ON ADMISSIBILITY CRITERIA FOR WEAK SOLUTIONS OF THE EULER EQUATIONS

CAMILLO DE LELLIS AND LÁSZLÓ SZÉKELYHIDI JR.

ABSTRACT. We consider solutions to the Cauchy problem for the incompressible Euler equations satisfying several additional requirements, like the global and local energy inequalities. Using some techniques introduced in an earlier paper we show that, for some bounded compactly supported initial data, none of these admissibility criteria singles out a unique weak solution.

As a byproduct we show bounded initial data for which admissible solutions to the p–system of isentropic gas dynamics in Eulerian coordinates are not unique in more than one space dimension.

1. Introduction

In this paper we consider the Cauchy problem for the incompressible Euler equations in n–space dimensions

$$\left\{\begin{array}{l} \partial_t v + \operatorname{div}(v\otimes v) + \nabla p \;=\; 0 \\ \operatorname{div} v \;=\; 0 \\ v(x,0) \;=\; v^0(x)\,, \end{array}\right. \tag{1}$$

where the initial data v^0 satisfies the compatibility condition

$$\operatorname{div} v^0 \;=\; 0\,. \tag{2}$$

A divergence–free vector field $v \in L^2_{loc}$ is a *weak solution* of (1) if

$$\int_{\mathbb{R}^n\times\mathbb{R}^+} [v\cdot\partial_t\varphi + \langle v\otimes v, \nabla\varphi\rangle]\,dx\,dt \;=\; \int_{\mathbb{R}^n} v^0(x)\varphi(x,0)\,dx \tag{3}$$

for every test function $\varphi \in C^\infty_c(\mathbb{R}^n\times\mathbb{R}_t,\mathbb{R}^n)$ with $\operatorname{div}\varphi = 0$. It is well–known that then the pressure is determined up to a function depending only on time (see [20]).

In his pioneering work [14] Scheffer showed that weak solutions to the 2–dimensional Euler equations are not unique. In particular Scheffer constructed a nontrivial weak solution compactly supported in space and time, thus disproving uniqueness for (1) even when $v^0 = 0$. A simpler construction was later proposed by Shnirelman in [16].

In a recent paper, we have shown a quite powerful approach to the construction of irregular solutions of (1), recovering Scheffer's and Shnirelman's counterexamples in all dimensions and with bounded velocity and pressure. Moreover, our construction yields as a simple corollary the existence of energy–decreasing solutions, thus recovering another groundbreaking result of Shnirelman [17], again with the additional features that our examples have bounded velocity and pressures and can be shown to exist in any dimension.

The aim of this note is to discuss the relations between our constructions and various admissibility criteria that could be imposed on weak solutions of Euler. With our methods we can show that none of these criteria implies uniqueness for general L^2 initial data. More precisely we prove the following theorem (for the relevant definitions of weak, strong and local energy inequalities, we refer to Sections 2.1 and 2.2).

Theorem 1.1. *Let $n \geq 2$. There exist bounded and compactly supported divergence–free vector fields v^0 for which there are*

(a) *infinitely many weak solutions of* (1) *satisfying both the strong and the local energy* equalities*;*
(b) *weak solutions of* (1) *satisfying the strong energy inequality but not the energy equality;*
(c) *weak solutions of* (1) *satisfying the weak energy inequality but not the strong energy inequality.*

Our examples display a very wild behavior, such as dissipation of the energy and amplitude of high–frequency oscillations. We will refer to them as *wild solutions*. Our analysis relies on some criteria on the initial data for the existence of (many) wild solutions satisfying the various admissibility conditions. We then exhibit initial data for which our criteria are satisfied and as a corollary we obtain several non–uniqueness results. We explicitly state these criteria in Section 3, see Proposition 3.3.

As a byproduct of our analysis we prove a similar non–uniqueness result for the p–system of isentropic gas dynamics in Eulerian coordinates, the oldest hyperbolic system of conservation laws. The unknowns of the system, which consists of $n+1$ equations, are the density ρ and the velocity v of the gas:

$$\begin{cases} \partial_t \rho + \mathrm{div}_x(\rho v) \;=\; 0 \\ \partial_t(\rho v) + \mathrm{div}_x(\rho v \otimes v) + \nabla[p(\rho)] \;=\; 0 \\ \rho(0,\cdot) \;=\; \rho^0 \\ v(0,\cdot) \;=\; v^0 \end{cases} \tag{4}$$

(cf. (3.3.17) in [4] and Section 1.1 of [15] p7). The pressure p is a function of ρ, which is determined from the constitutive thermodynamic relations of the gas in question and satisfies the assumption $p' > 0$. A typical example is $p(\rho) = k\rho^\gamma$, with constants $k > 0$ and $\gamma > 1$, which gives the constitutive relation for a polytropic gas (cf. (3.3.19) and (3.3.20) of [4]). Weak solutions of (4) are bounded functions which solve it in the sense of distributions. Admissible solutions have to satisfy an additional inequality, coming from the conservation law for the energy of the system. For the relevant definition we refer to Section 2.4.

Theorem 1.2. *Let $n \geq 2$. Then, for any given function p, there exist bounded initial data (ρ^0, v^0) with $\rho^0 \geq c > 0$ for which there are infinitely many bounded admissible solutions (ρ, v) of (4) with $\rho \geq c > 0$.*

The paper is organized as follows. Section 2 contains a survey of several admissibility conditions for (1) and the definition of admissible solutions for (4). Section 3 states a general criterion on the existence of wild solutions to (1) for a given initial data, see Proposition 3.3. In Section 4 we prove Proposition 3.3 and in Section 5 we construct initial data meeting the requirements of Proposition 3.3, see Proposition 5.1. Finally, in Section 6 we prove the non–uniqueness theorems 1.1 and 1.2 using Proposition 3.3 and Proposition 5.1.

2. An overview of the different notions of admissibility

In this section we discuss various admissibility criteria for weak solutions which have been proposed in the past years in the literature.

2.1. **Weak and strong energy inequalities.** All the admissibility criteria considered so far in the literature are motivated by approximating (1) with the Navier Stokes equations. We therefore consider the following vanishing viscosity approximation of (1)

$$\left\{\begin{array}{l} \partial_t v + \operatorname{div}(v\otimes v) + \nabla p \;=\; \varepsilon\Delta v\\ \operatorname{div} v \;=\; 0\\ v(x,0) \;=\; v^0(x)\,, \end{array}\right. \tag{5}$$

where the parameter ε is positive but small. A *weak solution* of (5) is then a divergence–free field $v \in L^2_{loc}$ satisfying

$$\int_{\mathbb{R}^n\times\mathbb{R}^+} \Big[v\cdot(\partial_t\varphi + \varepsilon\Delta\varphi) + \langle v\otimes v, \nabla\varphi\rangle\Big]\,dx\,dt \;=\; \int_{\mathbb{R}^n} v^0(x)\varphi(x,0)\,dx \tag{6}$$

for every test function $\varphi \in C^\infty_c(\mathbb{R}^n\times\mathbb{R}_t, \mathbb{R}^n)$ with $\operatorname{div}\varphi = 0$.

For smooth solutions, we can multiply (1) and (5) by v and derive corresponding partial differential equations for $|v|^2$, namely

$$\partial_t\frac{|v|^2}{2} + \operatorname{div}\left(v\left(\frac{|v|^2}{2}+p\right)\right) \;=\; 0 \tag{7}$$

and

$$\partial_t\frac{|v|^2}{2} + \operatorname{div}\left(v\left(\frac{|v|^2}{2}+p\right)\right) \;=\; \varepsilon\Delta\frac{|v|^2}{2} - \varepsilon|\nabla v|^2\,. \tag{8}$$

Recall that (1) and (5) model the movements of ideal incompressible fluids. If we assume that the constant density of the fluid is normalized to 1, then $|v|^2/2$ is the energy density and (7) and (8) are simply the laws of conservation of the energy, in local form.

Integrating (7) and (8) in time and space and assuming that p and v are decaying sufficiently fast at infinity, we achieve the following identities

$$\frac{1}{2}\int_{\mathbb{R}^n} |v|^2(x,t)\,dx \;=\; \frac{1}{2}\int_{\mathbb{R}^n} |v|^2(x,s)\,ds \qquad \text{for all } s,t\in\mathbb{R} \tag{9}$$

$$\frac{1}{2}\int_{\mathbb{R}^n} |v|^2(x,t)\,dx \;=\; \frac{1}{2}\int_{\mathbb{R}^n} |v|^2(x,s)\,dx - \varepsilon\int_s^t\int_{\mathbb{R}^3} |\nabla v|^2(x,\tau)\,dx\,d\tau\,. \tag{10}$$

The celebrated result of Leray [10] (see [8] for a modern introduction) shows the existence of weak solutions to (5) which satisfy a relaxed version of (10), the so–called *weak energy inequality*, namely (11) and (12) below.

Theorem 2.1 (Leray–Hopf)**.** *Let $v^0 \in L^2$ be a divergence–free vector field. Then there exists a weak solution v of* (5) *with $\nabla v \in L^2_{loc}$ which satisfies*

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{R}^n} |v|^2(x,t)\,dx \;\le\; &\frac{1}{2}\int_{\mathbb{R}^n} |v^0|^2(x)\,dx\\ &-\varepsilon\int_0^t\int_{\mathbb{R}^3} |\nabla v|^2(x,\tau)\,dx\,d\tau \qquad \textit{for almost all } t\end{aligned} \tag{11}$$

and

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{R}^n} |v|^2(x,t)\,dx \;\le\; &\frac{1}{2}\int_{\mathbb{R}^n} |v|^2(x,s)\,dx\\ &-\varepsilon\int_s^t\int_{\mathbb{R}^3} |\nabla v|^2(x,\tau)\,dx\,d\tau \qquad \textit{for a.a. pairs } (s,t) \textit{ with } s<t.\end{aligned} \tag{12}$$

In what follows, the solutions of Theorem 2.1 will be called *Leray solutions*. If a weak solution v of (1) is the strong limit of a sequence of Leray solutions v_k of (5) with vanishing viscosity $\varepsilon = \varepsilon_k \downarrow 0$, then v inherits in the limit (11) and (12). This justifies the following definition.

Definition 2.2. *A weak solution of* (1) *satisfies the* weak energy inequality *if*

$$\int_{\mathbb{R}^n} |v|^2(x,t)\,dx \;\le\; \int_{\mathbb{R}^n} |v^0|^2(x)\,dx \qquad \textit{for a.a. } t \tag{13}$$

and

$$\int_{\mathbb{R}^n} |v|^2(x,t)\,dx \;\le\; \int_{\mathbb{R}^n} |v|^2(x,s)\,dx \qquad \textit{for a.a. } (s,t) \textit{ with } s<t. \tag{14}$$

A weak solution of (1) *satisfies the* weak energy equality *if equality holds in* (13) *and* (14)*.*

An interesting feature of both (1) and (5) is that weak solutions have a natural notion of trace at *every* time t, i.e. they are weakly continuous in time.

Lemma 2.3. *Let v be a weak solution of (1) or a weak solution of (5), belonging to the space $L^\infty([0,T],L^2(\mathbb{R}^n))$. Then, v can be redefined on a set of t of measure zero so that $v \in C([0,T],L^2_w(\mathbb{R}^n))$, i.e. so that the function*

$$t \;\mapsto\; \int_{\mathbb{R}^n} v(x,t)\cdot\varphi(x)\,dx \tag{15}$$

is continuous for every $\varphi \in L^2(\mathbb{R}^n,\mathbb{R}^n)$.

This property (or a suitable variant of it) is common to all distributional solutions of evolution equations which can be written as balance laws (see for instance Theorem 4.1.1 in [4]) and can be proved by standard arguments. In the appendix we include, for the reader's convenience, a proof of a more general statement, which will be useful later. From now on, we will use the slightly shorter notation $C\big([0,T],L^2_w\big)$ for $C\big([0,T],L^2_w(\mathbb{R}^n)\big)$.

It follows that weak solutions satisfying the weak energy inequality have a well–defined notion of total energy $\frac{1}{2}\int |v|^2(x,t)dt$ at every time t. Moreover, it is easy to see that (11) and (13) are actually satisfied at all times $t>0$. Similarly, for a.a. s (12) and (14) holds for every $t>s$. Instead, the following is a stronger requirement.

Definition 2.4. *A weak solution $v \in C\big(([0,T],L^2_w\big)$ of (1) (resp. of (5)) satisfies the strong energy inequality if (14) (resp. (12)) holds for every pair (s,t) with $s<t$. Similarly, we say that it satisfies the strong energy inequality if the equality in (14) holds for every pair (s,t) with $s<t$.*

The strong energy inequality seems a very reasonable condition from the physical point of view, both for Euler and Navier–Stokes. Whether Leray's solutions satisfy the strong energy inequality is a long–standing open question (see [8]). As an outcome of our approach, we answer negatively the same question for weak solutions of (1) satisfying the weak energy inequality (see Theorem 1.1).

2.2. **The local energy inequality.** Consider next a Leray solution of (5). It turns out that $v \in L^\infty_t(L^2_x)$ and $\nabla v \in L^2_t(L^2_x)$. The Sobolev inequality and a simple interpolation argument shows that $v \in L^3_{loc}(\mathbb{R}^n\times\mathbb{R}^+)$ if the space dimension n is less or equal to 4 [1]. In this case, one could formulate a weak local form of the energy inequality, requiring that the natural inequality corresponding to (8) holds in the distributional sense. This amounts to the condition

$$\int_{\mathbb{R}^n\times\mathbb{R}^+} \frac{|v|^2}{2}(-\partial_t\varphi+\varepsilon\Delta\varphi)\,dx\,dt \;\le\; \int_{\mathbb{R}^n\times\mathbb{R}^+}\left(\frac{|v|^2}{2}+p\right)v\cdot\nabla\varphi\,dx\,dt \tag{16}$$

[1]Indeed, by the Sobolev embedding, we conclude that $v \in L^2_t(L^{2^*}_x)$. Interpolating between the spaces $L^\infty L^2$ and $L^2L^{2^*}$ we conclude that $u\in L^r_t(L^s_x)$ for every exponents r and s satisfying the identities

$$\frac{1}{r}=\frac{1-\alpha}{2} \quad \frac{1}{s}=\frac{\alpha}{2}+\frac{1-\alpha}{2^*} \;=\; \frac{1}{2}-\frac{1-\alpha}{n} \qquad \text{for some } \alpha\in[0,1].$$

Plugging $\alpha=2/(2+n)$ we obtain $r=s=2(1+\frac{2}{n})=:q$. Clearly, $q\geq 3$ for $n=2,3,4$.

for any nonnegative $\varphi \in C_c^\infty(\mathbb{R}^n \times \mathbb{R}^+)$. Note that, since $v \in L^3_{loc}$ and

$$\Delta p \;=\; \operatorname{div}\operatorname{div}(v \otimes v)\,, \tag{17}$$

by the Calderon–Zygmund estimates we have $p \in L^{3/2}_{loc}$. Therefore pv is a well–defined locally summable function.

It is not known whether the Leray solutions do satisfy (16). However, it is possible to construct global weak solutions satisfying the weak energy inequality and the local energy inequality. This fact has been proved for the first time by Scheffer in [13] (see also the appendix of [3]). The local energy inequality is a fundamental ingredient in the partial regularity theory initiated by Scheffer and culminating in the work of Caffarelli, Kohn and Nirenberg, see [3] and [11].

Theorem 2.5. *Let $n \leq 4$ and let $v^0 \in L^2(\mathbb{R}^n)$ be a divergence–free vector field. Then there exists a weak solution v of* (5) *with $\nabla v \in L^2_{loc}$ and which satisfies* (11)*,* (12) *and* (16)*.*

By analogy, for weak solutions of (1), Duchon and Robert in [7] have proposed to look at a local form of the energy inequality (14).

Definition 2.6 (Duchon–Robert). *Consider an L^3_{loc} weak solution v of* (1)*. We say that v satisfies the* local energy inequality *if*

$$\partial_t \frac{|v|^2}{2} + \operatorname{div}\left(v\left(\frac{|v|^2}{2} + p\right)\right) \;\leq\; 0 \tag{18}$$

in the sense of distributions, i.e. if

$$-\frac{1}{2}\int_{\mathbb{R}^n \times \mathbb{R}^+} |v|^2 \partial_t \varphi \;\leq\; \int_{\mathbb{R}^n \times \mathbb{R}^+} \left(\frac{|v|^2}{2} + p\right) v \cdot \nabla\varphi \tag{19}$$

for every nonnegative $\varphi \in C_c^\infty(\mathbb{R}^n \times \mathbb{R}^+)$.

Similarly, if the equality in (19) *holds for every test function, then we say that v satisfies the* local energy equality*.*

Since (17) holds even for weak solutions of (1), $v \in L^3_{loc}$ implies $p \in L^{3/2}_{loc}$, and hence the product pv is well–defined. Note, however, that, for solutions of Euler, the requirement $v \in L^3_{loc}$ is not at all natural, even in low dimensions: there is no apriori estimate yielding this property.

2.3. **Measure–valued and dissipative solutions.**

Two other very weak notions of solutions to incompressible Euler have been proposed in the literature: DiPerna–Majda's measure–valued solutions (see [6]) and Lions' dissipative solutions (see Chapter 4.4 of [12]).

Both notions are based on considering weakly convergent sequences of solutions of Navier-Stokes with vanishing viscosity.

On the one hand, the possible oscillations in the nonlinear term $v \otimes v$ lead to the appearance of an additional term in the limit, where this term is subject to a certain pointwise convexity constraint. This can be formulated by saying that the weak limit is the barycenter of a measure–valued solution (cf. [6] and also [1, 19] for alternative settings using Wigner- and H-measures). A closely related object is our "subsolution", defined in Section 4.1.

On the other hand, apart from the energy inequality, a version of the Gronwall inequality prevails in the weak limit, leading to the definition of dissipative solutions, cf. Appendix B. As a consequence, dissipative solutions coincide with classical solutions as long as the latter exist:

Theorem 2.7 (Proposition 4.1 in [12]). *If there exists a solution $v \in C([0,T], L^2(\mathbb{R}^n))$ of* (1) *such that $(\nabla v + \nabla v^T) \in L^1([0,T], L^\infty(\mathbb{R}^n))$, then any dissipative solution of* (1) *is equal to v on $\mathbb{R}^n \times [0,T]$.*

This is relevant for our discussion because of the following well–known fact.

Proposition 2.8. *Let $v \in C([0,T], L^2_w)$ be a weak solution of* (1) *satisfying the weak energy inequality. Then v is a dissipative solution.*

Our construction yields initial data for which the nonuniqueness results of Theorem 1.1 hold on any time interval $[0,\varepsilon[$. However, for sufficiently regular initial data, classical results give the local existence of smooth solutions. Therefore, Proposition 2.8 imply that, *a fortiori*, the initial data considered in our examples have necessarily a certain degree of irregularity.

Though Proposition 2.8 is well–known, we have not been able to find a reference for its proof and therefore we include one in Appendix B (see the proof of Proposition 8.2).

2.4. **Admissible solutions to the p-system.** As usual, by a weak solution of (4) we understand a pair $(\rho, v) \in L^\infty$ such that the following identities hold for every test functions $\psi \in C^\infty_c(\mathbb{R}^n \times \mathbb{R}), \varphi \in C^\infty_c(\mathbb{R}^n \times \mathbb{R})$:

$$\int_{\mathbb{R}^n\times\mathbb{R}^+} \Big[\rho\partial_t\psi + \rho v\cdot\nabla_x\psi\Big]\,dx\,dt \;=\; \int_{\mathbb{R}^n} \rho(x)\,\psi(x,0)\,dx \tag{20}$$

$$\int_{\mathbb{R}^n\times\mathbb{R}^+} \Big[\rho v\cdot\partial_t\varphi + \rho\langle v\otimes v, \nabla\varphi\rangle\Big]\,dx\,dt \;=\; \int_{\mathbb{R}^n} \rho^0(x)v^0(x)\cdot\varphi(x,0)\,dx\,. \tag{21}$$

Admissible solutions have to satisfy an additional constraint. Consider the internal energy $\varepsilon : \mathbb{R}^+ \to \mathbb{R}$ given through the law $p(r) = r^2\varepsilon'(r)$. Then admissible solutions of (20) have to satisfy the inequality

$$\partial_t\left[\rho\varepsilon(\rho) + \frac{\rho|v|^2}{2}\right] + \mathrm{div}_x\left[\left(\rho\varepsilon(\rho) + \frac{\rho|v|^2}{2} + p(\rho)\right)v\right] \;\leq\; 0 \tag{22}$$

(cf. (3.3.18) and (3.3.21) of [4]). More precisely

Definition 2.9. *A weak solution of* (4) *is admissible if the following inequality holds for every nonnegative* $\psi \in C_c^\infty(\mathbb{R}^n \times \mathbb{R})$*:*

$$\int_{\mathbb{R}^n\times\mathbb{R}^+} \left[\left(\rho\varepsilon(\rho) + \frac{\rho|v|^2}{2}\right)\partial_t\psi + \left(\rho\varepsilon(\rho) + \frac{\rho|v|^2}{2} + p(\rho)\right)\cdot\nabla_x\psi\right] + \int_{\mathbb{R}^n}\left(\rho^0\varepsilon(\rho^0) + \frac{\rho^0|v^0|^2}{2}\right)\psi(\cdot,0) \;\geq\; 0\,. \tag{23}$$

3. A criterion for the existence of wild solutions

In this section we state some criteria to recognize initial data v^0 which allow for many weak solutions of (1) satisfying the weak, strong and/or local energy inequality. In order to state it, we need to introduce some of the notation already used in [5].

3.1. **The Euler equation as a differential inclusion.** In particular, we state the following lemma (compare with Lemma 2.1 of [5]). Here and in what follows we denote by $\mathcal{S}^n$ the space of symmetric $n\times n$ matrices, by $\mathcal{S}^n_0$ the subspace of $\mathcal{S}^n$ of matrices with trace 0, and by I_n the $n\times n$ identity matrix.

Lemma 3.1. *Suppose* $v \in L^2(\mathbb{R}^n\times[0,T],\mathbb{R}^n)$*,* $u \in L^2(\mathbb{R}^n\times[0,T],\mathcal{S}^n_0)$*, and* q *is a distribution such that*

$$\begin{aligned} \partial_t v + div\ u + \nabla q = 0\,,\\ div\ v = 0\,. \end{aligned} \tag{24}$$

If (v,u,q) *solve* (24) *and in addition*

$$u = v\otimes v - \frac{1}{n}|v|^2 I_n \qquad a.e.\ in\ \mathbb{R}^n\times[0,T]\,, \tag{25}$$

then v *and* $p := q - \frac{1}{n}|v|^2$ *solve* (1) *distributionally. Conversely, if* v *and* p *solve* (1) *distributionally,* v*,* $u = v\otimes v - \frac{1}{n}|v|^2 I_n$ *and* $q = p + \frac{1}{n}|v|^2$ *solve* (24) *and* (25)*.*

Next, for every $r \geq 0$, we consider the set of *Euler states of speed* r

$$K_r \;:=\; \left\{(v,u)\in\mathbb{R}^n\times\mathcal{S}^n_0 :\ u = v\otimes v - \frac{r^2}{n}I_n,\ |v| = r\right\} \tag{26}$$

(cf. Section of [5], in particular (25) therein). Lemma 3.1 says simply that solutions to the Euler equations can be viewed as evolutions on the manifold of Euler states subject to the linear conservation laws (24).

Next, we denote by K_r^{co} the convex hull in $\mathbb{R}^n\times\mathcal{S}^n_0$ of K_r. In the following Lemma we give an explicit formula for K_r^{co}. Since it will be often used in the sequel, we introduce the following notation. For $v, w \in \mathbb{R}^n$ let $v\odot w$ denote the symmetrized tensor product, that is

$$v\odot w \;=\; \frac{1}{2}\big(v\otimes w + w\otimes v\big), \tag{27}$$

and let $v \bigcirc w$ denote its traceless part, that is

$$v \bigcirc w \;=\; \frac{1}{2}\big(v\otimes w + w\otimes v\big) - \frac{v\cdot w}{n} I_n. \tag{28}$$

Note that

$$v \bigcirc v \;=\; v\otimes v - \frac{|v|^2}{n} I_n$$

and hence K_r is simply

$$K_r \;=\; \{(v, v\bigcirc v) : |v| = r\}\;.$$

Lemma 3.2. *For any $w \in \mathcal{S}^n$ let $\lambda_{max}(w)$ denote the largest eigenvalue of w. For $(v,u) \in \mathbb{R}^n \times \mathcal{S}^n_0$ let*

$$e(v,u) := \frac{n}{2}\lambda_{max}(v\otimes v - u). \tag{29}$$

Then

(i) $e : \mathbb{R}^n \times \mathcal{S}^n_0 \to \mathbb{R}$ *is convex;*
(ii) $\frac{1}{2}|v|^2 \le e(v,u)$, *with equality if and only if* $u = v\otimes v - \frac{|v|^2}{n} I_n$*;*
(iii) $|u|_\infty \le 2\frac{n-1}{n}\, e(v,u)$, *where* $|u|_\infty$ *denotes the operator norm of the matrix;*
(iv) *The* $\frac{1}{2}r^2$*–sublevel set of* e *is the convex hull of* K_r*, i.e.*

$$K_r^{co} \;=\; \left\{(v,u) \in \mathbb{R}^n \times \mathcal{S}^n_0 : e(v,u) \le \frac{r^2}{2}\right\}\;. \tag{30}$$

(v) *If* $(u,v) \in \mathbb{R}^n \times \mathcal{S}^n_0$*, then* $\sqrt{2e(v,u)}$ *gives the smallest* ρ *for which* $(u,v) \in K^{co}_\rho$.

In view of (ii) if a triple (v,u,q) solving (24) corresponds a solution of the Euler equations via the correspondence in Lemma 3.1, then $e(v,u)$ is simply the energy density of the solution. In view of this remark, if (v,u,q) is a solution of (24), $e(v,u)$ will be called the *generalized energy density*, and $E(t) = \int_{\mathbb{R}^n} e(v(x,t), u(x,t))dx$ will be called the *generalized energy*.

We postpone the proof of Lemma 3.2 to the next subsection and we state now the criterion for the existence of wild solutions. Its proof, which is the core of the paper, will be given in Section 4.

Proposition 3.3. *Let $\Omega \subset \mathbb{R}^n$ be an open set (not necessarily bounded) and let*

$$\bar{e} \in C\big(\overline{\Omega}\times]0,T[\big) \cap C\big([0,T]; L^1(\Omega)\big).$$

Assume there exists (v_0, u_0, q_0) smooth solution of (24) on $\mathbb{R}^n\times]0,T[$ with the following properties:

$$v_0 \in C\big([0,T]; L^2_w\big), \tag{31}$$

$$\operatorname{supp}(v_0(\cdot,t), u_0(\cdot,t)) \subset\subset \Omega \text{ for all } t \in]0,T[, \tag{32}$$

$$e\big(v_0(x,t), u_0(x,t)\big) < \bar{e}(x,t) \text{ for all } (x,t) \in \Omega\times\,]0,T[\,. \tag{33}$$

Then there exist infinitely many weak solutions v of the Euler equations (1) *with pressure*

$$p = q_0 - \frac{1}{n}|v|^2 \tag{34}$$

such that

$$v \in C\big([0,T]; L^2_w\big), \tag{35}$$

$$v(\cdot,t) = v_0(\cdot,t) \qquad \text{for } t = 0, T, \tag{36}$$

$$\frac{1}{2}|v(\cdot,t)|^2 = \bar{e}(\cdot,t)\,\mathbf{1}_\Omega \qquad \text{for every } t \in]0,T[. \tag{37}$$

3.2. **Proof of Lemma 3.2.**

Proof. **(i)** Note that

$$\begin{aligned} e(v,u) &= \frac{n}{2}\max_{\xi\in S^{n-1}}\Big\langle \xi, (v\otimes v - u)\xi\Big\rangle = \frac{n}{2}\max_{\xi\in S^{n-1}}\Big\langle \xi, \langle \xi, v\rangle v - u\xi\Big\rangle \\ &= \frac{n}{2}\max_{\xi\in S^{n-1}}\Big[|\langle\xi,v\rangle|^2 - \langle \xi, u\xi\rangle\Big]. \end{aligned} \tag{38}$$

Since for every $\xi \in S^{n-1}$ the map $(v,u)\mapsto |\langle\xi,v\rangle|^2 - \langle\xi,u\xi\rangle$ is convex, it follows that e is convex.

(ii) Since $v\otimes v = v \bigcirc v + \frac{|v|^2}{n}I_n$, we have, similarly to above, that

$$\begin{aligned} e(v,u) &= \frac{n}{2}\max_{\xi\in S^{n-1}}\Big\langle \xi, (v\bigcirc v - u)\xi\Big\rangle + \frac{|v|^2}{2} \\ &= \frac{n}{2}\lambda_{max}(v\bigcirc v - u) + \frac{|v|^2}{2}. \end{aligned} \tag{39}$$

Observe that, since $v\bigcirc v - u$ is traceless, the sum of its eigenvalues is zero. Therefore $\lambda_{max}(v\bigcirc v - u)\geq 0$ with equality if and only if $v\bigcirc v - u = 0$. This proves the claim.

(iii) From (38) and (39) we deduce

$$e(v,u) \geq \frac{n}{2}\max_{\xi\in S^{n-1}}\Big(-\langle\xi,u\xi\rangle\Big) = -\frac{n}{2}\lambda_{min}(u)\,.$$

Therefore $-\lambda_{min}(u)\leq \frac{2}{n}e(v,u)$. Since u is traceless, the sum of its eigenvalues is zero, hence

$$|u|_\infty \leq (n-1)|\lambda_{min}(u)| \leq \frac{2(n-1)}{n}\,e(v,u).$$

(iv) Without loss of generality we assume $r=1$. Let

$$S_1 \;:=\; \left\{(v,u)\in\mathbb{R}^n\times\mathcal{S}^n_0 : e(v,u) \;\leq\; \frac{1}{2}\right\}. \tag{40}$$

Observe that $e(v,u)=\frac{1}{2}$ for all $(v,u)\in K_1$, hence - by convexity of e -

$$K_1^{co}\subset S_1.$$

To prove the opposite inclusion, observe first of all that S_1 is convex by (i) and compact by (ii) and (iii). Therefore S_1 is equal to the closed convex hull of its extreme points. In light of this observation it suffices to show that the convex extreme points of S_1 are contained in K_1.

To this end let $(v,u) \in S_1 \setminus K_1$. By a suitable rotation of the coordinate axes we may assume that $v \otimes v - u$ is diagonal, with diagonal entries $1/n \geq \lambda_1 \geq \cdots \geq \lambda_n$. Note that $(v,u) \notin K_1 \Longrightarrow \lambda_n < 1/n$. Indeed, if $\lambda_n = 1/n$, then we have the identity $u = v \otimes v - \frac{1}{n} I_n$. Since the trace of u vanishes, this identity implies $|v|^2 = 1$ and $u = v \otimes v - \frac{|v|^2}{n} I_n$, which give $(v,u) \in K_1$.

Let $e_1, \ldots, e_n$ denote the coordinate unit vectors, and write $v = \sum_i v^i e_i$. Consider the pair $(\bar{v}, \bar{u}) \in \mathbb{R}^n \times \mathcal{S}_0^n$ defined by

$$\bar{v} = e_n, \quad \bar{u} = \sum_{i=1}^{n-1} v^i (e_i \otimes e_n + e_n \otimes e_i).$$

A simple calculation shows that

$$(v + t\bar{v}) \otimes (v + t\bar{v}) - (u + t\bar{u}) = (v \otimes v - u) + (2t\, v^n + t^2) e_n \otimes e_n.$$

In particular, since $\lambda_n < 1/n$, $e(v+t\bar{v}, u+t\bar{u}) \leq 1/n$ for all sufficiently small $|t|$, so that $(v,u) + t(\bar{v},\bar{u}) \in S_1$. This shows that (v,u) cannot be a convex extreme point of S_1.

(v) is an easy direct consequence of (iv). □

4. Proof of Proposition 3.3

Although the general strategy for proving Proposition 3.3 is based on Baire category arguments as in [5], there are several points in which Proposition 3.3 differs, which give rise to technical difficulties. The main technical difficulty is given by the requirements (35) and (37), where we put a special emphasis on the fact that the equality in (37) must hold for *every* time t. The arguments in [5], which are based on the interplay between weak-strong convergence following [9], yield only solutions in the space $L^\infty\big([0,T]; L^2(\mathbb{R}^n)\big)$. Although such solutions can be redefined on a set of times of measure zero (see Lemma 2.3) so that they belong to the space $C\big([0,T]; L^2_w\big)$, this gives the equality

$$\frac{1}{2}|v(\cdot,t)|^2 = \bar{e}(\cdot,t)\, \mathbf{1}_\Omega \text{ for almost every } t \in]0,T[\,. \tag{41}$$

For the construction of solutions satisfying the strong energy inequality this conclusion is not enough. Indeed, a consequence of Theorem 1.1c) is precisely the fact that (37) does not follow automatically from (41).

This Section is split into five parts. In 4.1 we introduces the functional framework, we state Lemma 4.3, Lemma 4.4 and Proposition 4.5, and we show how Proposition 3.3 follows from them. The two lemmas are simple consequences of functional analytic facts, and they are proved in 4.2. The perturbation property of Proposition 4.5 is instead the main point of the argument. In 4.3 we introduce the waves which are the basic building blocks for proving Proposition 4.5. In 4.4 we introduce a suitable potential to localize the waves of 4.3. Finally, in 4.5 we use these two tools and a careful construction to prove Proposition 4.5.

4.1. **Functional setup.** We start by defining the space of "subsolutions" as follows. Let v_0 be a vectorfield as in Proposition 3.3 with associated modified pressure q_0, and consider velocity fields $v : \mathbb{R}^n \times [0,T] \to \mathbb{R}^n$ which satisfy

$$\operatorname{div} v = 0, \tag{42}$$

the initial and boundary conditions

$$\begin{aligned} v(x,0) &= v_0(x,0), \\ v(x,T) &= v_0(x,T), \\ \operatorname{supp} v(\cdot,t) &\subset\subset \Omega \text{ for all } t \in]0,T[, \end{aligned} \tag{43}$$

and such that there exists a smooth matrix field $u : \mathbb{R}^n \times]0,T[\to \mathcal{S}_0^n$ with

$$\begin{aligned} e\big(v(x,t),u(x,t)\big) &< \bar{e}(x,t) \text{ for all } (x,t) \in \Omega \times]0,T[\,, \\ \operatorname{supp} u(\cdot,t) &\subset\subset \Omega \text{ for all } t \in]0,T[\,, \\ \partial_t v + \operatorname{div} u + \nabla q_0 &= 0 \text{ in } \mathbb{R}^n \times [0,T]. \end{aligned} \tag{44}$$

Definition 4.1 (The space of subsolutions). *Let X_0 be the set of such velocity fields, i.e.*

$$X_0 = \Big\{ v \in C^\infty\big(\mathbb{R}^n \times]0,T[\big) \cap C\big([0,T], L^2_w\big) : \text{(42),(43),(44) } \textit{are satisfied} \Big\},$$

and let X be the closure of X_0 in $C\big([0,T]; L^2_w\big)$.

We assume that $\bar{e} \in C\big([0,T]; L^1(\Omega)\big)$, therefore there exists a constant c_0 such that $\int_\Omega \bar{e}(x,t)dx \le c_0$ for all $t \in [0,T]$. Since for any $v \in X_0$ we have

$$\frac{1}{2}\int_\Omega |v(x,t)|^2\, dx \le \int_\Omega \bar{e}(x,t)dx \qquad \text{for all } t \in [0,T],$$

we see that X_0 consists of functions $v : [0,T] \to L^2(\mathbb{R}^n)$ taking values in a bounded subset B of $L^2(\mathbb{R}^n)$. Without loss of generality we can assume that B is weakly closed. Let d_B be a metric on B which metrizes the weak topology. Then (B, d_B) is a compact metric space. Moreover, d_B induces naturally a metric d on the space $Y := C([0,T], (B, d_B))$ via the definition

$$d(w_1, w_2) \;=\; \max_{t\in[0,T]} d_B(w_1(\cdot,t), w_2(\cdot,t)). \tag{45}$$

The topology induced by d on Y is equivalent to the topology of Y as subset of $C\big([0,T]; L^2_w\big)$. Moreover, by Arzelà-Ascoli's theorem, the space (Y,d) is compact. Finally, X is the closure in (Y,d) of X_0, and hence (X,d) is as well a compact metric space.

Definition 4.2 (The functionals I_{ε,Ω_0}). *Next, for any $\varepsilon > 0$ and any bounded open set $\Omega_0 \subset \Omega$ consider the functional*

$$I_{\varepsilon,\Omega_0}(v) := \inf_{t\in[\varepsilon,T-\varepsilon]} \int_{\Omega_0} \Big[\frac{1}{2}|v(x,t)|^2 - \bar{e}(x,t)\Big]\, dx.$$

It is clear that on X each functional I_{ε,Ω_0} is bounded from below.

We are now ready to state the three important building blocks of the proof of Proposition 3.3. The first two lemmas are simple consequences of our functional analytic framework

Lemma 4.3. *The functionals I_{ε,Ω_0} are lower-semicontinuous on X.*

Lemma 4.4. *For all $v \in X$ we have $I_{\varepsilon,\Omega_0}(v) \le 0$. If $I_{\varepsilon,\Omega_0}(v) = 0$ for every $\varepsilon > 0$ and every bounded open set $\Omega_0 \subset \Omega$, then v is a weak solution of the Euler equations* (1) *with pressure*

$$p = q_0 - \frac{1}{n}|v|^2,$$

and such that (35),(36),(37) *are satisfied.*

The following proposition is the key point in the whole argument, and it is the only place where the particularities of the equations enter. It corresponds to Lemma 4.6 of [5], though its proof is considerably more complicated due to the special role played by the time variable in this context.

Proposition 4.5 (The perturbation property)**.** *Let Ω_0 and $\varepsilon > 0$ be given. For all $\alpha > 0$ there exists $\beta > 0$ (possibly depending on ε and Ω_0) such that whenever $v \in X_0$ with*

$$I_{\varepsilon,\Omega_0}(v) < -\alpha,$$

there exists a sequence $v_k \in X_0$ with $v_k \xrightarrow{d} v$ and

$$\liminf_{k\to\infty} I_{\varepsilon,\Omega_0}(v_k) \ge I_{\varepsilon,\Omega_0}(v) + \beta\,.$$

Remark 1. *In fact the proof of Proposition 4.5 will show that in case Ω is bounded and $\bar{e}$ is uniformly bounded in $\overline{\Omega} \times [0,T]$, the improvement β in the statement can be chosen to be*

$$\beta = \min\{\alpha/2, C\alpha^2\},$$

with C only depending on $|\Omega|$ and $\|\bar{e}\|_\infty$.

We postpone the proofs of these facts to the following subsections, and now show how Proposition 3.3 follows from them and the general Baire category argument.

Proof of Proposition 3.3. Since the functional I_{ε,Ω_0} is lower-semicontinuous on the compact metric space X and takes values in a bounded interval of $\mathbb{R}$, it can be written as a pointwise supremum of countably many continuous functionals, see Proposition 11 in Section 2.7 of Chapter IX of [2]. Therefore, I_{ε,Ω_0} is a Baire-1 map and hence its points of continuity form a residual set in X. We claim that if $v \in X$ is a point of continuity of I_{ε,Ω_0}, then $I_{\varepsilon,\Omega_0}(v) = 0$.

To prove the claim, assume the contrary, i.e. that there exists $v \in X$ which is a point of continuity of I_{ε,Ω_0} and $I_{\varepsilon,\Omega_0}(v) < -\alpha$ for some $\alpha > 0$. Choose a sequence $\{v_k\} \subset X_0$ such that $v_k \xrightarrow{d} v$. Then in particular $I_{\varepsilon,\Omega_0}(v_k) \to I_{\varepsilon,\Omega_0}(v)$ and so, by possibly renumbering the sequence, we may assume that

$I_{\varepsilon,\Omega_0}(v_k) < -\alpha$. Now we invoke Proposition 4.5 for each function v_k and by extracting a diagonal subsequence find a new sequence $\{\tilde{v}_k\} \subset X_0$ such that

$$\tilde{v}_k \stackrel{d}{\to} v \text{ in } X,$$
$$\liminf_{k\to\infty} I_{\varepsilon,\Omega_0}(\tilde{v}_k) \geq I_{\varepsilon,\Omega_0}(v) + \beta.$$

This is in contradiction with the assumption that v is a point of continuity of I_{ε,Ω_0}, thereby proving our claim.

Next, let Ω_k be an exhausting sequence of bounded open subsets of Ω. Consider the set Ξ which is the intersection of

$$\Xi_k \;:=\; \left\{v \in X \;:\; I_{1/k,\Omega_k} \text{ is continuous at } v\right\}\,.$$

Ξ is the intersection of countably many residual sets and hence it is residual. Moreover, if $v \in \Xi$, then $I_{\varepsilon,\Omega_0}(v) = 0$ for any $\varepsilon > 0$ and any bounded $\Omega_0 \subset \Omega$. By Lemma 4.4, any $v \in \Xi$ satisfies the requirements of Proposition 3.3. One can easily check that the cardinality of X is infinite and therefore the cardinality of any residual set in X is infinite as well. This concludes the proof. □

4.2. **Proofs of Lemma 4.3 and Lemma 4.4.**

Proof of Lemma 4.3. Assume for a contradiction that there exists $v_k, v \in X$ such that $v_k \stackrel{d}{\to} v$ in X, but

$$\begin{aligned}
&\lim_{k\to\infty} \inf_{t\in[\varepsilon,T-\varepsilon]} \int_{\Omega_0} \left[\frac{1}{2}|v_k(x,t)|^2 - \bar{e}(x,t)\right] dx \\
< \quad & \inf_{t\in[\varepsilon,T-\varepsilon]} \int_{\Omega_0} \left[\frac{1}{2}|v(x,t)|^2 - \bar{e}(x,t)\right] dx.
\end{aligned}$$

Then there exists a sequence of times $t_k \in [\varepsilon, T-\varepsilon]$ such that

$$\begin{aligned}
&\lim_{k\to\infty} \int_{\Omega_0} \left[\frac{1}{2}|v_k(x,t_k)|^2 - \bar{e}(x,t_k)\right] dx \\
< \quad & \inf_{t\in[\varepsilon,T-\varepsilon]} \int_{\Omega_0} \left[\frac{1}{2}|v(x,t)|^2 - \bar{e}(x,t)\right] dx. \qquad (46)
\end{aligned}$$

We may assume without loss of generality that $t_k \to t_0$. Since the convergence in X is equivalent to the topology of $C\big([0,T]; L^2_w\big)$, we obtain that

$$v_k(\cdot, t_k) \rightharpoonup v(\cdot, t_0) \text{ in } L^2(\Omega) \text{ weakly,}$$

and hence

$$\liminf_{k\to\infty} \int_{\Omega_0} \left[\frac{1}{2}|v_k(x,t_k)|^2 - \bar{e}(x,t_k)\right] dx \geq \int_{\Omega_0} \left[\frac{1}{2}|v(x,t_0)|^2 - \bar{e}(x,t_0)\right] dx.$$

This contradicts (46), thereby concluding the proof. □

Proof of Lemma 4.4. For $v \in X_0$ there exists $u : \mathbb{R}^n \times]0,T[\to \mathcal{S}_0^n$ such that (44) holds. Therefore

$$\frac{1}{2}|v(x,t)|^2 \leq e\big(v(x,t), u(x,t)\big) < \bar{e}(x,t)$$

for all $(x,t) \in \Omega \times]0,T[$ and hence $I_{\varepsilon,\Omega_0}(v) \leq 0$ for $v \in X_0$. For general $v \in X$ the inequality follows from the density of X_0 and the lower-semicontinuity of I_{ε,Ω_0}.

Next, let $v \in X$ and assume that $I_{\varepsilon,\Omega_0}(v) = 0$ for every $\varepsilon > 0$ and every bounded open $\Omega_0 \subset \Omega$. Let $\{v_k\} \subset X_0$ be a sequence such that $v_k \overset{d}{\to} v$ in X and let u_k be the associated sequence of matrix fields satisfying (44). The sequence $\{u_k\}$ satisfies the pointwise estimate

$$|u_k|_\infty \leq \frac{2(n-1)}{n}\, e(v_k, u_k) < \frac{2(n-1)}{n}\, \bar{e},$$

because of Lemma 3.2 (iii). Therefore $\{u_k\}$ is locally uniformly bounded in L^∞ and hence, by extracting a weakly convergent subsequence and relabeling, we may assume that

$$u_k \overset{*}{\rightharpoonup} u \ \text{ in } L^\infty_{loc}\big(\Omega \times [0,T]\big).$$

Since $v_k \to v$ in $C\big([0,T]; L^2_w\big)$ and $I_{\varepsilon,\Omega_0}(v) = 0$ for every choice of ε and Ω_0, we see that v satisfies (35), (36) and (37). Moreover, the linear equations

$$\begin{cases} \partial_t v + \operatorname{div} u + \nabla q_0 = 0, \\ \qquad\qquad \operatorname{div} v = 0 \end{cases}$$

hold in the limit, and – since e is convex – we have

$$e\big(v(x,t), u(x,t)\big) \leq \bar{e}(x,t) \text{ for a.e. } (x,t) \in \Omega \times [0,T]. \tag{47}$$

To prove that v is a weak solution of the Euler equations (1) with pressure $p = q_0 - \frac{1}{n}|v|^2$, in view of Lemma 3.1, it suffices to show that

$$u = v \otimes v - \frac{|v|^2}{n} I_n \ \text{ a.e. in } \Omega \times [0,T]. \tag{48}$$

Combining (37) and (47) we have

$$\frac{1}{2}|v(x,t)|^2 = e\big(v(x,t), u(x,t)\big) \text{ for almost every } (x,t) \in \Omega \times [0,T],$$

so that (48) follows from Lemma 3.2 (ii). □

4.3. **Geometric setup.** In this subsection we introduce the first tool for proving Proposition 4.5. The admissible segments defined below correspond to suitable plane-wave solutions of (24). More precisely, following L. Tartar [18], the directions of these segments belong to the wave cone Λ for the system of linear PDEs (24) (cf. Section 2 of [5] and in particular (7) therein).

Definition 4.6. *Given $r > 0$ we will call σ an* admissible segment *if σ is a line segment in $\mathbb{R}^n \times \mathcal{S}_0^n$ satisfying the following conditions:*

- *σ is contained in the interior of K_r^{co},*
- *σ is parallel to $(a, a\otimes a)-(b, b\otimes b)$ for some $a, b\in\mathbb{R}^n$ with $|a| = |b| = r$ and $b\neq \pm a$.*

The following lemma, a simple consequence of Carathéodory's theorem for convex sets, ensures the existence of sufficiently large admissible segments (cf. with Lemma 4.3 of [5]).

Lemma 4.7. *There exists a dimensional constant $C>0$ such that for any $r>0$ and for any $(v,u)\in int\, K_r^{co}$ there exists an admissible line segment*

$$\sigma = \Big[(v,u)-(\bar v,\bar u)\,,\,(v,u)+(\bar v,\bar u)\Big] \tag{49}$$

such that

$$|\bar v|\geq \frac{C}{r}(r^2-|v|^2).$$

Proof. Let $z=(v,u)\in \text{int } K_r^{co}$. By Carathéodory's theorem (v,u) lies in the interior of a simplex in $\mathbb{R}^n\times\mathcal{S}^n_0$ spanned by elements of K_r. In other words

$$z=\sum_{i=1}^{N+1}\lambda_i z_i,$$

where $\lambda_i\in]0,1[\,,\ \sum_{i=1}^{N+1}\lambda_i=1,\ N=n(n+3)/2-1$ is the dimension of $\mathbb{R}^n\times\mathcal{S}^n_0$ and

$$z_i=\big(v_i, v_i\otimes v_i-\frac{r^2}{n}I_n\big)$$

for some $v_i\in\mathbb{R}^n$ with $|v_i|=r$. By possibly perturbing the z_i slightly, we can ensure that $v_i\neq\pm v_j$ whenever $i\neq j$ (this is possible since (v,u) is contained in the interior of the simplex). Assume that the coefficients are ordered so that $\lambda_1=\max_i\lambda_i$. Then for any $j>1$

$$z\,\pm\,\frac12\lambda_j(z_j-z_1)\in \text{int } K_r^{co}.$$

Indeed,

$$z\,\pm\,\frac12\lambda_j(z_j-z_1)=\sum_i\mu_i z_i,$$

where $\mu_1=\lambda_1\mp\frac12\lambda_j$, $\mu_j=\lambda_j\pm\frac12\lambda_j$ and $\mu_i=\lambda_i$ for $i\neq 1,j$. It is easy to see that $\mu_i\in]0,1[$ for all $i=1\dots N$.

On the other hand $z-z_1=\sum_{i=2}^{N+1}\lambda_i(z_i-z_1)$, so that

$$|v-v_1|\leq N\max_{i=2\dots N+1}\lambda_i|v_i-v_1|. \tag{50}$$

Let $j>1$ be such that $\lambda_j|v_j-v_1|=\max_{i=2\dots N+1}\lambda_i|v_i-v_1|$, and let

$$\begin{aligned}(\bar v,\bar u)&=\frac12\lambda_j(z_j-z_1)\\ &=\frac12\lambda_j\big(v_j-v_1,\,v_j\otimes v_j-v_1\otimes v_1\big).\end{aligned}$$

Then σ, defined by (49), is contained in the interior of K_r^{co}, hence it is an admissible segment. Moreover, by the choice of j and using (50)

$$\frac{1}{4rN}(r^2-|v|^2) \le \frac{1}{4rN}(r+|v|)(r-|v|) \le \frac{1}{2N}|v-v_1| \le |\bar{v}|.$$

This finishes the proof. □

4.4. **Oscillations at constant pressure.** In this section we construct a potential for the linear conservation laws (24). Similar potentials were constructed in the paper [5] (see Lemma 3.4 therein). However, the additional feature of this new potential is that it allows to localize the oscillations at constant pressure, which are needed in the proof of Proposition 4.5.

As a preliminary step recall from Section 3 in [5] that solutions of (24) in $\mathbb{R}^n$ correspond to symmetric divergence–free matrix fields on $\mathbb{R}^{n+1}$ for which the $(n+1),(n+1)$ entry vanishes. To see this it suffices to consider the linear map

$$\mathbb{R}^n \times \mathcal{S}_0^n \times \mathbb{R} \ni (v,u,q) \quad \mapsto \quad U = \begin{pmatrix} u+qI_n & v \\ v & 0 \end{pmatrix}. \tag{51}$$

Note also that with this identification $q = \operatorname{tr} U$. Therefore solutions of (24) with $q \equiv 0$ correspond to matrix fields $U : \mathbb{R}^{n+1} \to \mathbb{R}^{(n+1)\times(n+1)}$ such that

$$\operatorname{div} U = 0, \quad U^T = U, \quad U_{(n+1),(n+1)} = 0, \quad \operatorname{tr} U = 0. \tag{52}$$

Furthermore, given a velocity vector $a \in \mathbb{R}^n$, the matrix of the corresponding Euler state is

$$U_a = \begin{pmatrix} a \otimes a - \frac{|a|^2}{n} I_n & a \\ a & 0 \end{pmatrix}.$$

The following proposition gives a potential for solutions of (24) oscillating between two Euler states U_a and U_b of equal speed at constant pressure.

Proposition 4.8. *Let $a,b \in \mathbb{R}^n$ such that $|a| = |b|$ and $a \neq \pm b$. Then there exists a matrix–valued, constant coefficient, homogeneous linear differential operator of order 3*

$$A(\partial) : C_c^\infty(\mathbb{R}^{n+1}) \to C_c^\infty\left(\mathbb{R}^{n+1}; \mathbb{R}^{(n+1)\times(n+1)}\right)$$

such that $U = A(\partial)\phi$ satisfies (52) for all $\phi \in C_c^\infty(\mathbb{R}^{n+1})$. Moreover there exists $\eta \in \mathbb{R}^{n+1}$ such that

- *η is not parallel to e_{n+1};*
- *if $\phi(y) = \psi(y \cdot \eta)$, then*

$$A(\partial)\phi(y) = (U_a - U_b)\, \psi'''(y \cdot \eta).$$

Proof. A matrix valued homogeneous polynomial of degree 3

$$A : \mathbb{R}^{n+1} \to \mathbb{R}^{(n+1)\times(n+1)}$$

gives rise to a differential operator required by the proposition if and only if $A = A(\xi)$ satisfies

$$A\xi = 0, \quad A^T = A, \quad Ae_{(n+1)} \cdot e_{(n+1)} = 0, \quad \operatorname{tr} A = 0 \tag{53}$$

for all $\xi \in \mathbb{R}^{n+1}$.

Define the $(n+1)\times(n+1)$ antisymmetric matrices

$$R = a\otimes b - b\otimes a,$$
$$Q(\xi) = \xi\otimes e_{n+1} - e_{n+1}\otimes \xi,$$

where in the definition of R we treat $a, b \in \mathbb{R}^n$ as elements of $\mathbb{R}^{n+1}$ by setting the $(n+1)$'s coordinate zero. The following facts are easily verified:

(i) $R\xi\cdot\xi = 0, \quad Q(\xi)\xi\cdot\xi = 0$, due to antisymmetry;
(ii) $R\xi\cdot e_{n+1} = 0$, since $a\cdot e_{n+1} = b\cdot e_{n+1} = 0$;
(iii) $R\xi\cdot Q(\xi)\xi = 0$, because by (i) and (ii) $R\xi$ is perpendicular to the range of Q.

Let

$$A(\xi) = R\xi\odot\big(Q(\xi)\xi\big) = \frac{1}{2}\Big(R\xi\otimes\big(Q(\xi)\xi\big) + \big(Q(\xi)\xi\big)\otimes R\xi\Big)$$

The properties (i),(ii),(iii) immediately imply (53).

Now define $\eta\in\mathbb{R}^{n+1}$ by

$$\eta = \frac{-1}{(|a||b| + a\cdot b)^{2/3}}\Big(a + b - (|a||b| + a\cdot b)e_{n+1}\Big).$$

Since $|a| = |b|$ and $a \neq \pm b$, $|a||b| + a\cdot b \neq 0$ so that η is well–defined and non–zero. Moreover, a direct calculation shows that

$$A(\eta) = \begin{pmatrix} a\otimes a - b\otimes b & a-b \\ a-b & 0\end{pmatrix} = U_a - U_b.$$

Finally, observe that if $\phi(y) = \psi(y\cdot\eta)$, then $A(\partial)\phi(y) = A(\eta)\psi'''(y\cdot\eta)$. $\square$

The following simple lemma ensures that the oscillations of the plane-waves produced by Proposition 4.8 have a certain size in terms of functionals of the type I_{ε,Ω_0}.

Lemma 4.9. *Let $\eta\in\mathbb{R}^{n+1}$ be a vector which is not parallel to e_{n+1}. Then for any bounded open set $B\subset\mathbb{R}^n$*

$$\lim_{N\to\infty}\int_B \sin^2\big(N\eta\cdot(x,t)\big)\,dx \;=\; \frac{1}{2}|B|$$

uniformly in $t\in\mathbb{R}$.

Proof. Let us write $\eta = (\eta', \eta_{n+1}) \in \mathbb{R}^n\times\mathbb{R}$, so that $\eta'\in\mathbb{R}^n\setminus\{0\}$. By elementary trigonometric identities

$$\sin^2\big(N\eta\cdot(x,t)\big) = \sin^2(N\eta'\cdot x) + \\ + \sin^2(N\eta_{n+1}t)\cos(2N\eta'\cdot x) + \frac{1}{2}\sin(2N\eta'\cdot x)\sin(2N\eta_{n+1}t).$$

For the second term we have

$$\Big|\int_B \sin^2(N\eta_{n+1}t)\cos(2N\eta'\cdot x)dx\Big| \le \Big|\int_B \cos(2N\eta'\cdot x)dx\Big| \to 0$$

as $N\to\infty$, and similarly the third term vanishes in the limit uniformly in t. The statement of the lemma now follows easily. $\square$

4.5. **Proof of the perturbation property.** We are now ready to conclude the proof of Proposition 4.5.

Step 1. Shifted grid. We start by defining a grid on $\mathbb{R}^n_x \times \mathbb{R}_t$ of size h. For $\zeta \in \mathbb{Z}^n$ let $|\zeta| = \zeta_1 + \cdots + \zeta_n$ and let $Q_\zeta, \tilde{Q}_\zeta$ be cubes in $\mathbb{R}^n$ centered at ζh with sidelength h and $\frac{3}{4}h$ respectively, i.e.

$$Q_\zeta := \zeta h + \left[-\frac{h}{2}, \frac{h}{2}\right]^n, \ \tilde{Q}_\zeta := \zeta h + \left[-\frac{3h}{8}, \frac{3h}{8}\right]^n.$$

Furthermore, for every $(\zeta, i) \in \mathbb{Z}^n \times \mathbb{Z}$ let

$$C_{\zeta,i} = \begin{cases} Q_\zeta \times [ih, (i+1)h] & \text{if } |\zeta| \text{ is even,} \\ Q_\zeta \times [(i-\frac{1}{2})h, (i+\frac{1}{2})h] & \text{if } |\zeta| \text{ is odd.} \end{cases}$$

Next, we let $0 \le \varphi \le 1$ be a smooth cutoff function on $\mathbb{R}^n_x \times \mathbb{R}_t$, with support contained in $[-h/2, h/2]^{n+1}$, identically 1 on $[-3h/8, 3h/8]^{n+1}$ and strictly less than 1 outside. Denote by $\varphi_{\zeta,i}$ the obvious translation of φ supported in $C_{\zeta,i}$, and let

$$\phi^h := \sum_{\zeta \in \mathbb{Z}^n, i \in \mathbb{Z}} \varphi_{\zeta,i}.$$

Given an open and bounded set Ω_0, let

$$\Omega_1^h = \bigcup \{\tilde{Q}_\zeta : |\zeta| \text{ even}, Q_\zeta \subset \Omega_0\}, \quad \Omega_2^h = \bigcup \{\tilde{Q}_\zeta : |\zeta| \text{ odd}, Q_\zeta \subset \Omega_0\}.$$

Observe that

$$\lim_{h \to 0} |\Omega_\nu^h| = \frac{1}{2}\left(\frac{3}{4}\right)^n |\Omega_0| \quad \text{for } \nu = 1, 2,$$

and for every fixed t the set $\{x \in \Omega_0 : \phi^h(x,t) = 1\}$ contains at least one of the sets Ω_ν^h, see Figure 1. Indeed, if

$$\tau_1^h = \bigcup_{i \in \mathbb{N}} \Big[(i + \frac{1}{4})h, (i + \frac{3}{4})h\Big[\text{ and } \tau_2^h = \bigcup_{i \in \mathbb{N}} \Big[(i - \frac{1}{4})h, (i + \frac{1}{4})h\Big[,$$

then $\tau_1^h \cup \tau_2^h = \mathbb{R}$, and for $\nu = 1, 2$

$$\phi^h(x,t) = 1 \text{ for all } (x,t) \in \Omega_\nu^h \times \tau_\nu^h.$$

Now let $v \in X_0$ with

$$I_{\varepsilon,\Omega_0}(v) < -\alpha$$

for some $\alpha > 0$, and let $u : \Omega \times]0, T[\to \mathcal{S}^n_0$ be a corresponding smooth matrix field satisfying (44). Let

$$M = \max_{\Omega_0 \times [\varepsilon/2, T - \varepsilon/2]} \bar{e}, \tag{54}$$

and let $E_h : \Omega_0 \times [\varepsilon, T - \varepsilon] \to \mathbb{R}$ be the step-function on the grid defined by

$$E_h(x,t) = E_h(\zeta h, ih) = \frac{1}{2}\big|v(\zeta h, ih)\big|^2 - \bar{e}(\zeta h, ih) \quad \text{for } (x,t) \in C_{\zeta,i}.$$

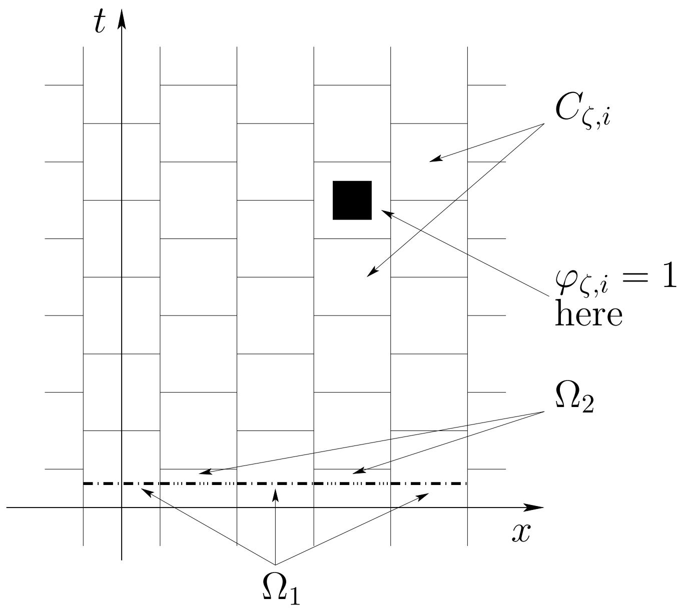

FIGURE 1. The "shifted" grid in dimension $1+1$.

This is well–defined provided $h < \varepsilon$. Since v and $\bar{e}$ are uniformly continuous on $\Omega_0 \times [\varepsilon/2, T-\varepsilon/2]$, for any $\nu \in \{1,2\}$

$$\lim_{h\to 0}\int_{\Omega_\nu^h} E_h(x,t)dx \;=\; \frac{1}{2}\left(\frac{3}{4}\right)^n \int_{\Omega_0}\Big[\frac{1}{2}|v(x,t)|^2 - \bar{e}(x,t)\Big]dx$$

uniformly in $t \in [\varepsilon, T-\varepsilon]$. In particular there exists a dimensional constant $c > 0$ such that, for all sufficiently small grid sizes h and for any $t \in [\varepsilon, T-\varepsilon]$, we have

$$\begin{aligned} &\int_{\Omega_\nu^h} |E_h(x,t)|dx \geq c\alpha \\ &\qquad \text{whenever } \int_{\Omega_0}\Big[\frac{1}{2}|v(x,t)|^2 - \bar{e}(x,t)\Big]dx \leq -\frac{\alpha}{2}. \end{aligned} \tag{55}$$

Next, for each $(\zeta, i) \in \mathbb{Z}^n \times \mathbb{Z}$ such that $C_{\zeta,i} \subset \Omega_0 \times [\varepsilon/2, T-\varepsilon/2]$ let

$$z_{\zeta,i} = \big(v(\zeta h, ih), u(\zeta h, ih)\big),$$

and, using Lemma 4.7, choose a segment

$$\sigma_{\zeta,i} = \big[z_{\zeta,i} - \bar{z}_{\zeta,i}, z_{\zeta,i} + \bar{z}_{\zeta,i}\big]$$

admissible for $r = \sqrt{2\bar{e}(\zeta h, ih)}$ (cf. Definition 4.6) with midpoint $z_{\zeta,i}$ and direction $\bar{z}_{\zeta,i} = \big(\bar{v}_{\zeta,i}, \bar{u}_{\zeta,i}\big)$ such that

$$|\bar{v}_{\zeta,i}|^2 \geq \frac{C}{\bar{e}(\zeta h, ih)}|E_h(\zeta h, ih)|^2 \geq \frac{C}{M}|E_h(\zeta h, ih)|^2. \tag{56}$$

Since $z := (v,u)$ and $\bar{e}$ are uniformly continuous, for sufficiently small h we have

$$e\big(z(x,t) + \lambda \bar{z}_{\zeta,i}\big) < \bar{e}(x,t) \quad \text{ for all } \lambda \in [-1,1] \text{ and } (x,t) \in C_{\zeta,i}. \tag{57}$$

Thus we fix the grid size $0 < h < \varepsilon/2$ so that the estimates (55) and (57) hold.

Step 2. The perturbation. Fix (ζ, i) for the moment. Corresponding to the admissible segment $\sigma_{\zeta,i}$, in view of Proposition 4.8 and the identification (51) there exists an operator $A_{\zeta,i}$ and a direction $\eta_{\zeta,i} \in \mathbb{R}^{n+1}$, not parallel to e_{n+1}, such that for any $N \in \mathbb{N}$

$$A_{\zeta,i}\left(N^{-3}\cos\left(N^3 \eta_{\zeta,i}\cdot(x,t)\right)\right) \; = \; \bar{z}_{\zeta,i}\sin\left(N^3\eta_{\zeta,i}\cdot(x,t)\right)\,,$$

and such that the pair $(v_{\zeta,i}, u_{\zeta,i})$ defined by

$$(v_{\zeta,i}, u_{\zeta,i})(x,t) \; := \; A_{\zeta,i}\Big[\varphi_{\zeta,i}(x,t)\, N^{-3}\cos\left(N^3\eta_{\zeta,i}\cdot(x,t)\right)\Big]$$

satisfies (24) with $q \equiv 0$. Note that $(v_{\zeta,i}, u_{\zeta,i})$ is supported in the cylinder $C_{\zeta,i}$ and that

$$\begin{aligned}
&\Big\|(v_{\zeta,i}, u_{\zeta,i}) - \varphi_{\zeta,i}\bar{z}_{\zeta,i}\sin\left(N^3\eta_{\zeta,i}\cdot(x,t)\right)\Big\|_\infty \\
&= \Big\|A_{\zeta,i}\Big[\varphi_{\zeta,i}\, N^{-3}\cos\left(N^3\eta_{\zeta,i}\cdot(x,t)\right)\Big] \\
&\qquad - \; \varphi_{\zeta,i}\, A_{\zeta,i}\Big[N^{-3}\cos\left(N^3\eta_{\zeta,i}\cdot(x,t)\right)\Big]\Big\|_\infty \\
&\leq C\big(A_{\zeta,i}, \eta_{\zeta,i}, \|\varphi_{\zeta,i}\|_{C^3}\big)\frac{1}{N},
\end{aligned} \tag{58}$$

since $A_{\zeta,i}$ is a linear differential operator of homogeneous degree 3. Let

$$(\tilde{v}_N, \tilde{u}_N) \; := \sum_{(\zeta,i): C_{\zeta,i}\subset\Omega_0\times[\varepsilon, T-\varepsilon]} (v_{\zeta,i}, u_{\zeta,i})$$

and

$$(v_N, u_N) = (v,u) + (\tilde{v}_N, \tilde{u}_N)\,.$$

Observe that the sum consists of finitely many terms. Therefore from (57) and (58) we deduce that there exists $N_0 \in \mathbb{N}$ such that

$$v_N \in X_0 \text{ for all } N \geq N_0. \tag{59}$$

Furthermore, recall that for all $(x,t) \in \Omega_\nu \times \tau_\nu$ we have $\phi^h(x,t) = 1$ and hence

$$|\tilde{v}_N(x,t)|^2 = |\bar{v}_{\zeta,i}|^2\sin^2(N^3\eta_{\zeta,i}\cdot(x,t))\,,$$

where $i \in \mathbb{N}$ is determined by the inclusion $(x,t) \in C_{\zeta,i}$. Since $\eta_{\zeta,i} \in \mathbb{R}^{n+1}$ is not parallel to e_{n+1}, from Lemma 4.9 we see that

$$\lim_{N\to\infty}\int_{\tilde{Q}_\zeta}|\tilde{v}_N(x,t)|^2 dx \; = \; \frac{1}{2}\int_{\tilde{Q}_\zeta}|\bar{v}_{\zeta,i}|^2 dx$$

uniformly in t. In particular, using (56) and summing over all (ζ, i) such that $C_{\zeta,i} \subset \Omega_0 \times [\varepsilon, T-\varepsilon]$, we obtain

$$\lim_{N\to\infty} \int_{\Omega_\nu^h} \frac{1}{2}|\tilde{v}_N(x,t)|^2 dx \geq \frac{c}{M} \int_{\Omega_\nu^h} |E_h(x,t)|^2 dx \tag{60}$$

uniformly in $t \in \tau_\nu \cap [\varepsilon, T-\varepsilon]$, where $c > 0$ is a dimensional constant.

Step 3. Conclusion. For each $t \in [\varepsilon, T-\varepsilon]$ we have

$$\begin{aligned}\int_{\Omega_0} \Big[\frac{1}{2}|v_N(x,t)|^2 - \bar{e}(x,t)\Big] dx &= \int_{\Omega_0} \Big[\frac{1}{2}|v(x,t)|^2 - \bar{e}(x,t)\Big] dx \\ &+ \int_{\Omega_0} \frac{1}{2}|\tilde{v}_N(x,t)|^2 dx + \int_{\Omega_0} \tilde{v}_N(x,t) \cdot v(x,t) dx.\end{aligned}$$

Since v is smooth on $\Omega_0 \times [\varepsilon/2, T-\varepsilon/2]$,

$$\int_{\Omega_0} \tilde{v}_N(x,t) \cdot v(x,t) dx \;\to\; 0 \text{ as } N \to \infty, \text{ uniformly in } t,$$

hence

$$\liminf_{N\to\infty} I_{\varepsilon,\Omega_0}(v_N) \geq \liminf_{N\to\infty} \inf_{t\in[\varepsilon,T-\varepsilon]} \left\{ \int_{\Omega_0} \Big[\frac{1}{2}|v|^2 - \bar{e}\Big] dx + \int_{\Omega_0} \frac{1}{2}|\tilde{v}_N|^2 dx \right\}.$$

Since the limit in (60) is uniform in t, it follows that

$$\begin{aligned}\liminf_{N\to\infty} I_{\varepsilon,\Omega_0}(v_N) &\geq \inf_{t\in[\varepsilon,T-\varepsilon]} \left\{ \int_{\Omega_0} \Big[\frac{1}{2}|v|^2 - \bar{e}\Big] dx + \frac{c}{M} \min_{\nu\in\{1,2\}} \int_{\Omega_\nu^h} |E_h|^2 dx \right\} \\ &\geq \inf_{t\in[\varepsilon,T-\varepsilon]} \left\{ \int_{\Omega_0} \Big[\frac{1}{2}|v|^2 - \bar{e}\Big] dx + \frac{c}{M|\Omega_0|} \min_{\nu\in\{1,2\}} \Big(\int_{\Omega_\nu^h} |E_h| dx \Big)^2 \right\},\end{aligned}$$

where we have applied the Cauchy-Schwarz inequality on the last integral. We conclude, using (55), that

$$\begin{aligned}\liminf_{N\to\infty} I_{\varepsilon,\Omega_0}(v_N) &\geq \min\left\{ -\frac{\alpha}{2}, -\alpha + \frac{c}{M|\Omega_0|}\alpha^2 \right\} \\ &\geq -\alpha + \min\left\{ \frac{\alpha}{2}, \frac{c}{M|\Omega_0|}\alpha^2 \right\}.\end{aligned}$$

On the other hand we recall from (59) that $v_N \in X_0$ for $N \geq N_0$ and furthermore clearly $v_N \overset{d}{\to} v$. This concludes the proof.

5. Construction of suitable initial data

In this section we construct examples of initial data for which we have a "subsolution" in the sense of Proposition 3.3. We fix here a bounded open set $\Omega \subset \mathbb{R}^n$.

Proposition 5.1. *There exist triples $(\bar{v}, \bar{u}, \bar{q})$ solving (24) in $\mathbb{R}^n \times \mathbb{R}$ and enjoying the following properties:*

$$\bar{q} = 0,\ (\bar{v}, \bar{u}) \text{ is smooth in } \mathbb{R}^n \times (\mathbb{R} \setminus \{0\}) \text{ and } \bar{v} \in C\big(\mathbb{R}; L^2_w\big)\,, \tag{61}$$

$$\operatorname{supp}(\bar{v}, \bar{u}) \subset \Omega \times]-T, T[\,, \tag{62}$$

$$\operatorname{supp}(\bar{v}(\cdot, t), \bar{u}(\cdot, t)) \subset\subset \Omega \text{ for all } t \neq 0\,, \tag{63}$$

$$e\big(\bar{v}(x,t), \bar{u}(x,t)\big) < 1 \text{ for all } (x,t) \in \mathbb{R}^n \times (\mathbb{R} \setminus \{0\})\,. \tag{64}$$

Moreover

$$\frac{1}{2}|\bar{v}(x,0)|^2 = 1 \text{ a.e. in } \Omega.$$

Proof. In analogy with Definition 4.1 we consider the space X_0, defined as the set of vector fields $v : \mathbb{R}^n \times]-T, T[\to \mathbb{R}^n$ in $C^\infty(\mathbb{R}^n \times]-T, T[)$ to which there exists a smooth matrix field $u : \mathbb{R}^n \times]-T, T[\to \mathcal{S}^n_0$ such that

$$\begin{aligned} \operatorname{div} v &= 0, \\ \partial_t v + \operatorname{div} u &= 0, \end{aligned} \tag{65}$$

$$\operatorname{supp}(v, u) \subset \Omega \times [-T/2, T/2[\,, \tag{66}$$

and

$$e\big(v(x,t), u(x,t)\big) < 1 \quad \text{for all } (x,t) \in \Omega \times]-T, T[\,. \tag{67}$$

This choice of X_0 corresponds - up to changing the time interval under consideration - in Section 4.1 to the choices $(v_0, u_0, q_0) \equiv (0,0,0)$ and $\bar{e} \equiv 1$. Similarly to before, X_0 consists of functions $v :]-T, T[\to L^2(\mathbb{R}^n)$ taking values in a bounded set $B \subset L^2(\mathbb{R}^n)$ (recall that in this section we assume Ω is bounded). On B the weak topology of L^2 is metrizable, and correspondingly we find a metric d on $C(]-T, T[\,, B)$ inducing the topology of $C(]-T, T[\,, L^2_w(\mathbb{R}^n))$.

Next we note that with minor modifications the proof of the perturbation property in Section 4.5 leads to the following claim (cf. Remark 1 following the statement of Proposition 4.5):

Claim: Let $\Omega_0 \subset\subset \Omega$ be given. Let $v \in X_0$ with associated matrix field u and let $\alpha > 0$ such that

$$\int_{\Omega_0} \left[\frac{1}{2}|v(x,0)|^2 - 1\right] dx < -\alpha.$$

Then for any $\varepsilon > 0$ there exists a sequence $v_k \in X_0$ with associated smooth matrix field u_k such that

$$\operatorname{supp}(v_k - v, u_k - u) \subset \Omega_0 \times [-\varepsilon, \varepsilon], \tag{68}$$

$$v_k \xrightarrow{d} v, \tag{69}$$

and

$$\liminf_{k\to\infty} \int_{\Omega_0} \frac{1}{2}|v_k(x,0)|^2\, dx \geq \int_{\Omega_0} \frac{1}{2}|v(x,0)|^2\, dx + \min\left\{\frac{\alpha}{2}, C\alpha^2\right\}, \tag{70}$$

where C is a fixed constant independent of $\varepsilon, \alpha, \Omega_0$ and v.

Fix an exhausting sequence of bounded open subsets $\Omega_k \subset \Omega_{k+1} \subset \Omega$, each compactly contained in Ω, and such that $|\Omega_{k+1} \setminus \Omega_k| \leq 2^{-k}$. Let also ρ_ε be a standard mollifying kernel in $\mathbb{R}^n$. Using the claim above we construct inductively a sequence of velocity fields $v_k \in X_0$, associated matrix fields u_k and a sequence of numbers $\eta_k < 2^{-k}$ as follows.

First of all let $v_1 \equiv 0$ and $u_1 \equiv 0$. Having obtained $(v_1, u_1), \dots, (v_k, u_k)$ and $\eta_1, \dots, \eta_{k-1}$ we choose $\eta_k < 2^{-k}$ in such a way that

$$\|v_k - v_k * \rho_{\eta_k}\|_{L^1} < 2^{-k}. \tag{71}$$

Furthermore, we define

$$\alpha_k = -\int_{\Omega_k} \left[\frac{1}{2}|v_k(x,0)|^2 - 1\right] dx.$$

Note that due to (67) we have $\alpha_k > 0$.

Then we apply the claim with Ω_k, $\alpha = \frac{3}{4}\alpha_k$ and $\varepsilon = 2^{-k}T$ to obtain $v_{k+1} \in X_0$ and associated smooth matrix field u_{k+1} such that

$$\text{supp}\,(v_{k+1} - v_k, u_{k+1} - u_k) \subset \Omega_k \times \left[-2^{-k}T, 2^{-k}T\right], \tag{72}$$

$$d(v_{k+1}, v_k) < 2^{-k}\,, \tag{73}$$

$$\int_{\Omega_k} \frac{1}{2}|v_{k+1}(x,0)|^2 dx \geq \int_{\Omega_k} \frac{1}{2}|v_k(x,0)|^2 dx + \frac{1}{4}\min\{\alpha_k, C\alpha_k^2\}, \tag{74}$$

and recalling that d induces the topology of $C(]-T,T[, L^2_w)$ we can prescribe in addition that

$$\|(v_k - v_{k+1}) * \rho_{\eta_j}\|_{L^2(\Omega)} < 2^{-k} \text{ for all } j \leq k \text{ for } t = 0. \tag{75}$$

From (73) we deduce that there exists $\bar{v} \in C(]-T,T[\,, L^2_w(\Omega))$ such that

$$v_k \xrightarrow{d} \bar{v}.$$

From (72) we see that for any compact subset of $\Omega \times]-T,0[\cup]0,T[$ there exists k_0 such that $(v_k, u_k) = (v_{k_0}, u_{k_0})$ for all $k > k_0$. Hence (v_k, u_k) converges in $C^\infty_{loc}(\Omega \times]-T,0[\cup]0,T[)$ to a smooth pair $(\bar{v}, \bar{u})$ solving the equations (65) in $\mathbb{R}^n \times]0,T[$ and such that (61), (62), (63) and (64) hold. It remains to show that $\frac{1}{2}|\bar{v}(x,0)|^2 = 1$ for almost every $x \in \Omega$.

From (74) we obtain

$$\alpha_{k+1} \leq \alpha_k - \frac{1}{4}\min\{\alpha_k, C\alpha_k^2\} + |\Omega_{k+1} \setminus \Omega_k| \leq \alpha_k - \frac{1}{4}\min\{\alpha_k, C\alpha_k^2\} + 2^{-k},$$

from which we deduce that

$$\alpha_k \to 0 \text{ as } k \to \infty. \tag{76}$$

Note that

$$0 \;\geq\; \int_\Omega \left[\frac{1}{2}|v_k(x,0)|^2 - 1\right] dx \geq -\big(\alpha_k + |\Omega \setminus \Omega_k|\big) \;\geq\; -(\alpha_k + 2^{-k})\,. \tag{77}$$

Therefore, by (76),

$$\lim_{k\uparrow\infty}\int_\Omega \left[\frac{1}{2}|v_k(x,0)|^2-1\right]dx \;=\; 0\,. \tag{78}$$

Finally, observe that, using (75), for $t=0$ for every k

$$\begin{aligned}\|v_k*\rho_{\eta_k}-\bar v*\rho_{\eta_k}\|_{L^2} &\le \sum_{j=0}^{\infty}\|v_{k+j}*\rho_{\eta_k}-v_{k+j+1}*\rho_{\eta_k}\|_{L^2}\\ &\le 2^{-k}+2^{-(k+1)}+\ldots\le 2^{-(k-1)}\end{aligned} \tag{79}$$

and on the other hand

$$\|v_k-\bar v\|_{L^2}\le\|v_k-v_k*\rho_{\eta_k}\|_{L^2}+\|v_k*\rho_{\eta_k}-\bar v*\rho_{\eta_k}\|_{L^2}+\|\bar v*\rho_{\eta_k}-\bar v\|_{L^2}.$$

Thus, (71) and (79) imply that $v_k(\cdot,0)\to\bar v(\cdot,0)$ strongly in $L^2(\mathbb{R}^n)$, which together with (78) implies that

$$\frac{1}{2}|\bar v(x,0)|^2=1 \text{ for almost every } x\in\Omega.$$

□

6. Proofs of Theorem 1.1 and Theorem 1.2

6.1. **Theorem 1.1. Proof of (a)** Let $T=1/2$, Ω be the open unit ball in $\mathbb{R}^n$, and $(\bar v,\bar u)$ be as in Proposition 5.1. Define $\bar e:=1$, $q_0:=0$,

$$v_0(x,t) \;:=\; \begin{cases}\bar v(x,t) & \text{for } t\in[0,1/2]\\ \bar v(x,t-1/2) & \text{for } t\in[1/2,1],\end{cases} \tag{80}$$

$$u_0(x,t) \;:=\; \begin{cases}\bar u(x,t) & \text{for } t\in[0,1/2]\\ \bar u(x,t-1/2) & \text{for } t\in[1/2,1].\end{cases} \tag{81}$$

It is easy to see that the triple (v_0,u_0,q_0) satisfies the assumptions of Proposition 3.3 with $\bar e\equiv 1$. Therefore, there exists infinitely many solutions $v\in C([0,1],L^2_w)$ of (1) in $\mathbb{R}^n\times[0,1]$ with

$$v(x,0)=\bar v(x,0)=v(x,1) \text{ for a.e. } x\in\Omega,$$

and such that

$$\frac{1}{2}|v(\cdot,t)|^2 \;=\; \mathbf{1}_\Omega \qquad \text{for every } t\in]0,1[\,. \tag{82}$$

Since $\frac{1}{2}|v_0(\cdot,0)|^2=\mathbf{1}_\Omega$ as well, it turns out that the map $t\mapsto v(\cdot,t)$ is continuous in the strong topology of L^2.

Each such v can be extended to $\mathbb{R}^n\times[0,\infty[$ 1-periodically in time, by setting $v(x,t)=v(x,t-k)$ for $t\in[k,k+1]$. Thus the energy

$$E(t) \;=\; \frac{1}{2}\int|v(x,t)|^2dx$$

is equal to $|\Omega|$ at *every* time t, i.e. v satisfies the strong energy equality in the sense specified in Section 2.

Next, notice that $\frac{1}{2}|v|^2 = \mathbf{1}_{\Omega\times[0,\infty[}$ and that $p = -|v|^2/n = -\frac{2}{n}\,\mathbf{1}_{\Omega\times[0,\infty[}$. Therefore

$$\partial_t \frac{|v|^2}{2} + \mathrm{div}\left[\left(\frac{|v|^2}{2} + p\right) v\right] \;=\; \frac{1}{2}\,\partial_t\,\mathbf{1}_{\Omega\times[0,\infty[} + \frac{n-2}{n}\,\mathrm{div}\, v \;=\; 0$$

in the sense of distributions. This gives infinitely many solutions satisfying both the strong energy equality and the local energy equality and all taking the same initial data.

Proof of (b) As in the proof of (a), let $T = 1/2$, Ω be the open unit ball in $\mathbb{R}^n$, and $(\bar v, \bar u)$ be as in Proposition 5.1. Again, as in the proof of (a) we set $q_0 = 0$. However we choose v_0, u_0 and $\bar e$ differently:

$$v_0(x,t) \;:=\; \begin{cases} \bar v(x,t) & \text{for } t\in[0,1/2] \\ 0 & \text{for } t\in[1/2,1], \end{cases} \tag{83}$$

and

$$u_0(x,t) \;:=\; \begin{cases} \bar u(x,t) & \text{for } t\in[0,1/2] \\ 0 & \text{for } t\in[1/2,1]. \end{cases} \tag{84}$$

Next consider the function

$$\tilde e(t) \;=\; \max_{x\in\Omega} e(v_0(x,t), u_0(x,t)) \qquad \text{for } t \in]0,1].$$

Clearly $\tilde e$ is continuous, takes values in $[0,1[$ and vanishes in a neighborhood of $t=1$. Moreover, it converges to 1 as $t\downarrow 0$: hence, we set $\tilde e(0)=1$. Define $\hat e : [0,1]\to\mathbb{R}$ as $\hat e(t) := (1-t) + t\max_{\tau\in[t,1]}\tilde e(\tau)$. Then $\hat e$ is a continuous decreasing function, with $\hat e(0)=1$, $\hat e(1)=0$ and $1 > \hat e(t) > \tilde e(t)$ for every $t\in]0,1[$.

Now, apply Proposition 3.3 to get a solution $v\in C([0,1],L^2_w)$ of (1) in Ω with $v(\cdot,0)=v_0(\cdot,0)$, $v(\cdot,1)=0$ and such that

$$\frac{1}{2}|v(\cdot,t)| \;=\; \hat e(t)\,\mathbf{1}_\Omega \qquad \text{for every } t\in]0,1[\,. \tag{85}$$

Arguing as in the proof of (a), we conclude that $t\mapsto v(\cdot,t)$ is a strongly continuous map. Since $v(\cdot,1)=0$, we can extend v trivially on $[0,\infty[\times\mathbb{R}^n$ in order to get a global weak solution. Clearly, this solution satisfies the strong energy inequality. However, it does not satisfy the energy equality. Note, in passing, that v satisfies the local energy inequality.

Proof of (c) As in the proof of (a) and (b), let $T=1$, Ω be the open unit ball in $\mathbb{R}^n$, and $(\bar v,\bar u)$ be as in Proposition 5.1. Again, as in the proof of (a) and (b) we set $q_0=0$. This time we choose v_0, u_0 as in (b) and $\bar e$ as in (a).

Let $v_1\in C([0,1],L^2_w)$ be the solution of (1) obtained in Proposition 3.3. Since $\frac{1}{2}|v_0(\cdot,0)| = \mathbf{1}_\Omega$, it turns out that the map $t\mapsto v_1(\cdot,t)$ is continuous in the strong topology of L^2 at every $t\in[0,1[$. However, this map is *not* strongly continous at $t=1$, because $v_1(1,\cdot)=0$.

Next, let $v_2\in C([0,1],L^2_w)$ be a solution of (1) obtained in Proposition 3.3 with $\bar e\equiv 1$ and $(v_0,u_0,q_0)\equiv(0,0,0)$. Since $v_1,v_2\in C([0,1],L^2_w)$ with

$v_1(\cdot,1) = v_2(\cdot,0) = v_2(\cdot,1) = 0$, the velocity field $v : \mathbb{R}^n \times [0,\infty[\to \mathbb{R}^n$ defined by

$$v(x,t) = \begin{cases} v_1(x,t) & \text{for } t \in [0,1] \\ v_2(x,t-k) & \text{for } t \in [k,k+1], k = 1,2,\ldots \end{cases} \tag{86}$$

belongs to the space $C([0,\infty[\,, L^2_w)$ and therefore v solves (1). Moreover

$$\frac{1}{2}\int |v(x,t)|^2\,dx = |\Omega| \qquad \text{for every } t \notin \mathbb{N}$$

and

$$\frac{1}{2}\int |v(x,t)|^2\,dx = 0 \qquad \text{for every } t \in \mathbb{N},\, t \geq 1.$$

Hence v satisfies the weak energy inequality but not the strong energy inequality.

6.2. **Theorem 1.2.** We recall that $p(\rho)$ is a function with $p'(\rho) > 0$. Let $\alpha := p(1)$, $\beta := p(2)$ and $\gamma = \beta - \alpha$. We let Ω be the unit ball. Arguing as in the proof of Theorem 1.1(a) we can find an initial data $v^0 \in L^\infty(\mathbb{R}^n)$ with $|v^0|^2 = n\gamma\,\mathbf{1}_\Omega$ and for which there exists infinitely many solutions (v,p) of (1) with the following properties:

- $v \in C([0,\infty[, L^2)$ and $|v|^2 = n\gamma\,\mathbf{1}_{\Omega\times[0,\infty[}$;
- $p = -|v|^2/n = -\gamma\,\mathbf{1}_{\Omega\times[0,\infty[}$.

Therefore, we conclude that

$$\partial_t v + \operatorname{div} v \otimes v + \nabla\big(\alpha\,\mathbf{1}_{\Omega\times[0,\infty[} + \beta\,\mathbf{1}_{\mathbb{R}^n\setminus\Omega\times[0,\infty[}\big) = 0\,.$$

Hence, if we set

$$\rho = \mathbf{1}_{\Omega\times[0,\infty[} + 2\,\mathbf{1}_{\mathbb{R}^n\setminus\Omega\times[0,\infty[}$$

for any such v, the pair (ρ, v) is a weak solution of (4) with initial data (ρ^0, v^0), where $\rho_0 = \mathbf{1}_\Omega + 2\,\mathbf{1}_{\mathbb{R}^n\setminus\Omega}$.

Each such solution is admissible. Indeed

$$\begin{aligned} & \partial_t\left[\rho\varepsilon(\rho) + \frac{\rho|v|^2}{2}\right] + \operatorname{div}_x\left[\left(\rho\varepsilon(\rho) + \frac{\rho|v|^2}{2} + p(\rho)\right)v\right] \\ = \;& \partial_t\left[\left(\varepsilon(1) + \frac{n\gamma}{2}\right)\mathbf{1}_{\Omega\times[0,\infty[} + 2\varepsilon(2)\,\mathbf{1}_{\mathbb{R}^n\setminus\Omega\times[0,\infty[}\right] \\ & + \left(\varepsilon(1) + p(1) + \frac{n\gamma}{2}\right)\operatorname{div} v = 0\,. \end{aligned} \tag{87}$$

This gives (22). In order to prove the stronger requirement (23) of Definition 2.9, it suffices to notice that $(\rho(\cdot,t), v(\cdot,t)) \to (\rho^0, v^0)$ strongly in L^2_{loc}.

7. Appendix A: Weak continuity in time for evolution equations

In this section we prove a general lemma on the weak continuity in time for certain evolution equations. Lemma 2.3 is a corollary of this Lemma and standard estimates for the Euler and Navier–Stokes equations.

Lemma 7.1. *Let $v \in L^\infty(]0,T[, L^2(\mathbb{R}^n,\mathbb{R}^n))$, $u \in L^1_{loc}(\mathbb{R}^n\times]0,T[,\mathbb{R}^{n\times n})$ and $q \in L^1_{loc}(]0,T[\times\mathbb{R}^n)$ be distributional solutions of*

$$\partial_t v + \mathrm{div}_x u + \nabla q \;=\; 0\,. \tag{88}$$

Then, after redefining v on a set of t's of measure zero, $v \in C(]0,T[, L^2_w)$.

Proof. Consider a countable set $\{\varphi_i\} \subset C^\infty_c(\mathbb{R}^n,\mathbb{R}^n)$ dense in the strong topology of L^2. Fix φ_i and any test function $\chi \in C^\infty_c(]0,T[)$. Testing (88) with $\chi(t)\varphi_i(x)$ we obtain the following identity:

$$\int_0^T \Phi_i \partial_t \chi \;=\; -\int_0^T \chi \int_{\mathbb{R}^n} \left[\langle u, \nabla\varphi_i\rangle + q\,\mathrm{div}\varphi_i\right], \tag{89}$$

where $\Phi_i(t) := \int \varphi_i(x)\cdot v(x,t)dx$. We conclude therefore that $\Phi_i' \in L^1$ in the sense of distributions. Hence we can redefine each Φ_i on a set of times $\tau_i \subset]0,T[$ of measure zero in such a way that Φ_i is continuous. We keep the same notation for these functions, and let $\tau = \cup_i \tau_i$. Then $\tau \subset]0,T[$ is of measure zero and for every $t \in]0,T[\setminus\tau$ we have

$$\Phi_i(t) \;=\; \int \varphi_i(x)\cdot v(x,t)\,dx \qquad \text{for every } i. \tag{90}$$

Moreover, with $c := \|v\|_{L^\infty_t(L^2_x)}$ we have that $|\Phi_i(t)| \le c\|\varphi_i\|_{L^2}$ for all $t \in]0,T[$. Therefore, for each $t \in]0,T[$ there exists a unique bounded linear functional L_t on $L^2(\mathbb{R}^n,\mathbb{R}^n)$ such that $L_t(\varphi_i) = \Phi_i(t)$. By the Riesz representation theorem there exists $\bar v(\cdot,t) \in L^2(\mathbb{R}^n)$ such that

- $\bar v(\cdot,t) = v(\cdot,t)$ for every $t \in]0,T[\setminus\tau$;
- $\|\bar v(\cdot,t)\|_{L^2} \le c$ for every t;
- $\int \bar v(x,t)\cdot\varphi_i(x)dx = \Phi_i(t)$ for every t.

To conclude we show that $\bar v \in C(]0,T[,\, L^2_w)$, i.e. that for any $\varphi \in L^2(\mathbb{R}^n,\mathbb{R}^n)$ the function $\Phi(t) := \int v(x,t)\cdot\varphi(x)dx$ is continuous on $]0,T[$. Since the set $\{\varphi_i\}$ is dense in $L^2(\mathbb{R}^n,\mathbb{R}^n)$, we can find a sequence sequence $\{j_k\}$ such that $\varphi_{j_k} \to \varphi$ strongly in L^2. Then

$$|\Phi(t) - \Phi_{j_k}(t)| \;\le\; c\|\varphi_{j_k} - \varphi\|_{L^2}\,. \tag{91}$$

Therefore Φ_{j_k} converges uniformly to Φ, from which we derive the continuity of Φ. This shows that $\bar v \in C(]0,T[, L^2_w)$ and concludes the proof. $\square$

8. Appendix B: Dissipative solutions

We follow here the book [12] and define dissipative solutions of (1). First of all, for any divergence–free vector field $v \in L^2_{loc}(\mathbb{R}^n \times [0,T])$ we consider the following two distributions:

- The symmetric part of the gradient $d(v) := \frac{1}{2}(\nabla v + \nabla v^t)$;
- $E(v)$ given by

$$E(v) \;:=\; -\partial_t v - P(\mathrm{div}\,(v\otimes v))\,. \tag{92}$$

Here P denotes the Helmholtz projection on divergence–free fields, so that if $p(x,t)$ is the potential–theoretic solution of $-\Delta p = \sum_{i,j} \partial^2_{ij}(v^i v^j)$, then

$$P(\mathrm{div}\,(v \otimes v)) \;=\; \mathrm{div}\,(v \otimes v) + \nabla p\,.$$

Finally, when $d(v)$ is locally summable, we denote by $d^-(v)$ the negative part of its smallest eigenvalue, that is $(-\lambda_{min}(d(v)))^+$.

P. L. Lions introduced the following definition in [12]:

Definition 8.1. *Let $v \in L^\infty([0,T], L^2(\mathbb{R}^n,\mathbb{R}^n)) \cap C([0,T], L^2_w)$. Then v is a dissipative solution of* (1) *if the following two conditions hold*

- $v(x,0) = v_0(x)$ *for* $x \in \mathbb{R}^n$*;*
- $\mathrm{div}\, v = 0$ *in the sense of distributions;*
- *whenever* $w \in C([0,T], L^2(\mathbb{R}^n,\mathbb{R}^n))$ *is such that* $d(w) \in L^1_t(L^\infty_x)$, $E(w) \in L^1_t(L^2_x)$ *and div* $w = 0$*, then*

$$\begin{aligned}
\|v(\cdot,t) - w(\cdot,t)\|^2_{L^2_x} &\leq e^{\int_0^t 2\|d^-(w)\|_{L^\infty_x}\,d\tau}\|v_0(\cdot) - w(\cdot,0)\|^2_{L^2_x}\\
&+ 2\int_0^t\int_{\mathbb{R}^n} e^{\int_s^t 2\|d^-(w)\|_{L^\infty_x}\,d\tau} E(w)(x,s)\cdot(v(x,s)-w(x,s))\,dx\,ds
\end{aligned}\tag{93}$$

for every $t \in [0,T]$*.*

We next come to the proof of Proposition 2.8 which we state again for the reader's convenience.

Proposition 8.2. *Let $v \in C([0,T], L^2_w)$ be a weak solution of* (1) *satisfying the weak energy inequality. Then v is a dissipative solution.*

Proof. As already remarked at page 156 of [12] it suffices to check Definition 8.1 for smooth w. This is achieved by suitably regularizing the test function w of (93) and observing that if $w \in C([0,T], L^2(\mathbb{R}^n,\mathbb{R}^n))$ is such that $d(w) \in L^1_t(L^\infty_x)$, then any approximation w_k such that

(a) $w_k \to w$ in $C([0,T], L^2)$;
(b) $d(w_k) \to d(w)$ a.e. in $\mathbb{R}^n \times [0,T]$;
(c) $\limsup_{k\to\infty} \|d(w_k)\|_{L^\infty_x} \leq \|d(w)\|_{L^\infty_x}$

also satisfies

$$E(w_k) \to E(w) \text{ in } L^1_t L^2_x$$

and hence one can pass to the limit in (93). Indeed, this follows from the observation that $P(E(w)) = 2P(d(w)\cdot w)$ (see the computations on page 155 of [12]).

Step 1. Next we show that it suffices to check Definition 8.1 when w is compactly supported in space. Indeed, fix w as above. We claim that we can approximate w with compactly supported divergence–free vector fields w_k such that (a),(b) and (c) above hold. The reader may consult Appendix A of [12] and jump directly to Step 2. Otherwise, the following is a short self-contained proof.

Fix a smooth cut–off function χ equal to 1 on the ball $B_1(0)$, supported in the ball $B_2(0)$, and taking values between 0 and 1, and set $\chi_r(x) = \chi(r^{-1}x)$.

Let ξ be the potential–theoretic solution of $\Delta\xi = \mathrm{curl}\ w$, so that $w = \mathrm{curl}\ \xi$. Recall that in dimension $n = 2$ the curl operator can be defined as curl $= (-\partial_2, \partial_1)$, in dimension $n = 3$ it is given by curl $w = \nabla \times w$ and ξ is obtained via the Biot-Savart law. Let $\langle\xi\rangle_k = \frac{1}{|B_{2k}\setminus B_k|}\int_{B_{2k}\setminus B_k} \xi\, dx$ and let

$$w_k = \mathrm{curl}\big(\chi_k(\xi - \langle\xi\rangle_k)\big).$$

Clearly w_k is compactly supported and divergence–free. Since ξ is smooth, and $\|\partial_i(\chi_k)\|_\infty \le Ck^{-1}$ and $\|\partial^2_{ij}(\chi_k)\|_\infty \le Ck^{-2}$, we see that

$$d(w_k)(\cdot, t) \to d(w)(\cdot, t) \text{ locally uniformly}$$

for every t. Thus (b),(c) follow easily. Moreover $\|\nabla\xi(\cdot, t)\|_{L^2_x} \le \|w(\cdot, t)\|_{L^2_x}$ and hence, using the Poincaré inequality, for every $t \in [0, T]$ we have

$$\begin{aligned}
\|w_k - w\|^2_{L^2_x} &\le C\int_{\mathbb{R}^n\setminus B_k(0)} |w|^2 dx + C\|\nabla\chi_{k^{-1}}\|^2_{C^0}\int_{B_{2k}(0)\setminus B_k(0)} |\xi - \langle\xi\rangle_k|^2 dx\\
&\le C\int_{\mathbb{R}^n\setminus B_k(0)} |w|^2 dx + \frac{C}{k^2}\int_{B_{2k}\setminus B_k(0)} |\nabla\xi|^2 dx\\
&\le C\int_{\mathbb{R}^n\setminus B_k(0)} |w|^2 dx + \frac{C}{k^2}\int_{\mathbb{R}^n} |w|^2 dx.
\end{aligned}$$

Since $w \in C([0, T], L^2(\mathbb{R}^n, \mathbb{R}^n))$, we deduce (a).

Step 2. We are now left with task of showing (93) when w is a smooth test function compactly supported in space. Consider the function

$$F(t) := \int_{\mathbb{R}^n} |w(x, t) - v(x, t)|^2 dx.$$

Since w is smooth and $v \in C([0, T], L^2_w)$, F is lower–semicontinuous. Moreover, due to the weak energy inequality $v(t, \cdot) \to v(0, \cdot)$ strongly in L^2_{loc} as $t \downarrow 0$. So F is continuous at 0. We claim that, in the sense of distributions,

$$\frac{dF}{dt} \le 2\int_{\mathbb{R}^n} \big[E(w)\cdot(v - w) - d(w)(v - w)\cdot(v - w)\big]\, dx\,. \tag{94}$$

From this inequality we infer

$$\frac{dF}{dt} \ \le\ 2\|d^-(w)(t, \cdot)\|_{L^\infty_x} F(t) + 2\int_{\mathbb{R}^n} \big[E(w)\cdot(v - w)\big]\, dx\,. \tag{95}$$

From the continuity of F at $t = 0$ and Gronwall's Lemma, we conclude (93) for a.e. t. By the lower semicontinuity of F, (93) actually holds for every t. Therefore it remains to prove (94). We expand F as

$$\begin{aligned}
F(t) &= \int_{\mathbb{R}^n} |v(x, t)|^2\, dx + \int_{\mathbb{R}^n} |w(x, t)|^2\, dx - 2\int_{\mathbb{R}^n} \big[v(x, t)\cdot w(x, t)\big]\, dx\\
&=: F_1(t) + F_2(t) + F_3(t)\,.
\end{aligned}$$

The weak energy inequality implies $\frac{d}{dt}F_1(t) \le 0$ and a standard calculation gives

$$\frac{dF_2}{dt}(t) \;=\; -2\int_{\mathbb{R}^n} \big[E(w)\cdot w\big]\,dx\,.$$

It remains to show that

$$\frac{dF_3}{dt} = 2\int_{\mathbb{R}^n} \big[E(w)\cdot v - d(w)(v-w)\cdot(v-w)\big]\,dx \tag{96}$$

We fix a smooth function $\psi \in C_c^\infty(]0,T[)$ and test (1) (or more precisely (3)) with $w(x,t)\psi(t)$. It then follows that

$$2\int_{\mathbb{R}}\int_{\mathbb{R}^n} v\cdot w\psi'\,dx\,dt = -2\int_{\mathbb{R}}\psi\int_{\mathbb{R}^n}\big[v\cdot\partial_t w + \langle v\otimes v, \nabla w\rangle\big]\,dx\,dt. \tag{97}$$

Inserting $\partial_t w = -E(w) - P(\mathrm{div}(w\otimes w))$ and taking into account that $\mathrm{div}\, v = 0$, we obtain

$$\begin{aligned}\int_{\mathbb{R}} F_3(t)\psi'(t)\,dt \;&=\; 2\int_{\mathbb{R}}\psi\int_{\mathbb{R}^n}\big[\langle v\otimes v,\nabla w\rangle - \mathrm{div}(w\otimes w)\cdot v\big]\,dx\,dt\\ &\quad -2\int_{\mathbb{R}}\psi\int_{\mathbb{R}^n} E(w)\cdot v\,dx\,dt\end{aligned} \tag{98}$$

Next, observe that $\mathrm{div}(w\otimes w)\cdot v = \sum_{j,i} v_j w_i \partial_i w_j$ and that $\langle v\otimes v, \nabla w\rangle = \sum_{j,i} v_j v_i \partial_i w_j$. Therefore we have

$$\langle v\otimes v,\nabla w\rangle - \mathrm{div}(w\otimes w)\cdot v \;=\; \nabla w\,(v-w)\cdot v\,. \tag{99}$$

On the other hand,

$$\nabla w\,(v-w)\cdot w \;=\; \sum_{i,j}(v_i - w_i)\partial_i w_j w_j \;=\; (v-w)\cdot\nabla\frac{1}{2}|w|^2\,.$$

Since $v-w$ is divergence–free in the sense of distributions and $|w|^2/2$ is a smooth function compactly supported in space, integrating by parts we get

$$\int_{\mathbb{R}}\psi\int_{\mathbb{R}^n}\big[\nabla w\,(v-w)\cdot w\big]\,dx\,dt \;=\; 0\,. \tag{100}$$

From (98), (99) and (100) we obtain

$$\begin{aligned}\int_{\mathbb{R}} F_3(t)\psi'(t)\,dt \;&=\; 2\int_{\mathbb{R}}\psi\int_{\mathbb{R}^n}\big[\nabla w\,(v-w)\cdot(v-w)\big]\,dx\,dt\\ &\quad -2\int_{\mathbb{R}}\psi\int_{\mathbb{R}^n} E(w)\cdot v\,dx\,dt\,.\end{aligned} \tag{101}$$

Finally, observe that

$$\nabla w(v-w)\cdot(v-w) = \big\langle \nabla w, (v-w)\otimes(v-w)\big\rangle = \big\langle d(w), (v-w)\otimes(v-w)\big\rangle,$$

since $(v-w)\otimes(v-w)$ is a symmetric matrix. Plugging this into (101), by the arbitrariness of the test function ψ, we obtain (96). $\square$

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Institut für Mathematik, Universität Zürich, CH-8057 Zürich
E-mail address: camillo.delellis@math.unizh.ch

Hausdorff Center for Mathematics, Universität Bonn, D-53115 Bonn
E-mail address: laszlo.szekelyhidi@hcm.uni-bonn.de