On the Dirichlet problem for first order quasilinear equations on a manifold.

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Abstract

We study the Dirichlet problem for a first order quasilinear equation on a smooth manifold with boundary. Existence and uniqueness of a generalized entropy solution are established. The uniqueness is proved under some additional requirement on the field of coefficients. It is shown that generally the uniqueness fails. The non-uniqueness occurs because of presence of the characteristics not outgoing from the boundary (including closed ones). The existence is proved in general case. Moreover, we establish that among generalized entropy solutions laying in the ball $\|u\|_\infty \le R$ there exist the unique maximal and minimal solutions. To prove our results, we use the kinetic formulation similar to one by C. Imbert and J. Vovelle.

Introduction.

Let M be a n-dimensional C^2-smooth compact manifold with boundary $S = \partial M$. Thus, in a neighborhood of each point $x_0 \in M$ we can define the local coordinates $(x_1, \dots, x_n) = j(x)$, $x \in U$ corresponding to the chart (U, j, V), where U is a neighborhood of the point x_0 (coordinate neighborhood), V is an open subset of the half-space

$$\Pi = \{ \ x = (x_1, \dots, x_n) \in \mathbb{R}^n \ \mid \ x_1 \ge 0, x' = (x_2, \dots, x_n) \in \mathbb{R}^{n-1} \ \},$$

and $j : U \to V$ is a C^2-diffeomorphism. Clearly, boundary points of U and V must correspond each other under the diffeomorphism j: $j(U \cap S) = V \cap \partial\Pi$. Denote by $\mathrm{T}M$, T^*M the tangent and cotangent bundles on M, and by T_xM, T^*_xM the corresponding fibres of these bundles at a point $x \in M$. Any vector field $a = a(x) \in \mathrm{T}_xM$, being a section of the tangent bundle $\mathrm{T}M$, acts on smooth functions $f(x) \in C^1(M)$ as a first order differential operator represented in local coordinates $x_1, \dots, x_n$ as $a = a^i(x)\partial/\partial x_i$, i.e.

$$\langle a, f\rangle = a^i(x)\frac{\partial f(x)}{\partial x_i}.$$

Here and below repeated indexes indicate summation from 1 to n. We denote by $\bigwedge^1 M$ the space of C^1 vector fields on M.

Now suppose that $a(x,u)$ is a family of C^1 vector fields on M depending continuously on the real parameter u. We also assume that this field is uniformly bounded. This means that for some continuous Riemann metric g on M the norm $|a(x,u)| = ((a(x,u), a(x,u))_g)^{1/2} \le const$ for all $x \in M$, $u \in \mathbb{R}$. Here $(\cdot,\cdot)_g$ denotes a scalar product of vectors in $\mathrm{T}_x M$ generated by the metric tensor g. Since M is compact the latter property does not depend on the choice of the metric g. As easy to verify $a(x,u)$ is uniformly bounded if and only if for any $f(x) \in C^1(M)$

$$\sup_{x\in M, u\in\mathbb{R}} |\langle a(x,u), f(x)\rangle| < \infty.$$

The family $a(x,u)$ generates the first order quasilinear equation

$$\langle a(x,u), u\rangle = 0, \tag{1}$$

where $u = u(x)$ is an unknown function on M. Observe that in local coordinates $x_1, \dots, x_n$ equation (1) acquires the standard form

$$a^i(x,u)\frac{\partial u}{\partial x_i} = 0.$$

We will study the Dirichlet problem for equation (1) with boundary condition at the boundary S

$$u(x) = u_b(x), \quad u_b(x) \in L^\infty(S). \tag{2}$$

The Cauchy problem for the evolutionary equation

$$u_t + \langle a(x,u), u\rangle = 0, \tag{3}$$

$u = u(t,x)$, $0 < t < T \le +\infty$, $x \in M$, with initial data

$$u(0,x) = u_0(x) \in L^\infty(M) \tag{4}$$

was firstly studied in [17]. In [17] the manifold M was assumed to be closed (i.e. $S = \emptyset$). In this paper the Kruzhkov-like notion of generalized entropy solution (g.e.s.) of problem (3), (4) was introduced, existence and uniqueness of g.e.s. were proved. The proofs are based on reduction of the problem to some "classical" Cauchy problem in an Euclidean space $x \in \mathbb{R}^N$. Recently, in [4, 2] the Cauchy problem was studied in the case of conservative equation

$$u_t + \operatorname{div} \varphi(x, u) = 0 \tag{5}$$

on a Riemann manifold M. In this equation the divergence operator is determined by the metric on M. The vector field $x \to f(x, u)$ is supposed to be geometrically compatible that is $\operatorname{div}_x f(x, u) = 0\ \forall u \in \mathbb{R}$. In [4, 2] also the special Cauchy problem for equation $\operatorname{div} \varphi(x, u) = 0$ was studied with time-like flux vector on Lorentzian manifold.

We underline that in the present paper as well as in [17] the manifold M is not endowed with any additional structure such as a metric or pseudo-metric. In particular we can not a priori consider equations in the divergence form but only in quasilinear forms like (1), (3). Moreover, as we demonstrate later in Example 5 for equations in the divergence form the Dirichlet problem can be ill-posed.

The Cauchy problem (3), (4) turns out to be the particular case of the general Dirichlet problem. To our knowledge the Dirichlet problem for first order quasilinear equations have not been studied yet in its pure form while the initial-boundary value problem for evolutionary equations in an Euclidean domain is widely investigated since celebrated paper of Bardos, LeRoux and Nédélec [3]. In this paper existence of the strong trace of the solution at the boundary was assumed. Then the boundary condition can be written in the simple geometric form (see relation (40) below for the Dirichlet problem). The weak formulation of initial-boundary value problem, which does not require existence of strong traces, was later developed by F. Otto in [13]. In this presentation we will follow the Otto's formulation. To prove our main results we extend the kinetic approach developed for initial-boundary value problems by Imbert and Vovelle in [10].

In this paper we prove existence and uniqueness of a generalized entropy solution of (1), (2). For uniqueness some additional assumption is necessary, see condition (U) below. Generally, the uniqueness may fail. Moreover, under condition (U) we prove the comparison principle, see Theorem 6. The existence is proved without any additional assumptions. Actually we prove existence of maximal and minimal solutions of (1), (2), see Theorem 11.

§ 1. Preliminaires.

In order to define solutions of equation (1) in the distributional sense we have to rewrite this equation in the conservative form. It can be done in the terms of smooth measures on M, as in [17]. In this section we give some necessary notions. The smooth measure μ on M is a non-negative finite measure such that for any chart (U,j,V) the restriction $\mu|_U$ is absolutely continuous with respect to the image of the Lebesgue measure dx on V under the map j^{-1}: $\mu|_U=\omega(x)(j^{-1})^*dx$ (later on, we will use the shorter form $\mu=\omega(x)dx$), and the density $\omega(x)\in C^1(U)$, $\omega(x)>0$. Smooth measures are defined on the σ-algebra of Lebesgue sets on M consisting of subsets $A\subset M$ such that $j(A\cap U)$ is Lebesgue measurable on V for all charts (U,j,V). Analogously, we can define smooth measures on the boundary S. As is easy to see, any smooth measures μ_1,μ_2 are absolutely continuous with respect to each other: $\mu_2=\alpha(x)\mu_1$ where $\alpha(x)\in C^1(M)$, $\alpha(x)>0$. In particular, on the manifold M one can always specify a metric g, and this metric in standard way induces a smooth measure μ^g. If the metric tensor has the form $g=g_{ij}dx_idx_j$ in a local coordinates $x_1,\dots,x_n$ then the corresponding smooth measure μ^g is represented in these coordinates as $\mu^g=\sqrt{\det g_{ij}}dx$. Clearly, the metric g induces also the smooth measure μ_b^g on the boundary S.

The presence of the smooth measure μ on M allows one to define the divergence div^μ of a C^1 smooth vector field $a(x)$ by the identity (e.g. [17])

$$\int_M f(x)\operatorname{div}^\mu a(x)d\mu=-\int_M\langle a(x),f(x)\rangle d\mu \quad \forall f(x)\in C_0^1(M_0), \qquad (6)$$

where $C_0^1(M_0)$ is the space of functions from $C^1(M)$ with compact supports contained in $M_0=\operatorname{Int}M=M\setminus S$. Taking in (6) tests functions supported in a coordinate neighborhood, we readily find that in local coordinates $x_1,\dots,x_n$

$$\operatorname{div}^\mu a(x)=\frac{1}{\omega(x)}\frac{\partial\omega(x)a^i(x)}{\partial x_i}, \qquad (7)$$

where $\mu=\omega(x)dx$, $a(x)=a^i(x)\partial/\partial x_i$. Conversely, the right-hand side of (7) is independent of the choice of local coordinates (this follows from (6)), hence the function $\operatorname{div}^\mu a(x)$ is well-defined on the entire manifold M. From local representation (7) it directly follows the identity

$$\operatorname{div}^\mu(\alpha a)=\alpha\operatorname{div}^\mu a+\langle a,\alpha\rangle \quad \forall a=a(x)\in \bigwedge^1 M, \alpha=\alpha(x)\in C^1(M). \quad (8)$$

Observe also that if μ_1, μ_2 are two smooth measure and $\mu_2 = \alpha(x)\mu_1$ then the corresponding divergence operators are connected by the relation

$$\operatorname{div}^{\mu_2} a = \frac{1}{\alpha}\operatorname{div}^{\mu_1} \alpha a = \operatorname{div}^{\mu_1} a + \frac{1}{\alpha}\langle a, \alpha\rangle. \tag{9}$$

Now, let $\bar\mu = (\mu, \mu_b)$ be a pair of smooth measures on M and S respectively. Then the following analog of the integration by parts formula holds.

Theorem 1. *There exist a C^1 co-vector $n = n_{\bar\mu}$ defined on S (i.e. a C^1 section $x \to n(x) \in \mathrm{T}^*_x M$, $x \in S$) such that for each $a = a(x) \in \bigwedge^1 M$, $f = f(x) \in C^1(M)$*

$$\int_M \operatorname{div}^{\mu}(af)d\mu = \int_M \langle a, f\rangle d\mu + \int_M f \operatorname{div}^{\mu} a d\mu = \int_S f(x)\langle n_{\bar\mu}(x), a(x)\rangle d\mu_b. \tag{10}$$

Proof. Let (U, j, V) be a chart and $f(x) \in C^1_0(V)$. The letter means that $\operatorname{supp} f$ is compact and contained in V (observe that $\operatorname{supp} f$ may intersect $\partial\Pi$). Suppose that in the corresponding local coordinates $a = a^i(x)\partial/\partial x_i$, $\mu = \omega(x)dx$, $\mu_b = \omega_b(x')dx'$, here $x' \in V' = \{ (x_2, \ldots, x_n) \mid (0, x_2, \ldots, x_n) \in V \}$. Then, integration by parts yields

$$\int_M \langle a, f\rangle d\mu = \int_V a^i(x)\frac{\partial f(x)}{\partial x_i}\omega(x)dx = -\int_V \frac{f(x)}{\omega(x)}\frac{\partial}{\partial x_i}(\omega(x)a^i(x))\omega(x)dx - \int_{V'} f(0,x')a^1(0,x')\frac{\omega(0,x')}{\omega_b(x')}\omega_b(x')dx' = -\int_M f(x)\operatorname{div}^{\mu} a(x)d\mu + \int_S f(x)\langle n(x), a(x)\rangle d\mu_b,$$

where

$$\langle n(x), a\rangle = -a^1(0,x')\omega(0,x')/\omega_b(x'), \quad (0,x') = j(x),\ x \in S \cap V. \tag{11}$$

By the construction equality (10) holds for every test function $f(j(x)) \in C^1_0(U)$ and since the right-hand side of (10) does not depend on the choice of the local coordinate (U, j, V) we see that the value of $\langle n(x), a\rangle$ in (11) does not depend on it either. Therefore, equality (11) correctly defines the unique co-vector $n = n_{\bar\mu}(x)$ on $U \cap S$. Finally, since U is an arbitrary coordinate neighborhood, $n(x)$ is well-defined on the entire boundary S. From local representation (11) it follows that $n(x)$ is C^1-smooth. Recall that (10) holds for each test function $f(x)$ with support in a coordinate neighborhood. The case of general $f(x) \in C^1(M)$ is treated in standard way by using of a partition of unity. The proof is complete.

Remark 1. If $\bar{\mu}' = (\mu', \mu_b')$ is another pair of smooth measures then, as directly follows from local representation (11), $n_{\bar{\mu}'} = \frac{\alpha(x)}{\alpha_b(x)} n_{\bar{\mu}}$, $x \in S$, where the positive C^1 functions $\alpha(x), \alpha_b(x)$ are taken from the relations $\mu' = \alpha(x)\mu$, $\mu_b' = \alpha_b(x)\mu_b$. In particular, the "direction" of the co-vector $n(x)$ does not depend on the choice of the pair $\bar{\mu}$.

Remark 2. As is directly verified, in the case when measures μ, μ_b are generated by a metric g $\quad \langle n_{\bar{\mu}}(x), a\rangle = (\mathbf{n}(x), a)_g$, where $\mathbf{n}(x)$ is the outward normal vector to S at a point $x \in S$, and $(\cdot,\cdot)_g$ is the scalar product in $\mathrm{T}_x M$ corresponding to the metric g.

Since all smooth measures are absolutely continuous with respect to each other, the σ-algebra of sets of null measure and the space $L^\infty(M, \mu)$ do not actually depend on the choice of the smooth measure μ. It allows us do denote this space simply by $L^\infty(M)$. In the same way we define the space $L^\infty(S)$. Recall that the latter space appeared in boundary condition (2). The spaces $L^p(M)$, $L^p(S)$ are also do not depend on the choice of smooth measures while their norms are mutually equivalent for different smooth measures. We will also denote by $L^\infty(M, \mathrm{T}M)$ the space of bounded measurable vector fields on M. This space consist of vector fields $a(x)$ such that for any $f(x) \in C^1(M)$ the function $\langle a(x), f(x)\rangle \in L^\infty(M)$. As is easily verified, the vector field $a(x) \in L^\infty(M, \mathrm{T}M)$ if and only if for each chart (U, j, V) $\quad a = a^i(x)\partial/\partial x_i$ with $a^i(x) \in L^\infty_{loc}(V)$. Since M is compact it follows from this property that $|a(x)|_g = ((a(x), a(x))_g)^{1/2} \in L^\infty(M)$ for each continuous metric tensor g. As is usual, vector fields, which differ on a set of null measure, are identified in $L^\infty(M, \mathrm{T}M)$.

We will need in the sequel the notion of a *divergence measure field* (see [5]).

Definition 1. A vector field $a(x) \in L^\infty(M, TM)$ is called the divergence measure field if there exists a finite Borel measure γ on $M_0 = M \setminus S$ (not necessarily nonnegative) such that $\forall f(x) \in C_0^1(M_0)$

$$\int_M \langle a(x), f(x)\rangle d\mu = -\int_M f(x) d\gamma, \tag{12}$$

here μ is some smooth measure on M.

The class of divergence measure fields does not depend on the choice of measure μ (while the measure γ depends on this choice). Indeed, if μ' is another smooth measure then $\mu' = \alpha(x)\mu$, $\alpha(x) \in C^1(M)$, $\alpha(x) > 0$ and

$$\int_M \langle a(x), f(x)\rangle d\mu' = \int_M \langle a(x), f(x)\rangle \alpha(x) d\mu =$$
$$\int_M \langle a(x), \alpha(x) f(x)\rangle d\mu - \int_M \langle a(x), \alpha(x)\rangle f(x) d\mu =$$
$$-\int_M \alpha(x) f(x) d\gamma - \int_M \langle a(x), \alpha(x)\rangle f(x) d\mu = -\int_M f(x) d\gamma',$$

where $\gamma' = \alpha(x)\gamma + \langle a(x), \alpha(x)\rangle\mu$. Remark that the total variation $|\gamma'|(M) \le \|\alpha(x)\|_\infty \cdot |\gamma|(M) + \|\langle a(x), \alpha(x)\rangle\|_\infty \mu(M) < \infty$.

Relation (12) may be formulated as $\operatorname{div}^\mu a = \gamma$ in the sense of distributions on M_0 (in $\mathcal{D}'(M_0)$). In a local coordinates (U, j, V) this relation yields

$$\int_V a^i(x) \frac{\partial f(x)}{\partial x_i} \omega(x) dx = -\int_V f(x) d\gamma,$$

where a^i are coordinates of a, $\mu = \omega(x)dx$, and γ is identified with $j^*(\gamma|_U)$. The obtained relation means that

$$\operatorname{div}(\omega a) = \omega \operatorname{div} a + \langle a, \omega\rangle = \gamma \tag{13}$$

in the sense of distributions on V (in $\mathcal{D}'(V)$), where $\operatorname{div} a = \dfrac{\partial}{\partial x_i} a^i(x)$ is the "classical" divergence. This implies that

$$\operatorname{div} a = \tilde{\gamma} \doteq \frac{1}{\omega(x)}(\gamma - \langle a(x), \omega(x)\rangle dx) \ \text{ in } \mathcal{D}'(V)$$

and both the fields $\omega(x)a(x)$, $a(x)$ are divergence measure fields on V in the sense of [5] (passing to a smaller set V if necessary we may assume that $a^i \in L^\infty(V)$, $i = 1, \dots, n$, and $|\tilde{\gamma}|(V) < \infty$). By the results of [5] and the arbitrariness of the coordinate neighborhood we see that there exists a weak trace of the "normal component" of a divergence measure field $a(x)$ at the boundary S. More precisely we have the following result.

Theorem 2. *Let $a(x) \in L^\infty(M, \mathrm{T}M)$ be a divergence measure field, and $\bar{\mu} = (\mu, \mu_b)$ be a pair of smooth measures on M and S respectively. Then* *there exists a function $v = a_{\bar{\mu}}(x) \in L^\infty(S)$ such that for any $f(x) \in C^1(M)$*

$$\int_M \langle a(x), f(x)\rangle d\mu + \int_M f(x)d\gamma = \int_S v(x)f(x)d\mu_b, \tag{14}$$

where the measure γ is taken from (12).

Proof. Let (U, j, V) be a chart such that $V = [0, h) \times W$, where $h > 0$, $W \subset \mathbb{R}^{n-1}$ is an open subset, and $a = a^i(x)\frac{\partial}{\partial x_i}$ be the coordinate representation of a (as usual, we identify the field a and its image j^*a under the map j). Here $x = (x_1, x')$, $x_1 \in [0, h)$, $x' = (x_2, \ldots, x_n) \in W$. Suppose also that $\mu = \omega(x)dx$, $\mu_b = \omega_b(x')dx'$, where $\omega(x) \in C^1(V)$, $\omega_b(x') \in C^1(W)$, and $\omega, \omega_b > 0$. Since $a(x) \in L^\infty(M, \mathrm{T}M)$ is a divergence measure field, we have $a^i(x) \in L^\infty_{loc}(V)$, $i = 1, \ldots, n$, and in view of (13)

$$\operatorname{div} \omega a = \frac{\partial}{\partial x_i}\omega(x)a^i(x) = \gamma \ \text{ in } \mathcal{D}'(V), \tag{15}$$

Passing to the smaller set V if necessary we can suppose that $a^i(x) \in L^\infty(V)$, $i = 1, \ldots, n$ and the functions ω, ω_b are bounded together with the functions $1/\omega$, $1/\omega_b$.

Now, we choose a function $\rho(s) \in C^1_0(\mathbb{R})$ such that $\rho(s) \geq 0$, $\operatorname{supp}\rho \subset (0, 1)$, $\int \rho(s)ds = 1$ and set for $\nu \in \mathbb{N}$ $\rho_\nu(s) = \nu\rho(\nu s)$, $\theta_\nu(t) = \int_{-\infty}^t \rho_\nu(s)ds$. Clearly, the sequence $\rho_\nu(s)$ converges as $\nu \to \infty$ to the Dirac δ-function $\delta(s)$ in $\mathcal{D}'(\mathbb{R})$ while the sequence $\theta_\nu(t)$ converges point-wise to the Heaviside function $\operatorname{sign}^+(t) = \begin{cases} 0, & t \leq 0, \\ 1, & t > 0 \end{cases}$. Let $f(x) \in C^1_0(V)$, $\tau \in E$, where

$$E = \{\ t \in (0, h) \mid (t, y) \text{ is a Lebesgue point of } a^1(t, y) \text{ for a.e. } y \in W\ \}. \tag{16}$$

From the known properties of Lebesgue points it follows that $E \subset (0, h)$ is a set of full measure. Let $g(x) = \theta_\nu(x_1 - \tau)f(x)$. Obviously, the function $g(x) \in C^1_0((0, h) \times W)$ for sufficiently large ν and applying (15) to this test function, we obtain that

$$-\int_0^h \left(\int_W \omega(x_1, x')a^1(x_1, x')f(x_1, x')dx'\right)\rho_\nu(x_1 - \tau)dx_1 =$$
$$\int_V \omega(x)a^i(x)f_{x_i}(x)\theta_\nu(x_1 - \tau)dx + \int_V \theta_\nu(x_1 - \tau)f(x)d\gamma.$$

Passing in this equality to the limit as $\nu \to \infty$ and taking into account that τ is a Lebesgue point of the function $I(t) = \displaystyle\int_W \omega(t, x')a^1(t, x')f(t, x')dx'$

(this easily follows from the definition of the set E) and that $\theta_\nu(x_1-\tau)$ point-wise converges to the function $\mathrm{sign}^+(x_1-\tau)$ while $0\le\theta_\nu(x_1-\tau)\le 1$, we arrive at

$$-\int_W \omega(\tau,x')a^1(\tau,x')f(\tau,x')dx' =$$
$$\int_{(\tau,h)\times W}\omega(x)a^i(x)f_{x_i}(x)dx+\int_{(\tau,h)\times W} f(x)d\gamma. \qquad (17)$$

Since the family $a^1(\tau,x)$, $\tau\in E$ is bounded in $L^\infty(W)$ we can choose a sequence $\tau_m\in E$, $\tau_m\to 0$ such that $a^1(\tau_m,x)$ converges as $m\to\infty$ to some function $w(x)$ weakly-$*$ in $L^\infty(W)$. Taking in (17) $\tau=\tau_m$ and passing to the limit as $m\to\infty$, we derive that

$$-\int_W \omega(0,x')w(x')f(0,x')dx'=\int_V \omega(x)a^i(x)f_{x_i}(x)dx+\int_V f(x)d\gamma.$$

Changing the variables in the obtained relation we can rewrite it in the form (here we keep the notation $f(x)$ for the function $f(j(x))$)

$$\int_S v(x)f(x)d\mu_b=\int_M\langle a(x),f(x)\rangle d\mu+\int_M f(x)d\gamma, \qquad (18)$$

where $v(x)=-\omega(0,x')w(x')/\omega_b(x')$, $(0,x')=j(x)$, $x\in S$. It is clear that $v(x)\in L^\infty(S\cap U)$ and (18) holds for all $f(x)\in C^1(M)$ with compact support in U. This implies that essential values $v(x)$ do not depend on the choice of a coordinate neighborhood of the boundary point x as well as on the choice of the sequence τ_m. Since coordinate neighborhoods with prescribed above properties cover the boundary, the function $v(x)$ is well-defined on the entire boundary and (due to compactness of S) $v(x)\in L^\infty(S)$. Identity (18) shows that (14) is satisfied for test functions with supports in the coordinate neighborhoods of boundary points. As directly follows from (12), equality (14) is also true for test functions $f(x)$ with compact support in M_0. With the help of the partition of unity we derive that (14) holds for all $f(x)\in C^1(M)$, which completes the proof.

Remark 3. As one can see from the proof of Theorem 2, $v(x)$ is the weak normal trace of $a(x)$ if and only if for any chart (U,j,V) such that U is a neighborhood of a boundary point and $V=[0,h)\times W$ with $h>0$, W being an open subset of $\mathbb{R}^{n-1}$

$$\operatorname*{ess\,lim}_{x_1\to 0}\langle n_{\bar\mu}(x'),a(x_1,x')\rangle=v(x')\doteq v(j^{-1}(0,x'))\ \text{ weakly-}*\text{ in } L^\infty_{loc}(W). \qquad (19)$$

Here one should take into account that (see the proof of Theorem 2) $v(x')$ is a weak limit of the sequence

$$\langle n_{\bar{\mu}}(x'), a(\tau_m, x')\rangle = -\omega(\tau_m, x')a^1(\tau_m, x')/\omega_b(x')$$

and this limit does not depend on the choice of the vanishing sequence τ_m from the set E of full measure defined by (16).

Definition 2. The function $v = a_{\bar{\mu}}(x)$ for which relation (19) is satisfied is called the *weak normal trace* of the field $a(x)$. If the limit relation (19) is valid in the space $L^1_{loc}(W)$ we will call $v(x)$ the *strong normal trace.*

Observe that by Theorem 1 in the case when $a(x) \in C^1(M, \mathrm{T}M)$ the weak normal trace of a coincides with $\langle n_{\bar{\mu}}(x), a(x)\rangle$. Since $a(x)$ is smooth, this normal trace is in fact strong.

We put below the following simple condition, which is sufficient for vector field to be divergence measure field.

Proposition 1. *Let $a(x) \in L^\infty(M, \mathrm{T}M)$, $c = c(x) \in L^1(M)$, μ be a smooth measure on M, and $\forall f = f(x) \in C^1_0(M_0)$, $f \geq 0$*

$$\int_M [\langle a, f\rangle + cf] d\mu \geq 0.$$

Then $a(x)$ is a divergence measure field.

Proof. By the assumption the functional $I(f) = \int_M [\langle a, f\rangle + cf] d\mu$ is a nonnegative linear functional on $C^1_0(M_0)$. Therefore, by the known representation property, this functional is given by integration with respect to some nonnegative locally finite Borel measure α on M_0: $I(f) = \int_M f(x) d\alpha$ (we extend α as a measure on the entire manifold M, setting its value being equalled $\alpha(A \cap M_0)$ for every Borel set $A \subset M$). Hence $\forall f = f(x) \in C^1_0(M_0)$, $f \geq 0$

$$\int_M [\langle a, f\rangle + cf] d\mu = \int_M f(x) d\alpha. \tag{20}$$

We have to show that the measure α is finite. Since M is compact and α is locally finite on M_0 it is sufficient to prove that α is finite in some neighborhood of arbitrary boundary point $x_0 \in S$. We choose a coordinate neighborhood U of x_0. Let (U, j, V) be the corresponding chart. We may suppose that $V = [0, h) \times W$, where $h > 0$ and W is an open subset of $\mathbb{R}^{n-1}$. Let $a^i(x)$, $i = 1, \ldots, n$ be coordinates of the vector $a(x)$, $x = (x_1, x') \in V$,

$\mu = \omega(x)dx$. We define the set E of full measure on $(0,h)$ as in (16) and suppose that $t \in E$, $h(x) \in C_0^1(V)$, $h(x) > 0$. We set $f(x) = \theta_\nu(x_1 - t)h(x)$, where the sequence θ_ν, $\nu \in \mathbb{N}$ was defined in the proof of Theorem 2. Applying (20) to the nonnegative test function $f(j(x)) \in C_0^1(M_0)$ (f is assumed to be equalled 0 out of U) and passing to the variables $x \in V$, we derive that

$$\int_0^h \int_W a^1(x_1,x')h(x_1,x')\omega(x_1,x')dx'\rho_\nu(x_1-t)dx_1 +$$
$$\int_V [\langle a(x), h(x)\rangle + c(x)h(x)]\omega(x)\theta_\nu(x_1-t)dx = \int_V h(x)\theta_\nu(x_1-t)d\alpha.$$

Passing in this equality to the limit as $\nu \to \infty$, and taking into account that $t \in E$ is a Lebesgue point of the function $t \to \int_W a^1(t,x')h(t,x')\omega(t,x')dx'$ and that the sequence $\theta_\nu(x_1-t)$ is uniformly bounded and converges pointwise to $\operatorname{sign}^+(x_1-t)$, we obtain that

$$\int_W a^1(t,x')h(t,x')\omega(t,x')dx' +$$
$$\int_{(t,h)\times W} [\langle a(x), h(x)\rangle + c(x)h(x)]\omega(x)dx = \int_{(t,h)\times W} h(x)d\alpha.$$

This implies the estimate

$$\int_{(t,h)\times W} h(x)d\alpha \le \int_{(t,h)\times W} [\langle a(x), h(x)\rangle + c(x)h(x)]\omega(x)dx + C,$$

where $C = \sup\limits_{t\in(0,h)} \int_W a^1(t,x')h(t,x')\omega(t,x')dx' < \infty$. From this estimate it follows in the limit as $t \to 0$, $t \in E$ that

$$\int_V h(x)d\alpha \le \int_V [\langle a(x), h(x)\rangle + c(x)h(x)]\omega(x)dx + C < \infty$$

and since the measure α is nonnegative and $h(x)$ is an arbitrary nonnegative function from $C_0^1(V)$ we conclude that α is locally finite on U as required. Hence, α is finite measure. Further, by (20)

$$\int_M \langle a, f\rangle d\mu = -\int_M f(x)d\gamma,$$

where $\gamma = c(x)\mu - \alpha$ is a finite measure, $\operatorname{Var}\gamma \le \alpha + |c(x)|\mu < \infty$. Thus, $a(x)$ is a divergence measure field. Proposition is proved.

Corollary 1. *Under the assumptions of Proposition 1 there exists the weak normal trace $a_{\bar\mu}(x)$ depending also on the choice of the measure μ_b on S such that* $\forall f = f(x) \in C^1(M)$, $f \ge 0$

$$\int_M [\langle a, f\rangle + cf]d\mu - \int_S a_{\bar\mu}(x)f(x)d\mu_b \ge 0. \tag{21}$$

Proof. By Proposition 1 $a(x)$ is a divergence measure field and the existence of a weak normal trace $a_{\bar\mu}(x)$ follows from the assertion of Theorem 2. As we show in the proof of Proposition 1, $\operatorname{div}^\mu a(x) = \gamma = c(x)\mu - \alpha$, where α is a nonnegative Borel measure. Then from (14) it follows that

$$\int_M [\langle a, f\rangle + cf]d\mu - \int_S a_{\bar\mu}(x)f(x)d\mu_b = \int_M f(x)d\alpha \ge 0 \tag{22}$$

for all nonnegative test functions $f \in C^1(M)$, as was to be proved.

§ 2. The notion of generalized entropy solution.

Introduce the vector field $\varphi(x,u) = \int_0^u a(x,\lambda)d\lambda$ in $\mathrm{T}_x M$ so that $\dfrac{\partial\varphi(x,u)}{\partial u} = a(x,u)$. If μ is a smooth measure on M then equation (1) can be written (at least formally) in the divergence form

$$\operatorname{div}^\mu\varphi(x,u) - \operatorname{div}_x^\mu\varphi(x,u) = 0, \tag{23}$$

where $\operatorname{div}_x^\mu\varphi(x,u) = \operatorname{div}^\mu\varphi(\cdot,u)$.

Indeed, if $u = u(x) \in C^1(M)$ then in the local coordinates $x_1,\dots,x_n$ we have $\mu = \omega(x)dx$ and in view of (7)

$$\begin{aligned}\langle a(x,u), u\rangle &= \frac{1}{\omega(x)}(\omega(x)a^i(x,u(x)))\frac{\partial u}{\partial x_i} = \frac{1}{\omega(x)}\frac{\partial}{\partial u}(\omega(x)\varphi^i(x,u))\Big|_{u=u(x)}\frac{\partial u}{\partial x_i}\\ &= \frac{1}{\omega(x)}\frac{\partial}{\partial x_i}(\omega(x)\varphi^i(x,u(x))) - \frac{1}{\omega(x)}\frac{\partial}{\partial x_i}(\omega(x)\varphi^i(x,u))\Big|_{u=u(x)} =\\ &\qquad \operatorname{div}^\mu\varphi(x,u(x)) - \operatorname{div}_x^\mu\varphi(x,u(x)).\end{aligned}$$

Equation (23), which is the divergence form of equation (1), allows to consider the equation from view point of the theory of distributions, and develop the Kruzhkov-like (see [11]) theory of generalized entropy solutions (g.e.s. for short). Using approaches of papers [10, 13, 17], we introduce below the notions of generalized entropy sub-solutions (g.e.sub-s.) and generalized entropy super-solutions (g.e.super-s) of the Dirichlet problem (1), (2). Let $\bar\mu=(\mu,\mu_b)$ be a pair of smooth measures on M and S respectively. Denote $u^+=\max(u,0)$, $u^-=\max(-u,0)$, $\operatorname{sign}^+(u)=(\operatorname{sign} u)^+$ (the Heaviside function), $\operatorname{sign}^-(u)=-(\operatorname{sign} u)^-$.

Definition 3. A function $u=u(x)\in L^\infty(M)$ is called a g.e.sub-s. of (1), (2) if there exists a positive constant L such that for every $k\in\mathbb{R}$, $\forall f=f(x)\in C^1(M)$, $f\ge 0$

$$\int_M \operatorname{sign}^+(u-k)[\langle\varphi(x,u)-\varphi(x,k),f\rangle+f\operatorname{div}_x^\mu(\varphi(x,u)-\varphi(x,k))]d\mu$$
$$+L\int_S (u_b-k)^+ f d\mu_b\ge 0; \quad (24)$$

a function $u=u(x)\in L^\infty(M)$ is called a g.e.super-s. of (1), (2) if there exists a positive constant L such that for every $k\in\mathbb{R}$, $\forall f=f(x)\in C^1(M)$, $f\ge 0$

$$\int_M \operatorname{sign}^-(u-k)[\langle\varphi(x,u)-\varphi(x,k),f\rangle+f\operatorname{div}_x^\mu(\varphi(x,u)-\varphi(x,k))]d\mu$$
$$+L\int_S (u_b-k)^- f d\mu_b\ge 0. \quad (25)$$

A function $u=u(x)\in L^\infty(M)$ is called a g.e.s. of (1), (2) if it is a g.e.sub-s. and g.e.super-s. of this problem simultaneously.

Let us firstly show that our definition actually does not depend on the choice of the pair of smooth measures $\bar\mu=(\mu,\mu_b)$. For this, suppose that $\bar\mu'=(\mu',\mu_b')$ is another pair of smooth measures on M and S respectively, and $u(x)$ is a g.e.sub-s. corresponding to the pair $\bar\mu$. Then $\mu'=\alpha(x)\mu$, $\mu_b'=\alpha_b(x)\mu_b$, where $\alpha(x)\in C^1(M)$, $\alpha_b(x)\in C^1(S)$, $\alpha(x),\alpha_b(x)>0$. Using relations (8), (9) we arrive at

$$\int_M \operatorname{sign}^+(u-k)[\langle\varphi(x,u)-\varphi(x,k),f\rangle+f\operatorname{div}_x^{\mu'}(\varphi(x,u)-\varphi(x,k))]d\mu'=$$
$$\int_M \operatorname{sign}^+(u-k)[\alpha(x)\langle\varphi(x,u)-\varphi(x,k),f\rangle+$$

$$f\mathrm{div}_x^\mu(\alpha(x)(\varphi(x,u)-\varphi(x,k)))]d\mu =$$

$$\int_M \mathrm{sign}^+(u-k)[\alpha\langle\varphi(x,u)-\varphi(x,k),f\rangle + f\langle\varphi(x,u)-\varphi(x,k),\alpha\rangle +$$

$$\alpha f\mathrm{div}_x^\mu(\varphi(x,u)-\varphi(x,k))]d\mu =$$

$$\int_M \mathrm{sign}^+(u-k)[\langle\varphi(x,u)-\varphi(x,k),\alpha f\rangle + \alpha f\mathrm{div}_x^\mu(\varphi(x,u)-\varphi(x,k))]d\mu.$$

Taking $L' = L\max\limits_{x\in S}[\alpha(x)/\alpha_b(x)]$ we obtain also the inequality

$$L'\int_S (u_b-k)^+ f d\mu_b' = L'\int_S (u_b-k)^+ f\alpha_b d\mu_b \ge L\int_S (u_b-k)^+\alpha f d\mu_b.$$

Together with the preceding equality this inequality implies that $\forall f \in C^1(M),\ f\ge 0$

$$\int_M \mathrm{sign}^+(u-k)[\langle\varphi(x,u)-\varphi(x,k),f\rangle +$$

$$f\mathrm{div}_x^{\mu'}(\varphi(x,u)-\varphi(x,k))]d\mu' + L'\int_S (u_b-k)^+ f d\mu_b' \ge$$

$$\int_M \mathrm{sign}^+(u-k)[\langle\varphi(x,u)-\varphi(x,k),\alpha f\rangle +$$

$$\alpha f\mathrm{div}_x^\mu(\varphi(x,u)-\varphi(x,k))]d\mu + L\int_S (u_b-k)^+\alpha f d\mu_b \ge 0$$

in view of relation (24) with the test function αf. This shows that $u(x)$ is a g.e.sub-s. corresponding to the pair $\bar\mu'$.

Replacing sign^+ by sign^- in the above reasoning, we claim that if $u(x)$ is a g.e.super-s. corresponding to some pair $\bar\mu$ of smooth measures then $u(x)$ is also a g.e.super-s. corresponding to any other pair $\bar\mu'$.

Thus, the class of g.e.sub-s. (g.e.super-s., g.e.s.) of the problem (1), (2) does not depend on the choice of smooth measures μ, μ_b.

From (24), (25) it follows that for a g.e.s. $u = u(x)$ the following condition is satisfied: $\forall k\in\mathbb{R}$, $\forall f = f(x)\in C_0^1(M_0)$, $f\ge 0$

$$\int_M \mathrm{sign}^\pm(u-k)[\langle\varphi(x,u)-\varphi(x,k),f\rangle + f\mathrm{div}_x^\mu(\varphi(x,u)-\varphi(x,k))]d\mu \ge 0. \tag{26}$$

Since $u = u(x)\in L^\infty(M)$, the fields $\mathrm{sign}^\pm(u-k)(\varphi(x,u)-\varphi(x,k)) \in L^\infty(M,\mathrm{T}M)$ for each $k\in\mathbb{R}$.

By Proposition 1 with $c(x) = \operatorname{sign}^{\pm}(u-k)\operatorname{div}^{\mu}_x(\varphi(x,u)-\varphi(x,k)) \in L^\infty(M)$ and Corollary 1 inequality (26) yields existence of weak normal traces $[\operatorname{sign}^{\pm}(u-k)(\varphi(x,u)-\varphi(x,k))]_{\bar\mu}$ of these fields. Recall that these traces depend on the choice of the pair $\bar\mu$ of smooth measures.

It is useful to reformulate the notion of g.e.s. in the following way.

Theorem 3. *A function $u = u(x) \in L^\infty(M)$ is a g.e.s. of (1), (2) if and only if*

1) for each $k \in \mathbb{R}$ $\forall f = f(x) \in C_0^1(M_0)$, $f \geq 0$

$$\int_M \operatorname{sign}(u-k)[\langle \varphi(x,u)-\varphi(x,k), f\rangle + f \operatorname{div}^{\mu}_x(\varphi(x,u)-\varphi(x,k))]d\mu \geq 0, \quad (27)$$

i.e.

$$\operatorname{div}^{\mu}[\operatorname{sign}(u-k)(\varphi(x,u)-\varphi(x,k))] - \operatorname{sign}(u-k)\operatorname{div}^{\mu}_x(\varphi(x,u)-\varphi(x,k)) \leq 0$$

in $\mathcal{D}'(M_0)$, and for some positive constant L for each $k \in \mathbb{R}$

$$[\operatorname{sign}^{\pm}(u-k)(\varphi(x,u)-\varphi(x,k))]_{\bar\mu} + L(u_b-k)^{\pm} \geq 0 \quad \textit{a.e. on } S. \quad (28)$$

Proof. Suppose that $u = u(x) \in L^\infty(M)$ is a g.e.s. of (1), (2), and $k \in \mathbb{R}$, $f = f(x) \in C_0^1(M_0)$, $f \geq 0$. Then, putting together inequalities (26)

$$\int_M \operatorname{sign}^{+}(u-k)[\langle \varphi(x,u)-\varphi(x,k), f\rangle + f \operatorname{div}^{\mu}_x(\varphi(x,u)-\varphi(x,k))]d\mu \geq 0,$$

$$\int_M \operatorname{sign}^{-}(u-k)[\langle \varphi(x,u)-\varphi(x,k), f\rangle + f \operatorname{div}^{\mu}_x(\varphi(x,u)-\varphi(x,k))]d\mu \geq 0,$$

we immediately obtain (27).

Further, we choose a function $g(x) \in C^1(S)$ and the sequence $V_m \subset M$ of open sets such that $S \subset V_{m+1} \subset V_m$ $\forall m \in \mathbb{N}$ and $\bigcap_{m=1}^{\infty} V_m = S$. Obviously, there exist functions $f_m \in C^1(M)$ with support $\operatorname{supp} f_m \subset V_m$ such that $f_m|_S = g$, $\|f_m\|_\infty \leq \|g\|_\infty$. For fixed $k \in \mathbb{R}$ the fields $\operatorname{sign}^{\pm}(u-k)(\varphi(x,u)-\varphi(x,k))$ are divergence measure fields, therefore for some finite Borel measures $\gamma_k^{\pm}$ on M_0 relations (14) hold. Putting in these relations $f = f_m$ we arrive at

$$\int \operatorname{sign}^{\pm}(u-k)\langle \varphi(x,u)-\varphi(x,k), f_m\rangle d\mu + \int_M f_m(x)d\gamma_k^{\pm} = \int_S v_k^{\pm} g d\mu_b, \quad (29)$$

where we denote

$$v_k^\pm = [\mathrm{sign}^\pm(u-k)(\varphi(x,u)-\varphi(x,k))]_{\bar\mu}$$

the weak normal traces of our vectors corresponding to a pair $\bar\mu$ of smooth measures. By relations (24), (25) we have that for some positive constant L

$$\int_M \mathrm{sign}^\pm(u-k)[\langle\varphi(x,u)-\varphi(x,k),f_m\rangle + f_m \mathrm{div}_x^\mu(\varphi(x,u)-\varphi(x,k))]d\mu$$
$$+L\int_S (u_b-k)^\pm g d\mu_b \ge 0.$$

Subtracting (29) from this inequality, we see that

$$\int_S [v_k^\pm + L(u_b-k)^\pm] g d\mu_b \ge \int_M f_m(x) d\gamma_k^\pm -$$
$$\int_M f_m\,\mathrm{sign}^\pm(u-k)\mathrm{div}_x^\mu(\varphi(x,u)-\varphi(x,k))d\mu \underset{m\to\infty}{\to} 0$$

because the absolute value of the right-hand side is bounded by $const\cdot(|\gamma_k^\pm|(V_m)+\mu(V_m))$ and this sequence tends to $const\cdot(|\gamma_k^\pm|(S)+\mu(S))=0$. Passing to the limit as $m\to\infty$ we obtain that $\int_S [v_k^\pm + L(u_b-k)^\pm] g d\mu_b \ge 0$ and since g is an arbitrary nonnegative smooth function on S this yields $v_k^\pm + L(u_b-k)^\pm \ge 0$ a.e. on S, which is precisely (28).

Conversely, assume that conditions (27), (28) are satisfied. Putting in (27) $k=\pm\|u\|_\infty$ and taking into account that by (6)

$$\int_M [\langle\varphi(x,k),f\rangle + f\mathrm{div}_x^\mu\varphi(x,k)]d\mu = 0, \tag{30}$$

we obtain that $\forall f=f(x)\in C_0^1(M_0)$, $f\ge 0$

$$\int_M [\langle\varphi(x,u),f\rangle + f\mathrm{div}_x^\mu\varphi(x,u)]d\mu = 0. \tag{31}$$

Since any function $f\in C_0^1(M_0)$ is a difference of two nonnegative functions from this space we see that (31) holds for all $f\in C_0^1(M_0)$, i.e. equality (23) is satisfied in $\mathcal{D}'(M_0)$. From (30), (31) it follows that for each $k\in\mathbb{R}$

$$\int_M [\langle\varphi(x,u)-\varphi(x,k),f\rangle + \mathrm{div}_x^\mu(\varphi(x,u)-\varphi(x,k))f]d\mu = 0.$$

If we add this equality multiplied by ± 1 to (27) and take into account that $\operatorname{sign} u \pm 1 = 2 \operatorname{sign}^{\pm} u$ for $u \neq 0$ then we derive identities (26), which hold for all real k and all nonnegative test functions $f \in C_0^1(M_0)$. As above, using Proposition 1 and Corollary 1 we find that there exist weak normal traces

$$v_k^{\pm} = [\operatorname{sign}^{\pm}(u-k)(\varphi(x,u) - \varphi(x,k))]_{\bar{\mu}}$$

on the boundary S, where $\bar{\mu} = (\mu, \mu_b)$ and μ_b is some smooth measure on S (in addition to already fixed μ).

We define as above a monotone sequence of open sets $V_m \subset M$ such that $S \subset V_{m+1} \subset V_m\ \forall m \in \mathbb{N}$, $\bigcap_{m=1}^{\infty} V_m = S$ and consider the partition of unity $1 = g_{1m} + g_{2m}$ corresponding to the covering $M = M_0 \cup V_m$. This means that $g_{1m} \in C_0^1(M_0)$, $g_{2m} \in C^1(M)$, $\operatorname{supp} g_{2m} \subset V_m$, $g_{1m}, g_{2m} \geq 0$. Then any nonnegative test function $f \in C^1(M)$ can be decomposed into the sum $f = f_{1m} + f_{2m}$, where $f_{1m} = f g_{1m} \in C_0^1(M_0)$, $f_{2m} = f g_{2m} \in C^1(M)$. Obviously, $\operatorname{supp} f_{2m} \subset V_m$, $f_{1m}, f_{2m} \geq 0$. By inequalities (26)

$$\begin{aligned}
\int_M \operatorname{sign}^{\pm}(u-k)[\langle \varphi(x,u) - \varphi(x,k), f\rangle + f \operatorname{div}_x^{\mu}(\varphi(x,u) - \varphi(x,k))]d\mu = \\
\int_M \operatorname{sign}^{\pm}(u-k)[\langle \varphi(x,u) - \varphi(x,k), f_{1m}\rangle + \\
f_{1m} \operatorname{div}_x^{\mu}(\varphi(x,u) - \varphi(x,k))]d\mu + \\
\int_M \operatorname{sign}^{\pm}(u-k)[\langle \varphi(x,u) - \varphi(x,k), f_{2m}\rangle + \\
f_{2m} \operatorname{div}_x^{\mu}(\varphi(x,u) - \varphi(x,k))]d\mu \geq \\
\int_M \operatorname{sign}^{\pm}(u-k)[\langle \varphi(x,u) - \varphi(x,k), f_{2m}\rangle + \\
f_{2m} \operatorname{div}_x^{\mu}(\varphi(x,u) - \varphi(x,k))]d\mu. \quad (32)
\end{aligned}$$

By Corollary 1 we have also

$$\int_M \operatorname{sign}^{\pm}(u-k)\langle \varphi(x,u) - \varphi(x,k), f_{2m}\rangle d\mu + \int_M f_{2m} d\gamma_k^{\pm} = \int_S f v_k^{\pm} d\mu_b, \quad (33)$$

where $\gamma_k^{\pm} = \operatorname{div}^{\mu}[\operatorname{sign}^{\pm}(u-k)(\varphi(x,u) - \varphi(x,k))]$ are finite Borel measures on M_0. We also take into account that $f_{2m}|_S = f$. In the limit as $m \to \infty$ equality (33) yields

$$\lim_{m\to\infty} \int_M \operatorname{sign}^{\pm}(u-k)\langle \varphi(x,u) - \varphi(x,k), f_{2m}\rangle d\mu = \int_S f v_k^{\pm} d\mu_b \quad (34)$$

because $\int_M f_{2m} d\gamma_k^\pm \to 0$. Passing in (32) to the limit as $m \to \infty$, we derive with the help of (34) and the obvious relation

$$\lim_{m\to\infty} \int_M \operatorname{sign}^\pm(u-k) \operatorname{div}_x^\mu(\varphi(x,u)-\varphi(x,k)) f_{2m} d\mu = 0$$

that

$$\int_M \operatorname{sign}^\pm(u-k)[\langle \varphi(x,u)-\varphi(x,k), f\rangle +$$

$$f \operatorname{div}_x^\mu(\varphi(x,u)-\varphi(x,k))] d\mu - \int_S f v_k^\pm d\mu_b \ge 0. \tag{35}$$

Further, in view of (28)

$$\int_S [v_k^\pm + L(u_b-k)^\pm] f d\mu_b \ge 0.$$

Adding this inequality to (35), we immediately obtain (24), (25). This means that u is a g.e.s. of (1), (2). The proof is complete.

Let us discuss the sense of boundary condition (28). Suppose that a g.e.s. $u(x)$ has the strong trace $u^*(x)$ on the boundary S. This means that for any chart (U, j, V) such that U is a neighborhood of a boundary point and $V = [0, h) \times W$ with $h > 0$ and W being an open subset of $\mathbb{R}^{n-1}$

$$\operatorname*{ess\,lim}_{x_1\to 0} u(x_1, x') = u^*(j^{-1}(0, x')) \ \text{ in } L^1_{loc}(W).$$

Then as follows from (19) the weak normal traces

$$[\operatorname{sign}^\pm(u-k)(\varphi(x,u)-\varphi(x,k))]_{\bar\mu} = \operatorname{sign}^\pm(u^*-k)(\varphi_{\bar\mu}(x,u^*)-\varphi_{\bar\mu}(x,k)),$$

where $\varphi_{\bar\mu}(x,u) = \langle n_{\bar\mu}(x), \varphi(x,u)\rangle$ is a strong normal trace of the vector field $\varphi(x,u)$ in the sense of Theorem 1. Therefore, in this case relation (28) reduces to the form: for a.e. $x \in S$

$$\operatorname{sign}^\pm(u^*-k)(\varphi_{\bar\mu}(x,u^*)-\varphi_{\bar\mu}(x,k)) + L(u_b(x)-k)^\pm \ge 0 \ \ \forall k \in \mathbb{R}. \tag{36}$$

Here we also take into account that the set $\tilde S \subset S$ of full μ_b-measure consisting of boundary points, for which inequality (36) holds, can be chosen common for all $k \in \mathbb{R}$ because the left-hand side of (36) depends continuously on the parameter k.

Since condition (36) remains valid after enlargement of L then, without lose of generality, we can assume that $L \geq L_{\bar{\mu}} \doteq \sup_{x\in S, u\in\mathbb{R}} |a_{\bar{\mu}}(x,u)|$ (recall that $a(x,u)$ is supposed to be uniformly bounded).

Now, we fix $x \in \tilde{S}$ and consider two possible cases $u^* \geq u_b$ and $u^* \leq u_b$. Here $u^* = u^*(x)$, $u_b = u_b(x)$. In the first case the inequality

$$\operatorname{sign}^+(u^* - k)(\varphi_{\bar{\mu}}(x,u^*) - \varphi_{\bar{\mu}}(x,k)) + L(u_b - k)^+ \geq 0 \tag{37}$$

immediately implies that

$$\varphi_{\bar{\mu}}(x,k) \leq \varphi_{\bar{\mu}}(x,u^*) \quad \forall k \in [u_b, u^*]. \tag{38}$$

Conversely, if condition (38) is satisfied then inequality (37) holds for each $k \in [u_b, u^*]$. Evidently, it also holds for $k > u^*$. If $k < u_b$ then (37) acquires the form $\varphi_{\bar{\mu}}(x,u^*) - \varphi_{\bar{\mu}}(x,k) + L(u_b - k) \geq 0$, and directly follows from the relation

$$\varphi_{\bar{\mu}}(x,u^*) - \varphi_{\bar{\mu}}(x,k) + L(u_b - k) = \varphi_{\bar{\mu}}(x,u^*) - \varphi_{\bar{\mu}}(x,u_b) + \int_k^{u_b} (a_{\bar{\mu}}(x,u) + L)du \geq \varphi_{\bar{\mu}}(x,u^*) - \varphi_{\bar{\mu}}(x,u_b) \geq 0$$

in view of (38) with $k = u_b$. Let us show that also for every $k \in \mathbb{R}$

$$\operatorname{sign}^-(u^* - k)(\varphi_{\bar{\mu}}(x,u^*) - \varphi_{\bar{\mu}}(x,k)) + L(u_b - k)^- \geq 0.$$

Indeed, if $k \leq u^*$ this inequality is evident while for $k > u^*$ it reduces to the inequality

$$\varphi_{\bar{\mu}}(x,k) - \varphi_{\bar{\mu}}(x,u^*) + L(k - u_b) = \int_{u^*}^{k} (a_{\bar{\mu}}(x,u) + L)du + L(u^* - u_b) \geq 0.$$

We see that condition (28) is satisfied.

By the similar reasons, in the case when $u^* \leq u_b$, the condition (28) is equivalent to the relation

$$\varphi_{\bar{\mu}}(x,k) \geq \varphi_{\bar{\mu}}(x,u^*) \quad \forall k \in [u^*, u_b]. \tag{39}$$

Conditions (38), (39) can be written in the unified form

$$(\operatorname{sign}(k - u^*) + \operatorname{sign}(u_b - k))(\varphi_{\bar{\mu}}(x,k) - \varphi_{\bar{\mu}}(x,u^*)) \geq 0 \quad \forall k \in \mathbb{R}, \tag{40}$$

known as (BLN)-condition (see [3]).

Thus, the boundary condition (28) can be written in the simple form (40) provided that there exists the strong trace u^* of the g.e.s. $u(x)$ on S. As follows from results of [23] (see also [22, 26]) the strong trace always exists under the following non-degeneracy condition:

for a.e. $x \in S$, $\forall \xi \in \mathrm{T}^*_x M$, $\xi \neq 0$ *the function* $u \to \langle \xi, a(x,u) \rangle$ *is not identically equalled zero on non-degenerate intervals.*

Remark also that it is sufficient to require that conditions (28) or (40) are satisfied for $k \in I$, where I is a segment, containing all essential values of both functions $u(x)$, $u_b(x)$.

Now, we introduce the set

$$S^- = \{ \ x \in S \mid a_{\bar{\mu}}(x,\lambda) < 0 \ \text{ for a.e. } \lambda \in I \ \}.$$

Observe that this set does not depend on the choice of a pair $\bar{\mu}$, and consists of boundary points x such that all characteristics (i.e. integral curves of the field $a(x,\lambda)$) are outgoing from x and transversal to S.

If $x \in S^-$ then the function

$$u \to \varphi_{\bar{\mu}}(x,u) = \int_0^u a_{\bar{\mu}}(x,\lambda) d\lambda$$

strictly decreases. Let us demonstrate that a g.e.s. $u(x)$ accepts the boundary data on S^- in the following strong sense.

Proposition 2. *For each chart* (U, j, V) *with* $V = [0,h) \times W$, $h > 0$, W *is an open set in* $\mathbb{R}^{n-1}$

$$\operatorname*{ess\,lim}_{x_1 \to 0} u(x_1, x') = u_b(x') \quad \textit{in } L^1_{loc}(\tilde{S}^-),$$

where $\tilde{S}^- = \{ \ x' \in W \mid j^{-1}(0,x') \in S^- \ \}$.

As usual, we identified u, u_b *with the corresponding functions of variables* x, x'.

Proof. Putting together both inequalities (28), we arrive at the relation: for a.e. $x \in S$

$$[\operatorname{sign}(u-k)(\varphi(x,u) - \varphi(x,k))]_{\bar{\mu}} + L|u_b - k| \geq 0 \quad \forall k \in \mathbb{R}.$$

Passing to our local coordinates and taking into account Remark 3 and the fact that the vector $\varphi(x,u)$ is smooth on $M\times\mathbb{R}$, we can transform the above inequality to the limit relation

$$-\operatorname*{ess\,lim}_{x_1\to 0}[\operatorname{sign}(u(x_1,x')-k)(\varphi^1(0,x',u(x_1,x'))-$$
$$\varphi^1(0,x',k))]\frac{\omega(0,x')}{\omega_b(x')}+L|u_b(x')-k|\ge 0 \quad \forall k\in\mathbb{R} \tag{41}$$

in the weak-$*$ topology of $L^\infty(W)$. Here $\varphi^1(x,u)$ is the first coordinate of the vector $\varphi(x,u)$ and $\omega(x)\in C^1(V)$, $\omega_b(x')\in C^1(W)$ are densities of the measures μ, μ_b: $\mu=\omega(x)dx$, $\mu_b=\omega_b(x')dx'$.

Obviously, relation (41) is also satisfied for $k=k(x')$, where $k'(x)=\sum_{i=1}^m k_i\chi_{A_i}(x)$ is a step function (χ_{A_i} are the indicator functions of measurable sets $A_i\subset W$, $i=1,\dots,m$). Since every function $k(x')\in L^\infty(W)$ can be uniformly approximated by a sequence of step functions we derive that (41) remains valid for $k=k(x')\in L^\infty(W)$. In particular, taking in (41) $k=u_b(x')$ we obtain that

$$\operatorname*{ess\,lim}_{x_1\to 0}[\operatorname{sign}(u(x_1,x')-u_b(x'))(\varphi^1(0,x',u(x_1,x'))-\varphi^1(0,x',u_b(x')))]\le 0 \tag{42}$$

weakly-$*$ in $L^\infty(W)$ and, therefore, also weakly-$*$ in $L^\infty(\tilde S^-)$.

By the definition of the set $\tilde S^-$ for a.e. $x'\in\tilde S^-$ the function $\varphi^1(0,x',u)$ is strictly increasing. Therefore, the function $F(x',u)=\operatorname{sign}(u-u_b(x'))(\varphi^1(0,x',u)-\varphi^1(0,x',u_b(x')))\ge 0$ and takes the zero value only if $u=u_b(x')$. Then from limit relation (42) we readily derive that $\operatorname*{ess\,lim}_{x_1\to 0}F(x',u(x_1,x'))=0$ in $L^1_{loc}(\tilde S^-)$ and this in turn implies that $\operatorname*{ess\,lim}_{x_1\to 0}u(x_1,x')=u_b(x')$ in $L^1_{loc}(\tilde S^-)$. This completes the proof.

Now, we introduce the set

$$S^+=\{\ x\in S\mid a_{\bar\mu}(x,\lambda)\ge 0\ \text{ for a.e. }\ \lambda\in\mathbb{R}\ \}$$

consisting of boundary points x such that for a.e. $\lambda\in\mathbb{R}$ the characteristics are incoming at x. It is natural to expect that the boundary values $u_b(x)$ do not matter at points of S^+. Namely, we have the following statement.

Proposition 3. *Suppose $u(x)$ is a g.e.s. of (1), (2). Then boundary condition (28) is satisfied for a.e. $x\in S^+$ independently of values of $u_b(x)$.*

Proof. In the local coordinates x_1, x' indicated in Proposition 2 condition (28) on the set S^+ reduces to the form similar to (41):

$$- \operatorname*{ess\,lim}_{x_1 \to 0} [\operatorname{sign}(u(x_1, x') - k)^{\pm}(\varphi^1(0, x', u(x_1, x')) - \varphi^1(0, x', k))] \frac{\omega(0, x')}{\omega_b(x')} + L(u_b(x') - k)^{\pm} \geq 0 \quad \forall k \in \mathbb{R} \tag{43}$$

in the weak-$*$ topology of $L^\infty(\tilde{S}^+)$, where $\tilde{S}^+ = \{ x' \in W \mid j^{-1}(0, x') \in S^+ \}$. Since for a.e. $x' \in \tilde{S}^+$ the function $\varphi^1(0, x', u) = \int_0^u a^1(0, x', \lambda) d\lambda$ decreases the functions $\operatorname{sign}(u(x_1, x') - k)^{\pm}(\varphi^1(0, x', u(x_1, x')) - \varphi^1(0, x', k)) \leq 0$ a.e. on $\tilde{S}^+$ for each x_1 and (43) is always satisfied. The proof is complete.

It is clear that generally the sets S^-, S^+ may be empty. Consider one particular case of the evolutionary equation

$$u_t + \langle a(t, x, u), u \rangle = 0, \tag{44}$$

where $(t, x) \in M = [0, T] \times \Omega$, Ω is a C^2 compact manifold without boundary, $T > 0$, $a(t, x, u)$ is a family of C^1 vector fields on Ω continuously depending on t and u as parameters. This equation has the form (1) with the vector $a = \partial/\partial t + a(t, x, u)$ being a vector field on M. As is easy to see here $S^- = \{0\} \times \Omega$, $S^+ = \{T\} \times \Omega$, and $S = \partial M = S^- \cup S^+$. Consider the Dirichlet data (2) for equation (44). By Proposition 3 the values $u_b(T, x)$ play no role and the problem (44), (2) reduces to the classic Cauchy problem with initial data

$$u(0, x) = u_0(x), \tag{45}$$

where $u_0(x) = u_b(0, x) \in L^\infty(\Omega)$.

Denotes $M_0 = \operatorname{Int} M = (0, T) \times \Omega$, $\varphi(t, x, u) = \int_0^u a(t, x, s) ds$. Let μ be a smooth measure on Ω. By Theorem 3 and Proposition 2 we claim that $u = u(t, x) \in L^\infty(M)$ is a g.e.s. of (44), (45) if and only if

1) $\forall k \in \mathbb{R}$, $\forall f = f(t, x) \in C_0^1(M_0)$, $f \geq 0$

$$\int_M \{|u - k| f_t + \operatorname{sign}(u - k)[\langle \varphi(t, x, u) - \varphi(t, x, k), f \rangle + f \operatorname{div}_x^\mu (\varphi(t, x, u) - \varphi(t, x, k))]\} dt d\mu(x) \geq 0.$$

2) $\operatorname*{ess\,lim}_{t \to 0} u(t, \cdot) = u_0$ in $L^1_{loc}(\Omega, \mu)$.

These are the classic Kruzhkov conditions adapted to the case when Ω is a manifold.

Remark that in the case then $\partial\Omega \neq \emptyset$ the initial boundary value problems arises when together with (44), (45) the boundary condition

$$u(t,x) = u_b(t,x) \text{ on } (0,T) \times \partial\Omega \tag{46}$$

is required. If to be precise, in this situation the manifold M has "angle points" in $\{0,T\} \times \partial\Omega$. But this is not a problem. In fact, we could initially treat the case of manifolds with angle points, changing the half-space Π by $(\mathbb{R}_+)^n$, $\mathbb{R}_+ = [0,+\infty)$ in the definition of a manifold.

The problem (44), (45) arises as a particular case of the following general situation. Suppose that $S = S^- \cup S^+$ (or, more generally, the complement $S \backslash (S^- \cup S^+)$ has null measure on S). Then, as follows from Propositions 2,3, boundary condition (2) reduces to the condition

$$u|_{S^-} = u_b \tag{47}$$

understood in the strong sense (as in Proposition 2). We will refer to problem (1), (47) as to *the generalized Cauchy problem.*

Remark 4. In the case of linear equation (1) when $a(x,u) = a(x)$ evidently $S = S^- \cup S^+$ and problem (1), (2) reduces to the generalized Cauchy problem (1), (47). If we suppose in addition that any point of M can be reached by a characteristic outgoing from a point of S^- then any g.e.s. is uniquely defined by the requirement that it must be constant along characteristics. Namely, $u(x) = u_b(y)$, where $x = x(s;y)$ is the characteristic outgoing from the point $y \in S^-$, i.e. $x(s) = x(s;y)$ is a solution of ODE $\dot{x} \doteq dx/ds = a(x)$ on M with initial condition $x(0) = y$.

The Dirichlet problem (1), (2) may be generally ill-posed. We confirm this by some examples.

Example 1. Let M be a segment $[0,1]$. Consider the problem

$$(u^2)' = 0, \quad u(0) = 1, u(1) = -1.$$

Then for any $\xi \in (0,1)$ the function $u(x) = 1 - 2\,\mathrm{sign}^+(x-\xi)$ is a g.e.s. of this problem. Indeed, $u(x)$ admits the boundary data in strong sense, and $[\mathrm{sign}(u-k)(u^2-k^2)]' = (1-k^2)(\mathrm{sign}(u-k))' \leq 0$ in $\mathcal{D}'((0,1))$ for all $k \in \mathbb{R}$ because the function $\mathrm{sign}(u(x)-k)$ is constant for $|k| > 1$ and decreases if

$|k| \leq 1$. In correspondence with Theorem 3 $u(x)$ is a g.e.s. of our problem for each ξ. Thus, we have constructed infinitely many g.e.s. of the problem under consideration.

In the above example $a(x,u) = 2u \equiv 0$ if $u = 0$. The next example shows that non-uniqueness may appear even if the vector field $a(x,u)$ does not degenerate at all points.

Example 2. Consider the 2-dimensional Dirichlet problem for the equation

$$(u^2)_x + g(u)_y = 0 \tag{48}$$

in the plain domain M determined by the inequality $|y| + ((|x|-1)^+)^4 \leq 1$ with boundary data

$$u_b(x,y) = \begin{cases} 1, & y \leq -x, \\ -1, & y > -x. \end{cases} \tag{49}$$

on $S = \partial M$. The rather complicated expression determining M is used to guarantee C^2-smoothness. If manifold with angle points are allowed we can take in this example M being the square $t, x \in [-1,1]$.

We will assume that the function $g(u) \in C^2(\mathbb{R})$ is convex and such that $g(-1) = g(1) = 0$, $g'(0) \neq 0$. Then the field $a = 2u\partial/\partial x + \varphi'(u)\partial/\partial y$ does not depend on (x,y) and is not degenerate for all $u \in \mathbb{R}$. In particular, characteristics of equation (48) are straight lines, and for each u the entire domain M is covered by characteristics going from the boundary. Nevertheless, problem (48), (49) has infinitely many g.e.s. $u(x,y) = 1 - 2\,\mathrm{sign}^+(x-\xi)$ (similar to ones in Example 1) depending on the parameter $\xi \in (-1,1)$. Indeed, in the same way as in Example 1 it is proved that u satisfies condition (27). Concerning the boundary condition, remark that u has strong trace $u^* = \pm 1$ at the boundary and one has to verify (BLN)-condition (40). In the case $y \leq -x$ either $u^* = u_b = 1$ or $u^* = -1 < u_b = 1$ and $\varphi_{\bar{\mu}}(u) = -g(u) \geq g(-1) = 0$ $\forall u \in [-1,1]$. In the case $y > -x$ either $u^* = u_b = -1$ or $u^* = 1 > u_b = -1$ and $\varphi_{\bar{\mu}}(u) = g(u) \leq g(1) = 0$ $\forall u \in [-1,1]$. Here $\bar{\mu}$ is a pair of smooth measures generated the Euclidean metric in $\mathbb{R}^2$. We see that (BLN)-condition is satisfied. Hence, $u(x,y)$ is a g.e.s. of (48), (49) for every $\xi \in (-1,1)$.

The next example shows that the generalized Cauchy problem may have infinitely many g.e.s. even for a linear equation degenerated at a single point.

Example 3. Let M be a disc $r^2 \doteq x^2+y^2 \le 1$ and $0 < r_0 < 1$. Consider the linear equation

$$\langle a(x,y), u\rangle \doteq (-x((r-r_0)^+)^2 - y)\frac{\partial u}{\partial x} + (-y((r-r_0)^+)^2 + x)\frac{\partial u}{\partial y} = 0. \quad (50)$$

Remark that the field $a(x,u) = 0$ only at the zero point. As is easy to see all points of the boundary circle S are outgoing for characteristics and $S^- = S$. In the small disk $r \le r_0$ the characteristics are circles $r = const \le r_0$. This implies that a g.e.s. $u(x,y)$ may coincide with arbitrary function $h(r) \in L^\infty([0,r_0])$ for $r \le r_0$. Setting $u(x,y) = 0$ for $r \in (r_0,1]$, $u(x,y) = h(r)$, $r \in [0,r_0]$, we obtain infinitely many g.e.s. of the generalized Cauchy problem for equation (50) with zero initial data. In particular, for the problem under consideration even the maximum principle is violated.

The following example explains why we are bounded by the case of homogeneous equations.

Example 4. In the same disc M as in the above example we consider the following generalized Cauchy problem for nonhomogeneous linear equation

$$\langle a(x,y), u\rangle \doteq -xr^\alpha \frac{\partial u}{\partial x} - yr^\alpha \frac{\partial u}{\partial y} = 1, \quad (51)$$

with zero initial data at the boundary circle. Here the parameter $\alpha \ge 0$. All points of M except zero can be reached by the characteristics

$$x(s) = x_0(1+\alpha s)^{-1/\alpha}, y(s) = y_0(1+\alpha s)^{-1/\alpha}, \quad \text{if } \alpha > 0,$$
$$x(s) = x_0 e^{-s}, y(s) = y_0 e^{-s}, \quad \text{if } \alpha = 0$$

outgoing from points (x_0, y_0) of the boundary (i.e. $x_0^2+y_0^2 = 1$) and defined for all $s \ge 0$. If $u(x,y)$ is a g.e.s. of the problem under consideration then it is uniquely defined by the condition that $\dot u = 1$ along characteristics. Easy computations yields

$$u(x,y) = s = \begin{cases} (r^{-\alpha} - 1)/\alpha & , \quad \alpha > 0, \\ -\ln r & , \quad \alpha = 0. \end{cases}$$

In any case $u \notin L^\infty(M)$, moreover $u \notin L^1_{loc}(M)$ if $\alpha \ge 2$. Thus, our problem does not generally admits weak solutions understood in the sense of distribution.

The following example illustrates that for conservative equations weak solutions may not exist.

Example 5. Consider the following conservative form of equation (51)

$$\frac{\partial}{\partial x}(-xr^\alpha u) + \frac{\partial}{\partial y}(-yr^\alpha u) = 0, \quad r^2 = x^2 + y^2 \le 1. \tag{52}$$

This equation can be rewritten as

$$-xr^\alpha \frac{\partial u}{\partial x} - yr^\alpha \frac{\partial u}{\partial y} = (2+\alpha) r^\alpha u.$$

Therefore, along characteristics, which are the same as in Example 4,

$$\dot{u} = (2+\alpha) r^\alpha u = (2+\alpha) u/(1+\alpha s).$$

If we set the initial data $u(x,y) \equiv 1$ for $r = 1$ then necessarily $u(x,y) = r^{-2-\alpha} \notin L^1_{loc}(M)$.

The above examples induce us to formulate additional conditions, which guarantee well-posedness of the Dirichlet problem. It turns out that one of such condition may be the following one: $\forall R > 0$ there exist a function $\rho(x) \in C^1(M)$, $\rho(x) \ge 0$ and a smooth measure μ on M such that

$$\operatorname{div}^\mu(a(x,\lambda)\rho(x)) < 0 \ \text{ for a.e. } x \in M, \lambda \in [-R,R]. \tag{U}$$

Observe that condition (U) is always satisfied for the evolutionary equation (44). Indeed, let $\mu = dt \times \mu_0$, where μ_0 is a smooth measure on Ω, $R > 0$ and $C > \max\limits_{(t,x)\in M, |\lambda|\le R} \operatorname{div}_x^{\mu_0} a(t,x,\lambda)$. Then the function $\rho(t,x) = e^{-Ct}$ satisfies (U) because

$$\operatorname{div}^\mu \rho \left(\frac{\partial}{\partial t} + a(t,x,u)\right) = \rho'_t + \rho \operatorname{div}_x^{\mu_0} a(t,x,\lambda) = (\operatorname{div}_x^{\mu_0} a(t,x,\lambda) - C)\rho < 0$$

for all $(t,x) \in M$, $\lambda \in [-R,R]$.

Concerning the above Examples 4,5, observe that for the linear equation $\langle a(x), u\rangle = -xr^\alpha \dfrac{\partial u}{\partial x} - yr^\alpha \dfrac{\partial u}{\partial y} = 0$ condition (U) is satisfied with $\rho(x,y) = r^2$. Indeed, $\operatorname{div}(a(x)\rho(x)) = -(4+\alpha)r^{2+\alpha} < 0$ for $r > 0$.

We give below one necessary condition for fulfillment of (U).

Proposition 4. *If condition (U) is satisfied then for each fixed λ for a.e. $x \in M$ there exists a characteristic outgoing from the boundary and passing through the point x.*

Proof. Let $\lambda \in \mathbb{R}$ be fixed and $x(s) = x(s; y)$ be a characteristic passing through $y \in M_0$ for $s = 0$. Thus, $x(s)$ is the unique solution of ODE $\dot{x} = a(x, \lambda)$ on M satisfying the initial condition $x(0) = y$. This solution is defined on some maximal interval $s \in (\alpha(y), \beta(y))$, where $-\infty \leq \alpha(y) < 0 < \beta(y) \leq +\infty$. As is known in the theory of ODEs the functions $\alpha(y), \beta(y)$ are upper and low semi-continuous, respectively. Therefore, the set $A = \{ y \in M_0 \mid \alpha(y) = -\infty \}$ is Borel as an intersection of the sequence $V_r = \{ y \in M_0 \mid \alpha(y) < -r \}$, $r \in \mathbb{N}$ of open sets. If $\alpha(y) > -\infty$ then taking into account compactness of M we conclude that $x(s; y)$ is defined for $s \in [\alpha(y), \beta(y))$ and $x(\alpha(y); y) \in S$. Thus, points of the complement $M_0 \setminus A$ can be reached by characteristics outgoing from S and to prove the proposition it suffices to show that $\mu(A) = 0$, where μ is a smooth measure on M. On the set A we can define the semigroup of shifting operators $T_s y = x(-s; y)$, $s \geq 0$. Clearly, T_s is a C^1 diffeomorphism of the neighborhood $\alpha(y) < c - s$ of A onto the open set $\alpha(y) < c$, $\beta(y) > -s$. In particular, $T_s(A)$ is a Borel set and $\mu(T_s(A)) = \int_A \gamma(s, x) d\mu$ with some density $\gamma(s, x) \in C^1$. Besides, as one can derive from the Liouville theorem, $\left.\dfrac{d\gamma(s, x)}{ds}\right|_{s=0} = -(\operatorname{div}^{\mu} a(x, \lambda))$. Indeed, let U be a coordinate neighborhood corresponding to some chart (U, j, V), and $\mu = \omega(x) dx$, $\omega(x) \in C^1(V)$, $\omega(x) > 0$. Then for each compact $K \subset U$ and for sufficiently small s (such that $T_s(K) \subset U$)

$$\mu(T_s(A \cap K)) = \int_{j(A \cap K)} \omega(x(-s; y)) \det\{\partial x_i(-s; y)/\partial y_j\} dy =$$
$$\int_{j(A \cap K)} \omega(x(-s; y)) \det\{\partial x_i(-s; y)/\partial y_j\}/\omega(y) d\mu(y),$$

where $x(s; y)$ is a solution of the problem $\dot{x} = a(x, \lambda)$, $x(0) = y$ considered on V. Thus, $\gamma(s, y) = \omega(x(-s; y)) \det\{\partial x_i(-s; y)/\partial y_j\}/\omega(y)$ and with using the Liouville theorem we obtain that

$$\left.\frac{d\gamma(s, y)}{ds}\right|_{s=0} = -\left(\langle a(y), \omega(y) \rangle + \omega(y) \frac{\partial a^i(y, \lambda)}{\partial y_i} \right) / \omega(y) =$$
$$-\frac{1}{\omega(y)} \frac{\partial (a^i(y, \lambda) \omega(y))}{\partial y_i} = -\operatorname{div}^{\mu} a(y).$$

Taking into account arbitrariness of the coordinate neighborhood U we conclude that $\left.\frac{d\gamma(s,y)}{ds}\right|_{s=0} = -(\operatorname{div}^{\mu} a(y,\lambda))$, as was announced.

By condition (U) there exist a smooth measure μ and a nonnegative function $\rho(x) \in C^1(M)$ such that $\operatorname{div}^{\mu}(a(x,\lambda)\rho(x)) < 0$ a.e. on M_0. Consider, the function

$$I(s) = \int_{T_s(A)} \rho(x)d\mu = \int_A \rho(T_s y)\gamma(s,y)d\mu(y).$$

We observe that the sets $T_s(A)$ are decreasing, i.e. $T_{s_2}(A) \subset T_{s_1}(A)$ for $s_2 > s_1 \geq 0$. Hence, $I'(0) \leq 0$. On the other hand,

$$I'(0) = -\int_A [\langle a(y,\lambda), \rho(y)\rangle + \rho(y)\operatorname{div}^{\mu} a(y,\lambda)\rho(y)]d\mu(y) =$$
$$-\int_A \operatorname{div}^{\mu}(a(y,\lambda)\rho(y))d\mu.$$

Hence, $\int_A \operatorname{div}^{\mu}(a(y,\lambda)\rho(y))d\mu \geq 0$ and since the integrand is negative for a.e. $y \in A$ we conclude that $\mu(A) = 0$ as required. The proof is complete.

§ 3. The kinetic formulation.

For a positive R we denote by F_R the class of functions $p(\lambda) = \nu((\lambda, +\infty))$, where ν is a probability measure with support in $[-R, R]$. Let $F = \bigcup_{R>0} F_R$. In other words, F consists of non-increasing distribution functions of probability measures with compact support on $\mathbb{R}$. We will refer to F as to *the kinetic class*. Obviously, a function $p(\lambda) \in F_R$ if and only if it is non-increasing, continuous from the right, and $p(\lambda) = 1$ for $\lambda < -R$, $p(\lambda) = 0$ for $\lambda \geq R$. The classes F_R, F are convex and closed subsets of $L^r(I)$ for any segment $I \subset \mathbb{R}$ and any $r \in [1, +\infty]$. We will consider functions $p(\lambda) \in F$ as elements of $L^\infty(\mathbb{R})$, so functions differing on a set of null measure, are identified. In particular, the values of $p(\lambda) \in F$ at discontinuity points may be chosen arbitrarily and the above requirement of right-continuity of $p(\lambda)$ may be removed. Observe that the functions $p(\lambda) = \operatorname{sign}^+(u - \lambda) \in F_R$ for each $u \in [-R, R]$. The classes F_R and F are invariant with respect to the involution $p \to \bar{p}$ defined as $\bar{p}(\lambda) = 1 - p(-\lambda)$. Clearly, this involution is a decreasing operator that is $\bar{p}_1 \leq \bar{p}_2$ whenever $p_1 \geq p_2$. Remark that for $p(\lambda) = \operatorname{sign}^+(u - \lambda)$ $\bar{p}(\lambda) = \operatorname{sign}^+(-u - \lambda)$. Thus, for this class of kinetic functions the involutions reduces to simple one: $u \to -u$. For the sequel we will use the following binary operation well-defined on the classes F_R and F: $p \circ q = 1-(1-p)(1-q) = p+q-pq$. This operation is uniquely defined by the property: $\overline{p \circ q} = \bar{p}\bar{q}$.

Now, we will denote by $\mathcal{F}_R(M)$ the space of strongly measurable functions with values in F_R. This space can be described as the subspace of functions $p \in L^\infty(M \times \mathbb{R})$ such that $p(x,\cdot) \in F_R$ for a.e. $x \in M$. We also use the notation $\mathcal{F}(M)$ for $\bigcup_{R>0} \mathcal{F}_R(M)$. Similarly we define spaces $\mathcal{F}_R(S)$, $\mathcal{F}(S)$.

We consider the following kinetic equation

$$\langle a(x,\lambda), p\rangle = 0, \quad p = p(x,\lambda) \tag{53}$$

associated with (1). We will study the Dirichlet problem for this equation with the boundary data

$$p(x,\lambda) = p_b(x,\lambda) = \operatorname{sign}^+(u_b(x)-\lambda) \ \text{ on } S \times \mathbb{R}, \tag{54}$$

where $u_b(x) \in L^\infty(S)$. Similarly to [10] we define notions of a kinetic sub- and super-solution (k.sub-s. and k.super-s. for short).

Definition 4. A function $p(x,\lambda) \in \mathcal{F}(M)$ is called a k.sub-s. of (53), (54) if there exists a constant $L > 0$ such that for each $q = q(\lambda) \in F$ $\forall f = f(x) \in C^1(M)$, $f \geq 0$

$$\int_{M\times\mathbb{R}} p(x,\lambda)(1-q(\lambda))\operatorname{div}^\mu(a(x,\lambda)f(x))d\mu d\lambda + \\ L\int_{S\times\mathbb{R}} p_b(x,\lambda)(1-q(\lambda))f(x)d\mu_b d\lambda \geq 0; \tag{55}$$

a function $p(x,\lambda) \in \mathcal{F}(M)$ is called a k.super-s. of (53), (54) if there exists a constant $L > 0$ such that for each $q = q(\lambda) \in F$ $\forall f = f(x) \in C^1(M)$, $f \geq 0$

$$\int_{M\times\mathbb{R}} (1-p(x,\lambda))q(\lambda)\operatorname{div}^\mu(a(x,\lambda)f(x))d\mu d\lambda + \\ L\int_{S\times\mathbb{R}} (1-p_b(x,\lambda))q(\lambda)f(x)d\mu_b d\lambda \geq 0. \tag{56}$$

We call $p(x,\lambda)$ a kinetic solution (a k.s.) of (53), (54) if it is a k.sub-s. and a k.super-s. of this problem simultaneously.

In relations (55), (56) μ, μ_b are smooth measures on M and S respectively. In the same way as for g.e.sub-s. and g.e.super-s. of the original problem, one can prove that our definition actually does not depend on the choice of these measures.

The following useful lemma allows to reduce statements concerning k.super-s. to ones for k.sub-s.

Lemma 1. *A function $p(x, \lambda)$ is a k.super-s. of (53), (54) if and only if $\bar{p}(x, \lambda) = 1 - p(x, -\lambda)$ is a k.sub-s. of the problem*

$$\langle a(x, -\lambda), p\rangle = 0, \quad p(x, \lambda)|_{S\times\mathbb{R}} = \overline{p_b}(x, \lambda) = 1 - p_b(x, -\lambda). \tag{57}$$

Proof. We write conditions (55) for problem (57) applied to the function $\bar{p}(x, \lambda)$: for each $q(\lambda) \in F$, $f(x) \in C^1(M)$, $f(x) \geq 0$

$$\int_{M\times\mathbb{R}} \bar{p}(x, \lambda)(1 - q(\lambda))\operatorname{div}^{\mu}(a(x, -\lambda)f(x))d\mu d\lambda + \\ L\int_{S\times\mathbb{R}} \overline{p_b}(x, \lambda)(1 - q(\lambda))f(x)d\mu_b d\lambda \geq 0. \tag{58}$$

Changing the variables $\lambda \to -\lambda$ in both integrals, we can rewrite it as follows

$$\int_{M\times\mathbb{R}} (1 - p(x, \lambda))\bar{q}(\lambda)\operatorname{div}^{\mu}(a(x, \lambda)f(x))d\mu d\lambda + \\ L\int_{S\times\mathbb{R}} (1 - p_b(x, \lambda))\bar{q}(\lambda)f(x)d\mu_b d\lambda \geq 0.$$

Since $\bar{q}(\lambda)$ runs the entire class F the latter relation coincide with (56). By the construction it is equivalent to (58), which completes the proof.

Our kinetic formulation is based on the following result.

Theorem 4. *Suppose that $p_b(x, \lambda) = \operatorname{sign}^+(u_b(x) - \lambda)$. Then $u(x) \in L^\infty(M)$ is a g.e.sub-s. (g.e.super-s.) of (1), (2) if and only if the function $p(x, \lambda) = \operatorname{sign}^+(u(x) - \lambda)$ is a k.sub-s. (respectively k.super-s.) of kinetic problem (53), (54).*

Proof. We prove the statement concerning sub-solutions. For super-solutions the proof is similar. Let $k \in \mathbb{R}$ and $q(\lambda) = \operatorname{sign}^+(k - \lambda)$. Then

$$\int p(x, \lambda)(1 - q(\lambda))\operatorname{div}^{\mu}(a(x, \lambda)f(x))d\lambda =$$

$$\operatorname{sign}^+(u(x)-k)\int_k^{u(x)} \operatorname{div}^{\mu}(a(x,\lambda)f(x))d\lambda =$$

$$\operatorname{sign}^+(u(x)-k)\left\{f(x)\int_k^{u(x)} \operatorname{div}^{\mu} a(x,\lambda)d\lambda + \int_k^{u(x)} \langle a(x,\lambda), f(x)\rangle d\lambda\right\} =$$

$$\operatorname{sign}^+(u(x)-k)[f(x)\operatorname{div}_x^{\mu}(\varphi(x,u)-\varphi(x,k)) + \langle \varphi(x,u)-\varphi(x,k), f\rangle],$$

$$\int p_b(x,\lambda)(1-q(\lambda))d\lambda = (u_b(x)-k)^+.$$

Hence relation (55) is equivalent to (24). We also take int account that the convex hull of functions $q(\lambda) = \operatorname{sign}^+(k-\lambda)$, $|k| \le R$ is dense in F_R (in the $L^1([-R,R])$-topology) for all $R > 0$. Therefore, it is sufficient to require in (55), (56) that $q(\lambda)$ is a function of the kind $q(\lambda) = \operatorname{sign}^+(k-\lambda)$. The proof is complete.

Using Theorem 4 and Lemma 1 we readily deduce the following statement.

Corollary 2. *A function $u = u(x) \in L^\infty(M)$ is a g.e.super-s. (g.e.sub-s., g.e.s.) of (1), (2) if and only if the function $-u$ is a g.e.s. (respectively g.e.super-s., g.e.s.) of the problem*

$$\langle a(x,-u), u\rangle = 0, \quad u|_S = -u_b(x).$$

If $p(x,\lambda)$ is a k.sub-s. of (53), (54) and $q \in F$ then as follows from (55) for each $f = f(x) \in C_0^1(M_0)$, $f \ge 0$

$$\int_M \left\{\left\langle \int a(x,\lambda)p(x,\lambda)(1-q(\lambda))d\lambda, f\right\rangle + \right.$$
$$\left. f\cdot \int \operatorname{div}^{\mu} a(x,\lambda)p(x,\lambda)(1-q(\lambda))d\lambda\right\} d\mu(x) \ge 0. \tag{59}$$

By Proposition 1 we see that $\int a(x,\lambda)p(x,\lambda)(1-q(\lambda))d\lambda$ is a divergence measure field for every $q(\lambda) \in F$. Therefore, there exists a weak normal trace $v = v_q(x) \in L^\infty(S)$ of this field at the boundary (for a fixed pair $\bar\mu = (\mu, \mu_b)$ of smooth measures). Passing to local coordinates $(x_1, x') \in (0,h)\times W$, indicated in Remark 3, we obtain with account of this Remark that

$$v_q(x') = \int a_{\bar\mu}(x',\lambda)p_\tau(x',\lambda)(1-q(\lambda))d\lambda,$$

where $p_\tau(x',\lambda)$ is an arbitrary weak-$*$ limit point of $p(x_1,x',\lambda)$ in $L^\infty(W\times\mathbb{R})$ as $x_1\to 0$ running over the set of full measure

$$E=\{\ t\in(0,h)\mid(t,y,\lambda)\text{ is a Lebesgue point of }\\ p(t,y,\lambda)\text{ for a.e. }y\in W,\lambda\in\mathbb{R}\ \}$$

defined similarly to (16). Since $v_q(x)$ does not depend on the choice of this limit point we conclude that $a_{\bar\mu}(x',\lambda)(p(x_1,x',\lambda)-p_\tau(x',\lambda))\underset{x_1\to 0,x_1\in E}{\to}0$ weakly-$*$ in $L^\infty(W\times\mathbb{R})$. This implies in particular that $p_\tau(x',\lambda)$ is uniquely defined on the set where $a_\mu(x',\lambda)\neq 0$ and does not depend on the choice of $\bar\mu$. Taking $R>0$ from the condition $p(x,\lambda)\in\mathcal{F}_R(M)$ we find also by the known property of weak limits that $p_\tau(x',\cdot)\in F_R$ because F_R is convex and closed subset of $L^\infty(\mathbb{R})$.

By the construction we have

$$\forall q\in F\quad v_q(x)=\int a_{\bar\mu}(x,\lambda)p_\tau(x,\lambda)(1-q(\lambda))d\lambda \tag{60}$$

a.e. on $S\cap U$. Here $p_\tau(x,\lambda)\doteq p_\tau(j(x),\lambda)$ on $S\cap U$, (U,j,V) being the chart corresponding to our local coordinates. By (60) we see that the product $a_{\bar\mu}(x,\lambda)p_\tau(x,\lambda)$ is uniquely determined in a coordinate neighborhood $U\cap S$ of an arbitrary boundary point. This allows us to define $p_\tau(x,\lambda)\in\mathcal{F}_R(S)$ in such a way that (60) is satisfied for a.e. $x\in S$. As follows from (59) and Corollary 1 for each $q(\lambda)\in F$ and all $f=f(x)\in C^1(M)$, $f\geq 0$

$$\int_{M\times\mathbb{R}}p(x,\lambda)(1-q(\lambda))\operatorname{div}^\mu(a(x,\lambda)f(x))d\mu d\lambda-\\ \int_{S\times\mathbb{R}}a_{\bar\mu}(x,\lambda)p_\tau(x,\lambda)(1-q(\lambda))f(x)d\mu_b d\lambda\geq 0. \tag{61}$$

Remark that the set $F_R\subset L^1([-R,R])$ is separable while the both sides of the equality $v_q(x)=\displaystyle\int a_{\bar\mu}(x,\lambda)p_\tau(x,\lambda)(1-q(\lambda))d\lambda$ are continuous with respect to $q\in L^1([-R,R])$ with values in $L^\infty(S)$. Therefore, we can choose a set $\tilde S_R\subset S$ of full measure in such a way that for $x\in\tilde S_R$ equality (60) holds for all $q\in F_R$. Taking $\tilde S=\bigcap_{R\in\mathbb{N}}\tilde S_R$ we obtain that $\tilde S$ has full measure on S and (60) is satisfied for all $x\in\tilde S$ and all $q\in F$. In other words, for a.e. $x\in S$ condition (60) holds.

In the same way, based on the relation

$$\forall f \in C_0^1(M_0), f \geq 0 \ \int_M \left\{ \left\langle \int a(x,\lambda)(1-p(x,\lambda))q(\lambda)d\lambda, f \right\rangle + f \cdot \int \operatorname{div}^\mu a(x,\lambda)(1-p(x,\lambda))q(\lambda)d\lambda \right\} d\mu(x) \geq 0, \quad (62)$$

we prove existence of weak trace $p_\tau(x,\lambda) \in \mathcal{F}(S)$ for a k.super-s. $p(x,\lambda)$ of (53), (54). This trace satisfies the condition: for a.e. $x \in S$

$$\forall q \in F \quad w_q(x) = \int a_{\bar{\mu}}(x,\lambda)(1-p_\tau(x,\lambda))q(\lambda)d\lambda, \quad (63)$$

where $w_q(x) \in L^\infty(S)$ is the weak trace of the divergence measure field $\int a(x,\lambda)(1-p(x,\lambda))q(\lambda)d\lambda$.

Using Corollary 1 again we derive from (62) that for each $q(\lambda) \in F$ and all $f = f(x) \in C^1(M)$, $f \geq 0$

$$\int_{M\times\mathbb{R}} (1-p(x,\lambda))q(\lambda)\operatorname{div}^\mu(a(x,\lambda)f(x))d\mu d\lambda - \int_{S\times\mathbb{R}} a_{\bar{\mu}}(x,\lambda)(1-p_\tau(x,\lambda))q(\lambda)f(x)d\mu_b d\lambda \geq 0. \quad (64)$$

As in Theorem 3 we derive that $p(x,\lambda)$ is a k.sub-s. of (53), (54) if and only if for each $q(\lambda) \in F$ and all $f = f(x) \in C_0^1(M_0)$, $f \geq 0$

$$\int_{M\times\mathbb{R}} p(x,\lambda)(1-q(\lambda))\operatorname{div}^\mu(a(x,\lambda)f(x))d\mu d\lambda \geq 0$$

and the following relation similar to (28) is satisfied a.e. on S:

$$\forall q(\lambda) \in F \ \int a_{\bar{\mu}}(x,\lambda)p_\tau(x,\lambda)(1-q(\lambda))d\lambda + L\int p_b(x,\lambda)(1-q(\lambda))d\lambda \geq 0. \quad (65)$$

Respectively, $p(x,\lambda)$ is a k.super-s. of (53), (54) if and only if for each $q(\lambda) \in F$ and all $f = f(x) \in C_0^1(M_0)$, $f \geq 0$

$$\int_{M\times\mathbb{R}} (1-p(x,\lambda))q(\lambda)\operatorname{div}^\mu(a(x,\lambda)f(x))d\mu d\lambda \geq 0$$

and for a.e. $x \in S$

$$\forall q(\lambda) \in F \ \int a_{\bar{\mu}}(x,\lambda)(1-p_\tau(x,\lambda))q(\lambda)d\lambda + L\int (1-p_b(x,\lambda))q(\lambda)d\lambda \geq 0. \quad (66)$$

Here L is a sufficiently large constant.

If $p_b(x,\lambda) = \mathrm{sign}^+(u_b(x)-\lambda)$ then boundary relations (65), (66) are equivalent the following conditions: for a.e. $x \in S$

$$\forall k \geq u_b(x) \quad \int_k^{+\infty} a_{\bar{\mu}}(x,\lambda)p_\tau(x,\lambda)d\lambda \geq 0; \tag{67}$$

$$\forall k \leq u_b(x) \quad \int_{-\infty}^{k} a_{\bar{\mu}}(x,\lambda)(1-p_\tau(x,\lambda))d\lambda \geq 0, \tag{68}$$

respectively. Indeed, these relations follows from (65), (66) with $q(\lambda) = \mathrm{sign}^+(k-\lambda)$. Conversely, suppose that $L \geq L_{\bar{\mu}} = \sup\limits_{x\in S, u\in\mathbb{R}} |a_{\bar{\mu}}(x,u)|$ and $p(x,\lambda)$ satisfies (67). Then for $q(\lambda) = \mathrm{sign}^+(k-\lambda)$ we have

$$\int a_{\bar{\mu}}(x,\lambda)p_\tau(x,\lambda)(1-q(\lambda))d\lambda + L\int p_b(x,\lambda)(1-q(\lambda))d\lambda =$$
$$\int_{\max(k,u_b(x))}^{+\infty} a_{\bar{\mu}}(x,\lambda)p_\tau(x,\lambda)d\lambda +$$
$$\mathrm{sign}^+(u_b(x)-k)\int_k^{u_b(x)} (a_{\bar{\mu}}(x,\lambda)p_\tau(x,\lambda)+L)d\lambda \geq 0.$$

Since the convex hull of functions $\mathrm{sign}^+(k-\lambda)$ is dense in F with respect to topology of $L^1_{loc}(\mathbb{R})$ we conclude that (65) fulfils for all $q(\lambda) \in F$. By similar reasons we prove that (68) implies (66).

Remark 5. As one can see from the above statements, we always may take in (55), (56) the constant $L = L_{\bar{\mu}}$.

§ 4. The comparison principle and uniqueness of g.e.s.

Using the Kruzhkov method of doubling variable we can establish (as in [10]) the following result.

Theorem 5. *Suppose that $p_1(x,\lambda)$ is a k.sub-s. and $p_2(x,\lambda)$ is a k.super-s. of (53), (54) with boundary data $p_{1b}(x,\lambda) = \mathrm{sign}^+(u_{1b}(x)-\lambda)$ and $p_{2b}(x,\lambda) = \mathrm{sign}^+(u_{2b}(x)-\lambda)$, respectively, and $\bar{\mu} = (\mu,\mu_b)$ is a pair of smooth measures. Then for some $L > 0$ and each $f = f(x) \in C^1(M)$, $f \geq 0$*

$$\int_{M\times\mathbb{R}} p_1(x,\lambda)(1-p_2(x,\lambda))\mathrm{div}^{\mu}(a(x,\lambda)f(x))d\mu d\lambda +$$
$$L\int_{S\times\mathbb{R}} p_{1b}(x,\lambda)(1-p_{2b}(x,\lambda))f(x)d\mu_b d\lambda \geq 0. \tag{69}$$

Proof. Since inequality (69) has local character we can assume that $\operatorname{supp} f$ contains in a coordinate neighborhood U corresponding to a chart (U, j, V). Suppose that in local coordinates $x_1, \dots, x_n$ $a = a^i(x, \lambda)\partial/\partial x_i$, $\mu = \omega(x)dx$, $\mu_b = \omega_b(x')dx'$, where $x' = (0, x_2, \dots, x_n) \in \partial V$. By relations (61), (64), with regards of formulas (7), (11), (60), (63), we have

$$\int_{V\times\mathbb{R}} p_1(x,\lambda)(1-q(\lambda))\operatorname{div}(a(x,\lambda)\omega(x)f(x))dxd\lambda +$$
$$\int_{\partial V\times\mathbb{R}} a^1(x',\lambda)(p_1)_\tau(x',\lambda)(1-q(\lambda))\omega(x')f(x')dx'd\lambda \ge 0; \tag{70}$$
$$\int_{V\times\mathbb{R}} q(\lambda)(1-p_2(x,\lambda))\operatorname{div}(a(x,\lambda)\omega(x)f(x))dxd\lambda +$$
$$\int_{\partial V\times\mathbb{R}} a^1(x',\lambda)q(\lambda)(1-(p_2)_\tau(x',\lambda))\omega(x')f(x')dx'd\lambda \ge 0 \tag{71}$$

for all nonnegative $f = f(x) \in C_0^1(V)$, where $(p_1)_\tau(x',\lambda)$, $(p_2)_\tau(x',\lambda)$ are weak traces of p_1, p_2 written in the local coordinates x' and $\operatorname{div} v$ means the "usual" divergence $\partial v^i/\partial x_i$ of a vector field v.

Putting in (70) $q(\lambda) = p_2(y,\lambda)$, $f = f(x;y) \in C_0^1(V\times V)$ and integrating over $y \in V$, we derive that

$$\int_{V\times V\times\mathbb{R}} p_1(x,\lambda)(1-p_2(y,\lambda))\operatorname{div}_x(a(x,\lambda)\omega(x)f(x;y))dxdyd\lambda +$$
$$\int_{\partial V\times V\times\mathbb{R}} a^1(x',\lambda)(p_1)_\tau(x',\lambda)(1-p_2(y,\lambda))\omega(x')f(x';y)dx'dyd\lambda \ge 0. \tag{72}$$

Similarly, changing the places of variables x and y, we derive from (71) with $q = p_1(x,\lambda)$, that

$$\int_{V\times V\times\mathbb{R}} p_1(x,\lambda)(1-p_2(y,\lambda))\operatorname{div}_y(a(y,\lambda)\omega(y)f(x;y))dxdyd\lambda +$$
$$\int_{V\times\partial V\times\mathbb{R}} a^1(y',\lambda)p_1(x,\lambda)(1-(p_2)_\tau(y',\lambda))\omega(y')f(x;y')dxdy'd\lambda \ge 0. \tag{73}$$

Putting (72), (73) together, we arrive at

$$\int_{V\times V\times\mathbb{R}} p_1(x,\lambda)(1-p_2(y,\lambda)) \times$$
$$\{\operatorname{div}_x(a(x,\lambda)\omega(x)f(x;y)) + \operatorname{div}_y(a(y,\lambda)\omega(y)f(x;y))\}dxdyd\lambda +$$

$$\int_{\partial V\times V\times\mathbb{R}} a^1(x',\lambda)(p_1)_\tau(x',\lambda)(1-p_2(y,\lambda))\omega(x')f(x';y)dx'dyd\lambda +$$
$$\int_{V\times\partial V\times\mathbb{R}} a^1(y',\lambda)p_1(x,\lambda)(1-(p_2)_\tau(y',\lambda))\omega(y')f(x;y')dxdy'd\lambda \ge 0. \qquad (74)$$

Now we apply (74) to the test function $f(x;y) = f(x)\delta_\nu(y-x)$, where $f(x)\in C_0^1(V)$, $f(x)\ge 0$, and $\delta_\nu(z)=\prod_{i=1}^n \rho_\nu(z_i)$ for $z=(z_1,\dots,z_n)\in\mathbb{R}^n$, $\nu\in\mathbb{N}$. Here the function $\rho_\nu(s)=\nu\rho(\nu s)$ was defined above in the proof of Proposition 1. Since $\operatorname{supp}\rho\subset(0,1)$ and $x_1\ge 0$ we see that $f(x;y')\equiv 0$ and the last integral in (74) disappears. We denote $r(x)=p_1(x,\lambda)$ for fixed $\lambda\in\mathbb{R}$ and observe that

$$\begin{aligned}
&\int_V r(x)[\operatorname{div}_x(a(x,\lambda)\omega(x)f(x;y)) + \operatorname{div}_y(a(y,\lambda)\omega(y)f(x;y))]dx = \\
&\qquad\int_V r(x)\operatorname{div}_x(f(x)\omega(x)a(x,\lambda))\delta_\nu(y-x)dx - \\
&\qquad\int_V r(x)f(x)\omega(x)a^i(x,\lambda)\frac{\partial\delta_\nu(y-x)}{\partial y_i}dx + \\
&\qquad\operatorname{div}_y\left(a(y,\lambda)\omega(y)\int_V f(x)r(x)\delta_\nu(y-x)dx\right) = \\
&\qquad(r\operatorname{div}(f\omega a(\cdot,\lambda)))*\delta_\nu(y) + \operatorname{div}(\omega(y)a(y,\lambda)(fr*\delta_\nu)(y)) - \\
&\qquad(\operatorname{div}(\omega a(\cdot,\lambda)fr))*\delta_\nu(y),
\end{aligned}$$

where $*$ is the convolution operation. Since the vector field $\omega(y)a(y,\lambda)$ is smooth then by the DiPerna-Lions commutation lemma (see [7])

$$\operatorname{div}(\omega(y)a(y,\lambda)(fr*\delta_\nu)(y)) - (\operatorname{div}(\omega a(\cdot,\lambda)fr))*\delta_\nu(y)\to 0$$

as $\nu\to\infty$ in $L^1_{loc}(V)$ for each $\lambda\in\mathbb{R}$ while by the known property of convolutions

$$(r\operatorname{div}(f\omega a(\cdot,\lambda)))*\delta_\nu(y)\to r(y)\operatorname{div}(f(y)\omega(y)a(y,\lambda)) \ \text{ in } L^1_{loc}(V).$$

Hence in the limit as $\nu\to\infty$ the first integral in (74) converges to

$$I_1=\int_{V\times\mathbb{R}} p_1(x,\lambda)(1-p_2(x,\lambda))\operatorname{div}(a(x,\lambda)\omega(x)f(x))dxd\lambda$$

(for convenience we place variable x instead of y).

If $\partial V \neq \emptyset$ we have to find the limit as $\nu \to \infty$ of the second integral in (74). In this case we may assume that $V = (0, h) \times W$. Then this integral can be rewritten as follows

$$I_2^\nu = \int_{W\times V\times\mathbb{R}} a^1(x',\lambda)(p_1)_\tau(x',\lambda)(1-p_2(y,\lambda))\omega(x')f(x')\delta_\nu(y-x')dx'dyd\lambda.$$

From the condition that $a^1(0,z,\lambda)(p_2(t,z,\lambda)-(p_2)_\tau(z,\lambda)) \to 0$ weakly-$*$ in $L^\infty(W\times\mathbb{R})$ as $t \to 0$ running over a set of full measure it easily follows that

$$a^1(x',\lambda)\int_V (1-p_2(y,\lambda))\delta_\nu(y-x')dy \underset{\nu\to\infty}{\to} a^1(x',\lambda)(1-(p_2)_\tau(x',\lambda))$$

weakly-$*$ in $L^\infty(W\times\mathbb{R})$ and therefore

$$I_2^\nu \underset{\nu\to\infty}{\to} I_2 = \int_{W\times\mathbb{R}} a^1(x',\lambda)(p_1)_\tau(x',\lambda)(1-(p_2)_\tau(x',\lambda))\omega(x')f(x')dx'd\lambda.$$

Hence from (74) in the limit as $\nu \to \infty$ it follows the relation $I_1 + I_2 \geq 0$ that is

$$\int_{V\times\mathbb{R}} p_1(x,\lambda)(1-p_2(x,\lambda))\operatorname{div}(a(x,\lambda)\omega(x)f(x))dxd\lambda +$$
$$\int_{W\times\mathbb{R}} a^1(x',\lambda)(p_1)_\tau(x',\lambda)(1-(p_2)_\tau(x',\lambda))\omega(x')f(x')dx'd\lambda \geq 0.$$

This relation could be rewritten as

$$\int_{M\times\mathbb{R}} p_1(x,\lambda)(1-p_2(x,\lambda))\operatorname{div}^\mu(a(x,\lambda)f(x))d\mu d\lambda -$$
$$\int_{S\times\mathbb{R}} a_{\bar\mu}(x,\lambda)(p_1)_\tau(x,\lambda)(1-(p_2)_\tau(x,\lambda))f(x)d\mu_b d\lambda \geq 0 \qquad (75)$$

for all nonnegative test functions $f(x)$ with supports in U. Since U is an arbitrary coordinate neighborhood then, with the help of a partition of unity, we conclude that (75) is satisfied for every $f(x) \in C^1(M)$, $f(x) \geq 0$. From relation (65) with $p = p_1$, $q = \max((p_2)_\tau(x,\lambda), p_{2b}(x,\lambda))$ it follows that for a.e. $x \in S$

$$\int_{u_{2b}(x)}^{+\infty} a_{\bar\mu}(x,\lambda)(p_1)_\tau(x,\lambda)(1-(p_2)_\tau(x,\lambda))d\lambda +$$
$$L\int p_{1b}(x,\lambda)(1-p_{2b}(x,\lambda))d\lambda \geq 0.$$

We take also into account that $q(\lambda) \geq p_{2b}(x,\lambda)$ so that $p_{1b}(x,\lambda)(1 - p_{2b}(x,\lambda)) \geq p_{1b}(x,\lambda)(1 - q(\lambda))$. Similarly, putting in (66) $p = p_2$, $q = \min((p_1)_\tau(x,\lambda), p_{2b}(x,\lambda))$ and taking into account that $q(\lambda)(1-p_{2b}(x,\lambda)) \equiv 0$ we derive that for a.e. $x \in S$

$$\int_{-\infty}^{u_{2b}(x)} a_{\bar{\mu}}(x,\lambda)(p_1)_\tau(x,\lambda)(1 - (p_2)_\tau(x,\lambda))d\lambda \geq 0.$$

Putting the above inequality together we obtain that for a.e. $x \in S$

$$\int a_{\bar{\mu}}(x,\lambda)(p_1)_\tau(x,\lambda)(1 - (p_2)_\tau(x,\lambda))d\lambda + L\int p_{1b}(x,\lambda)(1 - p_{2b}(x,\lambda))d\lambda \geq 0.$$

Integration over $f(x)\mu_b$ yields

$$\int_{S\times\mathbb{R}} a_{\bar{\mu}}(x,\lambda)(p_1)_\tau(x,\lambda)(1-(p_2)_\tau(x,\lambda))f(x)d\mu_b d\lambda + \\ L\int_{S\times\mathbb{R}} p_{1b}(x,\lambda)(1 - p_{2b}(x,\lambda))f(x)d\mu_b d\lambda \geq 0.$$

Adding this inequality to (75), we readily obtain (69). The proof is complete.

Corollary 3. *Suppose that condition (U) is satisfied. Then any k.s. of (53), (54) has the form $p(x,\lambda) = \operatorname{sign}^+(u(x) - \lambda)$, where $u(x)$ is a g.e.s. of the problem (1), (2).*

Proof. Since $p(x,\lambda)$ is a k.sub-s. and k.super-s. of (53), (54) simultaneously we can apply Theorem 5 to $p_1 = p_2 = p$. Since $p_b(x,\lambda)(1-p_b(x,\lambda)) \equiv 0$ we derive from (69) that

$$\int_{M\times\mathbb{R}} p(x,\lambda)(1 - p(x,\lambda))\operatorname{div}^\mu(a(x,\lambda)f(x))d\mu d\lambda \geq 0. \tag{76}$$

If $p(x,\lambda) \in \mathcal{F}_R(M)$ then $p(x,\lambda)(1-p(x,\lambda)) = 0$ out of the segment $[-R,R]$. By condition (U) we can find a test function $\rho(x) \in C^1(M)$, $\rho(x) \geq 0$ and smooth measure μ such that $\operatorname{div}^\mu(a(x,\lambda)f(x)) < 0$ for a.e. $x \in M$, $\lambda \in [-R,R]$. Putting $f = \rho(x)$ in (76) we find that $p(x,\lambda)(1 - p(x,\lambda)) = 0$ a.e. on $M \times \mathbb{R}$. Since $p(x,\lambda)$ is non-increasing with respect to λ this implies that this function has the required form $p(x,\lambda) = \operatorname{sign}^+(u(x) - \lambda)$, where $u(x) \in L^\infty(M)$, $\|u\|_\infty \leq R$ (the fact that $u(x)$ is measurable directly follows from measurability of $p(x,\lambda)$). By Theorem 4 we conclude that $u(x)$ is a g.e.s. of original problem (1), (2). This completes the proof.

Another consequence of Theorem 5 is the following comparison principle

Theorem 6. *Let $u_1(x)$ be a g.e.sub-s. and $u_2(x)$ be a g.e.super-s. of (1), (2) with boundary data u_{1b}, u_{2b} respectively, and $u_{1b} \le u_{2b}$ a.e. on S. Suppose also that condition (U) is satisfied. Then $u_1(x) \le u_2(x)$ a.e. on M.*

Proof. By Theorem 4 the function $p_1(x,\lambda) = \operatorname{sign}^+(u_1(x) - \lambda)$, $p_2(x,\lambda) = \operatorname{sign}^+(u_2(x) - \lambda)$ are a k.sub-s. and a k.super-s. of (53), (54) with the boundary data $p_{1b}(x,\lambda) = \operatorname{sign}^+(u_{1b}(x) - \lambda)$, $p_{2b}(x,\lambda) = \operatorname{sign}^+(u_{2b}(x) - \lambda)$ respectively. By the assumption $u_{1b} \le u_{2b}$ we have $p_{1b}(x,\lambda)(1 - p_{2b}(x,\lambda)) = 0$ a.e. on $S \times \mathbb{R}$. Then, by Theorem 5 for each smooth measure μ and every nonnegative test function $f = f(x) \in C^1(M)$

$$\int_{M\times\mathbb{R}} p_1(x,\lambda)(1 - p_2(x,\lambda))\operatorname{div}^{\mu}(a(x,\lambda)f(x))d\mu d\lambda \ge 0. \tag{77}$$

Using condition (U) with $R = \max(\|u_1\|_\infty, \|u_2\|_\infty)$ we find the nonnegative function $\rho(x) \in C^1(M)$ and the smooth measure μ such that $\operatorname{div}^{\mu}(a(x,\lambda)\rho(x)) < 0$ for a.e. $(x,\lambda) \in M \times [-R,R]$. Putting $f = \rho(x)$ in (77) we conclude that $p_1(x,\lambda)(1 - p_2(x,\lambda)) = 0$ a.e. on $M \times \mathbb{R}$ (we remark also that this function vanishes for $|\lambda| > R$). Since the latter is equivalent to the inequality $u_1(x) \le u_2(x)$ a.e. on M, this completes the proof.

Corollary 4. *Under condition (U) a g.e.s. of (1), (2) is unique.*

For the **proof** we apply the comparison principle for two g.e.s. $u_1(x)$, $u_2(x)$ and derive that $u_1(x) \le u_2(x)$, $u_2(x) \le u_1(x)$ a.e. on M. Thus, $u_1(x) = u_2(x)$ a.e. on M, as required.

Corollary 5 (maximum principle). *Assume that $c_1 \le u_b(x) \le c_2$ a.e. on S with some constants $c_1, c_2 \in \mathbb{R}$, and $u(x)$ is a g.e.s. of (1), (2). Then, under condition (U), $c_1 \le u(x) \le c_2$ a.e. on M.*

Proof. Let us first show that a constant function $u(x) \equiv c$ is a g.e.s. of problem (1), (2) with the same constant boundary data $u_b(x) \equiv c$. Indeed, let $\bar\mu = (\mu, \mu_b)$ be a pair of smooth measures and $L \ge \sup\limits_{x\in S, u\in\mathbb{R}} |a_{\bar\mu}(x,u)|$.

Then by relation (10) for each $f = f(x) \in C^1(M)$, $f \geq 0$

$$\int_M \operatorname{div}^\mu(a(x,\lambda)f(x))d\mu + L\int_S f(x)d\mu_b =$$
$$\int_S (a_{\bar{\mu}}(x,\lambda) + L)f(x)d\mu_b \geq 0.$$

Multiplying this inequality by the functions $p(\lambda)(1-q(\lambda))$, $(1-p(\lambda))q(\lambda)$ with $p(\lambda) = \operatorname{sign}^+(c-\lambda)$, $q(\lambda) \in F$ and integrating over λ, we derive that both conditions (55), (56) are satisfied, i.e. $p(\lambda)$ is a k.s. of the problem (53), (54) with boundary function $p_b = p(\lambda)$. By Theorem 4 this means that $u(x) \equiv c$ is a g.e.s. of (1), (2) with $u_b \equiv c$.

Further, applying the comparison principle for the three g.e.s. $u_1 \equiv c_1$, $u_2 \equiv u$, and $u_3 \equiv c_2$ we derive from the assumption $c_1 \leq u_b \leq c_2$ that $c_1 \leq u(x) \leq c_2$ a.e. on M, as was to be proved.

§ 5. Some properties of kinetic sub- and super-solutions.

In this section we are going to prove the existence of the maximal k.sub-s. and the minimal k.super-s. of the problem (53), (54).

Firstly, observe that if functions $p_1(\lambda), p_2(\lambda) \in F$ then

$$p_1(\lambda)p_2(\lambda) \leq \min(p_1(\lambda), p_2(\lambda)) \leq \max(p_1(\lambda), p_2(\lambda)) \leq p_1(\lambda) \circ p_2(\lambda).$$

Moreover if $p_1(\lambda) = \operatorname{sign}^+(u_1-\lambda)$, $p_2(\lambda) = \operatorname{sign}^+(u_2-\lambda)$ then $p_1(\lambda)p_2(\lambda) = \operatorname{sign}^+(\min(u_1,u_2)-\lambda)$, $p_1(\lambda) \circ p_2(\lambda) = \operatorname{sign}^+(\max(u_1,u_2)-\lambda)$.

The following statement takes place.

Proposition 5. *Suppose that $p_1(x,\lambda), p_2(x,\lambda)$ be a pair of k.sub-s. (k.super-s.) of (53), (54) with boundary data $p_{1b}(x,\lambda), p_{2b}(x,\lambda) \in \mathcal{F}(S)$. Then the function $p_1(x,\lambda) \circ p_2(x,\lambda)$ (respectively $p_1(x,\lambda)p_2(x,\lambda)$) is a k.sub-s. (k.super-s.) of this problem with the same boundary function $(p_{1b}(x,\lambda) + p_{2b}(x,\lambda))/2$.*

Proof. Suppose that $p_1(x,\lambda), p_2(x,\lambda)$ are k.sub-s. of (53), (54) with boundary data $p_{1b}(x,\lambda)$, $p_{2b}(x,\lambda)$. Then, as is easily verified, for each $x, y \in M$, $q(\lambda) \in F$

$$(p_1(x,\lambda) \circ p_2(y,\lambda))(1-q(\lambda)) = p_2(y,\lambda)(1-q(\lambda)) + p_1(x,\lambda)(1-p_2(y,\lambda) \circ q(\lambda)).$$

By (10) and relation (61) for the k.sub-s. p_1 with $q(\lambda)$ replaced by $p_2(y,\lambda) \circ q(\lambda)$, we arrive at

$$\int_{M\times\mathbb{R}} p_1(x,\lambda) \circ p_2(y,\lambda)(1-q(\lambda))\operatorname{div}_x^\mu(a(x,\lambda)f(x;y))d\mu(x)d\lambda -$$

$$\int_{S\times\mathbb{R}} a_{\bar\mu}(x,\lambda)((p_1)_\tau(x,\lambda)\circ p_2(y,\lambda))(1-q(\lambda))f(x;y)d\mu_b(x)d\lambda \ge 0,$$

where $f(x;y)$ is nonnegative test function, which belongs $C^1(M)$ with respect to each of variables x,y. Changing places of the variables x,y and the k.sub-s. p_1, p_2, we obtain the similar relation

$$\int_{M\times\mathbb{R}} p_1(x,\lambda)\circ p_2(y,\lambda)(1-q(\lambda))\operatorname{div}{}_y^\mu(a(y,\lambda)f(x;y))d\mu(y)d\lambda -$$
$$\int_{S\times\mathbb{R}} a_{\bar\mu}(y,\lambda)(p_1(x,\lambda)\circ (p_2)_\tau(y,\lambda))(1-q(\lambda))f(x;y)d\mu_b(y)d\lambda \ge 0.$$

The obtained relations allows us to apply the method of doubling variables, in the same way as in the proof of Theorem 5, and produce the inequality

$$\int_{M\times\mathbb{R}} p_1(x,\lambda)\circ p_2(x,\lambda)(1-q(\lambda))\operatorname{div}{}^\mu(a(x,\lambda)f(x))d\mu d\lambda -$$
$$\int_{S\times\mathbb{R}} a_{\bar\mu}(x,\lambda)((p_1)_\tau(x,\lambda)\circ (p_2)_\tau(x,\lambda))(1-q(\lambda))f(x)d\mu_b d\lambda \ge 0 \tag{78}$$

for all $f(x)\in C^1(M)$, $f(x)\ge 0$.

Now, we observe that

$$((p_1)_\tau(x,\lambda)\circ (p_2)_\tau(x,\lambda))(1-q(\lambda)) = (p_1)_\tau(x,\lambda)(1-q(\lambda)) +$$
$$(p_2)_\tau(x,\lambda)(1-(p_1)_\tau(x,\lambda)\circ q(\lambda)).$$

and, as follows from (65), for a.e. $x\in S$

$$\int a_{\bar\mu}(x,\lambda)((p_1)_\tau(x,\lambda)\circ (p_2)_\tau(x,\lambda))(1-q(\lambda))d\lambda +$$
$$L\int\Big(p_{1b}(x,\lambda)(1-q(\lambda)) + p_{2b}(x,\lambda)(1-(p_1)_\tau(x,\lambda)\circ q(\lambda))\Big)d\lambda \ge 0. \tag{79}$$

Now, remark that $(p_1)_\tau(x,\lambda)\circ q(\lambda)\ge q(\lambda)$. Therefore

$$p_{1b}(x,\lambda)(1-q(\lambda)) + p_{2b}(x,\lambda)(1-(p_1)_\tau(x,\lambda)\circ q(\lambda)) \le$$
$$(p_{1b}(x,\lambda)+p_{2b}(x,\lambda))(1-q(\lambda))$$

and (79) implies that

$$\int a_{\bar\mu}(x,\lambda)((p_1)_\tau(x,\lambda)\circ (p_2)_\tau(x,\lambda))(1-q(\lambda))d\lambda +$$
$$2L\int \frac{1}{2}(p_{1b}(x,\lambda)+p_{2b}(x,\lambda))(1-q(\lambda))d\lambda \ge 0.$$

Integrating this inequality over the measure $f(x)\mu_b$ and then adding the obtained inequality to (78), we arrive at

$$\int_{M\times\mathbb{R}} p_1(x,\lambda)\circ p_2(x,\lambda)(1-q(\lambda))\operatorname{div}_x^{\mu}(a(x,\lambda)f(x))d\mu d\lambda + \\ 2L\int_{S\times\mathbb{R}}\frac{1}{2}(p_{1b}(x,\lambda)+p_{2b}(x,\lambda))(1-q(\lambda))f(x)d\mu_b d\lambda \ge 0 \qquad (80)$$

Since (80) holds for each $q(\lambda)\in F$ and all $f(x)\in C^1(M)$, $f(x)\ge 0$ we see that $p_1(x,\lambda)\circ p_2(x,\lambda)$ is a k.sub-s. of (53), (54) with boundary data $(p_{1b}(x,\lambda)+p_{2b}(x,\lambda))/2$.

If $p_1(x,\lambda)$, $p_2(x,\lambda)$ are k.super-s. then by Lemma 1 the functions $\overline{p_1}(x,\lambda)$, $\overline{p_2}(x,\lambda)$ are k.sub-s. of the problem (57) with boundary data $\overline{p_{1b}}(x,\lambda)$, $\overline{p_{2b}}(x,\lambda)$ respectively. As it has already proved, $\overline{p_1}(x,\lambda)\circ\overline{p_2}(x,\lambda) = \overline{p_1\cdot p_2(x,\lambda)}$ is a k.sub-s. of (57) with boundary data $(\overline{p_{1b}}+\overline{p_{2b}})/2 = \overline{(p_{1b}+p_{2b})/2}$. By Lemma 1 again we conclude that $p_1(x,\lambda)p_2(x,\lambda)$ is a k.super-s. of the original problem (53), (54) with boundary data $(p_{1b}(x,\lambda)+p_{2b}(x,\lambda))/2$. The proof is complete.

We underline that in the above Proposition boundary data p_{1b}, p_{2b} are arbitrary functions from $\mathcal{F}(S)$. Suppose that in (54) $p_b(x,\lambda)\in\mathcal{F}_R(S)$. Then the functions $p_1(x,\lambda)=p_1(\lambda)=\operatorname{sign}^+(-R-\lambda)$ and $p_2(x,\lambda)=p_2(\lambda)=\operatorname{sign}^+(R-\lambda)$ are respectively a k.sub-s. and a k.super-s. of (53), (54). Indeed, since p_i, $i=1,2$ do not depend on x this functions are k.s. of (53), (54) with boundary data $p_i(\lambda)$ (see the proof of Corollary 5). Therefore for a pair $\bar\mu=(\mu,\mu_b)$ of smooth measures and some constant L we have the relations

$$\int_{M\times\mathbb{R}} p_1(\lambda)(1-q(\lambda))\operatorname{div}^{\mu}(a(x,\lambda)f(x))d\mu d\lambda + \\ L\int_{S\times\mathbb{R}} p_1(\lambda)(1-q(\lambda))f(x)d\mu_b d\lambda \ge 0;$$

$$\int_{M\times\mathbb{R}} (1-p_2(\lambda))q(\lambda)\operatorname{div}^{\mu}(a(x,\lambda)f(x))d\mu d\lambda + \\ L\int_{S\times\mathbb{R}} (1-p_2(\lambda))q(\lambda)f(x)d\mu_b d\lambda \ge 0$$

for each $q(\lambda)\in F$, $f(x)\in C^1(M)$, $f(x)\ge 0$. From these relations and the obvious inequality $p_1(\lambda) \le p_b(x,\lambda) \le p_2(\lambda)$ it follows that

$$\int_{M\times\mathbb{R}} p_1(\lambda)(1-q(\lambda))\operatorname{div}^\mu(a(x,\lambda)f(x))d\mu d\lambda +$$
$$L\int_{S\times\mathbb{R}} p_b(x,\lambda)(1-q(\lambda))f(x)d\mu_b d\lambda \ge 0;$$
$$\int_{M\times\mathbb{R}} (1-p_2(\lambda))q(\lambda)\operatorname{div}^\mu(a(x,\lambda)f(x))d\mu d\lambda +$$
$$L\int_{S\times\mathbb{R}} (1-p_b(x,\lambda))q(\lambda)f(x)d\mu_b d\lambda \ge 0,$$

i.e. $p_1(\lambda)$ is a k.sub-s. and $p_2(\lambda)$ is a k.super-s. of (53), (54), as required.

Remark also that, as follows from Proposition 5 for $p = p(\lambda) \in F$, the functions $p \circ p = 2p - p^2$ and p^2 are kinetic sub- and super-solution of (53), (54) respectively with the same boundary data $p(\lambda)$ while $2p - p^2 \ge p^2$ and $2p - p^2 = p^2$ only in the case when $p(\lambda) = \operatorname{sign}^+(k-\lambda)$ with some $k \in \mathbb{R}$. This demonstrates that the comparison principle is not satisfied for general kinetic boundary data.

Let $p_b(x,\lambda) = \operatorname{sign}^+(u_b(x)-\lambda)$, $u_b(x) \in L^\infty(M)$. Obviously $p_b(x,\lambda) \in \mathcal{F}_R(S)$ for all $R \ge \|u_b\|_\infty$. We will call the k.sub-s. $p_+(x,\lambda) \in \mathcal{F}_R(M)$ maximal (in the class $\mathcal{F}_R(M)$) if $p_+(x,\lambda) \ge p(x,\lambda)$ for each k.sub-s. $p(x,\lambda) \in \mathcal{F}_R(M)$. Analogously we define the notion of the minimal k.super-s.

We are ready to prove the following result.

Theorem 7. *There exist the maximal k.sub-s.* $p_+(x,\lambda) \in \mathcal{F}_R(M)$ *and the minimal k.super-s.* $p_-(x,\lambda) \in \mathcal{F}_R(M)$ *of (53), (54). This functions have the form* $p_+(x,\lambda) = \operatorname{sign}^+(u_+(x)-\lambda)$, $p_-(x,\lambda) = \operatorname{sign}^+(u_-(x)-\lambda)$, $u_\pm(x) \in L^\infty(M)$.

Proof. Denote by K_R^- the set of k.sub-s. $p \in \mathcal{F}_R(M)$ of (53), (54) and by K_R^+ the set of k.super-s. $p \in \mathcal{F}_R(M)$ of this problem. These sets are not empty because, as was shown above, $\operatorname{sign}^+(-R-\lambda) \in K_R^-$, $\operatorname{sign}^+(R-\lambda) \in K_R^+$. We set for $p = p(x,\lambda) \in \mathcal{F}_R(M)$ $I(p) = \displaystyle\int_{M\times[-R,R]} p(x,\lambda)d\mu d\lambda$, where μ is some smooth measure on M. Clearly, $I_0 = \sup\limits_{p\in K_R^-} I(p) \le I(1) = 2R\mu(M) < +\infty$. Hence, one can find a sequence $p_r = p_r(x,\lambda) \in K_R^-$, $r \in \mathbb{N}$ such that $I(p_r) \to I_0$ as $r \to \infty$. We construct the new sequence $\tilde{p}_r$ setting $\tilde{p}_1 = p_1$, $\tilde{p}_r = \tilde{p}_{r-1} \circ p_r$ for $r > 1$. Using Proposition 5 and induction in r

we find that $\tilde{p}_r \in K_R^-$. Since $\tilde{p}_r = \tilde{p}_{r-1} \circ p_r \geq \max(\tilde{p}_{r-1}, p_r)$ we see that $\tilde{p}_{r+1} \geq \tilde{p}_r \geq p_r$ for all $r \in \mathbb{N}$. In particular, $I(p_r) \leq I(\tilde{p}_r) \leq I_0\ \forall r \in \mathbb{N}$. This implies that $I(\tilde{p}_r) \to I_0$ as $r \to \infty$. Further, by monotonicity of the sequence $\tilde{p}_r$ and its boundedness from above ($\tilde{p}_r \leq 1$), there exists the point-wise limit $p_+(x,\lambda) = \lim\limits_{r\to\infty} \tilde{p}_r(x,\lambda) = \sup\limits_{r\in\mathbb{N}} \tilde{p}_r(x,\lambda)$. Obviously, $p_+(x,\lambda) \in \mathcal{F}_R(M)$. Passing to the limit as $r \to \infty$ in relations (55) corresponding to k.sub-s. $\tilde{p}_r$ and taking into account that, by Remark 5, we can take constant $L = L_{\bar{\mu}}$ in these relations, which is common for all $r \in \mathbb{N}$, we obtain that for each $q(\lambda) \in F$, $f(x) \in C^1(M)$, $f(x) \geq 0$

$$\int_{M\times\mathbb{R}} p_+(x,\lambda)(1-q(\lambda))\operatorname{div}^{\mu}(a(x,\lambda)f(x))d\mu d\lambda + \\ L\int_{S\times\mathbb{R}} p_b(x,\lambda)(1-q(\lambda))f(x)d\mu_b d\lambda \geq 0.$$

This means that $p_+(x,\lambda) \in K_R^-$. We have to show that $p_+(x,\lambda)$ is the maximal k.sub-s. For that, take an arbitrary k.sub-s. $p(x,\lambda) \in K_R^-$. Then by Proposition 5 the function $\tilde{p}(x,\lambda) = p_+(x,\lambda) \circ p(x,\lambda) \in K_R^-$ and $\tilde{p}(x,\lambda) \geq \max(p_+(x,\lambda), p(x,\lambda))$. Since $I_0 = I(p_+) \leq I(\tilde{p}) \leq I_0$ we conclude that $I(\tilde{p}) = I(p_+)$. This implies that

$$\int_{M\times[-R,R]} (\tilde{p}(x,\lambda) - p_+(x,\lambda))d\mu d\lambda = I(\tilde{p}) - I(p_+) = 0.$$

Therefore, $p_+(x,\lambda) = \tilde{p}(x,\lambda) \geq p(x,\lambda)$ a.e. on $M \times \mathbb{R}$ and $p_+(x,\lambda)$ is the maximal k.sub-s. By Proposition 5 we see that $p_+ \circ p_+ \in K_R^-$. Since $p_+ \circ p_+ \geq p_+$ then in view of maximality of p_+ we conclude that $p_+ \circ p_+ = p_+$, i.e. $p_+ = (p_+)^2$. This implies that p_+ necessarily has the form $p_+(x,\lambda) = \operatorname{sign}^+(u_+(x) - \lambda)$.

Existence of the minimal k.super-s. p_- easily follows from Lemma 1. Actually $p_- = \overline{p'_+}(x,\lambda) = \operatorname{sign}^+(u_-(x) - \lambda)$, where p'_+ is the maximal k.sub-s. of the problem (57). Clearly, functions $u_\pm(x) \in L^\infty(M)$, $\|u_\pm\|_\infty \leq R$. The proof is complete.

From Theorems 7 and 4 we derive the following

Corollary 6. *The function $u_+(x)$ is the maximal g.e.sub-s. of (1), (2) and the function $u_-(x)$ is the minimal g.e.super-s. of this problem among functions from the ball $\|u\|_\infty \leq R$.*

As will be established in the next section, functions $u_\pm$ are in fact g.e.s. of (1), (2). Remark that the similar result was established in [18, 19] for locally integrable g.e.s. of the Cauchy problem in the half-space $t > 0$ (in the situation when the uniqueness of g.e.s. may be violated).

Generally, the functions $u_\pm$ depend on R. For instance, consider the Dirichlet problem for equation (50) from Example 3 with zero boundary data. Then, as one can easily verify, $u^\pm(x) = 0$ for $r > r_0$ while $u^\pm(x) = \pm R$ for $r < r_0$.

§ 6. The existence of k.s.

We will study even more general nonhomogeneous transport equation

$$\langle a(x,\lambda), p\rangle = -H(x,\lambda,p), \quad p = p(x,\lambda), \tag{81}$$

where $H(x,\lambda,p)$ is a measurable function on $M \times \mathbb{R} \times [0,1]$ satisfying the following properties:

$$H(x,\lambda,\operatorname{sign}^+(-\lambda)) = 0 \ \text{ for } |\lambda| > R; \tag{82}$$

$$\forall p_1, p_2 \in [0,1], p_1 \ge p_2 \quad 0 \le H(x,\lambda,p_1) - H(x,\lambda,p_2) \le C(p_1 - p_2); \tag{83}$$

$$\forall p \in [0,1] \quad 0 \le p - \varepsilon H(x,\lambda,p) \le 1. \tag{84}$$

Here R, C, ε are some positive constants, and $R \ge \|u_b\|_\infty$. From (84) it follows that for $p \in [0,1]$

$$0 \le p - \delta H(x,\lambda,p) \le 1 \quad \forall \delta \in [0,\varepsilon]. \tag{85}$$

Indeed, that is readily derived from the relation

$$p - \delta H(x,\lambda,p) = \frac{\varepsilon - \delta}{\varepsilon} p + \frac{\delta}{\varepsilon}(p - \varepsilon H(x,\lambda,p)).$$

We need the notion of a measure valued function.

Let $(\Omega, \mathcal{M}, \mu)$ be a measure space, so that μ is a measure on a σ-algebra $\mathcal{M}$ in X. We recall (see [6, 25]) that a measure valued function on Ω is a weakly measurable map $x \mapsto \nu_x$ of Ω into the space $\mathrm{Prob}_0(\mathbb{R})$ of probability Borel measures with compact support in $\mathbb{R}$.

The weak measurability of ν_x means that for each continuous function $g(u)$ the function $x \to \langle g(u), \nu_x(u)\rangle = \int g(u) d\nu_x(u)$ is measurable on Ω.

We say that a measure valued function ν_x is *bounded* if there exists $R > 0$ such that $\operatorname{supp} \nu_x \subset [-R, R]$ for almost all $x \in \Omega$. We shall denote by $\|\nu_x\|_\infty$ the smallest of such R.

Finally, we say that measure valued functions of the kind $\nu_x(u) = \delta(u - u(x))$, where $u(x) \in L^\infty(\Omega)$ and $\delta(u - u^*)$ is the Dirac measure at $u^* \in \mathbb{R}$, are *regular*. We identify these measure valued functions and the corresponding functions $u(x)$, so that there is a natural embedding $L^\infty(\Omega) \subset MV(\Omega)$, where $MV(\Omega)$ is the set of bounded measure valued functions on Ω.

Measure valued functions naturally arise as weak limits of bounded sequences in $L^\infty(\Omega)$ in the sense of the following theorem of Tartar (see [25]).

Theorem T. *Let $u_m(x) \in L^\infty(\Omega)$, $m \in \mathbb{N}$ be a bounded sequence. Then there exist a subsequence $u_r(x)$ and a measure valued function $\nu_x \in \mathrm{MV}(\Omega)$ such that*

$$\forall g(u) \in C(\mathbb{R}) \quad g(u_r) \underset{r\to\infty}{\longrightarrow} \langle g(u), \nu_x(u)\rangle \qquad \textit{weakly-}* \textit{ in } L^\infty(\Omega). \tag{86}$$

Besides, ν_x is regular, i.e. $\nu_x = \delta(u - u(x))$ if and only if $u_r(x) \underset{r\to\infty}{\longrightarrow} u(x)$ in $L^1_{loc}(\Omega)$.

More generally, if $g(x, u)$ is a Caratheodory function bounded on the sets $\Omega \times [-R, R]$, $R > 0$ then for each $r \in \mathbb{N}$ the functions $g(x, u_r(x)) \in L^\infty(\Omega)$, $\int g(x, u)d\nu_x(u) \in L^\infty(\Omega)$, and

$$g(x, u_r(x)) \underset{r\to\infty}{\longrightarrow} \langle g(x, u), \nu_x(u)\rangle = \int g(x, u)d\nu_x(u) \qquad \text{weakly-}* \text{ in } L^\infty(\Omega). \tag{87}$$

This follows from the fact that any Caratheodory function is strongly measurable as a map $x \to g(x, \cdot) \in C(\mathbb{R})$ (see [9], Chapter 2) and, therefore, is a pointwise limit of step functions $g_m(x, u) = \sum\limits_i l_{mi}(x)h_{mi}(u)$ so that for $x \in \Omega \;\; g_m(x, \cdot) \underset{m\to\infty}{\longrightarrow} g(x, \cdot)$ in $C(\mathbb{R})$.

As was shown in [14] (see also [15]), for a measure valued function ν_x we can introduced the function

$$p(x, s) = \inf\{\, v \,|\, \nu_x((v, +\infty)) \le s \,\}$$

such that the measures ν_x is an image of the Lebesgue measure ds on $(0, 1)$ with respect to the map $s \to p(x, s)$: $\nu_x = p(x, \cdot)^*ds$. Moreover, the function $s \to p(x, s)$ is a unique non-increasing and right-continuous function with the property $\nu_x = p(x, \cdot)^*ds$. As is easy to verify (see [14, 15]) $p(x, s)$

is measurable on $\Omega \times (0,1)$, $p(x,s) \in L^\infty(\Omega \times (0,1))$, and $\|p\|_\infty = \|\nu_x\|_\infty$. Observe also that $p(x,s) = u(x)$ for a regular function $\nu_x \sim u(x)$. By the identity $\nu_x = p(x,\cdot)^* ds$ we have $\displaystyle\int g(x,u)d\nu_x(u) = \int_0^1 g(x,p(x,s))ds$ for each Caratheodory function $g(x,u)$. Therefore, the limit relation (87) can be rewritten as follows

$$g(x,u_r(x)) \underset{r\to\infty}{\to} \int g(x,p(x,s))ds \quad \text{weakly-}* \text{ in } L^\infty(\Omega). \tag{88}$$

Remark that the function $p(x,s)$ was used in [14, 15] in the definition of a *strong measure valued solution* for a scalar conservation law. In [16] the interesting connection between strong and statistical measure valued solutions was found.

The function $p(x,s)$ was called later in [8] *a bounded measurable process* on Ω (if to be exact, the non-decreasing version of p was used in [8] instead). We will use a shorter name *a process* in the sequel. Hence, a process on Ω is a function $p(x,s) \in L^\infty(\Omega \times (0,1))$, which is non-increasing and continuous from the right with respect to s. Clearly, correspondence $\nu_x = p(x,\cdot)^* ds$ between processes and measure valued functions is one to one.

Now we introduce the notions of a kinetic process solution to problem (81), (54). We will consider the space $\Omega = M \times \mathbb{R}$ as a measure space endowed with Lebesgue σ-algebra and measure $\mu \times d\lambda$ on it, where μ is a smooth measure on M.

Definition 5. A process $p(x,\lambda,s)$ on $M \times \mathbb{R}$ is called *a kinetic process sub-solution (super-solution)* of (81), (54) if $p(\cdot,s) \in \mathcal{F}_R(M)$ for some $R > 0$ and each $s \in (0,1)$, and the following relations similar to (55) and (56) are satisfied: $\forall q(\lambda) \in F$, $\forall f(x) \in C^1(M)$, $f(x) \geq 0$

$$\int_{M\times\mathbb{R}\times(0,1)} [p(x,\lambda,s)\operatorname{div}^\mu(a(x,\lambda)f(x)) - H(x,\lambda,p(x,\lambda,s))f(x)] \times \\ (1-q(\lambda))d\mu d\lambda ds + L\int_{S\times\mathbb{R}} p_b(x,\lambda)(1-q(\lambda))f(x)d\mu_b d\lambda \geq 0 \tag{89}$$

and, respectively,

$$\int_{M\times\mathbb{R}\times(0,1)} [(1-p(x,\lambda,s))\operatorname{div}^\mu(a(x,\lambda)f(x)) + H(x,\lambda,p(x,\lambda,s))f(x)] \times \\ q(\lambda)d\mu d\lambda ds + L\int_{S\times\mathbb{R}} (1-p_b(x,\lambda))q(\lambda)f(x)d\mu_b d\lambda \geq 0. \tag{90}$$

Here L is a sufficiently large constant, $\bar{\mu} = (\mu, \mu_b)$ is a pair of smooth measures on M and S.

We call $p(x, \lambda, s)$ a kinetic process solution of (81), (54) if it is a kinetic process sub- and super-solution of this problem simultaneously.

In the case when $p(x, \lambda, s) = p(x, \lambda) \in \mathcal{F}(M)$ (that corresponds to the regular measure valued function $\nu_{x,\lambda} \sim p(x, \lambda)$ relations (89), (90) are reduced to the following ones:

$$\int_{M\times\mathbb{R}} [p(x,\lambda)\operatorname{div}^{\mu}(a(x,\lambda)f(x)) - H(x,\lambda,p(x,\lambda))f(x)] \times$$
$$(1-q(\lambda))d\mu d\lambda + L\int_{S\times\mathbb{R}} p_b(x,\lambda)(1-q(\lambda))f(x)d\mu_b d\lambda \geq 0 \tag{91}$$

and, respectively,

$$\int_{M\times\mathbb{R}} [(1-p(x,\lambda))\operatorname{div}^{\mu}(a(x,\lambda)f(x)) + H(x,\lambda,p(x,\lambda))f(x)] \times$$
$$q(\lambda)d\mu d\lambda + L\int_{S\times\mathbb{R}} (1-p_b(x,\lambda))q(\lambda)f(x)d\mu_b d\lambda \geq 0. \tag{92}$$

In this case we call $p(x, \lambda)$ *a kinetic sub-solution* and, respectively, *a kinetic super-solution* of (81), (54). If $p(x, \lambda)$ is a kinetic sub- and super-solution simultaneously it is called *a kinetic solution*.

Observe that in the case when the function $p \to H(x, \lambda, p)$ is linear and $p(x, \lambda, s)$ is a kinetic process sub-solution (super-solution, solution) of (81), (54) the function $p(x, \lambda) = \int_0^1 p(x, \lambda, s)ds$ is a kinetic sub-solution (respectively - super-solution, solution) of this problem.

We are going to prove the existence of a kinetic process solution of (81), (54). For that, we consider the following equation

$$\langle a(x,\lambda), p\rangle = -r(p - P(\kappa_r p))(x,\lambda) - H(x,\lambda,P(\kappa_r p)(x,\lambda)) \tag{93}$$

containing the relaxation term $-r(p - P(\kappa_r p))$ with parameter $r \in \mathbb{N}$. Here the sequence $\kappa_r \in (0,1)$ is such that $r(1-\kappa_r) \to 0$ as $r \to \infty$; $Pw(x, \lambda) = (Pw(x, \cdot))(\lambda)$, where $P : L^2([-R, R]) \to F_R$ is a projection operator onto F_R so that for $w = w(\lambda) \in L^2([-R, R])$ $Pw = Pw(\lambda) \in F_R$ and $\|w - Pw\|_2 = \min\limits_{q\in F_R} \|w - q\|_2$. Here the constant $R > 0$ such that $p_b \in \mathcal{F}_R(S)$ is taken from condition (82).

Since F_R is a convex closed subset of the Hilbert space $L^2([-R,R])$ the projection P is well-defined, and

$$q = Pw \;\Leftrightarrow\; q \in F_R \;\text{ and }\; (w-q, p-q)_2 \le 0 \;\forall p \in F_R. \tag{94}$$

Any non-increasing function $q(\lambda)$ on $[-R,R]$ such that $0 \le q(\lambda) \le 1$ are assumed to be extended on $\mathbb{R}$ by equalities $g(\lambda) = 1$, $\lambda \le -R$; $g(\lambda) = 0$, $\lambda > R$ and will be considered as an element of F_R (obviously, after an improvement on a set of null measure it becomes also right-continuous).

We will need some properties of the projection. It is useful to introduce the projection $\tilde{P} : L^2([-R,R]) \to \tilde{F}_R$, where $\tilde{F}_R$ is a cone of all non-increasing functions from $L^2([-R,R])$.

Lemma 2 (properties of $\tilde{P}$).

a) $\|\tilde{P}v - \tilde{P}w\|_2 \le \|v-w\|_2$ *for all* $v, w \in L^2([-R,R])$*;*

b) if $v \le w$ *then* $\tilde{P}v \le \tilde{P}w$ *(monotonicity);*

c) $\|\tilde{P}v - \tilde{P}w\|_\infty \le \|v-w\|_\infty$ *for all* $v, w \in L^\infty([-R,R])$*;*

d) $(v - \tilde{P}v, q)_2 \le 0 \;\forall q \in \tilde{F}_R$*,* $(v - \tilde{P}v, \tilde{P}v)_2 = 0$*;*

e) if $0 \le v \le 1$ *then* $\tilde{P}v = Pv$*.*

Proof. By condition (94) (with P, F_R replaced by $\tilde{P}$, $\tilde{F}_R$) we have $(v-\tilde{P}v, \tilde{P}w-\tilde{P}v)_2 \le 0$, $(w-\tilde{P}w, \tilde{P}v-\tilde{P}w)_2 \le 0$. Putting these inequalities together, we arrive at $(v-w, \tilde{P}w-\tilde{P}v)_2 + \|\tilde{P}v-\tilde{P}w\|_2^2 \le 0$. Thus, $\|\tilde{P}v - \tilde{P}w\|_2^2 \le (v-w, \tilde{P}v-\tilde{P}w)_2 \le \|v-w\|_2 \|\tilde{P}v-\tilde{P}w\|_2$, which readily implies a).

To prove b), suppose firstly that $v < w$ a.e. on $[-R,R]$. Taking in the inequalities $(w-\tilde{P}w, q-\tilde{P}w)_2 \le 0$, $(v-\tilde{P}v, q-\tilde{P}v)_2 \le 0$ a function $q = \max(\tilde{P}v, \tilde{P}w)$ and $q = \min(\tilde{P}v, \tilde{P}w)$, respectively, we obtain that

$$(w-\tilde{P}w, (\tilde{P}v-\tilde{P}w)^+)_2 \le 0, \quad -(v-\tilde{P}v, (\tilde{P}v-\tilde{P}w)^+)_2 \le 0.$$

Putting together these inequality, we find that

$$\int_{-R}^{R} (w-v)(\tilde{P}v-\tilde{P}w)^+ d\lambda = (w-v, (\tilde{P}v-\tilde{P}w)^+)_2 \le$$
$$(\tilde{P}w-\tilde{P}v, (\tilde{P}v-\tilde{P}w)^+)_2 = \int_{-R}^{R} (\tilde{P}w-\tilde{P}v)(\tilde{P}v-\tilde{P}w)^+ d\lambda \le 0$$

and since $w - v > 0$ a.e. on $[-R,R]$ we conclude that $(\tilde{P}v-\tilde{P}w)^+ = 0$ a.e. on $[-R,R]$, i.e. $\tilde{P}v \le \tilde{P}w$ as required. In general case when $v \le w$ we replace w by $w_r = w + 1/r$, $r \in \mathbb{N}$. Then $v < w_r$ and, as we have already proved, $\tilde{P}v \leq \tilde{P}w_r$. It is clear that $w_r \to w$ as $r \to \infty$ in $L^2([-R, R])$ and in view of a) this implies that $\tilde{P}w_r \to \tilde{P}w$ in $L^2([-R, R])$ as $r \to \infty$. Therefore, we can pass to the limit in the inequality $\tilde{P}v \leq \tilde{P}w_r$ and derive the desired result $\tilde{P}v \leq \tilde{P}w$.

Now, suppose that $v = v(\lambda)$, $w = w(\lambda)$ are bounded and $C = \|v - w\|_\infty$. Then $w - C \leq v \leq w + C$ a.e. on $[-R, R]$ and by b) $\tilde{P}(w - C) \leq \tilde{P}v \leq \tilde{P}(w + C)$ a.e. on $[-R, R]$. Now observe that the cone $\tilde{F}_R$ is invariant under the transformations $v \to v \pm C$, therefore $\tilde{P}(w \pm C) = \tilde{P}w \pm C$. Thus, $\tilde{P}w - C \leq \tilde{P}v \leq \tilde{P}w + C$ a.e. on $[-R, R]$. This means that $\|\tilde{P}v - \tilde{P}w\|_\infty \leq C$, as was to be proved.

To prove d), observe that by (94) we have $(v - \tilde{P}v, q)_2 = (v - \tilde{P}v, q + \tilde{P}v - \tilde{P}v)_2 \leq 0$. Putting in (94) $p = c\tilde{P}v$ with $c > 0$, we obtain that $(c-1)(v - \tilde{P}v, \tilde{P}v)_2 \leq 0 \; \forall c > 0$. This obviously implies that $(v - \tilde{P}v, \tilde{P}v)_2 = 0$.

Finally, if $v = v(\lambda)$ is such that $0 \leq v \leq 1$ a.e. on $[-R, R]$ then by c) $0 = \tilde{P}0 \leq \tilde{P}v \leq \tilde{P}1 = 1$. Hence $\tilde{P}v \in K_R \subset \tilde{K}_R$. Obviously, this implies that $\tilde{P}v = Pv$. The proof is now complete.

Remark 6.

1) Since constant functions belongs to $\tilde{F}_R$ we derive from assertion d) with $q = \pm 1$ that $(v - \tilde{P}v, 1)_2 = 0$ that is $\int_{-R}^{R} v(\lambda)d\lambda = \int_{-R}^{R} \tilde{P}v(\lambda)d\lambda$;

2) Properties a)-c) are satisfied also for the projection P. Proofs of a), b) are exactly the same as in Lemma 2. To prove c), it is sufficient to show that $P(w + C) \leq Pw + C$ for $C > 0$. Indeed, then from the inequality $w - C \leq u \leq w + C$ it follows that $Pw - C \leq P(w - C + C) - C \leq P(w - C) \leq Pu \leq P(w + C) \leq Pw + C$ and we conclude as in Lemma 2.

To establish the inequality $P(w + C) \leq Pw + C$ we consider relations (94)

$$(w + C - P(w + C), p - P(w + C))_2 \leq 0, \;(w - Pw, p - Pw)_2 \leq 0 \;\forall p \in F_R$$

and put in the first inequality $p = \min(Pw + C, P(w + C))$ and in the second one $p = \max(Pw, P(w + C) - C)$ (as is easy to see the both functions $p \in F_R$). This yields the inequalities

$$-(w + C - P(w + C), (P(w + C) - Pw - C)^+)_2 \leq 0,$$

$$(w - Pw, (P(w + C) - Pw - C)^+)_2 \le 0$$

Putting them together, we arrive at

$$(P(w + C) - Pw - C, (P(w + C) - Pw - C)^+)_2 \le 0.$$

This implies that $(P(w+C) - Pw - C)^+ = 0$, i.e. $P(w+C) \le Pw + C$, as required.

It is important that our projection operator is a contraction not only in L^2 but also in L^∞. In BGK-like approximations, usually applied in kinetic models (see [10, 12, 24]), the corresponding operators do not satisfy this property.

In approximate equation (93) we need to be sure that $P(\kappa_r p)(x, \lambda) \in L^\infty(M \times \mathbb{R})$. This is proved in the following lemma.

Lemma 3. *Suppose that $w(x, \lambda) \in L^\infty(M \times [-R, R])$. Then the function $q(x, \lambda) = (Pw(x, \cdot))(\lambda) \in L^\infty(M \times [-R, R])$ as well.*

Proof. Since $w(x, \lambda) \in L^2(M \times [-R, R], \mu \times d\lambda)$ for a smooth measure μ on M the map $x \to W(x)(\lambda) \doteq w(x, \lambda)$ belongs to the space $L^2(M, X)$, where $X = L^2([-R, R])$. In view of Lemma 2,a) and Remark 6 the projection P is a continuous operator on X. Therefore, $Q(x) = PW(x) \in L^2(M, X)$, which implies that $q(x, \lambda) = Q(x)(\lambda) \in L^2(M \times [-R, R])$. Since $Q(x) \in F_R$ we see that $0 \le q(x, \lambda) \le 1$. Therefore, $q(x, \lambda) \in L^\infty(M \times [-R, R])$. The proof is complete.

We will understand solutions $p = p(x, \lambda) \in L^\infty(M \times \mathbb{R})$ of (93) as weak solutions of the nonhomogeneous transport equation on $M \times \mathbb{R}$

$$\langle a(x, \lambda), p \rangle + rp = h(x, \lambda)$$

with the source term $h(x, \lambda) = rP(\kappa_r p)(x, \lambda) - H(x, \lambda, P(\kappa_r p)(x, \lambda))$. Here $a(x, \lambda)$ may be considered as a vector field on the manifold $M \times \mathbb{R}$ (tangent to layers $M \times \{\lambda\}$). Let $\bar{\mu} = (\mu, \mu_b)$ be a pair of smooth measure on M and S, $a_{\bar{\mu}}(x, \lambda) = \langle n_{\bar{\mu}}(x), a(x, \lambda) \rangle$ be a normal trace of the field $a(\cdot, \lambda)$ on the boundary S. We introduce the set

$$D_- = \{\ (x, \lambda) \in S \times \mathbb{R} \mid a_{\bar{\mu}}(x, \lambda) < 0\}.$$

As is easy to see this set is open in $S \times \mathbb{R}$ and does not depend on the choice of $\bar{\mu}$. We will say that $p(x, \lambda) \in L^\infty(M \times \mathbb{R})$ is a weak solution of (93) if for all $f(x) \in C_0^1(M_0)$ and $g(\lambda) \in L^1(\mathbb{R})$

$$\int_{M\times\mathbb{R}} [p(x,\lambda)\mathrm{div}^{\mu}(a(x,\lambda)f(x)) + h(x,\lambda)f(x)]g(\lambda)d\mu d\lambda = 0. \qquad (95)$$

As is easy to verify, passing to local coordinates, $p(x,\lambda)$ is a weak solution of (93) if and only if it (after an improvement on a set of null measure in $M\times\mathbb{R}$, if necessary) satisfies the equation $\dot{p} + rp = h$ along the characteristics.

We will suppose also that a weak solution $p(x,\lambda)$ satisfies the boundary condition

$$p(x,\lambda) = p_b(x,\lambda) \ \text{ on } D_- \qquad (96)$$

in the strong sense.

Denote $x(s) = x(s;y,\lambda)$ a characteristic passing through $y \in M_0$ for $s = 0$. Thus, $x(s)$ is the unique solution of ODE $\dot{x} = a(x,\lambda)$ on M with Cauchy data $x(0) = y$. This solution is defined on some maximal interval $s \in (\alpha(y,\lambda), \beta(y,\lambda)) \ni 0$. Since along the characteristic $x(s) = x(s;y,\lambda)$

$$\dot{p} + rp = \frac{d}{ds}p(x(s),\lambda) + rp(x(s),\lambda) = h(x(s),\lambda) \qquad (97)$$

then, solving this ODE and taking into account condition (96) if $\alpha(y,\lambda) > -\infty$, we obtain the formula

$$p(y,\lambda) = p_b(x_0(y,\lambda))e^{r\alpha(y,\lambda)} + \int_{\alpha(y,\lambda)}^{0} e^{-rs}h(x(s),\lambda)ds, \qquad (98)$$

where $x_0(y,\lambda) = x(\alpha(y,\lambda))$. Remark (see the proof of Proposition 4) that in the case of finite $\alpha = \alpha(y,\lambda)$ the characteristic $x(s)$ is defined at $s = \alpha$, and $x_0 = x(\alpha) \in S$. If $\alpha(y,\lambda) = -\infty$ then we define

$$p(y,\lambda) = \int_{-\infty}^{0} e^{-rs}h(x(s),\lambda)ds, \qquad (99)$$

so that $p(y,\lambda) = p(0)$, where $p(s)$ is a unique bounded solution of (97). Since $h(x(s),\lambda)$ is a bounded function the integral in (99) exists. Both formulas (98), (99) hold for a.e. $(y,\lambda) \in M \times \mathbb{R}$ such that the function $h(x(s;y,\lambda),\lambda)$ is well-defined for a.e. $s \in (\alpha(y,\lambda), 0)$.

Remark that the function h depends on p and (98), (99) are integral equations with the unknown $p = p(x,\lambda)$. Let us show that these equations admit a unique solution.

Theorem 8. *For sufficiently large $r \in \mathbb{N}$ there exists a unique solution $p(x,\lambda) \in L^\infty(M \times \mathbb{R})$ of integral equations (98), (99). Moreover, $0 \le p(x,\lambda) \le 1$ and $p(x,\lambda) = \operatorname{sign}^+(-\lambda)$ for $|\lambda| > R$.*

Proof. We take $r > \max(C, 1/\varepsilon)$, where C, ε are the constants from conditions (83), (84). Let $q(x,\lambda) \in B$, where B is a closed subset of $L^\infty(M\times\mathbb{R})$ consisting of functions $q = q(x,\lambda)$ such that $0 \le q \le 1$, $q = \operatorname{sign}^+(-\lambda)$ for $|\lambda| > R$. Define the function $p(x,\lambda)$ by identities (98), (99) but with the function $\tilde{h}(x,\lambda) = rP(\kappa_r q)(x,\lambda) - H(x,\lambda,P(\kappa_r q)(x,\lambda))$ instead of h:

$$p(y,\lambda) = p_b(x_0(y,\lambda))e^{r\alpha(y,\lambda)} + \int_{\alpha(y,\lambda)}^0 e^{-rs}\tilde{h}(x(s),\lambda)ds \tag{100}$$

if $\alpha(y,\lambda) > -\infty$;

$$p(y,\lambda) = \int_{-\infty}^0 e^{-rs}\tilde{h}(x(s),\lambda)ds, \tag{101}$$

if $\alpha(y,\lambda) = -\infty$. The function $p = p(y,\lambda)$ is defined a.e. on $M \times \mathbb{R}$.

Since $\tilde{h}(x,\lambda) = r[P(\kappa_r q) - \frac{1}{r}H(x,\lambda,P(\kappa_r q))]$ and $1/r < \varepsilon$ it follows from (85) that $0 \le \tilde{h} \le r$. This and the condition $0 \le p_b \le 1$ imply that $0 \le p \le 1$. Indeed, in the case of (100)

$$0 \le p(y,\lambda) \le e^{r\alpha(y,\lambda)} + r\int_{\alpha(y,\lambda)}^0 e^{-rs}ds = 1,$$

while in the case of (101)

$$0 \le p(y,\lambda) \le r\int_{-\infty}^0 e^{-rs}ds = 1.$$

If $|\lambda| > R$ then $p_b(x,\lambda) = \operatorname{sign}^+(-\lambda)$, $\tilde{h}(x,\lambda) = r\operatorname{sign}^+(-\lambda)$ with account of (82) and the both equalities (100), (101) yield that $p(x,\lambda) = \operatorname{sign}^+(-\lambda)$. Thus, $p(x,\lambda) \in B$.

The correspondence $q \to p = Tq$ define an operator T on B. Let us show that this operator is a contraction. Indeed, since $0 \le \kappa_r q \le 1$ we derive from Lemma 2,e) that $P(\kappa_r q) = \tilde{P}(\kappa_r q)$. Then from Lemma 2,c) (or Remark 6) it follows that

$$\|P(\kappa_r q_1) - P(\kappa_r q_2)\|_\infty \le \kappa_r\|q_1 - q_2\|_\infty$$

for each $q_1 = q_1(x,\lambda)$, $q_2 = q_2(x,\lambda)$ from B. Now we fix (x,λ) and denote $v_i = P(\kappa_r q_i)(x\lambda)$, $\tilde{h}_i = rv_i - H(x,\lambda,v_i)$, $i = 1,2$. Suppose for definiteness that $v_1 \ge v_2$. Then by (83) and the assumption $r > C$ we have

$0 \leq \frac{1}{r}[H(x,\lambda,v_1) - H(x,\lambda,v_2)] \leq (v_1 - v_2)$. Since $\tilde{h}_1 - \tilde{h}_2 = r\Big(v_1 - v_2 - \frac{1}{r}(H(x,\lambda,v_1) - H(x,\lambda,v_2))\Big)$ we derive that $0 \leq \tilde{h}_1 - \tilde{h}_2 \leq r(v_1 - v_2)$. Thus,

$$|\tilde{h}_1(x,\lambda) - \tilde{h}_2(x,\lambda)| \leq r|P(\kappa_r q_1)(x,\lambda) - P(\kappa_r q_2)(x,\lambda)| \leq r\kappa_r \|q_1 - q_2\|_\infty.$$

Let $p_i = Tq_i$, $i = 1, 2$. Then, by equalities (100), (101) for a.e. (y,λ)

$$|p_1(y,\lambda) - p_2(y,\lambda)| \leq \int_{\alpha(y,\lambda)}^0 e^{-rs}|\tilde{h}_1(x(s),\lambda) - \tilde{h}_2(x(s),\lambda)|ds \leq$$
$$r\kappa_r \|q_1 - q_2\|_\infty \int_{\alpha(y,\lambda)}^0 e^{-rs} ds \leq \kappa_r \|q_1 - q_2\|_\infty.$$

Hence $\|p_1 - p_2\|_\infty \leq \kappa_r \|q_1 - q_2\|_\infty$ and since $\kappa_r < 1$ the operator T is a contraction on B. By the Banach theorem there exists a unique fixed point $p = p(x,\lambda) \in B$ of T. Then $p(x,\lambda)$ satisfies (98), (99). The proof is complete.

It is natural to consider the function $p(x,\lambda)$, constructed in the above theorem, as a weak solution to the approximate problem (93), (96). Now we fix a pair of smooth measures $\bar{\mu} = (\mu, \mu_b)$ and take $L = L_{\bar{\mu}} = \sup_{x\in S, u\in\mathbb{R}} |a_{\bar{\mu}}(x,u)|$. Let $p(x,\lambda)$ be a weak solution of (93), (96).

Proposition 6. *For a.e. $\lambda \in \mathbb{R}$ for each $f(x) \in C^1(M)$, $f(x) \geq 0$*

$$\int_M \{p(x,\lambda)\mathrm{div}^{\,\mu}(a(x,\lambda)f(x)) - f(x)[r(p(x,\lambda) - P(\kappa_r p)(x,\lambda)) +$$
$$H(x,\lambda,P(\kappa_r p)(x,\lambda))]\}d\mu + L\int_S p_b(x,\lambda)f(x)d\mu_b \geq 0; \quad (102)$$

$$\int_M \{(1-p(x,\lambda))\mathrm{div}^{\,\mu}(a(x,\lambda)f(x)) + f(x)[r(p(x,\lambda) - P(\kappa_r p)(x,\lambda)) +$$
$$H(x,\lambda,P(\kappa_r p)(x,\lambda))]\}d\mu + L\int_S (1 - p_b(x,\lambda))f(x)d\mu_b \geq 0. \quad (103)$$

Proof. We denote

$$Q(x,\lambda) = r(p(x,\lambda) - P(\kappa_r p)(x,\lambda)) + H(x,\lambda,P(\kappa_r p)(x,\lambda)).$$

Since for a.e. $\lambda \in \mathbb{R}$ the function $p(x,\lambda)$ satisfies (97) along characteristics $x(s;y,\lambda)$ (for almost all $y \in M$) it is a weak solution of the transport equation $\langle a(x,\lambda), p\rangle = -Q(x,\lambda)$. Therefore, for each $f(x) \in C_0^1(M_0)$

$$\int_M \{p(x,\lambda)\mathrm{div}^{\,\mu}(a(x,\lambda)f(x)) - f(x)Q(x,\lambda)\}d\mu = 0.$$

By Proposition 1 we see that $v(x) = p(x,\lambda)a(x,\lambda)$ is a divergence measure field. Therefore, there exists its weak normal trace $v_{\bar\mu}(x)$ on S, and by Corollary 1 for each $f(x) \in C^1(M)$, $f(x) \ge 0$

$$\int_M \{p(x,\lambda)\mathrm{div}^{\,\mu}(a(x,\lambda)f(x)) - f(x)Q(x,\lambda)\}d\mu - \int_S v_{\bar\mu}(x)f(x)d\mu_b \ge 0 \tag{104}$$

(in fact, as follows from (22), the left-hand side of this relation even equals 0). Passing to local coordinates and using (98) and (19) and the fact that $p \ge 0$ we find that $v_{\bar\mu}(x) = a_{\bar\mu}(x)p_b(x,\lambda) \ge -Lp_b(x,\lambda)$ for a.e. $x \in S$, such that $a_{\bar\mu}(x,\lambda) < 0$ while $v_{\bar\mu}(x) \ge 0$ for a.e. $x \in S$ such that $a_{\bar\mu}(x,\lambda) \ge 0$. In any case we have $v_{\bar\mu}(x) \ge -Lp_b(x,\lambda)$ a.e. on S and (104) implies (102).

In order to prove (103), we observe that $q = (1 - p(x,\lambda))$ is a weak solution of the equation $\langle a(x,\lambda), q\rangle = Q(x,\lambda)$. Repeating the arguments, used for the proof of (102), we readily derive (103). The proof is complete.

Corollary 7. *For any $q(\lambda) \in F$ and each $f(x) \in C^1(M)$, $f(x) \ge 0$*

$$\int_{M\times\mathbb{R}} (1-q(\lambda))\{p(x,\lambda)\mathrm{div}^{\,\mu}(a(x,\lambda)f(x)) - f(x)\times$$

$$H(x,\lambda,P(\kappa_r p)(x,\lambda))\}d\mu d\lambda + L\int_{S\times\mathbb{R}} p_b(x,\lambda)(1-q(\lambda))f(x)d\mu_b d\lambda \ge$$

$$\int_{M\times[-R,R]} f(x)r(1-\kappa_r)p(x,\lambda)(1-q(\lambda))d\mu d\lambda; \tag{105}$$

$$\int_{M\times\mathbb{R}} q(\lambda)\{(1-p(x,\lambda))\mathrm{div}^{\,\mu}(a(x,\lambda)f(x)) + f(x)\times$$

$$H(x,\lambda,P(\kappa_r p)(x,\lambda))\}d\mu d\lambda + L\int_{S\times\mathbb{R}} q(\lambda)(1-p_b(x,\lambda))f(x)d\mu_b d\lambda \ge$$

$$-\int_{M\times[-R,R]} f(x)r(1-\kappa_r)p(x,\lambda)q(\lambda)d\mu d\lambda. \tag{106}$$

Proof. By Theorem 8 the function $p(x,\lambda) \in B$ and Lemma 2,e) yields $P(\kappa_r p)(x,\lambda) = \tilde P(\kappa_r p)(x,\lambda)$. It is clear also that $p(x,\lambda) - P(\kappa_r p)(x,\lambda) = 0$ for $|\lambda| > R$. Using Lemma 2,d) and Remark 6,1) we find that

$$\int (1-q(\lambda))(p(x,\lambda) - P(\kappa_r p)(x,\lambda))d\lambda = \int_{-R}^{R} (1-\kappa_r)(1-q(\lambda))p(x,\lambda)d\lambda +$$

$$\int_{-R}^{R} (1-q(\lambda))(\kappa_r p(x,\lambda) - \tilde P(\kappa_r p)(x,\lambda))d\lambda \ge \int_{-R}^{R} (1-\kappa_r)(1-q(\lambda))p(x,\lambda)d\lambda,$$

$$\int q(\lambda)(p(x,\lambda)-P(\kappa_r p)(x,\lambda))d\lambda = \int_{-R}^{R}(1-\kappa_r)q(\lambda)p(x,\lambda)d\lambda +$$

$$\int_{-R}^{R} q(\lambda)(\kappa_r p(x,\lambda)-\tilde{P}(\kappa_r p)(x,\lambda))d\lambda \le \int_{-R}^{R}(1-\kappa_r)q(\lambda)p(x,\lambda)d\lambda.$$

Integrating (102), (103) over the measures $(1-q(\lambda))d\lambda$, $q(\lambda)d\lambda$, respectively and taking into account the above inequalities, we obtain (105), (106).

Now, we are going to pass to the limit as $r\to\infty$. Denote by $p_r(x,\lambda)$ the weak solution of (93), (96). Let $q_r(x,\lambda)=P(\kappa_r p_r)(x,\lambda)$.

Lemma 4. *As* $r\to\infty$ $p_r-q_r\to 0$ in $L^2(M\times\mathbb{R})$.

Proof. For a.e. $\lambda\in\mathbb{R}$ $p=p_r(x,\lambda)$ is a weak solution of the transport equation $\langle a(x,\lambda),p\rangle = -r(p-q)-H(x,\lambda,q)$ with $q=q_r(x,\lambda)$. Recall that the field $a(x,\lambda)$ is smooth. Then by the known renormalization property (see for example [7]) p^2 is a weak solution of this transport equation but with the source term $2p(-r(p-q)-H(x,\lambda,q))$. Hence, $\langle a(x,\lambda),p^2\rangle = -2rp(p-q)-2pH(x,\lambda,q)$ in the sense of distributions on M, i.e. for each test function $f(x)\in C_0^1(M_0)$

$$\int_M [p_r^2(x,\lambda)\operatorname{div}^{\mu}(a(x,\lambda)f(x)) - 2p_r(x,\lambda)H(x,\lambda,q_r(x,\lambda))f(x)]d\mu =$$
$$2r\int_M p_r(x,\lambda)(p_r(x,\lambda)-q_r(x,\lambda))f(x)d\mu.$$

Integrating this equality over $\lambda\in[-R,R]$ and taking into account the uniform estimates $0\le p_r\le 1$, $|H(x,\lambda,q_r)|\le 1/\varepsilon$ (the latter follows from (84)), we find that $\forall f(x)\in C_0^1(M_0)$, $f(x)\ge 0$

$$2r\int_{M\times[-R,R]} p_r(x,\lambda)(p_r(x,\lambda)-q_r(x,\lambda))f(x)d\mu d\lambda =$$
$$\int_{M\times[-R,R]}[p_r^2\operatorname{div}^{\mu}(a(x,\lambda)f(x)) - 2p_rH(x,\lambda,q_r)f(x)]d\mu d\lambda \le C_f, \quad (107)$$

where C_f is a constant independent of r. Now, observe that

$$\int_{-R}^{R} p_r(x,\lambda)(p_r(x,\lambda)-q_r(x,\lambda))d\lambda = \int_{-R}^{R}(p_r(x,\lambda)-q_r(x,\lambda))^2 d\lambda +$$
$$(1-\kappa_r)\int_{-R}^{R} p_r(x,\lambda)q_r(x,\lambda)d\lambda + \int_{-R}^{R} q_r(x,\lambda)(\kappa_r p_r(x,\lambda)-q_r(x,\lambda))d\lambda.$$

Since $q_r = P(\kappa_r p_r) = \tilde{P}(\kappa_r p_r)$ we derive from Lemma 2,d) that the last integral vanishes. Therefore,

$$\int_{-R}^{R} (p_r(x,\lambda) - q_r(x,\lambda))^2 d\lambda \le \int_{-R}^{R} p_r(x,\lambda)(p_r(x,\lambda) - q_r(x,\lambda)) d\lambda.$$

Integrating this inequality over M and taking into account (107) we arrive at

$$\int_{M\times\mathbb{R}} (p_r(x,\lambda) - q_r(x,\lambda))^2 f(x) d\mu d\lambda \le C_f/(2r) \underset{r\to\infty}{\to} 0.$$

We also use that $p_r(x,\lambda) - q_r(x,\lambda) = 0$ for $|\lambda| > R$. The obtained estimate and the fact that $\|p_r - q_r\| \le 1$ imply that $p_r - q_r \to 0$ in $L^2(M \times \mathbb{R})$, as was to be proved.

Since the sequence p_r, $r \in \mathbb{N}$ is bounded in $L^\infty(M \times \mathbb{R})$, $0 \le p_r \le 1$ there exists a subsequence (still denoted by p_r) and a process $p(x, \lambda, s)$ on $M \times \mathbb{R}$ such that $p_r \to p$ in the sense of relation (88):

$$g(x, \lambda, p_r(x,\lambda)) \underset{r\to\infty}{\to} \int_0^1 g(x, \lambda, p(x, \lambda, s)) ds \quad \text{weakly-}* \text{ in } L^\infty(M \times \mathbb{R}) \tag{108}$$

for every Caratheodory function $g(x, \lambda, p)$ on $M \times \mathbb{R} \times \mathbb{R}$. As is easy to verify, $0 \le p(x, \lambda, s) \le 1$. Further, from Lemma 4 it follows the relation

$$g(x, \lambda, q_r(x,\lambda)) \underset{r\to\infty}{\to} \int_0^1 g(x, \lambda, p(x, \lambda, s)) ds \quad \text{weakly-}* \text{ in } L^\infty(M \times \mathbb{R}), \tag{109}$$

i.e. q_r converges weakly as $r \to \infty$ to the same process p. From relations (108), (109) in particular follows that $p_r, q_r \to \bar{p}(x,\lambda) = \int_0^1 p(x, \lambda, s) ds$ weakly-$*$ in $L^\infty(M \times \mathbb{R})$. Since $q_r(x,\lambda) \in \mathcal{F}_R(M)$ and $\mathcal{F}_R(M)$ is a closed convex subset of $L^\infty(M \times \mathbb{R})$ the weak limit $\bar{p}(x,\lambda) \in \mathcal{F}_R(M)$. Let us show that, more generally, $p(\cdot, s) \in \mathcal{F}_R(M)$ for all $s \in (0,1)$.

Lemma 5. *For each* $s \in (0,1)$ $p(\cdot, s) \in \mathcal{F}_R(M)$.

Proof. Let $\nu_{x,\lambda} \in \mathrm{MV}(M \times \mathbb{R})$ be the limit measure valued function of the sequence $q_r(x,\lambda)$ in the sense of Theorem T. Then

$$p(x, \lambda, s) = \inf\{\, v \,|\, \nu_{x,\lambda}((v, +\infty)) \le s \,\} = \inf\{\, v \,|\, \langle g(u), \nu_{x,\lambda}(u)\rangle \le s \text{ for each nondecreasing continuous function } g(u) \le \operatorname{sign}^+(u - v) \,\}. \tag{110}$$

Since $\langle g(u), \nu_{x,\lambda}(u)\rangle$ is a weak-$*$ limit of the sequence $g(q_r(x,\lambda))$ and all these functions are non-increasing with respect to λ, the same is true for the limit function $\langle g(u), \nu_{x,\lambda}(u)\rangle$. From (110) it then follows that $p(x,\lambda,s)$ is non-increasing with respect to λ. Observe also that $0 \le p(x,\lambda,s) \le 1$ and $p(x,\lambda,s) = \mathrm{sign}^+(-\lambda)$ for $|\lambda| > R$ because this property holds for all functions $q_r(x,\lambda)$. We conclude that $p(x,\lambda,s) \in \mathcal{F}_R(M)$ for all fixed $s \in (0,1)$.

We are ready to prove the existence of a kinetic process solution.

Theorem 9. *The process $p(x,\lambda,s)$ is a kinetic process solution of (81), (54).*

Proof. It remains only to establish relations (89), (90). By Corollary 7 for any $q(\lambda) \in F$ and each $f(x) \in C^1(M)$, $f(x) \ge 0$

$$\int_{M\times\mathbb{R}} (1-q(\lambda))\{p_r(x,\lambda)\mathrm{div}^{\,\mu}(a(x,\lambda)f(x)) - f(x)\times$$
$$H(x,\lambda,q_r(x,\lambda))\}d\mu d\lambda + L\int_{S\times\mathbb{R}} p_b(x,\lambda)(1-q(\lambda))f(x)d\mu_b d\lambda \ge$$
$$\int_{M\times[-R,R]} f(x)r(1-\kappa_r)p_r(x,\lambda)(1-q(\lambda))d\mu d\lambda;$$
$$\int_{M\times\mathbb{R}} q(\lambda)\{(1-p_r(x,\lambda))\mathrm{div}^{\,\mu}(a(x,\lambda)f(x)) + f(x)\times$$
$$H(x,\lambda,q_r(x,\lambda))\}d\mu d\lambda + L\int_{S\times\mathbb{R}} q(\lambda)(1-p_b(x,\lambda))f(x)d\mu_b d\lambda \ge$$
$$-\int_{M\times[-R,R]} f(x)r(1-\kappa_r)p_r(x,\lambda)q(\lambda)d\mu d\lambda.$$

Passing in this inequalities to the limit as $r \to \infty$ with account of (108), (109) and the relation $r(1-\kappa_r) \to 0$, we arrive at (89), (90).

Corollary 8. *There exists a k.s. $p(x,\lambda) \in \mathcal{F}_R(M)$ of the problem (53), (54).*

Proof. We choose the function $H(x,\lambda,p) \equiv 0$. Obviously, this function satisfies all assumption (82)-(84). In this case (89), (90) reduce to (55), (56) with $p = \bar{p}(x,\lambda) = \int_0^1 p(x,\lambda,s)ds$. Thus, $\bar{p}(x,\lambda) \in \mathcal{F}_R(M)$ is a k.s. of (53), (54).

From Corollaries 8,3,4 we readily deduce the following

Corollary 9. *If condition (U) is satisfied then there exists a unique g.e.s. of (1), (2).*

But our aim is to prove the existence of g.e.s. without assumption (U). By Theorem 7 there exist the maximal k.sub-s. $p_+(x,\lambda) = \mathrm{sign}^+(u_+(x)-\lambda)$ and the minimal k.super-s. $p_-(x,\lambda) = \mathrm{sign}^+(u_-(x)-\lambda)$ of (53), (54), $u_\pm(x) \in L^\infty(M)$, $\|u_\pm\|_\infty \le R$. We will establish that the functions $u_\pm(x)$ are actually g.e.s. of (1), (2). In view of Theorem 4 it suffices to establish the following result.

Theorem 10. *The functions $p_\pm(x,\lambda)$ are k.s. of (53), (54).*

Proof. We introduce the function $H = H_l(x,\lambda,p) = l(p - p_-(x,\lambda))^+$, where $l > 0$. This function satisfies assumptions (82)-(84) with constants $C = l$, $\varepsilon = 1/l$. For instance (84) follows from the evident equality $p - (p-p_-)^+ = \min(p,p_-)$. By Theorem 9 for each $l > 0$ there exists a kinetic process solution $p_l(x,\lambda,s)$ of the problem (81), (89) with the indicated $H = H_l$ such that $p_l(\cdot,s) \in \mathcal{F}_R(M)$ for all $s \in (0,1)$. Denote

$$\bar{p}_l(x,\lambda) = \int_0^1 p_l(x,\lambda,s)ds \in \mathcal{F}_R(M), \quad \bar{H}_l(x,\lambda) = \int_0^1 H_l(x,\lambda,p_l(x,\lambda,s))ds.$$

Then in view of (89) for all $q(\lambda) \in F$ and each $f(x) \in C^1(M)$, $f(x) \ge 0$

$$\int_{M\times\mathbb{R}} [\bar{p}_l(x,\lambda)\mathrm{div}^{\,\mu}(a(x,\lambda)f(x)) - \bar{H}_l(x,\lambda)f(x)](1-q(\lambda))d\mu d\lambda + \\ L\int_{S\times\mathbb{R}} p_b(x,\lambda)(1-q(\lambda))f(x)d\mu_b d\lambda \ge 0. \quad (111)$$

From (111) it follows, in the same way as for kinetic solutions, the existence of a weak trace $(\bar{p}_l)_\tau(x,\lambda)$ at the boundary, and the relations like (61), (65): for each $q(\lambda) \in F$ and all $f = f(x) \in C^1(M)$, $f \ge 0$

$$\int_{M\times\mathbb{R}} [\bar{p}_l(x,\lambda)\mathrm{div}^{\,\mu}(a(x,\lambda)f(x)) - \bar{H}_l(x,\lambda)f(x)](1-q(\lambda))d\mu d\lambda - \\ \int_{S\times\mathbb{R}} a_{\bar{\mu}}(x,\lambda)(\bar{p}_l)_\tau(x,\lambda)(1-q(\lambda))f(x)d\mu_b d\lambda \ge 0; \quad (112)$$

for a.e. $x \in S$ $\forall q(\lambda) \in F$

$$\int a_{\bar{\mu}}(x,\lambda)(\bar{p}_l)_\tau(x,\lambda)(1-q(\lambda))d\lambda + L\int p_b(x,\lambda)(1-q(\lambda))d\lambda \ge 0. \quad (113)$$

Since $p_-(x,\lambda)$ is a k.super-s. of (53), (54) relations (64), (66) hold: for each $q(\lambda)\in F$ and all $f=f(x)\in C^1(M)$, $f\ge 0$

$$\int_{M\times\mathbb{R}}(1-p_-(x,\lambda))q(\lambda)\operatorname{div}^{\mu}(a(x,\lambda)f(x))d\mu d\lambda - \int_{S\times\mathbb{R}} a_{\bar\mu}(x,\lambda)(1-(p_-)_\tau(x,\lambda))q(\lambda)f(x)d\mu_b d\lambda \ge 0; \tag{114}$$

for a.e. $x\in S$ $\forall q(\lambda)\in F$

$$\int a_{\bar\mu}(x,\lambda)(1-(p_-)_\tau(x,\lambda))q(\lambda)d\lambda + L\int(1-p_b(x,\lambda))q(\lambda)d\lambda \ge 0. \tag{115}$$

Now, we apply the doubling variable method in the same way as in the proof of Theorem 5, namely we use (112) for $q(\lambda)=p_-(y,\lambda)$ and (114), written in variables (y,λ), for $q(\lambda)=\bar p_l(x,\lambda)$. Putting the obtained inequality together and applying the result to the test function $f(x;y)=f(x)\delta_\nu(y-x)$, we obtain in the limit as $\nu\to\infty$ (with account of (113), (115) as well) that

$$\int_{M\times\mathbb{R}}[\bar p_l(x,\lambda)\operatorname{div}^{\mu}(a(x,\lambda)f(x)) - \bar H_l(x,\lambda)f(x)](1-p_-(x,\lambda))d\mu d\lambda \ge -L\int_{S\times\mathbb{R}} p_b(x,\lambda)(1-p_b(x,\lambda))f(x)d\mu_b d\lambda = 0. \tag{116}$$

Since $p_-(x,\lambda)$ takes only values $0,1$ and $p_l(x,\lambda,s)\in(0,1)$ we have

$$\bar H_l(x,\lambda) = l\int_0^1 (p_l(x,\lambda,s)-p_-(x,\lambda))^+ds = l\int_0^1 p_l(x,\lambda,s)(1-p_-(x,\lambda))ds = l\bar p_l(x,\lambda)(1-p_-(x,\lambda)).$$

Therefore, (116) acquires the form (we use also that $(1-p_-(x,\lambda))^2 = 1-p_-(x,\lambda)$)

$$\int_{M\times\mathbb{R}}\bar p_l(x,\lambda)(1-p_-(x,\lambda))(\operatorname{div}^{\mu}(a(x,\lambda)f(x)) - lf(x))d\mu d\lambda \ge 0.$$

Taking $f(x)\equiv 1$ we derive that $\bar H_l(x,\lambda)\bar p_l(x,\lambda)(1-p_-(x,\lambda))=0$ a.e. on $M\times\mathbb{R}$ for each $l>\max_{M\times[-R,R]}\operatorname{div}^{\mu}a(x,\lambda)$. This means that $\bar p_l(x,\lambda)\le p_-(x,\lambda)$. Since $\bar H_l\equiv 0$ for such l we derive from (89), (90) that $\bar p_l$ is a k.s. of the homogeneous problem (53), (54). Recall that p_- is the minimal k.super-s. of this problem. Therefore, $p_- \le \bar{p}_l$. This, together with the inverse inequality $\bar{p}_l \le p_-$ yields $p_- = p_l$. Hence, $p_- = p_-(x, \lambda)$ is a k.s. of (53), (54), as was to be proved.

The fact that the maximal k.sub-s. $p_+(x, \lambda)$ is also a k.s. follows from Lemma 1. This statement can be also proved by the same arguments as above with using $H_l = -l(p_+(x, \lambda) - p)^+$.

Thus we proved the following general result.

Theorem 11. *There exists a g.e.s. $u(x)$ of (1), (2). Moreover, for each $R \ge \|u_b\|_\infty$ there exist the unique maximal g.e.s. $u_+(x)$ and minimal g.e.s. $u_-(x)$ among g.e.s. with norm $\|u\|_\infty \le R$.*

We conclude the paper by some remarks.

1) All the result of this paper can be generalized for the case of non-smooth fields $a(x, \lambda)$. For that, we have to utilize the well-posedness theory for transport equations with coefficients from Sobolev or BV classes developed in [7, 1]. We also can treat the case of non-compact manifolds. Of course, in this case some additional assumptions on the vector field $a(x, \lambda)$ are required;

2) In the case of the Cauchy problem (44), (45) one could use the more elegant L^2 kinetic formulation of strong measure valued solutions developed in [20, 21].

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