Singular solutions to systems of conservation laws: shocks, δ- and δ'-shocks

V. M. Shelkovich

ABSTRACT. Using the definitions of δ- and δ'-shocks for the systems of conservation laws [**12**], [**13**], [**39**], the Rankine–Hugoniot conditions for δ- and δ'-shocks are derived. We present a construction of solutions to the Cauchy problems admitting δ- and δ'-shocks. In particular, the Riemann problem admitting shocks, δ-shocks, δ'-shocks, and vacuum states is considered. The geometric aspects of δ- and δ'-shocks are studied. Balance relations connected with *area transportation*, in particular, *mass and momentum transportation* relations for the zero-pressure gas dynamics system, are derived. We also study the algebraic aspects of δ- and δ'-shocks. Namely, the flux-functions of δ- and δ'-shock solutions are computed. Though the flux-functions are nonlinear, they can be considered as *"right" singular superpositions of distributions* thus being well defined Schwartzian distributions. Therefore, singular solutions of the Cauchy problems generate algebraic relations between distributional components of these singular solutions. The validity and naturalness of the above-mentioned definitions of δ- and δ'-shocks are discussed.

CONTENTS

2000 *Mathematics Subject Classification.* Primary 35L65; Secondary 35L67, 76L05.

Key words and phrases. δ- and δ'-shock type solutions; the Rankine–Hugoniot conditions; balance relations connected with area transportation; flux-functions singularities.

The author was supported in part by DFG Project 436 RUS 113/823, DFG Project 436 RUS 113/895, and Grant 05-01-00912 of Russian Foundation for Basic Research.

[section]

1. Introduction

1.1. L^∞-type solutions. Let us recall some classical results. Consider the Cauchy problem for the system of conservation laws in one dimension space:

$$
\left\{\begin{array}{rcll} U_t+\big(F(U)\big)_x &=& 0, & \text{in}\, \mathbb{R}\times(0,\ \infty), \\ U &=& U^0, & \text{in}\, \mathbb{R}\times\{t=0\}, \end{array}\right. \tag{1.1}
$$

where $F:\mathbb{R}^m\to\mathbb{R}^m$ is called the *flux-function* associated with (1.1); $U^0:\mathbb{R}\to\mathbb{R}^m$ are given smooth vector-functions; $U=U(x,t)=(u_1(x,t),\dots,u_m(x,t))$ is the unknown function, $x\in\mathbb{R}$, $t\geq 0$.

As is well known, even in the case of smooth (and, certainly, in the case of discontinuous) initial data $U^0(x)$, we cannot in general find a smooth solution of (1.1). Quoting from Evans' book, "the great difficulty in this subject is discovering a proper notion of weak solution for the initial problem (1.1)" [**16**, 11.1.1.]. "We must devise some way to interpret a less regular function U as somehow "solving" this initial-value problem. But as it stands, the PDE does not even make sense unless U is differentiable. However, observe that if we *temporarily* assume U is smooth, we can as follows rewrite, so that the resulting expression does not directly involve the derivatives of U. The idea is to multiply the PDE in (1.1) by a smooth function φ and then to integrate by parts, thereby transferring the derivatives onto φ" [**16**, 3.4.1.a.]. According to the above reasoning, it is said that $U\in L^\infty\big(\mathbb{R}\times(0,\infty);\mathbb{R}^m\big)$ is a *generalized solution* of the Cauchy problem (1.1) if the integral identities

$$
\int_0^\infty\int\Big(U\cdot\widetilde{\varphi}_t+F(U)\cdot\widetilde{\varphi}_x\Big)\,dx\,dt+\int U^0(x)\cdot\widetilde{\varphi}(x,0)\,dx=0 \tag{1.2}
$$

hold for all compactly supported test vector-functions $\widetilde{\varphi}:\mathbb{R}\times[0,\infty)\to\mathbb{R}^m$, where $\cdot$ is the scalar product of vectors, and $\int f(x)\,dx$ denotes the improper integral $\int_{-\infty}^{\infty}f(x)\,dx$.

We would like to add to the aforesaid that the notion of a weak solution is to be such that one could use it to obtain the Rankine–Hugoniot conditions. Namely, the following classical theorem holds.

THEOREM 1.1. (see, e.g., [**16**, 11.1.1.]) *Let $\Omega\subset\mathbb{R}\times(0,\infty)$ be a region cut by a smooth curve Γ into a left- and right-hand parts $\Omega_\mp$. Let us assume that the integral solution U of* (1.1) *is smooth on either side of the curve Γ*

along which U has simple jump discontinuities. Then the Rankine–Hugoniot conditions

$$\big[F(U)\big]_{\Gamma}\nu_1 + \big[U\big]_{\Gamma}\nu_2 = 0, \tag{1.3}$$

hold along Γ, where $\mathbf{n} = (\nu_1, \nu_2)$ is the unit normal to the curve Γ pointing from Ω_- into Ω_+,

$$[F(U)] = F(U_-) - F(U_+)$$

and $[U] = U_- - U_+$ are jumps in $F(U)$ and in U across the discontinuity curve Γ, respectively. $U_\mp$ are respective left- and right-hand values of U on Γ.

If $\Gamma = \{(x,t) : x = \phi(t)\}$, where $\phi(t) \in C^1(0,+\infty)$, then

$$\mathbf{n} = (\nu_1, \nu_2) = \frac{1}{\sqrt{1 + (\dot{\phi}_i(t))^2}}\big(1, -\dot{\phi}_i(t)\big), \tag{1.4}$$

and (1.3) reads

$$\big[F(U)\big]_{\Gamma} = \dot{\phi}(t)\big[U\big]_{\Gamma}, \tag{1.5}$$

where $\dot{(\cdot)} = \frac{d}{dt}(\cdot)$.

It is well known that if $U \in L^\infty\big(\mathbb{R} \times (0,\infty); \mathbb{R}^m\big)$ is a generalized solution of the Cauchy problem (1.1) compactly supported with respect to x, then the integral of the solution on the whole space

$$\int U(x,t)\, dx = \int U^0(x)\, dx, \qquad t \ge 0 \tag{1.6}$$

is independent of time. These integrals can express the conservation laws of *total area, mass, momentum, energy*, etc.

1.2. δ-Shock wave type solutions. It is well known (see [**2**]– [**5**], [**10**] – [**19**], [**23**], [**27**], [**28**], [**44**], [**48**]– [**55**], [**57**] and the references therein) that there are "nonclassical" situations where, in contrast to Lax's and Glimm's results, the Cauchy problem for a system of conservation laws does not possess a weak L^∞-solution except for some particular initial data. In order to solve the Cauchy problem in this "nonclassical“ situation, it is necessary to introduce new singular solutions called *δ-shocks and singular shocks.* Roughly speaking, a *δ-shock* (*singular shock*) is a solution such that its components *contain Dirac measures* (see (1.13) and (4.1) below). The theory of δ-shocks singular shocks has been intensively developed in the last ten years.

As far as we know, all one-dimensional systems of conservation laws admitting δ-shocks are particular cases of systems:

$$L_1[u,v] = u_t + \big(F(u,v)\big)_x = 0, \quad L_2[u,v] = v_t + \big(G(u,v)\big)_x = 0, \tag{1.7}$$

and

$$L_1[u,v] = v_t + \big(G(u,v)\big)_x = 0, \quad L_2[u,v] = (uv)_t + \big(H(u,v)\big)_x = 0, \tag{1.8}$$

where $F(u,v)$, $G(u,v)$, $H(u,v)$ are smooth functions, *linear* with respect to v; $u = u(x,t),\ v = v(x,t) \in \mathbb{R};\ \ x \in \mathbb{R}$.

In particular, the δ-shock in the system

$$u_t + \big(f(u)\big)_x = 0, \quad v_t + \big(g(u)v\big)_x = 0, \tag{1.9}$$

(here $F(u,v) = f(u)$, $G(u,v) = vg(u)$) was studied in [**19**], [**15**], [**28**]. In numerous papers (see [**2**]– [**5**], [**12**], [**14**], [**44**], [**49**], [**57**]) the δ-shock in *zero-pressure gas dynamics*

$$v_t + \big(vu\big)_x = 0, \quad (vu)_t + \big(vu^2\big)_x = 0, \tag{1.10}$$

(here $G(u,v) = uv$, $H(u,v) = u^2v$) was studied. Here $v(x,t) \geq 0$ is density, and $u(x,t)$ is velocity. The models of "zero-pressure gas dynamics" were used for describing the formation of large-scale structures of the universe [**47**], [**58**] (see also [**1**]).

Several approaches to solving δ-shock problems are known (for details, see the review in [**2**, 1.1.], the above cited papers and the references therein). One of them was proposed in [**7**]– [**13**], [**48**] – [**50**]. In these papers the *weak asymptotics method*) for studying the *dynamics of propagation and interaction* of different singularities of quasi-linear differential equations and systems of conservation laws was developed. In [**11**]– [**13**], [**49**], in the framework of the *weak asymptotics method* Definitions 2.1, 2.2 of δ-shock wave type solutions by integral identities were introduced for two classes of systems of conservation laws (1.7) and (1.8). Using the *weak asymptotics method*, in [**2**], [**10**] – [**13**], [**48**]– [**50**], for some cases of systems (1.7), (1.8) with the initial data

$$\begin{aligned} u^0(x) &= u^0_+(x) + [u^0(x)]H(-x), \\ v^0(x) &= v^0_+(x) + [v^0(x)]H(-x) + e^0\delta(-x), \end{aligned} \tag{1.11}$$

the Cauchy problems were solved, where $[u^0] = u^0_- - u^0_+$, and $u^0_\pm(x)$, $v^0_\pm(x)$ are given smooth functions, e^0 is a given constant, $H(x)$ is the Heaviside function, $\delta(x)$ is the delta-function. It was observed in [**12**], that for the case of system (1.8) in addition to the initial data (1.11) we must consider the *initial velocity* of singularity

$$\dot\phi(0) = \phi^0, \tag{1.12}$$

where ϕ^0 is a given constant (see Definition 2.2 below).

According to [**10**]– [**13**], [**48**]– [**50**], the δ-shock wave type solutions of these Cauchy problems have the form

$$\begin{aligned} u(x,t) &= u_+(x,t) + [u(x,t)]H(-x+\phi(t)), \\ v(x,t) &= v_+(x,t) + [v(x,t)]H(-x+\phi(t)) + e(t)\delta(-x+\phi(t)), \end{aligned} \tag{1.13}$$

where $u_\pm(x,t)$, $v_\pm(x,t)$, $e(t)$, $\phi(t)$ are the desired functions, $x = \phi(t)$ is the discontinuity curve.

Remind that a *singular shock* was considered in [**22**]–[**25**], [**27**], [**45**], [**46**]. According to these papers, a model system admitting a *singular shock* is the well-known Keyfitz–Kranzer system

$$u_t + (u^2 - v)_x = 0, \quad v_t + \Big(\frac{1}{3}u^3 - u\Big)_x = 0 \tag{1.14}$$

(here $F(u,v) = u^2 - v$, $G(u,v) = \frac{1}{3}u^3 - u$), which was studied in [**23**], [**27**]. In the excellent paper [**23**], in order to construct *approximate solutions*, the Colombeau theory approach as well as the Dafermos–DiPerna regularization (under the assumption that Dafermos profiles exist) and the box approximations are used. However the notion of a *singular solution* has *not been defined*. Later, in [**43**], the existence of Dafermos profiles for singular shocks was proved. However it is not clear in which sense a singular shock satisfies the system (1.14). In the framework of the weak asymptotics method, in [**48**], it was first proved that the Cauchy problem (1.14), (1.11) admits an exact δ-shock solution (1.13) in the sense of Definition 2.1. Moreover, in [**52**], it was shown that both *singular shocks* and *δ-shocks are solutions of the same type* in the sense of Definition 2.1 (for details, see Sec. 4).

In [**48**], [**50**] (see also [**2**]), an exact δ-shock solution (1.13) of the Cauchy problem for the Keyfitz–Kranzer type system

$$u_t + \big(f(u) - v\big)_x = 0, \quad v_t + \big(g(u)\big)_x = 0, \tag{1.15}$$

with the initial data (1.11) was constructed, where $f(u)$ and $g(u)$ are polynomials of degree n and $n+1$, respectively, n is even (here $F(u,v) = f(u) - v$, $G(u,v) = g(u)$).

1.3. $\delta^{(n)}$-Shock wave type solutions, $n = 1, 2, \dots$.

In [**39**], a concept of $\delta^{(n)}$-shock wave type solutions was introduced, where $\delta^{(n)}$ is n-th derivative of the Dirac delta function, $n = 1, 2, \dots$. It is a *new type of singular solution* such that its components contain delta functions and their derivatives (for the exact structure of a δ'-shock see (1.19) below). In [**39**], [**41**] (for the short review see [**40**]), the theory of *δ'-shocks* was established. Definition 5.1 of a *δ'-shock wave type solution* for the system of conservation laws

$$\begin{aligned} L_1[u] = u_t + \big(f(u)\big)_x &= 0, \\ L_2[u,v] = v_t + \big(f'(u)v\big)_x &= 0, \\ L_3[u,v,w] = w_t + \big(f''(u)v^2 + f'(u)w\big)_x &= 0, \end{aligned} \tag{1.16}$$

was introduced, where $f(u)$ is a smooth function, $f''(u) > 0$, $u = u(x,t), v = v(x,t), w = w(x,t) \in \mathbb{R}$, $x \in \mathbb{R}$.

In [**39**], by using the *weak asymptotics method*, a δ'-shock wave type solution to the Cauchy problem of the system of conservation laws

$$u_t + \big(u^2\big)_x = 0, \quad v_t + 2\big(uv\big)_x = 0, \quad w_t + 2\big(v^2 + uw\big)_x = 0 \tag{1.17}$$

with the singular initial data

$$\begin{aligned} u^0(x) &= u^0_+(x) + [u^0(x)]H(-x), \\ v^0(x) &= v^0_+(x) + [v^0(x)]H(-x) + e^0\delta(-x), \\ w^0(x) &= w^0_+(x) + [w^0(x)]H(-x) + g^0\delta(-x) + h^0\delta'(-x), \end{aligned} \tag{1.18}$$

was constructed, where $u^0_\pm(x)$, $v^0_\pm(x)$, $w^0_\pm(x)$, are given smooth functions; e^0, g^0, h^0 are given constants, $\delta'(x)$ is the derivative of the delta function. This solution has the form

$$
\begin{aligned}
u(x,t) &= u_{+}(x,t) + [u(x,t)]H(-x+\phi(t)), \\
v(x,t) &= v_{+}(x,t) + [v(x,t)]H(-x+\phi(t)) + e(t)\delta(-x+\phi(t)), \\
w(x,t) &= w_{+}(x,t) + [w(x,t)]H(-x+\phi(t)) + g(t)\delta(-x+\phi(t)) \\
&\qquad + h(t)\delta'(-x+\phi(t))
\end{aligned}
\tag{1.19}
$$

where $u_{\pm}(x,t)$, $v_{\pm}(x,t)$, $w_{\pm}(x,t)$, $\phi(t)$, $e(t)$, $g(t)$, $h(t)$ are the desired functions.

In [**51**], using the *vanishing viscosity method*, a solution of the Riemann problem for the system (1.17) was constructed. This problem admits *δ-, δ'-shock wave type solutions and vacuum states.*

1.4. Resume. In order to deal with singular solutions like δ- and $\delta^{(n)}$-shocks, $n = 1, 2, \dots$, we need

• to discover a *proper notion of a singular solution* and to define *in which sense* it may satisfy a nonlinear system;

• to devise some way to define a *singular superposition (product)* of distributions (for example, a product of the Heaviside function and the delta function).

Unfortunately, using the above cited instruction from the Evans' book [**16**, 3.4.1.a.], δ- and δ'-shock wave type solutions *cannot be defined*. Indeed, as can be seen from (1.7), (1.8), (1.10), (1.14), (1.15), and (1.16), (1.17), if by integrating by parts we transfer the derivatives onto a test function φ, under the integral sign there still remain nonlinear terms $F(u,v)$, $G(u,v)$, $H(u,v)$, and $f'(u)v$, $f''(u)v^2 + f'(u)w$ *undefined in the distributional sense*, since the component v may contain Dirac measures, while the component w may contain the Dirac measures and their derivatives.

Fortunately, it appears that the *weak asymptotics method* is a proper technique to deal with δ- and δ'-shocks. Definitions 2.1, 2.2, and 5.1 derived in the framework of this method give *natural* generalizations of the classical definition of the weak L^∞-solutions (1.2) relevant for the structure of δ- and δ'-shocks. If a solution of the Cauchy problems contains no δ and δ'-terms then these definitions coincide with the classical definition (1.2). In [**2**], [**10**]–[**13**], [**39**], [**40**], [**48**]– [**51**], by using this technique, some Cauchy problems admitting δ- and δ'-shocks were solved. As far as we know, some problems related to δ- and δ'-shocks can be solved only by using the *weak asymptotics method*.

In the δ- and δ'-shock theories there are many open and complicated problems. The study of this area gives a new perspective in the theory of conservation law systems. In particular, the results [**39**], [**40**], [**51**] on δ'-shocks show that systems of conservation laws can develop not only Dirac measures (as in the case of δ-shocks) but their derivatives as well.

1.5. Main results. In Sec. 2, definitions of δ-shocks for system (1.7) and (1.8) are introduced (Definitions 2.1 and 2.2) and corresponding Rankine–Hugoniot conditions (2.7) and (2.13) are derived. As far as we know, all one-dimensional systems of conservation laws admitting δ-shocks, are particular cases of systems (1.7) and (1.8). In Sec. 3, we present a construction of the Cauchy problem admitting δ-shocks. Namely, in Subsec. 3.1, a notion of *weak asymptotic solution* is introduced, which is one of the most important in the *weak asymptotics method*. In the framework of the *weak asymptotics method*, we find a δ-shock wave type solution of the Cauchy problem as a weak limit (3.2) of the corresponding *weak asymptotic solutions* to this Cauchy problem. In Subsec. 3.2– 3.5, in Theorems 3.2– 3.5, solutions of the Cauchy problems (1.9), (1.11); (3.23), (1.11); and (1.10), (1.11), (1.12) are constructed. The Keyfitz–Kranzer system (1.14) and its generalization (1.15) are a particular case of system (3.23). In Sec. 4, a complicated problem related to the concept of *a singular shock* is considered. We prove that both *singular shock* (4.1 and δ*-shock* (1.13) *are solutions of the same type* in a sense of Definition 2.1.

In Sec. 5, a definition of δ'-shock for system (1.16) is introduced (Definition 5.1), and the corresponding Rankine–Hugoniot conditions (5.3)–(5.6) are derived. In Sec. 6, we present a construction of the Cauchy problems admitting δ'-shocks. In Subsec. 6.1, a notion of *a weak asymptotic solution* is introduced, and in Subsec. 6.2, Theorem 6.1, which give solutions of the Cauchy problem (1.17), (1.18) is proved. In Sec. 7, we present the results from [**51**]. In this paper the Riemann problem (1.17), (7.1) admitting shocks, δ-shocks, δ'-shocks, and vacuum states is considered. To solve this problem, the *vanishing viscosity method* was used. In fact, we also describe the formation of the δ'-shocks and the vacuum states from smooth solutions of the corresponding parabolic problem (7.2), (7.1).

In Sec. 8, geometric and physical aspects of δ- and δ'-shocks are studied. If $U \in L^\infty$ is a generalized solution of the Cauchy problem compactly supported with respect to x, then the conservation laws (1.6) hold. For δ- and δ'-shock wave type solutions this fact *does not hold*. Nevertheless, by Theorems 8.1– 8.3 "generalized" analogs of these conservation laws are derived. More precisely, the δ- and δ'-shock *balance relations* connected with *area transportation* are derived. In particular, we derive *mass and momentum transportation* relations for the zero-pressure gas dynamics system (1.10). According to our results, in zero-pressure gas dynamics *mass transfer* to the discontinuity curve $x = \phi(t)$ takes place.

In Subsec. 9.2, the algebraic aspects of δ- and δ'-shocks in systems (1.7), (1.8), (1.16) are studied. It is well known that in the general case, the product of distributions is either not a Schwartz distribution or it is a Schwartz distribution not uniquely defined. Nevertheless, we show that singular solutions of the Cauchy problems *generate algebraic relations* between their distributional components. If a system admits δ- or δ'-shock, then by using the *weak asymptotic solution* we can calculate flux-functions of δ- and δ'-shock solutions (see

Theorems 9.1– 9.4). As it follows from the proofs of Theorems 9.1– 9.4, it is the linear terms in systems (1.7), (1.8), (1.16) that determine corresponding flux-functions. Though the flux-functions are nonlinear, they can be considered as *"right" singular superpositions of distributions* and are *well defined Schwartzian distributions*. Thus a *"right" singular superposition* is determined only in the context of solving the Cauchy problem. Note that in our paper [**26**] the flux-functions of δ-shocks for system (1.7) with piecewise constant initial data were calculated. According to Theorems 9.1– 9.4 flux-function may be *very singular* and contain *δ-functions and their derivatives*.

In Subsec. 9.3, to illustrate the specific properties of the *"right" singular superpositions* we compare these superpositions for the Cauchy problem (1.9), (1.11) with those for the Cauchy problems (1.14), (1.11) and (1.15), (1.11). In fact, the first singular superpositions (9.30), (9.31), (9.36) can be reduced to the *unique product of the step function and the delta function*. The second singular superpositions (9.32), (9.33) and (9.34), (9.35) have "strange" specific properties. It is clear that "strange" singular superpositions (9.34), (9.35) for the Keyfitz–Kranzer system (1.14) can be calculated by using Dafermos profiles from [**43**] and the vanishing viscosity approach. Note that the Keyfitz–Kranzer system is an excellent model example which shows that δ-shocks constitute the universe with specific and "strange" properties. One of these "strange" properties is the fact that although the both terms $\lim_{\varepsilon\to+0} f''(u_\varepsilon)v_\varepsilon^2$ and $\lim_{\varepsilon\to+0} f'(u_\varepsilon)w_\varepsilon$ in (9.26) are *unbounded*, the right-hand side of (9.26) is a *well defined distribution*.

It remains to note that according to (9.32), (9.33); (9.34), (9.35), and (9.26), it is *impossible* to construct a δ-shock wave type solution for systems (1.14), (1.15) and a δ'-shock wave type solution for the system (1.16) by using the *nonconservative product* [**28**], [**29**], [**6**] (see Subsec. 9.1) (the above *singular superpositions* can not be reduced to terms of the form (9.1)) as well as the *measure-valued solutions approach* [**3**], [**55**], [**57**]. However, these approaches can be used for the case of the Cauchy problem (1.9), (1.11) (see formulas (9.30), (9.31, (9.36) and the paper [**28**], [**55**]).

In Sec. 10, using some passages from proofs of Theorems 9.1– 9.4, we discuss and substantiate the validity and naturalness of the above-mentioned definitions of δ- and δ'-shocks (see [**53**]). As these definitions differ from the classical definition of a weak L^∞-solution, this problem is important.

2. δ-Shock type solutions and the Rankine–Hugoniot conditions

2.1. The case of system (1.7). Suppose that $\Gamma = \{\gamma_i : i \in I\}$ is a graph in the upper half-plane $\{(x,t) : x \in \mathbb{R},\ t \in [0,\infty)\} \in \mathbb{R}^2$ containing smooth arcs $\gamma_i = \{(x,t) : S_i(x,t) = 0\}$, $S_i \in C^1$, $S_{i\,x} \neq 0$, $i \in I$, and I is a finite set (see Fig. 1.). By I_0 we denote a subset of I such that an arc γ_k for $k \in I_0$ starts from points of the x-axis. Denote by $\Gamma_0 = \{x_k^0 : k \in I_0\}$ the set of initial points of arcs γ_k, $k \in I_0$. Here arcs of a graph have the orientation corresponding to increasing time t.

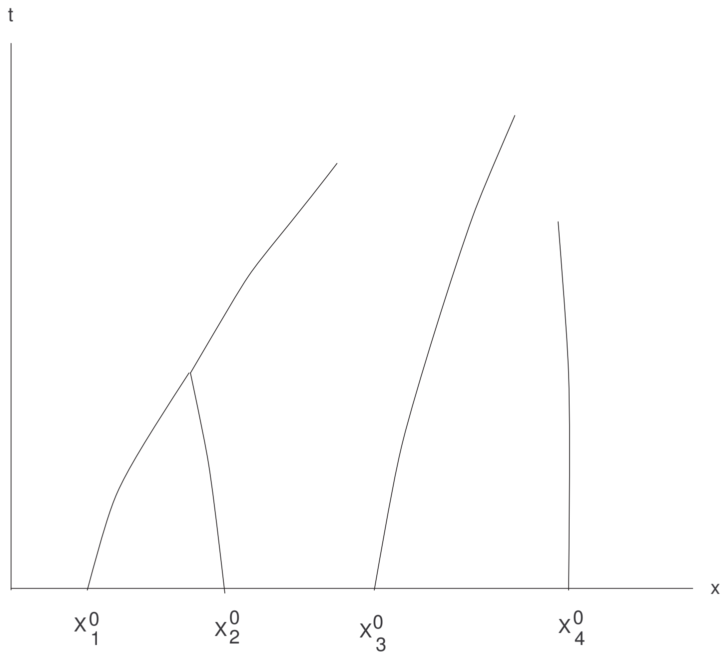

FIGURE 1. Graph Γ in the upper half-plane.

Consider the *δ-shock type initial data*

$$(u^0(x), v^0(x)), \quad \text{where} \quad v^0(x) = \widehat{v}^0(x) + e^0\delta(\Gamma_0), \tag{2.1}$$

$u^0, \widehat{v}^0 \in L^\infty\big(\mathbb{R};\mathbb{R}\big)$, $e^0\delta(\Gamma_0) \stackrel{def}{=} \sum_{k\in I_0} e_k^0\delta(x-x_k^0)$, e_k^0 are constants, $k \in I_0$.

Let us introduce the definition of a generalized *δ-shock wave type solution* for system (1.7).

DEFINITION 2.1. ([**11**]– [**13**]) A pair of distributions (u,v) and a graph Γ, where $v(x,t)$ has the form of the sum

$$v(x,t) = \widehat{v}(x,t) + e(x,t)\delta(\Gamma),$$

$u, \widehat{v} \in L^\infty\big(\mathbb{R}\times(0,\infty);\mathbb{R}\big)$, $e(x,t)\delta(\Gamma) \stackrel{def}{=} \sum_{i\in I} e_i(x,t)\delta(\gamma_i)$, $e_i(x,t) \in C(\Gamma)$, $i \in I$, is called a *δ-shock wave type solution* of the Cauchy problem (1.7), (2.1) if the integral identities

$$\begin{aligned}
\int_0^\infty \int \Big(u\varphi_t + F(u,\widehat{v})\varphi_x\Big)\,dx\,dt + \int u^0(x)\varphi(x,0)\,dx &= 0, \\
\int_0^\infty \int \Big(\widehat{v}\varphi_t + G(u,\widehat{v})\varphi_x\Big)\,dx\,dt + \sum_{i\in I}\int_{\gamma_i} e_i(x,t)\frac{\delta\varphi(x,t)}{\delta t}\frac{dl}{\sqrt{1+u_\delta^2}} & \\
+ \int \widehat{v}^0(x)\varphi(x,0)\,dx + \sum_{k\in I_0} e_k^0\varphi(x_k^0,0) &= 0,
\end{aligned} \tag{2.2}$$

hold for all test functions $\varphi(x,t) \in \mathcal{D}(\mathbb{R} \times [0,\infty))$, where

$$\frac{\delta\varphi}{\delta t}\Big|_{\gamma_i} = \left(\frac{\partial\varphi}{\partial t} - \frac{S_{i\,t}}{S_{i\,x}}\frac{\partial\varphi}{\partial x}\right)\Big|_{S_i(x,t)=0} \tag{2.3}$$

is a δ-*derivative* with respect to time [**21**, 5.2.(15)], which is the tangential derivative on the graph γ_i;

$$u_\delta(x,t)\big|_{\gamma_i} = -\frac{S_{i\,t}}{S_{i\,x}}\Big|_{\gamma_i}, \qquad i \in I, \tag{2.4}$$

is the velocity of a δ-shock on γ_i; $\int_{\gamma_i} \cdot \, dl$ is the line integral over the arc γ_i. Here the delta function $\delta(\gamma_i)$ on the curve γ_i is defined as in [**17**, ch.III,§1.3.], [**21**, 5.3.].

Suppose that the arcs of the graph $\Gamma = \{\gamma_i : i \in I\}$ have the form $\gamma_i = \{(x,t) : x = \phi_i\}$, $\phi_i(t) \in C^1(0,+\infty)$, $i \in I$. In this case

$$\frac{\delta\varphi}{\delta t}\Big|_{\gamma_i} = \sqrt{1+(\dot\phi_i(t))^2}\,\frac{\partial\varphi(x,t)}{\partial\mathbf{l}}\Big|_{\gamma_i} = \frac{d\varphi(\phi_i(t),t)}{dt}, \tag{2.5}$$

where $\frac{\partial\varphi}{\partial\mathbf{l}}$ is the tangential derivative on the graph γ_i along the unit vector $\mathbf{l} = (-\nu_2,\nu_1) = \frac{(\dot\phi_i(t),1)}{\sqrt{1+(\dot\phi_i(t))^2}}$, $\mathbf{n} = (\nu_1,\nu_2)$ is the unit oriented normal (1.4) to γ_i.

The integral identities (2.2) differ from the integral identities (1.2) (for $m = 2$) by the additional terms

$$\sum_{i\in I}\int_{\gamma_i} e_i(x,t)\frac{\delta\varphi(x,t)}{\delta t}\frac{dl}{\sqrt{1+u_\delta^2}}$$

in the second identity. This term appears due to the delta function in v.

Now, by using Definition 2.1, we derive the *δ-shock Rankine–Hugoniot conditions* for system (1.7).

THEOREM 2.1. ([**2**], [**49**], [**50**]) *Let us assume that $\Omega \subset \mathbb{R} \times (0,\infty)$ is a region cut by a smooth curve $\Gamma = \{(x,t) : S(x,t) = 0\}$ into the left- and right-hand parts $\Omega_\mp$. Let (u,v), Γ be a δ-shock wave type solution of system* (1.7), *and suppose that u, v are smooth in $\Omega_\pm$ and have one-sided limits $u_\pm$, $v_\pm$ on Γ. Then the* Rankine–Hugoniot conditions for the δ-shock

$$u_\delta = \frac{[F(u,v)]}{[u]}\Big|_\Gamma, \quad \frac{\delta e(x,t)}{\delta t}\Big|_\Gamma = \Big([G(u,v)]_\Gamma - [v]_\Gamma u_\delta\Big)\frac{S_x}{|S_x|}, \tag{2.6}$$

where $u_\delta(x,t)$ is the velocity (2.4) *of a δ-shock, $[a(u,v)] = a(u_-,v_-) - a(u_+,v_+)$ is, as usual, a jump of the function $a(u(x,t),v(x,t))$ across the discontinuity curve Γ.*

If $\Gamma = \{(x,t) : x = \phi(t)\}$, $\phi(t) \in C^1(0,+\infty)$, $\Omega_\pm = \{(x,t) : \pm(x-\phi(t)) > 0\}$, then the relations (2.6) *read*

$$\dot\phi(t) = \frac{[F(u,v)]}{[u]}\Big|_{x=\phi(t)}, \quad \dot e(t) = \Big([G(u,v)] - [v]\frac{[F(u,v)]}{[u]}\Big)\Big|_{x=\phi(t)}, \tag{2.7}$$

where e can be treated as a function of the single variable t, so that $e(t) \stackrel{def}{=} e(\phi(t), t)$.

Proof. Let $\mathbf{n} = (\nu_1, \nu_2) = \frac{(S_x, S_t)}{|\nabla_{(x,t)} S|}$ be a unit normal to the curve Γ oriented from Ω_- to Ω_+, $\mathbf{l} = (-\nu_2, \nu_1) = \frac{(-S_t, S_x)}{|\nabla_{(x,t)} S|}$ be a unit tangential vector to Γ, $\nabla_{(x,t)} S_i = (S_{ix}, S_{it})$.

For any test function $\varphi \in \mathcal{D}(\Omega)$ we have $\varphi(x,t) = 0$, if $(x,t) \notin G$, $\overline{G} \subset \Omega$. Selecting the test function $\varphi(x,t)$ with compact support in $\Omega_\pm$, we deduce from (2.2) that (1.7) hold in $\Omega_\pm$, respectively. Now, choosing the test function $\varphi(x,t)$ with support in Ω, we deduce from the first identity (2.2) that

$$\int_0^\infty \int \Big(u\varphi_t + F(u, \widehat{v})\varphi_x \Big)\, dx\, dt$$
$$= \int \int_{\Omega_- \cap G} \Big(u\varphi_t + F(u, \widehat{v})\varphi_x \Big)\, dx\, dt + \int \int_{\Omega_+ \cap G} \Big(u\varphi_t + F(u, \widehat{v})\varphi_x \Big)\, dx\, dt.$$

Since $u_t + \big(F(u,v)\big)_x = 0$ for $(x,t) \in \Omega_\pm$, integrating by parts, we obtain

$$\int \int_{\Omega_\pm \cap G} \Big(u\varphi_t + F(u, \widehat{v})\varphi_x \Big)\, dx\, dt = - \int \int_{\Omega_\pm \cap G} \Big(u_t + \big(F(u, \widehat{v})\big)_x \Big) \varphi\, dx\, dt$$
$$\mp \int_\Gamma \Big(\nu_2 u_\pm + \nu_1 F(u_\pm, v_\pm) \Big) \varphi\, dl - \int_{\Omega_\pm \cap G \cap \mathbb{R}} u^0(x) \varphi(x,0)\, dx$$
$$= \mp \int_\Gamma \Big(\nu_2 u_\pm + \nu_1 F(u_\pm, v_\pm) \Big) \varphi\, dl - \int_{\Omega_\pm \cap G \cap \mathbb{R}} u^0(x) \varphi(x,0)\, dx.$$

Adding the latter relations, we have

$$\int_0^\infty \int \Big(u\varphi_t + F(u, \widehat{v})\varphi_x \Big)\, dx\, dt + \int u^0(x)\varphi(x,0)\, dx$$
$$= \int_\Gamma \Big([F(u,v)]\nu_1 + [u]\nu_2 \Big) \varphi(x,t)\, dl = 0$$

for all $\varphi(x,t) \in \mathcal{D}(\Omega)$. This implies the first relation in (2.6).

In the same way as above, we obtain

$$\int_0^\infty \int \Big(\widehat{v}\varphi_t + G(u, \widehat{v})\varphi_x \Big)\, dx\, dt + \int \widehat{v}^0(x)\varphi(x,0)\, dx$$
$$= \int_\Gamma \Big([G(u,v)]\nu_1 + [v]\nu_2 \Big) \varphi(x,t)\, dl. \tag{2.8}$$

Next, integrating by parts, it is easy to see that

$$\sum_{i \in I} \int_{\gamma_i} e_i(x,t) \frac{\delta \varphi(x,t)}{\delta t} \frac{dl}{\sqrt{1 + u_\delta^2}}$$
$$= - \sum_{i \in I} \int_{\gamma_i} \frac{\delta e_i(x,t)}{\delta t} \varphi(x,t) \frac{dl}{\sqrt{1 + u_\delta^2}} - \sum_{k \in I_0} e^0 \varphi(x,0)\big|_{S_k(x,0)=0}, \tag{2.9}$$

where the δ-derivative $\frac{\delta\varphi}{\delta t}$ is defined in (2.3). Adding (2.8) and (2.9), we obtain

$$\int_{\Gamma}\Big([G(u,v)]\nu_1+[v]\nu_2-\frac{\delta e(x,t)}{\delta t}\frac{1}{\sqrt{1+u_\delta^2}}\Big)\varphi(x,t)\,dl=0,$$

for any $\varphi(x,t)\in\mathcal{D}(\Omega)$.

Thus the second relation in (2.6) holds.

If $\Gamma=\{(x,t)\,:\,x=\phi(t)\}$, $\phi(t)\in C^1(0,+\infty)$, in view of (1.4), (2.5), condition (2.6) can be rewritten as (2.7). $\square$

The first equation in (2.6) (or (2.7)) is the *standard* Rankine–Hugoniot condition (cf. (1.3) or (1.5)). The left-hand side of the second equation in (2.6) (or the right-hand side of the second equation in (2.7)) is called the *Rankine–Hugoniot deficit* in v.

2.2. The case of system (1.8). Let Γ be a graph introduced in Subsec. 2.1. For system (1.8) the *δ-shock type initial data* have the form

$$\big(u^0(x),v^0(x);\,u_\delta^0(x_k^0),\,k\in I_0\big),\qquad v^0(x)=\widehat{v}^0(x)+e^0\delta(\Gamma_0)\tag{2.10}$$

where $u^0,\widehat{v}^0\in L^\infty\big(\mathbb{R};\mathbb{R}\big)$, $u_\delta^0(x_k^0)$ is the *initial velocity* of δ-shock at the point x_k^0, and $e^0\delta(\Gamma_0)\stackrel{def}{=}\sum_{k\in I_0}e_k^0\delta(x-x_k^0)$, e_k^0 is a constant, $k\in I_0$.

DEFINITION 2.2. ([**2**]) A pair of distributions (u,v) and a graph Γ from Definition 2.1 is called a *δ-shock wave type solution* of the Cauchy problem (1.8), (2.10) if the integral identities

$$\begin{aligned}
&\int_0^\infty\int\Big(\widehat{v}\varphi_t+G(u,\widehat{v})\varphi_x\Big)\,dx\,dt+\sum_{i\in I}\int_{\gamma_i}e_i(x,t)\frac{\delta\varphi(x,t)}{\delta t}\frac{dl}{\sqrt{1+u_\delta^2}}\\
&\qquad\qquad+\int\widehat{v}^0(x)\varphi(x,0)\,dx+\sum_{k\in I_0}e_k^0\varphi(x_k^0,0)\;=\;0,\\
&\int_0^\infty\int\Big(u\widehat{v}\varphi_t+H(u,\widehat{v})\varphi_x\Big)\,dx\,dt\\
&\qquad\qquad+\sum_{i\in I}\int_{\gamma_i}e_i(x,t)u_\delta(x,t)\frac{\delta\varphi(x,t)}{\delta t}\frac{dl}{\sqrt{1+u_\delta^2}}\\
&\qquad\qquad+\int u^0(x)\widehat{v}^0(x)\varphi(x,0)\,dx+\sum_{k\in I_0}e_k^0u_\delta^0(x_k^0)\varphi(x_k^0,0)\;=\;0,
\end{aligned}\tag{2.11}$$

hold for all $\varphi(x,t)\in\mathcal{D}(\mathbb{R}\times[0,\,\infty))$, where $u_\delta(x,t)$ is the velocity of the δ-shock (2.4).

A definition of this type was first introduced for the zero-pressure gas dynamics system (1.10) in [**12**].

The first integral identity in (2.11) coincides with the second one in (2.2). The second integral identity in (2.11) differs from the integral identities (1.2) by the additional terms $\sum_{i\in I}\int_{\gamma_i} e_i(x,t)u_\delta(x,t)\frac{\delta\varphi(x,t)}{\delta t}\frac{dl}{\sqrt{1+u_\delta^2}}$.

Now, by using Definition 2.2, similarly to Theorem 2.1, one can derive the *δ-shock Rankine–Hugoniot conditions* for system (1.8).

THEOREM 2.2. ([**2**]) *Let us assume that $\Omega \subset \mathbb{R}\times(0,\ \infty)$ is some region cut by a curve $\Gamma = \{(x,t) : S(x,t) = 0\}$, into left- and right-hand parts $\Omega_\mp$. Let (u,v) and Γ be a generalized δ-shock wave type solution of system* (1.8) *and suppose that u,v are smooth in $\Omega_\pm$ and have one-sided limits $u_\pm$, $v_\pm$, on Γ. Then the* Rankine–Hugoniot conditions for the δ-shock

$$(2.12)\qquad \begin{aligned} \frac{\delta e(x,t)}{\delta t}\Big|_\Gamma &= \Big(\big[G(u,v)\big]_\Gamma - \big[v\big]_\Gamma u_\delta\Big)\frac{S_x}{|S_x|}, \\ \frac{\delta\big(e(x,t)u_\delta(x,t)\big)}{\delta t}\Big|_\Gamma &= \Big(\big[H(u,v)\big]_\Gamma - \big[v\big]_\Gamma u_\delta\Big)\frac{S_x}{|S_x|} \end{aligned}$$

hold along Γ.

If $\Gamma = \{(x,t) : x = \phi(t)\}$, $\phi(t)\in C^1(0,+\infty)$, $\Omega_\pm = \{(x,t) : \pm(x-\phi(t)) > 0\}$, then the relations (2.12) *read*

$$(2.13)\qquad \begin{aligned} \dot e(t) &= \Big([G(u,v)] - [v]\dot\phi(t)\Big)\Big|_{x=\phi(t)}, \\ \frac{d\big(e(t)\dot\phi(t)\big)}{dt} &= \Big([H(u,v)] - [uv]\dot\phi(t)\Big)\Big|_{x=\phi(t)}. \end{aligned}$$

where $e(t) \stackrel{def}{=} e(\phi(t),t)$.

According to (2.13), for the system

$$(2.14)\qquad v_t + \big(vf(u)\big)_x = 0, \qquad (vu)_t + \big(vuf(u)\big)_x = 0$$

(here $G(u,v) = f(u)v$, $H(u,v) = f(u)uv$) the Rankine–Hugoniot conditions have the form

$$(2.15)\qquad \begin{aligned} \dot e(t) &= [f(u)v] - [v]\dot\phi(t)\Big|_{x=\phi(t)}, \\ \frac{d\big(e(t)\dot\phi(t)\big)}{dt} &= [f(u)uv] - [uv]\dot\phi(t)\Big|_{x=\phi(t)}. \end{aligned}$$

In particular, for the zero-pressure gas dynamics system (1.10) the Rankine–Hugoniot conditions have the form

$$(2.16)\qquad \begin{aligned} \dot e(t) &= [uv] - [v]\dot\phi(t)\Big|_{x=\phi(t)}, \\ \frac{d\big(e(t)\dot\phi(t)\big)}{dt} &= [u^2v] - [uv]\dot\phi(t)\Big|_{x=\phi(t)}. \end{aligned}$$

The right-hand sides of the first and second equations in (2.13) are called the *Rankine–Hugoniot deficit* in v and uv, respectively.

REMARK 2.1. **(a)** The Rankine–Hugoniot conditions (2.12) for system (1.8) are *essentially different* from the Rankine–Hugoniot conditions (2.6) for system (1.7). The Rankine–Hugoniot conditions (2.6) is a system of *first-order equations*, while the Rankine–Hugoniot conditions (2.12) constitute a system of *second-order equations*. Thus, in the general case, to solve the Cauchy problem for system (1.8), we need to introduce the *initial velocity* $u_\delta(x_k^0,0)=-\frac{S_{k\,t}}{S_{k\,x}}\Big|_{x=x_k^0,t=0}$ $k\in I_0$ of δ-shock (2.4) in the initial data (2.10) (for details, see [**12**]).

(b) For system (2.14) the Rankine–Hugoniot conditions (2.15) are analogous to the Rankine–Hugoniot conditions

$$
\begin{aligned}
\frac{dx}{dt} &= \sigma,\\
\frac{dw}{dt} &= [vf(u)]-\sigma[v],\\
\frac{d(wu_\delta)}{dt} &= [vuf(u)]-\sigma[uv],
\end{aligned}
\tag{2.17}
$$

in the *measure-valued solution* approach [**3**], [**55**], [**57**]. Here u^-, u^+ and u_δ are the velocities before the discontinuity, after the discontinuity, and at the point of discontinuity, respectively, and $x(t)=\sigma t$ is the equation for the discontinuity line, $\sigma=f(u_\delta)$. Moreover, the formulas for the trajectory of a singularity $\phi(t)$ and for the coefficient $e(t)$ of the δ-function coincide with the analogous formulas in the *measure-valued solution* approach [**3**], [**55**], [**57**] if we *identify* the velocity $\dot\phi(t)$ at the discontinuity line $x=\phi(t)$ in formulas (2.15) with the quantity

$$
u_\delta=\sigma=f(u_\delta)=\dot\phi(t).
$$

in formulas (2.17).

3. The Cauchy problems admitting δ-shocks

3.1. Weak asymptotic solutions. We are going to introduce a notion of weak asymptotic solution, which is one of the most important in the weak asymptotics method.

Let $\alpha\in\mathbb{R}$. Denote by $O_{\mathcal{D}'}(\varepsilon^\alpha)$, $\varepsilon\to+0$ a collection of distributions (with respect to x) $f(x,t,\varepsilon)\in\mathcal{D}'(\mathbb{R}_x)$, $x\in\mathbb{R}$, $t\in[0,T]$, $\varepsilon>0$ such that

$$
\langle f(\cdot,t,\varepsilon),\ \psi(\cdot)\rangle=O(\varepsilon^\alpha),\quad \varepsilon\to+0,
$$

for any test function $\psi(x)\in\mathcal{D}(\mathbb{R})$, $x\in\mathbb{R}$. Moreover, $\langle f(\cdot,t,\varepsilon),\ \psi(\cdot)\rangle$ is a continuous function in t, and the estimate $O(\varepsilon^\alpha)$ is understood in the standard sense being uniform with respect to t in $[0,T]$. The notation $o_{\mathcal{D}'}(\varepsilon^\alpha)$, $\varepsilon\to+0$ is understood correspondingly.

DEFINITION 3.1. ([**11**]– [**13**]) A pair of functions $\big(u_\varepsilon(x,t),v_\varepsilon(x,t)\big)$ which are smooth as $\varepsilon>0$, $t\in[0,T]$ is called a *weak asymptotic solution* of the systems (1.7) with the initial data (2.1) (or the system (1.8) with the initial data (2.10)) if

$$\begin{aligned}
\int L_1[u_\varepsilon(x,t), v_\varepsilon(x,t)]\psi(x)\,dx &= o(1),\\
\int L_2[u_\varepsilon(x,t), v_\varepsilon(x,t)]\psi(x)\,dx &= o(1),\\
\int \Big(u_\varepsilon(x,0) - u^0(x)\Big)\psi(x)\,dx &= o(1),\\
\int \Big(v_\varepsilon(x,0) - v^0(x)\Big)\psi(x)\,dx &= o(1), \quad \varepsilon \to +0,
\end{aligned}$$

for all $\psi(x) \in \mathcal{D}(\mathbb{R})$, $x \in \mathbb{R}$, i.e.,

$$\begin{aligned}
L_1[u_\varepsilon(x,t), v_\varepsilon(x,t)] &= o_{\mathcal{D}'}(1),\\
L_2[u_\varepsilon(x,t), v_\varepsilon(x,t)] &= o_{\mathcal{D}'}(1),\\
u_\varepsilon(x,0) &= u^0(x) + o_{\mathcal{D}'}(1),\\
v_\varepsilon(x,0) &= v^0(x) + o_{\mathcal{D}'}(1), \quad \varepsilon \to +0,
\end{aligned} \tag{3.1}$$

where the first two estimates are uniform in $t \in [0,T]$.

In (3.1) all distributions in u, v depend on t as a parameter.

Recall that one of the methods for studying singular solutions of systems of conservation laws is the *vanishing viscosity method* which introduces *viscosity terms* in the right-hand sides of a system of conservation laws. In this case *viscosity terms* admit estimates of the form $o_{\mathcal{D}'}(1)$, and, consequently, a *viscosity solution* can be considered as a *weak asymptotic solution*. Thus a *viscosity solution* is a particular case of a *weak asymptotic solution* of the Cauchy problem, and our notation $o_{\mathcal{D}'}(1)$ in the right-hand sides of the equations (3.1) can be interpreted as a small viscosity.

Within the framework of the *weak asymptotics method*, we find a δ-shock wave type solution of the Cauchy problem as a weak limit

$$\begin{aligned}
u(x,t) &= \lim\nolimits_{\varepsilon\to+0} u_\varepsilon(x,t),\\
v(x,t) &= \lim\nolimits_{\varepsilon\to+0} v_\varepsilon(x,t),
\end{aligned} \tag{3.2}$$

of the *weak asymptotic solution* $(u_\varepsilon, v_\varepsilon)$ to the corresponding Cauchy problem.

Next, constructing the *weak asymptotic solution* of the Cauchy problem, multiplying the first two relations in (3.1) by a test function $\varphi(x,t) \in \mathcal{D}(\mathbb{R}\times[0,\,\infty))$, integrating these relations by parts and then passing to the limit as $\varepsilon \to +0$, we will see that the pair of distributions (u,v) in (1.13) satisfy the integral identities (2.2) or (2.11). In this way we will prove that the left-hand sides of the relations

$$\begin{aligned}
\lim_{\varepsilon\to+0}\int_0^\infty\int L_1[u_\varepsilon(x,t), v_\varepsilon(x,t)]\varphi(x,t)\,dx\,dt &= 0,\\
\lim_{\varepsilon\to+0}\int_0^\infty\int L_2[u_\varepsilon(x,t), v_\varepsilon(x,t)]\varphi(x,t)\,dx\,dt &= 0
\end{aligned}$$

coincide with the left-hand sides of the integral identities (2.2) or (2.11) for all test functions $\varphi(x,t) \in \mathcal{D}(\mathbb{R} \times [0,\infty))$.

According to [**10**]– [**13**], a *weak asymptotic solution* of the Cauchy problems (1.7), (1.11) or (1.8), (1.11), (1.12) admitting δ-shocks is constructed in the form of a smooth Ansatz

$$\begin{array}{rcl} u_\varepsilon(x,t) & = & \widetilde{u}_\varepsilon(x,t) + R_u(x,t,\varepsilon), \\ v_\varepsilon(x,t) & = & \widetilde{v}_\varepsilon(x,t) + R_v(x,t,\varepsilon), \quad \varepsilon > 0. \end{array}$$

Here the functions $(\widetilde{u}_\varepsilon, \widetilde{v}_\varepsilon)$ are regularizations of the singular Ansatz (1.13) with respect to singularities $H(-x+\phi(t))$, $\delta(-x+\phi(t))$, and the *corrections* $R_u(x,t,\varepsilon)$, $R_v(x,t,\varepsilon)$, are the desired functions which are assumed to admit the estimates:

$$R_j(x,t,\varepsilon) = o_{\mathcal{D}'}(1), \quad \frac{\partial R_j(x,t,\varepsilon)}{\partial t} = o_{\mathcal{D}'}(1), \quad \varepsilon \to +0, \qquad j = u,v,w. \tag{3.3}$$

Let us note that choosing the corrections is an essential part of the "right" construction of the *weak asymptotic solution* [**9**]– [**13**], [**39**], [**48**]– [**50**].

In order to construct a regularization $f(x,\varepsilon)$ of a distribution $f \in \mathcal{D}'(\mathbb{R})$ we use the representation

$$f(x,\varepsilon) = f(x) * \frac{1}{\varepsilon}\omega\left(\frac{x}{\varepsilon}\right), \quad \varepsilon > 0, \tag{3.4}$$

where $*$ is the convolution, and the mollifier ω has the following properties: (a) $\omega \in C^\infty(\mathbb{R})$, (b) $\omega(\eta)$ has a compact support or decreases sufficiently rapidly, as $|\eta| \to \infty$, (c) $\int \omega(\eta)\,d\eta = 1$, (d) $\omega(\eta) \ge 0$, (e) $\omega(-\eta) = \omega(\eta)$. It is known that $\lim\limits_{\varepsilon \to +0} \langle f(\cdot,\varepsilon), \varphi(\cdot)\rangle = \langle f(\cdot), \varphi(\cdot)\rangle$ for all $\varphi(x) \in \mathcal{D}(\mathbb{R})$.

Thus we seek a weak asymptotic solution of the Cauchy problems (1.7), (1.11) or (1.8), (1.11), (1.12) in the form:

$$\begin{array}{rcl} u_\varepsilon(x,t) & = & u_+(x,t) + [u(x,t)]H_u(-x+\phi(t),\varepsilon) + R_u(x,t,\varepsilon), \\ v_\varepsilon(x,t) & = & v_+(x,t) + [v(x,t)]H_v(-x+\phi(t),\varepsilon) \\ & & \qquad + e(t)\delta_v(-x+\phi(t),\varepsilon) + R_v(x,t,\varepsilon). \end{array} \tag{3.5}$$

Here, according to (3.4),

$$\delta_v(\xi,\varepsilon) = \frac{1}{\varepsilon}\omega_e\Big(\frac{\xi}{\varepsilon}\Big), \tag{3.6}$$

is a regularization of the δ-function,

$$H_j(\xi,\varepsilon) = \omega_{0j}\Big(\frac{\xi}{\varepsilon}\Big) = \int_{-\infty}^{\xi/\varepsilon} \omega_j(\eta)\,d\eta \tag{3.7}$$

are regularizations of the Heaviside function $H(\xi)$, where $\omega_{0j}(z) \in C^\infty(\mathbb{R})$, and $\lim_{z\to+\infty}\omega_{0j}(z) = 1$, $\lim_{z\to-\infty}\omega_{0j}(z) = 0$, $j = u,v,w$. The mollifiers ω_e, ω_g, ω_h, ω_j, $j = u,v$ have properties (a)–(e).

3.2. The Cauchy problems. In [**2**], [**10**]– [**13**], [**48**]– [**50**], by using the *weak asymptotics method*, the Cauchy problems (1.9), (1.11); (1.14), (1.11); (1.15), (1.11); (1.10), (1.11), (1.12) admitting δ-shocks were solved.

Solutions of some Cauchy problems admitting δ-shocks are given below.

As in [**15**], [**23**], [**55**], we use the "overcompression" condition (see [**35**])

$$\begin{array}{rcccl} \lambda_1(u_+,v_+) & \le & \dot\phi(t) & \le & \lambda_1(u_-,v_-), \\ \lambda_2(u_+,v_+) & \le & \dot\phi(t) & \le & \lambda_2(u_-,v_-) \end{array} \tag{3.8}$$

as the *admissibility condition* for the δ-shocks. Here $\lambda_1(u,v)$, $\lambda_2(u,v)$ are eigenvalues of the characteristic matrix of a hyperbolic system of conservation laws, $\dot\phi(t)$ is the velocity of propagation of the δ-shock wave, i.e., the velocity of motion of the δ-shock front, and u_-, v_- and u_+, v_+ are the respective left- and right-hand values of u, v on the discontinuity curve $x=\phi(t)$. This means that all characteristics on both sides of the discontinuity are in-coming.

3.3. System (1.9). Let us consider the Cauchy problem (1.9), (1.11), where $u_1^0(0)>0$. The eigenvalues of the characteristic matrix of system (1.9) are $\lambda_1(u)=f'(u)$, $\lambda_2(u)=g(u)$. We shall assume that

$$f''(u)>0, \quad g'(u)>0, \quad f'(u)\le g(u), \tag{3.9}$$

i.e., the "overcompression" conditions (3.8) are satisfied.

We will seek a *weak asymptotic solution* in the form (3.5), and choose corrections in the form

$$R_u(x,t,\varepsilon)=0, \qquad R_v(x,t,\varepsilon)=R(t)\frac{1}{\varepsilon}\Omega''\Big(\frac{-x+\phi(t)}{\varepsilon}\Big), \tag{3.10}$$

where $R(t)$ is a continuous function, $\varepsilon^{-3}\Omega''\big(x/\varepsilon\big)$ is a regularization of the distribution $\delta''(x)$, $\Omega(\eta)$ has the properties (a)–(c) (see Sec. 1). It is clear that estimates (3.3) hold.

Theorem 3.1. *Suppose that conditions* (3.9) *hold. Let* $[u^0(0)]>0$. *Then there exist* $T>0$ *and a zero neighborhood* $K\subset\mathbb{R}$ *such that, for* $(x,t)\in K\times[0,\,T)$, *the Cauchy problem* (1.9), (1.11) *has a weak asymptotic solution* (3.5), (3.10) *if and only if*

$$\begin{array}{rcl} u_{\pm t}+\big(f(u_\pm)\big)_x & = & 0, \quad \pm x>\pm\phi(t), \\ v_{\pm t}+\big(g(u_\pm)v_\pm\big)_x & = & 0, \quad \pm x>\pm\phi(t), \\ \dot\phi(t) & = & \frac{[f(u)]}{[u]}\Big|_{x=\phi(t)}, \\ \dot e(t) & = & \Big([vg(u)]-[v]\frac{[f(u)]}{[u]}\Big)\Big|_{x=\phi(t)}, \end{array} \tag{3.11}$$

$$R(t)=\frac{e(t)}{c(t)}\bigg(\frac{[f(u)]}{[u]}\Big|_{x=\phi(t)}-a(t)\bigg), \tag{3.12}$$

where $[h(u(x,t),v(x,t))]$ *is a jump in function* $h(u(x,t),v(x,t))$ *across the discontinuity curve* $x=\phi(t)$,

(3.13)

$$\begin{aligned}
a(t) &= \int g\big(u_-(x,t)\omega_{0u1}(\eta)+u_+(x,t)(1-\omega_{0u1}(\eta))\big)\Big|_{x=\phi(t)}\omega_{\delta 1}(\eta)\,d\eta,\\
c(t) &= \int g\big(u_-(x,t)\omega_{0u1}(\eta)+u_+(x,t)(1-\omega_{0u1}(\eta))\big)\Big|_{x=\phi(t)}\Omega''(\eta)\,d\eta\neq 0.
\end{aligned}$$

The initial data for system (3.11), (3.12) *are defined from* (1.11), *and*

$$\phi(0)=0,\quad R(0)=\frac{e^0}{c(0)}\bigg(\frac{[f(u^0)]}{[u^0]}\Big|_{x=0}-a(0)\bigg).$$

Proof. Let us substitute ansatz (3.5), (3.10), and asymptotics $f(u_\varepsilon(x,t))$ and $g(u_\varepsilon(x,t))v_\varepsilon(x,t)$ given by Lemma A.2 from Appendix A into system (1.9). Taking into account the estimates (3.3), we obtain up to $O_{\mathcal{D}'}(\varepsilon)$ the following relations

$$u_{\varepsilon t}+\big(f(u_\varepsilon)\big)_x=u_{+t}+\big(f(u_+)\big)_x+\Big\{\frac{\partial[u]}{\partial t}+\frac{\partial}{\partial x}\big[f(u)\big]\Big\}H(-x+\phi(t))$$

$$+\Big\{[u]\dot\phi(t)-\big[f(u)\big]\Big\}\delta(-x+\phi(t))+O_{\mathcal{D}'}(\varepsilon), \tag{3.14}$$

$$v_{\varepsilon t}+\big(g(u_\varepsilon)v_\varepsilon\big)_x=v_{+t}+\big(g(u_+)v_+\big)_x+\Big\{\frac{\partial[v]}{\partial t}+\frac{\partial}{\partial x}\big[vg(u)\big]\Big\}H(-x+\phi(t))$$

$$+\Big\{[v]\dot\phi(t)+\dot e(t)-\big[vg(u)\big]\Big\}\delta(-x+\phi(t))$$

$$+\Big\{e(t)\dot\phi(t)-e(t)a(t)-c(t)R(t)\Big\}\delta'(-x+\phi(t))+O_{\mathcal{D}'}(\varepsilon), \tag{3.15}$$

where $a(t)$, $c(t)$ are defined by formula (3.13). It is clear that mollifiers $\omega_{0u1}(\xi)$, $\Omega(\xi)$ can be chosen such that $\int\omega_{u1}(\eta)\Omega'(\eta)\,d\eta>0$. Consequently, taking into account that $g'(u)>0$, $[u^0(x)]>0$ and integrating by parts, we obtain

$$c(t)=-\int g'\big(u_0(x,t)+u_1(x,t)\omega_{0u1}(\eta)\big)u_1(x,t)\Big|_{x=\phi(t)}\omega_{u1}(\eta)\Omega'(\eta)\,d\eta\neq 0.$$

Setting the right-hand side of (3.14), (3.15) equal to zero, we obtain the necessary and sufficient conditions for the equalities $u_{\varepsilon t}+\big(f(u_\varepsilon)\big)_x=O_{\mathcal{D}'}(\varepsilon)$ and $v_{\varepsilon t}+\big(g(u_\varepsilon)v_\varepsilon\big)_x=O_{\mathcal{D}'}(\varepsilon)$, i.e. system (3.11), (3.12).

Now we consider the Cauchy problem

$$u_t+\big(f(u)\big)_x=0,\quad u(x,0)=u^0(x).$$

Since, according to (3.9), $f(u)$ is convex and $[u^0(0)]>0$, according to the results [**36**, Ch.4.2.], we extend $u^0_+(x)$ ($u^0_-(x)=u^0_+(x)+[u^0(x)]$) to $x\le 0$ ($x\ge 0$) in a bounded C^1 fashion and continue to denote the extended functions by $u^0_\pm(x)$. By $u_\pm(x,t)$ we denote the C^1 solutions of the problems

$$u_t+\big(f(u)\big)_x=0,\quad u_\pm(x,0)=u^0_\pm(x)$$

which exist for small enough time interval $[0,\ T_1]$ and are determined by integration along characteristics. The functions $u_\pm(x,t)$ determine a two-sheeted covering of the plane (x,t). Next, we define the discontinuity curve $x=\phi(t)$ as a solution of the problem

$$\dot\phi(t)=\frac{f\big(u_+(x,t)\big)-f\big(u_-(x,t)\big)}{u_+(x,t)-u_-(x,t)}\bigg|_{x=\phi(t)}, \quad \phi(0)=0.$$

It is clear that there exists a unique function $\phi(t)$ for sufficiently short times $[0,\ T_2]$. To this end, for $T=\min(T_1,T_2)$ we define the shock solution by

$$u(x,t)=\begin{cases} u_+(x,t), & x\ >\ \phi(t),\\ u_-(x,t), & x\ <\ \phi(t).\end{cases}$$

Thus the first, second and third equations of system (3.11) define a unique solution of the Cauchy problem $u_t+\big(f(u)\big)_x=0$, $u(x,0)=u^0(x)$ for $t\in[0,\ T)$.

Solving this problem, we obtain $u(x,t)$, $\phi(t)$. Then substituting these functions into system (3.11), we obtain $\widehat v(x,t)=\widehat v_+(x,t)+[\widehat v](x,t)H(-x+\phi(t))$, $e(t)$, and $v(x,t)=\widehat v(x,t)+e(t)\delta(-x+\phi(t))$. Moreover, for any functions $u_\pm(x,t)$, $e(t)$, $\phi(t)$, $t\in[0,\ T)$, there exists a function $R(t)$, which is defined by relation (3.12).

Thus for $t\in[0,\ T)$, the Cauchy problem (1.9), (1.11) has a weak asymptotic solution (3.5), (3.10) if and only if (3.11), (3.12) hold. $\square$

Now we obtain a δ-shock solution of the Cauchy problem (1.9), (1.11), as a weak limit of a weak asymptotic solution constructed by Theorem 3.1.

Theorem 3.2. ([**12**], [**13**]) *Let $[u^0(0)]>0$. Then there exist $T>0$ and a zero neighborhood $K\subset\mathbb{R}$ such that, for $(x,t)\in K\times[0,\ T)$, the Cauchy problem* (1.9), (1.11), (3.9), *has a unique solution* (1.13), *which satisfies the integral identities* (2.2), *where $\Gamma=\{(x,t)\,:\,x=\phi(t),\,t\in[0,\ T)\}$, and functions $u_\pm(x,t)$, $v_\pm(x,t)$, $\phi(t)$, $e(t)$ are defined by the system*

$$\begin{aligned} u_{\pm t}+\big(f(u_\pm)\big)_x &= 0, \quad \pm x>\pm\phi(t),\\ v_{\pm t}+\big(g(u_\pm)v_\pm\big)_x &= 0, \quad \pm x>\pm\phi(t),\\ \dot\phi(t) &= \frac{[f(u)]}{[u]}\bigg|_{x=\phi(t)},\\ \dot e(t) &= \Big([vg(u)]-[v]\frac{[f(u)]}{[u]}\Big)\bigg|_{x=\phi(t)}, \end{aligned} \tag{3.16}$$

with the initial data defined from (1.11), $\phi(0)=0$.

Proof. According to Lemma A.2 from Appendix A,

$$f\big(u_\varepsilon(x,t)\big)=f(u_+)+\big[f(u(x,t))\big]H(-x+\phi)+O_{\mathcal{D}'}(\varepsilon), \quad \varepsilon\to+0, \tag{3.17}$$

By substituting the relation (3.12), which determines $R(t)$, into the second relation of Lemma A.2 from Appendix A we obtain

$$v_\varepsilon(x,t)g\big(u_\varepsilon(x,t)\big)=v_+g(u_+)+\big[vg(u)\big]\Big|_{x=\phi(t)}H(-x+\phi(t))$$

$$+e(t)\frac{[f(u)]}{[u]}\Big|_{x=\phi(t)}\delta(-x+\phi(t))+O_{\mathcal{D}'}(\varepsilon), \quad \varepsilon\to+0. \tag{3.18}$$

By Theorem 3.1 we have $u_{\varepsilon t}+\big(f(u_\varepsilon)\big)_x=O_{\mathcal{D}'}(\varepsilon)$ and $v_{\varepsilon t}+\big(g(u_\varepsilon)v_\varepsilon\big)_x=O_{\mathcal{D}'}(\varepsilon)$. Applying the left-hand and right-hand sides of these relations to an arbitrary test function $\varphi(x,t)\in\mathcal{D}(\mathbb{R}\times[0,\,T))$, and integrating by parts, we obtain

$$\int_0^T\int\Big(u_\varepsilon(x,t)\varphi_t(x,t)+f(u_\varepsilon(x,t))\varphi_x(x,t)\Big)\,dxdt$$

$$+\int u_\varepsilon(x,0)\varphi(x,0)\,dx=O(\varepsilon), \tag{3.19}$$

$$\int_0^T\int\Big(v_\varepsilon(x,t)\varphi_t(x,t)+v_\varepsilon(x,t)g(u_\varepsilon(x,t))\varphi_x(x,t)\Big)\,dxdt$$

$$+\int v_\varepsilon(x,0)\varphi(x,0)\,dx=O(\varepsilon), \quad \varepsilon\to+0. \tag{3.20}$$

Substituting the asymptotics (3.5) and (3.17), (3.18) into relations (3.19), (3.20), passing to the limit as $\varepsilon\to+0$ in each of the integrals, and taking into account that

$$\lim_{\varepsilon\to+0}\int_0^T\int e(t)\delta_{v1}\big(-x+\phi(t),\varepsilon\big)\varphi(x,t)\,dxdt=\int_0^T e(t)\varphi(\phi(t),t)\,dt, \tag{3.21}$$

$$\lim_{\varepsilon\to+0}\int e(0)\delta_{v1}\big(-x,\varepsilon\big)\varphi(x,0)\,dx=e(0)\varphi(0,0), \tag{3.22}$$

we obtain the integral identities (2.2).

In view of Theorem 3.1, the Cauchy problem has a unique generalized solution. □

3.4. Keyfitz–Kranzer type system. Consider the problem of the propagation of a δ-shock in the Keyfitz–Kranzer type system system

$$\begin{aligned} u_t+\big(f(u)-\alpha v\big)_x &= 0,\\ v_t+\big(g(u)-\beta v\big)_x &= 0, \end{aligned} \tag{3.23}$$

where $f(u)=\sum_{k=0}^n A_k u^k$, $A_n\neq 0$, $g(u)=\sum_{k=0}^{n+1}B_k u^k$, $B_{n+1}\neq 0$, are polynomials; n is an even integer; α, β are constants; $u=u(x,t), v=v(x,t)\in\mathbb{R}$, $x\in\mathbb{R}$. The Keyfitz–Kranzer system (1.14) and its generalization are particular cases of system (3.23).

For system (3.23) the eigenvalues of the characteristic matrix are

$$\lambda_\pm(u)=\frac{1}{2}\Big(f'(u)-\beta\pm\sqrt{\big(f'(u)+\beta\big)^2-4\alpha g'(u)}\Big), \quad \big(f'(u)+\beta\big)^2\geq 4\alpha g'(u).$$

We shall assume that the "overcompression" conditions (3.8) are satisfied.

To construct a *weak asymptotic solution* (3.5) of the Cauchy problem (1.15), (1.11), we choose corrections in the form

$$
\begin{aligned}
R_u(x,t,\varepsilon) &= P(t)\frac{1}{\varepsilon^{1/n}}\Omega_P\Big(\frac{-x+\phi(t)}{\varepsilon}\Big) \\
&\qquad +Q(t)\frac{1}{\varepsilon^{1/(n+1)}}\Omega_Q\Big(\frac{-x+\phi(t)}{\varepsilon}\Big), \\
R_v(x,t,\varepsilon) &= 0,
\end{aligned}
\tag{3.24}
$$

where $P(t)$, $Q(t)$ are the desired functions, $\frac{1}{\varepsilon}\Omega_P^n\big(x/\varepsilon\big)$, $\frac{1}{\varepsilon}\Omega_Q^{n+1}\big(x/\varepsilon\big)$ are regularizations (3.6) of the delta function, mollifiers $\Omega_P(\eta)$, $\Omega_Q(\eta)$ have properties (a)–(c). Thus the estimates (3.3) hold. In addition to (3.24), we choose mollifiers $\Omega_P(\eta)$, $\Omega_Q(\eta)$ such that

$$
\begin{aligned}
&\int \Omega_P^k(\eta)\Omega_Q^{n+1-k}(\eta)\,d\eta = 0, \qquad k=1,2,\dots n+1, \\
&\qquad \int \Omega_Q^{n+1}(\eta)\,d\eta \neq 0, \qquad \int \Omega_P^{n}(\eta)\,d\eta \neq 0.
\end{aligned}
\tag{3.25}
$$

As first step, using (3.24), (3.25), we construct a weak asymptotic solution (3.5) of the Cauchy problem (1.15), (1.11).

THEOREM 3.3. *Let*

$$
\lambda_+(u_+^0(0)) \le \frac{[f(u^0)-\alpha v^0]}{[u^0]}\bigg|_{x=0} \le \lambda_-(u_-^0(0)).
\tag{3.26}
$$

Then there exist $T>0$ and a zero neighborhood $K\subset\mathbb{R}$ such that, for $(x,t)\in K\times[0,\,T)$, the Cauchy problem (3.23), (1.11) *has a weak asymptotic solution* (3.5), (3.24), (3.25) *if and only if*

$$
\begin{aligned}
u_{\pm t} + \big(f(u_\pm)-\alpha v_\pm\big)_x &= 0, \quad \pm x > \pm\phi(t), \\
v_{\pm t} + \big(g(u_\pm)-\beta v_\pm\big)_x &= 0, \quad \pm x > \pm\phi(t), \\
\dot\phi(t) &= \tfrac{[f(u)-\alpha v]}{[u]}\Big|_{x=\phi(t)}, \\
\dot e(t) &= \Big([g(u)-\beta v] - [v]\tfrac{[f(u)-\alpha v]}{[u]}\Big)\Big|_{x=\phi(t)},
\end{aligned}
\tag{3.27}
$$

$$
\begin{aligned}
P(t) &= \Big(\frac{\alpha e(t)}{aA_n}\Big)^{1/n}, \\
Q(t) &= \bigg\{\frac{e(t)}{cB_{n+1}}\Big(\frac{[f(u)-\alpha v]}{[u]} + \beta - \frac{\alpha}{A_n}\Big(B_n \\
&\qquad +\Big(\big(1-\frac{b}{a}\big)u_+ + \frac{b}{a}u_-\Big)(n+1)B_{n+1}\Big)\Big)\bigg|_{x=\phi(t)}\bigg\}^{1/(n+1)},
\end{aligned}
\tag{3.28}
$$

where

$$
\begin{aligned}
a &= \int \Omega_P^n(\eta)\,d\eta > 0,\\
b &= \int \omega_{0u}(\eta)\Omega_P^n(\eta)\,d\eta, \qquad (3.29)\\
c &= \int \Omega_Q^{n+1}(\eta)\,d\eta \neq 0.
\end{aligned}
$$

The initial data for system (3.27), (3.28) *are defined from* (1.11), *and*

$$
\begin{aligned}
e(0) &= e^0, \qquad \phi(0) = 0,\\
P(0) &= \Big(\frac{e^0}{aA_n}\Big)^{1/n},\\
Q(0) &= \bigg\{\frac{e^0}{cB_{n+1}}\Big(\frac{[f(u^0)-\alpha v^0]}{[u^0]} + \beta - \frac{\alpha}{A_n}\Big(B_n\\
&\qquad +\Big(\big(1-\frac{b}{a}\big)u_+^0 + \frac{b}{a}u_-^0\Big)(n+1)B_{n+1}\Big)\Big)\bigg\}^{1/(n+1)}\bigg|_{x=0}.
\end{aligned}
$$

Proof. Using the second and third relations in (A.1) from Lemma A.1, we calculate the following weak asymptotics

$$
\begin{aligned}
R_u^k(x,t,\varepsilon) &= o_{\mathcal{D}'}(1), \quad k \le n-1,\\
R_u^n(x,t,\varepsilon) &= aP^n(t)\delta(-x+\phi(t)) + o_{\mathcal{D}'}(1),\\
H(-x+\phi(t),\varepsilon)R_u^n(x,t,\varepsilon) &= bP^n(t)\delta(-x+\phi(t)) + o_{\mathcal{D}'}(1), \qquad (3.30)\\
R_u^{n+1}(x,t,\varepsilon) &= cQ^{n+1}(t)\delta(-x+\phi(t)) + o_{\mathcal{D}'}(1),
\end{aligned}
$$

where a, b, c are defined by (3.29). With help of (3.30) and Lemma A.1 one can calculate

$$
\begin{aligned}
\big(u_\varepsilon(x,t)\big)^k &= u_+^k + \big(u_-^k - u_+^k\big)H(-x+\phi(t))\\
&\qquad + o_{\mathcal{D}'}(1), \quad k \le n-1,\\
\big(u_\varepsilon(x,t)\big)^n &= u_+^n + \big(u_-^n - u_+^n\big)H(-x+\phi(t))\\
&\qquad + R_u^n(x,t,\varepsilon) + o_{\mathcal{D}'}(1), \qquad (3.31)\\
\big(u_\varepsilon(x,t)\big)^{n+1} &= u_+^{n+1} + \big(u_-^{n+1} - u_+^{n+1}\big)H(-x+\phi(t))\\
&\qquad +(n+1)\big(u_+ + [u]H(-x+\phi(t),\varepsilon)\big)R_u^n(x,t,\varepsilon)\\
&\qquad + R_u^{n+1}(x,t,\varepsilon) + o_{\mathcal{D}'}(1).
\end{aligned}
$$

Taking into account relations (3.30), (3.31), (3.5), (3.24), (3.25), we obtain the following weak asymptotics

$$
\begin{aligned}
&f\big(u_\varepsilon(x,t)\big) - \alpha v_\varepsilon(x,t) = f(u_+) - \alpha v_+ + \big[f(u) - \alpha v\big]H(-x+\phi(t))\\
&\qquad +\Big\{aA_nP^n(t) - \alpha e(t)\Big\}\delta(-x+\phi(t)) + o_{\mathcal{D}'}(1), \qquad (3.32)\\
&g\big(u_\varepsilon(x,t)\big) - \beta v_\varepsilon(x,t) = g(u_+) - \beta v_+ + \big[g(u) - \beta v\big]H(-x+\phi(t))\\
&\qquad +\Big\{aB_nP^n(t) + (n+1)\big(au_+ + b[u]\big)B_{n+1}P^n(t)
\end{aligned}
$$

$$+cB_{n+1}Q^{n+1}(t)-\beta e(t)\Big\}\delta(-x+\phi(t))+o_{\mathcal{D}'}(1),\quad \varepsilon\to+0. \tag{3.33}$$

Substituting the smooth ansatz (3.5) and relations (3.32), (3.33) into the left-hand side of system (3.23), and taking into account (3.3), we obtain

$$\begin{aligned}
&u_{\varepsilon t}+\big(f(u_\varepsilon)-\alpha v_\varepsilon\big)_x\\
&\quad=u_{+t}+\big(f(u_+)-\alpha v_+\big)_x+\Big\{\frac{\partial}{\partial t}[u]+\frac{\partial}{\partial x}\big[f(u)-\alpha v\big]\Big\}H(-x+\phi(t))\\
&\qquad+\Big\{[u]\dot\phi(t)-\big[f(u)-\alpha v\big]\Big\}\delta(-x+\phi(t))\\
&\qquad+\Big\{\alpha e(t)-aA_nP^n(t)\Big\}\delta'(-x+\phi(t))+o_{\mathcal{D}'}(1),
\end{aligned} \tag{3.34}$$

$$\begin{aligned}
&v_{\varepsilon t}+\big(g(u_\varepsilon)-\beta v_\varepsilon\big)_x\\
&\quad=v_{+t}+\big(g(u_+)-\beta v_+\big)_x+\Big\{\frac{\partial}{\partial t}[v]+\frac{\partial}{\partial x}\big[g(u)-\beta v\big]\Big\}H(-x+\phi(t))\\
&\qquad+\Big\{[v]\dot\phi(t)+\dot e(t)-\big[g(u)-\beta v\big]\Big\}\delta(-x+\phi(t))\\
&\qquad+\Big\{e(t)\dot\phi(t)+\beta e(t)-aB_nP^n(t)-(n+1)\big(au_++b[u]\big)B_{n+1}P^n(t)\\
&\qquad-cB_{n+1}Q^{n+1}(t)\Big\}\delta'(-x+\phi(t))+o_{\mathcal{D}'}(1),\quad \varepsilon\to+0.
\end{aligned} \tag{3.35}$$

Setting the left-hand side of (3.34), (3.35) equal to zero, we obtain the necessary and sufficient conditions for the equalities $u_{\varepsilon t}+\big(f(u_\varepsilon)-\alpha v_\varepsilon\big)_x=o_{\mathcal{D}'}(1)$, $v_{\varepsilon t}+\big(g(u_\varepsilon)-\beta v_\varepsilon\big)_x=o_{\mathcal{D}'}(1)$, i.e., systems (3.27), (3.28).

Now repeating the corresponding part of the proof of Theorem 3.1 almost word for word, we prove that there exists $T>0$ such that for $t\in[0,\,T)$, the Cauchy problem (1.15), (1.11) has a weak asymptotic solution (3.5), (3.10) if and only if (3.27), (3.28) hold. □

Now using this weak asymptotic solution constructed in Theorem 3.3, we prove the following theorem.

Theorem 3.4. *Let* (3.26) *holds. Then there exist $T>0$ and a zero neighborhood $K\subset\mathbb{R}$ such that, for $(x,t)\in K\times[0,\,T)$, the Cauchy problem* (3.23), (1.11) *has a unique solution* (1.13) *which satisfies the integral identities* (2.2) *where $\Gamma=\{(x,t):x=\phi(t),\,t\in[0,\,T)\}$, and functions $u_\pm(x,t)$, $v_\pm(x,t)$, $\phi(t)$, $e(t)$ are defined by the system*

$$\begin{aligned}
u_{\pm t}+\big(f(u_\pm)-\alpha v_\pm\big)_x&=0,\quad \pm x>\pm\phi(t),\\
v_{\pm t}+\big(g(u_\pm)-\beta v_\pm\big)_x&=0,\quad \pm x>\pm\phi(t),\\
\dot\phi(t)&=\frac{[f(u)]-\alpha[v]}{[u]}\Big|_{x=\phi(t)},\\
\dot e(t)&=\Big([g(u)-\beta v]-[v]\frac{[f(u)]-\alpha[v]}{[u]}\Big)\Big|_{x=\phi(t)},
\end{aligned} \tag{3.36}$$

with the initial data defined from (1.11), $\phi(0)=0$.

PROOF. Substituting the correction $P(t)$, $Q(t)$ given by (3.28) into expressions (3.32), (3.33), we obtain

$$f\big(u_\varepsilon(x,t)\big)-\alpha v_\varepsilon(x,t)$$

$$=f(u_+)-\alpha v_+ +\big[f(u)-\alpha v\big]H(-x+\phi(t))+o_{\mathcal{D}'}(1), \tag{3.37}$$

$$g\big(u_\varepsilon(x,t)\big)-\beta v_\varepsilon(x,t)=g(u_+)-\beta v_+ +\big[g(u)-\beta v\big]H(-x+\phi(t))$$

$$+e(t)\frac{\big[f(u)-\alpha v\big]}{[u]}\delta(-x+\phi(t))+o_{\mathcal{D}'}(1), \quad \varepsilon\to+0. \tag{3.38}$$

By Theorem 3.3 we have

$$u_{\varepsilon t}+\big(f(u_\varepsilon)-\alpha v_\varepsilon\big)_x=o_{\mathcal{D}'}(1), \quad v_{\varepsilon t}+\big(g(u_\varepsilon)-\beta v_\varepsilon\big)_x=o_{\mathcal{D}'}(1).$$

Applying the left-hand and right-hand sides of these relations to an arbitrary test function $\varphi(x,t)\in\mathcal{D}(\mathbb{R}\times[0,\,T))$, taking into account that $u_\varepsilon(x,t)$, $v_\varepsilon(x,t)$ are smooth, and integrating by parts, we obtain

$$\int_0^T\int\Big(u_\varepsilon(x,t)\varphi_t(x,t)+\Big(f(u_\varepsilon(x,t))-\alpha v_\varepsilon(x,t)\Big)\varphi_x(x,t)\Big)\,dxdt$$

$$+\int u_\varepsilon(x,0)\varphi(x,0)\,dx=o(1),$$

$$\int_0^T\int\Big(v_\varepsilon(x,t)\varphi_t(x,t)+\Big(g(u_\varepsilon(x,t))-\beta v_\varepsilon(x,t)\Big)\varphi_x(x,t)\Big)\,dxdt$$

$$+\int v_\varepsilon(x,0)\varphi(x,0)\,dx=o(1), \quad \varepsilon\to+0.$$

Substituting asymptotics (3.5), and (3.37), (3.38) into the above relations, passing to the limit as $\varepsilon\to+0$ in each of the integrals, and taking into account (3.21), (3.22), we obtain that the integral identities (2.2) hold.

In view of Theorem 3.3, system (3.36) has a unique solution. □

For the case $\alpha=1$, $\beta=0$ Theorem 3.4 was proved in [**48**], [**49**], [**2**, Theorem 3.2.].

COROLLARY 3.1. ([**49**], [**2**]) *Let* $u_+^0(0)+1\le\frac{[(u^0)^2]-[v^0]}{[u^0]}\big|_{x=0}\le u_-^0(0)-1$. *Then there exist* $T>0$ *and a zero neighborhood* $K\subset\mathbb{R}$ *such that, for* $(x,t)\in K\times[0,\,T)$, *the Cauchy for Keyfitz–Kranzer system* (1.14), (1.11) *has a unique generalized solution* (1.13) *which satisfies the integral identities* (2.2), *where* $\Gamma=\{(x,t): x=\phi(t),\, t\in[0,\,T)\}$, *and functions* $u_\pm(x,t)$, $v_\pm(x,t)$, $\phi(t)$, $e(t)$ *are defined by the system* (3.36), *where* $f(u)=u^2$, $g(u)=\frac{1}{3}u^3-u$.

3.5. Zero-pressure gas dynamics system (1.10). The characteristic matrix of zero-pressure gas dynamics (1.10) has the repeating eigenvalues $\lambda_1(u) = \lambda_2(u) = u$. We assume that the "overcompression" conditions (3.8) are satisfied, i.e., $u_+ \le \dot\phi(t) \le u_-$.

To construct a *weak asymptotic solution* (3.5) to the Cauchy problem (1.10), (1.11),(1.12), we choose corrections in the form

$$
\begin{aligned}
R_u(x,t,\varepsilon) &= Q(t)\Omega'\Big(\tfrac{-x+\phi(t)}{\varepsilon}\Big), \\
R_v(x,t,\varepsilon) &= R(t)\tfrac{1}{\varepsilon}\widetilde{\Omega}''\Big(\tfrac{-x+\phi(t)}{\varepsilon}\Big),
\end{aligned}
\tag{3.39}
$$

where $Q(t)$, $R(t)$ are desired functions, $\varepsilon^{-1}\Omega\big(x/\varepsilon\big)$ and $\varepsilon^{-1}\widetilde{\Omega}\big(x/\varepsilon\big)$ are regularizations (3.6) of the delta function, $\Omega(\eta)$ and $\widetilde{\Omega}(\eta)$ have properties (a)–(c). Thus, relations (3.3) hold. It is clear (see [**12**]) that in addition we can choose mollifiers $\Omega(\eta)$, $\widetilde{\Omega}(\eta)$ such that

$$
\begin{aligned}
\int \omega_\delta(\eta)\Omega'(\eta)\,d\eta &\ne 0, \\
\int \omega_{0u}(\eta)\widetilde{\Omega}''(\eta)\,d\eta &= 0, \\
\int \omega_{0u}^2(\eta)\widetilde{\Omega}''(\eta)\,d\eta &\ne 0, \\
\int \omega_{0u}(\eta)\Omega'(\eta)\widetilde{\Omega}''(\eta)\,d\eta &= 0, \\
\int \Omega'(\eta)\widetilde{\Omega}''(\eta)\,d\eta &= 0.
\end{aligned}
\tag{3.40}
$$

Now, using (3.39), (3.40) and repeating the constructions of Theorems 3.1, 3.3, we construct a weak asymptotic solution (3.5) of the Cauchy problem (1.10), (1.11), (1.12). Next, using this weak asymptotic solution and repeating the constructions of Theorems 3.2, 3.4 we prove the following theorem.

THEOREM 3.5. ([**12**]) *Let $u_+^0(0) \le \dot\phi(0) \le u_-^0(0)$. Then there exist $T > 0$ and a zero neighborhood $K \subset \mathbb{R}$ such that, for $(x,t) \in K \times [0,\,T)$, the Cauchy problem* (1.10), (1.11), (1.12) *has a unique solution* (1.13) *which satisfies the integral identities* (2.11) *where $\Gamma = \{(x,t) : x = \phi(t),\, t \in [0,\,T)\}$, and functions $u_\pm(x,t)$, $v_\pm(x,t)$, $\phi(t)$, $e(t)$ are defined by the system*

$$
\begin{aligned}
v_{\pm t} + \big(v_\pm u_\pm\big)_x &= 0, \quad \pm x > \pm\phi(t), \\
(v_\pm u_\pm)_t + \big(v_\pm u_\pm^2\big)_x &= 0, \quad \pm x > \pm\phi(t), \\
\dot e(t) &= \Big([uv] - [v]\dot\phi(t)\Big)\Big|_{x=\phi(t)}, \\
\frac{d\big(e(t)\dot\phi(t)\big)}{dt} &= \Big([u^2v] - [uv]\dot\phi(t)\Big)\Big|_{x=\phi(t)},
\end{aligned}
\tag{3.41}
$$

where the initial data are defined from (1.11), (1.12).

If in Theorems 3.1– 3.5 and Corollary 3.1, we consider the piecewise constant initial data (1.11) then $T = \infty$.

4. δ-Shock and singular shock

In this section a complicated problem related to the concept of *singular shock* is considered (see [**52**]).

As was mentioned above, a model system admitting a *singular shock* is the well-known Keyfitz-Kranzer system (1.14). In [**24**], [**25**], [**45**], [**46**] a *singular shock solution* for system of conservation laws

$$w_t + \big(q(w)\big)_x = 0, \quad x \in \mathbb{R}, \qquad w(x,t) \in \mathbb{R}^n,$$

is defined as a measure of the form

$$w(x,t) = \omega(x,t) + \sum_i M_i \chi_i(t) \delta(x - x_i(t)), \tag{4.1}$$

where $q : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth function, ω is a classical weak solution away from the singularities, χ_i is the characteristic function of interval $[A_i, B_i)$; $M_i \in W^\infty$ and $x_i \in W^{1,\infty}$. The function w is the weak limit of the sequence w^ε with $w^\varepsilon(\cdot,t) \in L^1_{loc}$ uniformly with respect to ε, point-wise in t, satisfying

$$\begin{array}{rcl} w^\varepsilon(\cdot,t) & \to & w(\cdot,t), \\ \big(w^\varepsilon(\cdot,t)\big)_t + \big(q(w^\varepsilon(\cdot,t))\big)_x - \varepsilon(A(w^\varepsilon(\cdot,t))_x)_x & \to & 0, \quad \varepsilon \to 0, \end{array} \tag{4.2}$$

weakly in the space of measures on $\mathbb{R}$, point-wise with respect to t, for some positive definite matrix A. In the above papers some modifications of this definition are also used. Note that since $w^\varepsilon \to w$ weakly, Definition (4.1), (4.2) can be used without the term $\varepsilon(A(w^\varepsilon(\cdot,t))_x)_x$ (this was done in [**45**]).

The authors ([**22**]– [**25**], [**45**], [**46**]) *distinguish* between δ-shocks and singular shocks. In fact, the main distinction of a *singular shock* is that its flux function *is not defined*. As said in [**22**, p.106], "unlike the delta-shocks..., the singular shocks which are needed to solve (1.14) are truly nonlinear objects which cannot defined in the context of classical distribution theory." According to [**22**]– [**25**], [**45**], [**46**], some *model problems for δ-shocks* are described in [**3**], [**14**], [**32**], [**55**], where for "zero-pressure gas dynamics" (1.10) the *measure-valued solution* approach is used, and flux-functions ρu, ρu^2 are *well-defined measures*.

We would like to stress that Definition (4.1), (4.2) of a *singular shock* and the other ones from [**24**], [**25**], [**45**], [**46**] *do not connect the limiting function* (4.1) with the system $w_t + \big(q(w)\big)_x = 0$; they only *connect the regularizing function* w^ε with the *regularizing system* (4.2). Thus it is not defined in which sense the singular shock (4.1) satisfies a nonlinear system. In this way only approximating (or viscosity) solutions and their structure can be studied. (A more general and strict definition of the type (4.1), (4.2) was introduced in [**10**].)

Using our results, we show that both *singular shock* (4.1) and *δ-shock* (1.13) *are solutions of the same type*. More precisely, there is no reasons to *distinguish* between δ-shocks and singular shocks.

To prove our assertion we compare singular solutions which have δ- singularities for the systems (1.14) and (1.15), and the system (1.9). According to [**22**]– [**25**], [**45**], [**46**], systems (1.14), (1.15) and (1.9) are model problems for *singular shocks* and δ-shocks, respectively. For these systems we consider the initial data of the form (1.11).

Our arguments are the following:

(i) General structure of δ*-shock* (see Definition 2.1) and *singular shock* (4.1) is identically.

(ii) Systems (1.14), (1.15), and (1.9) are particular cases of the same system (1.7), for which the δ-shock wave type solution is introduced by the Definition 2.1.

(iii) According to Corollary 3.1, Theorems 3.4, and Theorem 3.2, δ-shock wave type solutions of the Cauchy problems (1.14), (1.11); (1.15), (1.11); and (1.9), (1.11) have the same form (1.13):

$$\begin{aligned} u(x,t) &= u_+(x,t) + [u(x,t)]H(-x+\phi(t)), \\ v(x,t) &= v_+(x,t) + [v(x,t)]H(-x+\phi(t)) + e(t)\delta(-x+\phi(t)) \end{aligned}$$

and satisfy corresponding systems of conservation laws in the sense of the same Definition 2.1.

(iv) According to Theorem 2.1, the Rankine–Hugoniot conditions for the above δ-shock wave type solutions are given by the identical formula (2.7).

(v) For the Cauchy problems (1.14), (1.11); (1.15), (1.11); and (1.9), (1.11) the flux-functions of δ-shocks (9.34), (9.35); (9.32), (9.33); and (9.30), (9.31) are well-defined *Schwartz distributions* and have the identical structure (9.3) (see below in Subsec. 9.2, 9.3).

Nevertheless, flux–functions of δ–shocks for the Keyfitz–Kranzer system (1.14) and its generalization (1.15) have some specific and "strange" properties which are described below in Subsec. 9.3. The point is that δ-shocks constitute the universe with unusual and "strange" properties, and the Keyfitz-Kranzer system is an excellent model example which demonstrates this.

5. δ'-Shock type solutions and the Rankine–Hugoniot conditions

Denote by $\widetilde{\mathcal{C}}\big(\mathbb{R}\times(0,\infty);\mathbb{R}\big)$ the class of piecewise-smooth functions. Let $\Gamma=\{\gamma_i : i\in I\}$ be the graph introduced in Subsec. 2.1 (see Fig. 1.). The initial data

(5.1)
$$(u^0(x),v^0(x),w^0(x)), \quad \text{where} \quad \begin{aligned} v^0(x) &= \widehat{v}^0(x)+e^0\delta(\Gamma_0) \\ w^0(x) &= \widehat{w}^0(x)+g^0\delta(\Gamma_0)+h^0\delta'(\Gamma_0), \end{aligned}$$

and $u^0,\widehat{v}^0,\widehat{w}^0\in\widetilde{\mathcal{C}}\big(\mathbb{R};\mathbb{R}\big)$, will be called δ'*-shock type initial data*. Here, by definition, $e^0\delta(\Gamma_0)\stackrel{def}{=}\sum_{k\in I_0}e^0_k\delta(x-x^0_k)$, $g^0\delta(\Gamma_0)\stackrel{def}{=}\sum_{k\in I_0}g^0_k\delta(x-x^0_k)$, $h^0\delta(\Gamma_0)\stackrel{def}{=}\sum_{k\in I_0}h^0_k\delta'(x-x^0_k)$, where e^0_k, g^0_k, h^0_k are constants, $k\in I_0$.

DEFINITION 5.1. ([**39**]) A triple of distributions (u,v,w) and a graph Γ, where $v(x,t)$ and $w(x,t)$ have the form of the sums

$$v(x,t)=\widehat{v}(x,t)+e(x,t)\delta(\Gamma), \quad w(x,t)=\widehat{w}(x,t)+g(x,t)\delta(\Gamma)+h(x,t)\delta'(\Gamma),$$

and $u,\widehat{v},\widehat{w}\in\widetilde{\mathcal{C}}\big(\mathbb{R}\times(0,\infty);\mathbb{R}\big)$, $e\delta(\Gamma)\stackrel{def}{=}\sum_{i\in I}e_i\delta(\gamma_i)$, $g\delta(\Gamma)\stackrel{def}{=}\sum_{i\in I}g_i\delta(\gamma_i)$, $h\delta'(\Gamma)\stackrel{def}{=}\sum_{i\in I}h_i(x,t)\delta'(\gamma_i)$, $e_i(x,t),g_i(x,t),h_i(x,t)\in C^1(\Gamma)$, $i\in I$, is called a δ'-*shock wave type solution* of the Cauchy problem (1.16), (5.1) if the integral identities

(5.2)
$$\begin{aligned}
\int_0^\infty\int\Big(u\varphi_t+f(u)\varphi_x\Big)\,dx\,dt+\int u^0(x)\varphi(x,0)\,dx &= 0,\\
\int_0^\infty\int\widehat{v}\Big(\varphi_t+f'(u)\varphi_x\Big)\,dx\,dt+\sum_{i\in I}\int_{\gamma_i}e_i(x,t)\frac{\delta\varphi(x,t)}{\delta t}\frac{dl}{\sqrt{1+u_\delta^2}}&\\
+\int\widehat{v}^0(x)\varphi(x,0)\,dx+\sum_{k\in I_0}e_k^0\varphi(x_k^0,0) &= 0,\\
\int_0^\infty\int\Big(\widehat{w}\varphi_t+\big(f''(u)\widehat{v}^2+f'(u)\widehat{w}\big)\varphi_x\Big)\,dx\,dt&\\
+\sum_{i\in I}\bigg(\int_{\gamma_i}g_i(x,t)\frac{\delta\varphi(x,t)}{\delta t}\frac{dl}{\sqrt{1+u_\delta^2}}+\int_{\gamma_i}h_i(x,t)\frac{\delta\varphi_x(x,t)}{\delta t}\frac{dl}{\sqrt{1+u_\delta^2}}&\\
+\int_{\gamma_i}\frac{\frac{\delta e_i^2(x,t)}{\delta t}-h_i(x,t)\frac{\delta[u(x,t)]}{\delta t}}{[u(x,t)]}\varphi_x(x,t)\frac{dl}{\sqrt{1+u_\delta^2}}\bigg)&\\
+\int\widehat{w}^0(x)\varphi(x,0)\,dx+\sum_{k\in I_0}g_k^0\varphi(x_k^0,0)+\sum_{k\in I_0}h_k^0\varphi_x(x_k^0,0) &= 0,
\end{aligned}$$

hold for all test functions $\varphi(x,t)\in\mathcal{D}(\mathbb{R}\times[0,\infty))$. The derivative of the delta function $\delta'(\gamma_i)$ on the curve γ_i is defined in [**17**, ch.III,§1.5.], [**21**, 5.3.;5.5.].

THEOREM 5.1. ([**39**]) *Let us assume that* $\Omega\subset\mathbb{R}\times[0,\infty)$ *is some region cut by a curve* $\Gamma=\{(x,t):x=\phi(t)\}$, $\phi(t)\in C^1(0,+\infty)$ *into left- and right-hand parts* $\Omega_\pm=\{(x,t)\in\Omega:\pm(x-\phi(t))>0\}$. *Let* $(u(x,t),v(x,t),w(x,t))$, Γ *be a generalized* δ'*-shock wave type solution of system* (1.16). *Assume that* (u,v,w) *are smooth in the domains* $\Omega_\pm$ *and have one-sided limits* $u_\pm$, $v_\pm$, $w_\pm$

on Γ, *which are supposed to be continuous functions on* Γ. *Then the* Rankine–Hugoniot conditions for the δ'-shock

$$\dot\phi(t) = \frac{[f(u)]}{[u]}\Big|_{x=\phi(t)}, \tag{5.3}$$

$$\dot e(t) = \Big([f'(u)v] - [v]\frac{[f(u)]}{[u]}\Big)\Big|_{x=\phi(t)}, \tag{5.4}$$

$$\dot g(t) = \Big([f''(u)v^2 + f'(u)w] - [w]\frac{[f(u)]}{[u]}\Big)\Big|_{x=\phi(t)}, \tag{5.5}$$

$$\frac{d}{dt}\Big(h(t)[u(\phi(t),t)]\Big) = \frac{de^2(t)}{dt} \tag{5.6}$$

hold along Γ. *Here* e, g, h *can be treated as functions of the single variable* t, *so that* $e(t) \stackrel{def}{=} e(\phi(t),t)$, $g(t) \stackrel{def}{=} g(\phi(t),t)$, $h(t) \stackrel{def}{=} h(\phi(t),t)$.

Proof. Setting $F(u,v) = f(u)$, $G(u,v) = f'(u)v$ and repeating the proof of Theorem 2.1 word for word, we prove conditions (5.3) and (5.4).

Conditions (5.5), (5.6) are proved similarly. Let $\mathbf{n} = (\nu_1,\nu_2)$ be a unit normal to the curve Γ oriented from Ω_- to Ω_+, $\mathbf{l} = (-\nu_2,\nu_1)$ be a unit tangential vector to Γ. The triple $(u,\widehat v,\widehat w)$ being a smooth solution of the system (1.16) in the domains $\Omega_\pm$ by applying (5.2) to a test function $\varphi(x,t) \in \mathcal{D}(\mathbb{R}\times(0,+\infty))$, taking into account (1.4), (2.5), and integrating by parts, we obtain the identity

$$\begin{aligned}
0 &= \int_\Omega \Big(\widehat w\varphi_t + \big(f''(u)\widehat v^2 + f'(u)\widehat w\big)\varphi_x\Big)\,dx\,dt + \int_\Gamma g(x,t)\frac{\delta\varphi(x,t)}{\delta t}\frac{dl}{\sqrt{1+u_\delta^2}} \\
&\quad + \int_\Gamma h(x,t)\frac{\delta\varphi_x(x,t)}{\delta t}\frac{dl}{\sqrt{1+u_\delta^2}} + \int_\Gamma \frac{\frac{\delta e^2(x,t)}{\delta t} - h(x,t)\frac{\partial[u(x,t)]}{\partial \mathbf{l}}}{[u(x,t)]}\varphi_x(x,t)\frac{dl}{\sqrt{1+u_\delta^2}} \\
&= \int_\Gamma \Big([w]\nu_2 + [f''(u)v^2 + f'(u)w]\nu_1\Big)\varphi(x,t)\,dl + \int_\Gamma g(x,t)\frac{\delta\varphi(x,t)}{\delta t}\frac{dl}{\sqrt{1+(\dot\phi(t))^2}} \\
&\quad + \int_\Gamma h(x,t)\frac{\delta\varphi_x(x,t)}{\delta t}\frac{dl}{\sqrt{1+(\dot\phi(t))^2}} \\
&\qquad + \int_\Gamma \frac{\frac{\delta e^2(x,t)}{\delta t} - h(x,t)\frac{\partial[u(x,t)]}{\partial \mathbf{l}}}{[u(x,t)]}\varphi_x(x,t)\frac{dl}{\sqrt{1+(\dot\phi(t))^2}} \\
&= \int_0^\infty \Big(-[w]\dot\phi(t) + [f''(u)v^2 + f'(u)w]\Big)\varphi(\phi(t),t)\,dt \\
&\quad + \int_0^\infty g(t)\frac{d\varphi(\phi(t),t)}{dt}\,dt \\
&\quad + \int_0^\infty h(t)\frac{d\varphi_x(\phi(t),t)}{dt}\,dt + \int_0^\infty \frac{\frac{de^2(t)}{dt} - h(t)\frac{d[u(\phi(t),t)]}{dt}}{[u(\phi(t),t)]}\varphi_x(\phi(t),t)\,dt
\end{aligned}$$

$$= \int_0^\infty \Big(-[w]\dot{\phi}(t) + [f''(u)v^2 + f'(u)w] - \dot{g}(t) \Big) \varphi(\phi(t), t)\, dt$$

$$+ \int_0^\infty \left(\frac{\frac{de^2(t)}{dt} - h(t)\frac{d[u(\phi(t),t)]}{dt}}{[u(\phi(t),t)]} - \dot{h}(t) \right) \varphi_x(\phi(t), t)\, dt. \tag{5.7}$$

Here we take into account that integrating by parts we can easily see that

$$\int_\Gamma \psi(x,t) \frac{\delta \varphi(x,t)}{\delta t} \frac{dl}{\sqrt{1+u_\delta^2}} = -\int_\Gamma \frac{\delta \psi(x,t)}{\delta t} \varphi(x,t) \frac{dl}{\sqrt{1+u_\delta^2}}$$
$$= -\int_\Gamma \frac{d\psi(\phi(t),t)}{dt} \varphi(\phi(t), t)\, dt,$$

where $\psi(x,t)$ is a smooth function.

Since $\varphi(\phi(t),t)$ and $\varphi_x(\phi(t),t)$ are arbitrary smooth functions, and

$$\frac{\frac{de^2(t)}{dt} - h(t)\frac{d[u(\phi(t),t)]}{dt}}{[u(\phi(t),t)]} - \dot{h}(t) = \frac{\frac{de^2(t)}{dt} - \frac{dh(t)u(\phi(t),t)}{dt}}{[u(\phi(t),t)]},$$

we conclude that in view of (5.7) conditions (5.5), (5.6) hold along Γ. $\square$

The system of the Rankine–Hugoniot conditions (5.3)–(5.6) determines the trajectory $x = \phi(t)$ of a δ'-shock wave and the coefficients $e(t)$, $g(t)$, $h(t)$ of the singularities. The first equation in this system is the "standard" Rankine–Hugoniot condition for the *shock* (cf. (1.3) or (1.5)), while the first and second equations are the "standard" *Rankine–Hugoniot conditions for the δ-shock* (cf. (2.7)). The right-hand sides of equalities (5.4), (5.5) are the *first Rankine-Hugoniot deficits*, while the right-hand side of (5.6) is the *second Rankine-Hugoniot deficit.*

The integral identities (5.2) differ from the classical integral identities (1.2) (for $m = 3$) by the additional terms in the second and third identities. Here the terms

$$\sum_{i \in I} \int_{\gamma_i} e_i(x,t) \frac{\delta \varphi(x,t)}{\delta t} \frac{dl}{\sqrt{1+u_\delta^2}}, \quad \sum_{i \in I} \int_{\gamma_i} g_i(x,t) \frac{\delta \varphi(x,t)}{\delta t} \frac{dl}{\sqrt{1+u_\delta^2}}$$

appear due to the delta functions in v w, and the terms

$$\sum_{i \in I} \Big(\int_{\gamma_i} h_i(x,t) \frac{\delta \varphi_x(x,t)}{\delta t} \frac{dl}{\sqrt{1+u_\delta^2}}$$
$$+ \int_{\gamma_i} \frac{\frac{\delta e_i^2(x,t)}{\delta} - h_i(x,t)\frac{\partial [u(x,t)]}{\partial \mathbf{l}}}{[u(x,t)]} \varphi_x(x,t) \frac{dl}{\sqrt{1+u_\delta^2}} \Big)$$

appear due to the derivative of delta function in w. Moreover, the first integral identity in (5.2) is a "standard" type integral identity (cf. (1.2)), while the first and second integral identities in (5.2) constitute δ-shock type integral identities (cf. Definition 2.1). The third integral identity in (5.2) is a special type of δ'-shock type integral identity.

6. The Cauchy problems admitting δ'-shocks

6.1. Weak asymptotic solutions. Similarly to Subsec. 3.1, we introduce the following definition.

DEFINITION 6.1. ([**39**]) A triple of functions $\big(u_\varepsilon(x,t), v_\varepsilon(x,t), w_\varepsilon(x,t)\big)$, smooth as $\varepsilon > 0$, $t \in [0,T]$ is called a *weak asymptotic solution* of system (1.16) with the initial data $(u^0(x), v^0(x), w^0(x))$ if

$$\begin{aligned}
\int L_1[u_\varepsilon(x,t)]\psi(x)\,dx &= o(1),\\
\int L_2[u_\varepsilon(x,t), v_\varepsilon(x,t)]\psi(x)\,dx &= o(1),\\
\int L_3[u_\varepsilon(x,t), v_\varepsilon(x,t), w_\varepsilon(x,t)]\psi(x)\,dx &= o(1),\\
\int \Big(u_\varepsilon(x,0) - u^0(x)\Big)\psi(x)\,dx &= o(1),\\
\int \Big(v_\varepsilon(x,0) - v^0(x)\Big)\psi(x)\,dx &= o(1),\\
\int \Big(w_\varepsilon(x,0) - w^0(x)\Big)\psi(x)\,dx &= o(1), \quad \varepsilon \to +0,
\end{aligned}$$

for all $\psi(x) \in \mathcal{D}(\mathbb{R})$. The latter relations can be rewritten as

$$\begin{aligned}
L_1[u_\varepsilon(x,t)] &= o_{\mathcal{D}'}(1),\\
L_2[u_\varepsilon(x,t), v_\varepsilon(x,t)] &= o_{\mathcal{D}'}(1),\\
L_3[u_\varepsilon(x,t), v_\varepsilon(x,t), w_\varepsilon(x,t)] &= o_{\mathcal{D}'}(1),\\
u_\varepsilon(x,0) &= u^0(x) + o_{\mathcal{D}'}(1),\\
v_\varepsilon(x,0) &= v^0(x) + o_{\mathcal{D}'}(1),\\
w_\varepsilon(x,0) &= w^0(x) + o_{\mathcal{D}'}(1), \quad \varepsilon \to +0,
\end{aligned} \tag{6.1}$$

where the first three estimates are uniform in $t \in [0,T]$.

In (6.1) all distributions in u, v, w depend on t as a parameter.

Within the framework of the *weak asymptotics method*, we find a δ'-shock wave type solution of the Cauchy problem as a weak limit

$$\begin{aligned}
u(x,t) &= \lim_{\varepsilon\to+0} u_\varepsilon(x,t),\\
v(x,t) &= \lim_{\varepsilon\to+0} v_\varepsilon(x,t),\\
w(x,t) &= \lim_{\varepsilon\to+0} w_\varepsilon(x,t),
\end{aligned} \tag{6.2}$$

of the *weak asymptotic solution* $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$ to the corresponding Cauchy problem.

Next, using the *weak asymptotic solution* of the Cauchy problem, we need to prove that the left-hand sides of the relations

$$
\begin{aligned}
\lim_{\varepsilon\to+0}\int_0^\infty\int L_1[u_\varepsilon(x,t)]\varphi(x,t)\,dx\,dt &= 0,\\
\lim_{\varepsilon\to+0}\int_0^\infty\int L_2[u_\varepsilon(x,t),v_\varepsilon(x,t)]\varphi(x,t)\,dx\,dt &= 0,\\
\lim_{\varepsilon\to+0}\int_0^\infty\int L_3[u_\varepsilon(x,t),v_\varepsilon(x,t),w_\varepsilon(x,t)]\varphi(x,t)\,dx\,dt &= 0
\end{aligned}
$$

coincide with the left-hand sides of the integral identities (5.2) for all test functions $\varphi(x,t)\in\mathcal{D}(\mathbb{R}\times[0,\infty))$.

According to [**39**]– [**41**], a *weak asymptotic solution* of the Cauchy problem (1.16), (1.18) admitting δ'-shocks is constructed in the form of the smooth Ansatz

$$
\begin{aligned}
u_\varepsilon(x,t) &= \widetilde{u}_\varepsilon(x,t)+R_u(x,t,\varepsilon),\\
v_\varepsilon(x,t) &= \widetilde{v}_\varepsilon(x,t)+R_v(x,t,\varepsilon),\\
w_\varepsilon(x,t) &= \widetilde{w}_\varepsilon(x,t)+R_w(x,t,\varepsilon), \quad \varepsilon>0.
\end{aligned}
$$

Here the functions $(\widetilde{u}_\varepsilon,\widetilde{v}_\varepsilon,\widetilde{w}_\varepsilon)$ are regularizations of the singular Ansatz (1.19) with respect to singularities $H(-x+\phi(t))$, $\delta(-x+\phi(t))$, $\delta'(-x+\phi(t))$, and the *corrections* $R_u(x,t,\varepsilon)$, $R_v(x,t,\varepsilon)$, $R_w(x,t,\varepsilon)$ are desired functions which are assumed to admit the estimates (3.3).

We will seek a *weak asymptotic solution* of the Cauchy problem (1.16), (1.18) in the form:

$$
\begin{aligned}
u_\varepsilon(x,t) &= u_+(x,t)+[u(x,t)]H_u(-x+\phi(t),\varepsilon)+R_u(x,t,\varepsilon),\\
v_\varepsilon(x,t) &= v_+(x,t)+[v(x,t)]H_v(-x+\phi(t),\varepsilon)\\
&\qquad +e(t)\delta_v(-x+\phi(t),\varepsilon)+R_v(x,t,\varepsilon),\\
w_\varepsilon(x,t) &= w_+(x,t)+[w(x,t)]H_w(-x+\phi(t),\varepsilon)\\
&\quad +g(t)\delta_w(-x+\phi(t),\varepsilon)+h(t)\delta'_w(-x+\phi(t),\varepsilon)\\
&\qquad +R_w(x,t,\varepsilon).
\end{aligned}
\tag{6.3}
$$

Here $\delta_e(\xi,\varepsilon)$ given by (3.6) and

$$
\delta_w(\xi,\varepsilon)=\frac{1}{\varepsilon}\omega_g\Big(\frac{\xi}{\varepsilon}\Big)
\tag{6.4}
$$

are regularizations of the δ-function,

$$
\delta'_w(\xi,\varepsilon)=\frac{1}{\varepsilon^2}\omega'_h\Big(\frac{\xi}{\varepsilon}\Big)
\tag{6.5}
$$

is a regularization of the distribution δ', $H_j(\xi,\varepsilon)$ given by (3.7) are regularizations of the Heaviside function $H(\xi)$, $j=u,v,w$.

6.2. The Cauchy problems. In the papers [**39**], [**40**], [**51**], by using the *weak asymptotics method*, the Cauchy problem (1.17), (1.18) admitting δ'-shocks was solved.

We used the following admissibility condition for δ'-shocks:

$$f'(u_+) \le \dot{\phi}(t) \le f'(u_-), \tag{6.6}$$

where $\dot{\phi}(t)$ is the velocity of the δ'-shock wave, and u_-, u_+ are the respective left- and right-hand values of u at the discontinuity curve. Condition (6.6), as well as condition (3.8) mean that all characteristics on both sides of the discontinuity are in-coming.

For systems (1.17) the eigenvalues of the characteristic matrix are $\lambda_1(u) = \lambda_2(u) = \lambda_3(u) = 2u$. We assume that the "overcompression" condition (6.6) is satisfied.

To construct a *weak asymptotic solution* (6.3) of the Cauchy problem (1.17), (1.18), we choose corrections in the form

$$\begin{aligned} R_u(x,t,\varepsilon) &= 0, \\ R_v(x,t,\varepsilon) &= 0, \\ R_w(x,t,\varepsilon) &= P(t)\tfrac{1}{\varepsilon^2}\Omega'''_P\Big(\tfrac{-x+\phi(t)}{\varepsilon}\Big), \end{aligned} \tag{6.7}$$

where $P(t)$ is the desired function, $\frac{1}{\varepsilon^4}\Omega'''_P\big(\frac{x}{\varepsilon}\big)$ is a regularization of the distribution $\delta'''(x)$. Consequently, $R_w(x,t,\varepsilon) = \varepsilon^2 P(t)\delta'''_P(-x+\phi(t),\varepsilon) \in O_{\mathcal{D}'}(\varepsilon)$, i.e., estimates (3.3) hold.

Using (6.7) and repeating the scheme of the construction from Theorem 3.1, we obtain the weak asymptotic solution (6.3) of the Cauchy problem (1.17), (1.18). Next, using this weak asymptotic solution, one can prove the following theorem.

THEOREM 6.1. ([**39**]) *Let* $2u^0_+(0) \le \dot{\phi}(0) \le 2u^0_-(0)$, *i.e.,* (6.6) *holds. Then there exist* $T > 0$ *and a zero neighborhood* $K \subset \mathbb{R}$ *such that for* $(x,t) \in K \times [0,\,T)$ *the Cauchy problem* (1.17), (1.18) *has a unique solution* (1.19) *which satisfies the integral identities* (5.2) *where* $\Gamma = \{(x,t) : x = \phi(t),\ t \in [0,\ T)\}$, *and functions* $u_\pm(x,t)$, $v_\pm(x,t)$, $w_\pm(x,t)$, $\phi(t)$, $e(t)$, $g(t)$, $h(t)$ *are defined by* *the system*

$$
\begin{array}{rcl}
u_{\pm t}+\big(u_{\pm}^2\big)_x &=& 0, \quad \pm x>\pm\phi(t),\\
v_{\pm t}+2\big(u_{\pm}v_{\pm}\big)_x &=& 0, \quad \pm x>\pm\phi(t),\\
w_{\pm t}+2\big(v_{\pm}^2+u_{\pm}w_{\pm}\big)_x &=& 0, \quad \pm x>\pm\phi(t),\\
\dot\phi(t) &=& \frac{[u^2]}{[u]}\Big|_{x=\phi(t)}=(u_-+u_+)\big|_{x=\phi(t)},\\
\dot e(t) &=& \Big(2[vu]-[v]\frac{[u^2]}{[u]}\Big)\Big|_{x=\phi(t)}\\
&=& [u](v_-+v_+)\big|_{x=\phi(t)},\\
\dot g(t) &=& \Big(2[v^2+uw]-[w]\frac{[u^2]}{[u]}\Big)\Big|_{x=\phi(t)}\\
&=& \big(2[v](v_-+v_+)+[u](w_-+w_+)\big)\Big|_{x=\phi(t)},\\
\dfrac{d\big(h(t)[u(\phi(t),t)]\big)}{dt} &=& \dfrac{de^2(t)}{dt},
\end{array}
\tag{6.8}
$$

where the initial data are defined from (1.18), $\phi(0)=0$.

The last two equations in (3.16) and (3.36), constitute the corresponding Rankine–Hugoniot conditions for δ-shocks and are particular cases of (2.7). The last two equations in (3.41) constitute the Rankine–Hugoniot conditions for δ-shocks and are particular cases of (2.13).

The last four equations in (6.8) give the Rankine–Hugoniot conditions for δ'-shocks and are particular cases of (5.3)–(5.6).

If we consider piecewise constant initial data in Theorem 6.1 then $T=\infty$.

7. A Riemann problem admitting shocks, δ-shocks, δ'-shocks, and vacuum states

7.1. The vanishing viscosity approach. In this section, by using the *vanishing viscosity method*, we solve the Riemann problem for system (1.17) with the initial data

$$
\big(u^0(x),v^0(x),w^0(x)\big)=\begin{cases}(u_-,v_-,w_-), & x<0,\\ (u_+,v_+,w_+), & x>0,\end{cases}
\tag{7.1}
$$

where $u_+=u_0^0$, $v_+=v_0^0$, $w_+=w_0^0$, $u_-=u_0^0+u_1^0$, $v_-=v_0^0+w_1^0$, $w_-=w_0^0+w_1^0$ are given constants (for details, see [**51**]). The initial data (7.1) are a particular case of the initial data (1.18).

First, we construct solutions to the parabolic approximation of system (1.17)

$$
\begin{array}{rcl}
u_{\varepsilon\,t}+\big(u_\varepsilon^2\big)_x &=& \varepsilon u_{\varepsilon\,xx},\\
v_{\varepsilon\,t}+2\big(u_\varepsilon v_\varepsilon\big)_x &=& \varepsilon v_{\varepsilon\,xx},\\
w_{\varepsilon\,t}+2\big(v_\varepsilon^2+u_\varepsilon w_\varepsilon\big)_x &=& \varepsilon w_{\varepsilon\,xx}
\end{array}
\tag{7.2}
$$

with the initial data (7.1).

By the Hopf-Cole transformations

$$u_\varepsilon(x,t) = -\varepsilon\frac{A_{\varepsilon\,x}}{A_\varepsilon}, \ v_\varepsilon(x,t) = -\varepsilon\Big(\frac{B_\varepsilon}{A_\varepsilon}\Big)_x, \ w_\varepsilon(x,t) = -\varepsilon\Big(\frac{A_\varepsilon C_\varepsilon - B_\varepsilon^2}{A_\varepsilon^2}\Big)_x \tag{7.3}$$

system (7.2) is reduced to the linear system of heat equations

$$A_{\varepsilon\,t} = \varepsilon A_{\varepsilon\,xx}, \quad B_{\varepsilon\,t} = \varepsilon B_{\varepsilon\,xx}, \quad C_{\varepsilon\,t} = \varepsilon C_{\varepsilon\,xx}. \tag{7.4}$$

The initial data for the last system read off from the initial data (7.1) and the Hopf-Cole transformations (7.3) and have the form:

$$\big(A_\varepsilon^0(x),\, B_\varepsilon^0(x),\, C_\varepsilon^0(x)\big)$$

$$= \begin{cases} \Big(e^{-\frac{u_- x}{\varepsilon}},\, -\frac{v_- x}{\varepsilon}e^{-\frac{u_- x}{\varepsilon}},\, (\frac{v_-^2 x^2}{\varepsilon^2} - \frac{w_- x}{\varepsilon})e^{-\frac{u_- x}{\varepsilon}}\Big), & x \ < \ 0, \\ \Big(e^{-\frac{u_+ x}{\varepsilon}},\, -\frac{v_+ x}{\varepsilon}e^{-\frac{u_+ x}{\varepsilon}},\, (\frac{v_+^2 x^2}{\varepsilon^2} - \frac{w_+ x}{\varepsilon})e^{-\frac{u_- x}{\varepsilon}}\Big), & x \ > \ 0. \end{cases} \tag{7.5}$$

It is well known that a solution of the heat equation with the initial data

$$\Phi_{\varepsilon\,t} = \varepsilon\Phi_{\varepsilon\,xx}, \qquad \Phi_\varepsilon(x,0) = \Phi_\varepsilon^0(x)$$

has the following form

$$\Phi_\varepsilon(x,t) = \frac{1}{\sqrt{4\pi t\varepsilon}}\int_{-\infty}^{\infty}\Phi_\varepsilon^0(y)\exp\Big(-\frac{(x-y)^2}{4t\varepsilon}\Big)\,dy. \tag{7.6}$$

By substituting the initial data (7.5) into formula (7.6), we obtain a solution of the problem (7.4), (7.5):

$$\begin{array}{rcl} A_\varepsilon(x,t) & = & a_-^\varepsilon(x,t) + a_+^\varepsilon(x,t), \\ B_\varepsilon(x,t) & = & b_-^\varepsilon(x,t) + b_+^\varepsilon(x,t), \\ C_\varepsilon(x,t) & = & c_-^\varepsilon(x,t) + c_+^\varepsilon(x,t), \end{array} \tag{7.7}$$

where

$$\begin{array}{rcl} a_-^\varepsilon(x,t) & = & \displaystyle\frac{1}{\sqrt{4\pi t\varepsilon}}\int_{-\infty}^{0}\exp\Big(-\frac{(x-y)^2}{4t\varepsilon} - \frac{u_-}{\varepsilon}y\Big)\,dy, \\ a_+^\varepsilon(x,t) & = & \displaystyle\frac{1}{\sqrt{4\pi t\varepsilon}}\int_{0}^{\infty}\exp\Big(-\frac{(x-y)^2}{4t\varepsilon} - \frac{u_+}{\varepsilon}y\Big)\,dy, \end{array} \tag{7.8}$$

$$\begin{array}{rcl} b_-^\varepsilon(x,t) & = & \displaystyle\frac{1}{\sqrt{4\pi t\varepsilon}}\int_{-\infty}^{0}\Big(-\frac{v_-}{\varepsilon}y\Big)\,\exp\Big(-\frac{(x-y)^2}{4t\varepsilon} - \frac{u_-}{\varepsilon}y\Big)\,dy, \\ b_+^\varepsilon(x,t) & = & \displaystyle\frac{1}{\sqrt{4\pi t\varepsilon}}\int_{0}^{\infty}\Big(-\frac{v_+}{\varepsilon}y\Big)\,\exp\Big(-\frac{(x-y)^2}{4t\varepsilon} - \frac{u_+}{\varepsilon}y\Big)\,dy, \end{array} \tag{7.9}$$

$$
\begin{aligned}
c^{\varepsilon}_{-}(x,t) &= \frac{1}{\sqrt{4\pi t\varepsilon}}\int_{-\infty}^{0}\Big(\frac{v_{-}^{2}}{\varepsilon^{2}}y^{2}-\frac{w_{-}}{\varepsilon}y\Big)\exp\Big(-\frac{(x-y)^{2}}{4t\varepsilon}-\frac{u_{-}}{\varepsilon}y\Big)\,dy,\\
c^{\varepsilon}_{+}(x,t) &= \frac{1}{\sqrt{4\pi t\varepsilon}}\int_{0}^{\infty}\Big(\frac{v_{+}^{2}}{\varepsilon^{2}}y^{2}-\frac{w_{+}}{\varepsilon}y\Big)\exp\Big(-\frac{(x-y)^{2}}{4t\varepsilon}-\frac{u_{+}}{\varepsilon}y\Big)\,dy.
\end{aligned}
\tag{7.10}
$$

Using the above solution (7.7)–(7.10) of the heat problem (7.4), (7.5), we find a solution of the problem (7.2), (7.1) by the following lemma.

LEMMA 7.1. ([**51**, Lemma 4.1]) *A solution* $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$ *of the problem* (7.2), (7.1) *is represented in the form*

$$
u_\varepsilon(x,t)=\frac{u_{-}a^{\varepsilon}_{-}+u_{+}a^{\varepsilon}_{+}}{a^{\varepsilon}_{-}+a^{\varepsilon}_{+}},
\tag{7.11}
$$

$$
v_\varepsilon(x,t)=V_{\varepsilon\,x}(x,t),
\tag{7.12}
$$

$$
w_\varepsilon(x,t)=W_{\varepsilon\,x}(x,t),
\tag{7.13}
$$

where

$$
\begin{aligned}
V_\varepsilon(x,t) &= -\varepsilon\frac{B_\varepsilon}{A_\varepsilon}\\
&= \frac{v_{-}(x-2u_{-}t)a^{\varepsilon}_{-}+v_{+}(x-2u_{+}t)a^{\varepsilon}_{+}-(v_{-}-v_{+})\sqrt{\frac{t\varepsilon}{\pi}}e^{-\frac{x^2}{4t\varepsilon}}}{a^{\varepsilon}_{-}+a^{\varepsilon}_{+}},
\end{aligned}
\tag{7.14}
$$

$$
W_\varepsilon(x,t)=-\varepsilon\bigg(\frac{A_\varepsilon C_\varepsilon-B_\varepsilon^2}{A_\varepsilon^2}\bigg)=-\varepsilon\frac{C_\varepsilon}{A_\varepsilon}+\frac{1}{\varepsilon}\Big(V_\varepsilon\Big)^2,
\tag{7.15}
$$

where A_ε, B_ε, C_ε *are given by formulas* (7.7)–(7.10), *and*

$$
B_\varepsilon(x,t)=-\frac{1}{\varepsilon}\Big(v_{-}(x-2u_{-}t)a^{\varepsilon}_{-}+v_{+}(x-2u_{+}t)a^{\varepsilon}_{+}-(v_{-}-v_{+})\sqrt{\frac{t\varepsilon}{\pi}}e^{-\frac{x^2}{4t\varepsilon}}\Big),
\tag{7.16}
$$

$$
\begin{aligned}
C_\varepsilon(x,t) &= \frac{2t}{\varepsilon}\big(v_{-}^2a^{\varepsilon}_{-}+v_{+}^2a^{\varepsilon}_{+}\big)+\frac{1}{\varepsilon^2}\big(v_{-}^2(x-2u_{-}t)^2a^{\varepsilon}_{-}+v_{+}^2(x-2u_{+}t)^2a^{\varepsilon}_{+}\big)\\
&\quad -\frac{1}{\varepsilon^2}\sqrt{\frac{t\varepsilon}{\pi}}e^{-\frac{x^2}{4t\varepsilon}}\big(v_{-}^2(x-2u_{-}t)-v_{+}^2(x-2u_{+}t)\big)\\
&\quad -\frac{1}{\varepsilon}\Big(w_{-}(x-2u_{-}t)a^{\varepsilon}_{-}+w_{+}(x-2u_{+}t)a^{\varepsilon}_{+}-(w_{-}-w_{+})\sqrt{\frac{t\varepsilon}{\pi}}e^{-\frac{x^2}{4t\varepsilon}}\Big).
\end{aligned}
\tag{7.17}
$$

To solve the Riemann problem (1.17), (7.1), we have to calculate a weak limit of the solution to the parabolic problem (7.2), (7.1).

7.2. Weak limit of the solution to the problem (7.2), (7.1). Using the solution to the parabolic problem (7.2), (7.1) given by (7.11)–(7.17), one can prove the following theorems.

THEOREM 7.1. ([**51**, Theorem 5.1.]) *Let $u_+ \le u_-$. If $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$ is a solution of the parabolic problem (7.2), (7.1) then for $t \in [0, \infty)$ we have in the weak sense*

$$\begin{aligned}
u(x,t) &= \lim_{\varepsilon\to+0} u_\varepsilon(x,t) = u_+ + [u]H(-x+\phi(t)), \\
v(x,t) &= \lim_{\varepsilon\to+0} v_\varepsilon(x,t) = v_+ + [v]H(-x+\phi(t)) \\
&\qquad +e(t)\delta(-x+\phi(t)), \\
w(x,t) &= \lim_{\varepsilon\to+0} w_\varepsilon(x,t) = w_+ + [w]H(-x+\phi(t)) \\
&\qquad +g(t)\delta(-x+\phi(t)) + h(t)\delta'(-x+\phi(t)),
\end{aligned} \tag{7.18}$$

where

$$\begin{aligned}
\phi(t) &= ct = \tfrac{[u^2]}{[u]}t = \big(u_- + u_+\big)t, \\
e(t) &= \Big(2[uv] - [v]\dot\phi(t)\Big)t = [u](v_- + v_+)t, \\
g(t) &= \Big(2[v^2+uw] - [w]\dot\phi(t)\Big)t \\
&= \big(2[v](v_- + v_+) + [u](w_- + w_+)\big)t, \\
h(t) &= [u](v_- + v_+)^2 t^2.
\end{aligned} \tag{7.19}$$

Moreover,

$$h(t) = \frac{e^2(t)}{[u]}. \tag{7.20}$$

THEOREM 7.2. ([**51**, Theorem 6.1.]) *Let $u_+ > u_-$. If $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$ is a solution of the parabolic problem (7.2), (7.1) then for $t \in [0, \infty)$ we have in the weak sense*

$$\begin{aligned}
&\big(u(x,t), v(x,t), w(x,t)\big) = \lim_{\varepsilon\to+0} \big(u_\varepsilon(x,t), v_\varepsilon(x,t), w_\varepsilon(x,t)\big) \\
&= \begin{cases} (u_-, v_-, w_-), & x \le 2u_-t, \\ \big(\frac{x}{2t}, 0, 0\big), & 2u_-t < x < 2u_+t, \\ (u_+, v_+, w_+), & x \ge 2u_+t, \end{cases} \\
&= (u_+, v_+, w_+)\big(1 - H(-x+2u_+t)\big) + (u_-, v_-, w_-)H(-x+2u_-t) \\
&\qquad +\Big(\frac{x}{2t}, 0, 0\Big)\big(H(-x+2u_+t) - H(-x+2u_-t)\big).
\end{aligned} \tag{7.21}$$

7.3. δ- and δ'-shock in the Riemann problem (1.17), (7.1). Now we prove that the triple of distributions (7.18) constructed by Theorem 7.1 is a δ'-shock wave type solution of the Cauchy problem (1.17), (7.1) for $u_+ \le u_-$.

THEOREM 7.3. ([**51**, Theorem 7.1.]) *Let $u_+ \le u_-$. Then for $t \in [0,\ \infty)$, the Cauchy problem* (1.17), (7.1) *has a unique generalized δ'-shock wave type solution* (1.19) (*see* (7.18))

$$\begin{aligned}
u(x,t) &= u_+ + [u]H(-x+\phi(t)),\\
v(x,t) &= v_+ + [v]H(-x+\phi(t)) + e(t)\delta(-x+\phi(t)),\\
w(x,t) &= w_+ + [w]H(-x+\phi(t)) + g(t)\delta(-x+\phi(t))\\
&\qquad\qquad +h(t)\delta'(-x+\phi(t)),
\end{aligned}$$

which satisfies the integral identities (5.2)*:*

$$\begin{aligned}
\int_0^\infty\int\Big(u(x,t)\varphi_t + u^2(x,t)\varphi_x\Big)\,dx\,dt + \int u^0(x)\varphi(x,0)\,dx &= 0,\\
\int_0^\infty\int\Big(\widehat{v}(x,t)\varphi_t + 2u(x,t)\widehat{v}(x,t)\varphi_x\Big)\,dx\,dt&\\
+\int_\Gamma e(t)\frac{\partial\varphi(x,t)}{\partial\mathbf{l}}\,dl + \int\widehat{v}^0(x)\varphi(x,0)\,dx &= 0,\\
\int_0^\infty\int\Big(\widehat{w}(x,t)\varphi_t + 2\big(\widehat{v}^2(x,t)+u(x,t)\widehat{w}(x,t)\big)\varphi_x\Big)\,dx\,dt&\\
+\int_\Gamma g(t)\frac{\partial\varphi(x,t)}{\partial\mathbf{l}}\,dl + \int_\Gamma h(x,t)\frac{\partial\varphi_x(x,t)}{\partial\mathbf{l}}\,dl&\\
+\int_\Gamma\frac{\frac{\partial e^2(x,t)}{\partial\mathbf{l}}}{[u]}\varphi_x(x,t)\,dl + \int\widehat{w}^0(x)\varphi(x,0)\,dx &= 0,
\end{aligned}\tag{7.22}$$

for all $\varphi(x,t) \in \mathcal{D}(\mathbb{R}\times[0,\ \infty))$, where $e(t)$, $g(t)$, $h(t)$ are given by (7.19)*. Here $\Gamma = \{(x,t) : x = \phi(t) = ct,\ t \ge 0\}$, $\widehat{v}(x,t) = v_+ + [v]H(-x+\phi(t))$, $\widehat{w}(x,t) = w_+ + [w]H(-x+\phi(t))$, and* (*see* (2.5))

$$\begin{aligned}
\int_\Gamma e(x,t)\frac{\partial\varphi(x,t)}{\partial\mathbf{l}}\,dl &= \int_0^\infty e(t)\frac{d\varphi(\phi(t),t)}{dt}\,dt,\\
\int_\Gamma g(x,t)\frac{\partial\varphi(x,t)}{\partial\mathbf{l}}\,dl &= \int_0^\infty g(t)\frac{d\varphi(\phi(t),t)}{dt}\,dt,\\
\int_\Gamma h(x,t)\frac{\partial\varphi_x(x,t)}{\partial\mathbf{l}}\,dl &= \int_0^\infty h(t)\frac{d\varphi_x(\phi(t),t)}{dt}\,dt,\\
\int_\Gamma\frac{\frac{\partial e^2(x,t)}{\partial\mathbf{l}} - h(x,t)\frac{\partial[u(x,t)]}{\partial\mathbf{l}}}{[u(x,t)]}\varphi_x(x,t)\,dl &= \int_0^\infty\frac{\frac{de^2(t)}{dt}}{[u]}\varphi_x(\phi(t),t)\,dt.
\end{aligned}$$

Moreover, for this solution the admissibility condition (6.6) *holds.*

PROOF. Let $\Omega \subset \mathbb{R}\times[0,\infty)$ be some region and suppose that the curve $\Gamma = \{(x,t) : x = \phi(t) = ct,\ t \ge 0\}$ cuts it into a left- and right-hand parts $\Omega_\pm = \{(x,t) : \pm(x-ct) > 0\}$. Let $\mathbf{n} = (\nu_1,\nu_2) = \frac{(1,-\dot\phi(t))}{\sqrt{1+(\dot\phi(t))^2}} = \frac{(1,-c)}{\sqrt{1+c^2}}$ be the unit normal to the curve Γ pointing from Ω_- into Ω_+, and let $\mathbf{l} = (-\nu_2,\nu_1) = \frac{(c,1)}{\sqrt{1+c^2}}$ be the tangential vector to Γ (see (1.4)).

Choosing a test function $\varphi(x,t)$ with support in Ω, we deduce that the left-hand side of the first relation in (7.22) is transformed to the form

$$\int_0^\infty \int \Big(u(x,t)\varphi_t + u^2(x,t)\varphi_x\Big)\,dx\,dt + \int u^0(x)\varphi(x,0)\,dx$$

$$= \int\int_{\Omega_-} \big(u_-\varphi_t + u_-^2\varphi_x\big)\,dx\,dt + \int\int_{\Omega_+} \big(u_+\varphi_t + u_+^2\varphi_x\big)\,dx\,dt$$

$$+ \int_{-\infty}^0 u^0(x)\varphi(x,0)\,dx + \int_0^\infty u^0(x)\varphi(x,0)\,dx. \tag{7.23}$$

Next, integrating by parts and taking into account that $\frac{dx}{dt} = -\frac{\nu_2}{\nu_1} = -c$ and $\nu_1 dl = dt$, we obtain

$$\int\int_{\Omega_\pm} \big(u_\pm\varphi_t + u_\pm^2\varphi_x\big)\,dx\,dt$$

$$= \mp \int_\Gamma \big(\nu_2 u_\pm + \nu_1 u_\pm^2\big)\varphi\,dl \mp \int_0^{\pm\infty} u^0(x)\varphi(x,0)\,dx$$

$$= \mp \int_0^\infty \big(-cu_\pm + u_\pm^2\big)\varphi(ct,t)\,dt \mp \int_0^{\pm\infty} u^0(x)\varphi(x,0)\,dx. \tag{7.24}$$

Since according to the first equation in (7.19) $\dot\phi(t) = c = \frac{[u^2]}{[u]}$, relations (7.23), (7.24) imply

$$\int_0^\infty \int \Big(u(x,t)\varphi_t + u^2(x,t)\varphi_x\Big)\,dx\,dt + \int u^0(x)\varphi(x,0)\,dx$$

$$= \int_0^\infty \big(-c[u] + [u^2]\big)\varphi(ct,t)\,dt = 0. \tag{7.25}$$

Thus the first identity in (7.22) holds.

Applying the above calculations to the left-hand side of the second relation in (7.22), we obtain

$$\int_0^\infty \int \Big(\widehat{v}(x,t)\varphi_t + 2u(x,t)\widehat{v}(x,t)\varphi_x\Big)\,dx\,dt + \int \widehat{v}^0(x)\varphi(x,0)\,dx$$

$$= \int_\Gamma \big(\nu_2 v_- + \nu_1 2u_-v_-\big)\varphi\,dl - \int_\Gamma \big(\nu_2 v_+ + \nu_1 2u_+v_+\big)\varphi\,dl$$

$$= \int_\Gamma \big(\nu_2[v] + \nu_1 2[uv]\big)\varphi\,dl = \int_0^\infty \big(-c[v] + 2[uv]\big)\varphi(ct,t)\,dt. \tag{7.26}$$

Since integrating by parts, we have

$$\int_0^\infty t\frac{d\varphi(ct,t)}{dt}\,dt = -\int_0^\infty \varphi(ct,t)\,dt,$$

in view of the second equation in (7.19) and (2.5), we deduce that

$$\int_0^\infty \int \Big(\widehat{v}(x,t)\varphi_t + 2u(x,t)\widehat{v}(x,t)\varphi_x\Big)\,dx\,dt + \int \widehat{v}^0(x)\varphi(x,0)\,dx$$

$$= \int_0^\infty \big(-c[v] + 2[uv]\big)\varphi(ct,t)\,dt = -\int_0^\infty \big(-c[v] + 2[uv]\big)t\frac{d\varphi(ct,t)}{dt}\,dt$$

$$= -\int_0^\infty e(t)\frac{d\varphi(ct,t)}{dt}\,dt = -\int_\Gamma e(t)\frac{\partial\varphi(x,t)}{\partial\mathbf{l}}\,dl.$$

By substituting the last relation into the left-hand side of the second relation in (7.22), we see that the second identity in (7.22) holds.

Now, applying the above calculations to the left-hand side of the third relation in (7.22), we obtain

$$\int_0^\infty \int \Big(\widehat{w}(x,t)\varphi_t + 2\big(\widehat{v}^2(x,t) + u(x,t)\widehat{w}(x,t)\big)\varphi_x\Big)\,dx\,dt$$

$$+\int \widehat{w}^0(x)\varphi(x,0)\,dx = -\int_0^\infty g(t)\frac{d\varphi(ct,t)}{dt}\,dt = -\int_\Gamma g(t)\frac{\partial\varphi(x,t)}{\partial\mathbf{l}}\,dl,$$

where, according to (7.19), $g(t) = \big(2[v^2+uw] - [w]\frac{[u^2]}{[u]}\big)t$. Thus,

$$\int_0^\infty \int \Big(\widehat{w}(x,t)\varphi_t + 2\big(\widehat{v}^2(x,t) + u(x,t)\widehat{w}(x,t)\big)\varphi_x\Big)\,dx\,dt$$

$$+\int \widehat{w}^0(x)\varphi(x,0)\,dx + \int_\Gamma g(t)\frac{\partial\varphi(x,t)}{\partial\mathbf{l}}\,dl = 0. \tag{7.27}$$

According to (7.19), (7.20), $e(t) = [u](v_- + v_+)t$, $h(t) = \frac{e^2(t)}{[u]} = [u](v_- + v_+)^2t^2$. Consequently, taking into account that $[u]$ is a constant and integrating by parts, we have

$$\int_0^\infty h(t)\frac{d\varphi_x(ct,t)}{dt}\,dt = \int_0^\infty [u](v_- + v_+)^2t^2\frac{d\varphi_x(ct,t)}{dt}\,dt$$

$$= -\int_0^\infty 2[u](v_- + v_+)^2t\varphi_x(ct,t)\,dt = -\int_0^\infty \frac{\frac{de^2(t)}{dt}}{[u]}\varphi_x(ct,t)\,dt,$$

i.e., in view of (2.5),

$$\int_\Gamma h(x,t)\frac{\partial\varphi_x(x,t)}{\partial\mathbf{l}}\,dl + \int_\Gamma \frac{\frac{\partial e^2(x,t)}{\partial\mathbf{l}}}{[u]}\varphi_x(x,t)\,dl = 0. \tag{7.28}$$

By summing (7.27) and (7.28), we deduce that the third identity in (7.22) holds.

The proof is complete. □

Note that the functions in system (7.19) which determine the trajectory $x = \phi(t)$ of the δ'-shock wave and the coefficients $e(t)$, $g(t)$, $h(t)$ of the singularities constitute a solution to the system of the Rankine–Hugoniot conditions for δ'-shock (5.3)–(5.6).

If $u_+ \le u_-$, it follows from Theorems 7.1, 7.3 that $c = u_+ + u_- = \dot{\phi}(t)$ and $x = \phi(t) = ct$ are the velocity of motion and the trajectory of a δ'-shock wave, respectively. Moreover, Theorems 7.1, 7.3 imply the following statements.

Corollary 7.1. ([**51**, Corollary 7.1.]) *Let $u_+ < u_-$. The Riemann problem* (1.17), (7.1) *has*

(a.1) *a classical shock-solution* (1.19) *of the form*

$$\begin{aligned} u(x,t) &= u_+ + [u]H(-x+\phi(t)), \\ v(x,t) &= v_+ + [v]H(-x+\phi(t)), \\ w(x,t) &= w_+ + [w]H(-x+\phi(t)), \end{aligned} \tag{7.29}$$

if and only if $v_- + v_+ = 0$ and $w_- + w_+ = 0$;

(a.2) *a δ-shock solution* (1.19) *of the form*

$$\begin{aligned} u(x,t) &= u_+ + [u]H(-x+\phi(t)), \\ v(x,t) &= v_+ + [v]H(-x+\phi(t)), \\ w(x,t) &= w_+ + [w]H(-x+\phi(t)) + [u](w_- + w_+)t\delta(-x+\phi(t)). \end{aligned} \tag{7.30}$$

if $v_- + v_+ = 0$ and $w_- + w_+ \neq 0$, or

$$\begin{aligned} u(x,t) &= u_0, \\ v(x,t) &= v_+ + [v]H(-x+\phi_0(t)), \\ w(x,t) &= w_+ + [w]H(-x+\phi_0(t)) + 2[v^2]t\delta(-x+\phi_0(t)), \end{aligned} \tag{7.31}$$

if $u_+ = u_- = u_0$, where $\phi_0(t) = 2u_0 t$;

(a.3) *a δ'-shock wave type solution* (1.19) *only if $v_- + v_+ \neq 0$, $w_- + w_+ \neq 0$.*

Proof. Let $u_+ < u_-$. In this case, according to (1.19), (7.18), and (7.19), the Riemann problem (1.17), (7.1) has a classical shock-solution (7.29) if and only if $v_- + v_+ = 0$, $w_- + w_+ = 0$.

If $v_- + v_+ = 0$, $w_- + w_+ \neq 0$, in view of (7.19), the Riemann problem has a δ-shock wave type solution (1.19) of the form (7.30).

According to (7.19), the Riemann problem (1.17), (7.1) has a δ'-shock wave type solution (1.19) (see (7.18)) only if $v_- + v_+ \neq 0$, $w_- + w_+ \neq 0$.

Let $u_+ = u_- = u_0$. In this case the Riemann problem (1.17), (7.1) has a δ-shock wave type solution (1.19) of the form (7.31)), where $\phi_0(t) = 2u_0 t$. Here $x = \phi_0(t) = 2u_0 t$ is a characteristic line of the first equation $u_t + \big(u^2\big)_x = 0$ in system (1.17) issued from $(0,0)$. $\square$

7.4. Vacuum states in a solution of the Riemann problem (1.17), (7.1). Now we consider the case $u_+ > u_-$. Substituting the triple of distributions (7.21) constructed by Theorem 7.2 into the left-hand side of (7.22), it is easy to prove the following assertion.

THEOREM 7.4. ([**51**, Theorem 7.2.]) *Let* $u_+ > u_-$. *Then for* $t \in [0, \infty)$ *the triple of distributions* (7.21)

$$\big(u(x,t), v(x,t), w(x,t)\big) = \begin{cases} (u_-, v_-, w_-), & x \le 2u_-t, \\ \big(\frac{x}{2t}, 0, 0\big), & 2u_-t < x < 2u_+t, \\ (u_+, v_+, w_+), & x \ge 2u_+t, \end{cases}$$

is a unique generalized solution of the Cauchy problem (1.17), (7.1), *which satisfies the integral identities* (7.22), *where* $\widehat{v}(x,t) = v(x,t)$, $\widehat{w}(x,t) = w(x,t)$, *and* $e(t) \equiv 0$, $g(t) \equiv 0$, $h(t) \equiv 0$.

Here the first component u of the solution (7.21) is a rarefaction wave, while the second component v and the third component w contain the intermediate *vacuum states* $v = 0$ and $w = 0$.

8. Geometrical and physical aspects of singular solutions

8.1. δ-Shocks. The case of system (1.7). For a δ-shock wave type solution classical conservation laws (1.6) *do not hold*. However, there is a "generalized" analog of conservation laws (1.6).

Denote by

$$\begin{aligned} S_u(t) &= \int_{-\infty}^{\phi(t)} u(x,t)\,dx + \int_{\phi(t)}^{+\infty} u(x,t)\,dx, \\ S_v(t) &= \int_{-\infty}^{\phi(t)} v(x,t)\,dx + \int_{\phi(t)}^{+\infty} v(x,t)\,dx, \\ S_u(0) &= \int_{-\infty}^{0} u^0(x)\,dx + \int_{0}^{+\infty} u^0(x)\,dx, \\ S_v(0) &= \int_{-\infty}^{0} v^0(x)\,dx + \int_{0}^{+\infty} v^0(x)\,dx, \end{aligned} \tag{8.1}$$

the areas under the graphs $y = u(x,t)$, $y = \widehat{v}(x,t)$, and $y = u^0(x)$, $y = \widehat{v}^0(x)$, respectively, where $x = \phi(t)$ is a line in the upper half-plane $\{(x,t) : x \in \mathbb{R},\ t \in [0,\infty)\}$ issued from $\phi(0) = 0$.

THEOREM 8.1. ([**2**], [**49**]) *Let* (u, v) *be a δ-shock wave type solution of the Cauchy problem* (1.7) *with δ-shock initial data* (2.1), *where* $v(x,t) = \widehat{v}(x,t) + e(t)\delta(\Gamma)$, $\Gamma = \{(x,t) : x = \phi(t)\}$ *is the discontinuity curve, and* $u(x,t)$, $\widehat{v}(x,t)$ *are compactly supported functions with respect to* x (see Definition 2.1). Then the following balance relations hold:

$$\begin{aligned} \dot{S}_u(t) &= 0, \\ \dot{S}_v(t) &= -\dot{e}(t) = -\Big([G(u,v)] - [v]\tfrac{[F(u,v)]}{[u]}\Big)\Big|_{x=\phi(t)}, \end{aligned} \tag{8.2}$$

where $S_u(t)$, $S_v(t)$ are given by (8.1), and $\dot e(t)$ is the Rankine–Hugoniot deficit (2.7). Thus,

$$
\begin{aligned}
&\int_{-\infty}^{\phi(t)} u(x,t)\,dx + \int_{\phi(t)}^{+\infty} u(x,t)\,dx \\
&\qquad\qquad = \int_{-\infty}^{0} u^0(x)\,dx + \int_{0}^{+\infty} u^0(x)\,dx, \\
&\int_{-\infty}^{\phi(t)} v(x,t)\,dx + \int_{\phi(t)}^{+\infty} v(x,t)\,dx + e(t) \\
&\qquad\qquad = \int_{-\infty}^{0} v^0(x)\,dx + \int_{0}^{+\infty} v^0(x)\,dx + e^0,
\end{aligned}
\tag{8.3}
$$

where e^0 is the initial amplitude of the δ-function in the component v.

Proof. Let $v_\pm = \lim_{x\to\phi(t)\pm 0} v(x,t)$ denote the right- and left-hand side values of $v(x,t)$ on the curve Γ. Differentiating the second relation (8.1) and using the second equation of the system (1.7), we obtain

$$
\begin{aligned}
\dot S_v(t) &= v_-\dot\phi(t) - v_+\dot\phi(t) + \int_{-\infty}^{\phi(t)} v_t(x,t)\,dx + \int_{\phi(t)}^{+\infty} v_t(x,t)\,dx \\
&= [v]\Big|_{x=\phi(t)}\dot\phi(t) - \int_{-\infty}^{\phi(t)} \big(G(u,v)\big)_x\,dx - \int_{\phi(t)}^{+\infty} \big(G(u,v)\big)_x\,dx \\
&= [v]\Big|_{x=\phi(t)}\dot\phi(t) - [G(u,v)]\Big|_{x=\phi(t)} \\
&\qquad + G\big(u(-\infty,t),v(-\infty,t)\big) - G\big(u(+\infty,t),v(+\infty,t)\big).
\end{aligned}
$$

Since $G\big(u(-\infty,t),v(-\infty,t)\big) = G\big(u(+\infty,t),v(+\infty,t)\big) = G(0,0)$, using the Rankine–Hugoniot conditions (2.7), we obtain

$$
\dot S_v(t) = \left([v]\frac{[F(u,v)]}{[u]}\Big|_{x=\phi(t)} - [G(u,v)]\right)\Big|_{x=\phi(t)}.
$$

Thus the second relation (8.2) holds.

The proof of the first relation (8.2) is carried out in the same way. This relation is the well-known conservation law (1.6) for a $L^1\cap L^\infty$-generalized solution.

Integrating expressions (8.2), we obtain (8.3). $\square$

From the second relation (8.3), we can see that the meaning of the amplitude $e(t)$ of the δ function is the *"area" of the discontinuity curve.* Moreover, the *"total area" $S_v(t)+e(t)$ is independent of time.*

8.2. δ-Shocks. The case of system (1.8). Let

$$(8.4)\qquad \begin{aligned} S_{uv}(t) &= \int_{-\infty}^{\phi(t)} u(x,t)v(x,t)\,dx + \int_{\phi(t)}^{+\infty} u(x,t)v(x,t)\,dx,\\ S_{uv}(0) &= \int_{-\infty}^{0} u^0(x)v^0(x)\,dx + \int_{0}^{+\infty} u^0(x)v^0(x)\,dx, \end{aligned}$$

be the areas under the graphs $y = u(x,t)\widehat{v}(x,t)$, and $y = u^0(x)\widehat{v}^0(x)$, respectively, where $x = \phi(t)$ is a curve in the upper half-plane $\{(x,t) : x \in \mathbb{R},\ t \in [0,\infty)\}$ issued from $\phi(0) = 0$.

THEOREM 8.2. ([**2**], [**49**]) *Let (u,v) be a δ-shock wave type solution of the Cauchy problem* (1.8) *with the δ- shock initial data* (2.10)*, where $v(x,t) = \widehat{v}(x,t) + e(t)\delta(\Gamma)$, $\Gamma = \{(x,t) : x = \phi(t)\}$ is a discontinuity curve, and $u(x,t)$, $\widehat{v}(x,t)$ are compactly supported functions with respect to x* (see Definition 2.2). Then the following balance relations hold:

$$(8.5)\qquad \begin{aligned} \dot{S}_v(t) &= -\dot{e}(t) = -\Big([G(u,v)] - [v]\tfrac{[F(u,v)]}{[u]}\Big)\Big|_{x=\phi(t)},\\ \dot{S}_{uv}(t) &= -\frac{d\big(e(t)\dot{\phi}(t)\big)}{dt} = -\Big([H(u,v)] - [uv]\dot{\phi}(t)\Big)\Big|_{x=\phi(t)}, \end{aligned}$$

where $S_v(t)$, $S_{uv}(t)$ are given by (8.1), (8.4), and the Rankine–Hugoniot deficits $\dot{e}(t)$, $\frac{d\left(e(t)\dot{\phi}(t)\right)}{dt}$ are given by (2.13). Thus,

$$(8.6)\qquad \begin{aligned} &\int_{-\infty}^{\phi(t)} v(x,t)\,dx + \int_{\phi(t)}^{+\infty} v(x,t)\,dx + e(t)\\ &\qquad\qquad = \int_{-\infty}^{0} v^0(x)\,dx + \int_{0}^{+\infty} v^0(x)\,dx + e^0,\\ &\int_{-\infty}^{\phi(t)} u(x,t)v(x,t)\,dx + \int_{\phi(t)}^{+\infty} u(x,t)v(x,t)\,dx + e(t)\dot{\phi}(t)\\ &\qquad = \int_{-\infty}^{0} u^0(x)v^0(x)\,dx + \int_{0}^{+\infty} u^0(x)v^0(x)\,dx + e^0\dot{\phi}(0), \end{aligned}$$

where e^0 is the initial amplitude of the δ-function in the component v, $\dot{\phi}(0)$ is the initial velocity of the δ-shock.

According to Theorem 8.2, the *"total areas"* $S_v(t)+e(t)$ and $S_{uv}(t)+e(t)\dot{\phi}(t)$ are independent of time.

Relations (8.2), (8.3) and (8.5), (8.6) not only express δ-shock conservation laws but show that the *"area" transportation processes* between the areas under the graphs $y = \widehat{v}(x,t)$ and $y = u(x,t)\widehat{v}(x,t)$, and the δ-shock wave front Γ take place.

8.3. Zero-pressure gas dynamics (1.10). One-dimensional zero- pressure gas dynamics (1.10) is a particular case of system (1.8), where $G(u,v) = uv$, $H(u,v) = vu^2$. Here $v(x,t) \geq 0$ is the density and $u(x,t)$ is the velocity. The areas $S_v(t) = M(t)$ and $S_{uv}(t) = P(t)$ can be considered as mass and momentum, respectively, except for the discontinuity trajectory $x = \phi(t)$. In this case, Theorem 8.2 implies the following corollary.

COROLLARY 8.1. ([**2**]) *Let (u, v) be a δ-shock wave type solution of the "zero-pressure gas dynamics" system* (1.10) *with the δ-shock initial data* (2.10), *where $v(x,t) = \widehat{v}(x,t) + e(t)\delta(\Gamma)$, $\Gamma = \{(x,t) : x = \phi(t)\}$ is a discontinuity curve, and $u(x,t)$, $\widehat{v}(x,t)$ are compactly supported functions with respect to x. Then we have the following mass and momentum balance relations*

$$\dot{e}(t) = -\dot{M}(t) > 0, \qquad \frac{d\big(e(t)\dot{\phi}(t)\big)}{dt} = -\dot{P}(t), \tag{8.7}$$

and

$$\begin{aligned} M(t) + e(t) &= M(0) + e^0, \\ P(t) + e(t)\dot{\phi}(t) &= P(0) + e^0\phi(0), \end{aligned} \tag{8.8}$$

where $M(0) = S_v(0)$ and $P(0) = S_{uv}(0)$ are initial mass and momentum, respectively.

The latter system can be rewritten as

$$\begin{aligned} \dot{\phi}(t) &= \frac{P(0) + e^0\phi(0) - P(t)}{M(0) + e^0 - M(t)}, \\ e(t) &= M(0) + e^0 - M(t), \end{aligned}$$

In the special case of the initial data $M(0) = -e^0$, $P(0) = -e^0\phi(0)$ we can readily see that the discontinuity point $x = \phi(t)$ moves at the velocity

$$\dot{\phi}(t) = \frac{P(t)}{M(t)} = \frac{\int_{x \neq \phi(t)} u(x,t)v(x,t)\,dx}{\int_{x \neq \phi(t)} v(x,t)\,dx},$$

i.e., in such a way as if the total mass were concentrated at the point $x = \phi(t)$. Thus the point $x = \phi(t)$ can be considered, in a sense, as the system barycenter.

In view of inequality (8.7), in the case of "zero-pressure gas dynamics" *mass transportation* from area $S_v(t)$ to the discontinuity curve $x = \phi(t)$ takes place. Thus the transportation process is a concentration process.

8.4. δ'-Shocks. The case of system (1.16). First, we recall our results from [**39**, 3.3.] and then derive an analog of the balance relations (1.6), (8.2), (8.3) for δ'-shocks.

Denote by

$$\begin{aligned} S_w(t) &= \int_{-\infty}^{\phi(t)} w(x,t)\,dx + \int_{\phi(t)}^{+\infty} w(x,t)\,dx, \\ S_w(0) &= \int_{-\infty}^{0} w^0(x)\,dx + \int_{0}^{+\infty} w^0(x)\,dx \end{aligned} \tag{8.9}$$

the areas under the graphs $y = \widehat{w}(x,t)$ and $y = \widehat{w}^0(x)$, respectively, where $x = \phi(t)$ is a line in the upper half-plane $\{(x,t) : x \in \mathbb{R},\ t \in [0,\infty)\}$ issued from $\phi(0) = 0$.

Repeating the proof of Theorem 8.1 almost word for word, we derive the following assertion.

THEOREM 8.3. ([**39**, 3.3.]) *Let (u,v,w) be a δ'-shock wave type solution of the Cauchy problem* (1.16) *with the δ'-shock wave type initial data, where* $v(x,t) = \widehat{v}(x,t) + e(t)\delta(\Gamma)$, $w(x,t) = \widehat{w}(x,t) + g(x,t)\delta(\Gamma) + h(x,t)\delta'(\Gamma)$, $\Gamma = \{(x,t) : x = \phi(t)\}$ *is the discontinuity curve, and* $u(x,t)$, $\widehat{v}(x,t)$, $\widehat{w}(x,t)$ *are compactly supported functions with respect to* x (see Definition 5.1). Then the following balance relations hold:

$$
\begin{aligned}
\dot{S}_u(t) &= 0, \\
\dot{S}_v(t) &= -\dot{e}(t) = -\Big([f'(u)v)] - [v]\tfrac{[f(u)]}{[u]}\Big)\Big|_{x=\phi(t)}, \\
\dot{S}_w(t) &= -\dot{g}(t) = -\Big([f''(u)v^2 + f'(u)w] - [w]\tfrac{[f(u)]}{[u]}\Big)\Big|_{x=\phi(t)},
\end{aligned}
\tag{8.10}
$$

where $S_u(t)$, $S_v(t)$, $S_w(t)$ are given by (8.1), (8.9), and the first Rankine–Hugoniot deficits $\dot{e}(t)$, $\dot{g}(t)$ are given by (5.4), (5.5). Thus

$$
\begin{aligned}
&\int_{-\infty}^{\phi(t)} u(x,t)\,dx + \int_{\phi(t)}^{+\infty} u(x,t)\,dx \\
&\qquad = \int_{-\infty}^{0} u^0(x)\,dx + \int_{0}^{+\infty} u^0(x)\,dx, \\
&\int_{-\infty}^{\phi(t)} v(x,t)\,dx + \int_{\phi(t)}^{+\infty} v(x,t)\,dx + e(t) \\
&\qquad = \int_{-\infty}^{0} v^0(x)\,dx + \int_{0}^{+\infty} v^0(x)\,dx + e^0, \\
&\int_{-\infty}^{\phi(t)} w(x,t)\,dx + \int_{\phi(t)}^{+\infty} w(x,t)\,dx + g(t) \\
&\qquad = \int_{-\infty}^{0} w^0(x)\,dx + \int_{0}^{+\infty} w^0(x)\,dx + g^0,
\end{aligned}
\tag{8.11}
$$

where e^0 and g^0 are the initial amplitudes of the δ-functions in components v and w, respectively.

From relations (8.11), we see that the amplitudes $e(t)$ and $g(t)$ of the δ functions in v and w can be interpreted as *"areas" of the discontinuity curve.* Moreover, the *"total areas"* $S_v(t) + e(t)$ and $S_w(t) + g(t)$ *are independent of time.*

REMARK 8.1. The most unexpected result obtained by Theorem 8.3 is the fact that the *"area" balance relation* for w is independent of the *second Rankine-Hugoniot deficit* (5.6).

9. Algebraic aspects of singular solutions

9.1. The problem of multiplication of distributions. As was already mentioned above in 1.4, to introduce singular solutions to a nonlinear system, we need to solve the problem of *multiplication of distributions*. One of the approaches to this problem is the theory of *nonconservative product* [**6**], [**28**], [**29**], [**30**]. This approach generalizes the concept of Volpert's averaged superposition [**56**]. In [**6**], a general framework for the *nonconservative product*

$$g(u)\frac{du}{dx} \tag{9.1}$$

was introduced, where $g : \mathbb{R}^n \to \mathbb{R}^n$ is a locally bounded Borel function, and $u : (a, b) \to \mathbb{R}^n$ is a discontinuous function of bounded variation.

This approach can be used to solve the Cauchy problems for nonlinear hyperbolic systems in non-conservative form. In [**28**], to construct a δ-shock wave type solution of the system (1.9) for the case $g(u) = f'(u)$, the problem of *multiplication of distributions* is solved by using the *nonconservative product*. However, in the framework of this approach, the notion of a generalized solution *depends on the specific family of paths*, which *can not be derived from the hyperbolic system only.*

Another approach is the Colombeau theory. Applications of this approach are described in many papers and books (see, for example, [**18**], [**37**], [**38**]).

9.2. Flux-functions singularities. Now we show that singular solutions to systems of conservation laws generate *algebraic relations* between their distributional components. These algebraic relations *can be derived from the hyperbolic system only.*

It seems natural to introduce the product of the Heaviside function and the delta function as the weak limit of the product of their regularizations.

Let $\delta(x, \varepsilon) = \frac{1}{\varepsilon}\omega_\delta\Big(\frac{x}{\varepsilon}\Big)$ be the regularization of the delta function (6.4), and $H(x, \varepsilon) = \omega_0\Big(\frac{x}{\varepsilon}\Big) = \int_{-\infty}^{\frac{x}{\varepsilon}} \omega(\eta)\,d\eta$, be the corresponding regularization of the Heaviside function (3.7), $x \in \mathbb{R}$. Since the function $\omega_\delta(\eta)\omega_0(\eta)$ decreases sufficiently rapidly as $|\eta| \to \infty$, we have

$$\Big\langle \frac{1}{\varepsilon}\omega_\delta\Big(\frac{\cdot}{\varepsilon}\Big)\omega_0\Big(\frac{\cdot}{\varepsilon}\Big), \psi(\cdot)\Big\rangle = \int \omega_\delta(\eta)\omega_0(\eta)\psi(\varepsilon\eta)\,d\eta$$

$$= A\psi(0) + O(\varepsilon), \quad \varepsilon \to +0, \quad \forall\ \psi(x) \in \mathcal{D}(\mathbb{R}).$$

Thus one can define the product as

$$\overbrace{H(x)\delta(x)} \stackrel{def}{=} \lim_{\varepsilon\to+0} H(x, \varepsilon)\delta(x, \varepsilon) = A\delta(x), \tag{9.2}$$

where $A = A(\omega_0, \omega_\delta) = \int \omega_0(\eta)\omega_\delta(\eta)\,d\eta$. The product (9.2) defined in this way *depends* on the mollifiers ω, ω_δ, i.e., on the regularizations of the distributions $H(x)$, $\delta(x)$.

In a similar way, we can introduce the *singular superpositions* for *flux-functions* $F(u,v)$, $G(u,v)$, $H(u,v)$ associated with systems (1.7), (1.8). Let $u(x,t,\varepsilon)$, $v(x,t,\varepsilon)$ be the regularizations of the distributions $u(x,t)$, $v(x,t)$ in (1.13). Then we define *singular superpositions* by the following definition:

$$\begin{aligned}
\overbrace{F(u,v)} &\stackrel{def}{=} \lim_{\varepsilon\to+0} F(u(x,t,\varepsilon),v(x,t,\varepsilon)),\\
\overbrace{G(u,v)} &\stackrel{def}{=} \lim_{\varepsilon\to+0} G(u(x,t,\varepsilon),v(x,t,\varepsilon)),\\
\overbrace{H(u,v)} &\stackrel{def}{=} \lim_{\varepsilon\to+0} H(u(x,t,\varepsilon),v(x,t,\varepsilon)),
\end{aligned}$$

if the limits exist in the weak sense. Similarly we can introduce *singular superpositions* for the *flux-functions* $\overbrace{f'(u)v}$, $\overbrace{f''(u)v^2+f'(u)w}$ associated with system (1.16).

It is easy to see that these *singular superpositions* either *depend* on the regularizations of the distributions H, δ, δ' or do not exist in the sense of distributions (see [**2**], [**12**], [**13**], [**39**], [**50**]). This fact implies that the above introduced *singular superpositions* are *not unique.*

However, in the context of constructing δ- and δ'-shock solutions to the Cauchy problems we can define explicit formulas for the *"right" unique singular superpositions.*

Theorem 9.1. *Let (u,v) be a δ-shock type solution* (1.13) *to the Cauchy problem* (1.7), (1.11)*, and let $(u_\varepsilon,v_\varepsilon)$ be its weak asymptotic solution* (*see Definition* 3.1)*. Then for $t\in[0,T)$ we can define the explicit formulas for the "right" singular superpositions:*

$$F(u,v) \stackrel{def}{=} \lim_{\varepsilon\to+0} F(u_\varepsilon,v_\varepsilon) = F(u_+,v_+) + [F(u,v)]H(-x+\phi(t)), \tag{9.3}$$

$$\begin{aligned}
G(u,v) &\stackrel{def}{=} \lim_{\varepsilon\to+0} G(u_\varepsilon,v_\varepsilon)\\
&= G(u_+,v_+) + [G(u,v)]H(-x+\phi(t)) + e(t)\dot{\phi}(t)\delta(-x+\phi(t)),
\end{aligned} \tag{9.4}$$

where the limits are understood in the weak sense.

Proof. Let $(u_\varepsilon(x,t),v_\varepsilon(x,t))$ be a *weak asymptotic solution* to the Cauchy problem (1.7), (1.11). In view of (3.1), we have

$$u_{\varepsilon\,t} + \big(F(u_\varepsilon,v_\varepsilon)\big)_x = o_{\mathcal{D}'}(1), \quad v_{\varepsilon\,t} + \big(G(u_\varepsilon,v_\varepsilon)\big)_x = o_{\mathcal{D}'}(1), \quad \varepsilon\to+0. \tag{9.5}$$

Moreover, relations (6.2) hold, where $(u(x,t),v(x,t))$ is a δ-shock wave type solution (1.13) of the Cauchy problem (1.7), (1.11).

By definition, the *"right" singular superpositions* are defined as the weak limits

$$F(u,v) \stackrel{def}{=} \lim_{\varepsilon\to+0} F(u_\varepsilon,v_\varepsilon), \quad G(u,v) \stackrel{def}{=} \lim_{\varepsilon\to+0} G(u_\varepsilon,v_\varepsilon), \tag{9.6}$$

where the pair of distributions (u,v) is given by (1.13).

Next, according to (9.5), (6.2), we have

$$\begin{aligned} \lim_{\varepsilon\to+0}\langle u_{\varepsilon\,t},\varphi\rangle + \lim_{\varepsilon\to+0}\Big\langle\big(F(u_\varepsilon,v_\varepsilon)\big)_x,\varphi\Big\rangle = \lim_{\varepsilon\to+0}\big\langle o_{\mathcal{D}'}(1),\varphi\big\rangle &= 0,\\ \lim_{\varepsilon\to+0}\langle v_{\varepsilon\,t},\varphi\rangle + \lim_{\varepsilon\to+0}\Big\langle\big(G(u_\varepsilon,v_\varepsilon)\big)_x,\varphi\Big\rangle = \lim_{\varepsilon\to+0}\big\langle o_{\mathcal{D}'}(1),\varphi\big\rangle &= 0, \end{aligned} \tag{9.7}$$

for all $\varphi\in\mathcal{D}(\mathbb{R}\times[0,\infty))$. Thus (9.7), (9.6) imply

$$\begin{aligned} \Big\langle\big(F(u,v)\big)_x,\varphi\Big\rangle &= \lim\nolimits_{\varepsilon\to+0}\Big\langle\big(F(u_\varepsilon,v_\varepsilon)\big)_x,\varphi\Big\rangle = -\langle u_t,\varphi\rangle,\\ \Big\langle\big(G(u,v)\big)_x,\varphi\Big\rangle &= \lim\nolimits_{\varepsilon\to+0}\Big\langle\big(G(u_\varepsilon,v_\varepsilon)\big)_x,\varphi\Big\rangle = -\langle v_t,\varphi\rangle, \end{aligned} \tag{9.8}$$

for all $\varphi(x,t)\in\mathcal{D}(\mathbb{R}\times[0,\ \infty))$. Since u, v are distributions, the $\big(F(u,v)\big)_x$, $\big(G(u,v)\big)_x$ are distributions as well.

Using (9.8) and (1.13), we obtain in the weak sense

$$\begin{aligned} \big(F(u,v)\big)_x &= -u_t = -\big(u_+ + [u]H(-x+\phi(t))\big)_t\\ &= -u_{+t} - [u_t]H(-x+\phi(t)) - [u]\dot\phi(t)\delta(-x+\phi(t)), \end{aligned} \tag{9.9}$$

$$\begin{aligned} \big(G(u,v)\big)_x &= -v_t = -\big(v_+ + [v]H(-x+\phi(t)) + e(t)\delta(-x+\phi(t))\big)_t\\ &= -v_{+t} - [v_t]H(-x+\phi(t))\\ &\quad -\big([v]\dot\phi(t)+\dot e(t)\big)\delta(-x+\phi(t)) - e(t)\dot\phi(t)\delta'(-x+\phi(t)). \end{aligned} \tag{9.10}$$

Taking into account that

$$u_{\pm\,t} + \big(F(u_\pm,v_\pm)\big)_x = 0, \qquad v_{\pm\,t} + \big(G(u_\pm,v_\pm)\big)_x = 0 \tag{9.11}$$

for $\pm x > \pm\phi(t)$, and substituting (9.11) into (9.9), (9.10), we derive

$$\begin{aligned} \big(F(u,v)\big)_x &= \big(F(u_+,v_+)\big)_x + \big([F(u,v)]H(-x+\phi(t))\big)_x\\ &\qquad + \big([F(u,v)] - [u]\dot\phi(t)\big)\delta(-x+\phi(t)),\\ \big(G(u,v)\big)_x &= \big(G(u_+,v_+)\big)_x + \big([G(u,v)]H(-x+\phi(t))\big)_x\\ &\quad + \big([G(u,v)] - [v]\dot\phi(t) - \dot e(t)\big)\delta(-x+\phi(t)) - e(t)\dot\phi(t)\delta'(-x+\phi(t)). \end{aligned}$$

Integrating the last relations with respect to x, we have

$$\begin{aligned} F(u,v) &= F(u_+,v_+) + [F(u,v)]H(-x+\phi(t))\\ &\quad -\big([F(u,v)] - [u]\dot\phi(t)\big)\big|_{x=\phi(t)}H(-x+\phi(t)) + C_1(t), \end{aligned} \tag{9.12}$$

$$\begin{aligned} G(u,v) &= G(u_+,v_+) + [G(u,v)]H(-x+\phi(t))\\ &\quad -\big([G(u,v)] - [v]\dot\phi(t) - \dot e(t)\big)\big|_{x=\phi(t)}H(-x+\phi(t))\\ &\quad + e(t)\dot\phi(t)\delta(-x+\phi(t)) + C_2(t), \end{aligned} \tag{9.13}$$

where $C_1(t)$, $C_2(t)$ are functions.

Taking into account that

$$\begin{aligned} \lim_{\varepsilon\to+0} F(u_\varepsilon(x,t),v_\varepsilon(x,t)) &= F(u_\pm,v_\pm),\\ \lim_{\varepsilon\to+0} G(u_\varepsilon(x,t),v_\varepsilon(x,t)) &= G(u_\pm,v_\pm), \quad \pm x > \pm\phi(t), \end{aligned} \tag{9.14}$$

and using the Rankine–Hugoniot conditions for δ-shocks (2.7), we conclude that (9.12), (9.13) imply $C_1(t) = C_2(t) = 0$ and the fact that relations (9.3) and (9.4) hold. $\square$

In fact, for some particular cases of the Cauchy problems (1.9), (1.11) and (1.15), (1.11) Theorem 9.1 was proved directly, in the context of constructing δ-shock type solutions, in [**2**], [**13**], [**49**], i.e., flux-functions were constructed. Using relations (3.37), (3.38), one can directly construct flux-functions for the case of the Cauchy problem (3.23), (1.11).

THEOREM 9.2. *Let* (u, v) *be a* δ*-shock type solution* (1.13) *of the Cauchy problem* (1.10)*,* (1.11)*,* (1.12)*, and let* $(u_\varepsilon, v_\varepsilon)$ *be its weak asymptotic solution* (*see Definition* 3.1)*. Then for* $t \in [0, T)$ *we can define the explicit formulas for the "right" singular superpositions:*

$$uv \overset{def}{=} \lim_{\varepsilon\to+0} u_\varepsilon v_\varepsilon = u_+ v_+ + [uv]H(-x+\phi(t)) + e(t)\dot{\phi}(t)\delta(-x+\phi(t)), \tag{9.15}$$

$$u^2 v \overset{def}{=} \lim_{\varepsilon\to+0} u_\varepsilon^2 v_\varepsilon = u_+^2 v_+ + [u^2 v]H(-x+\phi(t)) + e(t)\big(\dot{\phi}(t)\big)^2\delta(-x+\phi(t)), \tag{9.16}$$

where the limits are understood in the weak sense.

PROOF. Let $(u_\varepsilon(x,t), v_\varepsilon(x,t))$ be a *weak asymptotic solution* of the Cauchy problem (1.10), (1.11), (1.12). Then in view of (3.1), we have

$$v_{\varepsilon\,t} + \big(u_\varepsilon v_\varepsilon\big)_x = o_{\mathcal{D}'}(1), \qquad \big(u_\varepsilon v_\varepsilon\big)_t + \big(u_\varepsilon^2 v_\varepsilon\big)_x = o_{\mathcal{D}'}(1), \quad \varepsilon \to +0. \tag{9.17}$$

Moreover, relations (6.2) hold, where $(u(x,t), v(x,t))$ is a δ-shock wave type solution (1.13) of the Cauchy problem (1.10), (1.11), (1.12).

By Theorem 9.1, relation (9.15) holds.

Next, repeating the proof of Theorem 9.1, and using the second equation in (9.17) and relation (9.15), we obtain

$$\begin{aligned}
\big\langle \big(u^2 v\big)_x, \varphi\big\rangle &= \lim_{\varepsilon\to+0}\big\langle \big(u_\varepsilon^2 v_\varepsilon\big)_x, \varphi\big\rangle = -\lim_{\varepsilon\to+0}\big\langle (u_\varepsilon v_\varepsilon)_t, \varphi\big\rangle = -\big\langle (uv)_t, \varphi\big\rangle \\
&= -\Big\langle (u_+ v_+)_t + [(uv)_t]H(-x+\phi(t)) + [uv]\dot{\phi}(t)\delta(-x+\phi(t)) \\
&\quad + \frac{d}{dt}\big(e(t)\dot{\phi}(t)\big)\delta(-x+\phi(t)) + e(t)\big(\dot{\phi}(t)\big)^2\delta'(-x+\phi(t)), \varphi\Big\rangle,
\end{aligned} \tag{9.18}$$

for all $\varphi \in \mathcal{D}(\mathbb{R} \times [0, \infty))$. Since the term uv given by relation (9.15) is a distribution, the term $u^2 v$ is a distribution as well.

Taking into account that

$$v_{\pm\,t} + \big(u_\pm v_\pm\big)_x = 0, \qquad (u_\pm v_\pm)_t + \big(u_\pm^2 v_\pm)\big)_x = 0 \tag{9.19}$$

for $\pm x > \pm\phi(t)$, and substituting the second relation from (9.19) into (9.18), we have

$$\big(u^2 v\big)_x = \big(u_+^2 v_+\big)_x + \big([u^2 v]H(-x+\phi(t))\big)_x$$

$$+\Big([u^2v]-[uv]\dot\phi(t)-\frac{d}{dt}\big(e(t)\dot\phi(t)\big)\Big)\delta(-x+\phi(t))$$
$$-e(t)\big(\dot\phi(t)\big)^2\delta'(-x+\phi(t)).$$

Integrating the last relations with respect to x, we have

$$\begin{aligned}u^2v &= u_+^2v_+ + [u^2v]H(-x+\phi(t))\\ &\quad -\Big([u^2v]+[uv]\dot\phi(t)-\frac{d}{dt}\big(e(t)\dot\phi(t)\big)\Big)\Big|_{x=\phi(t)}H(-x+\phi(t))\\ &\quad +e(t)\big(\dot\phi(t)\big)^2\delta(-x+\phi(t))+C(t),\end{aligned} \tag{9.20}$$

where $C(t)$ is a function.

Taking into account that $\lim_{\varepsilon\to+0}u_\varepsilon^2v_\varepsilon = u_\pm^2v_\pm$ for $\pm x>\pm\phi(t)$, and using the Rankine–Hugoniot conditions for δ-shocks (2.16), we conclude that (9.20) implies $C(t)=0$ and the fact that relation (9.16) holds. □

Theorem 9.2 can be proved by using the weak asymptotic solution of the Cauchy problem (1.10), (1.11), (1.12) constructed in [**12**].

Theorem 9.3. *Let* (u,v) *be a δ-shock type solution* (1.13) *of the Cauchy problem* (1.8), (1.11), (1.12), *and let* $(u_\varepsilon,v_\varepsilon)$ *be its weak asymptotic solution* (*see Definition* 3.1). *Then for* $t\in[0,T)$ *we can define the explicit formulas for the "right" singular superpositions:*

$$\begin{aligned}G(u,v) &\stackrel{def}{=} \lim_{\varepsilon\to+0}G(u_\varepsilon,v_\varepsilon)\\ &= G(u_+,v_+)+[G(u,v)]H(-x+\phi(t))+e(t)\dot\phi(t)\delta(-x+\phi(t)),\end{aligned} \tag{9.21}$$

$$\begin{aligned}H(u,v) &\stackrel{def}{=} \lim_{\varepsilon\to+0}H(u_\varepsilon,v_\varepsilon)\\ &= H(u_+,v_+)+[H(u,v)]H(-x+\phi(t))+e(t)\big(\dot\phi(t)\big)^2\delta(-x+\phi(t)),\end{aligned} \tag{9.22}$$

where the limits are understood in the weak sense.

Proof. By Theorem 9.1, relation (9.21) holds. Note that only the first and second terms in the right-hand side of relation (9.21) depend on the non-linearity $G(u,v)$. The third term containing a δ-singularity is independent of $G(u,v)$. Thus we have, in particular,

$$uv \stackrel{def}{=} \lim_{\varepsilon\to+0}u_\varepsilon v_\varepsilon = u_+v_+ + [uv]H(-x+\phi(t))+e(t)\dot\phi(t)\delta(-x+\phi(t)). \tag{9.23}$$

The latter relation coincides with relation (9.15) proved by Theorem 9.2.

It remains to point out that differentiating relation (9.23) with respect to t, and repeating the proof of Theorem 9.2 almost word for word, we derive relation (9.22). □

THEOREM 9.4. *Let* (u,v,w) *be a* δ'*-shock type solution* (1.19) *of the Cauchy problem* (1.16), (1.18), *and let* $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$ *be its weak asymptotic solution* (*see Definition* 6.1). *Then for* $t \in [0,\,T)$ *we can define the explicit formulas for the "right" singular superpositions:*

$$f(u) \stackrel{def}{=} \lim_{\varepsilon\to+0} f(u_\varepsilon) = f(u_+) + [f(u)]H(-x+\phi(t)), \tag{9.24}$$

$$f'(u)v \stackrel{def}{=} \lim_{\varepsilon\to+0} \big(f'(u_\varepsilon)v_\varepsilon\big)$$

$$= f'(u_+)v_+ + [f'(u)v]H(-x+\phi(t)) + e(t)\dot\phi(t)\delta(-x+\phi(t)), \tag{9.25}$$

$$f''(u)v^2 + f'(u)w \stackrel{def}{=} \lim_{\varepsilon\to+0} \big(f''(u_\varepsilon)v_\varepsilon^2 + f'(u_\varepsilon)w_\varepsilon\big)$$

$$= f''(u_+)v_+^2 + f'(u_+)w_+ + [f''(u)v^2 + f'(u)w]H(-x+\phi(t))$$

$$+\big(g(t)\dot\phi(t) + \dot h(t)\big)\delta(-x+\phi(t)) + h(t)\dot\phi(t)\delta'(-x+\phi(t)), \tag{9.26}$$

where the limits are understood in the weak sense.

PROOF. Let $(u_\varepsilon(x,t), v_\varepsilon(x,t))$ be a *weak asymptotic solution* of the Cauchy problem (1.16), (1.18). In view of (3.1), we have

$$\begin{aligned} u_{\varepsilon\,t} + \big(f(u_\varepsilon)\big)_x &= o_{\mathcal{D}'}(1), \\ v_{\varepsilon\,t} + \big(f'(u_\varepsilon)v_\varepsilon\big)_x &= o_{\mathcal{D}'}(1), \\ w_{\varepsilon\,t} + \big(f''(u_\varepsilon)v_\varepsilon^2 + f'(u_\varepsilon)w_\varepsilon\big)_x &= o_{\mathcal{D}'}(1), \quad \varepsilon\to+0. \end{aligned} \tag{9.27}$$

Moreover, relations (6.2) hold, where $(u(x,t), v(x,t), w(x,t))$ is a δ'-shock wave type solution (1.19) of the Cauchy problem (1.16), (1.18).

By Theorem 9.1, relations (9.24), (9.25) hold.

Just as above, using the third equation from (9.27) and the third relation from (1.19), we obtain in the weak sense

$$\big(f''(u)v^2 + f'(u)w\big)_x = \lim_{\varepsilon\to+0} \big(f''(u_\varepsilon)v_\varepsilon^2 + f'(u_\varepsilon)w_\varepsilon\big)_x = -\lim_{\varepsilon\to+0} w_{\varepsilon\,t} = -w_t$$

$$= -\big((w_+)_t + [w_t]H(-x+\phi(t)) + [w]\dot\phi(t)\delta(-x+\phi(t))$$

$$+\dot g(t)\delta(-x+\phi(t)) + g(t)\dot\phi(t)\delta'(-x+\phi(t))$$

$$+\dot h(t)\delta'(-x+\phi(t)) + h(t)\dot\phi(t)\delta''(-x+\phi(t))\big). \tag{9.28}$$

Taking into account that $w_{\pm\,t} + \big(f''(u_\pm)v_\pm^2 + f'(u_\pm)w_\pm\big)_x = 0$, for $\pm x > \pm\phi(t)$, and substituting the latter relation into (9.28), we obtain

$$\big(f''(u)v^2 + f'(u)w\big)_x$$

$$= \big(f''(u_+)v_+^2 + f'(u_+)w_+\big)_x + \big([f''(u)v^2 + f'(u)w]H(-x+\phi(t))\big)_x$$

$$+\big([f''(u)v^2 + f'(u)w] - [w]\dot\phi(t) - \dot g(t)\big)\delta(-x+\phi(t))$$

$$-\big(g(t)\dot\phi(t) + \dot h(t)\big)\delta'(-x+\phi(t)) - h(t)\dot\phi(t)\delta''(-x+\phi(t)).$$

Integrating the last relations with respect to x, we have

$$
\begin{aligned}
&f''(u)v^2 + f'(u)w \\
&\quad = f''(u_+)v_+^2 + f'(u_+)w_+ + [f''(u)v^2 + f'(u)w]H(-x+\phi(t)) \\
&\quad -\big([f''(u)v^2 + f'(u)w] - [w]\dot\phi(t) - \dot g(t)\big)\big|_{x=\phi(t)}H(-x+\phi(t)) \\
&\quad +\big(g(t)\dot\phi(t) + \dot h(t)\big)\delta(-x+\phi(t)) + h(t)\dot\phi(t)\delta'(-x+\phi(t)) + C(t),
\end{aligned}
\tag{9.29}
$$

where $C(t)$ is a function. Since according to (1.19), we have $\lim_{\varepsilon\to+0}\big(f''(u_\varepsilon)v_\varepsilon^2 + f'(u_\varepsilon)w_\varepsilon\big) = f''(u_\pm)v_\pm^2 + f'(u_\pm)w_\pm$ for $\pm x > \pm\phi(t)$, we conclude that (9.29) implies $C(t)=0$ and the fact that relation (9.26) holds. □

In particular, for the case of the Cauchy problem (1.17), (1.18), in [**39**] and [**51**], Theorem 9.4 was proved in the context of constructing δ'-shock type solutions.

9.3. Two significant examples. As mentioned above, the *"right" singular superpositions of distributions* are determined only in the *context* of solving the Cauchy problem. Moreover, they are unique Schwartz distributions. In order to illustrate the specific properties of the *"right" singular superpositions*, we consider two particular cases of Theorem 9.1.

(a) In [**12**], [**13**], a δ-shock wave type solution (1.13) of the Cauchy problem (1.9), (1.11) was constructed. In these papers, formulas (9.3), (9.4) defining the flux-functions of the δ-shock (for the case $F(u,v)=f(u)$, $G(u,v)=vg(u)$) were derived directly as the weak limit of the weak asymptotic solution to the Cauchy problem (1.9), (1.11):

$$
f(u) = f(u_+) + [f(u)]H(-x+\phi(t)),
\tag{9.30}
$$

$$
vg(u) = v_+g(u_+) + [vg(u)]H(-x+\phi(t)) + e(t)\dot\phi(t)\delta(-x+\phi(t)),
\tag{9.31}
$$

where the distributions $u(x,t)$, $v(x,t)$ are given by (1.13). Here, in view of (2.7), $\dot\phi(t) = \frac{[f(u)]}{[u]}$, $\dot e(t) = \big([vg(u)] - [v]\frac{[f(u)]}{[u]}\big)\big|_{x=\phi(t)}$.

(b) The Cauchy problems (1.14), (1.11) and (1.15), (1.11) for the Keyfitz-Kranzer system and its generalization were solved in [**48**], [**49**]. In [**49**] (see also [**2**], [**50**]), formulas (9.3), (9.4) defining flux-functions of the δ-shock (for the case $F(u,v)=f(u)-v$, $G(u,v)=g(u)$), i.e., the *unique "right" singular superpositions*, were directly derived as the weak limit of the weak asymptotic solution to the Cauchy problem (1.15), (1.11):

$$
f\big(u(x,t)\big) - v(x,t) = f(u_+) - v_+ + \Big[f(u) - v\Big]H(-x+\phi(t)),
\tag{9.32}
$$

$$
\begin{aligned}
g\big(u(x,t)\big) &= g(u_+) + \Big[g(u)\Big]H(-x+\phi(t)) \\
&\quad + e(t)\frac{\big[f(u)-v\big]}{[u]}\delta(-x+\phi(t)),
\end{aligned}
\tag{9.33}
$$

where the distributions $u(x,t)$, $v(x,t)$ are given by (1.13). Here, in view of (2.7), $\dot{\phi}(t)=\frac{[f(u)-v]}{[u]}$, $\dot{e}(t)=\big([g(u)]-[v]\frac{[f(u)-v]}{[u]}\big)\big|_{x=\phi(t)}$.

In particular, for the Keyfitz-Kranzer system the above formulas imply

$$u^2-v=u_+^2-v_++\big[u^2-v\big]H(-x+\phi(t)), \tag{9.34}$$

$$\frac{1}{3}u^3-u=\frac{1}{3}u_+^3-u_+$$

$$+\big[\frac{1}{3}u^3-u\big]H(-x+\phi(t))+e(t)\frac{\big[u^2-v\big]}{[u]}\delta(-x+\phi(t)), \tag{9.35}$$

where, in view of (2.7), $\dot{\phi}(t)=\frac{[u^2-v]}{[u]}$, $\dot{e}(t)=\big([\frac{1}{3}u^3-u]-[v]\frac{[u^2-v]}{[u]}\big)\big|_{x=\phi(t)}$.

Note that the *unique "right" singular superpositions* (9.30), (9.31) are *essentially different* from the *unique "right" singular superpositions* (9.32), (9.33) and (9.34), (9.35). The main distinction between them is the following.

Taking into account that $H(x)\cdot H(x)=H(x)$, one can see that in fact, by (9.31), the *unique "right" product of the step function and the delta function* is defined by:

$$e(t)\delta(-x+\phi(t))u(x,t)=e(t)\delta(-x+\phi(t))\cdot\begin{cases}u_-, & x\ <\ \phi(t),\\ u_+, & x\ >\ \phi(t),\end{cases}$$

$$=e(t)\frac{[f(u)]}{[u]}\delta(-x+\phi(t)). \tag{9.36}$$

In the case of the Keyfitz–Kranzer system (1.14) and its generalization (1.15), formulas (9.32), (9.33) and (9.34), (9.35) *do not define (!)* the *product of the Heaviside function and the δ-function.* Moreover, although according to (1.13), $u(x,t)$ *does not depend* on the terms $e(t)\delta(-x+\phi(t))$ and $[v(x,t)]\big|_{x=\phi(t)}$, the "right" singular superposition $g\big(u(x,t)\big)$ (or $\frac{1}{3}u^3-u$) determined by (9.33) (or (9.35)) *does depend (!)* on these terms. Thus one can say that the term

$$e(t)\frac{[f(u)-v]}{[u]}\delta(-x+\phi(t))\quad\text{or}\quad e(t)\frac{[u^2-v]}{[u]}\delta(-x+\phi(t))$$

"appears from nothing".

Similarly, the left-hand sides of relations (9.32) and (9.34) *depend* on the term $e(t)\delta(-x+\phi(t))$ while the right-hand sides in (9.32) and (9.34) are *independent* of this term. Nevertheless, in the context of solving the Cauchy problem, the flux-function is determined *uniquely.*

9.4. Commentary. Since the nonlinear terms in systems (1.14), (1.15), and (1.16) can not be reduced to terms of the form (9.1)), it is *impossible* to construct δ-shocks for systems (1.14), (1.15) and δ'-shocks for system (1.16) by using the *nonconservative product* [**28**], [**29**], [**6**]. It is also *impossible* to use the *measure-valued solutions approach* [**3**], [**55**], [**57**].

In [**20**], the system of conservation laws

$$u_t + \Big(\frac{u^2}{2}\Big)_x = 0, \quad v_t + (uv)_x = 0, \quad w_t + \Big(\frac{v^2}{2} + uw\Big)_x = 0 \tag{9.37}$$

was studied. This system has repeated eigenvalues. As stated in [**20**], system (9.37) *cannot be solved in the classical distributional sense*, therefore it is necessary to define a generalized solution in the Colombeau sense. In [**20**] this is motivated by the following arguments: if $v_- + v_+ \neq 0$ then the v component contains a δ measure along $x = 0$. Though the product uv does not make sense in the classical theory of distributions, it can be defined in the sense of the approach [**6**], but v^2 contains a square of δ measure and thus cannot be defined in this sense.

It is clear that by the change of variables $u \to 2u$, $v \to 2v$, $w \to w$ system (9.37) can be transformed into system (1.17). Thus, contrary to the assertion from the paper [**20**], according to Theorem 6.1, system (9.37) *admits a δ'-shock wave type solution*. This solution considered in the sense of Definition 5.1 is a *distributional solution*.

Thus we can see that the problem of introducing singular solutions to system (9.37) is reduced to the problem of the "right" definition of singular solutions. In the above-mentioned case a generalized solution of system (9.37) is represented by Schwartz distributions but not Colombeau generalized functions.

10. Validity and naturalness of δ- and δ'-shock definitions

Definitions 2.1, 2.2, and 5.1 derived in the framework of the *weak asymptotics method* give *natural* generalizations of the classical definition of weak L^∞-solutions (1.2) relevant for the structure of δ- and δ'-shocks.

Now we discuss and substantiate the validity and naturalness of the above-mentioned definitions.

First, if a solution of the Cauchy problems contains no δ and δ'-terms then these definitions coincide with the classical Definition (1.2). Second, using these definitions, one can derive the Rankine–Hugoniot conditions for δ- and δ'-shocks (2.7), (2.13), and (5.3)–(5.6). Below we *temporarily* suppose that Definitions 2.1, 2.2, 5.1 are not known, and will show that by using some passages from proofs of Theorems 9.1– 9.4 and some nonstrict reasoning, one can derive the same "right" Rankine–Hugoniot conditions for δ- and δ'-shocks which were derived by using these definitions.

We stress that, in fact, the proofs of Theorems 9.1– 9.4 do not use Definitions 2.1, 2.2, 5.1.

LEMMA 10.1. *Let $(u_\varepsilon, v_\varepsilon)$ be a weak asymptotic solution of the Cauchy problem* (1.7), (1.11) *such that it satisfies relation* (6.2)*, where the pair of distributions (u, v) is given by* (1.13). *Then for the components of* (1.13) *the Rankine–Hugoniot conditions* (2.7) *for δ-shock hold.*

Proof. Repeating the proof of Theorem 9.1 almost word for word, we obtain the relations (9.12), (9.13). Next, taking into account (9.14), we conclude that $C_1(t) = 0$, $C_2(t) = 0$, and the relations $\big([F(u,v)] - [u]\dot{\phi}(t)\big)\big|_{x=\phi(t)} = 0$, $\big([G(u,v)] - [v]\dot{\phi}(t) - \dot{e}(t)\big)\big|_{x=\phi(t)} = 0$ hold. Thus we derive the Rankine–Hugoniot conditions (2.7) for δ-shocks. □

Lemma 10.2. *Let $(u_\varepsilon, v_\varepsilon)$ be a weak asymptotic solution of the Cauchy problem* (1.10)*,* (1.11)*,* (1.12) *such that it satisfies relation* (6.2)*, where the pair of distributions (u,v) is given by* (1.13)*. Then for the components of* (1.13) *the Rankine–Hugoniot conditions* (2.16) *for δ-shock hold.*

Proof. By Lemma 10.2 one can derive the first Rankine–Hugoniot condition (2.16). Next, repeating the proof of Theorem 9.2 almost word for word, we obtain relation (9.20). Taking into account that $\lim_{\varepsilon\to+0} u_\varepsilon^2 v_\varepsilon = u_\pm^2 v_\pm$ for $\pm x > \pm\phi(t)$, one can conclude that $C(t) = 0$, and the relation $\big([u^2v] - [uv]\dot{\phi}(t) - \frac{d}{dt}\big(e(t)\dot{\phi}(t)\big)\big)\big|_{x=\phi(t)} = 0$ is valid. Thus, the second relation in (2.16) holds. □

Lemma 10.3. *Let $(u_\varepsilon, v_\varepsilon)$ be a weak asymptotic solution of the Cauchy problem* (1.8)*,* (1.11) *such that it satisfies relation* (6.2)*, where the pair of distributions (u,v) is given by* (1.13)*. Then for the components of* (1.13) *the Rankine–Hugoniot conditions* (2.13) *for δ-shock hold.*

Proof. By Lemma 10.1, the first Rankine–Hugoniot condition in (2.13) holds.

According to Theorem 9.1, the term uv is represented by relation (9.23). Next, differentiating this relation with respect to t and repeating the proof of Lemma 10.2 almost word for word, we derive the second Rankine–Hugoniot condition in (2.13). □

Lemma 10.4. *Let $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$ be a weak asymptotic solution of the Cauchy problem* (1.16)*,* (1.18) *such that it satisfies relation* (6.2)*, where the triple of distributions (u,v,w) is given by* (1.19)*. Then for* (1.19) *the first, second and third Rankine–Hugoniot conditions* (5.3)–(5.6) *for δ'-shock hold. The fourth Rankine–Hugoniot condition* (5.6) *cannot be derived in this way.*

Proof. According to Lemma 10.2, the first and second Rankine–Hugoniot conditions (5.3), (5.4) hold.

Next, repeating the proof of Theorem 9.1 almost word for word, we derive relation (9.29). Taking into account that $\lim_{\varepsilon\to+0} f''(u_\varepsilon)v_\varepsilon^2 + f'(u_\varepsilon)w_\varepsilon = f''(u_\pm)v_\pm^2 + f'(u_\pm)w_\pm$ for $\pm x > \pm\phi(t)$, one can conclude that $C(t) = 0$, and $\big([f''(u)v^2 + f'(u)w] - [w]\dot{\phi}(t) - \dot{g}(t)\big)\big|_{x=\phi(t)} = 0$ is valid. Thus, the third Rankine–Hugoniot condition (5.5) holds.

The fourth Rankine–Hugoniot condition (5.6) $\frac{d}{dt}\big(h(t)[u(\phi(t),t)]\big) = \frac{de^2(t)}{dt}$ can not be derived in the same way as (5.3)–(5.5). □

Nevertheless, the fourth Rankine–Hugoniot condition (5.6) can be derived by nonstrict reasoning without using Definition 5.1 (see [**39**, 2.2]).

We stress that the Rankine–Hugoniot conditions for δ- and δ'-shocks were derived by Lemmas 10.1– 10.4 without using corresponding Definitions 2.1–2.2, and 5.1. This fact shows that our definitions are "right" and natural.

Appendix A. Some weak asymptotic expansions.

LEMMA A.1. *Let $\delta(x,\varepsilon)=\frac{1}{\varepsilon}\omega_\delta\big(\frac{x}{\varepsilon}\big)$, $\frac{1}{\varepsilon}\omega_1\big(\frac{x}{\varepsilon}\big)$ be regularizations of the delta function, and $H(\xi,\varepsilon)=\omega_0\big(\frac{\xi}{\varepsilon}\big)=\int_{-\infty}^{\frac{x}{\varepsilon}}\omega(\eta)\,d\eta$, $j=1,2$ be a regularization of the Heaviside function $H(x)$, $x\in\mathbb{R}$ (see Subsec. 3.2). Then we have the following weak asymptotic expansions:*

$$\begin{aligned}
\big(H(\xi,\varepsilon)\big)^r &= H(\xi)+O_{\mathcal{D}'}(\varepsilon),\\
\Big(H(x,\varepsilon)\Big)^r\delta(x,\varepsilon) &= B_r\delta(x)+O_{\mathcal{D}'}(\varepsilon),\\
\delta(x,\varepsilon)\Big(\omega_1\Big(\frac{x}{\varepsilon}\Big)\Big)^r &= A_r\delta(x)+O_{\mathcal{D}'}(\varepsilon),\quad \varepsilon\to+0,
\end{aligned}\tag{A.1}$$

where $B_r=\int\omega_0^r(\eta)\omega_\delta(\eta)\,d\eta$, $A_r=\int\omega_\delta(\eta)\omega^r(\eta)\,d\eta$, $r=1,2,\dots$.

PROOF. It is clear that the first relation in (A.1) holds. Making the change of variables $x=\varepsilon\eta$, we obtain

$$\Big\langle\Big(\omega_0\Big(\frac{x}{\varepsilon}\Big)\Big)^r\frac{1}{\varepsilon}\omega_\delta\Big(\frac{x}{\varepsilon}\Big),\psi(x)\Big\rangle$$
$$=\int\omega_0^r(\eta)\omega_\delta(\eta)\psi(\varepsilon\eta)\,d\eta=B_r\psi(0)+O(\varepsilon),\quad\varepsilon\to+0,$$

for all $\psi(x)\in\mathcal{D}(\mathbb{R})$, i.e., the second relation in (A.1) is proved. Since $\omega_\delta(\eta)\omega^r(\eta)$ decreases sufficiently rapidly as $|\eta|\to\infty$, then following the same reasoning, we obtain the third relation in (A.1):

$$\Big\langle\frac{1}{\varepsilon}\omega_\delta\Big(\frac{x}{\varepsilon}\Big)\Big(\omega_1\Big(\frac{x}{\varepsilon}\Big)\Big)^r,\psi(x)\Big\rangle$$
$$=\int\omega_\delta(\eta)\omega^r(\eta)\psi(\varepsilon\eta)\,d\eta=A_r\psi(0)+O(\varepsilon),\quad\varepsilon\to+0,$$

for all $\psi(x)\in\mathcal{D}(\mathbb{R})$, $r=1,2,\dots$. □

LEMMA A.2. ([**9**, Corollary 1.1.], [**12**], [**13**]) *If $f(u)$, $g(u)$ are smooth functions, and $u(x,t,\varepsilon)$, $v(x,t,\varepsilon)$ are defined by* (3.5), (3.10) *then*

$$f\big(u_\varepsilon(x,t,)\big)=f(u_0)+\big[f(u)\big]H(-x+\phi(t))+O_{\mathcal{D}'}(\varepsilon),\quad\varepsilon\to+0,$$
$$\begin{aligned}
v_\varepsilon(x,t,)g\big(u_\varepsilon(x,t,)\big)&=g(u_0)v_0+\big[g(u)v\big]H(-x+\phi(t))\\
&\quad+\Big\{e(t)a(t)+R(t)c(t)\Big\}\delta(-x+\phi(t))+O_{\mathcal{D}'}(\varepsilon),\quad\varepsilon\to+0,
\end{aligned}$$

where $a(t)$, $c(t)$ are defined by (3.13).

References

[1] S. Albeverio, S. A. Molchanov, and D. Surgailis, *Stratified structure of the Universe and the Burgers' equation – a probabilistic approach*, Probab. Theory Relat. Fields, **100**, (1994), 457–484.

[2] S. Albeverio, V. M. Shelkovich, *On the delta-shock front problem*, in the book: "Analytical Approaches to Multidimensional Balance Laws", Ch. 2, (Ed. O. S. Rozanova), Nova Science Publishers, Inc., 2005, pp. 45–88.

[3] F. Bouchut, *On zero pressure gas dynamics*, in: "Advances in Kinetic Theory and Computing", Series on Advances in Mathematics for Applied Sciences, Vol. 22, World Scientific, Singapore, 1994, pp. 171–190.

[4] G. Q. Chen, H. Liu, *Formation of delta-shocks and vacuum states in the vanishing pressure limit of solutions to the Euler equations for isentropic fluids*, SIAM J. Math. Anal., **34**, (2003), 925–938.

[5] G. Q. Chen, H. Liu, *Concentration and cavitation in the vanishing pressure limit of solutions to the Euler equations for nonisentropic fluids*, Physica D, **189**, (2004), 141–165.

[6] G. Dal Maso, P. G. Le Floch, and F. Murat, *Definition and weak stability of nonconservative products*, J. Math. Pures Appl., **74**, (1995), 483–548.

[7] V. G. Danilov, G. A. Omel′yanov, V. M. Shelkovich, *Weak Asymptotics Method and Interaction of Nonlinear Waves*, in: Mikhail Karasev (Ed.), "Asymptotic Methods for Wave and Quantum Problems", Amer. Math. Soc. Transl., Ser. 2, **208**, 2003, pp. 33–165.

[8] V. G. Danilov, V. M. Shelkovich, *Propagation and interaction of nonlinear waves to quasilinear equations*, in: H. Freistühler, G. Warnecke (Eds.), Hyperbolic problems: Theory, Numerics, Applications (Eighth International Conference in Magdeburg, February/March 2000, vol. I). International Series of Numerical Mathematics, vol. 140, Birkhäuser Verlag Basel/Switzerland, 2001, pp. 267–276.

[9] V. G. Danilov and V. M. Shelkovich, *Propagation and interaction of shock waves of quasilinear equation*, Nonlinear Studies, **8**, no. 1, (2001), 135–169.

[10] V. G. Danilov, V. M. Shelkovich, *Propagation and interaction of delta-shock waves of a hyperbolic system of conservation laws*, in: T. Y. Hou, E. Tadmor (Eds.), Hyperbolic Problems: Theory, Numerics, Applications. Proceedings of the Ninth International Conference on Hyperbolic Problems held in CalTech, Pasadena, March 25-29, 2002, Springer Verlag, 2003, pp. 483–492.

[11] V. G. Danilov, V. M. Shelkovich, *Propagation and interaction of δ-shock waves to hyperbolic systems of conservation laws*, Dokl. Ross. Akad. Nauk, **394**, no. 1, (2004), 10–14. English transl. in Russian Doklady Mathematics., **69**, no. 1, (2004), 4–8.

[12] V. G. Danilov, V. M. Shelkovich, *Delta-shock wave type solution of hyperbolic systems of conservation laws*, Quarterly of Applied Mathematics, **63**, no. 3, (2005), 401–427.

[13] V. G. Danilov, V. M. Shelkovich, *Dynamics of propagation and interaction of delta-shock waves in conservation law systems*, Journal of Differential Equations, **211**, (2005), 333–381.

[14] Weinan E., Yu. Rykov, Ya. G. Sinai, *Generalized variational principles, global weak solutions and behavior with random initial data for systems of conservation laws arising in adhesion particle dynamics*, Comm. Math. Phys., **177**, (1996), 349–380.

[15] G. Ercole, *Delta-shock waves as self-similar viscosity limits*, Quart. Appl. Math., **LVIII**, no. 1, (2000), 177–199.

[16] L. C. Evans, *Partial Differential Equations*, Amer. Math. Soc. Providence, Road Island, 1998.

[17] I. M. Gel'fand and G. E. Shilov, *Generalized Functions*, vol. 1, Properties and Operations. New York, Acad. Press, 1964.

[18] M. Grosser, M. Kunzinger, M. Oberguggenberger, R. Steinbauer, *Geometric theory of generalized functions with applications to general relativity*, Kluver Academic Publ., Dordrecht, 2001.

[19] Feiming Huang, *Existence and uniqueness of discontinuous solutions for a class non-strictly hyperbolic systems*, In Chen, Gui-Qiang (ed.) et al. Advances in nonlinear partial differential equations and related areas. Proceeding of conf. dedicated to prof. Xiaqi Ding, China, 1997, 187–208.

[20] K. T. Joseph, *Explicit generalized solutions to a system of conservation laws*, Proc. Indian Acad. Sci. (Math. Sci.), **109** (4), (1999), 401–409.

[21] Ram P. Kanwal, *Generalized Functions: Theory and technique*, Birkhäuser Boston–Basel–Berlin, 1998.

[22] B. L. Keyfitz, *Conservation laws, delta-shocks ans singular shocks*, In: Grosser, M., Horman, G., Kunzinger, M., Oberguggenberger, M. (eds) Nonlinear theory of generalized functions. Boca Raton: Chapman and Hall/CRC press, (1999), 99–111.

[23] B. L. Keyfitz and H. C. Kranzer, *Spaces of weighted measures for conservation laws with singular shock solutions*, Journal of Differential Equations, **118**, (1995), 420–451.

[24] B. L. Keyfitz, R. Sanders, M. Sever, *Lack of hyperbolicity in the two-fluid model for two-phase incompreddible flow*, Discrete and continuous dynamical systems–Series B, **3**, no. 4, (2003), 541–563.

[25] B. L. Keyfitz, M. Sever, Fu Zhang, *Viscous singular shock structure for nonohyperbolic two-fluid model*, Nonlinearity, **17**, (2004), 1731–1747.

[26] A. Yu. Khrennikov, V. M. Shelkovich, *On the algebraic aspect of singular solutions to conservation laws systems*, In: “Mathematical Modelling of Wave Phenomena”: 2nd Conference on Mathematical Modelling of Wave Phenomena, Växjö (Sweden), 14-19 August 2005. Eds: B. Nilsson and L. Fishman, AIP Conference Proceedings – May 4, 2006, Vol. 834, Issue 1, pp. 206-213

[27] H. C. Kranzer and B. Lee Keyfitz, *A strictly hyperbolic system of conservation laws admitting singular shocks*, Nonlinear Evolution Equations That Change Type, Springer-Verlag, 1990, 107–125.

[28] P. Le Floch, *An existence and uniqueness result for two nonstrictly hyperbolic systems*, Nonlinear Evolution Equations That Change Type, IMA Vol. Math. Appl. 27 (1990), Springer-Verlag, 126–138.

[29] Philippe Le Floch, *Entropy weak solutions to nonlinear hyperbolic systems under non-conservative form*, Commun. in partial differential equations, **13**, no. 6, (1988), 669–727.

[30] Philippe Le Floch, Tai-Ping Liu, *Existense theory for nonlinear hyperbolic systems in nonconservative form*, Forum Math., **5**, (1993), no. 3, 261–280.

[31] J. Li, Tong Zhang, *On the initial-value problem for zero-pressure gas dynamics*, Hyperbolic problems: Theory, Numerics, Applications. Seventh International Conference in Zürich, February 1998, Birkhäuser Verlag, Basel, Boston, Berlin, 1999, 629–640.

[32] J. Li, G. Warnece, *On measure solutions to the zero-pressure gas model and their uniquenness*, Mathematica Bohemica, **127**, no. 2, 265–273 (2002)

[33] J. Li, Li Wei, *Riemann problem for zero-pressure flow in gas dynamics*, Progress in natural science, **11**, N 5, (2001), 331–344.

[34] J. Li, Hanchun Yang, *Delta-shocks as limit of vanishing viscosity for multidimensional zero-pressure gas dynamics*, Quart. Appl. Math., **LIX**, N 2, (2001), 315–342.

[35] Tai-Ping Liu, Zhouping Xin, *Overcompressive shock waves*, Nonlinear evolution equations that change type, Springer-Verlag, 1990, 139–145.

[36] A. Majda, *Compressible fluid flow and systems of conservation laws in several space variables*, Springer-Verlag New York, Berlin, Heidelberg, Tokyo, 1984.

[37] M. Oberguggenberger, *Multiplication of distributions and applications to partial differential equations*, Pitman Research Notes in Mathematics 259, Longman, Harlow 1992.

[38] M. Oberguggenberger, *Generalizef functions in nonlinear models – a survey*, Nonlinear Analysis **47** (2001), 5029–5040.

[39] E. Yu. Panov, V. M. Shelkovich, *δ'-Shock waves as a new type of solutions to systems of conservation laws*, Journal of Differential Equations, **228** , (2006), 49–86.

[40] E. Yu. Panov, V. M. Shelkovich, *δ'-Shock wave as a new type of singular solutions of hyperbolic systems of conservation laws*, Dokl. Ross. Akad. Nauk, **407**, no. 5, (2006), 595–599. English transl. in Russian Doklady Mathematics., **73**, no. 2, (2006), 264–268

[41] E. Yu. Panov, V. M. Shelkovich, *Definition of δ'-shock waves type solutions and the Rankine–Hugoniot conditions*, Proceedings of International Seminar Days on Difraction'2005, Faculty of Physics, St.Petersburg, 2005, pp. 187–199.

[42] D. G. Schaeffer and S. Schecter and M. Shearer, *Nonstrictly hyperbolic conservation laws with a parabolic line*, Journal of Differential Equations, **103**, (1993), 94–126.

[43] S. Schecter, *Existence of Dafermos profiles for singular shocks*, Journal of Differential Equations, **205**, (2004), 185–210.

[44] M. Sever, *An existence theorem in the large for zero-pressure gas dynamics*, Differential and integral equations, **14**, N 9, (2001), 1077–1092.

[45] M. Sever, *Viscous structure of singular shocks*, Nonlinearity, **15**, (2002), 705–725.

[46] M. Sever, *Distribution solutions of nonolinear systems of conservation laws*, Preprint Hebrew University, Jerusalem, (2003).

[47] S. F. Shandarin and Ya. B. Zeldovich, *The large-scale structure of the universe: turbulence, intermittence, structures in self-gravitating medium*, Rev. Mod. Phys., **61**, (1989), 185–220.

[48] V. M. Shelkovich, *Delta-shock waves of a class of hyperbolic systems of conservation laws*, in: A. Abramian, S. Vakulenko, V. Volpert (Eds.), "Patterns and Waves", AkademPrint, St. Petersburg, 2003, pp. 155–168.

[49] V. M. Shelkovich, *A specific hyperbolic system of conservation laws admitting delta-shock wave type solutions*, Preprint 2003-059 at the url: http://www.math.ntnu.no/conservation/2003/059.html

[50] V. M. Shelkovich, *Delta-shocks, the Rankine–Hugoniot conditions, and singular superposition of distributions*, Proceedings of International Seminar Days on Difraction'2004, June 29–July 2, 2004, St.Petersburg, 2004, pp. 175–196. (See as the Preprint 2004-059 at the url: http://www.math.ntnu.no/conservation/2004/059.html)

[51] V. M. Shelkovich, *The Riemann problem admitting δ-, δ'-shocks, and vacuum states* (*the vanishing viscosity approach*), Journal of Differential Equations, **231**, (2006), 459–500.

[52] V. M. Shelkovich, *On delta-shocks and singular shocks*, in: Sylvie Benzoni-Gavage, Denis Serre (Eds.), Hyperbolic Problems: Theory, Numerics, Applications: Proceedings of the XIth International Conference on Hyperbolic Problems held in Ecole Normale Superieure, Lyon, July 17-21, 2006, Springer Verlag, 2007, pp.971–980.

[53] V. M. Shelkovich, *Validity and naturalness of definitions of δ- and δ'-shock type solutions to systems of conservation laws*, Proceedings of International Seminar Days on Difraction'2006, Faculty of Physics, St.Petersburg, 2006, pp. 266–279.

[54] Wancheg Shen, Tong Zhang, *The Riemann problem for the transportaion equations in gas dynamics*, Memoirs of the Amer. Math. Soc., **137**, N 654, (1999), 1–77.

[55] Dechun Tan, Tong Zhang and Yuxi Zheng, *Delta-shock waves as limits of vanishing viscosity for hyperbolic systems of conservation laws*, Journal of Differential Equations, **112**, (1994), 1–32.

[56] A. I. Volpert, *The space BV and quasilinear equations*, Math. USSR Sb. **2**, (1967), 225–267.

[57] Hanchun Yang, *Riemann problems for class of coupled hyperbolic systems of conservation laws*, Journal of Differential Equations, **159**, (1999), 447–484.

[58] Ya. B. Zeldovich, *Gravitationnal instability: An approximate theory for large density perturbations*, Astron. Astrophys., **5**, (1970), 84–89.

Department of Mathematics, St.-Petersburg State Architecture and Civil Engineering University, 2 Krasnoarmeiskaya 4, 190005, St. Petersburg, Russia.

E-mail address: `shelkv@vs1567.spb.edu`