MULTIDIMENSIONAL DELTA-SHOCKS AND THE TRANSPORTATION AND CONCENTRATION PROCESSES

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Abstract. We introduce the definitions of a *δ-shock wave type solution* for the multidimensional system of conservation laws

$$\rho_t + \nabla\cdot(\rho F(U)) = 0, \qquad (\rho U)_t + \nabla\cdot(\rho N(U)) = 0, \quad x \in \mathbb{R}^n,$$

where $F = (F_j)$ is a given vector field, $N = (N_{jk})$ is a given tensor field, $F_j, N_{kj} : \mathbb{R}^n \to \mathbb{R}$, $j,k = 1,\dots,n$. The well-known particular cases of this system are zero-pressure gas dynamics in a standard form

$$\rho_t + \nabla\cdot(\rho U) = 0, \quad (\rho U)_t + \nabla\cdot(\rho U \otimes U) = 0,$$

and in the relativistic form

$$\rho_t + \nabla\cdot(\rho C(U)) = 0, \quad (\rho U)_t + \nabla\cdot(\rho U \otimes C(U)) = 0,$$

where $C(U) = \frac{c_0 U}{\sqrt{c_0^2+|U|^2}}$, c_0 is the speed of light. Using this definition, the Rankine–Hugoniot conditions for δ-shocks are derived. We also derive the δ-shock balance laws describing mass and momentum transportation between the volume outside the wave front and the wave front. In the case of zero-pressure gas dynamics the transportation process is the concentration process.

1. Introduction

1.1. L^∞-type solutions. As is well known, even in the case of smooth (and, certainly, in the case of discontinuous) initial data $U^0(x)$, we cannot in general find a smooth solution of a one dimensional system of conservation laws:

$$\begin{cases} U_t + \big(F(U)\big)_x &= 0, \qquad \text{in}\, \mathbb{R}\times(0,\ \infty), \\ U &= U^0, \qquad \text{in}\, \mathbb{R}\times\{t=0\}, \end{cases} \tag{1.1}$$

where $F : \mathbb{R}^m \to \mathbb{R}^m$ is called the *flux-function* associated with (1.1); $U^0 : \mathbb{R} \to \mathbb{R}^m$ are given smooth vector-functions; $U = U(x,t) = (u_1(x,t),\dots,u_m(x,t))$ is the unknown function, $x \in \mathbb{R}$, $t \ge 0$.

Quoting from Evans' book, "the great difficulty in this subject is discovering a proper notion of weak solution for the initial problem (1.1)" [17, 11.1.1.]. "We must devise some way to interpret a less regular function U as somehow "solving" this initial-value problem. But as it stands, the PDE does not even make sense unless U is differentiable. However, observe that if we *temporarily* assume U is smooth, we can as follows rewrite, so that the resulting expression does not directly involve the derivatives of U. The idea is to multiply the PDE in (1.1) by a smooth function φ and then to integrate by parts, thereby transferring the derivatives onto φ" [17,

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3.4.1.a.]. According to the above reasoning, it is said that $U \in L^\infty\big(\mathbb{R} \times (0,\infty); \mathbb{R}^m\big)$ is a *generalized solution* of the Cauchy problem (1.1) if the integral identities

$$\int_0^\infty \int \Big(U \cdot \widetilde{\varphi}_t + F(U) \cdot \widetilde{\varphi}_x\Big)\,dx\,dt + \int U^0(x) \cdot \widetilde{\varphi}(x,0)\,dx = 0 \qquad (1.2)$$

hold for all compactly supported test vector-functions $\widetilde{\varphi} : \mathbb{R} \times [0,\infty) \to \mathbb{R}^m$, where $\cdot$ is the scalar product of vectors, and $\int f(x)\,dx$ denotes the improper integral $\int_{-\infty}^{\infty} f(x)\,dx$.

Using Definition (1.2), one can derive the classical Rankine–Hugoniot conditions for shocks (see, e.g., [17, 11.1.1.]).

1.2. $\delta^{(n)}$**-Shock wave type solutions,** $n = 0, 1, \dots$. It is well known that there are "nonclassical" situations where, in contrast to Lax's and Glimm's classical results, the Cauchy problem for a system of conservation laws does not possess a weak L^∞-solution except for some particular initial data. In order to solve the Cauchy problem in this "nonclassical" situation, it is necessary to introduce new singular solutions called *δ-shocks* (see [1], [6]– [8], [11]– [15], [20], [22]– [27], [34]– [36], [39]– [44] and the references therein). Roughly speaking, a *δ-shock* (*singular shock*) is a solution such that its components *contain Dirac measures*. The theory of δ-shocks singular shocks has been intensively developed in the last ten years.

Recently, in [31] (see also [32], [38]), a concept of $\delta^{(n)}$-shock wave type solutions was introduced, $n = 1, 2, \dots$. It is a *new type of singular solution* of a system of conservation laws such that its components contain delta functions and their derivatives up to n-th order. In [31] the theory of *δ'-shocks* was established. The results [31], [32], [38] show that systems of conservation laws can develop not only Dirac measures (as in the case of δ-shocks) but their derivatives as well.

The above singular solutions are connected with *transportation processes and concentration processes* [1], [7], [8], [31], [40].

δ- and $\delta^{(n)}$-shocks, $n = 1, \dots$, do not satisfy standard (L^∞) integral identities. Consequently, to deal with these singular solutions, we need

• to discover a *proper notion of a singular solution* and to define *in which sense* it may satisfy a nonlinear system;

• to devise some way to define a *singular superposition* (*product*) of distributions (for example, a product of the Heaviside function and the delta function).

Unfortunately, using the above cited instructions from the Evans' book [17, 3.4.1.a.], $\delta^{(n)}$-shock wave type solutions *cannot be defined.* Indeed, if by integrating by parts we transfer the derivatives onto a test function φ, under the integral sign there still remain nonlinear terms *undefined in the distributional sense*, since the components of a solution may contain Dirac measures and their derivatives.

Thus we need to develop a special technique.

In numerous cited above papers the δ-shocks in the *system of zero-pressure gas dynamics* were studied.

In [15], for one-dimensional case of zero-pressure gas dynamics

$$\rho_t + (\rho u)_x = 0, \qquad (\rho u)_t + (\rho u^2)_x = 0 \qquad (1.3)$$

the global δ-shock wave type solution in the sense of Radon measures was obtained. In [20], for this system the uniqueness of the weak solution is proved for the case when the initial value is a Radon measure.

The multidimensional zero-pressure gas dynamics has the form

$$\rho_t + \nabla \cdot (\rho U) = 0, \qquad (\rho U)_t + \nabla \cdot (\rho U \otimes U) = 0, \qquad (1.4)$$

where $\rho = \rho(x,t) \ge 0$ is the density, $U = (u_1(x,t), \dots, u_n(x,t)) \in \mathbb{R}^n$ is the velocity, $\nabla = \big(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\big)$, $\cdot$ is the scalar product of vectors, $\otimes$ is the usual tensor product of vectors.

In [24], [26], [27], [41], [44] the *planar* δ-shock wave type solution in (1.4) is defined as a *measure-valued solution*. The *measure-valued solution* is defined in the following way. Let $BM(\mathbb{R}^n)$ be the space of bounded Borel measures on $\mathbb{R}^n$. A pair (ρ, U), where $\rho(x,t) \in C\big(BM(\mathbb{R}^n), [0,\infty)\big)$, $U(x,t) \in \big(L^\infty\big(L^\infty(\mathbb{R}^n), [0,\infty)\big)\big)^n$, and U is measurable with respect to ρ at almost all $t \ge 0$, is said to be a *measure-valued solution* of (1.4) in the sense of the measures if

$$\begin{aligned} \int_0^\infty \int_{\mathbb{R}^n} \Big(\varphi_t + U \cdot \nabla \varphi_x\Big)\, d\rho\, dt &= 0, \\ \int_0^\infty \int_{\mathbb{R}^n} U\Big(\varphi_t + U \cdot \nabla \varphi_x\Big)\, d\rho\, dt &= 0, \end{aligned} \tag{1.5}$$

hold for all $\varphi(x,t) \in \mathcal{D}(\mathbb{R}^n \times [0, \infty))$.

In this approach a smooth discontinuity surface Σ is parametrized as $X = X(s)$, $t = t(s)$ $(s \in \mathbb{R}^n)$, separating (X,t)-space into two infinite parts Ω_1 and Ω_2, $N = (N_X, N_t)$ is the space-time normal to the surface Σ. The delta-shock solution takes the form

$$\big(\rho, U\big)(X,t)) = \begin{cases} \big(\rho_1, U_1\big)(X,t), & (X,t) \in \Omega_1, \\ \big(w(s,t)\delta(X - X(s,t),t), U_\delta(s,t)\big), & (X,t) \in \Sigma, \\ \big(\rho_2, U_2\big)(X,t), & (X,t) \in \Omega_2. \end{cases} \tag{1.6}$$

Here U_δ is the velocity at the points of discontinuity, (ρ_1, U_1) and (ρ_2, U_2) are smooth solutions of (1.4) in regions Ω_1 and Ω_2 respectively.

In [34], [35], for the 2-D case of system (1.4) the notion of generalized solutions in terms of Radon measures is introduced, and the problem of the propagation of δ-shock waves is considered. The existence of a global weak solution for the multidimensional system of "zero-pressure gas dynamics" is obtained in [36]. The approach of the latter paper is based on the introducing of Lagrangian coordinates and on the Dafermos entropy condition. In [9], for multidimensional continuity equation (the first equation in system (1.4)) the possibility of existence of δ-shock was considered.

To study zero-pressure gas dynamics and its generalization are important for applications. The zero-pressure gas dynamics can be considered as a model of the "sticky particle dynamics". These models are used in many different areas of physics such that in cosmology [37], [45] (to describe the formation of large-scale structures of the universe), in a mathematical modelling of pressureless mediums, in models of dusty gases (see the excellent papers [23], [30]), in mathematical description of granular gases [16] [18]. In these mediums the concentration processes are going on.

1.3. **Contents of the paper.** In this paper we study the problems related with the δ-shock in multidimensional system of conservation laws

$$\rho_t + \nabla \cdot (\rho F(U)) = 0, \qquad (\rho U)_t + \nabla \cdot (\rho N(U)) = 0, \tag{1.7}$$

where $F = (F_1, \dots, F_n)$ is a given vector field, $N = (N_1, \dots, N_n)$ is a given tensor field, $N_k = (N_{k1}, \dots, N_{kn})$, $k = 1, \dots, n$; $F_j, N_{kj} : \mathbb{R}^n \to \mathbb{R}$; $\rho = \rho(x,t)$, $U = (u_1(x,t), \dots, u_n(x,t)) \in \mathbb{R}^n$ are the unknown function; $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, $t \ge 0$. System (1.7) can be rewritten as

$$\rho_t + \sum_{j=1}^n \frac{\partial}{\partial x_j}(\rho F_j(U)) = 0, \quad (\rho u_k)_t + \sum_{j=1}^n \frac{\partial}{\partial x_j}(\rho N_{kj}(U)) = 0, \quad k = 1, \dots, n.$$

The well-known particular cases of this system are zero-pressure gas dynamics in a standard form (1.4) (here $F(U) = U$, $N(U) = U \otimes U$) and in a relativistic form

$$\rho_t + \nabla \cdot (\rho C(U)) = 0, \quad (\rho U)_t + \nabla \cdot (\rho U \otimes C(U)) = 0, \tag{1.8}$$

(here $F(U) = C(U)$, $N(U) = U \otimes C(U)$), where $C(U) = \frac{c_0 U}{\sqrt{c_0^2 + |U|^2}}$, c_0 is the speed of light. The relativistic form (1.8) of zero-pressure gas dynamics was presented in [33].

In Sec. 2, we introduce the integral identities (2.2) which constitute Definition 2.1 of δ-shocks for system (1.7). Next, using this definition the Rankine–Hugoniot conditions (2.7) for curvilinear δ-shocks are derived. The Rankine–Hugoniot conditions (2.13) for zero-pressure gas dynamics (1.4) and the Rankine–Hugoniot conditions (2.14) for its relativistic form (1.8) are particular cases of (2.7).

In this section we also consider a *spherically symmetric case of zero-pressure gas dynamics* (2.16) and derive the Rankine–Hugoniot conditions (2.18) for a δ-shock type solution of system (2.16). Recall that a *spherically symmetric case of the gas dynamics* admits a solution which describes the *heavy shock*. This solution related with investigation of atomic bomb explosion was found by L. I. Sedov, J. von Neumann, and G. I. Taylor [42, 6.16.]. It is clear that a δ-shock type solution of system (2.16) can describe the explosion of a super bomb.

In Sec. 3, geometric and physical aspects of δ-shocks are studied. It is well-known that if $U \in L^\infty$ is a generalized solution of the Cauchy problem compactly supported with respect to x, then the integral $\int_{\mathbb{R}^n} U(x,t)\,dx$ is independent of time. For δ-shock wave type solutions this fact *does not hold*. Nevertheless, by Theorems 3.1 "generalized" analogs of these conservation laws are derived. We prove that the "mass" and "momentum" *transportation processes* between the volume outside the moving δ-shock front Γ_t and the front Γ_t *are going on*. Moreover, we derive the δ-shock *balance relations* (3.3) which show that the total "mass" $M(t) + m(t)$ and "momentum" $P(t) + p(t)$ are independent of time, where $M(t)$, $P(t)$ are "mass" and "momentum" of the domain outside the wave front, and $m(t)$, $p(t)$ are "mass" and "momentum" of the wave front Γ_t. For zero-pressure gas dynamics system (1.4), "mass" and "momentum" have really a sense of mass and momentum. In this case the *mass transportation process* described by Theorem 3.1 is the *mass concentration process* on the moving front Γ_t.

In this section we also consider the possibility of the *effect of kinematic self-gravitation* and the *effect of dimensional bifurcations of δ-shock.*

In Appendix A, some auxiliary facts are given. In particular, we give results related with moving surfaces and distributions defined on these surfaces, prove the surface transport theorems.

2. δ-Shock type solutions and the Rankine–Hugoniot conditions

2.1. **δ-Shock type solutions.** Let $\Gamma = \{(x,t) : S(x,t) = 0\}$ be a hypersurface of codimension 1 in the upper half-space $\{(x,t) : x \in \mathbb{R}^n,\ t \in [0,\infty)\} \subset \mathbb{R}^{n+1}$, $S \in C^\infty(\mathbb{R}^n \times [0,\infty))$, $\nabla S(x,t)\big|_{S=0} \neq 0$ for any fixed t, where $\nabla = \big(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\big)$. Let $\Gamma_t = \{x : S(x,t) = 0\}$ be a moving surface in $\mathbb{R}^n$. Denote by ν the unit space normal to the surface Γ_t pointing (in the positive direction) from $\Omega_t^- = \{x \in \mathbb{R}^n : S(x,t) < 0\}$ to $\Omega_t^+ = \{x \in \mathbb{R}^n : S(x,t) > 0\}$ such that $\nu_j = \frac{S_{x_j}}{|\nabla S|}$, $j = 1, \dots, n$. The direction of the vector ν coincides with the direction in which the function S increases, i.e., inward the domain Ω_t^+. Denote by $-G = \frac{S_t}{|\nabla S|}$ the velocity (along the normal ν) of the moving wave front Γ_t (see Appendix A.1).

For system (1.7) we consider the *δ-shock type initial data*

$$(U^0(x), \rho^0(x);\, U^0_\delta(x),\, x \in \Gamma_0), \quad \text{where} \quad \rho^0(x) = \widehat{\rho}^0(x) + e^0(x)\delta(\Gamma_0), \tag{2.1}$$

$U^0 \in L^\infty\big(\mathbb{R}^n;\mathbb{R}^n\big)$, $\widehat{\rho}^0 \in L^\infty\big(\mathbb{R}^n;\mathbb{R}\big)$, $e^0 \in C(\Gamma_0)$, $\Gamma_0 = \big\{x : S^0(x) = 0\big\}$ is the initial position of the δ-shock front, $\nabla S^0(x)\big|_{S^0=0} \neq 0$, $U^0_\delta(x)$, $x \in \Gamma_0$ is the *initial velocity* of the δ-shock, $\delta(\Gamma_0)$ ($\equiv \delta(S^0)$) is the Dirac delta function concentrated on the surface Γ_0. The facts related to distributions defined on surfaces can be found in Appendix A.2.

Let us introduce the definition of a *δ-shock wave type solution* for system (1.4).

Definition 2.1. Distributions (U, ρ) and a hypersurface Γ, where $\rho(x,t)$ has the form of the sum

$$\rho(x,t) = \widehat{\rho}(x,t) + e(x,t)\delta(\Gamma),$$

and $U \in L^\infty\big(\mathbb{R}^n \times (0,\infty);\mathbb{R}^n\big)$, $\widehat{\rho} \in L^\infty\big(\mathbb{R}^n \times (0,\infty);\mathbb{R}\big)$, $e \in C(\Gamma)$, is called a *δ-shock wave type solution* of the Cauchy problem (1.7), (2.1) if the integral identities

$$\begin{aligned}
&\int_0^\infty \int \widehat{\rho}\Big(\varphi_t + F(U)\cdot\nabla\varphi\Big)\,dx\,dt + \int_\Gamma e\frac{\delta\varphi}{\delta t}\frac{d\mu(x,t)}{\sqrt{1+G^2}} \\
&\qquad + \int \widehat{\rho}^0(x)\varphi(x,0)\,dx + \int_{\Gamma_0} e^0(x)\varphi(x,0)\,d\mu(x) \;=\; 0, \\
&\int_0^\infty \int \widehat{\rho}\Big(U\varphi_t + N(U)\cdot\nabla\varphi\Big)\,dx\,dt + \int_\Gamma eU_\delta\frac{\delta\varphi}{\delta t}\frac{d\mu(x,t)}{\sqrt{1+G^2}} \\
&\qquad + \int U^0(x)\widehat{\rho}^0(x)\varphi(x,0)\,dx + \int_{\Gamma_0} e^0(x)U^0_\delta(x)\varphi(x,0)\,d\mu(x) \;=\; 0,
\end{aligned} \tag{2.2}$$

hold for all $\varphi \in \mathcal{D}(\mathbb{R}^n \times [0,\infty))$, where $\int f(x)\,dx$ denotes the improper integral $\int_{\mathbb{R}^n} f(x)\,dx$;

$$U_\delta = \nu G = -\frac{S_t\nabla S}{|\nabla S|^2} \tag{2.3}$$

is the δ-shock velocity, $-G = \frac{S_t}{|\nabla S|}$, $\frac{\delta\varphi}{\delta t}$ is the δ-derivative with respect to the time variable (A.5).

Remark 2.1. It is natural to generalize Definition 2.1 and introduce a concept of a multidimensional δ-shock type solution (U, ρ) to system (1.7), where $\rho(x,t)$ has the form of the sum

$$\rho(x,t) = \widehat{\rho}(x,t) + \sum_{j=1}^n e_j(x,t)\delta(\Gamma^{(j)}), \tag{2.4}$$

$U \in L^\infty\big(\mathbb{R}^n \times (0,\infty);\mathbb{R}^n\big)$, $\widehat{\rho} \in L^\infty\big(\mathbb{R}^n \times (0,\infty);\mathbb{R}\big)$, $e_j \in C(\Gamma^{(j)})$, $\Gamma^{(j)}$ is a hypersurface of codimension j, $\delta(\Gamma^{(j)})$ the Dirac delta function concentrated on the hypersurface $\Gamma^{(j)}$, $j = 1, 2, \ldots, n$. For this purpose we need to derive special integral identities analogous to (2.2).

Let S^0 be a given smooth function. Denote by $\Omega_0^- = \{x : S^0(x) < 0\}$ and $\Omega_0^+ = \{x : S^0(x) > 0\}$ the domains on the one side and on the other side of the hypersurface $\Gamma_0 = \big\{x : S^0(x) = 0\big\}$. In order to study the δ-shock front-problem, i.e., to describe the propagation of a singular front Γ starting from the initial position Γ_0, we need to solve the Cauchy problem for system (1.7) with the initial data

$$\begin{aligned}
\rho^0(x) &= \rho^{+0}(x) + [\rho^0(x)]H(-\Gamma_0) + e^0(x)\delta(\Gamma_0), \\
U^0(x) &= U^{+0}(x) + [U^0(x)]H(-\Gamma_0), \\
U_\delta(x,0) &= U^0_\delta(x),
\end{aligned} \tag{2.5}$$

where $[U^0(x)] = U^{-0}(x) - U^{+0}(x)$ is a jump of the function U^0 across the discontinuity hypersurface Γ_0; $U^0 = U^{0+}$, $\rho^0 = \rho^{0+}$ if $x \in \Omega_0^+$, and $U^0 = U^{0-} = U^{0+} + [U^0]$, $\rho^0 = \rho^{0-} = \rho^{0+} + [\rho^0]$ if $x \in \Omega_0^-$; e^0 and $\rho^{0\pm}$ are given functions, $U^{0\pm}$ are given vectors; $H(-\Gamma_0)$ ($\equiv H(-S^0)$)is the Heaviside function defined on the surface Γ_0, $H(-\Gamma_0) = 1$ if $S^0(x) < 0$, $H(-\Gamma_0) = 0$ if $S^0(x) > 0$. We assume that for the initial data (2.5) the geometric entropy condition

$$U^{0+}(x) \cdot \nu^0\big|_{\Gamma_0} < U^0_\delta(x) \cdot \nu^0\big|_{\Gamma_0} < U^{0-}(x) \cdot \nu^0\big|_{\Gamma_0} \tag{2.6}$$

holds, where $\nu^0 = \frac{\nabla S^0(x)}{|\nabla S^0(x)|}$ is the unit space normal of Γ_0 oriented from $\Omega_0^- = \{x \in \mathbb{R}^n : S^0(x) < 0\}$ to $\Omega_0^+ = \{x \in \mathbb{R}^n : S^0(x) > 0\}$.

2.2. **Rankine–Hugoniot conditions.** Using Definition 2.1, we derive the *δ-shock Rankine–Hugoniot conditions* for system (1.7).

Theorem 2.1. *Let us assume that $\Omega \subset \mathbb{R}^n \times (0,\infty)$ is a region cut by a smooth hypersurface $\Gamma = \big\{(x,t) : S(x,t) = 0\big\}$ into a left- and right-hand parts $\Omega_\mp$. Let (U,ρ), Γ be a δ-shock wave type solution of system* (1.7) (*in the sense of Definition* 2.1), *and suppose that (U,ρ) is smooth in $\Omega_\pm$ and has one-sided limits $U^\pm$, $\widehat{\rho}^\pm$ on Γ. Then the* Rankine–Hugoniot conditions for the δ-shock

$$\begin{aligned} \frac{\delta e}{\delta t} + \nabla_{\Gamma_t} \cdot (eU_\delta) &= \big([\rho F(U)],\, [\rho]\big) \cdot \mathbf{n}, \\ \frac{\delta (eU_\delta)}{\delta t} + \nabla_{\Gamma_t} \cdot (eU_\delta \otimes U_\delta) &= \big([\rho N(U)],\, [\rho U]\big) \cdot \mathbf{n}, \end{aligned} \tag{2.7}$$

hold on the discontinuity hypersurface Γ, where $\mathbf{n} = (\nu, -G) = \frac{\nabla_{(x,t)} S}{|\nabla S|}$ is the space-time normal to the surface Γ, $\nabla_{(x,t)} = \big(\nabla, \frac{\partial}{\partial t}\big)$, $\big[f(U,\rho)\big] = f(U^-,\rho^-) - f(U^+,\rho^+)$ is a jump of the function $f(U,\rho)$ across the discontinuity hypersurface Γ, $\frac{\delta}{\delta t}$ is the δ-derivative (A.5) *with respect to t, and the tangent gradient $\nabla_{\Gamma_t} = \Big(\frac{\delta}{\delta x_1}, \dots, \frac{\delta}{\delta x_n}\Big)$ to the surface Γ_t is defined by* (A.5), (A.6). *The equivalent forms of* (2.7) *are the following:*

$$\begin{aligned} \frac{\delta e}{\delta t} + \nabla_{\Gamma_t} \cdot (eU_\delta) &= \big([\rho F(U)] - [\rho] U_\delta\big) \cdot \nu, \\ \frac{\delta (eU_\delta)}{\delta t} + \nabla_{\Gamma_t} \cdot (eU_\delta \otimes U_\delta) &= \big([\rho N(U)] - [\rho U] U_\delta\big) \cdot \nu, \end{aligned} \tag{2.8}$$

or

$$\begin{aligned} \frac{\delta e}{\delta t} - 2\mathcal{K} G e &= \big([\rho F(U)] - [\rho] U_\delta\big) \cdot \nu, \\ \frac{\delta (eU_\delta)}{\delta t} - 2\mathcal{K} G e U_\delta &= \big([\rho N(U)] - [\rho U] U_\delta\big) \cdot \nu. \end{aligned} \tag{2.9}$$

Proof. For any test function $\varphi \in \mathcal{D}(\Omega)$ we have $\varphi(x,t) = 0$ for $(x,t) \notin G$, $\overline{G} \subset \Omega$. Selecting the test function $\varphi(x,t)$ with compact support in $\Omega_\pm$, we deduce from (2.2) that (1.7) hold in $\Omega_\pm$, respectively. Now, if the test function $\varphi(x,t)$ has the support in Ω, then

$$\begin{aligned} &\int_0^\infty \int \widehat{\rho}\Big(\varphi_t + F(U) \cdot \nabla\varphi\Big)\, dx\, dt \\ &= \int_{\Omega_- \cap G} \widehat{\rho}\Big(\varphi_t + F(U) \cdot \nabla\varphi\Big)\, dx\, dt + \int_{\Omega_+ \cap G} \widehat{\rho}\Big(\varphi_t + F(U) \cdot \nabla\varphi\Big)\, dx\, dt. \end{aligned}$$

Using integrating-by-parts formula, we obtain

$$\int_{\Omega_\pm \cap G} \widehat{\rho}\Big(\varphi_t + F(U) \cdot \nabla\varphi\Big)\, dx\, dt = -\int_{\Omega_\pm \cap G} \Big(\rho_t + \nabla \cdot (\rho F)\Big) \varphi(x,t)\, dx\, dt$$

$$\mp \int_{\Gamma\cap G} \Big(\frac{\widehat{\rho}^{\pm} S_t}{|\nabla_{(x,t)} S|} + \frac{\widehat{\rho}^{\pm} F(U^{\pm})\cdot\nabla S}{|\nabla_{(x,t)} S|}\Big)\varphi(x,t)\,d\mu(x,t) - \int_{\Omega_{\pm}\cap G\cap\mathbb{R}^n} \widehat{\rho}^0(x)\varphi(x,0)\,dx,$$

where $d\mu(x,t)$ is the surface measure on Γ. Next, adding the latter relations, and taking into account that $\rho_t + \nabla\cdot(\rho F) = 0$, $(x,t)\in\Omega_{\pm}$, we have

$$\int_0^{\infty}\int \widehat{\rho}\Big(\varphi_t + F(U)\cdot\nabla\varphi\Big)\,dx\,dt + \int \widehat{\rho}^0(x)\varphi(x,0)\,dx$$
$$= \int_{\Gamma}\Big(-[\rho]G + [\rho F(U)]\cdot\nu\Big)\varphi(x,t)\frac{d\mu(x,t)}{\sqrt{1+G^2}}. \tag{2.10}$$

Now, using the second integrating-by-parts formula in (A.13), one can see that

$$\int_{\Gamma} e\frac{\delta\varphi}{\delta t}\frac{d\mu(x,t)}{\sqrt{1+G^2}} + \int_{\Gamma_0} e^0(x)\varphi(x,0)\,d\mu(x) = -\int_{\Gamma}\frac{\delta^* e}{\delta t}\varphi\frac{d\mu(x,t)}{\sqrt{1+G^2}},$$

where the adjoint operator $\frac{\delta^* e}{\delta t}$ is defined in (A.14). Thus

$$\int_{\Gamma} e\frac{\delta\varphi}{\delta t}\frac{d\mu(x,t)}{\sqrt{1+G^2}} + \int_{\Gamma_0} e^0(x)\varphi(x,0)\,d\mu(x)$$
$$= -\int_{\Gamma}\Big(\frac{\delta e}{\delta t} + \nabla_{\Gamma_t}\cdot(eG\nu)\Big)\varphi\frac{d\mu(x,t)}{\sqrt{1+G^2}}. \tag{2.11}$$

Adding (2.10) and (2.11), we derive

$$\int_{\Gamma}\Big(-[\rho]G + [\rho F(U)]\cdot\nu - \frac{\delta e}{\delta t} - \nabla_{\Gamma_t}\cdot(eG\nu)\Big)\varphi(x,t)\,\frac{d\mu(x,t)}{\sqrt{1+G^2}} = 0,$$

for all $\varphi(x,t)\in\mathcal{D}(\Omega)$. Taking into account formula (2.3) for the δ-shock velocity, one can see that the last relation implies the first relation in (2.7).

In the same way as above, we obtain the second relation in (2.7).

In view of (2.3) and (A.14), the Rankine–Hugoniot conditions (2.7) can be rewritten as (2.8). Since, according to the proof of Lemma A.1,

$$\nabla_{\Gamma_t}\cdot(eU_\delta) = -2\mathcal{K}Ge, \qquad \nabla_{\Gamma_t}\cdot(eU_\delta\otimes U_\delta) = -2\mathcal{K}GeU_\delta, \tag{2.12}$$

the Rankine–Hugoniot conditions (2.8) can be rewritten also in the form (2.9), where $\mathcal{K}$ is the mean curvature (A.7) of the surface Γ_t. $\Box$

The right-hand sides of the first and second equations in (2.7) or (2.8) are called the *Rankine–Hugoniot deficits* in ρ and ρU, respectively.

Zero-pressure gas dynamics. According to (2.7) and (2.8), for zero-pressure gas dynamics (1.4) and (1.8) the Rankine–Hugoniot conditions have the forms

$$\begin{aligned} \frac{\delta e}{\delta t} + \nabla_{\Gamma_t}\cdot(eU_\delta) &= \Big([\rho U] - [\rho]U_\delta\Big)\cdot\nu, \\ \frac{\delta(eU_\delta)}{\delta t} + \nabla_{\Gamma_t}\cdot(eU_\delta\otimes U_\delta) &= \Big([\rho U\otimes U] - [\rho U]U_\delta\Big)\cdot\nu \end{aligned} \tag{2.13}$$

and

$$\begin{aligned} \frac{\delta e}{\delta t} + \nabla_{\Gamma_t}\cdot(eU_\delta) &= \Big([\rho C(U)] - [\rho]U_\delta\Big)\cdot\nu, \\ \frac{\delta(eU_\delta)}{\delta t} + \nabla_{\Gamma_t}\cdot(eU_\delta\otimes U_\delta) &= \Big([\rho U\otimes C(U)] - [\rho U]U_\delta\Big)\cdot\nu. \end{aligned} \tag{2.14}$$

respectively.

In this case the *Rankine–Hugoniot deficits* in ρ and ρU are the *currents of mass and momentum, respectively.*

Remark 2.2. **(a)** The Rankine–Hugoniot conditions (2.7) constitute a system of *second-order* PDEs. According to this fact, to solve the Cauchy problem for system (1.7), we use the initial data of the form (2.1), where the *initial velocity* $U_\delta^0(x)$ of a δ-shock is specified.

(b) For system (1.4) the Rankine–Hugoniot conditions (2.7) are analogous to the Rankine–Hugoniot conditions

$$\begin{aligned} \frac{\partial X}{\partial t} &= U_\delta(s,t), \\ \frac{\partial w}{\partial t} &= \big([\rho U],[\rho]\big)\cdot(N_X,N_t), \\ \frac{\partial(wU_\delta)}{\partial t} &= \big([\rho U\otimes U],[\rho U]\big)\cdot(N_X,N_t), \end{aligned} \tag{2.15}$$

in the *measure-valued solution* approach [26], [27], [44] (see Subsec. 1.2), where (N_X,N_t) is the space-time normal to the δ-shock front.

Spherically symmetric case of zero-pressure gas dynamics. It is easy to see that the solution of (1.4) with spherical symmetry $\rho=\rho(r,t)$, $U=u(r,t)\frac{x}{r}$, where $r=|x|$, $x\in\mathbb{R}^n$, satisfies the following system of equations

$$\rho_t+(\rho u)_r+\frac{n-1}{r}\rho u=0, \qquad (\rho u)_t+(\rho u^2)_r+\frac{n-1}{r}\rho u^2=0. \tag{2.16}$$

In this case $\Gamma=\big\{(x,t)\in\mathbb{R}^n\times[0,\infty): S(r,t)=0\big\}$, $\Gamma_t=\big\{x\in\mathbb{R}^n: S(r,t)=0\big\}$; $\nabla S(r,t)=S_r\frac{x}{r}$, $|\nabla S(r,t)|=|S_r|$; $\nu=\frac{S_r}{|S_r|}\frac{x}{r}$; $G=-\frac{S_t}{|S_r|}$; the δ-shock velocity (2.3) is represented as $U_\delta=\nu G=-\frac{S_t}{|S_r|}\frac{x}{r}$; $x\in\mathbb{R}^n$. It is easy to see if $f=f(r,t)$ then formulas (A.5) have the form

$$\frac{\delta f}{\delta t}=\frac{\partial f}{\partial t}-\frac{S_t}{S_r}\frac{\partial f}{\partial r}, \qquad \frac{\delta f}{\delta x_j}=0, \quad j=1,\dots,n. \tag{2.17}$$

Now the formulas (2.12) read

$$\nabla_{\Gamma_t}\cdot(eU_\delta)=-e\frac{S_t}{S_r}\frac{n-1}{r}, \qquad \nabla_{\Gamma_t}\cdot(eU_\delta\otimes U_\delta)=e\frac{S_t^2}{S_r^2}\frac{(n-1)x}{r^2}, \quad (x,t)\in\Gamma.$$

Taking into account the above formula, we observe that the Rankine–Hugoniot conditions (2.13) take the form

$$\begin{aligned} e_t-\frac{S_t}{S_r}e_r-e\frac{S_t}{S_r}\frac{n-1}{r} &= [\rho u]\frac{S_r}{|S_r|}+[\rho]\frac{S_t}{|S_r|}, \\ \Big(-e\frac{S_t}{S_r}\Big)_t-\frac{S_t}{S_r}\Big(-e\frac{S_t}{S_r}\Big)_r+e\Big(\frac{S_t}{S_r}\Big)^2\frac{n-1}{r} &= [\rho u^2]\frac{S_r}{|S_r|}+[\rho u]\frac{S_t}{|S_r|}, \end{aligned} \tag{2.18}$$

for $(x,t)\in\Gamma$.

If $S(r,t)=-r+\phi(t)$, the Rankine–Hugoniot conditions (2.18) can be rewritten as

$$\begin{aligned} e_t+\dot\phi(t)e_r+e\dot\phi(t)\frac{n-1}{r} &= -[\rho u]+[\rho]\dot\phi(t), \\ \Big(e\dot\phi(t)\Big)_t+\dot\phi(t)\Big(e\dot\phi(t)\Big)_r+e\Big(\dot\phi(t)\Big)^2\frac{n-1}{r} &= -[\rho u^2]+[\rho u]\dot\phi(t). \end{aligned} \tag{2.19}$$

3. Geometrical and physical aspects of δ-shocks

3.1. **Volume, mass, and momentum balance relations.** It is well known that if $U\in L^\infty\big(\mathbb{R}\times(0,\infty);\mathbb{R}^m\big)$ is a generalized solution of the Cauchy problem (1.1) compactly supported with respect to x, then the integral of the solution on the whole space

$$\int U(x,t)\,dx = \int U^0(x)\,dx, \qquad t \geq 0 \tag{3.1}$$

is independent of time. These integrals can express the conservation laws of *total area, mass, momentum, energy*, etc. For a δ-shock wave type solution the classical conservation laws (3.1) *do not hold*. However, there is a "generalized" analog of conservation laws (3.1). In one-dimensional case these "generalized" analogs were derived in [1], [31], [40]. Now we derive multidimensional generalization of these laws.

Let us assume that a moving surface $\Gamma_t = \big\{x : S(x,t) = 0\big\}$ permanently separates $\mathbb{R}^n_x$ into two parts $\Omega_t^- = \{x \in \mathbb{R}^n : S(x,t) < 0\}$ and $\Omega_t^+ = \{x \in \mathbb{R}^n : S(x,t) > 0\}$, $\Omega_0^\pm = \{x \in \mathbb{R}^n : \pm S^0(x) > 0\}$. Let (U,ρ) be compactly supported with respect to x. Denote by

$$M(t) = \int_{\Omega_t^- \cup \Omega_t^+} \rho(x,t)\,dx \qquad M(0) = \int_{\Omega_0^- \cup \Omega_0^+} \rho^0(x)\,dx,$$

$$P(t) = \int_{\Omega_t^- \cup \Omega_t^+} \rho(x,t) U(x,t)\,dx, \qquad P(0) = \int_{\Omega_0^- \cup \Omega_0^+} \rho^0(x) U^0(x)\,dx,$$

and

$$m(t) = \int_{\Gamma_t} e(x,t)\,d\mu(x), \qquad m(0) = \int_{\Gamma_0} e^0(x)\,d\mu(x),$$

$$p(t) = \int_{\Gamma_t} e(x,t) U_\delta(x,t)\,d\mu(x), \qquad p(0) = \int_{\Gamma_0} e^0(x) U_\delta^0(x)\,d\mu(x),$$

"masses" and "momentums" of the domains $\Omega_t^- \cup \Omega_t^+$, $\Omega_0^- \cup \Omega_0^+$ and the "masses" and "momenta" of the hypersurfaces Γ_t, Γ_0, respectively, where $d\mu(x)$ is the surface measure on Γ_t. The quantities $M(t)$ and $P(t)$ can be interpreted as the volumes under the graphs $y = \widehat{\rho}(x,t)$ and $Y = \widehat{\rho}(x,t) U(x,t)$, $x \in \Omega_t^- \cup \Omega_t^+$.

Theorem 3.1. *Let* (U,ρ) *and the discontinuity hypesurface* $\Gamma = \big\{(x,t) : S(x,t) = 0\big\}$ *be a* δ*-shock wave type solution* (*in the sense of Definition* 2.1) *of the Cauchy problem* (1.7), (2.1), *compactly supported with respect to* x, *where* $\rho(x,t) = \widehat{\rho}(x,t) + e(x,t)\delta(\Gamma)$. *Then the following* "mass" and "momentum" balance relations *hold:*

$$\dot{M}(t) = -\dot{m}(t), \qquad \dot{P}(t) = -\dot{p}(t); \tag{3.2}$$

$$M(t) + m(t) = M(0) + m(0), \qquad P(t) + p(t) = P(0) + p(0). \tag{3.3}$$

Thus the "mass" and "momentum" transportation processes *between the volume* $\Omega_t^- \cup \Omega_t^+$ *and the moving front* Γ_t are going on. *Moreover, the total "mass"* $M(t) + m(t)$ *and "momentum"* $P(t) + p(t)$ *are independent of time.*

Proof. Let us assume that the supports of $U(x,t)$ and $\rho(x,t)$ with respect to x belong to a compact $K \Subset \mathbb{R}^n_x$ bounded by ∂K. Let $K_t^\pm = \Omega_t^\pm \cap K$. By ν we denote the space normal to Γ_t pointing from Ω_t^- to Ω_t^+. Differentiating $M(t)$ and using the volume transport Theorem A.1, we obtain

$$\dot{M}(t) = \int_{K_t^- \cup K_t^+} \frac{\partial \rho}{\partial t}\,dx + \int_{\partial K_t^- \cup \partial K_t^+} G\rho\,d\mu(x),$$

where $G = -\frac{S_t}{|\nabla S|}$. Since $\rho_t^\pm + \nabla \cdot (\rho F(U^\pm)) = 0$, $x \in K^\pm$ and the vectors $U^\pm$ and functions $\rho^\pm$ are equal to zero on the surface $\partial K_t^\pm$ except Γ_t, applying Gauss's divergence theorem, we transform the last relation to the form

$$\dot{M}(t) = -\int_{K_t^-} \nabla \cdot (\rho^- F(U^-))\,dx - \int_{K_t^+} \nabla \cdot (\rho^+ F(U^+))\,dx + \int_{\Gamma_t} G[\rho]\,d\mu(x)$$

$$= -\int_{\Gamma_t} \rho^- F(U^-)\cdot\nu\, d\mu(x) + \int_{\Gamma_t} \rho^+ F(U^+)\cdot\nu\ \ d\mu(x) + \int_{\Gamma_t} G[\rho]\, d\mu(x)$$
$$= -\int_{\Gamma_t} \big([\rho F(U)]\cdot\nu - [\rho]G\big)\, d\mu(x). \tag{3.4}$$

Using the first Rankine–Hugoniot condition (2.8) and taking into account that $G = U_\delta\cdot\nu$, relation (3.4) can be rewritten as

$$\dot M(t) = -\int_{\Gamma_t} \big([\rho F(U)] - [\rho]U_\delta\big)\cdot\nu\, d\mu(x) = -\int_{\Gamma_t} \Big(\frac{\delta e}{\delta t} + \nabla_{\Gamma_t}\cdot(eU_\delta)\Big)\, d\mu(x).$$

According to the surface transport Theorem A.2, we have

$$\dot m(t) = \int_{\Gamma_t} \Big(\frac{\delta e}{\delta t} + \nabla_{\Gamma_t}\cdot(eU_\delta)\Big)\, d\mu(x).$$

Thus the first balance relation in (3.2) is proved.

Repeating the proof of the first balance relation in (3.2) almost word for word, we derive the second balance relation in (3.2).

To complete the proof of the theorem, it remains to integrate (3.2) with respect to t. $\square$

3.2. **The case of zero-pressure gas dynamics.** In this case $\rho \ge 0$ and U can be considered as the gas density and the gas velocity, respectively.

To solve the Cauchy problem, we assume that for its solution the geometric entropy condition

$$U^+(x,t)\cdot\nu\big|_{\Gamma_t} < U_\delta(x,t)\cdot\nu\big|_{\Gamma_t} < U^-(x,t)\cdot\nu\big|_{\Gamma_t}, \tag{3.5}$$

holds, where U_δ is the velocity (2.3) of the δ-shock front Γ_t, $U^\pm$ is the velocity behind the δ-shock wave front and ahead of it, respectively. Condition (3.5) implies that all characteristics on both sides of the initial discontinuity Γ_t must overlap. For $t = 0$ the condition (3.5) coincides with (2.6) .

Corollary 3.1. *In the case of zero-pressure gas dynamics* (1.4) *the* transportation process *described by Theorem* 3.1 *is the* mass concentration process *on the moving front* Γ_t*:*

$$\begin{aligned} &\dot M(t) = -\dot m(t), \qquad \dot m(t) > 0, && \dot P(t) = -\dot p(t),\\ &M(t) + m(t) = M(0) + m(0), && P(t) + p(t) = P(0) + p(0). \end{aligned} \tag{3.6}$$

Proof. Now it remains to prove the inequality $\dot m(t) > 0$. Since the solution (U,ρ) of the Cauchy problem (1.4), (2.1) satisfies the entropy condition (3.5) and $\rho^\pm \ge 0$, we have for the first relation in (2.13)

$$\frac{\delta e}{\delta t} + \nabla_{\Gamma_t}\cdot(eU_\delta) = \big([\rho U] - [\rho]U_\delta\big)\cdot\nu\big|_{\Gamma_t}$$
$$= \big(\rho^-(U^- - U_\delta)\cdot\nu + \rho^+(U_\delta - U^+)\cdot\nu\big)\big|_{\Gamma_t} \ge 0.$$

This inequality and Theorem 3.1 imply that $\dot m(t) = \int_{\Gamma_t}\big(\frac{\delta e}{\delta t} + \nabla_{\Gamma_t}\cdot(eU_\delta)\big)\, d\mu(x) > 0$ and $\dot M(t) < 0$. $\square$

The above results imply the possibility of the following interesting effects.

The effect of kinematic self-gravitation. According to (3.6), the mass concentration process on the moving discontinuity surface Γ_t is going on. Moreover, the second δ-shock Rankine–Hugoniot condition in (3.6) is the momentum conservation law. Using this fact, one can introduce an "effective" gravitational potential in a neighbourhood of discontinuity surface and describe the *concentration process* *in terms of gravitational interaction.* Since in system (1.4) there is no term related with gravitational interaction, this "gravitational effect" has *kinematic character.*

Dimensional bifurcations of δ-shock. It follows from Corollary 3.1 that in the n-dimensional zero-pressure gas dynamics (1.4) the mass transportation process from the volume $\Omega_t^- \cup \Omega_t^+$ onto the $n-1$-dimensional moving δ-shock front Γ_t is going on. Let us suppose that in *finite time* $\tilde{t}$ the whole initial mass $M(0)$ may be concentrated on Γ_t. Then, according to (2.13), for $t > \tilde{t}$, *instead* of the *whole "initial" n-dimensional* system of zero-pressure gas dynamics (1.4) we obtain a *"surface" $(n-1)$-dimensional version of this system*

$$\frac{\delta e}{\delta t} + \nabla_{\Gamma_t} \cdot (eU_\delta) = 0, \qquad \frac{\delta(eU_\delta)}{\delta t} + \nabla_{\Gamma_t} \cdot (eU_\delta \otimes U_\delta) = 0, \tag{3.7}$$

where instead of the gas velocity U we have the velocity U_δ of the moving δ-shock front Γ_t, and instead of the gas volume density ρ we have the *surface density of the front mass* e. Moreover, the quantities U_δ, e are defined only on the moving front Γ_t. Since system (3.7) is an $n-1$-dimensional analog of system (1.4) on the $(n-1)$– dimensional surface Γ_t as on a Riemannian manifold, therefore its solution can develop singularities within a finite time period, and the whole mass concentrates on the manifold of dimension $n-2$, and so on. Thus, it may happen that after a finite number of bifurcations the whole initial mass will be concentrated at the singular point. To describe this effect we need to deal with the δ-shock in the form (2.4). As a consequence of this fact we need to develop the theory of such type solutions.

Appendix A. Some auxiliary facts

A.1. **Moving surfaces of discontinuity.** Let us present some results concerning moving surfaces from [21, 5.2.], [2], [3]. Let Γ_t be a smooth moving surface of codimension 1 in the space $\mathbb{R}^n$. Such a surface can be represented locally either in the form $\Gamma_t = \big\{x \in \mathbb{R}^n : S(x,t) = 0\big\}$, or in terms of the curvilinear Gaussian coordinates $s = (s_1, \dots, s_{n-1})$ on the surface:

$$x_j = x_j(s_1, \dots, s_{n-1}, t), \qquad s \in \mathbb{R}^{n-1}.$$

We also consider the surface $\Gamma = \big\{(x,t) \in \mathbb{R}^{n+1} : S(x,t) = 0\big\}$ as a submanifold of the space-time $\mathbb{R}^n \times \mathbb{R}$. We shall assume that $\nabla S(x,t)\big|_{\Gamma_t} \neq 0$ for all fixed values of t, where $\nabla = \big(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\big)$. Let ν be the unit space normal to the surface Γ_t pointing in the positive direction such that $\frac{\partial S}{\partial x_j} = |\nabla S|\nu_j$, $j = 1, \dots, n$.

Let $f(x,t)$ be a function defined on the surface Γ_t for some time interval, and let $\frac{\delta f}{\delta t}$ to denote the derivative with respect to time as it would be computed by an observer moving with the surface. This derivative has the following geometrical interpretation. Let M_0 be a point on the surface at the time $t = t_0$. Construct the normal line to the surface at M_0. At the time $t = t_0 + \Delta t$, Δt is an infinitesimal, this normal meets the surface $\Gamma_{t+\Delta t}$ at the point $M = M(t + \Delta t)$. Then the δ-derivative is defined as

$$\frac{\delta f(M_0, t_0)}{\delta t} = \lim_{\Delta t \to 0} \frac{f(M) - f(M_0)}{\Delta t}. \tag{A.1}$$

If Δs is the distance between M_0 and M, then

$$G = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} \tag{A.2}$$

is the *normal velocity of the moving surface* Γ_t and

$$\frac{\delta x_j}{\delta t} = \lim_{\Delta t \to 0} \frac{\Delta x_j}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} \frac{\Delta x_j}{\Delta s} = G\nu_j, \quad j = 1, \dots, n. \tag{A.3}$$

Since it is essential that the δ-derivative is computed on a surface, and S remains constant on this surface then $\frac{\delta S}{\delta t}=0$. Thus we have

$$0=\frac{\delta S}{\delta t}=\frac{\partial S}{\partial t}+\sum_{j=1}^{n}\frac{\delta S}{\delta x_j}\frac{\delta x_j}{\delta t}=\frac{\partial S}{\partial t}+\sum_{j=1}^{n}G|\nabla S|\nu_j^2,$$

i.e.,

$$S_t=-G|\nabla S|. \tag{A.4}$$

From this formula we can see that $-G=\frac{S_t}{|\nabla S|}$ can be interpreted as the time component of the normal vector.

The space-time unit normal to the surface Γ is given by $\mathbf{n}=\frac{(\nu,-G)}{\sqrt{1+G^2}}$, where $\sqrt{1+G^2}=\frac{|\nabla_{(x,t)}S|}{|\nabla S|}$, $\nabla_{(x,t)}=\big(\nabla,\frac{\partial}{\partial t}\big)$.

If $f(x,t)$ is a function defined only on Γ, its first order δ-derivatives with respect to the time and space variables are defined by the following formulas [21, 5.2.(15),(16)]:

$$\frac{\delta f}{\delta t}\stackrel{def}{=}\frac{\partial\widetilde{f}}{\partial t}+G\frac{\partial\widetilde{f}}{\partial\nu},\qquad \frac{\delta f}{\delta x_j}\stackrel{def}{=}\frac{\partial\widetilde{f}}{\partial x_j}-\nu_j\frac{\partial\widetilde{f}}{\partial\nu},\quad j=1,\dots,n, \tag{A.5}$$

where $\widetilde{f}$ is a smooth extension of f to a neighborhood of Γ in $\mathbb{R}^n\times\mathbb{R}$, $j=1,\dots,n$, and $\frac{\partial\widetilde{f}}{\partial\nu}=\nu\cdot\nabla\widetilde{f}$ is a normal derivative. Thus the gradient tangent to the surface Γ_t is defined as

$$\nabla_{\Gamma_t}=\nabla-\nabla_\nu=\Big(\frac{\delta}{\delta x_1},\dots,\frac{\delta}{\delta x_n}\Big), \tag{A.6}$$

where $\nabla_\nu=\nu\big(\nu\cdot\nabla\big)$ is the gradient along the normal direction to the surface Γ_t.

Note that the δ-derivatives (A.5) depend only on the values of f on Γ, i.e., if $f=0$ on Γ then $\frac{\delta f}{\delta x_j}$ and $\frac{\delta f}{\delta t}$ on Γ, $j=1,\dots,n$. Indeed, let $(x_0,t_0)\in\Gamma$. If $\nabla_{(x,t)}f(x_0,t_0)=0$ then $\nabla_{\Gamma_t}f(x_0,t_0)=0$ and $\frac{\delta f}{\delta t}(x_0,t_0)=0$, where $\nabla_{(x,t)}=\big(\nabla,\frac{\partial}{\partial t}\big)$. If $\nabla f(x_0,t_0)\neq 0$ then in a neighborhood of the point (x_0,t_0) the surface Γ_t has the unit space normal $\nu=\frac{\nabla f}{|\nabla f|}$ and $G=-\frac{\frac{\partial f}{\partial t}}{|\nabla f|}$. Consequently, $\nabla_{\Gamma_t}f(x_0,t_0)=0$ and $\frac{\delta f}{\delta t}(x_0,t_0)=0$. In the sequel we shall drop tilde from f.

For a vector $A(x,t)=(A_1(x,t),\dots,A_n(x,t))$ defined only on Γ_t, we introduce the *surface* (*tangent*) *divergence* by the following formula

$$\mathrm{div}_{\Gamma_t}A=\nabla_{\Gamma_t}\cdot A=\sum_{j=1}^{n}\frac{\delta A_j}{\delta x_j}.$$

The *mean curvature* of the surface Γ_t is defined as

$$\mathcal{K}\stackrel{def}{=}-\frac{1}{2}\nabla_{\Gamma_t}\cdot\nu=-\frac{1}{2}\sum_{j=1}^{n}\frac{\delta\nu_j}{\delta x_j}=-\frac{1}{2}\nabla\cdot\nu. \tag{A.7}$$

A.2. **Distributions defined on a surface.** Consider some facts about distributions defined on a surface [21, 5.2.], [19, ch.III,§1.], [2], [3]. The Heaviside function $H(S)$ is introduced by the following definition:

$$\big\langle H(S),\ \varphi(x,t)\big\rangle=\int_{S\geq 0}\varphi(x,t)\,dx\,dt,\quad \forall\,\varphi\in\mathcal{D}(\mathbb{R}^n\times\mathbb{R}).$$

According to [21, 5.3.(1),(2)], we now introduce the delta function $\delta(S)$ on the surface Γ, whose action on a test function $\varphi(x,t)\in\mathcal{D}(\mathbb{R}^n\times\mathbb{R})$ is given by

$$\big\langle\delta(S),\ \varphi(x,t)\big\rangle=\int_{-\infty}^{\infty}\int_{\Gamma_t}\varphi(x,t)\,d\mu(x)\,dt=\int_{\Gamma}\varphi(x,t)\frac{d\mu(x,t)}{\sqrt{1+G^2}}, \tag{A.8}$$

where $d\mu$ is the surface measure on the corresponding surface. According to [21, 5.5.Theorem 1.], we have

$$\frac{\partial H(S)}{\partial x_j} = \nu_j \delta(S), \qquad \frac{\partial H(S)}{\partial t} = -G\delta(S).$$

Now we introduce the derivative of the delta function $\partial_\nu \delta(S)$ along the space normal ν by the formula [21, 5.3.(7)]

$$\big\langle \partial_\nu \delta(S),\ \varphi \big\rangle = -\Big\langle \delta(S),\ \frac{\partial \varphi}{\partial \nu} \Big\rangle = -\int_{-\infty}^{\infty} \int_{\Gamma_t} \frac{\partial \varphi}{\partial \nu}\, d\mu(x)\, dt, \quad \forall\, \varphi \in \mathcal{D}(\mathbb{R}^n \times \mathbb{R}), \tag{A.9}$$

where $\frac{\partial \varphi}{\partial \nu} = \nu \cdot \nabla \varphi$ is the normal derivative of φ. If $f(x,t)$ is a continuous function defined on Γ which is a a restriction of some continuous function defined in a neighborhood of Γ in $\mathbb{R}^n \times \mathbb{R}$, then the distribution $\partial_\nu(f\delta(S))$ (the so-called *double layer*) is a functional acting by the rule

$$\big\langle \partial_\nu \big(f\delta(S)\big),\ \varphi \big\rangle = -\Big\langle \delta(S),\ f\frac{\partial \varphi}{\partial \nu} \Big\rangle, \quad \forall\, \varphi \in \mathcal{D}(\mathbb{R}^n \times \mathbb{R}).$$

According to [21, 5.3.(6)], we have

$$\delta'(S) = \sum_{i=1}^{n} \nu_i \frac{\partial}{\partial x_i} \delta(S) = 2\mathcal{K}\delta(S) + \partial_\nu \delta(S)$$

and

$$\frac{\partial}{\partial t}\delta(S) = -G\big(2\mathcal{K}\delta(S) + \partial_\nu \delta(S)\big), \qquad \frac{\partial}{\partial x_j}\delta(S) = \nu_j\big(2\mathcal{K}\delta(S) + \partial_\nu \delta(S)\big), \tag{A.10}$$

where $\mathcal{K}$ is the mean curvature (A.7) of the surface Γ_t.

If $f(x,t)$ is a differentiable function, using (A.5), (A.10), one can prove the following relations [21, 12.6.(15),(16)]

$$\frac{\partial}{\partial x_j}\big(f\delta(S)\big) = \Big(\frac{\partial f}{\partial x_j} - \nu_j \frac{\partial f}{\partial \nu} + 2\mathcal{K}\nu_j f\Big)\delta(S) + \nu_j f \partial_\nu \delta(S), \quad j = 1, \dots, n, \tag{A.11}$$

$$\frac{\partial}{\partial t}\big(f\delta(S)\big) = \Big(\frac{\partial f}{\partial t} + G\frac{\partial f}{\partial \nu} - 2\mathcal{K}Gf\Big)\delta(S) - Gf\partial_\nu \delta(S). \tag{A.12}$$

A.3. **One integrating-by-parts formula.**

Lemma A.1. *Suppose that $e(x,t)$ is a compactly supported smooth function defined only on the surface $\Gamma = \big\{(x,t) : S(x,t) = 0\big\}$, and $e(x,t)$ is the restriction of some smooth function defined in a neighborhood of Γ in $\mathbb{R}^n \times \mathbb{R}$, and $\Gamma_0 = \big\{x : S(x,0) = 0\big\}$. Then the formula for integration by parts holds:*

$$\int_\Gamma e\frac{\delta \varphi}{\delta t} \frac{d\mu(x,t)}{\sqrt{1+G^2}} = -\int_\Gamma \frac{\delta^* e}{\delta t}\varphi \frac{d\mu(x,t)}{\sqrt{1+G^2}} - \int_{\Gamma_0} e(x,0)\varphi(x,0)\, d\mu(x), \tag{A.13}$$

for any $\varphi \in \mathcal{D}(\mathbb{R}^n \times [0,\infty))$, where $\frac{\delta^*}{\delta t}$ is the adjoint operator defined as

$$\frac{\delta^* e}{\delta t} = \frac{\delta e}{\delta t} - 2\mathcal{K}Ge = \frac{\delta e}{\delta t} + \nabla_{\Gamma_t} \cdot (eG\nu), \tag{A.14}$$

$\mathcal{K}$ is the mean curvature (A.7) of the surface Γ_t.

Proof. With the help of formulas (A.8), (A.9), (A.10), (A.11), (A.12), by simple calculations, we obtain

$$\int_\Gamma e\frac{\delta \varphi}{\delta t} \frac{d\mu(x,t)}{\sqrt{1+G^2}} = \Big\langle e\delta(S),\ \frac{\delta \varphi}{\delta t} \Big\rangle = \Big\langle e\delta(S)H(t),\ \frac{\partial \varphi}{\partial t} + G\frac{\partial \varphi}{\partial \nu} \Big\rangle$$

$$= -\Big\langle \frac{\partial}{\partial t}\big(e\delta(S)H(t)\big),\ \varphi \Big\rangle - \Big\langle \partial_\nu\big(Ge\delta(S)\big)H(t),\ \varphi \Big\rangle$$

$$= -\Big\langle \frac{\delta e}{\delta t}\delta(S) - eG\big(2\mathcal{K}\delta(S) + \partial_\nu\delta(S)\big),\ \varphi\Big\rangle - \Big\langle e\delta(S)\delta(t),\ \varphi\Big\rangle$$
$$-\Big\langle \delta(S)\sum_{k=1}^{n}\frac{\delta(Ge)}{\delta x_k}\nu_k + eG\partial_\nu\delta(S),\ \varphi\Big\rangle$$
$$= -\Big\langle \Big(\frac{\delta e}{\delta t} - 2\mathcal{K}Ge\Big)\delta(S),\ \varphi\Big\rangle - \Big\langle e(x,0)\delta(S(x,0)),\ \varphi(x,0)\Big\rangle,$$

where $H(t)$ is the Heaviside function. Here we use the obvious relation

$$\sum_{k=1}^{n}\frac{\delta(Ge)}{\delta x_k}\nu_k = 0.$$

Using the last relation and formula (A.7), we calculate

$$\frac{\delta e}{\delta t} - 2\mathcal{K}Ge = \frac{\delta e}{\delta t} + \sum_{j=1}^{n}\frac{\delta \nu_j}{\delta x_j}Ge = \frac{\delta e}{\delta t} + \sum_{j=1}^{n}\frac{\delta(eG\nu_j)}{\delta x_j}.$$

□

A.4. **Transport theorems.** Here we give the following *transport theorems*.

Theorem A.1. ([21, 12.8.(3)], [2], [4], [5]) *Let $f(x,t)$ be a sufficiently smooth function defined in a moving solid Ω_t and let a moving hypersurface $\partial\Omega_t$ be its boundary. Let ν be the outward unit space normal to the surface $\partial\Omega_t$ and $W(x,t)$ be the velocity of the point x in Ω_t. Then the volume transport theorem holds:*

$$\frac{d}{dt}\int_{\Omega_t} f(x,t)\,dx = \int_{\Omega_t}\frac{\partial f}{\partial t}\,dx + \int_{\partial\Omega_t} fW\cdot\nu\,d\mu(x)$$
$$= \int_{\Omega_t}\Big(\frac{\partial f}{\partial t} + \operatorname{div}(fW)\Big)\,dx. \tag{A.15}$$

Theorem A.2. ([21, 12.8.(9)]) *If $e(x,t)$ is a smooth function defined only on the moving surface $\Gamma_t = \big\{x : S(x,t) = 0\big\}$ (which is the restriction of some smooth function defined in a neighborhood of Γ_t), then the surface transport theorem holds:*

$$\frac{d}{dt}\int_{\Gamma_t} e(x,t)\,d\mu(x)$$
$$= \int_{\Gamma_t}\Big(\frac{\delta e}{\delta t} - 2\mathcal{K}Ge\Big)\,d\mu(x) = \int_{\Gamma_t}\Big(\frac{\delta e}{\delta t} + \nabla_{\Gamma_t}\cdot(eU_\delta)\Big)\,d\mu(x), \tag{A.16}$$

where U_δ is the velocity of Γ_t.

Proof. Since according to definition (A.8),

$$m(t) = \int_{\Gamma_t} e(x,t)\,d\mu(x) = \big\langle e(x,t)\delta(S),\ 1\big\rangle_x,$$

using (A.12), we obtain

$$\dot m(t) = \Big\langle \frac{\partial}{\partial}\big(e(x,t)\delta(S)\big),\ 1\Big\rangle_x = \Big\langle \Big(\frac{\delta e}{\delta t} - 2\mathcal{K}Ge\Big)\delta(S) - Ge\partial_\nu\delta(S),\ 1\Big\rangle_x$$
$$= \Big\langle \Big(\frac{\delta e}{\delta t} - 2\mathcal{K}Ge\Big)\delta(S),\ 1\Big\rangle_x = \int_{\Gamma_t}\Big(\frac{\delta e}{\delta t} - 2\mathcal{K}Ge\Big)\,d\mu(x).$$

To complete the proof of the theorem, it remains to use the formulas (A.14) and (2.3). □

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