(x, m_\alpha^{\varepsilon, \pm}(y))) H_\omega(x-y) \partial_y J dt dx dy.$$

By the hypothesis (H3) on the flux function, $F_\varepsilon(\cdot, \rho)$ is a Lipschitz function from $(-\infty, \rho_m]$ and $[\rho_m, \infty)$ to $[M_0, \infty)$ (or $(-\infty, M_0]$), which implies

$$|\operatorname{sgn}(\rho^\varepsilon(t, x) - \rho^\varepsilon(t, y)) (F_\varepsilon(x, \rho^\varepsilon(t, x)) - F_\varepsilon(x, \rho^\varepsilon(t, y)))| \leq C |\rho^\varepsilon(t, x) - \rho^\varepsilon(t, y)|.$$

One can show in a similar way as in the first term that

$$\lim_{\omega \rightarrow 0} \int |\rho^\varepsilon(t, x) - \rho^\varepsilon(t, y)| H_\omega(x-y) \partial_y J dt dx dy = 0.$$

This means that, in the second term of (3.4), one can replace $F_\varepsilon(x, \rho^\varepsilon(t, x))$ by $F_\varepsilon(x, \rho^\varepsilon(t, y))$. Since F_ε is also a smooth function with respect to the first variable, the second term converges to

$$\int \operatorname{sgn}(\rho^\varepsilon(t, y) - m_\alpha^{\varepsilon, \pm}(y)) (F_\varepsilon(y, \rho^\varepsilon(t, y)) - \alpha) \partial_y J(t, y) dt dy.$$

In the third and fourth term in (3.4), for $z \in \mathbb{R}$, we have

$$\begin{aligned}
& \lim_{\omega \rightarrow 0} \int (\partial_x + \partial_y) F_\varepsilon(x, m_\alpha^{\varepsilon, \pm}(y)) \omega H_\omega(\omega z) J(t, y + \frac{1}{2}\omega z) dz \\
& = \lim_{\omega \rightarrow 0} \int (\partial_x + \partial_y) F_\varepsilon(x, m_\alpha^{\varepsilon, \pm}(y)) H(z) J(t, y + \frac{1}{2}\omega z) dz \\
& = J(t, y) \partial_y F_\varepsilon(y, m_\alpha^{\varepsilon, \pm}(y)) = J(t, y) \partial_y \alpha = 0.
\end{aligned}$$

With these results, as $\omega \rightarrow 0$, inequality (3.4) becomes (2.3) for $F_\varepsilon(x, \rho) = (F(\cdot, \rho) * \theta^\varepsilon)(x)$ with steady-state solutions $m_\alpha^\pm = m_\alpha^{\varepsilon, \pm}$. \square

Thus we conclude the existence of an entropy solution $\rho_\varepsilon(t, x)$ in the sense of Definition 2.1 for each F_ε with fixed $\varepsilon > 0$.

Remark 3.1. Notice that the sequence of approximate entropy solutions converges to a measure-valued entropy solution as $\varepsilon \rightarrow 0$: First, since $\rho_0 \in L^\infty$, we find that, for α big enough,

$$m_{\alpha}^{\varepsilon,-}(x) \leq \rho_0(x) \leq m_{\alpha}^{\varepsilon,+}(x) \quad \text{for all } x \in \mathbb{R}.$$

From [1], it then follows that

$$m_{\alpha}^{\varepsilon,-}(x) \leq \rho^{\varepsilon}(t, x) \leq m_{\alpha}^{\varepsilon,+}(x),$$

which implies the uniform boundedness of $\rho^{\varepsilon}(t, x)$ in ε since $m_{\alpha}^{\varepsilon,\pm}(x)$ are uniformly bounded in ε . Then there exists a compactly supported family of probability measures $\pi_{t,x}$ on \mathbb{R} (i.e. Young measures; see Tartar [22]) and a subsequence (still denoted by) $\rho^{\varepsilon}(t, x)$ such that, for any continuous function $f(\rho)$,

$$f(\rho^{\varepsilon}(t, x)) \xrightarrow{*} \langle \pi_{t,x}, f(k) \rangle \quad \text{as } \varepsilon \rightarrow 0. \quad (3.5)$$

On the other hand, by Section 3.1, the sequence $\rho^{\varepsilon}(t, x)$ satisfies the entropy inequality (2.3) for $F_{\varepsilon}(x, \rho)$ and the steady-state solutions $m_{\alpha}^{\pm} = m_{\alpha}^{\varepsilon,\pm}$. In particular, we use (3.5) and the definition of the sequence $F_{\varepsilon}(x, \rho)$ in (3.1) to conclude that, as $\varepsilon \rightarrow 0$, the compactly supported family of probability measures $\pi_{t,x}$ satisfies that, for any test function $J : \mathbb{R}_+^2 \mapsto \mathbb{R}_+$,

$$\begin{aligned} & \int (\langle \pi_{t,x}; |k - m_{\alpha}^{\pm}(x)| \rangle \partial_t J + \langle \pi_{t,x}; \text{sgn}(k - m_{\alpha}^{\pm}) (F(x, k) - \alpha) \rangle \partial_x J) dx dt \\ & + \int |\rho_0(x) - m_{\alpha}^{\pm}(x)| J(0, x) dx \geq 0. \end{aligned} \quad (3.6)$$

Thus, $\pi_{t,x}$ is a measure-valued entropy solution of (1.1)–(1.2) with compact support for a.e. $(t, x) \in \mathbb{R}_+^2$ in the sense of Definition 2.2.

3.2. Existence of entropy solutions when F is discontinuous in x . We are now ready to state the main theorem of this section.

Theorem 3.1. *Let $F(x, \rho)$ be strictly convex or concave in ρ for a.e. $x \in \mathbb{R}$ and satisfy (H1)–(H3), or let $F(x, \rho)$ satisfy (H1)–(H2) and (H3'). Let $\rho_0(x) \in L^\infty$. Then the sequence of entropy solutions ρ^{ε} of the Cauchy problem (3.2) (in the sense of Definition 2.1) converges to the unique entropy solution of the Cauchy problem (1.1)–(1.2) in the sense of Definition 2.1.*

Proof. We consider the two cases separately.

For the case (H1)–(H2) and (H3'), that is, the flux function F is monotone in ρ , we apply the compactness framework established in Section 2 to establish the convergence. For this case, the existence of entropy solutions has been established in [3]. In Remark 3.1, we have shown that the limit of the entropy solutions ρ^{ε} is determined by a measure-valued entropy solution $\pi_{t,x}$. Then, by Theorems 2.1–2.2, $\pi_{t,x}$ is the Dirac measure concentrated on the unique entropy solution $\rho(t, x)$ of (1.1)–(1.2) in the sense of Definition 2.1, which implies the whole sequence converges.

For the case (H1)–(H3), since we have not established the existence of an entropy solution, we employ the compensated compactness method to establish the convergence of the entropy solutions of the Cauchy problem (3.2), which also yields the existence of a unique entropy solution of the Cauchy problem (1.1)–(1.2).

From Remark 3.1, we have known that ρ^{ε} is uniformly bounded in L^∞ which implies that there exists a subsequence ρ^{ε} converging weakly to a compactly supported family of probability measures $\nu_{t,x}$ on \mathbb{R}_+ such that, for any function $f(\rho, t, x)$ that is continuous in ρ for a.e. (t, x) ,

$$f(\rho^{\varepsilon}(t, x), t, x) \xrightarrow{*} \langle \nu_{t,x}, f(k, t, x) \rangle \quad \text{as } \varepsilon \rightarrow 0. \quad (3.7)$$

In particular,

$$\rho^{\varepsilon}(t, x) \xrightarrow{*} \langle \nu_{t,x}, k \rangle =: \rho(t, x) \in L^\infty. \quad (3.8)$$

Our goal is to prove the strong convergence of $\rho^\varepsilon(t, x)$ to $\rho(t, x)$ a.e., equivalently, $\nu_{t,x} = \delta_{\rho(t,x)}$, which implies that $\rho(t, x)$ is an entropy solution of (1.1)–(1.2), that is, $\rho(t, x)$ satisfies the entropy inequality in Definition 2.1.

By Section 3.1, we have known that the sequence ρ^ε exists and satisfies

$$E^\varepsilon := \partial_t |\rho_\varepsilon(t, x) - \hat{\rho}^\varepsilon(s, y, x)| + \partial_x (\text{sgn}(\rho^\varepsilon(t, x) - \hat{\rho}^\varepsilon(s, y, x)) (F_\varepsilon(x, \rho^\varepsilon(t, x)) - \gamma(s, y))) \leq 0$$

in the sense of distributions, where

$$\hat{\rho}^\varepsilon(s, y, x) := m_{\gamma(s,y)}^{+,\varepsilon}(x) \text{sgn}_+(\rho(s, y) - \rho_m(y)) + m_{\gamma(s,y)}^{-,\varepsilon}(x) \text{sgn}_-(\rho(s, y) - \rho_m(y)).$$

Notice that $\gamma(s, y) := F(y, \rho(s, y))$ is independent of ε . Thus, for fixed (s, y) , we have the strong convergence of $m_{\gamma(s,y)}^{\pm,\varepsilon}(x)$ to a steady-state solution $m_{\gamma(s,y)}^\pm(x)$ of (1.1)–(1.2) as $\varepsilon \rightarrow 0$. In particular,

$$\|\hat{\rho}^\varepsilon\|_{L^\infty} \leq M, \quad M \text{ independent of } \varepsilon;$$

and, for a.e. $(s, y, x) \in \mathbb{R}_+^2 \times \mathbb{R}$,

$$\hat{\rho}^\varepsilon(s, y, x) \rightarrow \hat{\rho}(s, y, x) := m_{\gamma(s,y)}^+(x) \text{sgn}_+(\rho(s, y) - \rho_m(y)) + m_{\gamma(s,y)}^-(x) \text{sgn}_-(\rho(s, y) - \rho_m(y)),$$

as $\varepsilon \rightarrow 0$. By Schwartz's lemma, E^ε is a sequence of measures; by Murat's lemma [20], E^ε is uniformly bounded measure sequence in the measure space. This implies that

$$E^\varepsilon \quad \text{is compact in } W_{loc}^{-1,p}(\mathbb{R}_+^2) \quad \text{for any } p \in (1, 2). \quad (3.9)$$

On the other hand, since the vector field sequence

$$(|\rho^\varepsilon(t, x) - m_{\gamma(s,y)}^{\pm,\varepsilon}(x)|, \text{sgn}(\rho^\varepsilon(t, x) - m_{\gamma(s,y)}^{\pm,\varepsilon}(x)) (F_\varepsilon(x, \rho^\varepsilon(t, x)) - \gamma(s, y)))$$

is uniformly bounded in ε for any fixed (s, y) , it follows that

$$E^\varepsilon \quad \text{is bounded in } W_{loc}^{-1,\infty}(\mathbb{R}_+^2). \quad (3.10)$$

With (3.9)–(3.10), we obtain by a compact interpolation theorem in [4, 11] that

$$E^\varepsilon \quad \text{is compact in } H_{loc}^{-1}(\mathbb{R}_+^2). \quad (3.11)$$

On the other hand,

$$\partial_t \rho^\varepsilon + \partial_x F_\varepsilon(x, \rho^\varepsilon) = 0 \quad \text{which is automatically compact in } H_{loc}^{-1}(\mathbb{R}_+^2). \quad (3.12)$$

Moreover, since $\hat{\rho}^\varepsilon(s, y, x)$ strongly converges a.e., then we find that, as $\varepsilon \rightarrow 0$,

$$\begin{aligned} \eta_1^\varepsilon(\rho^\varepsilon, t, x, s, y) &:= |\rho^\varepsilon(t, x) - \hat{\rho}^\varepsilon(s, y, x)| \\ &\stackrel{*}{\rightharpoonup} \langle \nu_{t,x}(k); |k - \hat{\rho}(s, y, x)| \rangle \\ &=: \langle \nu_{t,x}; \eta_1(k, t, x, s, y) \rangle, \\ q_1^\varepsilon(\rho^\varepsilon, t, x, s, y) &:= \text{sgn}(\rho^\varepsilon(t, x) - \hat{\rho}^\varepsilon(s, y, x)) (F_\varepsilon(x, \rho^\varepsilon) - \gamma(s, y)) \\ &\stackrel{*}{\rightharpoonup} \langle \nu_{t,x}(k); \text{sgn}(k - \hat{\rho}(s, y, x)) (F(x, k) - \gamma(s, y)) \rangle \\ &=: \langle \nu_{t,x}; q_1(k, t, x, s, y) \rangle, \\ \eta_2^\varepsilon(\rho^\varepsilon(t, x)) &:= \rho^\varepsilon(t, x) \\ &\stackrel{*}{\rightharpoonup} \langle \nu_{t,x}(k); k \rangle = \rho(t, x) \\ &=: \langle \nu_{t,x}; \eta_2(k) \rangle, \\ q_2^\varepsilon(\rho^\varepsilon(t, x), x) &:= F_\varepsilon(x, \rho^\varepsilon) \\ &\stackrel{*}{\rightharpoonup} \langle \nu_{t,x}(k); F(x, k) \rangle \\ &=: \langle \nu_{t,x}; q_2(k, x) \rangle, \end{aligned} \quad (3.13)$$

and

$$\left| \begin{array}{cc} \eta_1(\rho^\varepsilon(t, x), s, y, x) & q_1(\rho^\varepsilon(t, x), s, y, x) \\ \eta_2(\rho^\varepsilon(t, x)) & q_2(\rho^\varepsilon(t, x), x) \end{array} \right| \stackrel{*}{\rightharpoonup} \left\langle \nu_{t,x}; \left| \begin{array}{cc} \eta_1(k, s, y, x) & q_1(k, s, y, x) \\ \eta_2(k) & q_2(k, x) \end{array} \right| \right\rangle, \quad (3.14)$$

where

$$\begin{aligned} (\eta_1(k, t, x, s, y), q_1(k, t, x, s, y)) &= (|k - \hat{\rho}(s, y, x)|, \operatorname{sgn}(k - \hat{\rho}(s, y, x))) (F(x, k) - \gamma(s, y)), \\ (\eta_2(k), q_2(k, x)) &= (k, F(x, k)). \end{aligned}$$

Together (3.11)–(3.12) with (3.13)–(3.14), we apply the Div-Curl lemma (see Tartar [22] and Murat [19]) to obtain

$$\left\langle \nu_{t,x}; \begin{vmatrix} \eta_1(k, s, y, x) & q_1(k, s, y, x) \\ \eta_2(k) & q_2(k, x) \end{vmatrix} \right\rangle = \begin{vmatrix} \langle \nu_{t,x}; \eta_1(k, s, y, x) \rangle & \langle \nu_{t,x}; q_1(k, s, y, x) \rangle \\ \langle \nu_{t,x}; \eta_2(k) \rangle & \langle \nu_{t,x}; q_2(k, x) \rangle \end{vmatrix}$$

for all $(s, y), (t, x) \in \mathbb{R} \setminus \mathcal{M}$ with \mathcal{M} a set of measure zero in \mathbb{R}_+^2 . Thus, we have

$$\begin{aligned} &\langle \nu_{t,x}; |k - \hat{\rho}(s, y, x)| F(x, k) - k \operatorname{sgn}(k - \hat{\rho}(s, y, x)) (F(x, k) - \gamma(s, y)) \rangle \\ &= \langle \nu_{t,x}; |k - \hat{\rho}(s, y, x)| \rangle \langle \nu_{t,x}; F(x, k) \rangle - \langle \nu_{t,x}; k \rangle \langle \nu_{t,x}; \operatorname{sgn}(k - \hat{\rho}(s, y, x)) (F(x, k) - \gamma(s, y)) \rangle. \end{aligned}$$

Equivalently, we have

$$\begin{aligned} &\langle \nu_{t,x}; |k - \hat{\rho}(s, y, x)| (F(x, k) - \langle \nu_{t,x}; F(x, k) \rangle) \rangle \\ &- \langle \nu_{t,x}; (k - \rho(t, x)) \operatorname{sgn}(k - \hat{\rho}(s, y, x)) (F(x, k) - F(x, \rho(t, x))) \rangle = 0. \end{aligned}$$

Since this is true for all (s, y) and (t, x) except on a set \mathcal{M} of measure zero, we then choose $(s, y) = (t, x)$ for $(t, x) \in \mathbb{R} \setminus \mathcal{M}$ to obtain

$$\begin{aligned} &\langle \nu_{t,x}; |k - \rho(t, x)| (F(x, k) - \langle \nu_{t,x}; F(x, k) \rangle) \rangle \\ &- \langle \nu_{t,x}; (k - \rho(t, x)) \operatorname{sgn}(k - \rho(t, x)) (F(x, k) - F(x, \rho(t, x))) \rangle = 0, \end{aligned}$$

that is,

$$\langle \nu_{t,x}; |k - \rho(t, x)| (F(x, \rho(t, x)) - \langle \nu_{t,x}; F(x, k) \rangle) \rangle = 0. \quad (3.15)$$

There are two possibilities:

When $\langle \nu_{t,x}; |k - \rho(t, x)| \rangle = 0$, then we have $\nu_{t,x}(k) = \delta_{\rho(t, x)}(k)$.

When $\langle \nu_{t,x}; F(x, k) \rangle - F(x, \rho(t, x)) = 0$, we note that

$$\begin{aligned} \langle \nu_{t,x}; F(x, k) \rangle - F(x, \rho(t, x)) &= \langle \nu_{t,x}; F(x, k) - F(x, \rho(t, x)) \rangle \\ &= \langle \nu_{t,x}; F_\rho(x, \rho)(k - \rho) + \frac{1}{2} F_{\rho\rho}(x, \xi)(k - \rho)^2 \rangle \\ &= F_\rho(x, \rho) \langle \nu_{t,x}; k - \rho \rangle + \frac{1}{2} \langle \nu_{t,x}; F_{\rho\rho}(x, \xi)(k - \rho)^2 \rangle \\ &= \frac{1}{2} \langle \nu_{t,x}; \int_0^1 \theta F_{\rho\rho}(x, \theta\rho + (1 - \theta)k) d\theta (k - \rho)^2 \rangle. \end{aligned}$$

Since $F(x, \rho)$ is strictly convex or concave in ρ , we conclude

$$\nu_{t,x}(k) = \delta_{\rho(t, x)}(k) \quad \text{for } (t, x) \text{ a.e.} \quad (3.16)$$

Therefore, we have

$$\rho^\varepsilon(t, x) \rightarrow \rho(t, x) \quad \text{a.e. as } \varepsilon \rightarrow 0.$$

Since the limit is unique via the uniqueness result in [1], the whole sequence $\rho^\varepsilon(t, x)$ strongly converges to $\rho(t, x)$ a.e. It is easy to check that $\rho(t, x)$ is the unique entropy solution of the Cauchy problem (1.1)–(1.2) in the sense of Definition 2.1. \square

Remark 3.2. The conditions on the flux function $F(x, \rho)$ in Theorem 3.1 for the non-monotone case can be relaxed as follows: $F(x, \rho)$ satisfies (H1)–(H3) and is convex or concave with

$$\mathcal{L}^1\{\rho : F_{\rho\rho}(x, \rho) = 0\} = 0 \quad \text{for a.e. } x \in \mathbb{R},$$

where \mathcal{L}^1 is the one-dimensional Lebesgue measure.

4. HYDRODYNAMIC LIMIT OF A ZERO RANGE PROCESS WITH DISCONTINUOUS SPEED-PARAMETER

In Section 2, we have established a compactness framework for approximate solutions via the reduction of measure-valued entropy solutions of (1.1)–(1.2) in the sense of Definition 2.1. In this section we focus on a microscopic particle system for a Zero Range Process (ZRP) with discontinuous speed-parameter $\lambda(x)$. We apply the compactness framework to show the hydrodynamic limit for the particle system, when the distance between particles tend to zero, to the unique entropy solution of the Cauchy problem

$$\partial_t \rho + \partial_x (\lambda(x)h(\rho)) = 0 \quad (4.1)$$

and with initial data:

$$\rho|_{t=0} = \rho_0(x) \geq 0, \quad (4.2)$$

where $h(\rho)$ is a monotone function of ρ , and $\lambda(x)$ is continuous in $x \in \mathbb{R}$ with $0 < \lambda_1 \leq \lambda(x) \leq \lambda_2 < \infty$ for some constants λ_1 and λ_2 , except on a closed set \mathcal{N} of measure zero. Then $m_\alpha^+ = m_\alpha^- := m_\alpha$ for $\alpha \in [0, \infty)$.

Rezakhanlou in [21] first established the hydrodynamic limit of the processus des misanthropes (PdM) with constant speed-parameter. Covert-Rezakhanlou [10] provided a proof of the hydrodynamic limit of a PdM with nonconstant continuous speed-parameter λ . In both proofs, the most important step is to show an entropy inequality at microscopic level, which then implies the (macroscopic) Kruzkov entropy inequality, when the distance between particles tends to zero, and thereby implies the uniqueness of limit points. In this section, we generalize this to the case when the speed-parameter λ has jumps for the attractive Zero Range Process (ZRP). In §4.1, we analyze some properties of the ZRP. In §4.2, we prove the one-dimensional microscopic entropy inequality letting $\varepsilon = \varepsilon(N) = N^{-\sigma}$, $\sigma \in (0, 1)$, for a ZRP with discontinuous speed-parameter as $N \rightarrow \infty$. In §4.3, we show the existence of measure-valued solutions via the microscopic entropy inequality and how inequality (2.3) follows.

4.1. Some properties of the microscopic interacting particle system. We consider a system of particles with conserved total mass and evolving on a one-dimensional lattice \mathbb{Z} according to a Markovian law. With the Euler scaling factor N , the microscopic particle density is expected to converge to a deterministic limit as $N \rightarrow \infty$, which is characterized by a solution of a conservation law. Under the Euler scaling, $\frac{1}{N}$ represents the distance between sites. Obviously we have two space scales: The discrete lattice \mathbb{Z} as embedded in \mathbb{R} with “vertices” $\frac{u}{N}$ and $u \in \mathbb{Z}$. In this way, the distances between particles tend to zero if N increases to infinity. Sites of the microscopic scale \mathbb{Z} are denoted by the letters u, v and correspond to the points $\frac{u}{N}, \frac{v}{N}$ in the macroscopic scale \mathbb{R} . Points of the macroscopic space scale \mathbb{R} are denoted by the letters x, y and correspond to the sites $[xN], [yN]$ in the microscopic space scale, where $[z]$ is the integer part of z . We denote by $\eta_t(u)$ the number of particles at time $t > 0$ at site u . Then the vector $\eta_t = (\eta_t(u) : u \in \mathbb{Z})$ is called a configuration at time t with configuration space $\mathbb{N}^{\mathbb{Z}}$.

In general, the ZRP can be described as follows: Infinitely many indistinguishable particles are distributed on a 1-dimensional lattice. Any site of the lattice may be occupied by a finite number of particles. Associated to a given site u there is an exponential clock with rate $\lambda_\varepsilon(\frac{u}{N})g(\eta(u))$ depending on the macroscopic spatial coordinates. Each time the clock rings on the site u , one of the particles jumps to the site v chosen with probability $p(u, v)$. The elementary transition probabilities $p: \mathbb{Z} \mapsto [0, 1]$ are supposed to be

- (i) translation invariant: $p(x, y) = p(0, y - x) =: p(y - x)$;
- (ii) normalized: $\sum_y p(x, y) = 1$, $p(x, x) = 0$;
- (iii) assumed to be of finite range: $p(x, y) = 0$ for $|y - x|$ sufficiently large;
- (iv) irreducible: $p(0, 1) > 0$.

Without loss of generality, we assume that $\sum_z p(z)z = \gamma = 1$; otherwise, for $\gamma \neq 1$, we replace the function $h(\rho)$ by $h(\rho)/\gamma$ in the following argument. The rate $g : \mathbb{N} \rightarrow \mathbb{R}_+$ is a positive, nondecreasing function with $g(0) = 0$, $g(+\infty) = +\infty$, and

$$\frac{g(k)}{k^2} \rightarrow 0 \quad \text{as } k \rightarrow \infty. \quad (4.3)$$

With this description, the Markov process η_t is generated by

$$NL_\varepsilon^N f(\eta) = N \sum_{u,v} \lambda_\varepsilon\left(\frac{u}{N}\right) g(\eta(u)) p(v-u) (f(\eta^{u,v}) - f(\eta)). \quad (4.4)$$

Here N comes from the Euler scaling factor speeding the generator, thus η_t denotes a configuration on which this speeded generator NL_ε^N has acted for time t , and $\eta^{u,v}$ represents the configuration η where one particle jumped from u to v :

$$\eta^{u,v}(w) = \begin{cases} \eta(w) & \text{if } w \neq u, v, \\ \eta(u) - 1 & \text{if } w = u, \\ \eta(v) + 1 & \text{if } w = v. \end{cases}$$

For any $\varepsilon = \varepsilon(N) > 0$ and for any constant $\alpha \geq 0$, we define a product measure given by

$$\tilde{\nu}_\alpha^N(\eta) := \prod_u \frac{1}{Z(\alpha/\lambda_\varepsilon(\frac{u}{N}))} \frac{\alpha^{\eta(u)}}{(\lambda_\varepsilon(\frac{u}{N}))^{\eta(u)} g(\eta(u))!} := \prod_u \tilde{\nu}_\alpha^N(\eta(u)), \quad (4.5)$$

where Z is a partition function equal to

$$Z\left(\frac{\alpha}{\lambda_\varepsilon(\frac{u}{N})}\right) = \sum_{n=0}^{\infty} \frac{\alpha^n}{(\lambda_\varepsilon(\frac{u}{N}))^n g(n)!}. \quad (4.6)$$

Then the expected value of the occupation variable $\eta(u)$ is equal to

$$E_{\tilde{\nu}_\alpha^N}[\eta(u)] = \frac{\alpha}{\lambda_\varepsilon(\frac{u}{N})} \frac{Z'(\frac{\alpha}{\lambda_\varepsilon(\frac{u}{N})})}{Z(\frac{\alpha}{\lambda_\varepsilon(\frac{u}{N})})} := R\left(\frac{\alpha}{\lambda_\varepsilon(\frac{u}{N})}\right).$$

Now let h be the inverse function of R to obtain

$$h\left(R\left(\frac{\alpha}{\lambda_\varepsilon(\frac{u}{N})}\right)\right) = \frac{\alpha}{\lambda_\varepsilon(\frac{u}{N})} \Rightarrow \lambda_\varepsilon\left(\frac{u}{N}\right) h\left(E_{\tilde{\nu}_\alpha^N}[\eta(u)]\right) = \alpha \Leftrightarrow E_{\tilde{\nu}_\alpha^N}[\eta(u)] = m_\alpha\left(\frac{u}{N}\right),$$

where m_α is a steady-state solution to

$$\partial_t \rho + \partial_x (\lambda_\varepsilon(x) h(\rho)) = 0. \quad (4.7)$$

Furthermore, it follows that

$$E_{\tilde{\nu}_\alpha^N}[g(\eta(u))] = h\left(m_\alpha\left(\frac{u}{N}\right)\right).$$

From now on, we set

$$\mu_{m_\alpha}^N(\eta) = \prod_u \nu_{m_\alpha(\frac{u}{N})}(\eta(u)) := \prod_u \tilde{\nu}_{\lambda_\varepsilon(\frac{u}{N}) h(m_\alpha(\frac{u}{N}))}^N(\eta(u)). \quad (4.8)$$

The important attribute of the ZRP with nonconstant speed-parameter is that the *product* measure $\mu_{m_\alpha}^N(\eta)$ is invariant under the generator NL_ε^N , i.e.,

$$\int L_\varepsilon^N(f(\eta)) d\mu_{m_\alpha}^N(\eta) = 0. \quad (4.9)$$

As a reasonable initial distribution, we choose *the product measure* $\mu_0^N(\eta)$ associated to a bounded density profile defined as follows: For a bounded density profile $\rho_0 \geq 0$, the probability that particles at time $t = 0$ are distributed with configuration η is equal to

$$\mu_0^N(\eta) := \prod_u \frac{1}{Z(h(\rho_{u,N})/\lambda_\varepsilon(\frac{u}{N}))} \frac{(h(\rho_{u,N}))^{\eta(u)}}{(\lambda_\varepsilon(\frac{u}{N}))^{\eta(u)} g(\eta(u))!}, \quad (4.10)$$

where $\rho_{u,N} \geq 0$ is a sequence satisfying $\lim_{N \rightarrow \infty} \int |\rho_{[Nx],N} - \rho_0(x)| dx = 0$ for $[Nx]$ as the integer part of Nx . With this definition, we say that a sequence of probability measures μ^N is associated to a density profile $\rho \geq 0$ if

$$\lim_{N \rightarrow \infty} \langle \mu^N(\eta); \left| \frac{1}{N} \sum_u J(\frac{u}{N}) \eta(u) - \int J(x) \rho(x) dx \right| \rangle = 0 \quad \text{for every test function } J.$$

Furthermore, let

$$\mu_t^N = S_t^N * \mu_0^N, \quad (4.11)$$

where $S_t^N = e^{tNL_\varepsilon^N}$ is the semigroup corresponding to the generator NL_ε^N . Then the attractiveness for two initial measures $\mu_{\rho_0}^N$ and $\mu_{\omega_0}^N$ with profiles ρ_0 and ω_0 , respectively, implies that

$$\mu_{\rho_0}^N \leq \mu_{\omega_0}^N \Rightarrow \mu_{\rho_t}^N \leq \mu_{\omega_t}^N$$

is satisfied by the assumption that g is a nondecreasing function. Moreover, it is easy to prove that $\mu_{\rho_0}^N \leq \mu_{\omega_0}^N$ if $\rho_0 \leq \omega_0$. It then follows by attractiveness that, for any constant α such that $m_\alpha(x) \geq \rho_0(x)$, we obtain that the inequality $\mu_0^N \leq \mu_{m_\alpha}^N$ implies

$$S_t^N \mu_0^N \leq S_t^N \mu_{m_\alpha}^N = \mu_{m_\alpha}^N. \quad (4.12)$$

Since our initial distribution has a bounded density profile, then the density profile remains bounded at later time t .

The goal in proving the hydrodynamic limit of a ZRP is that, if we start from a configuration η_0 distributed with an initial measure μ_0^N associated to the bounded density profile ρ_0 , then the configuration η_t at later time t is distributed with the measure μ_t^N defined by (4.11) and having density profile $\rho(t, \cdot)$, where ρ is the solution of the Cauchy problem (4.1)–(4.2) in the sense of Definition 2.1. In other words, our main theorem in this section is the following.

Theorem 4.1 (Hydrodynamic limit of an attractive ZRP with discontinuous speed-parameter). *Let η_t be an attractive ZRP with (4.3) initially distributed by the measure μ_0^N associated to a bounded density profile $\rho_0 : \mathbb{R}_+^2 \mapsto \mathbb{R}_+$ as defined in (4.10). Let $\varepsilon = \varepsilon(N) = N^{-\sigma}$, $\sigma \in (0, 1)$. Then, at later time t ,*

$$\lim_{N \rightarrow \infty} \langle \mu_t^N(\eta); \left| \frac{1}{N} \sum_u J(\frac{u}{N}) \eta_t(u) - \int J(x) \rho(t, x) dx \right| \rangle = 0 \quad (4.13)$$

for any test function $J : \mathbb{R}_+^2 \rightarrow \mathbb{R}$, where ρ is the unique solution of the Cauchy problem (4.1)–(4.2) in the sense of Definition 2.1.

To achieve this, we have to establish an entropy inequality in the sense of Definition 2.1 at microscopic level. This will be done in §4.2 by using the scaling relation $\varepsilon = \varepsilon(N) = N^{-\sigma}$, $\sigma \in (0, 1)$. Associated to each configuration η_t , we may define the empirical measure viewed as a random measure on \mathbb{R} by

$$\chi_t^N(x) := \frac{1}{N} \sum_u \eta_t(u) \delta_{\frac{u}{N}}(x). \quad (4.14)$$

Then $\langle \chi_t^N(\cdot), J(\cdot) \rangle = \frac{1}{N} \sum_u J(\frac{u}{N}) \eta_t(u)$, and we can rewrite (4.13) by

$$\lim_{N \rightarrow \infty} \langle \mu_t^N(\eta); |\langle \chi_t^N(\cdot), J(\cdot) \rangle - \int J(x) \rho(t, x) dx| \rangle = 0. \quad (4.15)$$

4.2. The entropy inequality at microscopic level. The following proposition is essential towards the hydrodynamic limit.

Proposition 4.1 (Entropy inequality at microscopic level for $\varepsilon = N^{-\sigma}$ with $\sigma \in (0, 1)$ as $N \rightarrow \infty$). *Let m_α^ε be the steady-state solutions of (3.2) as defined in (1.3) with $F_\varepsilon(x, \rho) = \lambda_\varepsilon(x)h(\rho)$. Let η_t be the ZRP generated by $N\bar{L}_\varepsilon^N$ defined by (4.4) and initially distributed by the measure μ_0^N defined by (4.10). Let $\eta^l(u)$ be the average density of particles in large microscopic boxes of size $2l+1$ and centered at u :*

$$\eta^l(u) := \frac{1}{2l+1} \sum_{|u-v| \leq l} \eta(v).$$

Then, for every test function $J : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$,

$$\begin{aligned} \lim_{l \rightarrow \infty} \lim_{N \rightarrow \infty} \mu_t^N \left\{ \int_0^t \frac{1}{N} \sum_u \left(\partial_s J(s, \frac{u}{N}) \left| \eta_s^l(u) - m_\alpha^\varepsilon\left(\frac{u}{N}\right) \right| + \partial_x J(s, \frac{u}{N}) \left| \lambda_\varepsilon\left(\frac{u}{N}\right) h(\eta_s^l(u)) - \alpha \right| \right) ds \right. \\ \left. + \frac{1}{N} \sum_u J(0, \frac{u}{N}) \left| \eta_0^l(u) - m_\alpha^\varepsilon\left(\frac{u}{N}\right) \right| \geq -\delta \right\} = 1. \end{aligned} \quad (4.16)$$

Inequality (4.16) is the entropy inequality (2.3) with ρ replaced by the average density of particles in the microscopic boxes of length $2l+1$. To prove the microscopic entropy inequality, we consider the coupled process (η_t, ξ_t) generated by $N\bar{L}_\varepsilon^N$, where \bar{L}_ε^N is defined by

$$\begin{aligned} \bar{L}_\varepsilon^N f(\eta, \xi) &= \sum_{u,v} p(v-u) \lambda_\varepsilon\left(\frac{u}{N}\right) \min\{g(\eta(u)), g(\xi(u))\} (f(\eta^{u,v}, \xi^{u,v}) - f(\eta, \xi)) \\ &+ \sum_{u,v} p(v-u) \lambda_\varepsilon\left(\frac{u}{N}\right) \{g(\eta(u)) - g(\xi(u))\}_+ (f(\eta^{u,v}, \xi) - f(\eta, \xi)) \\ &+ \sum_{u,v} p(v-u) \lambda_\varepsilon\left(\frac{u}{N}\right) \{g(\xi(u)) - g(\eta(u))\}_+ (f(\eta, \xi^{u,v}) - f(\eta, \xi)). \end{aligned} \quad (4.17)$$

Furthermore, denote the initial distribution of (η_t, ξ_t) by $\bar{\mu}_0^N = \mu_0^N \times \mu_{m_\alpha^\varepsilon}^N$, where μ_0^N is the initial measure with density profile ρ_0 defined by (4.10) and $\mu_{m_\alpha^\varepsilon}^N$ denotes the invariant measure as defined in (4.8).

Then, to prove Proposition 4.1, it suffices to prove the following proposition.

Proposition 4.2. *Let (η_t, ξ_t) be the coupled process, starting from $\bar{\mu}_0^N$, generated by $N\bar{L}_\varepsilon^N$ as defined by (4.17). Let $\bar{\mu}_t^N = \bar{S}_t^N * \bar{\mu}_0^N$, where \bar{S}_t^N is the semigroup corresponding to the generator $N\bar{L}_\varepsilon^N$. Then, for every test function $J : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$ and every $\varepsilon = N^{-\sigma}$ with $\sigma \in (0, 1)$,*

$$\begin{aligned} \lim_{l \rightarrow \infty} \lim_{N \rightarrow \infty} \bar{\mu}_t^N \left\{ \int_0^T \frac{1}{N} \sum_u \left\{ \partial_s J(s, \frac{u}{N}) \left| \eta_s^l(u) - \xi_s^l(u) \right| + \partial_x J(s, \frac{u}{N}) \lambda_\varepsilon\left(\frac{u}{N}\right) \left| h(\eta_s^l(u)) - h(\xi_s^l(u)) \right| \right\} ds \right. \\ \left. + \frac{1}{N} \sum_u J(0, \frac{u}{N}) \left| \eta_0^l(u) - \xi_0^l(u) \right| \geq -\delta \right\} = 1. \end{aligned}$$

Recall that a microscopic entropy inequality leading to the Kruzkov entropy inequality has been proved in [10] for the process of PdM with nonconstant but continuous speed-parameter λ_ε . Since there does not exist an invariant product measure for a PdM in general such that $E_{\mu_{m_\alpha^\varepsilon}^N}[\xi(u)] = m_\alpha^\varepsilon\left(\frac{u}{N}\right)$, to replace the process ξ by the process $m_\alpha^\varepsilon\left(\frac{u}{N}\right)$, one has to apply the relative entropy method of Yau [23].

In our case of a space-dependent ZRP, the invariant product measure is available so that we can approximate the steady-state solution m_α^ε by a process ξ distributed by the invariant measure $\mu_{m_\alpha^\varepsilon}^N$ for any $\alpha \in (0, \infty)$. Then, Proposition 4.1 indeed directly follows from Proposition 4.2.

4.3. Proof of Proposition 4.2. We split the proof in three steps.

Step 1: Lower bound for the martingale. For a test function J with compact support in \mathbb{R}_+^2 , define by M_t^J the martingale vanishing at time $t = 0$:

$$\begin{aligned} M_t^J &= \frac{1}{N} \sum_u J(t, \frac{u}{N}) |\eta_t(u) - \xi_t(u)| - \frac{1}{N} \sum_u J(0, \frac{u}{N}) |\eta_0(u) - \xi_0(u)| \\ &\quad - \int_0^t (\partial_s + N\bar{L}_\varepsilon^N) \left(\frac{1}{N} \sum_u J(s, \frac{u}{N}) |\eta_s(u) - \xi_s(u)| \right) ds. \end{aligned}$$

Since J has compact support, then, for t large enough,

$$M_t^J = -\frac{1}{N} \sum_u J(0, \frac{u}{N}) |\eta_0(u) - \xi_0(u)| - \int_0^t (\partial_s + N\bar{L}_\varepsilon^N) \left(\frac{1}{N} \sum_u J(s, \frac{u}{N}) |\eta_s(u) - \xi_s(u)| \right) ds.$$

We now calculate

$$\begin{aligned} &\bar{L}_\varepsilon^N |\eta(u) - \xi(u)| \\ &= \sum_{v,w} p(w-v) \lambda_\varepsilon \left(\frac{v}{N} \right) \left\{ \min\{g(\eta(v)), g(\xi(v))\} (|\eta^{v,w}(u) - \xi^{v,w}(u)| - |\eta(u) - \xi(u)|) \right. \\ &\quad \left. + \{g(\eta(v)) - g(\xi(v))\}_+ (|\eta^{v,w}(u) - \xi(u)| - |\eta(u) - \xi(u)|) \right. \\ &\quad \left. + \{g(\xi(v)) - g(\eta(v))\}_+ (|\eta(u) - \xi^{v,w}(u)| - |\eta(u) - \xi(u)|) \right\} \\ &= \sum_v (1 - G_{u,v}(\eta, \xi)) \left(-p(v-u) \lambda_\varepsilon \left(\frac{u}{N} \right) |g(\eta(u)) - g(\xi(u))| + p(u-v) \lambda_\varepsilon \left(\frac{v}{N} \right) |g(\eta(v)) - g(\xi(v))| \right) \\ &\quad - \sum_v G_{u,v}(\eta, \xi) \left(p(v-u) \lambda_\varepsilon \left(\frac{u}{N} \right) |g(\eta(u)) - g(\xi(u))| + p(u-v) \lambda_\varepsilon \left(\frac{v}{N} \right) |g(\eta(v)) - g(\xi(v))| \right), \quad (4.18) \end{aligned}$$

where $G_{u,v}$ is the indicator function that equals to 1 if η and ξ are not ordered, i.e.,

$$G_{u,v}(\eta, \xi) = \mathbf{1} \{ \eta(u) < \xi(u); \eta(v) > \xi(v) \} + \mathbf{1} \{ \eta(u) > \xi(u); \eta(v) < \xi(v) \}.$$

Notice that the second sum is nonpositive. Therefore, plugging in the last expression in the martingale M_t^J and then interchange u and v in the last term, we can bound the martingale below by

$$\begin{aligned} &-\frac{1}{N} \sum_u J(0, \frac{u}{N}) |\eta_0(u) - \xi_0(u)| - \int_0^t \frac{1}{N} \sum_u \partial_s J(s, \frac{u}{N}) |\eta_s(u) - \xi_s(u)| ds \\ &+ \int_0^t \sum_{u,v} (J(s, \frac{u}{N}) - J(s, \frac{v}{N})) p(v-u) (1 - G_{u,v}(\eta_s, \xi_s)) \lambda_\varepsilon \left(\frac{u}{N} \right) |g(\eta_s(u)) - g(\xi_s(u))| ds. \end{aligned}$$

Since the transition probability p is of finite range, i.e. $p(z) = 0$ if $|z| > r$ for some r , then

$$\left(J(s, \frac{u}{N}) - J(s, \frac{v}{N}) \right) p(v-u) = -\frac{1}{N} (v-u) p(v-u) \partial_x J(s, \frac{u}{N}) + O\left(\frac{1}{N^2}\right).$$

With $v = u + y$, it then follows that the martingale is bounded below by

$$\begin{aligned} & - \int_0^t \frac{1}{N} \sum_u \left\{ \partial_s J(s, \frac{u}{N}) |\eta_s(u) - \xi_s(u)| \right. \\ & \quad \left. + \partial_x J(s, \frac{u}{N}) \lambda_\varepsilon(\frac{u}{N}) \tau_u \left(\sum_y yp(y)(1 - G_{0,y}) \right) |g(\eta_s(0)) - g(\xi_s(0))| \right\} ds \\ & - \frac{1}{N} \sum_u J(0, \frac{u}{N}) |\eta_0(u) - \xi_0(u)| + O\left(\frac{1}{N}\right). \end{aligned}$$

Step 2: We show

$$\lim_{N \rightarrow \infty} E_{\bar{\mu}_t^N} [(M_t^J)^2] = 0. \quad (4.19)$$

Recall that

$$N_t^J := (M_t^J)^2 - \int_0^t \left(N \bar{L}_\varepsilon^N (A^J(s, \eta, \xi))^2 - 2A^J(s, \eta, \xi) N \bar{L}_\varepsilon^N (A^J(s, \eta, \xi)) \right) ds$$

is a martingale vanishing at time $t = 0$, where A^J is defined by

$$A^J(t, \eta, \xi) = \frac{1}{N} \sum_u J(t, \frac{u}{N}) |\eta_t(u) - \xi_t(u)|.$$

Then, by definition, $E_{\bar{\mu}_s^N} [N_s^J] = 0$ for all $0 \leq s \leq t$. Thus, it suffices to show that the expectation of the integral term of N_t^J converges to zero as $N \rightarrow \infty$. In order to prove this, we first find that, by careful calculation,

$$\begin{aligned} & N \bar{L}_\varepsilon^N (A^J(s, \eta, \xi))^2 - 2NA^J(s, \eta, \xi) \bar{L}_\varepsilon^N (A^J(s, \eta, \xi)) \\ & = \sum_{v,w} p(w-v) N \lambda_\varepsilon\left(\frac{v}{N}\right) \left\{ |g(\eta_s(v)) - g(\xi_s(v))| \frac{1}{N^2} (1 - G_{v,w}(\eta_s, \xi_s)) \left(J(s, \frac{w}{N}) - J(s, \frac{v}{N}) \right)^2 \right. \\ & \quad \left. + |g(\xi_s(v)) - g(\eta_s(v))| \frac{1}{N^2} G_{v,w}(\eta_s, \xi_s) \left(J(s, \frac{v}{N}) + J(s, \frac{w}{N}) \right)^2 \right\}. \end{aligned}$$

Since J is a smooth function, the first term of this expression is less $\mathcal{O}\left(\frac{g(CN)}{N^2}\right)$ for some constant C depending on the total initial mass and therefore converges to zero as $N \rightarrow \infty$ by (4.3). For the second term, we know that $(J(s, \frac{v}{N}) + J(s, \frac{w}{N}))^2 \leq 4 \|J\|_\infty^2$, which implies

$$\begin{aligned} & N \bar{L}_\varepsilon^N (A^J(s, \eta, \xi))^2 - 2NA^J(s, \eta, \xi) \bar{L}_\varepsilon^N (A^J(s, \eta, \xi)) \\ & = \mathcal{O}\left(\frac{g(CN)}{N^2}\right) + \frac{4 \|J\|_\infty^2}{N} \sum_{v,w} G_{v,w}(\eta_s, \xi_s) p(w-v) \lambda_\varepsilon\left(\frac{v}{N}\right) |g(\xi_s(v)) - g(\eta_s(v))|. \end{aligned}$$

Then, to conclude the proof of (4.19), it suffices to show

$$E_{\bar{\mu}_t^N} \left[\int_0^t \left(\sum_{v,w} G_{v,w}(\eta_s, \xi_s) p(w-v) \lambda_\varepsilon\left(\frac{v}{N}\right) |g(\xi_s(v)) - g(\eta_s(v))| \right) ds \right] = \mathcal{O}(1). \quad (4.20)$$

For this, we use the martingale M_t^J vanishing at 0 with $J \equiv 1$, that is,

$$M_t := \frac{1}{N} \sum_u |\eta_t(u) - \xi_t(u)| - \frac{1}{N} \sum_u |\eta_0(u) - \xi_0(u)| - \int_0^t \frac{1}{N} \sum_u N \bar{L}_\varepsilon^N |\eta_s(u) - \xi_s(u)| ds.$$

By (4.18), the integral term of the martingale is equal to

$$\int_0^t \frac{2}{N} \sum_{u,v} N G_{u,v}(\eta_s, \xi_s) p(v-u) \lambda_\varepsilon\left(\frac{u}{N}\right) |g(\eta_s(u)) - g(\xi_s(u))| ds,$$

by interchanging u and v in some terms. Then we find

$$\begin{aligned} & E_{\bar{\mu}_t^N} \left[\int_0^t 2 \sum_{u,v} G_{u,v}(\eta_s, \xi_s) p(v-u) \lambda_\varepsilon\left(\frac{u}{N}\right) |g(\eta_s(u)) - g(\xi_s(u))| ds \right] \\ &= E_{\bar{\mu}_t^N} \left[\int_0^t \frac{1}{N} \sum_u |\eta_0(u) - \xi_0(u)| ds \right] - E_{\bar{\mu}_t^N} \left[\int_0^t \frac{1}{N} \sum_u |\eta_t(u) - \xi_t(u)| ds \right] \\ &\leq E_{\bar{\mu}_t^N} \left[\int_0^t \frac{1}{N} \sum_u |\eta_0(u) - \xi_0(u)| ds \right]. \end{aligned}$$

Since we assumed that both marginals of $\bar{\mu}_t^N$ are bounded, (4.20) follows, which leads to (4.19).

With the result of Step 1 and (4.19) and using the Chebichev inequality, we obtain

$$\begin{aligned} & \bar{\mu}_t^N \left\{ \frac{1}{N} \sum_u J\left(0, \frac{u}{N}\right) |\eta_0(u) - \xi_0(u)| + \int_0^t \frac{1}{N} \sum_u \left\{ \partial_s J\left(s, \frac{u}{N}\right) |\eta_s(u) - \xi_s(u)| \right. \right. \\ & \quad \left. \left. + \partial_x J\left(s, \frac{u}{N}\right) \lambda_\varepsilon\left(\frac{u}{N}\right) \tau_u\left(\sum_y yp(y)(1 - G_{0,y})(\eta, \xi)\right) |g(\eta_s(0)) - g(\xi_s(0))| \right\} ds + O\left(\frac{1}{N}\right) < -\delta \right\} \\ & \leq \bar{\mu}_t^N \{M_t^J > \delta\} \leq \bar{\mu}_t^N \{|M_t^J| > \delta\} \leq \frac{1}{\delta^2} E_{\bar{\mu}_t^N} [(M_t^J)^2], \end{aligned} \quad (4.21)$$

which converges to 0 as $N \rightarrow \infty$, for all $\delta > 0$.

Step 3. We next use the following summation by parts formula: For any bounded function a of $\eta(\cdot)$ with $a(0) = 0$ and for any smooth test function $J : \mathbb{R} \rightarrow \mathbb{R}$, we obtain that, for any $L > 0$,

$$\frac{1}{N} \sum_{|u| \leq LN} J\left(\frac{u}{N}\right) a(\eta(u)) = \frac{1}{N} \frac{1}{(2l+1)} \sum_{|u| \leq LN} J\left(\frac{u}{N}\right) \sum_{|u-v| \leq l} a(\eta(v)) + \mathcal{O}\left(\frac{l \|J\|_{Lip}}{N}\right). \quad (4.22)$$

Since we restrict $\varepsilon = N^{-\sigma}$, $\sigma \in (0, 1)$, then $\|\lambda_\varepsilon\|_{Lip} \leq C/\varepsilon = CN^\sigma$ and $\mathcal{O}\left(\frac{l \|\lambda_\varepsilon\|_{Lip}}{N}\right) = \mathcal{O}\left(\frac{l}{N^{1-\sigma}}\right) \rightarrow 0$ as $N \rightarrow \infty$ so that we can use this summation by parts formula (4.22) to replace inequality (4.21) by

$$\begin{aligned} & \lim_{l \rightarrow \infty} \lim_{N \rightarrow \infty} \bar{\mu}_t^N \left\{ \frac{1}{N} \sum_u J\left(0, \frac{u}{N}\right) \frac{1}{2l+1} \sum_{|z-u| \leq l} |\eta_0(z) - \xi_0(z)| \right. \\ & \quad + \int_0^t \frac{1}{N} \sum_u \partial_s J\left(s, \frac{u}{N}\right) \frac{1}{2l+1} \sum_{|z-u| \leq l} |\eta_s(z) - \xi_s(z)| ds \\ & \quad + \int_0^t \frac{1}{N} \sum_u \partial_x J\left(s, \frac{u}{N}\right) \lambda_\varepsilon\left(\frac{u}{N}\right) \frac{1}{2l+1} \\ & \quad \quad \times \sum_{|z-u| \leq l} \tau_z\left(\sum_y yp(y)(1 - G_{0,y})(\eta_s, \xi_s)\right) |g(\eta_s(0)) - g(\xi_s(0))| ds < -\delta \left. \right\} = 0. \end{aligned} \quad (4.23)$$

Notice that, in (4.23), since J is a positive function, by the triangle inequality, we can remove the sum inside the absolute value in the first line. Following the same argument as in [10, 21] (also [9]), since we first set $\varepsilon = \frac{1}{N^\sigma}$, independent of $\lambda_\varepsilon(x)$, we can obtain the following *one block estimates*:

$$\lim_{l \rightarrow \infty} \lim_{N \rightarrow \infty} E_{\bar{\mu}_t^N} \left\{ \int_0^t \frac{1}{N} \sum_u \left| \frac{1}{2l+1} \sum_{|u-z| \leq l} |\eta_s(z) - \xi_s(z)| - |\eta_s^l(u) - \xi_s^l(u)| \right| ds \right\} = 0, \quad (4.24)$$

and

$$\lim_{l \rightarrow \infty} \lim_{N \rightarrow \infty} E_{\bar{\mu}_t^N} \left\{ \int_0^t \frac{1}{N} \sum_u \tau_u \left| \frac{1}{2l+1} \sum_{|z| \leq l} \tau_z \left(\sum_y yp(y)(1 - G_{0,y})(\eta_s, \xi_s) \right) \left| g(\eta_s(0)) - g(\xi_s(0)) \right| - \left| h(\eta_s^l(0)) - h(\xi_s^l(0)) \right| \right| ds \right\} = 0. \quad (4.25)$$

Combining (4.23) with (4.24)–(4.25), we complete the proof of Proposition 4.2.

4.4. Existence of measure-valued entropy solutions. In this section, we prove that Theorem 4.1 implies the existence of a measure-valued entropy solution associated to the configuration η_t . We recall the empirical measure $\chi_t^N(x)$ associated to a configuration η_t in (4.14). We define the Young measures associated to η_t as follows:

$$\pi_t^{N,l}(x, k) := \frac{1}{N} \sum_u \delta_{\frac{u}{N}}(x) \delta_{\eta_t^l(u)}(k), \quad (4.26)$$

which implies $\langle \pi_t^{N,l}; J \rangle = \frac{1}{N} \sum_u J(\frac{u}{N}, \eta_t^l(u))$ for any $J \in C_0(\mathbb{R} \times \mathbb{R}_+)$. If E is the configuration space, then these two measures are finite positive measures on E and, for any $J \in C_0(\mathbb{R})$, they are related by the formula

$$\langle \pi_t^{N,l}; kJ(x) \rangle \approx \langle \chi_t^N(\cdot); J(\cdot) \rangle. \quad (4.27)$$

Notice that, since there are jumps, the probability measure μ_t^N defined by (4.11) must be defined on the Skorohod space $D[(0, \infty), E]$, which is the space of right continuous functions with left limits taking values in E . Then, using the one to one correspondence between the configuration η_t and the empirical measure $\chi_t^N(\cdot)$, the law of χ^N with respect to μ_t^N will give us a probability measure Q^N on the Skorohod space $D[(0, \infty), \mathcal{M}_+(\mathbb{R})]$, for the space $\mathcal{M}_+(\mathbb{R})$ of finite positive measures on \mathbb{R} endowed with the weak topology.

In the same way, we can associate a probability measure $\tilde{Q}^{N,l}$ on the space $D[(0, \infty), \mathcal{M}_+(\mathbb{R}_+^2)]$. With these definitions, we can state the main theorem of this section as follows.

Theorem 4.2 (Law of large numbers for the Young measures). *Let $(\mu^N)_{N \geq 1}$ be a sequence of probability measures, as defined by (4.10), associated to a bounded density profile $\rho_0 : \mathbb{R} \rightarrow \mathbb{R}_+$. Then the sequence $\tilde{Q}^{N,l}$ converges, as $N \rightarrow \infty$ first and $l \rightarrow \infty$ second, to the probability measure \bar{Q} concentrated on the measure-valued entropy solution $\pi_{t,x}$ in the sense of Definition 2.2.*

Proof. In order to be allowed to take the limit points Q and \tilde{Q} of Q^N and $\tilde{Q}^{N,l}$ respectively, we must know that the sequences are tight (weakly relatively compact). If $Q^{N,l}$ is weakly relatively compact, we can take \tilde{Q}^l as a limit point if $N \rightarrow \infty$ for each l . Denote by \tilde{Q} a limit point of $\tilde{Q}^{N,l}$ if $N \rightarrow \infty$ first and $l \rightarrow \infty$ second. Therefore, the proof consists in two main steps: The first is to show that $\tilde{Q}^{N,l}$ is weakly relatively compact and the second is to show the uniqueness of limit points. The key point in the proof is that these can be achieved independent of the choice of mollification λ_ε of the discontinuous speed-parameter λ with our choice of the notion of measure-valued entropy solutions.

These can be achieved by following exactly the standard argument in [10, 21, 16] since it requires only the uniform boundedness of λ_ε in the proof. That is, we can conclude the following: Let μ_t^N be a measure defined by (4.11). Then

- (i) The sequence Q^N defined above is tight in $D[(0, \infty), \mathcal{M}_+(\mathbb{R})]$ and all its limit points Q are concentrated on weakly continuous paths $\chi(t, \cdot)$;
- (ii) Similarly, the sequence $\tilde{Q}^{N,l}$ is tight in $D[(0, \infty), \mathcal{M}_+(\mathbb{R} \times \mathbb{R}_+)]$ and all its limit points \tilde{Q} are concentrated on weakly continuous paths $\pi(t, \cdot, \cdot)$;

- (iii) For every $t \geq 0$, $\pi(t, x, k) := \pi_t(x, k)$ is absolutely continuous with respect to the Lebesgue measure on \mathbb{R} , \tilde{Q} a.s.. That is, \tilde{Q} a.s.

$$\pi_t(x, k) = \pi_{t,x}(k) \otimes dx; \quad (4.28)$$

- (iv) For every $t \in [0, T]$, $\pi_{t,x}(k)$ is compactly supported, that is, there exists $k_0 > 0$ such that

$$\pi_{t,x}([0, k_0]^c) = 0 \quad \forall (t, x) \in [0, T] \times \mathbb{R}.$$

- (v) $\pi_{t,x}$ is a measure-valued entropy solution in the sense of Definition 2.2 for any $\alpha \in [M_0, \infty)$, i.e.,

$$\partial_t \langle \pi_{t,x}; |k - m_\alpha(x)| \rangle + \partial_x \langle \pi_{t,x}; |h(k)\lambda(x) - \alpha| \rangle \leq 0 \quad (4.29)$$

on \mathbb{R}_+^2 in the sense of distributions for any $\alpha \in [M_0, \infty)$ or $\alpha \in (-\infty, M_0]$.

The last result follows from the entropy inequality at microscopic level in Theorem 4.1. Indeed, in terms of the Young measures, the expression (4.16) of Proposition 4.1:

$$\begin{aligned} \lim_{l \rightarrow \infty} \lim_{N \rightarrow \infty} \mu_t^N \left\{ \int_0^\infty \frac{1}{N} \sum_u \left\{ \partial_t H(t, \frac{u}{N}) \left| \eta_t^l(u) - m_\alpha\left(\frac{u}{N}\right) \right| \right. \right. \\ \left. \left. + \partial_x H(t, \frac{u}{N}) \left| \lambda\left(\frac{u}{N}\right) h(\eta_t^l(u)) - \alpha \right| \right\} dt \geq -\delta \right\} = 1 \end{aligned}$$

can be restated as

$$\begin{aligned} \lim_{l \rightarrow \infty} \lim_{N \rightarrow \infty} \tilde{Q}^{N,l} \left\{ \int_0^T \left(\langle \pi_t(x, k); |k - m_\alpha(x)| \partial_t H(t, x) \rangle \right. \right. \\ \left. \left. + \langle \pi_t(x, k); |\lambda(x)h(k) - \alpha| \partial_x H(t, x) \rangle \right) dt \geq -\delta \right\} = 1, \end{aligned}$$

for every smooth function $H : (0, T) \times \mathbb{R} \rightarrow \mathbb{R}_+$ with compact support, any $\alpha \in [M_0, \infty)$ or $\alpha \in (-\infty, M_0]$, and any $\delta > 0$. Since \tilde{Q} is a weak limit point concentrated on absolutely continuous measures and since we already proved that $\pi_{t,x}$ is concentrated on a compact set (and therefore the integrand is a bounded function), we obtain from the last expression that

$$\tilde{Q} \left\{ \int_0^T \int \left(\langle \pi_{t,x}; |k - m_\alpha(x)| \rangle \partial_t H(t, x) + \langle \pi_{t,x}; |\lambda(x)h(k) - \alpha| \rangle \partial_x H(t, x) \right) dx dt \geq -\delta \right\} = 1.$$

Letting $\delta \rightarrow 0$, we have that \tilde{Q} a.s. (4.29) holds on $(0, T) \times \mathbb{R}$ in the sense of distributions for every $\alpha \in [0, \infty)$. This proves the uniqueness of \tilde{Q} and thereby concludes the proof of Proposition 4.2. \square

Then Theorem 4.1 follows immediately from this result since the measure-valued entropy solution reduces to the Dirac mass concentrated on the unique entropy solution $\rho(t, x)$ of (4.1)–(4.2) as we noticed in §3.2.

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