CAR-FOLLOWING AND THE MACROSCOPIC AW–RASCLE TRAFFIC FLOW MODEL

MARTE GODVIK AND HARALD HANCHE-OLSEN

Abstract. We consider a semi-discrete car-following model and the macroscopic Aw–Rascle model for traffic flow given in Lagrangian form. The solution of the car-following model converges to a weak entropy solution of the system of hyperbolic balance laws with Cauchy initial data. For the homogeneous system, we allow vacuum in the initial data. By using properties of the semi-discrete model, we show that this solution of the hyperbolic system is stable in the L^1-norm.

1. Introduction

We consider a semi-discrete model for traffic flow proposed by Aw, Klar, Materne and Rascle in [1],

$$\left.\begin{aligned}\dot\tau_k^\delta(t) &= \frac{1}{\delta}\left(v_{k+1}^\delta(t) - v_k^\delta(t)\right)\\ \dot w_k^\delta(t) &= R\left(\tau_k^\delta(t), w_k^\delta(t)\right)\end{aligned}\right\} \qquad \text{where } w_k^\delta = v_k^\delta + Q(\tau_k^\delta). \tag{1}$$

The functions $\tau_k^\delta(t)$ and $v_k^\delta(t)$ are the inverse density (the distance to car $k+1$) and velocity of car k, respectively. The Lagrangian position of car k on the roadway is $y = k\delta$ for some $\delta > 0$, and t denotes the time variable. The function Q is a "pressure" function and R is a relaxation term.

After investigating the above model, we show that the solution converges to a weak entropy solution of the macroscopic Aw–Rascle model for traffic flow [2] as $\delta \to 0$. This model is a system of hyperbolic balance laws. In Lagrangian form the system is given as

$$\left.\begin{aligned}\tau_t - v_y &= 0\\ w_t &= R(\tau, w)\end{aligned}\right\} \qquad \text{where } w = v + Q(\tau), \tag{2}$$

$\tau(y,t)$ and $v(y,t)$ are the inverse density and the velocity of cars on the roadway, and $y \in \mathbb{R}$ and $t \in \mathbb{R}^+$ are the Lagrangian mass and time variable, respectively. For simplicity we write the system as

$$u_t + f(u)_y = r(u),$$

where $u = (\tau, w)$, $r(u) = (0, R(\tau, w))$ and $f(u) = (-v, 0)$. The eigenvalues of the homogeneous system of hyperbolic conservation laws are $\lambda_1 = Q'(\tau)$ and $\lambda_2 = 0$, and for $\tau < \infty$ the system is strictly hyperbolic. Further, the first wave family is genuinely nonlinear and the second family is linearly degenerate. The system is of Temple class, that is the curves of the shock and the rarefaction coincide. For $\tau = \infty$ the eigenvalues coincide. A solution of the Cauchy problem with initial data in the domain

$$\mathcal{D} = \{u \in \mathcal{U} \subset \mathbb{R}^2 : w_- \le w(u) \le w_+, 0 \le v_- \le v(u) \le v_+, \}, \tag{3}$$

where $w_- > v_+$, will not contain vacuum. Consider the Riemann problem with initial data in $\mathcal{D}$. In general, a solution consists of a left state that connects to a middle state by a shock or rarefaction

2000 *Mathematics Subject Classification.* 35L45, 35L65, 35L67, 35L80, 90B20.

Key words and phrases. Traffic flow, car-following, non–strictly hyperbolic system, vacuum.

wave with negative speed, and a contact wave with zero speed that connects the middle state to the right state. When the initial Riemann data takes values in the domain

$$\mathcal{D}_V = \{u \in \mathcal{U} \subset \mathbb{R}^2 : 0 \leq v(u) \leq w(u) \leq v_+\}, \tag{4}$$

vacuum will occur if $v_R > w_L$. A rarefaction wave connects the left state to vacuum and vacuum is connected to the right state by a contact wave with speed zero. As we will show in the final section, not all inadmissible discontinuities will violate the entropy condition given by an entropy inequality. However, we will show that our solutions connects to a vacuum state continuously from the left. A discussion of the solution of the Riemann problem, in Eulerian form, can be found in [2]. The system in Lagrangian form is equivalent to the system in Eulerian form, see [15].

The macroscopic model (2) was introduced by Aw and Rascle in 2000 [2]. In [1] a connection between a microscopic model and a semi-discretization of the macroscopic Aw–Rascle model is established. In the homogeneous case it is showed that the semi-discretization of the macroscopic model is the limit of the time discretization of the microscopic model. Our approach is different. We derive the Aw–Rascle model by considering the semi-discrete car-following model, and show that the limit is a weak entropy solution of this system.

The Aw–Rascle system is most commonly studied in Eulerian form. As long as vacuum is excluded, existence of a weak entropy solution to the hyperbolic Aw–Rascle system with initial data in $\mathcal{D}$ follows from the Temple property and the Glimm scheme, see [14, Chapter 5]. Also, the hyperbolic balance laws with initial data in $\mathcal{D}$ satisfy the assumptions made by Colombo and Corli in [5], which yields well-posedness for strictly hyperbolic Temple system with source, assuming the eigenvalues are separated on every compact subset of $\mathcal{D}$. When vacuum is included, the homogeneous system is not strictly hyperbolic, the eigenvalues are not separated in $\mathcal{D}_V$ and there is no bound on the total variation of the (inverse) density. In [8] we show the existence of a weak entropy solution for Cauchy initial data in $\mathcal{D}_V$ in the Eulerian formulation. In this paper we show that the weak entropy solution of the hyperbolic system of conservation laws obtained as a limit of the car-following model, is stable in the L^1-norm. Our technique is to find and describe properties of the semi-discrete model, and then use these properties to obtain results for the macroscopic model. Under assumptions on the flux function and initial data which do not correspond to the assumption in this paper, a similar L^1-stability estimate, for weak solutions of scalar conservations laws with a flux function depending discontinuously on the space variable, is proved in [13].

For further discussions of the Aw–Rascle model, see [6], [7], [11] and [12]. In [3], [9] and [10] the system is considered in Lagrangian form.

1.1. **Assumptions and notation.** The function $Q(\tau)$ is a smooth, positive and strictly decreasing function. The prototype of this function is

$$Q(\tau) \propto \tau^{-\gamma}, \quad \gamma > 0. \tag{5}$$

We will assume Q satisfies

$$Q(\infty) = 0, \quad Q'(\tau) < 0 \;\text{ and }\; Q''(\tau) > 0 \;\text{ for }\; \tau < \infty. \tag{6}$$

On the domains $\mathcal{D}$ and $\mathcal{D}_V$ the function τ has a lower, positive bound $\tau_{\min}$ given by $Q(\tau_{\min}) = w^+ - v^-$. Thus, it follows that for $\tau_1, \tau_2 \geq \tau_{\min} > 0$, $Q(\tau)$ is Lipschitz in its argument,

$$|Q(\tau_1) - Q(\tau_2)| \leq L|\tau_1 - \tau_2|, \tag{7}$$

for some constant L. The above assumptions are satisfied for the prototype function. The relaxation term $R(\tau, w)$ is assumed to be smooth and Lipschitz continuous in w and v,

$$|R(\tau_1, w_1) - R(\tau_2, w_2)| \leq L_R\big(|w_1 - w_2| + |v_1 - v_2|\big), \tag{8}$$

where L_R is some positive constant. Remember that $w = v + Q(\tau)$. Further, the domains $\mathcal{D}$ and $\mathcal{D}_V$ should be invariant domains for the relaxation part of the system, i.e. the second equation given in (2). We consider the sign of w_t along the edges of the domains and thus we assume $R(\tau, w)$ satisfies

$$R(\tau, w) \begin{cases} \geq 0, & v_- \leq v \leq v_+, w = w_- \text{ or } v = v_-, w_- \leq w \leq w_+, \\ \leq 0, & v_- \leq v \leq v_+, w = w_+ \text{ or } v = v_+, w_- \leq w \leq w_+. \end{cases} \tag{9}$$

It is not clear how to interpret the above technical requirement from a traffic point of view. However, we may note that it is satisfied by a commonly used relaxation term on the form $R = \kappa(V(\tau) - v)$ so long as $0 \leq V(\tau) \leq v_+$.

We now introduce some notation. Let Ω denote any bounded measurable subset of $\mathbb{R}$. The purpose is to compute in $L^1(\Omega)$ and tacitly draw conclusions about $L^1_{\mathrm{loc}}(\mathbb{R})$. The norm in $L^1(\Omega)$ is denoted by $\|\cdot\|$, and also when μ is a measure we write $\|\mu\| = |\mu|(\Omega)$. By "weak convergence" we shall mean convergence in the sense of distributions, and we use the harpoon $\rightharpoonup$ for this. Further, we denote $u^\delta_k = u^\delta(k\delta) = u^\delta(y_k)$ and $u^\delta = u^\delta(y)$ for some y in $\mathbb{R}$. In the discrete case we denote $\|\tau\| = \delta \sum_k |\tau_k|$. For simplicity we will sometimes write u_k instead of u^δ_k and $u(t)$ instead of $u(\cdot, t)$. Furthermore, we let $a \vee b$ and $a \wedge b$ denote $\max(a, b)$ and $\min(a, b)$, respectively.

1.2. **Overview and main results.** First, in section 2 we consider the semi-discrete model. We investigate properties of the model and obtain results which will be useful later. Then, we start to consider the macroscopic model.

In section 3 we let the initial data take values in the domain $\mathcal{D}$. Thus

$$w_+ - v_- \leq Q(\tau) \leq w_- - v_+,$$

which implies

$$0 < Q^{-1}(w_- - v_+) \leq \tau \leq Q^{-1}(w_+ - v_-) < \infty,$$

and vacuum is not included in the solution. We find a bound on the total variation of (v^δ, w^δ) and thus there is a bound on the total variation of the solution (τ^δ, w^δ) of the semi-discrete system.

When it comes to convergence we will employ filters instead of subsequences and we let $\mathcal{U}$ denote a free ultrafilter on $\mathbb{N}$. In particular, whenever speaking of convergence of u^δ, we really mean $u^\delta \to u$ as $\delta \to \mathcal{U}$, i.e. for all $\epsilon > 0$, there exists $U \subset \mathcal{U}$ such that for δ in U we have $\|u^\delta - u\| < \epsilon$. As an alternative, and with only minimal modifications, we could employ more conventional diagonal arguments. This works just as well as the filters for single solutions. However, we are interested in obtaining solution semigroups, and for this purpose, the diagonal argument does not work because the space of initial functions $\bar{w}$ is not separable with respect to the TV norm. If we had proved uniqueness, this point would of course be moot.

The following theorem is proved in section 3.

Theorem 1. *Given initial data $(\bar{\tau}^\delta, \bar{w}^\delta)$ in $\mathrm{BV}(\mathbb{R})^2$ taking values in $\mathcal{D}$. Assume $(\bar{\tau}^\delta, \bar{w}^\delta)$ converges to a function $(\bar{\tau}, \bar{w})$ in $L^1_{loc}(\mathbb{R})^2$ taking values in $\mathcal{D}$. Let $Q(\cdot)$ and $R(\cdot,\cdot)$ satisfy (6)–(7) and (8)–(9), respectively. Then, the solution (τ^δ, w^δ) of the semi-discrete car-following model (1) converges in $L^1_{loc}(\mathbb{R} \times \mathbb{R}^+)^2$. The limit (τ, w) is a weak entropy solution of the macroscopic Aw–Rascle model (2) with Cauchy initial data $(\bar{\tau}, \bar{w})$.*

We do not consider the problem of letting the relaxation time go to zero. This would amount to replacing R by κR and letting $\kappa \to \infty$. With this change, our total variation estimates in section 2.1 would contain a multiplicative factor $e^{2\kappa L_R t}$, so other techniques (and possibly other assumptions) would be needed.

In section 4 we consider the homogeneous system, i.e. we assume $R = 0$. By using monotonicity properties of the solution for the semi-discrete system we prove stability of the solution in the L^1-norm. The proof is inspired by [4].

Theorem 2. *Let $(\bar{\tau}^1, w^1)$ and $(\bar{\tau}^2, w^2)$ be two sets of initial data in $\mathrm{BV}(\mathbb{R})^2$ taking values in $\mathcal{D}$. Assume the initial data is constant for $y \geq Y$ and for $y \leq -Y$. Let $Q(\cdot)$ satisfy (6)–(7) and $R = 0$.*

Then, the two solutions (τ^1, w^1) *and* (τ^2, w^2) *of system* (2)*, obtained as a limit of the car-following model* (1)*, satisfy*

$$\|\tau^1(t) - \tau^2(s)\| \le \|\bar{\tau}^1 - \bar{\tau}^2\| + (t \wedge s)\,\mathrm{TV}(w^1 - w^2) + (t \wedge s)\|Q'\|_\infty |\bar{\tau}^1(Y) - \bar{\tau}^2(Y)| + C|t-s|,$$

where $C = \mathrm{TV}(\bar{v}^1) \vee \mathrm{TV}(\bar{v}^2)$*.*

The presence of the initial data at Y, and more notably, the absence of initial data at $-Y$, in the stability estimate is due to negative characteristic speed.

We have proved existence of a semigroup

$$S : [0,\infty) \times \Gamma \to \Gamma,$$

where

$$\Gamma = \{(\tau, w) \in \mathrm{BV}(\mathbb{R})^2 : (\tau, w) \in \mathcal{D}\} \tag{10}$$

such that for each $t > 0$, the function $(y,t) \mapsto S_t(\bar{\tau}, w) = (S_t^w \bar{\tau}, w)$ is a weak entropy solution of system (2) with initial data $(\bar{\tau}, w)$ in Γ. Further, for each $t > 0$, the map $S_t : \Gamma \to \Gamma$ is stable with respect to the initial data.

In section 5 we include vacuum in the solution by letting the initial data take values in $\mathcal{D}_V$. Let V be a discrete subset of $\mathbb{R}$ and let $\Delta = \sum_{y \in V} \delta_y$ be the counting measure on V and m the Lebesgue measure on $\mathbb{R}$. Consider locally finite measures that are absolutely continuous with respect to $m+\Delta$ on $\mathbb{R}$. We write these measures as

$$d\hat{\tau} = \tau\, dm + h\, d\Delta = \tau\, dy + \sum_{y \in V} h(y)\, d\delta_y. \tag{11}$$

So if $y \in V$ then $h(y) \ge 0$ is the size of the vacuum at y (physically, the length of an empty road section). Define

$$\begin{aligned} \Gamma_V = \{(\hat{\tau}, w) : \;& \hat{\tau} \text{ is a positive, locally finite measure given as in (11)} \\ & \text{and absolutely continuous with respect to } m + \Delta, \\ & (\tau, w) \in \mathcal{D}_V, Q(\tau) \in \mathrm{BV}(\mathbb{R}), w \in \mathrm{BV}(\mathbb{R}), \\ & w(b) - w(a) \le L_w(b-a) \text{ for all } a < b \text{ such that} \\ & [a,b] \cap V = \emptyset \text{ where } L_w \text{ is a constant.}\} \end{aligned} \tag{12}$$

In order to handle vacuum in the initial data, we consider the Eulerian space coordinate $x = x(y,t)$ and the definition of the inverse density, $\tau = \partial x / \partial y$. We prove existence and stability of the semigroup

$$S : [0,\infty) \times \Gamma_V \to \Gamma_V,$$

and show that its trajectories are weak entropy solutions of the macroscopic system.

Theorem 3. *Given initial data* $(\bar{\tau}, w)$ *in* Γ_V*, and let* $Q(\cdot)$ *satisfy* (6)–(7) *and* $R(\tau, w) = 0$*. Assume* w^δ *converges to* w *in* $L^1_{loc}(\mathbb{R})$ *and let* $\bar{\tau}^\delta$ *be given by* $\bar{x}^\delta = \partial \bar{\tau}^\delta / \partial y$ *where* $\bar{x}^\delta(y)$ *is a function which converges to* $\bar{x}(y) = \partial \bar{\tau}^\delta / \partial y$ *pointwise a.e.*

Then, the solution $\tau^\delta(y,t)$ *of the semi-discrete car-following model* (1) *converges to a weak entropy solution of the macroscopic Aw–Rascle model* (2) *with Cauchy initial data* $(\bar{\tau}, w)$*, and the limit* $(\hat{\tau}, w)$ *is in* Γ_V*.*

Further, let $(\bar{\tau}^1, w^1)$ *and* $(\bar{\tau}^2, w^2)$ *be two sets of initial data satisfying the assumptions above and assume they are constant for* $|y| \ge Y$*. Then, the solutions* $\hat{\tau}^1$ *and* $\hat{\tau}^2$*, obtained as the limits of the* *car-following model* (1), *satisfy*

$$\|\hat{\tau}^1(t) - \hat{\tau}^2(s)\| \le \|\bar{\tau}^1 - \bar{\tau}^2\| + (t \wedge s)\,\mathrm{TV}(w^1 - w^2) \\ + (t \wedge s)\|Q'\|_\infty |\bar{\tau}^1(Y) - \bar{\tau}^2(Y)| + C|t-s|, \quad (13)$$

where $C = \mathrm{TV}(\bar{v}^1) \vee \mathrm{TV}(\bar{v}^2)$.

We may note that Bagnerini and Rascle [3] obtained a similar L^1 contraction principle with $w^1 = w^2$ and no vacuum. Their proof is based on a family of Kružkov-type entropies originally described by Baiti and Jenssen [4]. We intend to tackle the uniqueness question using these entropies in a forthcoming paper.

In [2], Aw and Rascle note a necessary lack of stability of the Riemann problem solutions near vacuum. This is in apparent contradiction to the above stability result. However, [2] considers the Eulerian model, while we consider the Lagrangian model. In Eulerian coordinates, a near vacuum is close to an actual vacuum, while this is not so Lagrangian coordinates. As an example, consider the Eulerian density ρ_ϵ and corresponding Lagrangian inverse density τ_ϵ:

$$\rho_\epsilon(x) = \begin{cases} 1 & \text{if } |x| > 1, \\ \epsilon & \text{if } |x| \le 1, \end{cases} \qquad \tau_\epsilon(y) = \begin{cases} 1 & \text{if } |y| > 1/\epsilon, \\ 1/\epsilon & \text{if } |y| \le \epsilon. \end{cases}$$

Now $\|\rho_\epsilon - \rho_0\| \to 0$ when $\epsilon \to 0$, but since we must write $\tau_0(y) = 2\delta(y) + 1$, we find $\|\tau_\epsilon - \tau_0\| = 4 - 2\epsilon \not\to 0$. This discontinuity in the conversion between Eulerian and Lagrangian coordinates explains the disparity.

2. The semi-discrete model

Consider individual cars driving on a roadway and let the Lagrangian position of a car denoted by k be given as $y = k\delta$ for some $\delta > 0$. Let the set of indices k be given as $I = \{-K, -K+1, \dots, K\}$ such that $\Omega = [-K, K]\delta = [-Y, Y]$.

In order to define the model (1) for all values of y, let $k\delta < y \le (k+1)\delta$, where $k \in I$. Define $u^\delta(y,t) = (\tau^\delta(y,t), w^\delta(y,t)) = (\tau_k^\delta(t), w_k^\delta(t))$ and similarly for $v^\delta(y,t)$. Thus the system turns into

$$\begin{aligned} \tau_t^\delta(y,t) &= \frac{v^\delta(y+\delta,t) - v^\delta(y,t)}{\delta} \\ w_t^\delta(y,t) &= R\left(\tau^\delta(y,t), w^\delta(y,t)\right). \end{aligned} \qquad (14)$$

Further, by integration of the first equation of the semi-discrete system with respect to t and use of the definition of v we get the equations

$$\begin{aligned} \tau_k^\delta(t) &= \frac{x_{k+1}^\delta(t) - x_k^\delta(t)}{\delta} \\ v_k^\delta(t) &= \dot{x}_k^\delta(t), \end{aligned} \qquad (15)$$

with the initial data given by

$$\tau_k^\delta(0) = \bar{\tau}_k^\delta = \frac{\bar{x}_{k+1}^\delta - \bar{x}_k^\delta}{\delta}, \quad v_k^\delta(0) = \bar{v}_k^\delta. \qquad (16)$$

The above expressions will be useful in the final section.

The initial data $\bar{u}^\delta(y)$ takes values in $\mathcal{D}$ or $\mathcal{D}_V$. By assumption, $\bar{w}^\delta$ and $\bar{v}^\delta$ are in $\mathrm{BV}(\mathbb{R})$. Further, we assume $\bar{u}^\delta(y) = \bar{u}^\delta(-Y)$ for all $y < -Y$ and $\bar{u}^\delta(y) = \bar{u}^\delta(Y)$ for all $y > Y$. Note that if $[y_{k-1}, y_k] \cap V = \emptyset$, then $w_k - w_{k-1} \le L_w\delta$ for some positive constant L_w.

2.1. **Properties of the model with relaxation.** Consider the semi-discrete system (14) with initial data as given above. First, we establish that the domains $\mathcal{D}$ and $\mathcal{D}_V$ given by (3) and (4), respectively, are invariant regions for the system of ordinary differential equations.

Lemma 1. *If the initial data* $(\bar{\tau}^\delta, \bar{w}^\delta)$ *takes values in* $\mathcal{D}$ *or* $\mathcal{D}_V$*, then so does the solution* (τ^δ, w^δ)*.*

Proof. Consider the boundary of $\mathcal{D}$. For $w = w_-$ we want $w_t \geq 0$. Thus we require $R(\tau, w_-) \geq 0$. Further, for $w = w_+$ we want $w_t \leq 0$, which is satisfied for $R(\tau, w_+) \leq 0$. We rewrite the last equation in (1) as

$$\dot{v}_k = R(\tau_k, w_k) - Q'(\tau_k)\dot{\tau}_k = R(\tau_k, w_k) - \frac{1}{\delta}Q'(\tau_k)(v_{k+1} - v_k).$$

Remember that Q is decreasing as a function of τ. For $v_k = v_-$, we have $v_k \leq v_{k+1}$ and the last term in the above equality is non-negative. Thus, in order to have $\dot{v}_k \geq 0$ we require $R(\tau, w) \geq 0$. Further, when $v_k = v_+$ the last term is non-positive, and $v_t \leq 0$ if $R(\tau_k, w) \leq 0$. Thus, requiring (9) yields invariance of the domain $\mathcal{D}$ with respect to the ordinary differential equations (1).

In order to show that $\mathcal{D}_V$ is invariant, it remains to consider the states where $w = v$. Since $(v - w)_t \to 0$ when $v - w \to 0$, we will never obtain a state where $v > w$. □

Now we find a bound on the total variation of the variables (v^δ, w^δ).

Lemma 2.

$$\mathrm{TV}\left(v^\delta(t), w^\delta(t)\right) \leq e^{2L_R t}\,\mathrm{TV}\left(\bar{v}^\delta, \bar{w}^\delta\right),$$

where $\mathrm{TV}(\bar{v}^\delta, \bar{w}^\delta) \leq \mathrm{TV}(\bar{v}, \bar{w}) = \mathit{const.}$

Proof. Our goal is to find a bound on the total variation of $w^\delta(t)$ and $v^\delta(t)$. Adding the equations for $\dot{w}_{k+1}$ and $-\dot{w}_k$ and multiplying by $\operatorname{sign}(w_{k+1} - w_k)$ yields

$$\begin{aligned} |w_{k+1} - w_k|_t &\leq |R(\tau_{k+1}, w_{k+1}) - R(\tau_k, w_k)| \\ &\leq L_R\left(|v_{k+1} - v_k| + |w_{k+1} - w_k|\right). \end{aligned} \tag{17}$$

We want to obtain a similar estimate for v^δ. The strategy is to estimate the difference between local extremes for v^δ. Let $\{k_i\}$ give the points at which v^δ attains local extreme values and thus we want to measure $|v_{k_i} - v_{k_{i+1}}|$. For a local minimum in v_k we have $v_k < v_{k+1}$ and for a local maximum in v_k we have $v_k > v_{k+1}$. Now, let v_{k_i} be a local minimum and $v_{k_{i+1}}$ be a local maximum. Thus

$$\begin{aligned} |v_{k_{i+1}} - v_{k_i}|_t &= \left(v_{k_{i+1}} - v_{k_i}\right)_t \\ &= R(\tau_{k_{i+1}}, w_{k_{i+1}}) - R(\tau_{k_i}, w_{k_i}) \\ &\quad - \frac{1}{\delta}Q'(\tau_{k_{i+1}})\left(v_{k_{i+1}+1} - v_{k_{i+1}}\right) + \frac{1}{\delta}Q'(\tau_{k_i})\left(v_{k_i+1} - v_{k_i}\right) \\ &\leq L_R\left(|v_{k_{i+1}} - v_{k_i}| + |w_{k_{i+1}} - w_{k_i}|\right) \end{aligned} \tag{18}$$

The inequality follows from the Lipschitz property of R as given by (8), and $Q' < 0$ along with assumptions that makes each of the two terms containing Q non-positive.

We sum over k in (17) and over k_i in (18). Adding the two and using

$$\sum_i |w_{k_{i+1}} - w_{k_i}| \leq \sum_k |w_{k+1} - w_k| \quad \text{and} \quad \sum_i |v_{k_{i+1}} - v_{k_i}| = \sum_k |v_{k+1} - v_k|$$

yields

$$\sum_k |w_{k+1} - w_k|_t + \sum_{k_i} |v_{k_{i+1}} - v_{k_i}|_t \leq 2L_R\Big(\sum_k |w_k - w_{k+1}| + \sum_{k_i} |v_{k_{i+1}} - v_{k_i}|\Big).$$

The set $\{k_i\}$ depends on the time. However, the function

$$f(t) = \sum_k |w_{k+1} - w_k| + \sum_{k_i} |v_{k_{i+1}} - v_{k_i}|$$

is continuous with respect to t. Further, we let f_t be the lower right hand Dini derivative of f at t. What we just proved should be interpreted as $f_t \leq 2L_R f$. Then, by an obvious extension of Grönwall's inequality using Dini derivatives,

$$f(t) \leq e^{2L_R t} f(0),$$

for all t in $[0, T]$ and we are done. □

A bound on the total variation of $Q(\tau^\delta)$ follows from $Q(\tau) = w - v$. Next, we show that $\tau^\delta(t)$ and $w^\delta(t)$ are Lipschitz in time as functions into $L^1_{\mathrm{loc}}(\mathbb{R})$.

Lemma 3.

$$\|\tau^\delta(s) - \tau^\delta(t)\| \leq C e^{2L_R(t \vee s)} |t - s|, \tag{19}$$

and

$$\|w^\delta(s) - w^\delta(t)\| \leq \|R(\tau^\delta, w^\delta)\|_\infty |\Omega| \, |t - s|, \tag{20}$$

where $C = \mathrm{TV}(\bar{v}^\delta, \bar{w}^\delta)$.

Proof. Equation (1) yields

$$\begin{aligned}\delta \sum_k |\tau_k(s) - \tau_k(t)| &= \sum_k \left| \int_t^s \left(v_{k+1}(\xi) - v_k(\xi)\right) \, d\xi \right| \\ &\leq \int_t^s \mathrm{TV}\big(v^\delta(\xi)\big) \, d\xi \leq C e^{2L_R(t \vee s)} |t - s|,\end{aligned}$$

where the last inequality follows by Lemma 2, and

$$\begin{aligned}\delta \sum_k |w_k(s) - w_k(t)| &= \delta \sum_k \left| \int_t^s R\left(\tau_k(\xi), w_k(\xi)\right) \, d\xi \right| \\ &\leq \|R(\tau^\delta, w^\delta)\|_\infty |\Omega| \, |t - s|.\end{aligned}$$

□

We want to find a discrete entropy inequality for the solution u^δ. For hyperbolic systems an entropy/entropy flux pair (η, q) is a convex entropy $\eta : \mathcal{U} \to \mathbb{R}$ and a flux $q : \mathcal{U} \to \mathbb{R}$ satisfying

$$\nabla q = df \cdot \nabla \eta. \tag{21}$$

However, when we include vacuum, there does not exist any strictly convex entropy/entropy flux pair, see [8]. We define a *semiconvex* entropy η with corresponding flux q as an entropy satisfying $\eta_{\tau\tau} > 0$ for $\tau < \infty$. Further, an entropy solution is a weak solution that satisfies an entropy inequality for all such entropy/entropy flux pairs. The semiconvex entropy/entropy flux pairs of system (2) are

$$q(\tau, w) = g\left(w - Q(\tau)\right), \quad \eta(\tau, w) = -\int_{\tau_0}^{\tau} g'\left(w - Q(\xi)\right) \, d\xi, \tag{22}$$

where τ_0 is a constant value and $g = g(w - Q(\tau))$ is a smooth function in $L^\infty(\mathbb{R})$ having continuous third order derivatives. In order to get strict convexity of η, we require $g'' < 0$. Consider discontinuities that do not connect to vacuum. If they are admissible they will satisfy the entropy inequality for all semiconvex entropies with corresponding entropy fluxes and the entropy inequality will fail for the inadmissible discontinuities [8].

We have the following lemma.

Lemma 4. *The solution u_k satisfies the discrete entropy inequality*

$$\eta(u_k)_t + \frac{1}{\delta}\big(q(u_{k+1}) - q(u_k)\big) \le \nabla_u \eta(u_k) r(u_k), \tag{23}$$

where q and η is given by (22).

Proof. Since $\eta_t = \eta_\tau \tau_t + \eta_w w_t$ and $\nabla_u \eta\, r = \eta_w R = \eta_w w_t$, we can rewrite the desired inequality as

$$\eta_\tau(u_k)\dot\tau_k \le \frac{1}{\delta}\big(q(u_k) - q(u_{k+1})\big).$$

Using (1) and (22) we rewrite this as $(v_k - v_{k+1})g'(v_k) \le g(v_k) - g(v_{k+1})$, or

$$g(v_{k+1}) \le g(v_k) + g'(v_k)\left(v_{k+1} - v_k\right),$$

Since $g(v)$ is concave, the above inequality holds, and the solution u^δ of (1) satisfies (23). □

2.2. **Properties of the model with $R = 0$.** In Theorem 2 and Theorem 3 we assume the systems are homogeneous. Then, w^δ is constant in time, and system (14) turns into the equation

$$\tau^\delta_t(y,t) = \frac{1}{\delta}\left(w^\delta(y+\delta) - w^\delta(y) - Q(\tau^\delta(y+\delta,t)) + Q(\tau^\delta(y,t))\right), \tag{24}$$

Let the function $(y,t) \mapsto S^{\delta,w}_t \bar\tau^\delta(y)$ be the solution of the above equation with initial data $\bar\tau^\delta(y)$ and $w^\delta(y)$. We abuse the notation and denote the corresponding v by $S^{\delta,w}_t \bar v^\delta$. Sometimes we simplify the notation and write $\tau_k(t)$ instead of $S^{\delta,w}_t \bar\tau_k$.

Consider two different sets of initial data $(\bar\tau^{1,\delta}, w^{1,\delta})$ and $(\bar\tau^{2,\delta}, w^{2,\delta})$, in $\mathrm{BV}(\mathbb{R})^2$ satisfying the assumption given in the beginning of this section. For simplicity, we will sometimes write $\bar\tau^i$ and w^i for $\bar\tau^{i,\delta}$ and $w^{i,\delta}$, respectively. An important property of the semi-discrete system is the monotonicity of $S^{\delta,w}_t \bar\tau$ with respect to the initial data . In order to prove two such results we will need the following trivial lemma.

Lemma 5. *Assume $\dot\alpha = a(\alpha,t)$ and $\dot\beta = b(\beta,t)$, $\alpha(0) \le \beta(0)$ and $a(\xi,t) < b(\xi,t)$ for all ξ, t. Then, $\alpha < \beta$ for all $t > 0$.*

We now show that the solution $S^{\delta,w}_t \bar\tau(y)$ is monotone with respect to $\bar\tau(y)$ and the difference $w(y+\delta) - w(y)$.

Lemma 6. *Assume*

$$\bar\tau^1_k \le \bar\tau^2_k,\ \ w^1_{k+1} - w^1_k \le w^2_{k+1} - w^2_k \quad \textit{for all } k.$$

Then

$$S^{\delta,w^1}_t \bar\tau^1 \le S^{\delta,w^2}_t \bar\tau^2.$$

Proof. We will use backwards induction in the space variable to prove the lemma. Since τ^1_k and τ^2_k are constant for $k \ge K$, we have $\tau^1_K \le \tau^2_K$ for all time t. Assuming $\tau^1_{k+1} \le \tau^2_{k+1}$ for all t, it remains to show that $\tau^1_k \le \tau^2_k$ for all t. Equation (24) yields

$$\begin{aligned}\delta\dot\tau^1_k &= w^1_{k+1} - w^1_k - Q(\tau^1_{k+1}) + Q(\tau^1_k)\\ &\le w^2_{k+1} - w^2_k - Q(\tau^2_{k+1}) + Q(\tau^1_k) = \delta\dot\tau^2_k - Q(\tau^2_k) + Q(\tau^1_k).\end{aligned}$$

Consider the cases where $w^1_{k+1} - w^1_k < w^2_{k+1} - w^2_k$ or $w^1_{k+1} - w^1_k \le w^2_{k+1} - w^2_k$ and $\tau^1_{k+1} < \tau^2_{k+1}$, which yields strict inequality in the calculation above,

$$\delta(\dot\tau^1_k - \dot\tau^2_k) < Q(\tau^1_k) - Q(\tau^2_k).$$

Initially we have $\tau^1_k \le \tau^2_k$ and thus $\dot\tau^1_k < \dot\tau^2_k$. If τ^1_k equals τ^2_k we still have $\dot\tau^1_k < \dot\tau^2_k$. Thus, Lemma 5 yields $\tau^1_k < \tau^2_k$ for all time t.

Then, consider the case where $w^1_{k+1} - w^1_k = w^2_{k+1} - w^2_k$ and $\bar\tau^1_{k+1} = \bar\tau^2_{k+1}$. The ordinary differential equations for τ^1_k and τ^2_k are identical. Thus, the uniqueness of solutions implies $\tau^1_k \le \tau^2_k$ for all time t. $\square$

Further, the solution operator $S^{\delta,w}_t$ is monotonicity preserving.

Lemma 7. *Assume*

$$\bar\tau_k \le \bar\tau_{k+1} \ \ \textit{and} \ \ w_{k+1} - w_k \le w_{k+2} - w_{k+1} \ \textit{for all} \ k. \tag{25}$$

Then,

$$S^{\delta,w}_t \bar\tau_k \le S^{\delta,w}_t \bar\tau_{k+1} \quad \textit{for all} \quad k.$$

Proof. The proof is exactly the same as the proof of Lemma 6, replacing τ^1_k by τ_k, and τ^1_{k+1} and τ^2_k by τ_{k+1} and τ^2_{k+1} by τ_{k+2} and similarly for the difference $w_{k+1} - w_k$. $\square$

Before stating and proving the next lemma, we need to introduce some notation. Define

$$\omega^+_k = \max\{w^1_{k+1} - w^1_k, w^2_{k+1} - w^2_k\}, \ \ \omega^-_k = \min\{w^1_{k+1} - w^1_k, w^2_{k+1} - w^2_k\}, \tag{26}$$

and let $S^{\delta,\omega^+}_t \bar\tau_k$ denote the solution of

$$\delta \dot\tau_k = \omega^+_k - Q(\tau_{k+1}) + Q(\tau_k) \tag{27}$$

with initial data $\bar\tau_k$ and similarly for $S^{\delta,\omega^-}_t \bar\tau_k$. Further, we denote

$$\tau^+_k = S^{\delta,\omega^+}_t \left(\bar\tau^1_k \vee \bar\tau^2_k\right) = S^{\delta,\omega^+}_t \bar\tau^+_k, \quad \tau^-_k = S^{\delta,\omega^-}_t \left(\bar\tau^1_k \wedge \bar\tau^2_k\right) = S^{\delta,\omega^-}_t \bar\tau^-_k. \tag{28}$$

Then, the above assumptions and Lemma 6 implies the following results,

$$|\bar\tau^1 - \bar\tau^2| = \bar\tau^+ - \bar\tau^-, \tag{29}$$

$$|S^{\delta,w^1}_t \bar\tau^1 - S^{\delta,w^2}_t \bar\tau^2| \le S^{\delta,w^+}_t \bar\tau^+ - S^{\delta,w^-}_t \bar\tau^-, \tag{30}$$

and

$$\mathrm{TV}(w^1 - w^2) = \sum_k (\omega^+_k - \omega^-_k), \tag{31}$$

which will be useful.

The waves of a hyperbolic system move with finite speed. Now we want to find a result for the discrete system which is analogous to the feature of finite speed of propagation.

Lemma 8. *Given two sets of initial data* $(\bar\tau^1, w)$ *and* $(\bar\tau^2, w)$ *such that* $\bar\tau^1(y) = \bar\tau^2(y)$ *for* $y \le 0$ *and an arbitrary* $\epsilon > 0$*. Let* $k_{j\epsilon}$ *be given such that*
$-Lt - j\epsilon - \delta \le k_{j\epsilon}\delta \le -Lt - j\epsilon$*. Then*

$$\delta \sum_{k \le k_{j\epsilon}} |\tau^2_k - \tau^1_k| \le \frac{\delta^j}{\epsilon^{2j}} \frac{(4L)^j}{2(j+1)!} w_+ t^{j+1}.$$

In particular

$$|\tau^{2,\delta}(y) - \tau^{1,\delta}(y)| \le \frac{\delta}{\epsilon^4} \frac{8L^2}{3!} w_+ t^3, \quad \textit{for} \quad y \le -Lt - 2\epsilon,$$

where $|Q'(\tau)| \le L$*.*

Proof. The final estimate follows from the first one by setting $j = 2$ and replacing the sum on the left by a single term.

Our goal is to find a bound for $|\tau^1(k\delta, t) - \tau^2(k\delta, t)|$ which gives, as δ becomes small, that $|\tau^1(k\delta, t) - \tau^2(k\delta, t)| \approx 0$ for $k\delta < -Lt$. Consider two sets of initial data $(\bar\tau^1, w)$ and $(\bar\tau^2, w)$ such that $\bar{\tau}^1(y) = \bar{\tau}^2(y)$ for $y \le 0$. We use the notation and result introduced in (26)–(31). The idea is to approximate the integral

$$\int_0^T \int_{-K}^{-Lt} (\tau_k^+ - \tau_k^-)\, dy\, dt.$$

Consider a test function $\psi(y)$ in C^∞ satisfying $\psi' \le 0$, and

$$\psi(y) = \begin{cases} 1, & y \le -1, \\ 0, & y \ge 0. \end{cases}$$

We shall pretend that $\|\psi''\| = 4$. This is not possible, but we can get as close as we wish, which is all that is needed for the estimate. Given an $\epsilon > \delta$. Let $\phi(y) = \psi(y/\epsilon)$ and define the test function

$$\phi_k^j = \phi(\delta k + j\epsilon + Lt).$$

Denote

$$I_j(t) = \delta \sum_{k \le k_{j\epsilon}} (\tau_k^+ - \tau_k^-) \quad \text{and} \quad J_j(t) = \delta \sum_k (\tau_k^+ - \tau_k^-)\phi_k^j.$$

Notice that

$$J_{j-1}(t) \ge I_j(t) \ge J_j(t). \tag{32}$$

We first find an upper bound for $I_0(t)$. Since Q is decreasing with respect to τ and $\|Q\|_\infty \le w_+$ we get

$$\begin{aligned}
\delta \sum_{k \le 0} (\dot{\tau}_k^+ - \dot{\tau}_k^-) &= \sum_{k \le 0} \big(-Q(\tau_{k+1}^+) + Q(\tau_k^+) + Q(\tau_{k+1}^-) - Q(\tau_k^-) \big) \\
&= -Q(\tau_1^+) + Q(\tau_{-K}^+) + Q(\tau_1^-) - Q(\tau_{-K}^-) \\
&\le w_+.
\end{aligned}$$

Thus, by integration we obtain an upper bound for $I_0(t)$,

$$I_0(t) \le \delta \sum_{k \le 0} \big(\tau_k^+ - \tau_k^-\big) \le w_+ t.$$

Next we want to find an estimate for $I_{j+1}(t)$. For simplicity, in the calculation below we write ϕ_k for ϕ_k^j. Consider

$$\begin{aligned}
\frac{d}{dt} J_j(t) &= \frac{d}{dt} \delta \sum_k (\tau_k^+ - \tau_k^-)\, \phi_k \\
&= \delta \sum_k \big[(\dot{\tau}_k^+ - \dot{\tau}_k^-)\phi_k + (\tau_k^+ - \tau_k^-)\dot{\phi}_k \big] \\
&= \sum_k \big[\big(-Q(\tau_{k+1}^+) + Q(\tau_{k+1}^-) + Q(\tau_k^+) - Q(\tau_k^-) \big)\phi_k + \delta(\tau_k^+ - \tau_k^-)\dot{\phi}_k \big] \\
&= \sum_k \big[(Q(\tau_k^+) - Q(\tau_k^-))(\phi_k - \phi_{k-1}) + \delta(\tau_k^+ - \tau_k^-)\dot{\phi}_k \big] \\
&\le \sum_k (\tau_k^+ - \tau_k^-) \big[L(\phi_{k-1} - \phi_k) + \delta\dot{\phi}_k \big]
\end{aligned}$$

The last equality follows by summation by parts and the inequality is a consequence of the Lipschitz continuity of $Q(\tau)$. Note that Q is decreasing with respect to τ. The mean value theorem implies that $\phi_{k-1} - \phi_k = -\delta\phi'_{k-\theta}$ for some value $\theta \in (0,1)$. Further, $\dot{\phi} = L\phi'$. Hence

$$\begin{aligned}\frac{d}{dt}J_j(t) &= \delta L \sum_k \left(\tau_k^+ - \tau_k^-\right)\left[(-\phi'_{k-\theta}) + \phi'_k\right] \\ &\leq \delta^2 L \sum_k (\tau_k^+ - \tau_k^-)\|\phi''\| \leq \delta L \|\phi''\|_\infty I_j(t) = 4\delta\epsilon^{-2} L I_j(t)\end{aligned}$$

where we once more have used the mean value theorem. Thus, integration with respect to time yields

$$J_j(t) \leq 4L\frac{\delta}{\epsilon^2}\int_0^t I_j(t')\,dt',$$

since $J_j(0) = 0$. Inequality (32) combined with the bound of $J_j(t)$ in the previous inequality yields,

$$I_1(t) \leq J_0(t) \leq \frac{2L\delta}{\epsilon^2} w_+ t^2$$

and

$$I_{j+1}(t) \leq J_j(t) \leq 4L\frac{\delta}{\epsilon^2}\int_0^t I_j(t')\,dt'.$$

Thus, it follows by induction that

$$I_j(t) \leq \frac{(4L)^j}{2(j+1)!}\frac{\delta^j}{\epsilon^{2j}} w_+ t^{j+1}.$$

Next we extend the result to arbitrary initial data $(\bar\tau^1, w)$ and $(\bar\tau^2, w)$ satisfying $\bar\tau^1 = \bar\tau^2$ for $y \leq 0$. By inequality (30) we have $\tau^- \leq \tau^1, \tau^2 \leq \tau^+$. Thus

$$\delta \sum_{k \leq k_{j\epsilon}} |\tau_k^2 - \tau_k^1| \leq \delta \sum_{k \leq k_{j\epsilon}} \left(\tau_k^+ - \tau_k^-\right) \leq \frac{\delta^j}{\epsilon^{2j}}\frac{(4L)^j}{2(j+1)!} w_+\, t^{j+1}.$$

□

Next we show stability in the L^1-norm of $S_t^{\delta,w}\bar\tau$ with respect to the initial data.

Lemma 9.

$$\begin{aligned}\|S_t^{\delta,w^{1,\delta}}\bar\tau^{1,\delta} - S_t^{\delta,w^{2,\delta}}\bar\tau^{2,\delta}\| &\leq \|\bar\tau^{1,\delta} - \bar\tau^{2,\delta}\| \\ &\quad + t\,\mathrm{TV}\left(w^{1,\delta} - w^{2,\delta}\right) + t\|Q'\|_\infty |\bar\tau^{1,\delta}(Y) - \bar\tau^{2,\delta}(Y)| \qquad (33)\end{aligned}$$

Proof. We make use of the notation and the results given in (26)–(31). Thus,

$$\begin{aligned}&\frac{d}{dt}\|S_t^{\delta,\omega^+}\left(\bar\tau^1 \vee \bar\tau^2\right) - S_t^{\delta,\omega^-}\left(\bar\tau^1 \wedge \bar\tau^2\right)\| \\ &\qquad = \delta \sum_k \frac{d}{dt}\left(S_t^{\delta,\omega^+}(\bar\tau_k^1 \vee \bar\tau_k^2) - S_t^{\delta,\omega^-}(\bar\tau_k^1 \wedge \bar\tau_k^2)\right).\end{aligned}$$

By using equation (27) on the right hand side we get

$$\begin{aligned}&\frac{d}{dt}\|S_t^{\delta,\omega^+}\left(\bar\tau^1 \vee \bar\tau^2\right) - S_t^{\delta,\omega^-}\left(\bar\tau^1 \wedge \bar\tau^2\right)\| \\ &\quad = \sum_k \left(\omega_k^+ - \omega_k^-\right) + \sum_k \left[-Q(\tau_{k+1}^+) + Q(\tau_{k+1}^-) + Q(\tau_k^+) - Q(\tau_k^-)\right] \\ &\quad = \sum_k \left(\omega_k^+ - \omega_k^-\right) - Q(\tau_{K+1}^+) + Q(\tau_{K+1}^-) + Q(\tau_{-K}^+) - Q(\tau_{-K}^-) \\ &\quad \leq \mathrm{TV}\left(w^1 - w^2\right) + \|Q'\|_\infty |\bar\tau^1(Y) - \bar\tau^2(Y)|\end{aligned}$$

The inequality follows by $\tau_K^- = \bar\tau_K^-$ and $\tau_K^+ = \bar\tau_K^+$, the result for the total variation given in (31) and $Q(\tau)$ being decreasing with respect to τ. Integration and inequality (30) yields

$$\|S_t^{\delta,w^1}\bar\tau^1 - S_t^{\delta,w^2}\bar\tau^2\| \le \|\bar\tau^1 - \bar\tau^2\| \\ + t\,\mathrm{TV}\left(w^1 - w^2\right) + t\|Q'\|_\infty|\bar\tau^1(Y) - \bar\tau^2(Y)|.$$

□

We now find an upper bound for the solution τ_k. It will be used later to find necessary conditions for the appearance of a vacuum in the solution.

Lemma 10. *Given some $y < 0$. Assume $\bar\tau < M$ on $[y, 0]$ and $w_k - w_{k-1} < \delta L_w$ for $y \le k\delta < 0$. Then,*

$$\tau^\delta(y) \le M - \frac{t}{y}w_+ + L_w t. \tag{34}$$

Proof. Fix an $y < 0$ such that $y \le -\delta$ and let k_y be given such that $y \in (k_y, k_y + 1]\delta$. First, assume the initial data $\bar\tau_k$ and the difference $w_{k+1} - w_k$ are monotone for all $k = k_y, \ldots, -2$, i.e. they satisfy (25). Summing from $k = k_y$ to $k = -2$ in equation (24) yields

$$\delta \sum_{k=k_y}^{-2} \dot\tau_k = w_{-1} - w_{k_y} - Q(\tau_{-1}) + Q(\tau_{k_y}) \le w_+ + L_w(-y),$$

since $Q(\tau_{k_y}) \le w_{k_y} \le w_+$. By integration we get

$$\delta \sum_{k=k_y}^{-2} \tau_k \le \delta \sum_{k=k_y}^{-2} \bar\tau_k + tw_+ + tL_w(-y) \le (-y)M + tw_+ + tL_w(-y),$$

and the monotonicity property in Lemma 7 implies

$$(-y)\tau_{k_y} \le (1 - k_y)\delta\tau_{k_y} \le \delta \sum_{k=k_y}^{-2} \tau_k \le M(-y) + tw_+ + tL_w(-y).$$

Next, consider initial data which is not monotone on $[y, 0]$, and hence they do not satisfy (25). However, we define a constant function $\tilde\tau = M$ such that $\tau_k \le M$ and a function $\tilde w$ so that $\tilde w_{k+1} - \tilde w_k = L_w\delta$ for all $k = k_y, \ldots, -2$. Thus, the estimate in (34) holds for $S_t^{\tilde w,\delta}\tilde\tau$. Since the solution is monotone with respect to the initial data as given in Lemma 6, we get $S_t^{w,\delta}\bar\tau \le S_t^{\tilde w,\delta}\tilde\tau$, which concludes the proof. □

3. Proof of Theorem 1

We want to prove convergence of u^δ to a weak entropy solution of the macroscopic system. By assumption, the initial data $(\bar\tau, \bar w)$ is in $\mathrm{BV}(\mathbb{R})^2$ and takes values in $\mathcal{D}$. Further, the piecewise constant initial data $\bar\tau^\delta$ and w^δ are constructed such that $\bar\tau^\delta \to \bar\tau$ in $L^1_{\mathrm{loc}}(\mathbb{R})$ and $\bar w^\delta \to \bar w$ in $L^1_{\mathrm{loc}}(\mathbb{R})$ as $\delta \to \mathcal{U}$.

First we show convergence of the sequence $u^\delta(t)$ to some limit $u(t)$ in $L^1_{\mathrm{loc}}(\mathbb{R})^2$. By Lemma 1 the domain $\mathcal{D}$ given by (3) is an invariant region in the sense that if the initial data $\bar u^\delta(y)$ lies in $\mathcal{D}$, then so does the solution u^δ. Since the domain $\mathcal{D}$ is bounded, the solution $(\tau^\delta(t), w^\delta(t))$ is bounded in $L^\infty(\mathbb{R})^2$. By Lemma 2, the total variation of w^δ and v^δ is bounded for finite times, which implies boundedness of the total variation of $Q(\tau^\delta)$. Since $0 < \tau_{\min} \le \tau^\delta \le \tau_{\max} < \infty$ and $Q'(\tau) < 0$, the function $(Q^{-1})'$ is bounded. Thus, we obtain a uniform bound on the total variation of $u^\delta(t)$,

$$\mathrm{TV}\left(\tau^\delta(t), w^\delta(t)\right) \le C\ \mathrm{TV}\left(\bar\tau^\delta, \bar w^\delta\right), \tag{35}$$

where $C = e^{2LT}(2\|1/Q'\|_\infty + 1)$. For any time t in $[0,T]$ the sequence $u^\delta(t)$ is bounded in $L^\infty(\mathbb{R})^2$ and has uniformly bounded total variation. Thus the closure of the set $\{u^\delta\}$ is compact in $L^1(\Omega)^2$, and $u^\delta(y,t)$ converges to some limit $u(y,t)$ in $L^1(\Omega\times[0,T])^2$ as $\delta\to\mathcal{U}$. Also, by (35) the limit $u(t)$ is in $\mathrm{BV}(\mathbb{R})^2$ and by (19)–(20) it is Lipschitz in time.

Our next goal is to show that the limit $u(y,t)$ is a weak solution of the macroscopic system (2). Multiply system (1) by δ and a test function $\phi(y_k,t)$ in $C_0^\infty(\mathbb{R}\times\mathbb{R})$, integrate over time and sum over k. We obtain

$$
\begin{aligned}
0 &= \int_{\mathbb{R}^+}\sum_k \left(-\delta\tau_t^\delta(y_k,t) + v^\delta(y_{k+1},t) - v^\delta(y_k,t)\right)\,\phi(y_k,t)\,dt\\
&= \int_{\mathbb{R}^+}\delta\sum_k \tau^\delta(y_k,t)\phi_t(y_k,t)\,dt + \delta\sum_k \bar\tau^\delta(y_k)\phi_t(y_k,0)\\
&\quad-\int_{\mathbb{R}^+}\sum_k v^\delta(y_k,t)\big(\phi(y_k,t)-\phi(y_{k-1},t)\big)\,dt,
\end{aligned}
$$

using integration and summation by parts. Now $v^\delta(y,t) = v^\delta(y_k,t)$ for $y\in(y_{k-1},y_k]$, so

$$
v^\delta(y_k,t)\big(\phi(y_k,t)-\phi(y_{k-1},t)\big) = \int_{y_{k-1}}^{y_k} v^\delta(y,t)\phi_y(y,t)\,dt.
$$

Defining ϕ^δ so that $\phi^\delta(y,t)=\phi(y_k,t)$ when $y\in(y_k,y_{k+1}]$, the first two terms are similarly rewritten, with the final result

$$
\int_{\mathbb{R}^+}\int_{\mathbb{R}}\tau^\delta(y,t)\phi_t^\delta(y,t)\,dy\,dt + \int_{\mathbb{R}}\bar\tau^\delta(y)\,\phi^\delta(y,0)\,dy
- \int_{\mathbb{R}^+}\int_{\mathbb{R}} v^\delta(y,t)\phi_y(y,t)\,dy\,dt = 0. \tag{36}
$$

As $\delta\to\mathcal{U}$, $\phi^\delta\to\phi$ and $\phi_t^\delta\to\phi_t$ uniformly, and the other terms converge in L^1. Thus

$$
\int_{\mathbb{R}}\int_{\mathbb{R}^+}\left(\tau\phi_t - v\phi_y\right)\,dy\,dt + \int_{\mathbb{R}}\bar\tau(y)\,\phi(y,0)\,dy = 0.
$$

A similar, somewhat simpler calculation leads from

$$
\int_{\mathbb{R}^+}\delta\sum_k\left(w^\delta(y_k,t) - R(\tau^\delta(y_k,t),w^\delta(y_k,t))\right)\,\phi(y_k,t)\,dy\,dt = 0
$$

to

$$
\int_{\mathbb{R}}\int_{\mathbb{R}^+}\left(w\phi_t + R\phi\right)\,dy\,dt + \int_{\mathbb{R}}\bar w(y)\,\phi(y,0)\,dy = 0,
$$

using that $R(\tau,w)$ is Lipschitz in its arguments. Hence the limit is a weak solution of system (2) with initial data $(\bar\tau,\bar w)$.

It remains to prove that this weak solution (τ,w) of (2) satisfies an entropy condition. We do this by showing that the discrete entropy inequality (23) implies an entropy inequality for macroscopic system as $\delta\to 0$. Multiply the discrete entropy inequality (23) by δ and a non-negative test function $\phi(y_k,t)$ in $C_0^\infty(\mathbb{R}\times\mathbb{R}^+)$, sum over k and integrate in time. Then

$$
\begin{aligned}
0 &\le \int_{\mathbb{R}^+}\sum_k\big(-\delta\eta(u_k)_t - q(u_{k+1}) + q(u_k) + \delta\nabla_u\eta(u_k)r(u_k)\big)\,\phi(y_k,t)\,dt\\
&= \int_{\mathbb{R}^+}\Big(\delta\sum_k\eta(u_k)\phi_t(y_k,t) + \delta\sum_k\eta(\bar u_k)\phi(y_k,0)\Big)\,dt\\
&\quad+\int_{\mathbb{R}^+}\sum_k q(u_k)\big(\phi(y_k)-\phi(y_{k-1})\big)\,dt + \int_{\mathbb{R}^+}\delta\sum_k\nabla_u\eta(u_k)r(u_k)\,\phi(y_k,t)\,dt.
\end{aligned}
$$

by integration and summation by parts. We make use of the same technique as previously to arrive at

$$\int_{\mathbb{R}^+}\int_{\mathbb{R}} \eta(u^\delta)\phi_t^\delta(y,t)\,dy\,dt + \int_{\mathbb{R}} \eta(\bar{u}^\delta)\phi^\delta(y,0)\,dy + \int_{\mathbb{R}^+}\int_{\mathbb{R}} q(u^\delta)\phi_y(y)\,dy\,dt + \int_{\mathbb{R}^+}\int_{\mathbb{R}} \nabla_u\eta(u^\delta)r(u^\delta)\,\phi^\delta(y,t)\,dy\,dt \geq 0. \quad (37)$$

Since the first and second partial derivatives of η and q are uniformly bounded, it follows that $\eta(u^\delta)$, $q(u^\delta)$ and $\nabla_u\eta(u^\delta)$ converges to $\eta(u)$, $q(u)$ and $\nabla_u\eta(u)$ in L^1_{loc}. Further, the function $r(u^\delta)$ is Lipschitz in its argument. Thus, as $\delta \to \mathcal{U}$, we get

$$\int_{\mathbb{R}^+}\int_{\mathbb{R}} \big(\eta(u)\phi_t + q(u)\,\phi_x\big)\,dy\,dt + \int_{\mathbb{R}} \eta^\delta(u_0)\phi(x,0)\,dy \qquad \int_{\mathbb{R}^+}\int_{\mathbb{R}} \nabla_u\eta(u)r(u)\,dy\,dt \geq 0, \quad (38)$$

and the limit $u(y,t)$ is a weak entropy solution of the macroscopic system (2).

4. Proof of Theorem 2

In this section we assume the macroscopic system is a system of hyperbolic conservation laws, i.e we assume $R(\tau,w)=0$. The goal is to show that the solution $\tau(y)$ is stable in L^1 with respect to the initial data. Consider two different set of initial data, $(\bar{\tau}^1,w^1)$ and $(\bar{\tau}^2,w^2)$, in $\mathrm{BV}(\mathbb{R})^2$ taking values in $\mathcal{D}$. The initial data is constant for $y \leq -Y$ and $y \geq Y$. Further, for $i=1,2$, the piecewise constant initial data $\bar{\tau}^{i,\delta}$ and $w^{i,\delta}$ are constructed such that $\bar{\tau}^{i,\delta} \to \bar{\tau}^i$ in $L^1_{\mathrm{loc}}(\mathbb{R})$, $\tau_K^{i,\delta} \to \bar{\tau}^i(Y)$ pointwise and $w^{i,\delta} \to w^i$ in $L^1_{\mathrm{loc}}(\mathbb{R})$ as $\delta \to \mathcal{U}$.

Consider the discrete L^1-stability estimate given by inequality (33) in Lemma 9. By construction, $\mathrm{TV}(w^{1,\delta}-w^{2,\delta}) \leq \mathrm{TV}(w^1-w^2)$. The functions $\bar{\tau}^{1,\delta}, \bar{\tau}^{2,\delta}, S_t^{w^{1,\delta}}\bar{\tau}^{1,\delta}$ and $S_t^{w^{2,\delta}}\bar{\tau}^{2,\delta}$ converge in $L^1(\Omega)$. Let $\delta \to \mathcal{U}$. We obtain

$$\|S_t^{w^1}\bar{\tau}^1 - S_t^{w^2}\bar{\tau}^2\| \leq \|\bar{\tau}^1-\bar{\tau}^2\| + t\,\mathrm{TV}\left(w^1-w^2\right) + t\|Q'\|_\infty|\bar{\tau}^1(Y)-\bar{\tau}^2(Y)|. \quad (39)$$

for $\bar{\tau}^1, \bar{\tau}^2, w^1$ and w^2 in $L^1_{\mathrm{loc}}(\mathbb{R}) \cap \mathrm{BV}(\mathbb{R})$ taking values in $\mathcal{D}$. By the same argument, the discrete estimate for Lipschitz continuity in time given in (19) also holds for the limit. However, since $R=0$, the constant L_R also equals zero. Thus, the proofs and estimates in Lemma 2 and Lemma 3 simplify, and we get $\mathrm{TV}(v) \leq \mathrm{TV}(\bar{v})$ and

$$\|S_t^w\bar{\tau} - S_s^w\bar{\tau}\| \leq \mathrm{TV}(\bar{v})|t-s|.$$

Finally, combining this with (39) by the triangle inequality yields

$$\begin{aligned}\|S_t^{w^1}\bar{\tau}^1 - S_s^{w^2}\bar{\tau}^2\| &\leq \|S_{t\wedge s}^{w^1}\bar{\tau}^1 - S_{t\wedge s}^{w^2}\bar{\tau}^2\| + \|S_{t\wedge s}^{w^2}\bar{\tau}^2 - S_{t\vee s}^{w^2}\bar{\tau}^2\| \\ &\leq \|\bar{\tau}^1-\bar{\tau}^2\| + (t\wedge s)\,\mathrm{TV}(w^1-w^2) \\ &\quad + (t\wedge s)\|Q'\|_\infty|\bar{\tau}^1(Y)-\bar{\tau}^2(Y)| + C|t-s|,\end{aligned}$$

where $C = \mathrm{TV}(\bar{v}^1) \vee \mathrm{TV}(\bar{v}^2)$, which concludes the proof of Theorem 2.

5. Proof of Theorem 3

We make use of the same notation and assumptions as given in the two previous sections. However, we allow vacuum to occur and thus we replace the domain $\mathcal{D}$ by the domain $\mathcal{D}_V$ given by (4). Further, since vacuum is included, the initial $\hat{\bar{\tau}}$ and the solution $\hat{\tau}$ are locally finite measures. For simplicity we will denote the initial data as $\bar{\tau}$.

When vacuum is included in the solution, τ is not bounded in L^∞. Therefore, in addition to τ we will also consider the Eulerian coordinate $x = x(y,t)$ given by

$$\tau = \frac{\partial x}{\partial y}, \qquad v = \frac{\partial x}{\partial t} \quad \text{a.e.}$$

By integration we get

$$x(y,t) = x(Y,t) - \int_y^Y \tau(y',t)\,dy'. \tag{40}$$

The semi-discrete $x_k^\delta(t) = x^\delta(k\delta,t)$ is given by equation (15). By rewriting and recursion we obtain the definition of x^δ,

$$x_k^\delta = x_{k+1}^\delta + \delta\tau_k^\delta = x_K^\delta - \delta\sum_{k'=k}^{K-1}\tau_{k'}^\delta \tag{41}$$

and for $k\delta < y \le (k+1)\delta$,

$$x^\delta(y) = x^\delta(y+\delta) + \delta\tau^\delta(y) = x^\delta(Y) - \int_y^{Y-\delta}\tau^\delta(y')\,dy'. \tag{42}$$

Clearly, by considering the above formulas it follows that $x^\delta(y,t)$ is a continuous, piecewise linear, strictly increasing and bounded function. Further, $\tau^\delta(y)$ is in $L^1_{\rm loc}(\mathbb{R})$ and by (40) so is $x^\delta(y)$. Moreover, $x^\delta(t)$ is uniformly Lipschitz continuous with respect to t. This follows by integrating the second equation in (15) with respect to time,

$$|x^\delta(y,t) - x^\delta(y,s)| = |\int_s^t v^\delta(y,\xi)\,d\xi| \le v_+\,|t-s|.$$

On the other hand, $x^\delta(y)$ is strictly increasing and differentiable a.e. with respect to y, and $\tau^\delta(y)$ is piecewise constant and measurable.

Fix y and t and consider the sequence $x^\delta(y,t)$. Since the closure of $\{x^\delta\}$ is compact, it follows that $x^\delta(t) \to x(t)$ pointwise as $\delta \to \mathcal{U}$. Further, since $x^\delta(y)$ is uniformly bounded in L^∞ and has uniformly bounded total variation, Helly's theorem yields convergence of x^δ to x in $L^1_{\rm loc}(\mathbb{R})$. By considering the same ultrafilter $\mathcal{U}$, the convergence extends to all y in Ω and all t in $[0,T]$. By a standard argument we have that the derivative of $x^\delta(y)$ converges weakly to the derivative of $x(y)$, that is $\tau^\delta \rightharpoonup \hat{\tau}$. Our goal is next to show that this limit $\hat{\tau}$ is a weak solution of system (2). By Lemma 1 and 2 it is clear that v^δ and w^δ are uniformly bounded in L^∞ and have uniformly bounded total variation, even when vacuum is included. Thus, by Helly's theorem, $v^\delta \to v$, and $w^\delta \to w$ in $L^1_{\rm loc}(\mathbb{R})$ for all times t in $[0,T]$ as $\delta \to \mathcal{U}$. This implies that also the function $Q(\tau^\delta)$ converges to the limit $\chi = w - v$ in $L^1_{\rm loc}(\mathbb{R}\times\mathbb{R}^+)$. From section 3 we know that the discrete function τ^δ satisfies (36), that is

$$\int_{\mathbb{R}^+}\int_{\mathbb{R}} \tau^\delta(y,t)\phi_t^\delta(y,t)\,dy\,dt + \int_{\mathbb{R}} \bar{\tau}(y)\,\phi^\delta(y,0)\,dy - \int_{\mathbb{R}^+}\int_{\mathbb{R}} v^\delta(y,t)\phi_y(y,t)\,dy\,dt = 0.$$

Assume $\bar{\tau}^\delta \rightharpoonup \bar{\tau}$ and let $\delta \to \mathcal{U}$ in the above expression. The same argument as in the proof of Theorem 1, except that we now have to consider weak convergence of τ^δ and $\bar{\tau}^\delta$, yields

$$\int_{\mathbb{R}}\int_{\mathbb{R}^+} \hat{\tau}(y)\phi_t(y,t)\,dy\,dt + \int_{\mathbb{R}} \bar{\tau}(y)\,\phi(y)\,dy - \int_{\mathbb{R}}\int_{\mathbb{R}^+} v(y)\phi_y(y,t)\,dy\,dt = 0. \tag{43}$$

It remains to show that the limit χ equals $Q(\tau)$. In order to handle this we will have to put some restrictions on the function w, namely that the set V, at which $w(y^-) < w(y^+)$, is discrete. Also we will have to show that, when vacuum is presented in the initial data, we do get weak convergence of $\bar{\tau}^\delta$ to $\bar{\tau}$.

5.1. **The vacuum state.** We will now discuss some features related to the vacuum state and derive some properties of the hyperbolic system.

In Lagrangian coordinates a vacuum state in the solution of the macroscopic system is concentrated at a single point y. In order to see this, consider the Riemann problem. In a solution that contains vacuum, from the left a state u_L will be connected to vacuum by a rarefaction and to the right vacuum will be connected to a right state u_R by a contact discontinuity. The eigenvalues coincide for $\tau = \infty$, and thus $\lambda_1 = \lambda_2 = 0$.

We now assume that the initial data $(\bar{\tau}, w)$ is in Γ_V as defined in (12). The piecewise constant data $w^\delta(y)$ is constructed such that $w^\delta(y)$ converges to $w(y)$ pointwise and in $L^1_{\text{loc}}(\mathbb{R})$ as $\delta \to \mathcal{U}$. Since the set V of points at which $w(y^-) < w(y^+)$ is countable and discrete it follows that, if $[y_k, y_{k+1}] \cap V = \emptyset$, then $w(y_k) - w(y_{k-1}) \leq L_w \delta$ for some positive constant L_w. Then, given the assumptions on the initial data specified above, we will now show that the possible locations of vacuum in the solution are exactly the points where $w(y^-) < w(y^+)$, or where vacuum occurs initially. This will enable us to control the positions of vacuum.

Lemma 11. *Vacuum can not occur outside V.*

Proof. Consider a value $y \notin V$. Since V is discrete there exist an interval (a, b) such that $(a, b) \cap V = \emptyset$ and y in in (a, b). Further, by assumptions, we have $w(b) - w(a) \leq L_w(b - a)$ and thus Lemma 10 yields boundedness of $\tau(y)$ as $\delta \to 0$. On the other hand, if $y \in V$, that is $w(y^-) < w(y^+)$, Lemma 10 implies that $\tau(y, t)$ may go to infinity as $\delta \to 0$. $\square$

5.2. **Vacuum in the initial data.** If $\bar{\tau}(y) = \infty$ at some values y in Ω, it is not possible to find a piecewise continuous function $\bar{\tau}^\delta(y)$ that converges to $\bar{\tau}(y)$ in $L^1_{\text{loc}}(\mathbb{R})$. Therefore, we will now consider the function $\bar{x}(y)$. Since $\bar{x}(y)$ given as in equation (40) is strictly increasing, $\bar{x}(y)$ is differentiable a.e. and $\frac{\partial \bar{x}}{\partial y} = \bar{\tau}(y)$ is measurable. In order to discretize $\bar{x}(y)$, we will need the following general result.

Lemma 12. *Let $\Omega \subset \mathbb{R}$ be compact and $x : \Omega \to \mathbb{R}$ be strictly increasing and bounded with discontinuities located at values in a set V. Then the set V is countable and there exists a piecewise linear strictly increasing function x^δ such that $x^\delta(y)$ converges pointwise to $x(y)$ on $\Omega \setminus V$. Further, the convergence is uniform on $\Omega \setminus U$, where $U \supset V$ is open.*

Proof. Given $\delta > 0$ and a constant K such that $|\Omega| = K\delta$. Let $y_k = k\delta$, $x^\delta(y_k) = x(y_k^-)$ for $k = 0, 1, \ldots, K$ and interpolate linearly. Thus, x^δ converges pointwise to x on $\Omega \setminus V$ as $\delta \to 0$.

Let $\epsilon > 0$ and let U denote an open set $U \supset V$. Further, $\Omega \setminus U \subseteq \cup_{i=1}^n (a_i, b_i)$ such that $x(b_i) - x(a_i) < \epsilon$. Since x^δ converges pointwise to x on $\Omega \setminus U$, we have, for some $\delta > 0$,

$$|x^\delta(a_i) - x(a_i)| < \epsilon, \quad |x^\delta(b_i) - x(b_i)| < \epsilon.$$

If $a_i < y < b_i$, the monotonicity of x^δ yields

$$x(a_i) - \epsilon < x^\delta(a_i) < x^\delta(y) < x^\delta(b_i) < x(b_i) + \epsilon,$$

and $x(a_i) < x(y) < x(b_i)$. Finally, by combining the above inequalities, we obtain

$$x^\delta(y) - x(y) < x(b_i) + \epsilon - x(a_i) < 2\epsilon,$$

which implies uniform convergence of x^δ on $\Omega \setminus U$.

Since the function $x(y)$ is increasing, the set at which it is discontinuous is countable. Thus the set V is countable. $\square$

The initial $\bar{x}$ is given by the initial $\bar{\tau}$. Approximate $\bar{x}$ by a piecewise linear function $\bar{x}^\delta$ satisfying $\bar{x}^\delta(y_k) = \bar{x}(y_k)$ where $y_k = k\delta$ for $k = -K, \dots, K$. By the proof of the above lemma, $\bar{x}^\delta$ converges to $\bar{x}$ pointwise a.e. Further, $\bar{x}^\delta$ is uniformly bounded on Ω and $\bar{x}^\delta$ is in $L^1(\Omega)$. Now we want to show convergence of $\bar{x}^\delta$ in $L^1(\Omega)$. The uniform convergence yields convergence in $L^1(\Omega \setminus U)$. Further,

$$\int_U |\bar{x}^\delta(\xi) - \bar{x}(\xi)|\, d\xi \le 2\|\bar{x}\|_\infty |U|, \tag{44}$$

and the above estimate can be made small since x and x^δ are bounded and $|U|$ is arbitrary small. Thus we get $\|\bar{x}^\delta - \bar{x}\| \to 0$ as $\delta \to 0$. Consider $\bar{\tau}^\delta$. The above convergence result implies $\bar{x}^\delta \rightharpoonup \bar{x}$, and we get $\bar{\tau}^\delta \rightharpoonup \bar{\tau}$, which is the result we need for the discretization of the initial data. Note that including vacuum in the initial data does not put any restrictions on w or the set V.

5.3. **Existence of a weak solution.** We will now use the properties of x^δ discussed above to complete the proof of existence of a weak solution. It remains to show that $Q(\tau^\delta) \rightharpoonup Q(\tau)$. First, we show convergence of $x^\delta(y,t)$ in $L^1_{\text{loc}}(\mathbb{R} \times \mathbb{R}^+)$.

Lemma 13. *As $\delta \to \mathcal{U}$, $x^\delta(y,t) \to x(y,t)$ uniformly on $(\Omega \setminus U) \times [0,T]$, where $U \supset V$ is a neighborhood of V. Further, the convergence is in $L^1_{loc}(\mathbb{R} \times \mathbb{R}^+)$, and the limit $x(y,t)$ is a monotone and piecewise continuous function of y.*

Proof. First, fix y and t and consider the sequence $x^\delta(y,t)$. The closure of $\{x^\delta\}$ is compact, and hence $x^\delta(t) \to x(t)$ pointwise as $\delta \to \mathcal{U}$. By considering the same ultrafilter $\mathcal{U}$ the pointwise convergence extends to all y in Ω and all t in $[0,T]$. Since $x^\delta(y,t)$ is monotone the same is true for the limit $x(y,t)$.

Further, the set V is countable and discrete. Let $V \subset U$, where U is an open set. By Lemma 10 $\tau^\delta(y)$ is uniformly bounded on $\Omega \setminus U$. Then, x^δ is uniformly Lipschitz, that is $x^\delta(y+\epsilon) - x^\delta(y) \le \text{const.} \cdot \epsilon$, on $[y, y+\epsilon] \subset \Omega \setminus U$. The same is true for the limit. Since x^δ is increasing, uniformly Lipschitz and $x^\delta \to x$ pointwise on $\Omega \setminus U$, it follows, by an argument similar to the one given in the proof of Lemma 12, that $x^\delta \to x$ uniformly on $\Omega \setminus U$. Hence the limit $x(y,t)$ is continuous.

It remains to show convergence in the L^1-norm. The uniform convergence on $\Omega \setminus U$ yields converges of x^δ to x in $L^1_{\text{loc}}(\Omega \setminus U)$. The sequence $x^\delta(y)$ is in $L^1_{\text{loc}}(\Omega)$ and uniformly bounded. Thus, for a fixed value of t the convergence of $x^\delta(y)$ to $x(y)$ in $L^1_{\text{loc}}(\Omega)$ follows by an argument similar to the one given in (44), which yields convergence in $L^1_{\text{loc}}(\Omega)$ of $\bar{x}^\delta$. Further, the convergence extends to all times t in $[0,T]$ by considering the same ultrafilter $\mathcal{U}$. $\square$

Since $x(y)$ is monotone and piecewise continuous, it is differentiable a.e. on $\mathbb{R}$ and $\hat{\tau}$ is a positive Radon measure consisting of an absolutely continuous and a singular part,

$$d\hat{\tau} = \tau\, dy + d\acute{\tau}.$$

For our specific choice of V, that is V is a discrete set, the singular part of the measure is discrete. We write the measure as

$$d\hat{\tau} = \tau\, dy + h\, d\Delta. \tag{45}$$

The absolutely continuous part of the measure $d\hat{\tau}$ is $d\tau = \tau\, dy$, where τ is in $L^1_{\text{loc}}(\mathbb{R} \setminus V \times \mathbb{R}^+)$. The discrete part of the measure is

$$h\, d\Delta = \sum_{y \in V} h(y,t)\, d\delta_y. \tag{46}$$

Notice that $\Delta(\Omega \setminus V) = 0$ and $\tau(V) = 0$.

In order to find $h(y,t)$ we assume for simplicity that $V = \{0\}$. By inserting the expression $\hat{\tau}(y,t) = h(y,t)\delta(y)$ into the weak formulation of (2) and integrating across $y = 0$ we find $\dot{h}(0,t) =$

$v(0^+,t) - v(0^-,t)$. By assumption the set V is discrete. Extending the above result to the given set V yields

$$\dot{h}(y,t) = v(y^+,t) - v(y^-,t). \tag{47}$$

Further, since,

$$\hat{\tau}[y_1,y_2] = \int_{y_1}^{y_2} \tau\,dy = \int_{y_1}^{y_2} \frac{\partial x}{\partial y}\,dy = x(y_2) - x(y_1), \tag{48}$$

the limit $\hat{\tau}(y,t)$ is a finite Radon measure on compact subsets of $\mathbb{R}\times\mathbb{R}^+$.

Next we consider the convergence of τ^δ in a distributional sense. Since $x^\delta \rightharpoonup x$ on $\Omega\times[0,T]$ we have $\tau^\delta \rightharpoonup \hat{\tau}$ on $\Omega\times[0,T]$ as well. Consider the domain $\mathbb{R}\setminus V$. The bounded monotone function $x(y)$ is continuous on $\mathbb{R}\setminus V$, and thus it is differentiable a.e. Since $x^\delta(y)$ converges pointwise to $x(y)$ on $\mathbb{R}\setminus V$ and uniformly on $\mathbb{R}\setminus U$, we have $\tau^\delta = \frac{\partial x^\delta}{\partial y} \to \frac{\partial x}{\partial y} = \tau$ pointwise a.e. and uniformly on $\mathbb{R}\setminus U$. Further, τ is in $L^1_{\mathrm{loc}}(\mathbb{R}\setminus V)$ and by an argument similar to the one given in (44) we get $\tau^\delta \to \tau$ strongly in $L^1_{\mathrm{loc}}((\mathbb{R}\setminus V)\times\mathbb{R}^+)$ as $\delta\to\mathcal{U}$. The convergence extends to all times t in $[0,T]$.

Since Q is Lipschitz, it follows that $Q(\tau^\delta)\to Q(\tau)$ in $L^1_{\mathrm{loc}}((\mathbb{R}\setminus V)\times\mathbb{R}^+)$ as well. But then this convergence holds in $L^1_{\mathrm{loc}}(\mathbb{R}\times\mathbb{R}^+)$ as well, since V has measure zero and $Q(\tau^\delta)$ is uniformly bounded. As noted in the text after (43), this is what remained to prove (the other requirement was taken care of at the end of section 5.2). This completes the proof of Theorem 3.

We conclude this section with an auxiliary result. Assume for simplicity that $V=\{0\}$ and let $\phi(y)\in C_0(\mathbb{R})$. Our next goal is to show that

$$h(0,t) = \lim_{\epsilon\to 0}\lim_{\delta\to 0}\int_{[-\epsilon,0]} \tau^\delta(y',t)\,dy', \tag{49}$$

Let k^δ_ϵ be given such that $k^\delta_\epsilon\delta < -\epsilon \le (k^\delta_\epsilon+1)\delta$ and consider the interval $[-\epsilon,0]$,

$$\Big|\int_{[-\epsilon,0]} \tau^\delta(y',t)\,dy' - h(0,t)\Big| = \\ \Big|\delta\sum_{k=k^\delta_\epsilon}^{-1}\tau_k(t) - \int_0^t \big(v(0^+,t') - v(0^-,t')\big)\,dt' - h(0,0)\Big|.$$

Equation (1) yields $\delta\tau_k = \delta\bar{\tau}_k + \int_0^t (v_{k+1}-v_k)\,dt$. Thus

$$\Big|\int_{[-\epsilon,0]} \tau(y',t)\,dy' - h(0,t)\Big| \\ \le \Big|\delta\sum_{k=k^\delta_\epsilon}^{-1}\bar{\tau}_k - h(0,0)\Big| + \int_0^t \big|v_0(t') - v_{k^\delta_\epsilon}(t') - v(0^+,t') + v(0^-,t')\big|\,dt'.$$

First, let $\delta\to 0$ in the above inequality. By the pointwise convergence of $\tau^\delta(y)$ and $w^\delta(y)$ and the boundedness of Q', we obtain convergence of v_0 and $v_{k^\delta_\epsilon}$ to $v(0^+)$ and $v(-\epsilon)$, respectively. In the limit we get

$$\Big|\int_{[-\epsilon,0]} \tau(y',t)\,dy' - h(0,t)\Big| \le \Big|\int_{[-\epsilon,0]} \bar{\tau}(y')\,dy' - h(0,0)\Big| \\ + \int_0^t \big|v(0^+,t') - v(-\epsilon,t') - v(0^+,t') + v(0^-,t')\big|\,dt'.$$

Next, let $\epsilon\to 0$. The first term on the right hand side goes to 0, since $\bar{\tau}$ is a locally integrable function plus $h(0,0)$ times a delta at $y=0$. Since $v(-\epsilon)\to v(0^-)$, the last term on the right hand side goes to 0 and we have proved (49).

5.4. **Entropy.** Our goal is now to show that the limit $\hat{\tau} = S_t^w \bar{\tau}$ is entropy admissible. By section 3 we know that the absolutely continuous part τ is a weak entropy solution of (2) on $\mathbb{R} \setminus V$, that is

$$\int_{\mathbb{R}\setminus V} \int_{\mathbb{R}^+} \big(\eta(\tau, w)\phi_t + q(\tau, w)\phi_y\big)\, dy\, dt + \int_{\mathbb{R}\setminus V} \eta(\bar{\tau}, w)\phi_t\, dy \geq 0,$$

for all non-negative $\phi(y,t)$ in $C_0^\infty(\mathbb{R} \setminus V \times \mathbb{R}^+)$ and all semiconvex entropy/entropy flux pairs (η, q) given by (22).

Since $g'' < 0$ and g is defined in the range of $w - Q(\tau)$ over $\mathcal{D}_V$, a closed bounded interval, g and g' are bounded. Clearly, $\|q(\tau^\delta, w^\delta)\|$ is uniformly bounded on compact subsets of $\mathbb{R}$. Consider the formula for η given by (22). Since

$$\left.\begin{aligned} \eta &\sim -g'(w)\tau, \\ \eta_\tau &\sim -g'(w), \end{aligned}\right\} \qquad \text{as} \quad \tau \to \infty,$$

we consider, instead of η, a measure $\hat{\eta}$ given by

$$d\hat{\eta} = \eta\, dy + d\acute{\eta} = \eta\, dy - g'(w)\, d\acute{\tau},$$

where $\eta\, dy$ and $d\acute{\eta}$ are the absolutely continuous and singular part, respectively. According to the discussion of the discrete model, we choose w to be continuous from the left, that is $w(y) = w(y^-)$ for all y. Recall that $\acute{\eta}$ is supported on V. When we assume the set V is discrete, the entropy is given as

$$d\hat{\eta} = \eta(\tau, w)\, dy - \sum_{y\in V} g'(w)h(y)\, d\delta_y.$$

By the expression of η given by (22) and the estimate for $\hat{\tau}(\Omega)$ in (48) we get

$$\begin{aligned} \hat{\eta}(\Omega) = \int_{-Y}^{Y} \eta(\tau, w)\, dy = -\int_{-Y}^{Y}\int_{\tau_0}^{\tau} g'(w - Q(\xi))\, d\xi\, dy& \\ \leq \|g'\|_\infty \left(x(Y) - x(-Y) + |\Omega|\tau_0\right),& \end{aligned}$$

and $\hat{\eta}$ is a locally finite measure. From the discussion in section 3 where vacuum is not included we obtained an entropy inequality (37) for the solution of the discrete system,

$$\begin{aligned} \int_{\mathbb{R}^+}\int_{\mathbb{R}} \eta(\tau^\delta, w^\delta)\phi_t^\delta(y,t)\, dy\, dt + \int_{\mathbb{R}} \eta(\bar{\tau}^\delta, w^\delta)\phi^\delta(y,0)\, dy& \\ + \int_{\mathbb{R}^+}\int_{\mathbb{R}} q(\tau^\delta, w^\delta)\phi_y(y)\, dy\, dt \geq 0.& \end{aligned} \tag{50}$$

We have weak convergence of τ^δ and $\bar{\tau}^\delta$ and convergence of v^δ and w^δ in $L^1_{\text{loc}}(\mathbb{R} \times \mathbb{R}^+)$. Since $q(\tau^\delta, w^\delta) = g(v^\delta)$, g' is bounded and $w = v + Q(\hat{\tau})$, we also have convergence of $q(\tau^\delta, w^\delta)$ to $q(\hat{\tau}, w)$ in $L^1_{\text{loc}}(\mathbb{R} \times \mathbb{R})$.

The entropy $\eta^\delta = \eta(\tau^\delta, w^\delta)$ needs to be considered more carefully. Since η^δ is locally bounded in the L^1-norm,

$$\|\eta(\tau^\delta, w^\delta)\| \leq \|g'\|_\infty \|\tau^\delta - \tau_0\| \leq \|g'\|_\infty \left(\|\tau^\delta\| + |\Omega|\tau_0\right),$$

$\eta(\tau^\delta, w^\delta)$ converges weakly to some limit $\tilde{\eta}$ as $\delta \to \mathcal{U}$. We have to show that the limit $\tilde{\eta}$ equals $\hat{\eta}(\hat{\tau}, w)$. For simplicity assume $V = \{0\}$, so vacuum is only possible at $y = 0$. For $\epsilon > 0$ consider the domains $\mathbb{R} \setminus [-\epsilon, 0]$ and $[-\epsilon, 0]$. Thus,

$$\begin{aligned} \lim_{\delta\to 0}\langle \eta(\tau^\delta, w^\delta), \phi\rangle &= \lim_{\epsilon\to 0}\lim_{\delta\to 0}\left(\int_{\mathbb{R}\setminus[-\epsilon,0]} \eta(\tau^\delta, w^\delta)\, \phi\, dy + \int_{[-\epsilon,0]} \eta(\tau^\delta, w^\delta)\, \phi\, dy\right) \\ &= \int_{\mathbb{R}\setminus\{0\}} \eta(\tau, w)\, \phi\, dy + \lim_{\epsilon\to 0}\lim_{\delta\to 0}\int_{[-\epsilon,0]} \eta(\tau^\delta, w^\delta)\, \phi\, dy, \end{aligned} \tag{51}$$

where the final equality is a consequence of $\eta(\tau^\delta, w^\delta) \to \eta(\tau, w)$ in $L^1_{\text{loc}}(\mathbb{R} \setminus [-\epsilon, 0])$. It remains to consider the last term on the right hand side, that is the limit of $\eta(\tau^\delta, q^\delta)$ on the interval $[-\epsilon, 0]$. We do that by comparing the terms $\int_{-\epsilon}^0 \eta(\tau^\delta, w^\delta)\phi\, dy$ and $-\int_{-\epsilon}^0 g'(w^\delta)\tau^\delta\phi\, dy$. By subtracting and taking the absolute value we get

$$\begin{aligned}
&\Big|\int_{-\epsilon}^0 \big(\eta(\tau^\delta, w^\delta) + g'(w^\delta)\tau^\delta\big)\phi\, dy\Big| \\
&\qquad = \Big|\int_{-\epsilon}^0 \left[\int_{\tau_0}^{\tau^\delta} \big(-g'(w^\delta - Q(\xi)) + g'(w^\delta)\big)\, d\xi + \tau_0 g'(w^\delta)\right]\phi\, dy\Big| \\
&\qquad \le \int_{-\epsilon}^0 \int_{\tau_0}^{\tau^\delta} |-g'(w^\delta - Q(\xi)) + g'(w^\delta)|\, d\xi\, \phi\, dy + \epsilon\tau_0 \|g'\|_\infty \|\phi\|_\infty.
\end{aligned}$$

Consider the integral with respect to ξ in the first term on the right hand side in the above inequality. Let ξ_0 be an arbitrary value of τ. For $\xi > \xi_0$ we have $|-g'(w^\delta - Q(\xi)) + g'(w^\delta)| < \epsilon_1$, where $\epsilon_1 > 0$. We split the integral into two terms. For the integral from τ_0 to ξ_0 we get an upper bound by using the Lipschitz continuity of g' and Q and the boundedness of τ_0 and ξ_0. On the other hand, the integral from ξ_0 to τ^δ is bounded by $\epsilon_1\tau^\delta$. Thus,

$$\begin{aligned}
&\Big|\int_{-\epsilon}^0 \big(\eta(\tau^\delta, w^\delta) + g'(w^\delta)\tau^\delta\big)\phi\, dy\Big| \\
&\qquad \le \epsilon\|g'\|_\infty \|Q'\|_\infty \|\phi\|_\infty |\xi_0 - \tau_0| + \epsilon_1 \|\phi\|_\infty \int_{-\epsilon}^0 \tau^\delta\, dy + \epsilon\tau_0\|g'\|_\infty\|\phi\|_\infty.
\end{aligned}$$

We let $\delta \to 0$ in the above estimate. By the weak convergence of τ^δ to $\hat\tau$ we obtain

$$\begin{aligned}
&\Big|\int_{-\epsilon}^0 \big(\tilde\eta(\hat\tau, w) + g'(w)\hat\tau\big)\phi\, dy\Big| \\
&\qquad \le \epsilon\|g'\|_\infty \|Q'\|_\infty \|\phi\|_\infty |\xi_0 - \tau_0| + \epsilon_1 \|\phi\|_\infty \lim_{\delta\to 0}\int_{-\epsilon}^0 \tau^\delta\, dy + \epsilon\tau_0\|g'\|_\infty\|\phi\|_\infty.
\end{aligned}$$

Next, let $\epsilon \to 0$. The middle term converges to $\epsilon_1\|\phi\|_\infty h(0)$ by using (49). Further, the value ϵ_1 could be made arbitrary small by choosing ξ_0 big. Thus we get

$$\lim_{\epsilon\to 0}\lim_{\delta\to 0}\int_{-\epsilon}^0 \eta(\tau^\delta, w^\delta)\,\phi\, dy = -g'(w(0))h(0)\phi(0),$$

which equals the discrete part of $\hat\eta$. By combining this result and inequality (51) we see that the limit $\tilde\eta$ equals $\hat\eta(\hat\tau, w)$ and hence $\eta(\tau^\delta, w^\delta) \rightharpoonup \hat\eta$ as $\delta \to 0$.

Now we return to the discrete entropy inequality (50). The weak convergence result discussed above and the uniform convergence of ϕ^δ and ϕ^δ_t yields

$$\begin{aligned}
&\int_{\mathbb{R}^+}\int_{\mathbb{R}} \hat\eta(\hat\tau, w)\phi_t(y,t)\, dy\, dt + \int_{\mathbb{R}} \hat\eta(\bar\tau, w)\phi(y,0)\, dy \\
&\qquad + \int_{\mathbb{R}^+}\int_{\mathbb{R}} q(\hat\tau, w)\phi_y(y,t)\, dy\, dt \ge 0, \qquad (52)
\end{aligned}$$

and the weak solution $\hat\tau$ satisfies an entropy inequality.

Next we investigate the entropy production across a vacuum state. By physical considerations and by the solution of the Riemann problem, it should not be admissible to connect a left state to a vacuum state by a discontinuity. Assume a vacuum is located in $y = 0$. Consider a non-physical solution given by a vacuum state in $y = 0$ and a constant state (τ_L, w_L) for $y < 0$. To the right of the vacuum the solution is the constant state (τ_R, w_R). We insert the expressions of $\hat{\eta}$ and q into the entropy inequality given by (52),

$$\begin{aligned}0 &\le \int_{\mathbb{R}^+}\int_{\mathbb{R}} \left[(\eta(\tau,w) - g'(w)h(y,t)\delta(y))\,\phi_t + g(v)\phi_y\right]\,dy\,dt\\ &= \left[g'(w_L)\big(v_R - v_L\big) - \big(g(v_R) - g(v_L)\big)\right]\int_0^t \phi(0,t)\,dt.\end{aligned}$$

The equality follows by partial integration with respect to time and integration with respect to y, the expression of $h(y,t)$ as given by (47) and, since τ is constant in time, $\eta_t = 0$. The above inequality holds if

$$g(v_R) - g(v_L) \le g'(w_L)(v_R - v_L).$$

First, consider the case where $v_L \ge v_R$. Since $v_L \le w_L$ it follows by the concavity of $g(v)$ that the above inequality is satisfied. Next, assume $v_L < v_R$. The inequality holds for $w_L = v_L$. For a given value $w_L \in (v_L, v_R]$ there exists a function g such that the entropy inequality fails. Furthermore, for $w_L > v_R$ the inequality fails by the concavity of g. In conclusion, the above discussion shows that not all the inadmissible discontinuities violate the entropy condition. However, we have the following lemma.

Lemma 14. *Assume vacuum is present at $y = 0$, that is $h(0) > 0$ in (11), for $t > 0$. Then*

$$\lim_{y\to 0^-} \tau(y,t) = \infty.$$

It follows that the solution $\hat{\tau}$ is not bounded to the left of a vacuum. Hence a left state can not be connected directly to a vacuum state by a contact discontinuity. Note that this result and the entropy inequality given by (52) tell us that the weak solution $d\hat{\tau} = \tau\,dy + d\check{\tau}$ is an entropy admissible solution.

Proof. We start by proving that $\limsup_{y\to 0^-} \hat{\tau}(y) = \infty$ by a contradiction. Let $0 \in V$, and assume $\bar{\tau} < M < \infty$ where M is a positive constant. Fix a $y \le -\delta$ and let k_y be given such that $y \in (k_y, k_y + 1]\delta$. Consider the discrete entropy inequality (23) with $r = 0$. By summing from $k = k_y$ to $k = -2$ we achieve

$$\delta \sum_{k=k_y}^{-2} \dot{\eta}(\tau_k, w_k) \le \sum_{k=k_y}^{-2} \big(q(\tau_k, w_k) - q(\tau_{k+1}, w_{k+1})\big) = q(\tau_{k_y}, w_{k_y}) - q(\tau_{-1}, w_{-1}).$$

We integrate the above inequality in time from 0 to t. Thus,

$$\delta \sum_{k=k_y}^{-2} \eta(\tau_k, w_k) - \delta \sum_{k=k_y}^{-2} \eta(\bar{\tau}_k, w_k) \le \int_0^t \big(q(\tau_{k_y}, w_{k_y}) - q(\tau_{-1}, w_{-1})\big)\,dt.$$

Next, insert the expressions of η and q given by (22) and obtain

$$-\delta \sum_{k=k_y}^{-2} \int_{\bar{\tau}_k}^{\tau_k} g'\big(w - Q(\xi)\big)\,d\xi \le \int_0^t \left[g\big(w_{k_y} - Q(\tau_{k_y})\big) - g\big(w_{-1} - Q(\tau_{-1})\big)\right]\,dt.$$

Since g' is bounded we get

$$-\|g'\|_\infty \delta \sum_{k=k_y}^{-2} |\tau_k - \bar{\tau}_k| \le \int_0^t \left[g\big(w_{k_y} - Q(\tau_{k_y})\big) - g\big(w_{-1} - Q(\tau_{-1})\big)\right]\,dt.$$

Let $\delta \to \mathcal{U}$. By assumption, $\tau_{-1} \to \infty$. Since τ^δ converges pointwise on $\Omega \setminus V$ and in $L^1_{\mathrm{loc}}(\mathbb{R} \setminus V)$ we have

$$-\|g'\|_\infty \int_{[y,0)} |\tau(\xi) - \bar{\tau}(\xi)|\,d\xi \le \int_0^t \left[g\big(w(y) - Q(\tau(y,t))\big) - g\big(w(0)\big)\right]\,dt.$$

Next, we find an upper bound for the integral to the left by taking the supremum of the integrand over y. Further, let $y \to 0$ from the left. By assumption w is continuous from the left. Thus,

$$- \lim_{y\to 0^-} |y| \cdot \|g'\|_\infty \left(\limsup_{y\le b<0} |\tau(b,t)| + \limsup_{y \le b < 0} |\bar\tau(b)| \right) \\ \le \int_0^t \left[g\big(w(0) - Q(\tau(0^-,t))\big) - g\big(w(0)\big) \right] dt$$

If we assume $\limsup_{y\to 0^-} \tau(y) < M < \infty$, the left hand side in the above inequality goes to 0. Further, we can choose g such that $g'(w(0)) = 0$. Since $-Q < 0$ and g is concave, the integrand to the right is negative and the right hand side of the inequality is bounded by 0. Hence

$$\begin{aligned} 0 &\le \int_0^t \left[g\big(w(0) - Q(\tau(0^-,t))\big) - g\big(w(0)\big) \right] dt \\ &\le t \left[g\big(w(0) - Q(\tau(0^-,t))\big) - g\big(w(0)\big) \right] \le -tC \end{aligned}$$

where C is a positive constant. The above inequality is a contradiction, and thus $\limsup_{y\to 0^-} \tau(y) = \infty$.

Consider the initial data $(\bar\tau, w)$ and assume $\bar\tau(0) = \infty$. If $\limsup_{y\to 0^-} \bar\tau(y) < \infty$, it follows by the above proof that, if there is still a vacuum present at $y = 0$ at some time t, that $\limsup_{y\to 0^-} \tau(y) < \infty$. On the other hand, if $\limsup_{y\to 0^-} \bar\tau(y) = \infty$ the above proof does not hold. However, consider two sets of initial data, $(\bar\tau^1, w)$ and $(\bar\tau^2, w)$ such that $\bar\tau^1 \le \bar\tau^2$ and $\bar\tau^1(0) = \bar\tau^2(0) = \infty$. Assume $\limsup_{y\to 0^-} \bar\tau^1(y) < \limsup_{y\to 0^-} \bar\tau^2(y) = \infty$, and thus by the above result $\limsup_{y\to 0^-} \tau^1(y,t) = \infty$. The monotonicity property given in Lemma 6 yields that also $\limsup_{y\to 0^-} \tau^2(y,t) = \infty$.

Now, we use $\limsup_{y\to 0^-} \tau(y) = \infty$ and $\mathrm{TV}(Q(\tau)) < M < \infty$ to conclude that $\lim_{y\to 0^-} \tau(y) = \infty$. The function $Q(\tau)$ is decreasing with respect to τ. Hence, $\limsup_{y\to 0^-} \tau(y) = \infty$ implies that $\liminf_{y\to 0^-} Q(\tau(y)) = 0$. Since the total variation of $Q(\tau)$ is bounded, the limit $\lim_{y\to 0^-} Q(\tau(y))$ exists and it follows by the the limit inferior at 0^- that it equals 0. Hence $\lim_{y\to 0^-} \tau(y) = \infty$. □

5.5. **Continuous dependence on the initial data.** We want to show stability of the solution $\hat\tau$ with respect to the initial data. The proof is similar to the one given in section 4 and we consider the discrete L^1-estimate given by inequality (33). Since we allow vacuum, we now do not have convergence of $\tau^\delta(y,t)$ in $L^1_{\mathrm{loc}}(\mathbb{R} \times \mathbb{R}^+)$. However, since $S_t^{\delta,w^1}\bar\tau^1 - S_t^{\delta,w^2}\bar\tau^2$ converges weakly to $\hat\tau^1 - \hat\tau^2$, we have

$$\|\hat\tau^1 - \hat\tau^2\| \le \liminf_{\delta\to\mathcal{U}} \|S_t^{\delta,w^1}\bar\tau^1 - S_t^{\delta,w^2}\bar\tau^2\|.$$

Further, by construction, $\mathrm{TV}(w^{1,\delta} - w^{2,\delta}) \le \mathrm{TV}(w^1 - w^2)$. We need to find a way to handle the term $\|\bar\tau^{1,\delta} - \bar\tau^{2,\delta}\|$.

We consider initial data $(\bar\tau^1, w^1)$ and $(\bar\tau^2, w^2)$ taking values in $\mathcal{D}_V$ with corresponding $\bar x^1$ and $\bar x^2$, respectively. Since we are interested in τ, i.e. the derivative of $x(y)$ with respect to y, the value $\bar x_K$ is of no importance. We choose $\bar x_K^1 = \bar x_K^2 = \bar x_K$. By construction,

$$\delta\bar\tau_k^\delta = \bar x_{k+1}^\delta - \bar x_k^\delta = \int_{k\delta}^{(k+1)\delta} \frac{\partial \bar x}{\partial y}\, dy = \int_{k\delta}^{(k+1)\delta} \bar\tau \, dy = \hat{\bar\tau}(k\delta, (k+1)\delta],$$

from which we obtain

$$\|\bar\tau^{1,\delta} - \bar\tau^{2,\delta}\| = \sum_k \left| \left(\hat{\bar\tau}^1 - \hat{\bar\tau}^2\right)(k\delta, (k+1)\delta] \right| \le \|\hat{\bar\tau}^1 - \hat{\bar\tau}^2\|.$$

Now we turn back to the discrete L^1-stability estimate. According to the construction of the discrete initial data and the above estimate we get

$$\|\tau^{1,\delta} - \tau^{2,\delta}\| \le \|\bar\tau^1 - \bar\tau^2\| + t\,\mathrm{TV}(w^1 - w^2) + t\|Q'\|_\infty |\bar\tau^1(Y) - \bar\tau^2(Y)|,$$

Letting $\delta \to \mathcal{U}$ yields

$$\|\hat{\tau}^1 - \hat{\tau}^2\| \leq \|\bar{\tau}^1 - \bar{\tau}^2\| + t\,\mathrm{TV}(w^1 - w^2) + t\|Q'\|_\infty|\bar{\tau}^1(Y) - \bar{\tau}^2(Y)|,$$

and the solution is stable with respect to the L^1-norm.

It remains to consider the stability in time. A discrete Lipschitz estimate in time is given by inequality (19). Since $R = 0$, the results in Lemma 2 and Lemma 3 simplify. By the same argument as above, we get, by letting $\delta \to \mathcal{U}$,

$$\|\hat{\tau}(t) - \hat{\tau}(s)\| \leq \mathrm{TV}(\bar{v})|t - s|.$$

Combining the above results by the triangle inequality yields

$$\begin{aligned}\|\hat{\tau}^1 - \hat{\tau}^2\| \leq \|\bar{\tau}^1 - \bar{\tau}^2\| &+ (t \wedge s)\,\mathrm{TV}(w^1 - w^2) \\ &+ (t \wedge s)\|Q'\|_\infty|\bar{\tau}^1(Y) - \bar{\tau}^2(Y)| + \big(\mathrm{TV}(\bar{v}^1) \vee \mathrm{TV}(\bar{v}^2)\big)|t - s|,\end{aligned}$$

which completes the proof of Theorem 3.

6. Conclusion and acknowledgment

We have derived the macroscopic Aw–Rascle model for traffic flow (2) in Lagrangian form directly from the microscopic car-following model (1). By carefully investigating the discrete model and taking the limit $\delta \to 0$ directly we obtain the existence of a continuous semigroup whose trajectories are weak entropy solutions of the macroscopic system. We have not, however, proved the uniqueness of this semigroup.

When including vacuum in the initial data and in the solution we have had to restrict the functions w in $\mathrm{BV}(\mathbb{R})$ to have positive jumps only in a discrete set V, and to satisfy a one sided Lipschitz condition limiting the rate of growth between the points of V. In [8] we considered the macroscopic model in Eulerian form and showed existence of weak entropy solutions including vacuum when only assuming w in BV. On the other hand, for homogeneous system, that is $R = 0$, the semigroup obtained in this paper depends continuously on the initial data.

The authors would like to thank Nils Henrik Risebro for suggesting various improvements to the paper.

References

[1] A. Aw, A. Klar, T. Materne, and M. Rascle. Derivation of continuum traffic flow models from microscopic follow-the-leader models. *SIAM J. Appl. Math.*, 63(1):259–278 (electronic), 2002.

[2] A. Aw and M. Rascle. Resurrection of "second order" models of traffic flow. *SIAM J. Appl. Math.*, 60(3):916–938 (electronic), 2000.

[3] P. Bagnerini and M. Rascle. A multiclass homogenized hyperbolic model of traffic flow. *SIAM J. Math. Anal.*, 35(4):949–973 (electronic), 2003.

[4] P. Baiti and H. K. Jenssen. Well-posedness for a class of 2×2 conservation laws with L^∞ data. *J. Differential Equations*, 140(1):161–185, 1997.

[5] R. M. Colombo and A. Corli. On a class of hyperbolic balance laws. *J. Hyperbolic Differ. Equ.*, 1(4):725–745, 2004.

[6] M. Garavello and B. Piccoli. *Traffic Flow on Networks.* AIMS on Applied Math. American Series of Mathematical Sciences, 2006.

[7] P. Goatin. The Aw-Rascle vehicular traffic flow model with phase transitions. *Math. Comput. Modelling*, 44(3-4):287–303, 2006.

[8] M. Godvik and H. Hanche-Olsen. Existence of solutions for the Aw–Rascle traffic flow model with vacuum. *J. Hyperbolic Differ. Equ.*, 2008. To appear.

[9] J. M. Greenberg. Extensions and amplifications of a traffic model of Aw and Rascle. *SIAM J. Appl. Math.*, 62(3):729–745 (electronic), 2001/02.

[10] J. M. Greenberg. Congestion redux. *SIAM J. Appl. Math.*, 64(4):1175–1185 (electronic), 2004.

[11] J. M. Greenberg, A. Klar, and M. Rascle. Congestion on multilane highways. *SIAM J. Appl. Math.*, 63(3):818–833 (electronic), 2003.

[12] M. Herty and M. Rascle. Coupling conditions for a class of second-order models for traffic flow. *SIAM J. Math. Anal.*, 38(2):595–616 (electronic), 2006.

[13] R. A. Klausen and N. H. Risebro. Stability of conservation laws with discontinuous coefficients. *J. Differential Equations*, 157(1):41–60, 1999.

[14] D. Serre. *Systems of conservation laws. 1*. Cambridge University Press, Cambridge, 1999. Hyperbolicity, entropies, shock waves, Translated from the 1996 French original by I. N. Sneddon.

[15] D. H. Wagner. Equivalence of the Euler and Lagrangian equations of gas dynamics for weak solutions. *J. Differential Equations*, 68(1):118–136, 1987.

Department of Mathematical Sciences, Norwegian University of Science and Technology, N-7491 Trondheim, Norway

E-mail address: godvik@math.ntnu.no
URL: www.math.ntnu.no/~godvik/

Department of Mathematical Sciences, Norwegian University of Science and Technology, N-7491 Trondheim, Norway

E-mail address: hanche@math.ntnu.no
URL: www.math.ntnu.no/~hanche/