Dynamics of the density discontinuity interface in compressible viscous flows. A transversal contact problem.

Abstract

The Navier-Stokes equations for the motion of compressible, viscous fluids in the half-space $\mathbb{R}^2_+$ with the no-slip boundary condition are considered. We study the problem of determining the evolution of the interface of discontinuity of a piece-wise $W^{1,p}$, $p > 2$, density when the interface is in transversal contact with the boundary of the domain. A unique global solution that exists near a constant equilibrium case is constructed that preserves $C^{1+\alpha}$ regularity of the interface.

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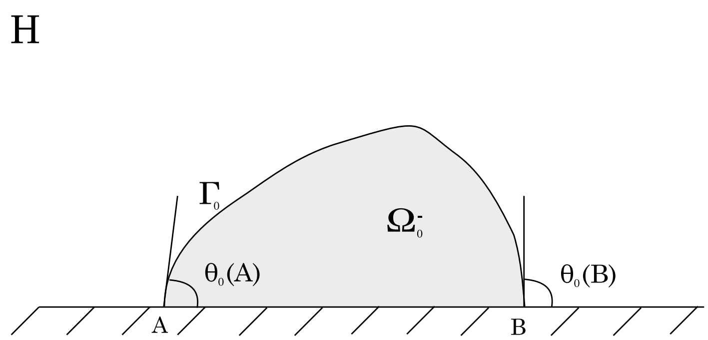

Figure 1: Transversal contact problem

0.1 Introduction

We consider a model for the motion of a compressible, viscous flow based on the Navier-Stokes equations. With $\rho(t,x)$ and $\mathbf{u}(t,x)$ being the density and the velocity of the fluid, the model consist of the equations:

$$\frac{\partial}{\partial t}\rho + \, div \, (\rho\mathbf{u}) \, = 0, \tag{1}$$

$$\frac{\partial}{\partial t} (\rho\mathbf{u}) \, + \, div \, (\rho\mathbf{u}\otimes\mathbf{u}) - (\lambda+\mu) \, div \, \mathbf{u} - \mu\Delta\mathbf{u} + \nabla P = 0, \tag{2}$$

$$\lambda + \mu \geq 0, \quad \mu > 0, \tag{3}$$

$$(t, \, x) \in \mathbb{R}_+ \times \Omega, \, \Omega \subset \mathbb{R}^2$$

and a set of initial and boundary conditions:

$$(\rho(0,x), \, \mathbf{u}(0,x)) = (\rho_0(x), \, \mathbf{u}_0(x)) \, , \quad x \in \Omega, \tag{4}$$

$$\mathbf{u}(t,x) = 0, \quad (t,x) \in \mathbb{R}_+ \times \partial\Omega. \tag{5}$$

We consider the problem of determining the dynamics of the surface of density discontinuity as it is modeled by system (1)–(5). Shortly, we consider initial density as being a piecewise smooth function, having jump discontinuity on the interface Γ_0 – smooth manifold of co-dimension 1. The problem in question is to construct a weak solution $\rho, \mathbf{u}$ and to study the properties of the interface Γ_t – the set where $\rho(t,\cdot)$ has jump discontinuity, as well as other regularity and stability properties of the solution. The problem can be viewed also as a prototype problem for multi-phase flows, i.e. flows in which λ, m are discontinuous as well.

We will distinguish several types of the geometries of Γ_0. We call the problem *contact* problem, if $\Gamma_0 \cap \partial H \neq \emptyset$, and *interior* problem otherwise. The latter case was considered in work [11] and, in a more general setting of the two-phase, heat conductive flows, in [16]. The arguments in these works are distinctively different and the theory developed in [11] allows for more general class of initial data. We will comment on it in more details in the next section.

In this work we concentrate on the study of the contact problem.

0.2 Statement of the result

Let H be a half-space. Let $\phi_0(x) \in C^{1+\alpha}(H)$, a level function, be such that $\Gamma_0 = \{\phi_0(x) = 0\} \cap \overline{H}$ is a simple, compact $C^{1+\alpha}$ curve and $\Gamma_0 \cap \partial H$ consists of two points A, B, see Figure 1. Γ_0 divides H into open subsets Ω_0^+, Ω_0^-. We designate the latter to be a bounded domain. We assume that

$$|\nabla\phi_0|_{inf} = \inf_{x\in\Gamma_0} |\nabla\phi_0(x)| > 0$$

and for definiteness we say that $\nabla\phi_0$ points into Ω_0^+. Let $\theta_0(A)$, $\theta_0(B)$ denote the contact angles, i.e. the angles that $\nabla^\perp\phi_0(x)$, $x = A, B$, make with ∂H. In this work we consider a *transversal* type of a contact problems, i.e.

$$0 < \theta_0(A), \theta_0(B) < \pi.$$

We measure curve Γ by the functional

$$|\Gamma|_{C^{1+\alpha}} = |\mathbf{W}|_{C^\alpha(\Gamma)} + (\inf_{x\in\Gamma} |\mathbf{W}(x)|)^{-1},$$

where $\mathbf{W}(x)$ is a tangent vector to Γ at point $x \in \Gamma$.

For various functional spaces appearing in this paper we will often write $f \in X(\Omega_0^\pm)$ to mean that the restriction of f to Ω_0^+ belongs to $X(\Omega_0^+)$ and the restriction of f to Ω_0^- belongs to $X(\Omega_0^+)$. The norm $|\cdot|_{X(\Omega_0^\pm)}$ will mean $|\cdot|_{X(\Omega_0^+)} + |\cdot|_{X(\Omega_0^-)}$.

Let us as fix $\bar\rho > 0$. The initial density, ρ_0 is defined as an element of

$$\rho_0 - \bar\rho \in L^2(H) \cap L^4(H) \cap W^{-1,2}(H) \cap W^{1,p}(\Omega_0^+) \cap W^{1,p}(\Omega_0^-),\ p > 2.$$

Such densities ρ_0 have continuous traces on Γ_0 and we set

$$[\rho_0] = \rho_0^+ - \rho_0^-,$$

to denote the jump of ρ_0 across Γ_0, which we assume to be non zero.

There is a “stability condition” that we are going to use in our theorem (see Remark 3 below):

$$\sup_{x\in\Gamma_0} |[\rho_0](x) d_x^{-\beta}| < +\infty, \tag{6}$$

where $d_x = \mathrm{dist}\{x, \partial H\}$, $\beta > 1 - 2p^{-1}$. For the initial velocity we assume that

$$\mathbf{u}_0 \in W_0^{1,2}(H).$$

We let

$$I[\rho_0,\mathbf{u}_0] = |\rho-\bar\rho|_{L^2(H)} + |\rho_0-\bar\rho|_{L^4(H)} + |\rho_0-\bar\rho|_{W^{1,p}(\Omega_0^\pm)} + \sup_{x\in\Gamma_0}|[\rho_0](x)d_x^{-\beta}| + |\mathbf{u}_0|_{W_0^{1,2}(H)}. \quad (7)$$

We restrict ourselves to the study of isothermal regime:

$$P = a\rho,\ a>0,$$

and assume that

$$\frac{\lambda}{\mu} > [\cot(\pi/8)]^2 - 3. \quad (8)$$

Theorem 1. *There is $p_0 = p_0(\lambda,\mu)$, $p_0\in(2,8/3)$, such that for any $p\in(2,p_0]$, $\alpha = 1-2p^{-1}$, any $\beta>p$ and $\Gamma_0\in C^{1+\alpha}$ defined by function ϕ_0 there is*

$$c = c(\lambda,\mu,\beta,p_0,\bar\rho,a,|\nabla\phi_0|_{C^\alpha(H)},|\nabla\phi_0|_{\inf})$$

such that if

$$I[\rho_0,\mathbf{u}_0] < c$$

then, there is a unique weak solution $\rho,\mathbf{u}$ of the problem (1)–(5) in the sense of Definition 1. Moreover,

$$\rho\in C([0,T]\,:\,L^q(B_R\cap H)),$$

$$\rho-\bar\rho\in L^\infty((0,T)\,:\,L^2(H))\cap L^\infty((0,T)\times H),$$

$$\mathbf{u}\in L^\infty((0,T)\,:\,W_0^{1,2}(H)),$$

$$\nabla\mathbf{u}\in L^{\omega(p)}((0,T)\,:\,L^\infty(H)),$$

for any B_R – ball of radius R, any $q\in[1,\infty)$, any $T>0$ and some $\omega(p)>1$. Also, the flow $X^t(x)$ is uniquely defined and satisfies the following estimates.

$$\sup_{t\in[0,T]}|X^t(x_1)-X^t(x_2)|\le C(T)|x_1-x_2|,$$

$$\sup_{x\in H}|X^{t_1}(x)-X^{t_2}(x)|\le C(T)|t_1-t_2|^{\gamma(p)},\ t_1<t_2<T,$$

for some $C(T)$ and $0<\gamma(p)<1$.

For any $t \geq 0$, $\rho(t,\cdot)$ has a jump discontinuity across interface $\Gamma_t = X^t[\Gamma_0]$ which is of the class $C^{1+\alpha}$. Contact angles

$$\theta_t(A), \theta_t(B) \in (0,\pi),\ t > 0.$$

Additionally,

$$\rho - \bar{\rho} \in L^\infty((0,T) \, : \, W^{1,p}(\Omega_t^\pm)),$$
$$\mathbf{u} \in C((0,T) \times H),$$
$$\nabla \mathbf{u}(t) \in C^\gamma(\Omega_t^\pm \cap B_R),\ t > 0,$$

for some $0 < \gamma < \alpha$. Norms of $\rho - \bar{\rho}$ in spaces $L^\infty((0,T) : W^{1,p}(\Omega_t^\pm))$, $L^\infty((0,T) : L^2(H))$ and $L^\infty((0,T) \times (H))$ are bounded independently on T, while there is c_0, independent of T such that

$$\left.\begin{array}{c} \int_0^T |\nabla \mathbf{u}|_\infty^{\omega(p)} \leq c_0 T, \\ \\ \sin\theta_t(x_0) \geq \sin\theta_0(x_0) e^{-c_0 T},\ x_0 = A, B, \\ \\ \sup_{t\in[0,T]} |\Gamma_t|_{C^{1+\alpha}} \leq |\Gamma_0|_{C^{1+\alpha}} e^{c_0 T}. \end{array}\right\} \tag{9}$$

Remark 1. *Condition* (8) *is crucial in obtaining long time stability but it seems to be an artifact of doing analysis in 2 dimensions and in unbounded domains.*

Remark 2. *The solution constructed in the theorem is unique within its regularity class, see theorem 4 of section 3. It is easy generalization of the uniqueness theorem proved in [13].*

Remark 3. *The stability condition is assumed in order to prove that $\rho(t,\cdot)$ remains with in the class $W^{1,p}(\Omega_t^\pm)$ and that $\Gamma_t \in C^{1+\alpha}$ remains transversal for positive time. It can also be viewed as a compatibility condition; if $\mathbf{u}(t,\cdot) \in C^{1+\alpha}\left(\Omega_t^\pm\right)$ and $\rho(t\cdot) \in C^\alpha\left(\Omega_t^\pm\right)$, then the Rankin-Hugoniot conditions on the curve Γ_t read as (see [10])*

$$\left.\begin{array}{r} (\lambda + 2\mu)[\,div\,\mathbf{u}] = [P], \\ [\,curl\,\mathbf{u}] = 0, \end{array}\right\} \tag{10}$$

assuming that

$$[\mathbf{u}] = 0. \tag{11}$$

If Γ_t intersects ∂H transversally, then (10),(11) *imply that* $[\nabla \mathbf{u}](x_0) = 0$, *where* x_0 *is a point of contact. Consequently, the first condition in* (10) *together with the regularity assumptions implies that:*

$$|[P](x)| \approx d_x^\alpha,\ x \in \Gamma_t.$$

Condition (6) *is slightly stronger.*

The proof follows the framework developed in [11]. At the first place we obtain several energy-type estimates (sections 1.2–1.4) Then, we view the equations (2) as an elliptic system

$$L\langle \mathbf{u} \rangle = \rho \dot{\mathbf{u}} + \nabla(\rho - \bar{\rho}), \tag{12}$$

$L = (\lambda + \mu)\nabla + \mu\Delta$, $\dot{\mathbf{u}}$ is a material derivative. We use Lichtenstein's method, see [15], to derive representation formulas for $div\, \mathbf{u}$. In particular,

$$div\, \mathbf{u} = div\, L^{-1}\langle \rho \dot{\mathbf{u}} \rangle + \alpha_1(\rho - \bar{\rho}) + \alpha_2 A\langle \rho - \bar{\rho} \rangle, \tag{13}$$

where $A\langle \cdot \rangle$ is a certain singular integral operator whose kernel has singularity on ∂H. This term appears in the formula due to the reflections of sounds waves on the the no-slip boundary ∂H. For a generic discontinuous densities, this operator is unbounded in L^∞ norm. Consequently, the interior smoothness properties of the viscous flux, $(\lambda + 2\mu)\, div\, \mathbf{u} - P$, are being lost at the boundary of the domain. This is one of the places where the stability assumption (6) is being used to secure the non-singular behavior of the solution.

We use the above representation formula for $div\, \mathbf{u}$, to obtain estimates on ρ and $\nabla\rho$ from equation (1), see Section 2.5. At that point we need bounds on $\nabla A\langle \rho \rangle$ in L^p spaces. Moreover, to assure that norms of $\rho, \nabla\rho$ do not grow in time we need to show that L^p norms of $\nabla A\langle \rho \rangle$ do not exceed $\frac{\alpha_1}{\alpha_2}$ and can be absorbed by pressure. However, the L^p norms do grow with p and for this reason we have a restriction on the range of p in the main theorem. One could suggest that for evolution equations it is rather the spectral bounds, which are known to be independent of p for this particular operator, that determine stability. However, the nonlinearity of the equations and low regularity of the solution prevent the application of any existing stability theory here.

Formula (13) also allows us to rewrite the Lamé equations (12) as vector Poissons equations:

$$-\mu\Delta\mathbf{u} = -\rho\dot{\mathbf{u}} + (\lambda+\mu)\nabla\, div\, L^{-1}\langle\rho\dot{\mathbf{u}}\rangle + (\alpha_1(\lambda+\mu)-1)\nabla(\rho-\bar{\rho}) + \alpha_2(\lambda+\mu)A\langle\rho-\bar{\rho}\rangle,$$

and obtain representation formulas for $\mathbf{u}$ in terms of the Green's function of the Laplace's operator in a half-space. With these formulas at hand we trace the regularity of Γ_t, see Section 2.7. We follow the approach developed in [3] for the study of the regularity of vortex patches in the framework of incompressible Euler equations, see also [2],[11]. The representation formulas in our problem are somewhat lengthy and technical due to the presence of terms describing reflection, but all additional terms can effectively treated under condition (6) and classical theory of singular integral operators.

The stability of the solution around a static equilibrium state $(\bar{\rho}, 0)$ is obtained as a consequence of viscous dissipation and propagation of sound waves. In particular, these two phenomena imply exponential in time decay of the jump $[\rho(t)]$. This fact is crucial in balancing the effect of deteriorating geometry of the interface that is present in the solution, see estimate (9).

The solution is built as a limit of solutions of two-level approximation scheme for the problem (1)–(5), that was introduced in [11].

The paper organized as follows: In the first few subsections we collect various analytical tools that we will use thought the paper.

In Section 1 we prove the existence of a local weak solution $(\tilde{\rho}, \tilde{u})$ of (1)–(5) on a time interval $[0, T]$. This solution is less regular than the one we are looking for in theorem 1. In particular, L^∞ norm of ρ is shown to be finite, but there is no information on piece-wise smooth structure of ρ. Solutions of that regularity class were studied in [18] for flows in $\mathbb{R}^3_+$, see also [10]. Here, the arguments are almost identical.

With that $(\tilde{\rho}, \tilde{u})$, in Section 2, we build a solution of an elliptic-hyperbolic problem

$$\left.\begin{aligned} L\langle \mathbf{u}\rangle &= \tilde{\rho}\dot{\tilde{u}} + \nabla(\rho - \bar{\rho}), \\ \rho_t + \, div\,(\rho \mathbf{u}) &= 0, \\ \rho(0,x) &= \rho_0(x). \end{aligned}\right\} \tag{14}$$

The solution $(\rho, \mathbf{u})$ verifies all the properties stated in the theorem 1, except for the fact that it might not be the solution of problem (1)–(5).

In Section 3 we state two uniqueness theorems; one for the problem (14), the other one for solutions of (1)–(5) within the regularity class of the theorem 1. Proofs can be derived from the results of [10, 11]. The proof of the theorem 1 is concluded by noticing first, that $(\tilde{\rho}, \tilde{u})$ is also a solution of (14) and thus $(\rho, \mathbf{u}) = (\tilde{\rho}, \tilde{u})$. And secondly, since the crucial bounds on $(\rho, \mathbf{u})$ bounds obtained in Section 2 are independent of the interval of the existence T and there is a uniqueness of solutions, $(\rho, \mathbf{u})$ can be continued for all times $T > 0$.

Section Appendix contains proofs of some technical lemmas used in the paper.

0.3 Functional setting

We use symbol ∇ to denote the spacial gradient of a function and D^2 the set of all spacial second derivatives. The norm $|\cdot|_p$ is the usual L^p norm. We use the standard notation $W^{k,p}(H)$, $k \in \mathbb{N}$, $1 \le p < +\infty$ for the space of weakly differentiable, up to the order k, functions, with derivatives from $L^p(H)$ space. We will use Besov spaces of traces $W^{s,p}(\Gamma)$ with $s = 1 - p^{-1}$, $p > 2$ as well. The definition and basic properties of these spaces are given in the Appendix. In this paper we will abbreviate $L^p(H)$ to L^p and use the same notation for norms of scalar and vector functions. Denote by

$$\langle \mathbf{u}\rangle_{C^\alpha(\Omega)} = \sup_{x,y\in\Omega,\, x\neq y} \frac{|\mathbf{u}(x) - \mathbf{u}(y)|}{|x-y|^\alpha},\ \alpha \in]0,1[.$$

The following estimates are well-known, see [5], Theorem 7.10, Theorem 7.17.

Lemma 1. *Let $u \in W^{1,2}(H)$. Then, $u \in L^p(H)$ and there is $c(p) > 0$, independent of u, such that*

$$|u|_{L^p} \le c|u|_2^{2/p} |\nabla u|_2^{1-2/p}.$$

Lemma 2. *Let u be a locally integrable function with $\nabla u \in L^p(H)$, $p > 2$. Then, there is $c = c(p)$ such that for a.e. $x, y \in H$ it holds*

$$|u(x) - u(y)| \leq c|x - y|^\alpha |\nabla u|_{L^p},$$

where $\alpha = 1 - 2p^{-1}$.

Definition 1. *A pair of functions*

$$(\rho,\, \mathbf{u}) = (\rho(t,x),\, u_1(t,x),\, u_2(t,x))$$

is called a weak solution of (1)-(5) if

$$\rho,\, \rho u_i,\, \nabla u_i \in L^1_{loc}(\mathbb{R}_+ \times H),\, i = 1,2,$$

$$\rho u_k \otimes u_l \in L^1_{loc}(\mathbb{R}_+ \times H),\, i,k,l = 1,2,$$

$$\left.\begin{array}{rcl} \nabla \mathbf{u} & \in & L^2(\mathbb{R}_+ \times H), \\ \mathbf{u} & = & 0, \ \ on\ \partial H, \end{array}\right\}$$

and for all test functions $\phi,\, \psi_i \in C^\infty([t,\, T] : C_0^\infty(H))$, $i = 1,2$, with $0 \leq t < T < +\infty$ it holds (summation over the repeated indexes is assumed)

$$\int\!\!\int_{\mathbb{R}_+ \times H} \rho \partial_t \phi + \rho \mathbf{u} \cdot \nabla \phi - \int_H \rho(\tau,\cdot)\phi(\tau,\cdot)\Big|_t^T = 0,$$

$$\begin{aligned} &\int\!\!\int_{\mathbb{R}_+ \times H} \rho u_k \partial_t \psi_k + \rho u_k u_j \partial_k \psi_j \\ &\qquad - \int\!\!\int_{\mathbb{R}_+ \times H} (\lambda + \mu)\, div\, \mathbf{u}\, div\, \psi + \mu \partial_k u_l \partial_k \psi_l + A(\rho - \bar{\rho})\partial_k \psi_k \\ &\qquad\qquad - \int_H \rho(\tau,\cdot) u_k(\tau,\cdot)\psi(\tau,\cdot)\Big|_t^T = 0. \end{aligned}$$

To simplify the presentation we assume that the constant $a = 1$ in (2). It is always possible to reduced to this case through the substitution $(t, x, \rho, \mathbf{u}) \to (\alpha^2 t,\, \alpha x,\, \rho,\, \alpha u)$, $\alpha = a^{-\frac{1}{2}}$, without changing the viscosity coefficients.

We will often use the non-conservative form of the equations (1)-(2). They are equivalent for smooth solutions.

$$\dot{\rho} = \, - \rho\, div\, \mathbf{u}, \tag{15}$$

$$\rho \dot{\mathbf{u}} - L\mathbf{u} + \nabla(\rho - \bar{\rho}) = 0, \tag{16}$$

where $L = (\lambda + \mu)\nabla\, div\, + \mu \Delta$ and $\dot{} = \partial_t + \mathbf{u} \cdot \nabla$.

0.4 Lamé equations

The principal part of (2) is an elliptic system of Lamé equations (17). Consider the problem

$$\left.\begin{array}{r}(\lambda+\mu)\nabla\, div\, \mathbf{u}+\mu\Delta\mathbf{u}=\mathbf{f}, \quad x\in H,\\ u=0, \quad x\in\partial H,\end{array}\right\} \tag{17}$$

with the conditions $\mu > 0$, $\lambda + 2\mu > 0$. Here, $\mathbf{f} = (f_1(x),\, f_2(x))$. The system is $W_0^{1,2}(H)$ – elliptic, see Chap. 3, sec. 7 of [17], meaning that the bilinear form

$$a(\mathbf{u},\mathbf{v})=\int_H(\lambda+\mu)\, div\, \mathbf{u}\, div\, \mathbf{v}+\mu\nabla\mathbf{u}:\nabla\mathbf{v},$$

is coercive, i.e.

$$a(\mathbf{u},\mathbf{u})\geq c|\nabla\mathbf{u}|^2_{L^2(H)}.$$

This condition is sufficient to imply the existence of the strong solution, [17], Theorem 2.1. Moreover, the higher regularity results hold:

Lemma 3 (Elliptic estimates I)**.** *Let* $\mathbf{f} \in L^2(H) \cap W^{-1,2}(H)$. *Then, there is a unique strong solution of* (17), *such that*

$$|\nabla\mathbf{u}|_{L^2(H)}\leq c|\mathbf{f}|_{W^{-1,2}(H)},$$

$$|D^2\mathbf{u}|_{L^2(H)}\leq c\left(|\mathbf{f}|_{L^2(H)}+|\mathbf{f}|_{W^{-1,2}(H)}\right).$$

The proof of the next lemma can be found in §8 of [14].

Lemma 4 (Elliptic estimates II)**.** *If* $\mathbf{f} \in L^p(H)$, *(or* $L^p_{loc}(H)$ *),* $1 < p < \infty$ *then all weak derivatives* $D^2\mathbf{u}$ *exist and there is* $c = c(p)$ *such that*

$$|D^2\mathbf{u}|_{L^p(H)}\leq c|\mathbf{f}|_{L^p(H)},$$

or

$$|D^2\mathbf{u}|_{p,B_1(x)\cap H}\leq c\left\{|\mathbf{f}|_{p,B_2(x)\cap H}+|\mathbf{f}|_{W^{-1,2}(H)}\right\},$$

for any $x \in H$.

Lemma 5 (Elliptic estimates III). *Suppose that in the problem* (17) $\mathbf{f} = \nabla P$, *for some* $P \in L^p(H)$, $p \in]1,\infty[$. *Then, the problem has a unique weak solution* $\mathbf{u} \in L^{\frac{2p}{2-p}}$, $p < 2$, $\mathbf{u} \in L^{\infty}_{\text{loc}}$, $p > 2$, *such that*

$$|\nabla \mathbf{u}|_{L^p(H)} \leq c|P|_{L^p(H)},$$

with $c = c(p)$.

The proof of this lemma is based on Calderon-Zygmund theory of singular integrals. Without claiming the originality of the result it follows as a byproduct of formula (25) and lemma 6.

0.5 Operator $div\, \Delta^{-1} \nabla$

Consider a problem

$$\Delta \mathbf{v} = \nabla P, \quad \mathbf{v}\Big|_{\partial H} = 0.$$

If $H^{x,y} = \frac{1}{2\pi} \ln|x-y|$ – a fundamental solution for the Laplace's equation, then there is a representation formula for

$$div\, \mathbf{v} = P + \int_H \nabla_y \cdot \nabla_x H^{x,y^*} P(y)\, dy,$$

where $y^* = (y_1, -y_2)$. Let

$$A\langle P \rangle := \int_H \nabla_y \cdot \nabla_x H^{x,y^*} P(y)\, dy.$$

Lemma 6. *Operator* A,

$$A : L^p(H) \to L^p(H)$$

is a bounded linear operator $1 < p < \infty$. *Moreover, for* $p \geq 2$,

$$|A|_{L^p \to L^p} \leq \frac{1}{2} \left[\cot \frac{\pi}{2p} \right]^2.$$

Proof. The proof of the first statement is an application of a classical Calderon-Zygmund theory of SIO. It is a direct computation to verify that

$$A\langle P \rangle = 2 \int_H \partial^2_{x_2} H^{x,y^*} P(y)\, dy = 2 \int_{\mathbb{R}^2} \frac{1}{2\pi} \frac{(x_1-y_1)^2 - (x_2-y_2)^2}{|x-y|^4} \chi_{\{x_2 - y_2 > 0\}} P(y^*)\, dy$$

where by $P(y)$ we mean the zero extension of P to the lower half space. The kernel $K(z) = \frac{1}{2\pi}\frac{(z_1)^2-(z_2)^2}{|z|^4}\chi_{\{z_2>0\}}$ has a zero mean on spheres $|z| = const.$ and one can compute its Fourier Transform:

$$\widehat{K}(\xi) = \frac{\xi_1^2 - \xi_2^2}{4|\xi|^2}.$$

Consider operators A_1, A_2 given by symbols:

$$\widehat{A_1} = \frac{\xi_1^2 - \xi_2^2}{2|\xi|^2}, \quad \widehat{A_2} = \frac{\xi_1 \xi_2}{|\xi|^2}.$$

Note, that the symbol of A_1 is obtained from the symbol of A_2 by a $\pi/4$ rotation of coordinates. Consequently,

$$|A_1|_{L^p \to L^p} = |A_2|_{L^p \to L^p}, \quad p \in]1, \infty[.$$

On the other hand, $A_2 = -R_1 \circ R_2$, where R_i – i^{th} Riesz transform, i.e.

$$\widehat{R_i} = -i\frac{\xi_i}{|\xi|}.$$

It follows from Remark 4.1.8 and Theorem 4.2.7 of [8] that

$$|R_i|_{L^p \to L^p} = |H|_{L^p \to L^p} = \cot\frac{\pi}{2p}, \quad p \geq 2,$$

here H is Hilbert transform. We conclude that

$$|A|_{L^p \to L^p} = \frac{1}{2}|A_1|_{L^p \to L^p} = \frac{1}{2}|A_2|_{L^p \to L^p} \leq \frac{1}{2}|R_1|^2_{L^p \to L^p}.$$

□

0.6 Representation formulas for solutions of Lamé equations

We split the velocity $\mathbf{u}$ as follows:

$$\mathbf{u} = \mathbf{v} + \mathbf{w}, \quad \mathbf{v} = L^{-1}\langle\rho\dot{\mathbf{u}}\rangle, \, \mathbf{w} = L^{-1}\langle\nabla\rho\rangle. \tag{18}$$

Now, we compute a representation formula for $div\,\mathbf{w}$. We will not write argument t in functions explicitly until the moment it becomes necessary to include it.

For $x = (x_1, x_2) \in H$, let $x^* = (x_1, -x_2)$. Let

$$H^{x,y} = \frac{1}{2\pi} \log |x - y|,$$

and denote by

$$G^{x,y} = H^{x,y} - H^{x,y^*} \tag{19}$$

the Green's function for the Laplace's equation in H. Consider an elliptic problem (17) where we set

$$\mathbf{f} = \nabla P.$$

Let

$$F = (\lambda + 2\mu)\, div\, \mathbf{w} - P, \tag{20}$$

be the notation for the viscous flux. Applying div to (17) we derive:

$$\Delta F = 0, \tag{21}$$

and the following integral representation holds (the dependence of functions on t is not written for notational convenience).

$$F(x) = \int_{\partial H} \partial_{n_y} G^{x,\cdot} F(\cdot). \tag{22}$$

Using (21), equations (17) can be written in the following form.

$$\Delta \left[\frac{\lambda + \mu}{2(\lambda + 2\mu)} F x + \mu \mathbf{w} \right] = \frac{\mu}{\lambda + 2\mu} \nabla_y P(y)$$

and so,

$$\begin{aligned} \mu \mathbf{w}(x) + \frac{\lambda + \mu}{2(\lambda + 2\mu)} F(x) x = \frac{\lambda + \mu}{2(\lambda + 2\mu)} \int_{\partial H} \partial_{n_y} G^{x,y} F(y) y \, dS_y \\ + \int_H G^{x,y} \left[\frac{\mu}{\lambda + 2\mu} \nabla_y P(y) \right] dy. \end{aligned} \tag{23}$$

We set

$$\alpha = \frac{\lambda + \mu}{\lambda + 2\mu},\ \beta = \mu + \frac{\alpha}{2}(\lambda + 2\mu) = \frac{3\mu + \lambda}{2}. \tag{24}$$

We take div of the last equations and use integral representation (22) for F to get the following equation (here and below the summation over repeated indexes is assumed).

$$\mu\, div\, \mathbf{w} + \alpha F = \\ + \alpha/2 \int_{\partial H} \partial_{n_y} \partial_{x_i} G^{x,y} (y_i - x_i) F(y)\, dS_y + (1-\alpha) \int_H \nabla_x G^{x,y} \cdot \nabla_y P\, dy.$$

One can easily verify that

$$\partial_{n_y} \partial_{x_i} G^{x,y} (y_i - x_i) = \partial_{n_y} G^{x,y}, \quad y \in \partial H.$$

We use this identity in the last equation together with (22) to obtain the following representation formula for $div\, \mathbf{w}$.

$$\beta\, div\, \mathbf{w}(x) = \frac{\alpha}{2} P(x) - (1-\alpha) \int_H \nabla_y \cdot \nabla_x G^{x,y} P\, dy \\ = \frac{2-\alpha}{2} P(x) + (1-\alpha) A\langle P \rangle (x), \quad (25)$$

where we used the definition of A given in lemma 6.

1 Local solutions

1.1 Statement of the result

Definition 2. *Let $L^\infty_\alpha(H)$ be a closure of the space $C^1(H)$ in the norm $[\cdot]_\alpha + |\cdot|_{L^\infty(H)}$*

$$[f]_\alpha = \sup_{x \neq y,\, x \in H, y \in \partial H} \frac{|f(x) - f(y)|}{|x-y|^\alpha}. \quad (26)$$

Let us state a local existence result.

Theorem 2 (Local existence of weak solutions)**.** *Let $\bar\rho > 0$. Let $\mathbf{u}_0 \in W^{1,2}_0(H)$,*

$$\rho_0 - \bar\rho \in L^\infty_\alpha(H) \cap L^2(H) \cap L^4(H) \cap W^{-1,2}(H),$$

$$\rho_0 > m > 0 \quad a.e.\, H,$$

$\alpha \in]0, \frac{1}{4}[$. *Then, there are*

$$T = T(\lambda, \mu, \bar{\rho}, \alpha), \quad c = c(\lambda, \mu, \bar{\rho}, \alpha)$$

and a weak solution, $(\rho, \mathbf{u})$, *of the problem* (1) – (5), *defined for* $t \in [0, T]$ *such that*

$$\left.\begin{array}{l} \rho - \bar{\rho} \in C([0,T] \,:\, L^2(H)), \\ \rho - \bar{\rho} \in L^\infty((0,T) \,:\, W^{-1,2}(H)), \\ \mathbf{u} \in C([0,T] \times H), \\ \mathbf{u} \in L^\infty((0,T) \,:\, W_0^{1,2}(H)), \\ \mathbf{u}_t \in L^2((0,T) \times H), \\ \dot{\mathbf{u}} \in L^\infty((0,T) \,:\, L^2(H)), \\ \sqrt{t}\dot{\mathbf{u}} \in L^2((0,T) \,:\, W_0^{1,2}(H)) \end{array}\right\} \tag{27}$$

and

$$\sup_{[0,T]} \left\{ |\rho(t,\cdot) - \bar{\rho}|_{L^2} + |\rho(t,\cdot) - \bar{\rho}|_{L^4} + |\rho(t,\cdot) - \bar{\rho}|_{L^\infty_\alpha} \right\}$$
$$\leq 2 \left\{ |\rho_0 - \bar{\rho}|_{L^2} + |\rho_0 - \bar{\rho}|_{L^4} + |\rho_0 - \bar{\rho}|_{L^\infty_\alpha} \right\}, \tag{28}$$

for any $t \in [0, T]$,

$$\rho(t, x) > m/2, \quad a.e. \, H.$$

Moreover, $\mathbf{u}$ *is a weak solution of equations:*

$$L\langle \mathbf{u} \rangle = \rho \dot{\mathbf{u}} + \nabla(\rho - \bar{\rho}), \quad on \ (0, T) \times H.$$

To prove the theorem we take $(\rho_0^\epsilon, \mathbf{u}_0^\epsilon)$ such that

$$\rho_0^\epsilon \bar{\rho} \in W^{-1,2} \cap W^{3,2}, \quad \mathbf{u}_0^\epsilon \in W_0^{3,2},$$

$$\rho_0^\epsilon - \bar{\rho} \to \rho_0 - \bar{\rho}, \quad \text{as } \epsilon \to 0$$

in $W^{-1,2} \cap L^2 \cap L^4 \cap L^\infty_\alpha$ and

$$\mathbf{u}_0^\epsilon \to \mathbf{u}_0,$$

in $W^{1,2}$ (all spaces are defined on H). For problem (1)–(5) with the initial conditions

$$(\rho(0, \cdot), \mathbf{u}(0, \cdot)) = (\rho_0^\epsilon(\cdot), \mathbf{u}_0^\epsilon(\cdot))$$

there is a unique, smooth solution, $(\rho^\epsilon, \mathbf{u}^\epsilon)$ that lives on time interval $[0, T_1]$,

$$T_1 = T_1(|\rho_0^e - \bar{\rho}|_{W^{3,2}}, |\mathbf{u}_0^\epsilon|_{W^{3,2}}, \inf \rho_0^e),$$

see for example [16]. We are going to derive *a priori* estimates for $(\rho^\epsilon, \mathbf{u}^\epsilon)$ in norms appearing in theorem 2 on some interval $T = T(\lambda, \mu, \bar{\rho}, \alpha)$. In case $T_1 < T$, local solutions $(\rho^\epsilon, \mathbf{u}^\epsilon)$ can be continued to interval $(0, T)$, retaining its smoothness. Given bounds in spaces appearing in theorem 2, standard functional analytic methods imply that there is a convergent subsequence of $(\rho^\epsilon, \mathbf{u}^\epsilon)$ that has a limit $(\rho, \mathbf{u})$ – the weak solution of (1)–(5) with initial conditions $(\rho_0, \mathbf{u}_0)$, which verifies all the estimates and inclusions of the theorem.

In deriving the estimates we abusively abbreviate $(\rho^\epsilon, \mathbf{u}^\epsilon)$ as $(\rho, \mathbf{u})$.

1.2 1^{st} energy estimate

In all subsequent estimates we assume

Hypothesis 1. *For all* (t, x),

$$\left.\begin{aligned} \rho(t,x) < M := 10\bar{\rho}, \\ \rho(t,x) > m := .1\bar{\rho}. \end{aligned}\right\} \tag{29}$$

Lemma 7. *Let*

$$\Phi(\rho) = \rho \int_{\bar{\rho}}^{\rho} s^{-2}(s - \bar{\rho})\, ds, \quad \rho \geq 0$$

and

$$E(t) = \int_H \rho(t,\cdot)|\mathbf{u}(t,\cdot)|^2/2 + \Phi(\rho(t,\cdot)).$$

Then, for any smooth solution $(\rho, \mathbf{u})$ *of the problem* (1)–(5) *the following equality holds.*

$$E(t) + \int_0^t \int_H (\lambda + 2\mu)|\,div\,\mathbf{u}(t,\cdot)|^2 + \mu|\,curl\,\mathbf{u}(t,\cdot)|^2 = E(0). \tag{30}$$

The proof of this Lemma is well-known and can be found, for example, in [11].

1.3 L^4 estimate on the density

From this point on we will use the following generic notation for the quantities that appear in estimates.

$$I_1 = I_1\left(\lambda, \mu, \bar\rho, m, M, |\rho_0-\bar\rho|_2, |\rho_0-\bar\rho|_4, |\mathbf{u}_0|_2, |\nabla \mathbf{u}_0|_2\right), \tag{31}$$

such that $I_1 \to 0$, as

$$|\rho_0-\bar\rho|_2 + |\rho_0-\bar\rho|_4 + |\mathbf{u}_0|_2 + |\nabla \mathbf{u}_0|_2 \to 0.$$

We allow I_1 to change from line to line.

From the system of equations (15)–(16) one can derive that

$$\dot\rho = \, div\, L^{-1}\langle \rho \dot{\mathbf{u}}\rangle + \, div\, L^{-1}\langle \nabla(\rho-\bar\rho)\rangle.$$

We multiply it by $4sign(\rho-\bar\rho)|\rho-\bar\rho|^3$ and derive the following inequality

$$\begin{aligned}\frac{d}{dt}\int \rho|\rho-\bar\rho|^4 + 4\kappa\int \rho|\rho-\bar\rho|^4 \le 4M^{\frac14}|\, div\, L^{-1}\langle \rho\dot{\mathbf{u}}\rangle|_4 |\rho^{1/4}(\rho-\bar\rho)|_4^3 \\ + 4M^{\frac14}|\, div\, L^{-1}\langle \nabla P\rangle - \kappa P|_4 |\rho^{1/6}\rho-\bar\rho|_4^3,\end{aligned} \tag{32}$$

where $\kappa > 0$ will be chosen later. We set $\mathbf{v} = L^{-1}\langle \rho\dot{\mathbf{u}}\rangle$. An estimate of lemma 3 and (16) imply that

$$|\nabla \mathbf{v}|_2 \le c|\rho\dot{\mathbf{u}}|_{W^{-1,2}} \le c\left\{|\nabla \mathbf{u}|_2 + |\rho-\bar\rho|_2\right\}. \tag{33}$$

With the notation $\mathbf{w} = L^{-1}\langle \nabla P\rangle$, formula (25), lemma 6 and $\kappa = \frac{2-a}{2\beta}$ we can estimate

$$4M^{\frac14}|\, div\, L^{-1}\langle\nabla P\rangle - \kappa P|_2 \le c_0(4)\frac{1-\alpha}{\beta}\left[\frac{M}{m}\right]^{\frac14} |\rho^{1/4}(\rho-\bar\rho)|_4, \tag{34}$$

$$c_0(p) = \frac12[\cot\frac{\pi}{8}]^2.$$

Moreover, by lemma 1, lemma 4 and (33)

$$|\nabla \mathbf{v}|_4^4 \le |\nabla \mathbf{v}|_2^2 |D^2\mathbf{v}|_2^2 \le |\nabla \mathbf{v}|_2^2 |\dot{\mathbf{u}}|_2^2 \le (|\nabla \mathbf{u}|_2 + |\rho-\bar\rho|_2)^2 |\dot{\mathbf{u}}|_2^2 \le I_1|\dot u|_2^2. \tag{35}$$

Substituting the estimates (34) and (35) in (32) and postulating

Hypothesis 2. *$M > m$ are chosen to verify the inequality*

$$\frac{2-\alpha}{2\beta} - c_0(4)\frac{1-\alpha}{\beta}\left[\frac{M}{m}\right]^{\frac{1}{4}} > 0,$$

we get

$$\sup_{[0,t]} \int \rho|\rho-\bar{\rho}|^4 + c\int_0^t \int \rho|\rho-\bar{\rho}|^4 \leq \int \rho_0|\rho_0-\bar{\rho}|^4 + I_1\int_0^t |\dot{\mathbf{u}}|_2^2. \tag{36}$$

Note that Hypothesis 2 is not void if

$$\frac{2-\alpha}{2\beta} - c_0(4)\frac{1-\alpha}{\beta} > 0,$$

which in turn true if (recall formulas (24))

$$\frac{\lambda}{\mu} > [\cot(\pi/8)]^2 - 3.$$

1.4 Energy estimates of higher order

The following lemmas are proved in exactly the same way as the corresponding estimates in Section 2 of [10].

Lemma 8. *Under Hypothesis 1, for $t > 0$ it holds:*

$$\sup_{s\in[0,t]} \int_H |\nabla\mathbf{u}(s,\cdot)|^2 + \int_0^t \int_H |\dot{\mathbf{u}}(s,\cdot)|^2 \leq c(|\nabla\mathbf{u}_0|_2, |\mathbf{u}_0|_2, |\rho_0-\bar{\rho}|_2) + \int_0^t \int_H |\nabla\mathbf{u}|^3.$$

Corollary 1. *For any $\epsilon > 0$ there is a C_ϵ such that*

$$\begin{aligned}\sup_{s\in[0,t]} |\nabla\mathbf{u}(s,\cdot)|_2^2 + |\rho(s,\cdot)-\bar{\rho}|_4^4 + \int_0^t |\dot{\mathbf{u}}(s,\cdot)|_2^2 + |\rho(s,\cdot)-\bar{\rho}|_4^4\, ds \\ \leq c(|\nabla\mathbf{u}_0|_2, |\mathbf{u}_0|_2, |\rho_0-\bar{\rho}|_2, |\rho_0-\bar{\rho}|_4) + \epsilon\int_0^t |\dot{\mathbf{u}}(s,\cdot)|_2^2\, ds \\ + C_\epsilon \int_0^t \left\{|\nabla\mathbf{u}(s,\cdot)|_2^6 + |\nabla\mathbf{u}(s,\cdot)|_2^2 + |\nabla\mathbf{u}(s,\cdot)|_2^3\right\}.\end{aligned} \tag{37}$$

Proof. Indeed, by the Hölder inequality and elliptic estimates of lemmas 3, 5 we have:

$$|\nabla \mathbf{u}|_3^3 \leq |\nabla \mathbf{u}|_2^{3/2} |\nabla \mathbf{u}|_6^{3/2} \leq c |\nabla \mathbf{u}|_2^{3/2} (|\nabla \mathbf{v}|_2^{1/2} |\dot{\mathbf{u}}|_2 + |\rho - \bar{\rho}|_6)^{3/2}, \tag{38}$$

where $\mathbf{v} = L^{-1}\langle \rho \dot{\mathbf{u}} \rangle$ for which the estimate (33) holds. Thus, we can continue the previous estimate:

$$|\nabla \mathbf{u}|_3^3 \leq \epsilon |\dot{\mathbf{u}}|_2^2 + \epsilon |\rho - \bar{\rho}|_4^4 + C_\epsilon \left\{ |\nabla \mathbf{u}|_2^6 + |\nabla \mathbf{u}|_2^4 + |\nabla \mathbf{u}|_2^3 \right\}$$

and the corollary follows from the previous lemma and estimate (36). □

Combining the estimate of lemma 30, (36) with the previous corollary and restricting $|\nabla \mathbf{u}_0|_2 + |\mathbf{u}_0|_2 + |\rho_0 - \bar{\rho}|_2 + |\rho_0 - \bar{\rho}|_4$ in a suitable way we obtain the next estimate.

$$\begin{aligned} \sup_{s \in [0,t]} |\nabla \mathbf{u}(s, \cdot)|_2^2 + |\rho(s, \cdot) - \bar{\rho}|_4^4 + \int_0^t |\dot{\mathbf{u}}(s, \cdot)|_2^2 + |\rho(s, \cdot) - \bar{\rho}|_4^4 \, ds \\ \leq c(|\nabla \mathbf{u}_0|_2, |\mathbf{u}_0|_2, |\rho_0 - \bar{\rho}|_2, |\rho_0 - \bar{\rho}|_4, M, m, \lambda, \mu) = I_1. \end{aligned} \tag{39}$$

Lemma 9. *For $\sigma(t) = \min\{1, t\}$ it holds:*

$$\begin{aligned} \sup_{s \in [0,t]} \sigma(s) \int_H |\dot{\mathbf{u}}|^2 + \int_0^t \int_H \sigma(s) |\nabla \dot{\mathbf{u}}(s, \cdot)|^2 \leq c + \int_0^t \int_H |\nabla \mathbf{u}(s, \cdot)|^3 \\ + \int_0^t \int_H \sigma(s) |\nabla \mathbf{u}(s, \cdot)|^4, \end{aligned}$$

with $c = c(|\nabla \mathbf{u}_0|_2, |\mathbf{u}_0|_2, |\rho_0 - \bar{\rho}|_2, |\rho_0 - \bar{\rho}|_4)$.

Corollary 2. *For any $\epsilon > 0$ there is C_ϵ such that*

$$\sup_{s \in [0,t]} \sigma(s) \int_H |\dot{\mathbf{u}}|^2 + \int_0^t \int_H \sigma(s) |\nabla \dot{\mathbf{u}}(s, \cdot)|^2 \leq I_1 \tag{40}$$

Proof. Using (18), (5), (35) and (36) we obtain

$$|\nabla \mathbf{u}|_4^4 \leq c |\nabla \mathbf{v}|_4^4 + c |\rho - \bar{\rho}|_4^4 \leq I_1 |\dot{u}|_2^2. \tag{41}$$

Integrating in time and using the estimate of corollary 1 we conclude.

□

1.5 Proof of the local existence theorem

Now, we investigate the regularity of $\mathbf{v}$ that was defined in (18).

Lemma 10. *For any $s > 0$ it holds*

$$|\nabla\mathbf{v}(s,\cdot)|_\infty \le ca_1(s),$$

$$\langle\nabla\mathbf{v}(s,\cdot)\rangle_{C^\alpha(H)} \le ca_1(s),\ p > 2,$$

where

$$a_1(s) = |\rho(s,\cdot)\dot{\mathbf{u}}(s,\cdot)|_p + |\nabla\mathbf{u}(s,\cdot)|_2 + |\rho(s,\cdot) - \bar\rho|_2 \tag{42}$$

with c depending on the same parameters as the constant in (40). *Moreover, for any $t - s > 1, s > 0,\ \epsilon > 0$ it holds*

$$\int_s^t a_1(\tau)d\tau \le c(\epsilon)I_1 + (\epsilon + I_1)(t-s). \tag{43}$$

Moreover, for all $t > 0,\ a > 0:$

$$e^{-at}\int_0^t e^{as}a_1(s) \le c(a, M, m, \lambda, \mu, |\nabla\mathbf{u}_0|_2, |\mathbf{u}_0|_2, |\rho_0 - \bar\rho|_2, |\rho_0 - \bar\rho|_4) = I_1. \tag{44}$$

For

$$1 < \omega < \frac{2p}{2p-2},$$

$$\int_s^t |\rho\dot{\mathbf{u}}|_p^\omega \le C(t,\omega) + I_1. \tag{45}$$

Proof. The proof goes by the classical embedding lemmas 1, 2. Indeed, for any $x \in H$ we have

$$|\nabla\mathbf{v}(x) - (\nabla\mathbf{v})_{B_1(x)\cap H}| \le c|D^2\mathbf{v}|_{p,B_1(x)\cap H},$$

$$\langle\nabla\mathbf{v}\rangle_{C^\alpha(H)} \le c(p)|D^2\mathbf{v}|_p,\ p > 2,$$

and

$$|(\nabla\mathbf{v})_{B_1(x)\cap H}|_{L^\infty} \le c|\nabla\mathbf{v}|_{2,B_1(x)\cap H}.$$

On the other hand, elliptic estimates of lemma 4 imply (note, that it follows from the energy estimates (7) and the equation (16) that $|\rho\dot{\mathbf{u}}(s,\cdot)|_{W^{-1,2}} \le |\nabla\mathbf{u}(s,\cdot)|_2 + |\rho(s,\cdot) - \bar\rho|_2$)

$$|\nabla\mathbf{v}|_2 \le c|\rho\dot{\mathbf{u}}|_{W^{-1,2}} \le c(|\nabla\mathbf{u}|_2 + |\rho - \bar\rho|_2),$$

$$|D^2\mathbf{v}|_{p,B_1(x)\cap H} \le c\left(|\rho\dot{\mathbf{u}}|_{p,B_2(s)} + |\rho\dot{\mathbf{u}}|_{W^{-1,2}}\right) \le c\left(|\rho\dot{\mathbf{u}}|_p + |\nabla\mathbf{u}|_2 + |\rho-\bar\rho|_2\right),$$

$$|D^2\mathbf{v}|_p \le c(p)|\rho\dot{\mathbf{u}}|_p.$$

The first statement of the lemma follows. To prove the second we notice that

$$|\dot{\mathbf{u}}|_p \le |\dot{\mathbf{u}}|_2^{2/p}|\nabla\dot{\mathbf{u}}|_2^{(p-2)/p}$$

and for $t-s>1$, $s>0$,

$$\int_s^t |\rho\dot{\mathbf{u}}|_p \le c\int_s^t |\dot{\mathbf{u}}|_2^{2/p}|\nabla\dot{\mathbf{u}}|_2^{(p-2)/p}\sigma^{(p-2)/2p}\sigma^{-(p-2)/2p} \le c\int_s^t \left\{\epsilon^{-1}|\dot{\mathbf{u}}|_2^2 + \epsilon^{-1}\sigma|\nabla\dot{\mathbf{u}}|_2^2\right\} + \int_s^t \epsilon\sigma^{-(p-2)/p} \le c(\epsilon)I_1 + \epsilon(t-s) \quad (46)$$

and

$$\int_s^t |\dot{\mathbf{u}}|_2 + |\rho-\bar\rho|_2 \le I_1(t-s).$$

In the same way one can easily verify that for

$$1<\omega<\frac{2p}{2p-2},$$

$$\int_s^t |\rho\dot{\mathbf{u}}|_p^\omega \le c\int_s^t \left\{|\dot{\mathbf{u}}|_2^2 + \sigma|\nabla\dot{\mathbf{u}}|_2^2\right\} + \int_s^t \sigma^{-\frac{(p-2)\omega}{p(2-\omega)}} \le C(t,\omega,p) + I_1.$$

□

Now, we turn to $\mathbf{w}$ part of the velocity, see (18). We state the following lemmas without proof. Proofs can be found in [18], lemmas 13, 14.

Lemma 11. *For any $\alpha\in\,]0,1[$ and $\delta>0$ there are $c>0$, $c_\delta>0$, independent of $(\lambda,\,\mu,\,\bar\rho,\,T)$, such, that*

$$[A\langle\rho-\bar\rho\rangle]_\alpha \le c\left(|\rho-\bar\rho|_2 + [\rho-\bar\rho]_\alpha\right),$$

and for any $x\in H$,

$$|A\langle\rho-\bar\rho\rangle(x)| \le \delta[\rho-\bar\rho]_\alpha + c_\delta|\rho-\bar\rho|_2.$$

The flow generated by the velocity field $\mathbf{u} = \mathbf{v} + \mathbf{w}$ is Lipschitz at the boundary of H as it is described in the next lemma. Let $X_1^s = X^t(x_1)$, $X_2^s = X^t(x_2)$ and $x_1 \in \partial H$, $x_2 \in H$.

Lemma 12. *There is* I_1*, as defined in* (31)*, for which*

$$|X_1^s - X_2^s| \leq |x_1 - x_2|(1 + I_1)e^{(1+I_1+\sup_{[0,T]}[\rho(t,\cdot)-\bar{\rho}]_\alpha)s}, \quad s \in]0, T[.$$

Integrating equation (1) along trajectories and using the representation (18) and (25) we derive

$$\log \rho(t, X_i^t) - \log \rho_0(x_i) + \frac{1}{\lambda + 2\mu} \int_0^t (\rho(s, X_i^s) - \bar{\rho}) = \int_0^t div\, \mathbf{v}(s, X_i^s)$$
$$+ \frac{2\mu}{\lambda + 3\mu} \int_0^t A\langle \rho(s) - \bar{\rho} \rangle (X_i^s), \quad i = 1, 2.$$

Combining lemma 10, lemma 11 and lemma 12 we can derive, similarly as it was done in [18], section 0.12, that for $0 < t < T < 1$,

$$|\rho(t, \cdot) - \bar{\rho}|_{L_\alpha^\infty} \leq c|\rho_0 - \bar{\rho}|_{L_\alpha^\infty} + I_1 + (1 + I_1) \int_0^T |\rho(s, \cdot) - \bar{\rho}|_{L_\alpha^\infty}, \tag{47}$$

where I_1 was defined in (31). The estimates of the theorem follow when we suitably restrict I_1, (i.e., $|\mathbf{u}_0|_{W^{1,2}}, |\rho_0 - \bar{\rho}|_{L^2}, |\rho_0 - \bar{\rho}|_{L^4}$) and T.

2 Auxiliary problem

2.1 Statement of the result

Let $(\tilde{\rho}, \tilde{u})$ be solution constructed in the Theorem 2, with the initial conditions $(\rho_0, \mathbf{u}_0)$ defined in section 0.2. Note, that such $(\rho_0, \mathbf{u}_0)$ also verify the requirements of Theorem 2 since it holds that

$$|\rho_0 - \bar{\rho}|_{L_\alpha^\infty} \leq c(|\Gamma_0|_{C^{1+\alpha}})(|\rho_0 - \bar{\rho}|_{W^{1,p}(\Omega_0^\pm)} + \sup_{x \in \Gamma_0} |d_x^{-\beta}|[\rho_0](x)|).$$

Consider the following system of equations.

$$L\langle \mathbf{u} \rangle = \tilde{\rho}\dot{\tilde{u}} + \nabla(\rho - \bar{\rho}), \tag{48}$$

$$\rho_t + \, div\,(\rho \mathbf{u}) = 0, \tag{49}$$

$$\rho(0,\cdot) = \rho,\, \mathbf{u}(t,x)\big|_{\partial H} = 0. \tag{50}$$

We prove the following lemma. Note, that it mostly repeats the statement of theorem 1.

Lemma 13. *There is $p_0 = p_0(\lambda, \mu)$, $p_0 \in (2, 8/3)$, such that for any $p \in (2, p_0]$, $\alpha = 1 - 2p^{-1}$, any $\beta > p$ and $\Gamma_0 \in C^{1+\alpha}$ defined by function ϕ_0 there is*

$$c = c(\lambda, \mu, \beta, p_0, \bar{\rho}, a, |\nabla \phi_0|_{C^\alpha(H)}, |\nabla \phi_0|_{\inf})$$

such that if

$$I[\rho_0, \mathbf{u}_0] < c,$$

see (7), then, there is a weak solution $(\rho, \mathbf{u})$ of (48)–(50). Moreover,

$$\begin{aligned} &\nabla \mathbf{u} \in L^2((0,T) \,:\, L^2(H)), \\ &\rho - \bar{\rho} \in L^2((0,T) \,:\, W^{-1,2}(H)), \end{aligned} \tag{51}$$

Also, the flow $X^t(x)$ is uniquely defined and satisfies the following estimates.

$$\sup_{t\in[0,T]} |X^t(x_1) - X^t(x_2)| \le C(T)|x_1 - x_2|,$$

$$\sup_{x\in H} |X^{t_1}(x) - X^{t_2}(x)| \le C(T)|t_1 - t_2|^{\gamma(p)},\ t_1 < t_2 < T,$$

for some $C(T)$ and $0 < \gamma(p) < 1$.

For any $t \ge 0$, $\rho(t,\cdot)$ has a jump discontinuity across interface $\Gamma_t = X^t[\Gamma_0]$ which is of the class $C^{1+\alpha}$. Contact angles

$$\theta_t(A), \theta_t(B) \in (0, \pi),\ t > 0.$$

There is c_1, independent of T such that

$$\int_0^T |\nabla \mathbf{u}|_\infty^{\omega(p)} \le c_1 T,$$

$$\sin \theta_t(x_0) \ge \, \sin \theta_0(x_0) e^{-c_1 T},\ x_0 = A, B,$$

$$\sup_{t\in[0,T]} |\Gamma_t|_{C^{1+\alpha}} \le |\Gamma_0|_{C^{1+\alpha}} e^{c_1 T}.$$

Additionally, there is

$$c_2 = c_2(\lambda, \mu, \bar{\rho}, \alpha, \beta, |\Gamma_0|_{C^{1+\alpha}}, |\rho_0 - \bar{\rho}|_2, |\rho_0 - \bar{\rho}|_4, |\rho_0 - \bar{\rho}|_{W^{1,p}(\Omega_0^{\pm})}),$$

non decreasing in $|\rho_0 - \bar{\rho}|_2, |\rho_0 - \bar{\rho}|_4, |\rho_0 - \bar{\rho}|_{W^{1,p}(\Omega_0^{\pm})}$ *and such that* $c_2 \to 0,$ *as*

$$|\rho_0 - \bar{\rho}|_2 + |\rho_0 - \bar{\rho}|_4 + |\rho_0 - \bar{\rho}|_{W^{1,p}(\Omega_0^{\pm})} \to 0,$$

and

$$\sup_{[0,T]} \left\{ |\rho(t,\cdot) - \bar{\rho}|_2 + |\rho(t,\cdot) - \bar{\rho}|_4 + |\rho(t,\cdot) - \bar{\rho}|_{W^{1,p}(\Omega_t^{\pm})} \right\} \leq c_2. \tag{52}$$

To prove this lemma, first, we derive a priori estimates, assuming that solution with needed regularity properties exists. Then, given these estimates, a solution $(\rho, \mathbf{u})$ can constructed by a suitable approximation procedure, as it was done in lemmas 3.1, 4.1, 4.2 of [11]. This step carries over to our problem with one modification that we now mention. Lamé equations

$$L\langle \mathbf{u} \rangle = \tilde{\rho}\dot{\tilde{u}} + \nabla(\rho - \bar{\rho}),$$

can be written as Poisson's equations

$$\mu \Delta \mathbf{u} = \tilde{\rho}\dot{\tilde{u}} + \nabla(\rho - \bar{\rho}) - (\lambda + \mu)\, div\, \mathbf{u}.$$

Using representation formulas (25) the equations become

$$\mu \Delta \mathbf{u} = \tilde{\rho}\dot{\tilde{u}} + \gamma_1 \nabla(\rho - \bar{\rho}) + \gamma_2 A\langle \rho - \bar{\rho} \rangle,$$

for some constants γ_i. Now, if $G^{x,y}$ is a Green's function for $\Delta \mathbf{u}$ on H, problem (48)–(50) can be approximated by the problem

$$\mu \mathbf{u}^a = (G^{x,y} \nu \left(\frac{|x-y|}{a} \right) * \left\{ \tilde{\rho}\dot{\tilde{u}} + \gamma_1 \nabla(\rho - \bar{\rho}) + \gamma_2 A\langle \rho - \bar{\rho} \rangle \right\},$$

$$\rho_t + \, div\, (\rho \mathbf{u}^a) = 0,$$

$$\rho(0,x) = \rho_0,$$

where ν is a standard molifier, see [11] for details.

2.2 A note on notation

We assume the following hypothesis.

Hypothesis 3.

$$\left.\begin{array}{c}|\nabla\rho(t)|_{W^{1,p}(\Omega_t^\pm)} \le 4|\nabla\rho_0|_{W^{1,p}(\Omega_0^\pm)} =: M_p, \\[2ex] e^{-\varepsilon t}\sup_{[0,t]}|\widehat{\mathbf{W}}(t)|_{C^\alpha\left(\Omega_t^-\right)} \le 4|\nabla\phi_0|_{C^\alpha\left(\Omega_0^-\right)}, \\[2ex] e^{\varepsilon t}\inf_{[0,t]}|\widehat{\mathbf{W}}(t)|_{\mathrm{inf}} \ge \frac{1}{4}|\nabla\phi_0|_{\mathrm{inf}}, \\[2ex] \int_0^t|\nabla\mathbf{u}(t)|_\infty \le \varepsilon t,\end{array}\right\}$$

for some $\varepsilon > 0$ and all $t > 0$. $\widehat{\mathbf{W}}$ is determined in (99).

We will use the following generic notation to write estimates throughout the paper.

$$\begin{gathered} c = c(\lambda,\mu,p,\beta,|\Omega_0|,|\Gamma_0|,m,M,M_p), \\ c_\delta = c(\delta,\lambda,\mu,p,\beta,|\Omega_0|,|\Gamma_0|,m,M,M_p), \\ c^*(t) = \sum_1^N c_k \inf_{[0,t]}|\widehat{\mathbf{W}}|_{inf}^{-m_k}\sup_{[0,t]}|\widehat{\mathbf{W}}|_{\alpha,\Omega_t^-}^{n_k}|\Omega_t^-|^{l_k}, \\ c_k = c_k(\lambda,\mu,p,\beta,|\Omega_0|,|\Gamma_0|,m,M,M_p,|\nabla\phi_0|_{\alpha,\Omega_0^-},|\nabla\phi_0|_{inf}), \end{gathered}$$

where $N, l_k, m_k, n_k \ge 0$ are functions of (p,β,λ,μ).

We denote by

$$\begin{aligned} I_2 = I_2\Big(&\lambda,\mu,\bar\rho,\beta,p,m,M,M_p,|\phi_0|_\alpha,|\phi_0|_{\mathrm{inf}},|\rho_0-\bar\rho|_2,|\rho_0-\bar\rho|_{W^{1,p}(\Omega_0^\pm)}, \\ &\sup_{x\in\Gamma_0}|[\rho_0](x)d_x^{-\beta}|,|\mathbf{u}_0|_2,|\nabla\mathbf{u}_0|_2\Big), \end{aligned} \tag{53}$$

such that $I_2 \to 0$, as

$$|\rho_0-\bar\rho|_2 + |\rho_0-\bar\rho|_{W^{1,p}(\Omega_0^\pm)} + \sup_{x\in\Gamma_0}|[\rho_0](x)d_x^{-\beta}| + |\mathbf{u}_0|_2 + |\nabla\mathbf{u}_0|_2 \to 0.$$

We allow I_2 to change from line to line.

The velocity $\mathbf{u}$ is split just it was done in (18):

$$\mathbf{u} = \mathbf{v} + \mathbf{w}, \tag{54}$$

$$\mathbf{v} = L^{-1}\langle \tilde{\rho}\dot{\tilde{u}}\rangle, \quad \mathbf{w} = L^{-1}\langle \rho - \bar{\rho}\rangle.$$

Moreover, $\mathbf{v}$ verifies the estimates of lemma (10).

2.3 Some estimates on $[\rho]$.

Define $[\rho] = \rho^+ - \rho^-$ on Γ_t. The time evolution of $[\rho]$ can be obtained from the equation of continuity and Rankin-Hugoniot conditions. Indeed, conditions (10) used in (15) imply that

$$\frac{d}{dt}[\log \rho] + \frac{1}{\lambda + 2\mu}[\rho - \bar{\rho}] = 0,$$

on Γ_t. Upon the integration of the above equation we obtain the identity

$$[\rho](t,x) = [\rho_0](X^{-t}(x)) exp\{-\frac{1}{\lambda + 2\mu}\int_0^t \frac{[\rho]}{[\log \rho]}(\tau, X^{-t+\tau})\, d\tau\} \tag{55}$$

and we derive the estimate

$$|[\rho](t,x)| \leq \frac{M}{m}|[\rho_0]|(X^{-t}(x))e^{-\frac{m}{\lambda+2\mu}t}, \tag{56}$$

where M, m are defined in Hypothesis (1). From the last estimate we can derive the following one:

$$\begin{aligned}\sup_{y\in\Gamma_t}|[\rho]d_y^{-\beta}| &\leq \frac{M}{m}\sup_{x\in\Gamma_0}|[\rho_0]d_x^{-\beta}|\sup_{y\in\Gamma_t}\left(\frac{d_{X^{-t}(y)}}{d_y}\right)^\beta e^{-\frac{m}{\lambda+2\mu}t} \\ &\leq c(M/m)\sup_{x\in\Gamma_0}|[\rho_0]d_x^{-\beta}|e^{-at+\beta\int_0^t|\nabla\mathbf{u}|_\infty},\end{aligned} \tag{57}$$

where $a = \frac{m}{\lambda+2\mu}$. In a similar way we obtain:

$$\begin{aligned}\int_{\Gamma_t}[\rho](y)d_y^{-2+2/p}dS_y &\leq \frac{M}{m}\sup_{x\in\Gamma_0}|[\rho_0]d_x^{-\beta}|e^{-at+(2-2/p)\int_0^t|\nabla\mathbf{u}|_\infty}\int_{\Gamma_t}d_{X^{-t}(y)}^{\beta-2+2/p}dS_y \\ &\leq \frac{M}{m}\sup_{x\in\Gamma_0}|[\rho_0]d_x^{-\beta}|e^{-at+(3-2/p+)\int_0^t|\nabla\mathbf{u}|_\infty}\int_{\Gamma_0}d_x^{\beta-2+2/p}dS_x \\ &\leq c(|\Gamma_0|_{1+\alpha}, p, \beta)\sup_{x\in\Gamma_0}|[\rho_0]d_x^{-\beta}|e^{-at+c(p)\int_0^t|\nabla\mathbf{u}|_\infty},\end{aligned} \tag{58}$$

provided that $\beta > 1 - \frac{2}{p}$. Moreover, we obtain the estimate on the $W^{s,p}(\Gamma_t)$ norm of $[\rho]$. Proofs of the following three lemmas can be found in the Appendix.

Lemma 14. *Under Hypotheses 1–3 there is I_2 with the properties defined in* (53) *such that*

$$|[\rho](t,\cdot)|_{W^{s,p}(\Gamma_t)} \leq I_2 e^{-3at/4}. \tag{59}$$

$[\rho]$ can be extended to Ω_t^- as an element of $W^{1,p}(\Omega_t^-)$ for which we keep the same notation $[\rho]$.

Lemma 15. *Let $[\rho] \in W^{s,p}(\Gamma_t)$ and vanishes at points of $\Gamma_t \cap \partial H$. Then, there is an extension $[\rho] \in W^{1,p}(\Omega_t^-)$, such that*

$$|[\rho](t,\cdot)|_{W^{1,p}(\Omega_t^-)} \leq \tilde{c}^* |[\rho](t,\cdot)|_{W^{s,p}(\Gamma_t)},\ s = 1 - p^{-1}. \tag{60}$$

The classical imbedding lemma 2 can be applied to obtain a C^α estimate on $[\rho]$.

Lemma 16. *For $p > 2$ there is c^* such that*

$$|[\rho](t,\cdot)|_{\infty,\Omega_t^-} \leq c^* |[\rho](t,\cdot)|_{W^{1,p}(\Omega_t^-)} \tag{61}$$

and

$$|[\rho](t,\cdot)|_{C^\alpha(\Omega_t^-)} \leq c^* |[\rho](t,\cdot)|_{W^{1,p}(\Omega_t^-)}, \quad \alpha = 1 - 2p^{-1}. \tag{62}$$

Combining three previous lemmas we get

Corollary 3. *There is c^* such that*

$$|[\rho](t,\cdot)|_\infty + |[\rho](t,\cdot)|_{C^\alpha(\Omega_t^-)} + |[\rho](t,\cdot)|_{W^{1,p}(\Omega_t^-)} \leq c^* I_2 e^{-3at/4 + c\varepsilon t}, \tag{63}$$

where $s = 1 - p^{-1}$, $a = m(\lambda + 2\mu)^{-1}$.

2.4 Some SIO – type representation formulas and estimates

Let $\omega = \frac{\gamma_1}{\mu}[\rho]$ be an element of $W^{1,p}(\Omega_t^-)$ extended by 0 to the whole half-space H. Let

$$\mathbf{w}_1 = \int \nabla_y G^{x,y} \omega(y), \quad \mathbf{w}_1 = \langle w_{1,1},\, w_{1,2} \rangle.$$

Then, $\mathbf{w}_1$ is weakly differentiable and a.e. $x \in H$

$$\partial_{x_j} w_{1,i} = pv \int \partial_{x_j} \partial_{y_i} G^{x,y} \omega(y) - \frac{1}{2}\delta_{ij}\omega(x). \tag{64}$$

From this we obtain by integrating by parts:

$$\partial_{x_j} w_{1,i} = \int_{\Gamma_t} \partial_{x_j} G^{x,y} \omega(y) N_{y_i} dS_y - \int_{\Omega_t^-} \partial_{x_j} G^{x,y} \partial_{y_i} \omega(y). \tag{65}$$

Let $\mathbf{W} \in C^{\alpha}\left(\Omega_t^-\right)$, $div\, \mathbf{W} = 0$, $\mathbf{W}(x)$ tangent to Γ_t at point $x \in \Gamma_t$ and $\widehat{\mathbf{W}} = \mathbf{W}\gamma$, $\gamma > 0$, $\gamma \in W^{1,p}\left(\Omega_t^-\right)$. For $x \in \Omega_t^-$ consider

$$\begin{aligned}
pv \int \partial_{x_j}\partial_{y_i} G^{x,y} \widehat{W}_j(y)\omega(y) = &- pv \int \partial_{y_j}\partial_{y_i} G^{x,y} \widehat{W}_j(y)\omega(y) \\
&+ \int (\partial_{x_j} + \partial_{y_j})\partial_{y_i} G^{x,y} \widehat{W}_j(y)\omega(y) = pv \int \partial_{y_j}\partial_{x_i} G^{x,y} \widehat{W}_j(y)\omega(y) \\
&- \int \partial_{y_j}(\partial_{y_i} + \partial_{x_i}) G^{x,y} \widehat{W}_j(y)\omega(y) \\
&+ \int (\partial_{x_j} + \partial_{y_j})\partial_{y_i} G^{x,y} \widehat{W}_j(y)\omega(y) \\
= \frac{1}{2}\delta_{ij}\widehat{W}_j(x)\omega(x) &- \int \partial_{x_i} G^{x,y} \widehat{\mathbf{W}}(y) \cdot \nabla_y \omega(y) - \int \partial_{x_i} G^{x,y} \widehat{\mathbf{W}} \cdot \nabla_y \ln\gamma(y)\omega(y) \\
&- \int \partial_{y_j}(\partial_{y_i} + \partial_{x_i}) G^{x,y} \widehat{W}_j(y)\omega(y) \\
&+ \int (\partial_{x_j} + \partial_{y_j})\partial_{y_i} G^{x,y} \widehat{W}_j(y)\omega(y).
\end{aligned} \tag{66}$$

Thus, we obtain:

$$\begin{aligned}
\nabla w_{1,i}(x) \cdot \widehat{\mathbf{W}}(x) = &\int \partial_{y_i} \nabla_x G^{x,y} \cdot (\widehat{\mathbf{W}}(x) - \widehat{\mathbf{W}}(y))\omega(y) \\
&- \int_{\Omega_t^-} \partial_{x_i} G^{x,y} \widehat{\mathbf{W}}(y) \cdot \nabla_y \omega(y) - \int_{\Omega_t^-} \partial_{x_i} G^{x,y} \widehat{\mathbf{W}}(y) \cdot \nabla_y \ln\gamma(y)\omega(y) \\
&- \int \partial_{y_j}(\partial_{y_i} + \partial_{x_i}) G^{x,y} \widehat{W}_j(y)\omega(y) \\
&+ \int (\partial_{x_j} + \partial_{y_j})\partial_{y_i} G^{x,y} \widehat{W}_j(y)\omega(y).
\end{aligned} \tag{67}$$

For any i, j it holds

$$\partial_{y_j}(\partial_{y_i} + \partial_{x_i}) G^{x,y} = 2\delta_{2i}\partial_{y_i}\partial_{y_j} H^{x,y^*}.$$

It follows that

$$
\begin{aligned}
&-\int \partial_{y_j}(\partial_{y_i}+\partial_{x_i})G^{x,y}\widehat{W}_j(y)\omega(y)+\int(\partial_{x_j}+\partial_{y_j})\partial_{y_i}G^{x,y}\widehat{W}_j(y)\omega(y)\\
&\qquad\qquad=\int 2\partial_{y_1}\partial_{y_2}H^{x,y^*}\widehat{W}_i^{\perp}(y)\omega(y)\\
&\quad=\int 2\partial_{y_1}\partial_{y_2}H^{x,y^*}(\widehat{W}_i^{\perp}(y)-\widehat{W}_i^{\perp}(x))\omega(y)+\widehat{W}_i^{\perp}(x)\int 2\partial_{y_1}\partial_{y_2}H^{x,y^*}\omega(y)\\
&\qquad\qquad=\int 2\partial_{y_1}\partial_{y_2}H^{x,y^*}(\widehat{W}_i^{\perp}(y)-\widehat{W}_i^{\perp}(x))\omega(y)\\
&\qquad+\widehat{W}_i^{\perp}(x)\int_{\Gamma_t}\partial_{y_2}H^{x,y^*}\omega(y)N_{y_1}dS_y-\widehat{W}_i^{\perp}(x)\int_{\Omega_t^-}\partial_{y_2}H^{x,y^*}\partial_{y_1}\omega(y). \qquad (68)
\end{aligned}
$$

where $\widehat{\mathbf{W}}^{\perp}=\langle\widehat{W}_2,-\widehat{W}_1\rangle$. Finally, substituting this in (67) we obtain

$$
\begin{aligned}
\nabla w_{1,i}(x)\cdot\widehat{\mathbf{W}}(x)&=\int\partial_{y_i}\nabla_x G^{x,y}\cdot(\widehat{\mathbf{W}}(x)-\widehat{\mathbf{W}}(y))\omega(y)\\
&+\int 2\partial_{y_1}\partial_{y_2}H^{x,y^*}(\widehat{W}_i^{\perp}(y)-\widehat{W}_i^{\perp}(x))\omega(y)-\int_{\Omega_t^-}\partial_{x_i}G^{x,y}\widehat{\mathbf{W}}\cdot\nabla_y\omega(y)\\
&-\int_{\Omega_t^-}\partial_{x_i}G^{x,y}\widehat{\mathbf{W}}\cdot\nabla_y\ln\gamma(y)\omega(y)-W_i^{\perp}(x)\int_{\Omega_t^-}\partial_{y_2}H^{x,y^*}\partial_{y_1}\omega(y)\\
&+W_i^{\perp}(x)\int_{\Gamma_t}\partial_{y_2}H^{x,y^*}\omega(y)N_{y_1}dS_y. \qquad (69)
\end{aligned}
$$

Now, we will estimate terms appearing in the above representation formulas. Let, Γ_t^e be a $C^{1+\alpha}$ extension of Γ^t constructed in Appendix and ω^e – the zero extension to $\Gamma_t^e\setminus\Gamma_t$ of ω defined on Γ_t. Then,

$$
|\omega^e|_{C^\alpha(\Gamma_t^e)}+|\omega^e|_{L^\infty(\Gamma_t^e)}\le|\omega|_{C^\alpha(\Gamma_t)}+|\omega|_{L^\infty(\Gamma_t)}
$$

and the following lemma holds.

Lemma 17.

$$
\begin{aligned}
\sup_x\Big|\int_{\Gamma_t}\partial_{x_j}G^{x,y}N_i(y)\omega(y)\Big|&=\sup_x\Big|\int_{\Gamma_t^e}\partial_{x_j}G^{x,y}N_i(y)\omega^e(y)\Big|\\
&\le c(\alpha)\left\{1+\frac{|\widehat{\mathbf{W}}|_\alpha}{|\widehat{\mathbf{W}}|_{inf}}|\Gamma_t|\right\}(|\omega|_{L^\infty(\Gamma_t)}+|\omega|_{C^\alpha(\Gamma_t)}). \qquad (70)
\end{aligned}
$$

Proof. The proof of the Lemma is well-known and contained, for example, in [2]. Here, $\mathbf{N}(y) = \nabla^{\perp}\phi(y)$. $\square$

Let

$$Q_0 = \int_{\Omega_t^-} \partial_{x_i} G^{x,y} \partial_{y_j} \omega(y),$$

$$Q_1 = \int_{\Omega_t^-} \partial_{x_i} H^{x,y^*} \partial_{y_j} \omega(y),$$

$$Q_2 = \int_{\Omega_t^-} \partial_{x_i} G^{x,y} \widehat{\mathbf{W}}(y) \cdot \nabla \ln \gamma(y) \omega(y).$$

Lemma 18. *There is $c(p)$, $p > 2$ such that for $i = 0, 1$,*

$$|Q_i|_\infty \leq c(p)(1 + |\Omega_t^-|^{(p-1)/p}) |\nabla \omega|_{p,\Omega_t^-},$$

$$\langle Q_i \rangle_{C^\alpha(H)} \leq c(p) |\omega|_{W^{1,p}(\Omega_t^-)},\ \alpha = 1 - 2p^{-1}$$

and

$$|Q_2|_\infty \leq c(p)(1 + |\Omega_t^-|^{(p-1)/p}) |\widehat{\mathbf{W}}|_\infty |\omega|_\infty |\nabla \ln \gamma|_{p,\Omega_t^-},$$

$$\langle Q_2 \rangle_{C^\alpha(H)} \leq c(p) |\widehat{\mathbf{W}}|_\infty |\omega|_\infty |\nabla \ln \gamma|_p.$$

Proof. We prove only estimates for Q_0. Others follow from the same arguments. Then, a.e. $x \in H$,

$$\partial_{x_k} Q_0 = pv \int_{\Omega_t^-} \partial_{x_k} \partial_{x_i} G^{x,y} \partial_{y_j} \omega(y) + \frac{1}{2} \delta_{ki} \chi_{\Omega_t^-} \partial_{x_j} \omega.$$

Moreover, for $p > 2$,

$$|\nabla Q_0|_{p,H} \leq c(p) |\nabla \omega|_{p,\Omega_t^-}. \tag{71}$$

Estimates of the lemma follow once we apply lemmas 2,1 and use the fact that ω is supported on Ω_t^-. $\square$

Now, combining estimates two previous lemmas and Hypotheses 1–3 we obtain that

$$|\nabla w_{1,i}|_\infty \leq c(\alpha) \left\{ 1 + \frac{|\nabla \phi|_\alpha}{|\nabla \phi|_{inf}} |\Gamma_t| \right\} (|\omega|_{\infty,\Gamma_t} + |\omega|_{C^\alpha(\Gamma_t)}) + c(p, |\Omega_t^-|) |\omega|_{W^{1,p}(\Omega_t^-)}$$
$$\leq c^*(|\omega|_{\infty,\Gamma_t} + |\omega|_{C^\alpha(\Gamma_t)} + |\omega|_{W^{1,p}(\Omega_t^-)}) \tag{72}$$

Consider now the integral

$$P = \int_{\Gamma_t} \partial_{x_j} H^{x,y^*} \omega(y) N_{y_k} dS_y.$$

Lemma 19. *There are $c(p)$, $p > 2$, and $c_1(\beta)$ such that*

$$\langle P \rangle_{C^\alpha(H)} \leq c(p) |\nabla P|_{p,H} \leq c(p) \left\{ |\omega|_{\infty,\Gamma_t} |\Gamma_t| + c(p) \int_{\Gamma_t} |\omega(y)| d_y^{-2+\frac{2}{p}} dS_y \right\}, \tag{73}$$

$$|P|_\infty \leq c_1(\beta) \sup_{y \in \Gamma_t} |\omega(y) d_y^{-\beta}| \frac{|\widehat{\mathbf{W}}|_\alpha}{|\widehat{\mathbf{W}}|_{\inf}} |\Gamma_t|. \tag{74}$$

Proof. It follows that

$$\nabla P = \int_{\Gamma_t} \nabla_x \partial_{x_j} H^{x,y^*} \omega(y) N_{y_k} dS_y,$$

and the second estimate in the first line follows by estimating the L^p norm of ∇P on B and $(H \setminus B)$, separately, where B is a ball containing Ω_t^-.
The L^∞ estimate follows easily from the inequality

$$|P|_\infty \leq c \sup_{y \in \Gamma_t} |\omega(y) d_y^{-\beta}| \int_{\Gamma_t} \frac{1}{|x - y^*|^{1-\beta}} dS_y.$$

□

To estimate terms

$$R_{1,i}(x) = \int \partial_{y_i} \nabla_x G^{x,y} \cdot (\widehat{\mathbf{W}}(x) - \widehat{\mathbf{W}}(y)) \omega(y)$$

and

$$R_{2,i}(x) = \int 2\partial_{y_1} \partial_{y_2} H^{x,y^*} (\widehat{W}_i^\perp(y) - \widehat{W}_i^\perp(x)) \omega(y)$$

we are going to use the following lemma, see p. 26 of [3].

Lemma 20. *Let K be a Calderon-Zygmund kernel, homogeneous of degree $-n$, with mean zero on spheres, satisfying $|\nabla K| \leq C|x|^{-n-1}$. There is a constant C_0 so that all $f \in C^\alpha(\mathbb{R}^n)$ and $\omega \in L^\infty(\mathbb{R}^n)$ satisfy*

$$|R|_{C^\alpha(\mathbb{R}^n)} \leq C_0(\alpha, n) |f|_\alpha (|K * \omega|_\infty + |\omega|_\infty),$$

where

$$R(x) = pv \int_{\mathbb{R}^n} K(x - y)(f(x) - f(y)) \omega(y) dy.$$

To apply this lemma to terms $R_{j,i}$ we notice that $\partial_{x_i}\partial_{y_j} G^{x,y} = \frac{1}{2\pi}(\partial_{x_i}\partial_{y_j} \log |x - y| - \partial_{x_i}\partial_{y_j} \log |x - y^*| = K_{ij}(x - y) + (-1)^j K_{ij}(x - y^*)$, where K_{ij} as described in the above lemma and

$$\int_H K_{ij}(x - y^*)(\widehat{W}_j(x) - \widehat{W}_j(y))\omega(y) = -\int_{H^*} K_{ij}(x - y)(\widehat{W}_j(x) - \widehat{W}_j(y^*))\omega(y^*)$$

Restricting $(x, y) \to \Omega_t^- \times \Omega_t^-$ we obtain the following estimates.

$$|R_{1,i}|_{\alpha,\Omega_t^-} \leq C_0(\alpha)|\widehat{\mathbf{W}}|_\alpha(|pv\int \partial_{y_i}\nabla_x G^{x,y}\omega(y)|_\infty + |\omega|_\infty)$$

and

$$|R_{2,i}|_{\alpha,\Omega_t^-} \leq C_0(\alpha)|\widehat{\mathbf{W}}|_\alpha(|pv\int 2\partial_{y_1}\partial_{y_2} H^{x,y^*}\omega(y)|_\infty + |\omega|_\infty).$$

On the other hand, by (65),

$$pv\int \partial_{y_i}\partial_{x_j} G^{x,y}\omega(y) = \partial_{x_j} w_{1,i} + \delta_{ij}\frac{1}{2}\omega(x),$$

and

$$pv\int 2\partial_{y_1}\partial_{y_2} H^{x,y^*}\omega(y) = -\int_{\Omega_t^-} \partial_{y_2} H^{x,y^*}\partial_{y_1}\omega(y) + \int_{\Gamma_t} \partial_{y_2} H^{x,y^*}\omega(y) N_{y_1} dS_y.$$

Combining above estimates with lemma 19 and estimates on Q_i we can estimates all terms in (69).

Lemma 21. *There are functions c^* as defined in subsection 2.2 such that*

$$|R_{1,i}|_{C^\alpha(\Omega_t^-)} \leq c^*(|\omega|_\infty + |\omega|_{\alpha,\Gamma_t} + |\omega|_{W^{1,p}(\Omega_t^-)}, \tag{75}$$

$$|R_{2,i}|_{C^\alpha(\Omega_t^-)} \leq c^*(\sup_{y\in\Gamma_t} |\omega(y)| d_y^{-\beta} + |\omega|_{W^{1,p}(\Omega_t^-)}. \tag{76}$$

Finally, we obtain the estimate on $\langle \nabla \mathbf{w}_1 \cdot \mathbf{W} \rangle_{\alpha,\Omega_t^-}$; there is c^* such that

$$\begin{aligned} \langle \nabla \mathbf{w}_1 \cdot \widehat{\mathbf{W}} \rangle_{C^\alpha(\Omega_t^-)} \leq c^*(|\omega|_\infty + |\omega|_{\alpha,\Gamma_t} + |\omega|_{W^{1,p},\Omega_t^-} + \sup_{y\in\Gamma_t} |\omega(y)| d_y^{-\beta} \\ + \int_{\Gamma_t} |\omega(y)| d_y^{-2+\frac{2}{p}} dS_y), \end{aligned} \tag{77}$$

$\alpha = 1 - 2p^{-1}$, $p > 2$. Recall now that $\omega = \frac{\gamma_1}{\mu}[\rho]$, where $[\rho]$ is defined in lemma 15 and γ_1, μ – numbers. Applying the estimates (63) we get

Corollary 4. *There is c^* and c such that*

$$|\nabla \mathbf{w}_1|_\infty + \langle \nabla \mathbf{w}_1 \cdot \widehat{\mathbf{W}} \rangle_{\alpha,\Omega_t^-} \le c^*(|[\rho_0]_{W^{s,p},\Gamma_0} + \sup_{y\in\Gamma_0} |[\rho_0](y)| d_y^{-\beta}) e^{-3at/4}$$
$$\le c^* I_2 e^{-3at/4}, \quad (78)$$

where $a = \frac{m}{\lambda+2\mu}$.

Also, we need estimates on $\nabla A\langle \rho - \bar\rho\rangle$, where A was introduced in lemma 6. We will use symbol $\nabla\rho$ to denote *the absolutely continuous* part of $\nabla\rho$. Consider

$$\begin{aligned}
\partial_{x_i} A\langle \rho-\bar\rho\rangle &= \partial_{x_i}\int \nabla_y\cdot\nabla_x H^{x,y^*}(\rho-\bar\rho)\\
&= -\int \partial_{y_i}\nabla_y\cdot\nabla_x H^{x,y^*}(\rho-\bar\rho) + \int(\partial_{x_i}+\partial_{y_i})\nabla_y\cdot\nabla_x H^{x,y^*}(\rho-\bar\rho)\\
&= -\int_{\partial H}\nabla_y\cdot\nabla_x H^{x,y^*}(\rho-\bar\rho)N_{y_i} + \int_{\partial H}\mathbf{N}_y\cdot(\partial_{x_i}+\partial_{y_i})\cdot\nabla_x H^{x,y^*}(\rho-\bar\rho)\\
&\quad -\int_{\Gamma_t}\nabla_y\cdot\nabla_x H^{x,y^*}[\rho]N_{y_i} + \int_{\Gamma_t}\mathbf{N}_y\cdot(\partial_{x_i}+\partial_{y_i})\cdot\nabla_x H^{x,y^*}[\rho]\\
&\quad +\int_{\Omega_t^+\cup\Omega_t^-}\nabla_y\cdot\nabla_x H^{x,y^*}\partial_{y_i}\rho - \int_{\Omega_t^+\cup\Omega_t^-}(\partial_{x_i}+\partial_{y_i})\nabla_x H^{x,y^*}\cdot\nabla_y\rho. \quad (79)
\end{aligned}$$

And consequently we obtain

$$\partial_{x_1}A\langle\rho-\bar\rho\rangle = -\int_{\Gamma_t} N_{y_1}\nabla_y\cdot\nabla_x H^{x,y^*}[\rho] + \int_{\Omega_t^+\cup\Omega_t^-} 2\partial_{x_2}^2 H^{x,y^*}\partial_{y_1}\rho, \qquad (80)$$

$$\begin{aligned}
\partial_{x_2}A\langle\rho-\bar\rho\rangle = &\int_{\Gamma_t}(-2\partial_{x_2}^2 H^{x,y^*}N_{y_2} + 2\partial_{x_2}\nabla_x H^{x,y^*}\cdot\mathbf{N}_y)[\rho]\\
&-\int_{\Omega_t^+\cup\Omega_t^-} 2\partial_{x_1}\partial_{x_2}H^{x,y^*}\partial_{y_1}\rho. \quad (81)
\end{aligned}$$

Now, for a smooth $f\in C_0^\infty(H)$ and $x\in H$,

$$\int_H \partial_{x_1}\partial_{x_2}H^{x,y^*}f(y) = \int_{\mathbb{R}^2}\frac{1}{2\pi}\frac{2(x_1-y_1)(x_2-y_2)}{|x-y|^4}\chi_{\{x_2-y_2>0\}}f^e(y^*),$$

where f^e is a zero extension of f to $\mathbb{R}^2$. Similarly,

$$\int_H \partial_{x_2}\partial_{x_2}H^{x,y^*}f(y) = \int_{\mathbb{R}^2}\frac{1}{2\pi}\frac{(x_1-y_1)^2-(x_2-y_2)^2}{|x-y|^4}\chi_{\{x_2-y_2>0\}}f^e(y^*).$$

Kernels $K_1(z) = \frac{1}{2\pi}\frac{2z_1z_2}{|z|^4}\chi_{\{z_2>0\}}$ and $K_2(z) = \frac{1}{2\pi}\frac{z_1^2-z_2^2}{|z|^4}\chi_{\{z_2>0\}}$ are Calderon-Zygmund kernels with zero mean on the sphere $|z| = 1$. Moreover, one can easily compute their Fourier Transforms:

$$\widehat{K_1} = \frac{1}{2}\frac{\xi_1\xi_2}{|\xi|^2}, \quad \widehat{K_2} = \frac{1}{4}\frac{\xi_1^2-\xi_2^2}{|\xi|^2}.$$

Lemma 22. *There is a* $c = c(p)$, $1 < p < \infty$, *with the property that* $c \to 1/2$ *as* $p \to 2$ *so that*

$$|\int_{\Omega_t^+\cup\Omega_t^-} \partial_{x_1}\partial_{x_2}H^{x,y^*}\partial_{y_1}\rho|_p \le c|\partial_{y_1}\rho|_{p,\Omega_t^\pm}, \tag{82}$$

$$|\int_{\Omega_t^+\cup\Omega_t^-} \partial_{x_2}\partial_{x_2}H^{x,y^*}\partial_{y_1}\rho|_p \le c|\partial_{y_1}\rho|_{p,\Omega_t^\pm}. \tag{83}$$

2.5 The estimates on $\nabla\rho$ and $\operatorname{osc}\rho$.

Let us consider the continuity equation written in the form

$$\log\rho_t + \mathbf{u}\cdot\nabla\log\rho = -\gamma P + \alpha_0 A\langle\rho-\bar\rho\rangle + \mathit{div}\,\mathbf{v}, \tag{84}$$

$$\gamma^{-1} = \lambda + 2\mu,\ \alpha_0 = \frac{2\mu}{(\lambda+3\mu)(\lambda+2\mu)}.$$

We set $\sigma_k = \partial_{x_k}\log\rho$. We differentiate the equation in x_k and multiply the it by $p\sigma_k|\sigma|^{p-2}\rho$ to derive

$$(\rho|\sigma_k|^p)_t + \mathit{div}\,(\rho|\sigma_k|^p\mathbf{u}) + \gamma p\rho^2|\sigma_k|^p = \alpha_0 p\rho\sigma_k|\sigma_k|^{p-2}\partial_{x_k}A\langle\rho-\bar\rho\rangle + \\ p\rho\sigma_k|\sigma_k|^{p-2}\partial_{x_k}\,\mathit{div}\,\mathbf{v} - p\rho\sigma_k|\sigma_k|^{p-2}\partial_{x_k}\mathbf{u}\cdot\sigma, \tag{85}$$

from which we obtain ($\sigma = \langle\sigma_1,\sigma_2\rangle$):

$$\frac{d}{dt}|\rho^{1/p}\sigma|_p + \gamma m|\rho^{1/p}\sigma|_p \le \alpha_0 M^{1/p}|\nabla A\langle\rho-\bar\rho\rangle|_p + M^{1/p}|\nabla\mathbf{u}|_\infty + M^{1/p}|\nabla\,\mathit{div}\,\mathbf{v}|_p. \tag{86}$$

Using the estimates of lemma 19 to estimate integrals over Γ_t in (80) and (81) as well as estimates of lemma 22 and lemma 10 we obtain that for $p > 2$ and $\beta > 1 - 2/p$ there are c^*, c and $c^0(p)$ such that

$$|\partial_{x_i}A\langle\rho-\bar\rho\rangle|_p \le c^*\sup_{x\in\Gamma_0}|[\rho_0]d_x^{-\beta}|e^{-at} + c^0(p)|\partial_{y_1}\rho|_{p,\Omega_t^\pm}, \quad i = 1, 2,$$

where $c^0(p) \to 1$, when $p \to 2$. Setting $\sigma_1 = \partial_{x_1}\rho$ we also get

$$|\partial_{x_i} A\langle \rho - \bar{\rho}\rangle|_p \le c^* \sup_{x\in\Gamma_0} |[\rho_0] d_x^{-\beta}| e^{-at} + \frac{c^0(p)M}{m^{1/p}} |\rho^{1/p}\sigma_1|_p, \quad i = 1, 2. \tag{87}$$

Using the above estimates in (86) we obtain

$$\frac{d}{dt}|\rho^{1/p}\sigma|_p + \gamma m|\rho^{1/p}\sigma|_p \le \alpha_0 c^0(p) M \left(M/m\right)^{1/p} |\rho^{1/p}\sigma|_p$$
$$+ c^* I_2 e^{-at} + M^{1/p}|\nabla \mathbf{u}|_\infty + M^{1/p}|\nabla\, div\, \mathbf{v}|_p. \tag{88}$$

To obtain estimates on $|\rho|_\infty$ we notice that from lemma 6 it follows that

$$|A\langle \rho(t) - \bar{\rho}\rangle(x)| \le |B_\delta(x)|^{-1} \int_{B_\delta(x)} |A\langle \rho(t) - \bar{\rho}\rangle| + \delta^\alpha c(p) |\nabla A\langle \rho(t) - \bar{\rho}\rangle|_p,$$

where $0 < \alpha = 1 - 2p^{-1}$. Moreover,

$$\int_{B_\delta} |A\langle \rho(t) - \bar{\rho}\rangle| \le \delta^{1/2} |A\langle \rho(t) - \bar{\rho}\rangle|_2 \le \delta^{1/2} |\rho(t) - \bar{\rho}|_2.$$

Thus,

$$|A\langle \rho(t) - \bar{\rho}\rangle|_\infty \le \delta^{-3/2} I_0 + c(p)\delta^{1/2} |\nabla A\langle \rho(t) - \bar{\rho}\rangle|_p.$$

It can be derived from the (84) then, that

$$\frac{d}{dt}|\rho(t, X^t(x)) - \bar{\rho}| + \gamma m|\rho(t, X^t(x)) - \bar{\rho}| \le \alpha_0 M \big(\delta^{-3/2} I_0$$
$$+ c(p)\delta^{1/2} |\nabla A\langle \rho(t) - \bar{\rho}\rangle|_p\big) + M|\, div\, \mathbf{v}|_\infty \tag{89}$$

and using (87) in it (choosing different $\delta > 0$)

$$\frac{d}{dt}|\rho(t, X^t(x)) - \bar{\rho}| + \gamma m|\rho(t, X^t(x)) - \bar{\rho}| \le \delta |\nabla \rho|_p + c(\delta) I_0 + c^* I_2 e^{-at}$$
$$+ M|\, div\, \mathbf{v}|_\infty. \tag{90}$$

Notice that $\gamma < \alpha_0$. Thus, there are m, M, such that $M > \bar{\rho} > m$ and

$$\gamma - \alpha_0 \left(M/m\right)^{3/2} > 0$$

then there are $p > 2$ and $\delta > 0$ such that

$$\kappa = m\gamma - \alpha_0 c^0(p) M \left(M/m\right)^{1/p} - \delta > 0.$$

It follows that after adding and integrating equations we can obtain (we also use lemma 10 and Hypotheses 1–3)

$$|\nabla\rho(t)|_p + \operatorname{osc}\rho(t) \le c|\nabla\rho_0|_p + c\operatorname{osc}\rho_0 + I_2. \tag{91}$$

2.6 Estimates on $div\,\mathbf{w}$.

According to the formulas (54), (25)

$$div\,\mathbf{u} = div\,\mathbf{v} + div\,\mathbf{w},$$

and

$$div\,\mathbf{w}(x) = \frac{2-\alpha}{2\beta}(\rho(x)-\bar\rho) + \frac{1-\alpha}{\beta}A\langle\rho-\bar\rho\rangle(x).$$

For any $t>0$ we can represent the density as

$$\rho(t,\cdot) = \rho^r(t,\cdot) + [\rho](t,\cdot),$$

where $\rho^r = \rho - [\rho]$ and $[\rho] \in W^{1,p}(\Omega_t^-)$ was defined lemma 15 as an extension of $[\rho(t)]$ defined on Γ_t. The following estimates hold

Lemma 23. *There is a constant c such that*

$$|\rho^r - \bar\rho|_2 \le I_1 + c|\Omega_t^-||[\rho_0]|_\infty e^{-at},$$

$$|\rho^r - \bar\rho|_\infty \le c|\rho-\bar\rho|_\infty + c|[\rho_0]|_\infty e^{-at},$$

$$|\nabla\rho^r|_p \le c|\nabla\rho|_p + c^* I_2 e^{-3at/4}, \tag{92}$$

$$|\rho-\bar\rho|_{L^\infty_\alpha} \le c|\nabla\rho|_p + c^* I_2 e^{-3at/4}, \tag{93}$$

where c^* is defined in subsection 2.2. (Recall that $\nabla\rho$ denotes the absolutely continuous part of $\nabla\rho$.)

Proof. All estimates, except the last one are straightforward, given the estimates of subsection 2.3. Let us prove (93). We can write

$$|\rho-\bar\rho|_{L^\infty_\alpha} \le |\rho^r-\bar\rho|_{L^\infty_\alpha} + |[\rho]|_{L^\infty_\alpha}.$$

Lemma 2 and (92) imply that

$$|\rho^r - \bar{\rho}|_{L^\infty_\alpha} \le c|\nabla \rho^r|_p \le c|\nabla \rho|_p + c^* I_2 e^{-3at/4}.$$

For $x \in \partial\Omega$, $y \in \Omega$ consider the ratio

$$\frac{|[\rho](x) - [\rho](y)|}{|x-y|^\alpha}.$$

If, in addition, $x, y \in \Omega_t^+$ the ratio is zero. If $x, y \in \Omega_t^-$ then the ratio is not greater than $c^* I_2 e^{-3at/4}$ as it was shown in lemma 16 and estimates (63). If $x \in \Omega_t^- \cap \partial H$, $y \in \Omega_t^+$ then there is a point $z \in \Gamma_t$ such that $|z-x| \le |y-x|$, $d_z < |z-x|$ and

$$\frac{|[\rho](x) - [\rho](y)|}{|x-y|^\alpha} = \frac{|[\rho](x)|}{|x-y|^\alpha} \le \frac{|[\rho](x) - [\rho](z)|}{|z-x|^\alpha} + \frac{|[\rho](z)|}{d_z^\alpha}.$$

The first term on the right-hand side of the above inequality had been already treated, while the second term is less than $c^* I_2 e^{-3at/4}$ by (57) and Hypotheses 1–3. □

According to formulas (54) we can represent $\mathbf{w}$ as a solution of Poissons equations

$$\Delta \mathbf{w} = \frac{\gamma_1}{\mu} \nabla(\rho^r + [\rho]) - \frac{\gamma_2}{\mu} \nabla A \langle \rho - \bar{\rho} \rangle,$$

where

$$\gamma_1 = 1 - \lambda \frac{2-\alpha}{2\beta}, \quad \gamma_2 = \lambda \frac{1-\alpha}{\beta}.$$

It is convenient to split $\mathbf{w} = \mathbf{w}_1 + \mathbf{w}_2$, where

$$\Delta \mathbf{w}_1 = \frac{\gamma_1}{\mu} \nabla [\rho],$$

$$\Delta \mathbf{w}_2 = \frac{\gamma_1}{\mu} \nabla \rho^r - \frac{\gamma_2}{\mu} \nabla A \langle \rho - \bar{\rho} \rangle,$$

and $\mathbf{w}_i$ satisfy the no-slip boundary conditions on ∂H.

Lemma 24. *There are c^*, $c_\delta = c(\delta, \lambda, \mu, p, |\Omega_0^-|)$, $\delta > 0$ and c such that*

$$|\nabla \mathbf{w}_2|_\infty \le c^* I_2 e^{-3at/4 + c\varepsilon t} + \delta |\nabla \rho|_p + c_\delta I_2 e^{-3at/4 + c\varepsilon t}, \tag{94}$$

$$\langle \nabla \mathbf{w}_2 \rangle_{C^\alpha(H)} \le c^* I_2 e^{-3at/4 + c\varepsilon t} + c|\nabla \rho|_p \tag{95}$$

and

$$\langle \nabla \mathbf{w}_2 \widehat{\mathbf{W}} \rangle_{\alpha, \Omega_t^-} \le (\langle \widehat{\mathbf{W}} \rangle_{\alpha, \Omega_t^-} + |\widehat{\mathbf{W}}|_\infty) \left(c^* I_2 e^{-3at/4 + c\varepsilon t} + c|\nabla \rho|_p \right) \tag{96}$$

Proof. Estimates are straightforward, given the estimates of the previous lemma and lemmas 1,2. $\square$

Recall now that $\mathbf{u} = \mathbf{v} + \mathbf{w}_1 + \mathbf{w}_2$. Combining the estimates of the last lemma, lemma 10 and lemma 78 on can obtain the following estimates

$$|\nabla \mathbf{u}|_\infty \leq |\nabla \mathbf{v}|_\infty + c(\delta)c^* I_2 e^{-3at/4 + c\int_0^t |\nabla \mathbf{u}|_\infty} + \delta |\nabla \rho|_p, \tag{97}$$

$$\begin{aligned}\langle \nabla \mathbf{u} \cdot \widehat{\mathbf{W}} \rangle_\alpha \leq (\langle \widehat{\mathbf{W}} \rangle_{\alpha, \Omega_t^-} + |\widehat{\mathbf{W}}|_\infty) \Big(c^* I_2 e^{-3at/4 + c\int_0^t |\nabla \mathbf{u}|_\infty} + c|\nabla \rho|_p \\ + c\langle \nabla \mathbf{v} \rangle_\alpha + c|\nabla \mathbf{v}|_\infty \Big)\end{aligned} \tag{98}$$

2.7 Dynamics of the interface and *a priori* estimates

Let $\phi(t,x) = \phi_0(X^t(x))$ and $\mathbf{W} = \nabla^\perp \phi$. $\mathbf{W}$ solves the evolution equation

$$\mathbf{W}_t + \mathbf{u} \cdot \nabla \mathbf{W} + \mathbf{W}\, div\, \mathbf{u} = \nabla \mathbf{u} \mathbf{W},$$

or

$$\frac{d}{dt} \Big[\mathbf{W}(t, X^t(x)) e^{\int_0^t div\, \mathbf{u}(\tau, X^\tau(x))} \Big] = \nabla \mathbf{u} \mathbf{W}(t, X^t(x)) e^{\int_0^t div\, \mathbf{u}(\tau, X^\tau(x))}.$$

Since $e^{\int_0^t div\, \mathbf{u}(\tau, X^\tau(x))} = \frac{\rho_0(x)}{\rho(t, X^t(x))}$ we obtain that the vector

$$\widehat{\mathbf{W}}(t,x) = \mathbf{W}(t,x) \frac{\rho_0(X^{-t}(x))}{\rho(t,x)} \tag{99}$$

solves the integral equation

$$\widehat{\mathbf{W}}(t,x) = \mathbf{W}_0(X^{-t}(x)) + \int_0^t \nabla \mathbf{u} \widehat{\mathbf{W}}(s, X^{-t+s}(x)). \tag{100}$$

From (97), (98) we can derive an estimate on $\widehat{\mathbf{W}}$:

$$\begin{aligned}\langle \widehat{\mathbf{W}} \rangle_\alpha + |\widehat{\mathbf{W}}|_\infty \leq (\langle \mathbf{W}_0 \rangle_\alpha + |\mathbf{W}_0|_\infty) e^{\int_0^t |\nabla \mathbf{u}|_\infty} + \int_0^t c|\nabla \rho|_p (\langle \widehat{\mathbf{W}} \rangle_\alpha + |\widehat{\mathbf{W}}|_\infty) e^{\int_s^t |\nabla \mathbf{u}|_\infty} \\ + \int_0^t c(\langle \nabla \mathbf{v} \rangle_\alpha + c|\nabla \mathbf{v}|_\infty)(\langle \widehat{\mathbf{W}} \rangle_\alpha + |\widehat{\mathbf{W}}|_\infty) e^{\int_s^t |\nabla \mathbf{u}|_\infty} \\ + \int_0^t c^* I_2 e^{-3at/4 + c\varepsilon t}.\end{aligned} \tag{101}$$

Applying Gronowall type inequality for $(\langle\widehat{\mathbf{W}}\rangle_\alpha + |\widehat{\mathbf{W}}|_\infty)e^{-\int_0^t |\nabla \mathbf{u}|_\infty}$ we obtain

$$\langle\widehat{\mathbf{W}}\rangle_\alpha + |\widehat{\mathbf{W}}|_\infty \le (\langle\mathbf{W}_0\rangle_\alpha + |\mathbf{W}_0|_\infty + I_2)e^{c\varepsilon t + I_1 + c\int_0^t |\nabla\rho|_p}.$$

Moreover

$$\int_0^t |\nabla \mathbf{u}|_\infty \le \epsilon + I_1 + c(\epsilon, M_p)c^* I_2 \int_0^t e^{-3at/4 + c\varepsilon t},$$

and assuming that $c\varepsilon < a/4$ we derive the next estimate.

$$\int_0^t |\nabla \mathbf{u}|_\infty \le \epsilon + I_1 + c(\epsilon, M_p)c^* I_2.$$

We list here estimates that we obtained so far:

$$\sup_{[0,t]} |\nabla\rho(s)|_p + \operatorname{osc}\rho(t) \le I_2,$$

$$\int_0^t |\nabla \mathbf{u}|_\infty \le \epsilon t + I_1 + c(\epsilon, M_p) I_2,$$

$$\sup_{[0,t]} \langle\widehat{\mathbf{W}}(s)\rangle_\alpha + |\widehat{\mathbf{W}}(s)|_\infty \le (\langle\mathbf{W}_0\rangle_\alpha + |\mathbf{W}_0|_\infty + I_2)e^{c\varepsilon t + I_1 + cM_p t},$$

$$\inf_{[0,t]} |\widehat{\mathbf{W}}(s)|_{\inf} \ge |\phi_0|_{\inf} e^{-c\varepsilon t}.$$

It follows from the above estimates that for any $\delta > 0$ we can always choose I_2 so small that

$$\sup_t \operatorname{osc}\rho(t) < \delta < M - m,$$

$$\sup_t |\nabla\rho(t)|_p < \delta < M_p,$$

$$\int_0^t |\nabla \mathbf{u}|_\infty \le \delta t < \varepsilon t,$$

$$\sup_{[0,t]} \langle\widehat{\mathbf{W}}(s)\rangle_\alpha + |\widehat{\mathbf{W}}(s)|_\infty \le 2(\langle\mathbf{W}_0\rangle_\alpha + |\mathbf{W}_0|_\infty)e^{\delta t} < 2(\langle\mathbf{W}_0\rangle_\alpha + |\mathbf{W}_0|_\infty)e^{\varepsilon t},$$

$$\inf_{[0,t]} |\widehat{\mathbf{W}}(s)|_{\inf} \ge \frac{1}{2}|\nabla\phi_0|_{inf} e^{-\delta t} > \frac{1}{2}|\nabla\phi_0|_{inf} e^{-\varepsilon t}.$$

Comparing the last statement with Hypotheses 1–3 we conclude that the estimates listed in Hypotheses 1–3 are indeed hold.

2.8 Non-degeneracy of the contact angle

Let $(0,0)$ be one of the contact points. Consider the ratio

$$\sin\alpha(t,x) = \frac{X_2^t(x)}{|X^t(x)|}.$$

It holds:

$$\frac{\partial}{\partial t}\sin\alpha = \sin\alpha\left(\frac{u_2}{X_2^t}\left(\frac{X_1^t}{|X^t|}\right)^2 - \frac{u_1 X_1^t}{|X^t|^2}\right),$$

where $u_i = u_i(t, X^t(x))$. Moreover,

$$|u_2(t, X^t(x))| \leq |\nabla \mathbf{u}(t)|_\infty X_2^t,$$

and we deduce that

$$\sin\alpha(t,x) \geq \sin\alpha(0,x) e^{-2\int_0^t |\nabla \mathbf{u}|_\infty}.$$

In the above formula we take $x \in \Gamma_0$, pass to the limit $x \to 0$ and derive the estimate on the contact angle $\theta(t)$:

$$\sin\theta(t) \geq \sin\theta_0 e^{-2\int_0^t |\nabla \mathbf{u}|_\infty},$$

which, as it is easy to see from the estimate $\int_0^t |\nabla \mathbf{u}|_\infty < \delta t$, that it is always in the range $(0,\pi)$.

2.9 Hölder continuity of the $\nabla \mathbf{u}(t)$ in $\Omega_t^\pm$

Recall a decomposition of

$$\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2 + \mathbf{v}.$$

According to (10) and (94) $\nabla \mathbf{w}_2 + \nabla \mathbf{v} \in C^\alpha(\Omega_t^\pm \cap B_R)$. Representation formula (65) reads:

$$\partial_{x_j} w_{1,i} = \frac{\gamma_1}{\mu}\int_{\Gamma_t} \partial_{x_j} G^{x,y}[\rho] N_{y_i} dS_y - \frac{\gamma}{\mu}\int_{\Omega_t^-} \partial_{x_j} G^{x,y} \partial_{y_i}[\rho].$$

The last term on the right belongs to $W^{1,p}$ as was derived in lemma 18. The first term can be written as

$$\frac{\gamma_1}{\mu}\int_{\Gamma_t} \partial_{x_j} G^{x,y}[\rho] N_{y_i} dS_y = \frac{\gamma_1}{\mu}\int_{\Gamma_t^e} \partial_{x_j} G^{x,y}[\rho] N_{y_i} dS_y,$$

where $\Gamma_t^e - C^{1+\alpha}$ curve, the extension of Γ_t, and $[\rho]$ is a C^α function on Γ_t^e, extended by zero beyond Γ_t. (see the Appendix). There is a classical result on the derivative of the single layer potential over the curve of class $C^{1+\alpha}$, see [9], p. 285, which implies that the above integral belongs to

$$C^\gamma\left(\Omega_t^\pm \cap B_R\right),$$

for some $0 < \gamma < \alpha$, and any ball B_R.

3 Proof of theorem 1

In this section we state two uniqueness theorems and conclude the proof of the main theorem.

Theorem 3 ([11], p.1402)**.** *If* $(\rho_1, \mathbf{u}_1)$ *and* $(\rho_2, \mathbf{u}_2)$ *are two weak solutions of problem* (48)–(50) *such that*

$$\left.\begin{array}{l}\mathbf{u}_i \in L^\infty((0,T) \,:\, W_0^{1,2}(H)),\\ \rho_i - \bar{\rho} \in L^\infty((0,T)\times H) \cap C([0,T] \,:\, L^2(H)) \cap L^\infty((0,T) \,:\, W^{-1,2}(H)),\\ \int_0^T |\nabla \mathbf{u}_1(t,\cdot)|_{L^\infty(H)} < \infty.\end{array}\right\} \quad (102)$$

Then,

$$(\rho_1, \mathbf{u}_1) = (\rho_2, \mathbf{u}_2),\ a.e.(0,T)\times H.$$

Proof. Proof is a straightforward adaptation of arguments of [11], p.1402, where the case of $H = \mathbb{R}^2$ is considered. We only have to notice that because of lemma 4, $|D^2\psi|_2 \le C|L\langle\psi\rangle|_2$, for any vector function $\psi \in W_0^{1,2}(H)$. □

We state here the uniqueness theorem for solutions of (1)–(5).

Theorem 4. *There is only one solution* $(\rho, \mathbf{u})$ *of* (1)–(5) *on* $[0,T]$ *in the regularity class*

$$\rho(t,x) < M < \infty, \quad a.e.\,(0,T)\times H,$$

$$\int_0^T |\nabla \mathbf{u}(t,\cdot)|_\infty < \infty,$$

$$
\left.\begin{array}{l}
\rho-\bar{\rho}\in C([0,T]\,:\,L^2(H)),\\
\tilde{\rho}-\bar{\rho}\in L^\infty((0,T)\,:\,W^{-1,2}(H)),\\
\mathbf{u}\in L^q((0,T)\times H),\, q>2,\\
\mathbf{u}\in C([0,T]\times H),\\
\mathbf{u}\in L^\infty((0,T)\,:\,W^{1,2}(H)),\\
\mathbf{u}_t\in L^2((0,T)\times H),\\
\nabla\mathbf{u}\in L^\infty((0,T)\,:\,L^2(H)),\\
\dot{\mathbf{u}}\in L^\infty((0,T)\,:\,L^2(H)),\\
\sqrt{t}\dot{\mathbf{u}}\in L^2((0,T)\,:\,W^{1,2}(H)),\\
div\,\mathbb{S}\in L^\infty((0,T)\,:\,L^2(H))
\end{array}\right\}\tag{103}
$$

and provided that a non-conservative form of equations (2) *hold, i.e., a.e.* $(0,T)\times H$

$$
\rho\dot{\mathbf{u}}=\,div\,\mathbb{S},
$$

where $\mathbb{S}=((\rho-\bar{\rho})+\lambda\,div\,\mathbf{u})\mathbb{I}+\mu(\nabla\mathbf{u}+\nabla^t\mathbf{u})$ *– Cauchy's stress tensor.*

Proof. The proof is essentially a repetition of the one given in [13], theorem 1 for the case for $\mathbb{R}^2$. Let us explain the differences. From the proof of theorem 1 in [13] we notice that the condition (1.12) of the cited theorem,

$$
\int_0^T t|\nabla F,\,\nabla\omega|_2^2+t^{2/3}|\nabla F,\,\nabla\omega|_{L^4}^3<\infty
$$

can be substituted by

$$
\int_0^T t|\,div\,\mathbb{S}|_2^2+t^{2/3}|\,div\,\mathbb{S}|_{L^4}^3<\infty.
$$

On the other hand we require that $div\,\mathbb{S}=\rho\dot{\mathbf{u}}$ and the needed integrability of $div\,\mathbb{S}$ follows from (103). Indeed, by the boundedness of ρ and lemma 1,

$$
|\rho\dot{\mathbf{u}}|\le c|\dot{\mathbf{u}}|_2^{1/2}|\nabla\dot{\mathbf{u}}|_2^{1/2}
$$

and thus, using (103),

$$
\int_0^T t^{2/3}|\,div\,\mathbb{S}|_4^3\le c\int_0^T t^{3/2}|\nabla\dot{\mathbf{u}}|_2^{3/2}<\infty.
$$

All other arguments are unchanged. □

Now we use the uniqueness theorem 3 and conclude that

$$(\tilde{\rho}, \tilde{u}) = (\rho, \mathbf{u})$$

is a solution of (1)–(5) on time interval $[0, T]$. It verifies all properties listed in the theorem 1 provided that initial data are suitable restricted. Notice, however that from (39), (52), Hypotheses 1–3 it follows that

$$\sup_{[0,T]} \left\{ \sup \rho(t,\cdot), \inf \rho(t,\cdot), |\rho(t,\cdot) - \bar{\rho}|_{L^2 \cap L^4}, |\rho(t,\cdot) - \bar{\rho}|_{L^\infty_\alpha}, |\mathbf{u}(t,\cdot)|_{W^{1,2}} \right\}$$

are bounded only in terms of initial data, not T. Thus, using the uniqueness theorem 4 the solution $(\rho, \mathbf{u})$ can be continued to arbitrary interval $[0, T]$, $T > 0$.

4 Appendix

Lemma 25. *There is an extension of Γ_t to a closed $C^{1+\alpha}$ curve Γ_t^e and there is a number $c > 1$ such that*

$$|\Gamma_t^e|_{1+\alpha} \leq c|\nabla\phi|_\alpha,$$

and for $l^t = \left(\frac{|\nabla\phi|_{inf}}{c|\nabla\phi|_\alpha}\right)^{2/\alpha}$ there is a collection of boxes $B_j^t(x_j)$, $j = 1..N^t$, $x_j^t \in \Gamma_t^e$, of side length l^t with the property that $\Gamma_t^e \subset \cup_j B_j^t$. $B_j^t \cap \Gamma_t^e$ in the local coordinate system centered at x_j^t is represented by a curve with the absolute value of its slope less then $1/2$. The number of boxes $N^t = c\frac{|\Gamma_t|}{l^t}$.

Proof. The proof is straightforward as we only require that $\Gamma_t^e \cap H^*$ joins smoothly to Γ_t at two points: $\Gamma_t \cap \partial H$. $\square$

We call Ω_t^e – the domain enclosed by Γ_t^e. By doubling the number of boxes if necessary we can always assume that

$$\Omega_t^e \subset \cup_j B_j \cup (\Omega_t^e)_{l/4}\,,$$

where $(\Omega_t^e)_{l/4}$ denotes the set of points of Ω_t^e lying at the distance $l^t/4$ from Γ_t^e. Given an open cover $\Gamma_t^e \subset \{B_j^t\}_1^{N^t}$ with the properties described in the above lemma there is a partition of unity, $\{\omega_j^t\}_1^{N^t}$ for Γ_t^e and C^1 maps $\Phi_j^t : B_j^t \to U_j^t$, U_j^t – open subsets of $\mathbb{R}^2$,

$$\Phi_j^t(B_j^t \cap \Omega_t^-) = \{y \in U_j^t \,:\, y_2 > 0\}$$

and $\Psi_j^t = \left(\Phi_j^t\right)^{-1}$ such that there is a number c that verifies the estimates

$$\left.\begin{aligned} &|\nabla \omega_j^t|_\infty \le c(l^t)^{-1}, \\ &|\nabla \Phi_j^t|_\infty \le c|\nabla \phi|_\infty, \\ &|\nabla \Psi_j^t|_\infty \le c\tfrac{|\nabla \phi|_\infty}{|\nabla \phi|_{\inf}}. \end{aligned}\right\} \tag{104}$$

Let now $f(x)$ be a continuous function on Γ_t and $f = 0$ at the points of contact: $\Gamma_t^e \cap \partial H$. We extend f to Γ_t^e by zero. The $W^{s,p}$, $s = 1 - \frac{1}{p}$, norm of f is defined as

$$|f|^p_{W^{s,p}(\Gamma_t)} = \sum_1^{N^t} |\omega_j^t f(\Psi_j(r,0))|^p_{W^{s,p}(\mathbb{R})}$$

and

$$|f(\cdot)|^p_{W^{s,p}(\mathbb{R})} = |f|^p_{p,\mathbb{R}} + \int\int_{\mathbb{R}\times\mathbb{R}} \frac{|f(x) - f(y)|^p}{|x-y|^p}\, dxdy.$$

Moreover $W^{s,p}(\mathbb{R})$ is a space of traces of functions $W^{1,p}(\mathbb{R}^2_+)$, see Section 7.51 of [1].

Lemma 26. *Under Hypotheses 1–3,*

$$|[\rho(t)]|_{W^{s,p}(\Gamma_t)} \le I_2 e^{-3at/4}, \tag{105}$$

$a = m(\lambda + 2\mu)^{-1}$.

Proof. With the notation introduced above one easily verifies that

$$\begin{aligned} |[\rho(t)]|_{W^{s,p}(\Gamma_t)} &\le c(p,M,m)|[\rho_0](X^{-t}(x))|_{W^{s,p}(\Gamma_t)} e^{-at} \\ &\quad + c(p,M,m)|[\rho_0]|_\infty e^{-at} \int_0^t |[\rho](\tau, X^{-t+\tau}(x))|_{W^{s,p}(\Gamma_t)}. \end{aligned} \tag{106}$$

We are going to show now that

$$|[\rho](\tau, X^{-t+\tau}(\cdot))|_{W^{s,p}(\Gamma_t)} \le \tilde{c}^*(t)|[\rho](\tau,\cdot)|_{W^{s,p}(\Gamma_\tau)}. \tag{107}$$

Set $f(x) = [\rho](t, X^{-t+\tau}(x))$.

$$|f \circ X^{-t+\tau}|^p_{W^{s,p}(\Gamma_t)} = \sum_1^{N^t} |(\omega_j^t f \circ X^{-t+\tau}) \circ \Psi_j^t(r,0)|^p_{W^{s,p}(mathbbR)}$$

$$\leq \sum_1^{N^t}\sum_1^{N^\tau} |(\omega_j^t\omega_k^\tau f \circ X^{-t+\tau}) \circ \Psi_j^t(r,0)|^p_{W^{s,p}(\mathbb{R})} \leq \sum_1^{N^t}\sum_1^{N^\tau} |(\omega_k^\tau f \circ X^{-t+\tau}) \circ \Psi_j^t(r,0)|^p_{L^p(\mathbb{R})}$$

$$+ \sum_1^{N^t}\sum_1^{N^\tau} \int\int_{\mathbb{R}\times\mathbb{R}} \frac{|(\omega_j^t\omega_k^\tau f \circ X^{-t+\tau}) \circ \Psi_j^t(r,0) - (\omega_j^t\omega_k^\tau f \circ X^{-t+\tau}) \circ \Psi_j^t(s,0)|^p}{|r-s|^p}\, drds$$

$$\leq (1 + |\nabla\omega_j^t|^p_\infty l^t + 1) \sum_1^{N^t}\sum_1^{N^\tau} |(\omega_k^\tau f \circ X^{-t+\tau}) \circ \Psi_j^t(r,0)|^p_{L^p(\mathbb{R})}$$

$$+ \sum_1^{N^t}\sum_1^{N^\tau} \int\int_{\mathbb{R}\times\mathbb{R}} \frac{|(\omega_k^\tau f \circ X^{-t+\tau}) \circ \Psi_j^t(r,0) - (\omega_k^\tau f \circ X^{-t+\tau}) \circ \Psi_j^t(s,0)|^p}{|r-s|^p}\, drds. \tag{108}$$

We make a change of variables in the integral:

$$r = \Phi_{j,1}^t \circ X^{t-\tau} \circ \Psi_k^\tau(a,0).$$

According to estimates (104) and a trivial bound $\left|\frac{\partial X^t(x)}{\partial x}\right| \leq e^{\int_0^t |\nabla \mathbf{u}|_\infty}$ we have

$$\left|\frac{dr}{da}\right| \leq ce^{\int_\tau^t |\nabla \mathbf{u}|_\infty} \frac{|\nabla\phi|^2_\infty}{|\nabla\phi|_{inf}},$$

$$|a-b| \leq ce^{\int_\tau^t |\nabla \mathbf{u}|_\infty} \frac{|\nabla\phi|_\infty}{|\nabla\phi|^2_{inf}} |r-s|.$$

Thus,

$$|f \circ X^{-t+\tau}|^p_{W^{s,p},\Gamma_t} \leq (1 + |\nabla\omega_j^t|^p_\infty l^t + 1) N^t \left|\frac{dr}{da}\right|_\infty \sum_1^{N^\tau} |(\omega_k^\tau f) \circ \Psi_k^\tau(r,0)|^p_{L^p(\mathbb{R})}$$

$$+ N^t \left|\frac{dr}{da}\right|_\infty e^{p\int_\tau^t |\nabla \mathbf{u}|_\infty} \frac{|\nabla\phi|^p_\infty}{|\nabla\phi|^{2p}_{inf}} \sum_1^{N\tau} \int\int_{\mathbb{R}\times\mathbb{R}} \frac{|(\omega_k^\tau f) \circ \Psi_k^\tau(a,0) - (\omega_k^\tau f)\Psi_k^\tau(b,0)|^p}{|a-b|^p} \tag{109}$$

and this proves (107) with

$$c^* = c|\Gamma_0| \left(\frac{|\nabla\phi|_\alpha^{(2\alpha+p\alpha+2)/p\alpha}}{|\nabla\phi|_{inf}^{(\alpha+2p\alpha+2)/p\alpha}} + \frac{|\nabla\phi|_\alpha^{(2\alpha+2(p-1)\alpha+2)/p\alpha}}{|\nabla\phi|_{inf}^{(\alpha+2(p-1)\alpha+2)/p\alpha}} \right) e^{2p\int_0^t |\nabla\mathbf{u}|_\infty}.$$

□

Substituting (107) in the inequality (106) and using Hypotheses 1–3 we obtain

$$|[\rho(t)]|_{W^{s,p}(\Gamma_t)} \le c^*|[\rho_0]|_{W^{s,p}(\Gamma_0)}e^{-at+c\varepsilon t} + c^*|[\rho_0]|_\infty e^{-at+c\varepsilon t}\int_0^t |[\rho(\tau)]|_{W^{s,p}(\Gamma_\tau)}. \quad (110)$$

Assuming that

$$c\varepsilon < a/4$$

we can obtain

$$|[\rho(t)]|_{W^{s,p}(\Gamma_t)} \le c(|[\rho_0]|_{W^{s,p}(\Gamma_0)})^{1/2}e^{-3at/4} \le I_2 e^{-3at/4}, \quad (111)$$

$c = c(\lambda, \mu, p)$.

Now we look at the extension of $[\rho]$ to $\Omega_t^{e,-}$.

Lemma 27. *Let* $[\rho] \in W^{s,p}(\Gamma_t)$ *and vanishes at points of* $\Gamma)t \cap \partial H$. *Then, there is an extension* $[\rho] \in W^{1,p}(\Omega_t^-)$, *such that*

$$|[\rho(t)]|_{W^{1,p}(\Omega_t^-)} \le \tilde{c}^*(t)|[\rho(t)]|_{W^{s,p}(\Gamma_t)},\ s = 1 - p^{-1}. \quad (112)$$

Proof. Clearly, it is enough to build an extension of $[\rho]$ defined on Γ_t^e to $\Omega_t^{e,-}$. Let $\{\omega_j^t\}_0^{N^t}$ be a partition of unity for

$$\cup_j B_j \cup\ (\Omega_t^e)_{l/4}\,.$$

Every function $(\omega_j^t[\rho]) \circ \Psi_j^t(r, 0)$ is a trace of $f_j \in W^{1,p}(\mathbb{R}_+^2)$, see Section 7.56 of [1], and

$$|f_j|_{W^{1,p}(\mathbb{R}_+^2)} \le c(p)|(\omega_j^t[\rho]) \circ \Psi_j^t(r, 0)|_{W^{s,p}(\mathbb{R})}.$$

The required extension is given by the formula

$$[\rho](x) = \sum_1^{N^t} \omega_j^t(x) f_j \circ \Phi_j^t(x),\ x \in \Omega_t^{e,-}$$

and the estimate on the norm is easily computed. □

Lemma 28. *There is* $c(p)$, $p > 2$, *such that*

$$||[\rho]|_{L^\infty(\Omega_t^-)} \leq \tilde{c}^*(t)|[\rho]|_{W^{1,p}(\Omega_t^-)} \tag{113}$$

and

$$|[\rho]|_{C^\alpha(\overline{\Omega}_t^-)} \leq \tilde{c}^*(t)|[\rho]|_{W^{1,p}(\Omega_t^-)}, \quad \alpha = 1 - 2p^{-1}, \tag{114}$$

where $\beta_i = \beta_i(p) < 0$.

Proof. We notice that at the scale $\min\{l^t, |\nabla\phi|_{inf}\}$, $\Omega_t^{e,-}$ is approximately a half-space. Then, we can apply classical arguments from lemma 7.16, theorem 7.17 of [5]. $\square$

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