Spectral stability of weak relaxation shock profiles

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Abstract: Using a combination of Kawashima- and Goodman-type energy estimates, we establish spectral stability of general small-amplitude relaxation shocks of symmetric dissipative systems. This extends previous results obtained by Plaza and Zumbrun [9] by singular perturbation techniques under an additional technical assumption, namely, that the background equation be noncharacteristic with respect to the shock.

1. Introduction

Let us consider the one-dimensional hyperbolic system with relaxation

$$w_t + F(w)_x = Q(w) \tag{1}$$

for the unknown $w = w(x,t) \in \mathbb{R}^N$, $x \in \mathbb{R}, t > 0$. Here $F \in C^2(\mathbb{R}^N; \mathbb{R}^N)$ is such that $\mathrm{d}F(w)$ has N real distinct eigenvalues for any state w under consideration and $Q \in C^1(\mathbb{R}^N; \mathbb{R}^N)$ has the structure $Q(w) = (0_n, q(w))$ where $q \in C^1(\mathbb{R}^N; \mathbb{R}^r)$, $r = N - n$. Additionally, we assume the function q to have a relaxation structure: let $w = (u,v) \in \mathcal{U} \times \mathcal{V} \subseteq \mathbb{R}^n \times \mathbb{R}^r$,

i. there exists a C^1 function $v^* : \mathcal{U} \subset \mathbb{R}^n \to \mathbb{R}^r$ such that $q(w) = 0$ in $\mathcal{U} \times \mathcal{V}$ if and only if $v = v^*(u)$ where $w = (u,v) \in \mathbb{R}^n \times \mathbb{R}^r$;

ii. for any $u \in \mathcal{U}$, all of the eigenvalues of $\mathrm{d}_u q(u, v^*(u))$ have negative real part.

As a consequence, it is natural to introduce the corresponding relaxed hyperbolic system of conservation laws, formally obtained by considering the first n equations of (1) and substituting the variable v with the equilibrium $v = v^*(u)$

$$u_t + f^*(u)_x = 0 \qquad \text{where} \quad f^*(u) := f(u, v^*(u)). \tag{2}$$

System (1) possesses smooth traveling wave solutions corresponding to shock waves of the relaxed system (2) at least in the small–amplitude case.[1] Existence of such special solutions has been given for specific models in the large-amplitude case — for example, the Broadwell model, [1] —, or for general relaxation system in the small–amplitude case, see [11, 7]. By changing frame, such travelling wave can be assumed, without loss of generality, as stationary solution of (1), i.e. solution of the form

$$W = W(x), \qquad W(\pm\infty) = W_\pm. \tag{3}$$

where $W_\pm = (u_\pm, v^*(u_\pm))$ with $u_\pm$ denoting the state connected by the corresponding relaxed shock wave.

The next natural question to answer is whether such steady states are stable or unstable. The equation for the perturbation $w := \tilde{w} - W$ is

$$w_t + \big(F(W+w) - F(W)\big)_x = Q(W+w) - Q(W).$$

and the corresponding linearized equation is

$$w_t = \mathcal{L}w := -\big(\mathrm{d}F(W)w\big)' + \mathrm{d}Q(W)w. \tag{4}$$

Thus, the linearized eigenvalue equation is

$$(\lambda\, I - \mathcal{L})w = \lambda\, w + \big(\mathrm{d}F(W)w\big)' - \mathrm{d}Q(W)w = 0. \tag{5}$$

From now on, we consider equation (5) in the Sobolev space H^1 and we say that $\lambda \in \mathbb{C}$ is an eigenvalue of $\mathcal{L}$ if there exists a function $w \in H^1 \setminus \{0\}$, such that (5) holds.[2]

By differentiating the equation satisfied by the profile W, we get

$$\big(\mathrm{d}F(W)W'\big)' - \mathrm{d}Q(W)W' = 0.$$

In the noncharacteristic case, i.e. $\mathrm{d}F(W_\pm)$ invertible, W' decays exponentially fast to zero as $|x| \to \infty$. Thus, $\lambda = 0$ is an eigenvalue of the linearized operator $\mathcal{L}$. Hence, instability is related to the presence of non-zero eigenvalues with non-negative real part.

[1] In all of the paper, we will not consider the case of nonsmooth relaxation shock profiles, i.e. exhibiting subshocks. This is not restrictive since the smallness assumption of the profile, usually, guarantees also smoothness.

[2] On the region $\mathrm{Re}\,\lambda \geq 0$ and $\lambda \neq 0$ that we will consider, and under our hypotheses (A1)–(A2) below, H^1 spectrum agrees with L^p spectrum for any $1 \leq p \leq \infty$; see [8].

Definition 1.1. The stationary solution W is spectrally stable in H^1, if for any $\lambda \in \mathbb{C} \setminus \{0\}$, $\operatorname{Re}\lambda \geq 0$, whenever $w \in H^1$ solves the resolvent equation (5) then $w \equiv 0$.

Results in [7, 8] show that, under additional assumptions, spectral stability implies both linear and nonlinear orbital stability. Hence, determining whether spectral stability holds or not is the key issue for determining nonlinear stability of relaxation shock profiles. The aim of the present paper is to prove a general result on spectral stability of relaxation shock profiles assuming smallness of $\varepsilon := |W_+ - W_-|$.

We divide the assumptions into three groups: A, B and C, referring, respectively, to the relaxation system (1), to the the relaxed system (2), to the relaxation shock profile (3). These are imposed on a small neighborhood $\mathcal{W} \subset \mathbb{R}^n$ about an equilibrium point $w_0 = (u_0, v_0)$, $q(w_0) = 0$, or equivalently $v_0 = v_*(u_0)$, and a neighborhood $\mathcal{U}^*$ of u_0 such that the graph of v_* over $\mathcal{U}^*$ is contained in $\mathcal{W}$.

A1. *There exists a smooth function $A_0 = A_0(W)$ from $\mathcal{W} \subset \mathbb{R}^n$ to the set of real symmetric positive definite matrices such that $(A_0 \mathrm{d}F)(W)$ is symmetric for any W under consideration, and*

$$\operatorname{Re}\langle w, (A_0\, \mathrm{d}Q)(W_\pm)\, w\rangle \leq -c|\Pi_\pm w|^2_{L^2}, \tag{6}$$

for some $c > 0$, where $\Pi_\pm\, w := \mathrm{d}Q(W_\pm)\, w$.

A2. *(Shizuta–Kawashima condition) There exists a smooth skew-symmetric matrix–valued function K, depending on $\mathrm{d}F, \mathrm{d}Q$ and A_0 such that*

$$\operatorname{Re}\,(K\mathrm{d}F - A_0\, \mathrm{d}Q) > 0. \tag{7}$$

B1. *There exists a smooth function $a_0 = a_0(u)$ from $\mathcal{U}^* \subset \mathbb{R}^p$ to the set of real symmetric positive definite matrices such that $a_0\, \mathrm{d}f^*(u)$ is symmetric for any u under consideration and $a_0\, b_0$ is positive semidefinite, $\operatorname{Re} a_0 b_0 \geq 0$, where b_0 as defined in (3) is the associated Chapman–Enskog viscosity.*

B2. *(reduced Shizuta–Kawashima condition) There exists a smooth skew-symmetric matrix–valued function k, depending on $\mathrm{d}f^*, b_0$ and a_0 such that*

$$\operatorname{Re}\,(k\, \mathrm{d}f^* - a_0\, b_0) > 0. \tag{8}$$

B3. *(Simplicity, genuine nonlinearity of principle equilibrium characteristic) There exists $c_0 > 0$ such that there is a single eigenvalue $\alpha_0(u)$ of $\mathrm{d}f^*(u)$ that has absolute* *value* $< c_0$ *on* $\mathcal{U}^*$*, with all others of absolute value* $\geq 2c_0$*. Moreover,*

$$\text{(9)} \qquad d\alpha_0(u) \cdot r_0(u) =: \eta(u) \neq 0$$

on $\mathcal{U}^*$*, where* $r_0(u)$ *denotes the unit right eigenvector associated with* $\alpha_0(u)$*.*
C. *There exists* $C > 0$ *such that for any* $x \in \mathbb{R}$ *there hold*

$$\text{(10)} \qquad |W'|_{L^\infty} \leq C|W_+ - W_-|^2, \qquad |W''(x)| \leq C|W_+ - W_-|\,|W'(x)|,$$

and

$$\text{(11)} \qquad \left|\frac{W'}{|W'|} + \text{sgn}\,(\eta) r_0\right| \leq C\,|W_+ - W_-|^2.$$

Remark 1.2. The apparently restrictive A1–A2, B1–B2 in fact all follow from the standard assumptions that (i) there exist a positive definite symmetrizer A_0, $A_0 dF$ symmetric, that *at equilibrium points* simultaneously symmetrizes dQ, $A_0 dQ$ symmetric (weak simultaneous symmetrizability), and (ii) at equilibrium points, no eigenvector of dF is in the kernel of dQ (genuine coupling); see Lemma A.1, Appendix A. These two assumptions hold quite generally in applications, in particular for discrete kinetic equations and moment closure systems [10]. Assumption B3 is standard and easily checked.

Remark 1.3. Assumption C is satisfied for a family of profiles near w_0 if:
(i) dF is invertible and $\alpha_0(u_0) = 0$ (see Appendix of [7]).
(ii) dF is constant, $\alpha_0(u_0) = 0$, and dimension N bounded, e.g. in the case of discrete kinetic models with upper bound on the number of modes (see [8]). It has been shown to hold also for the infinite-dimensional case of the Boltzmann equation [6].

With these assumptions, our main result is as follows.

Theorem 1.4 (Spectral stability)**.** *Under assumptions* A1–A2, B1–B2–B3*, and* C*, for* $\varepsilon := |W_+ - W_-|$ *sufficiently small, the relaxation shock* W *is spectrally stable.*

As the argument is somewhat complicated, it may be helpful to outline here the structure of the proof. We start by carrying out the following by-now-standard "Kawashima-type" energy estimates on the relaxation system.

Proposition 1.5. *Assume hypothesis* A1-A2 *and* C*. Let $\lambda \in \mathbb{C}$ such that $\operatorname{Re}\lambda \geq 0$ and let w be a solution of (5). Then for ε sufficiently small, there hold:*

$$\operatorname{Re}\,\lambda\,|w|^2_{L^2} + |\Pi\, w|^2_{L^2} + |w'|^2_{L^2} \leq C\varepsilon^2\,|u|^2_{L^2} \qquad \textit{(Kawashima estimate)} \tag{12}$$

$$\operatorname{Re}\lambda \leq C\,\varepsilon^2, \qquad |\operatorname{Im}\lambda| \leq C\,\varepsilon. \tag{13}$$

where $\Pi := \Pi_+ + \Pi_-$.

Evidently, it remains only to obtain estimates on the equilibrium variable $|u|^2_{L^2}$. To this end, we carry out an approximate Chapman–Enskog expansion, keeping track of error terms, to obtain an effective viscous system for u of the same symmetric dissipative type, but with error terms coming from higher derivatives. Applying Goodman-type energy estimates to the integrated version of this reduced system, following [5], we obtain the desired bounds on $|u|^2_{L^2}$ modulo errors consisting of higher-derivative and dissipative terms (denoted by $\hat{v}$ in Section 3). Observing that these, by (12), may be absorbed in lower-order and equilibrium terms, we are done.

More precisely, we establish the following bounds on the reduced system. Here we use the following notation for the W'–weighted L^2–norm

$$|z|_{W'} := |\sqrt{|W'|}\,z|_{L^2} = \left(\int_{\mathbb{R}} |z|^2\,|W'|(x)\,dx\right)^{1/2}$$

The space of functions with bounded $|\cdot|_{W'}$ will be denoted by $L^2_{W'}$. Since W' is bounded, there holds $|z|_{W'} \leq C\,|z|_{L^2}$ for some $C > 0$ (a natural choice is $C := |W'|^{1/2}_{L^\infty}$). Hence $L^2 \subset L^2_{W'}$ with continuous injection. The opposite inequality is false since W' decays (exponentially fast) to zero as $|x| \to \infty$.

Proposition 1.6. *Assume hypothesis* B1-B2 *and* C*. Let z be defined as follows*

$$z(x) := \int_{-\infty}^{x} u(y)\,dy.$$

Then, for ε sufficiently small, there holds

$$\operatorname{Re}\lambda\,|z|^2_{L^2} + |u|^2_{L^2} \leq C|z|^2_{W'}. \tag{14}$$

Thanks to (14), it is sufficient to control the weighted norm $|z|_{W'}$.

Proposition 1.7. *Under assumptions* B1–B2–B3 *and* C*, for ε sufficiently small, there holds:*

$$\operatorname{Re}\lambda|z|^2_{L^2} + |z|^2_{W'} \leq C\big(\varepsilon|u|^2_{L^2} + \varepsilon|\Pi w|^2_{L^2} + \varepsilon^{-1}|w'|^2_{L^2}\big) \quad \textit{(Goodman estimate)} \tag{15}$$

where $\Pi := \Pi_+ + \Pi_-$.

Theorem 1.4 is an immediate consequence of Propositions 1.5, 1.6 and 1.7.

Proof of Theorem 1.4. Combining (12) and (15), we obtain

$$|z|^2_{W'} \leq C\,\varepsilon\,|u|^2_{L^2}.$$

With (14), this gives

$$|u|^2_{L^2} \leq C\,|z|^2_{W'} \leq C\,\varepsilon^2\,|u|^2_{L^2},$$

showing that, if $\varepsilon = |W_+ - W_-|$ is small enough, $w \equiv 0$. □

Discussion and open problems. We remark briefly on the setting of these results. Small-amplitude existence and stability were shown in [11, 7] and [9] under the additional noncharacteristicity assumption $\det \mathrm{d}F \neq 0$, or, equivalently, the condition that characteristic speeds of the background system do not vanish relative to the shock speed. This hypothesis suffices to treat simple model problems such as the Broadwell or Jin–Xin equations. However, as discussed in [8], it is unrealistic for models derived by discretization or moment closure from kinetic equations, since these may possess characteristics of any speed. Thus, it is highly desirable to remove this technical hypothesis, as we do here. The combination of Goodman- and Kawashima-type energy estimates was used in [5] to treat stability of viscous shock profiles for systems with real viscosity. A similar, but more complicated argument combining these ingredients was used in [6] to treat stability of Boltzmann profiles. These results motivate the present analysis, which essentially interpolates between the two.

Interesting open problems are verification of linearized and nonlinear stability in the same setting, assuming spectral stability, and the direct verification of C using stability estimates together with known bounds on the profile for the reduced system following the philosophy set out in [6].

Notations and (very) basic tools. Given $w_1, w_2 : \mathbb{R} \to \mathbb{C}^n$, we denote by $\langle w_1, w_2 \rangle$ the scalar product defined as follows

$$\langle w_1, w_2 \rangle := \int_{\mathbb{R}} \overline{w_1(x)} \cdot w_2(x)\,dx$$

where $\bar{w}$ denotes the complex conjugate vector of w. Given A, $n \times n$ matrix with complex entries, there holds

$$\mathrm{Re}\,\langle w, Aw\rangle = \frac{1}{2}\left(\langle w, Aw\rangle + \overline{\langle w, Aw\rangle}\right) = \langle w, A^* w\rangle$$

where $A^* := (A + \overline{A^t})/2$.

If $S : \mathbb{R} \to \mathbb{R}^{n\times n}$ is such that $S(x)$ is symmetric for any x, then

$$\langle w, S\, w'\rangle = \int_{\mathbb{R}} \overline{w} \cdot Sw'\, dx = -\int_{\mathbb{R}} \overline{Sw'} \cdot w\, dx - \int_{\mathbb{R}} \overline{w} \cdot S'\, w\, dx = -\overline{\langle w, S\, w'\rangle} - \langle w, S'\, w\rangle$$

Hence

$$\mathrm{Re}\,\langle w, S\, w'\rangle = -\frac{1}{2}\langle w, S'\, w\rangle. \tag{16}$$

Similarly, if K is skew–symmetric, then

$$\mathrm{Im}\,\langle w', Kw\rangle = -\frac{1}{2}\langle w, K'\, w\rangle \tag{17}$$

In particular, if K is constant, $\langle w', Kw\rangle$ is a real number.

From here on, we will denote with $O(1)$ any function of x, W and λ, locally bounded in $\{(x, W, \lambda) : \mathrm{Re}\,\lambda \geq 0\}$. As a consequence, given the functions $f, g \in L^2$ and $h \in L^2_{W'}$, the following estimates hold

$$|\langle f, O(1)\, W' g\rangle| \leq C|W'|_{L^\infty} \left(\eta |f|^2_{L^2} + \eta^{-1}|g|^2_{L^2}\right) \tag{18}$$

$$|\langle h, O(1)\, W' g\rangle| \leq C \left(\eta |h|^2_{W'} + \eta^{-1}|W'|_{L^\infty}|g|^2_{L^2}\right) \tag{19}$$

where η is any strictly positive constant and C is a constant independent on η.

2. Estimates for the full system

Lemma 2.1. *Let* $\varepsilon := |W_+ - W_-|$ *and assume hypothesis* A1 *and* C. *Let* $\lambda \in \mathbb{C}$ *such that* $\mathrm{Re}\,\lambda \geq 0$ *let* w *be a solution of (5) and let* K *be any constant skew-symmetric matrix. Then for* $\varepsilon, \eta > 0$ *both sufficiently small, there hold*

$$\mathrm{Re}\ \lambda\, |w|^2_{L^2} + |\Pi\, w|^2_{L^2} \leq C\, \varepsilon^2\, |w|^2_{L^2}, \tag{20}$$

$$\mathrm{Re}\,\lambda |w'|^2_{L^2} - \mathrm{Re}\,\langle w', A_0\, \mathrm{d}Q\, w'\rangle \leq C\, \varepsilon^2\, |w|^2_{H^1}; \tag{21}$$

$$\mathrm{Re}\ \langle w', K\, \mathrm{d}F\, w'\rangle \leq C \left(\varepsilon^2 |w|^2_{L^2} + \eta^{-1}\, |\Pi w|^2_{L^2} + (\varepsilon^2 + \eta)|w'|^2_{L^2}\right) \tag{22}$$

where $\Pi := \Pi_+ + \Pi_-$ *and* C *denotes a constant independent on* ε *and* η.

Proof. Taking the scalar product of $A_0(W)\,w$ against (5), we obtain

$$\lambda\langle A_0\,w,w\rangle+\langle A_0\,w,\big(\mathrm{d}F\,w\big)'\rangle-\langle A_0\,w,\mathrm{d}Q\,w\rangle=0. \tag{23}$$

Hence, using (16), we get

$$\mathrm{Re}\,\lambda\langle A_0\,w,w\rangle-\mathrm{Re}\,\langle w,A_0\,\mathrm{d}Q\,w\rangle\le-\mathrm{Re}\,\langle A_0\,w,\mathrm{d}^2F\,W'\,w\rangle+\frac{1}{2}\mathrm{Re}\,\langle w,\mathrm{d}(A_0\,\mathrm{d}F)\,W'\,w\rangle.$$

Since A_0 is positive definite, there holds for some $C>0$

$$\mathrm{Re}\,\lambda\,|w|^2_{L^2}-\mathrm{Re}\,\langle w,A_0\,\mathrm{d}Q\,w\rangle\le C\,|W'|_{L^\infty}|w|^2_{L^2}. \tag{24}$$

Let us set

$$\Phi(W):=\frac{|W-W_+|}{|W_+-W_-|}(A_0\,\mathrm{d}Q)(W_-)+\frac{|W-W_-|}{|W_+-W_-|}(A_0\,\mathrm{d}Q)(W_+).$$

Then there holds, for some $C>0$,

$$|\Phi(W)-(A_0\,\mathrm{d}Q)(W)|\le C|W-W_-||W-W_+|.$$

Therefore

$$\mathrm{Re}\,\langle w,A_0\,\mathrm{d}Q\,w\rangle\le\mathrm{Re}\,\langle w,\Phi(W)\,w\rangle+C|W-W_-|_{L^\infty}|W-W_+|_{L^\infty}|w|^2_{L^2}$$

Thanks to (6), we get

$$\mathrm{Re}\,\langle w,A_0\,\mathrm{d}Q\,w\rangle\le-c|\Pi w|^2_{L^2}+C|W-W_-|_{L^\infty}|W-W_+|_{L^\infty}|w|^2_{L^2}$$

for some $C,c>0$. Hence, using (24), we obtain

$$\mathrm{Re}\,\lambda\,|w|^2_{L^2}+|\Pi\,w|^2_{L^2}\le C\,\big(|W-W_+|_{L^\infty}|W-W_-|_{L^\infty}+|W'|_{L^\infty}\big)|w|^2_{L^2}. \tag{25}$$

In term of ε, we get the *0-th order Friedrichs estimate* (20).

Differentiating (5) with respect to x,

$$\lambda\,w'+\big(\mathrm{d}F(W)w\big)''-(\mathrm{d}Q(W)w)'=0. \tag{26}$$

Taking the scalar product of $A_0(W)\,w'$ against (26), we get

$$\lambda\langle A_0\,w',w'\rangle-\langle w',A_0\,\mathrm{d}Q\,w'\rangle=-\langle A_0\,w',\big(\mathrm{d}F\,w\big)''\rangle+\langle A_0\,w',\mathrm{d}^2Q\,W'\,w\rangle. \tag{27}$$

Since

$$\begin{aligned}\langle A_0\,w',\big(\mathrm{d}F\,w\big)''\rangle&=\langle A_0\,w',\mathrm{d}^3F\,W'\,W'\,w\rangle+\langle A_0\,w',\mathrm{d}^2F\,W''\,w\rangle\\&\quad+2\langle A_0\,w',\mathrm{d}^2F\,W'\,w'\rangle+\langle w',A_0\,\mathrm{d}F\,w''\rangle,\end{aligned}$$

taking the real part and using (16), we obtain

$$\mathrm{Re}\,\langle A_0\,w',\big(\mathrm{d}F\,w\big)''\rangle=\langle w',O(1)W'\,w\rangle+\langle w',O(1)W''\,w\rangle$$

$$+\langle w', O(1) W' \, w' \rangle - \frac{1}{2} \langle w', \mathrm{d}(A_0 \, \mathrm{d}F) \, W' \, w' \rangle.$$

Hence, the following estimates holds

$$\left| \mathrm{Re} \, \langle A_0 \, w', \big(\mathrm{d}F \, w \big)'' \rangle \right| \leq C \left(|W'|_{L^\infty} + |W''|_{L^\infty} \right) |w|^2_{H^1}$$

$$\left| \mathrm{Re} \, \langle A_0 \, w', \mathrm{d}^2 Q \, W' \, w \rangle \right| \leq C \, |W'|_{L^\infty} |w|^2_{H^1}.$$

Therefore, from (27), using (10), we deduce

$$\mathrm{Re} \, \lambda |w'|^2_{L^2} - \mathrm{Re} \, \langle w', A_0 \, \mathrm{d}Q \, w' \rangle \leq C \, |W'|_{L^\infty} \, |w|^2_{H^1}.$$

Thus in term of ε, we obtain the *1-st order Friedrichs estimate* (21).

Now, let K be any constant skew-symmetric matrix. Applying K to the resolvent equation (5) and multiplying by w', we get

$$\lambda \, \langle w', K w \rangle + \langle w', K \big(\mathrm{d}F \, w \big)' \rangle - \langle w', K \, \mathrm{d}Q \, w \rangle = 0.$$

Taking the real parts and rearranging the terms, we obtain

$$\mathrm{Re} \, \langle w', K \, \mathrm{d}F \, w' \rangle = -\mathrm{Re} \, \big(\lambda \, \langle w', K w \rangle \big) - \mathrm{Re} \, \langle w', K \, \mathrm{d}^2 F \, W' \, w \rangle + \mathrm{Re} \, \langle w', K \, \mathrm{d}Q \, w \rangle.$$

Hence, thanks to (17), there holds $\mathrm{Im} \, \langle w', K w \rangle = 0$, since K is constant. Therefore, for $\mathrm{Re} \, \lambda \geq 0$, we obtain

$$|\mathrm{Re} \, \big(\lambda \, \langle w', K w \rangle \big)| = \mathrm{Re} \, \lambda \, |\mathrm{Re} \, \langle w', K w \rangle| \leq C \, \mathrm{Re} \, \lambda \, |w|^2_{H^1}.$$

Let us set

$$\Psi(W) := \frac{|W - W_+|}{|W_+ - W_-|} K \, \mathrm{d}Q(W_-) + \frac{|W - W_-|}{|W_+ - W_-|} K \, \mathrm{d}Q(W_+).$$

Then there holds, for some $C > 0$,

$$|\Psi(W) - K \, \mathrm{d}Q(W)| \leq C |W - W_-| |W - W_+|.$$

Therefore

$$\mathrm{Re} \, \langle w', K \, \mathrm{d}Q \, w \rangle \leq \mathrm{Re} \, \langle w', \Psi(W) \, w \rangle + C |W - W_-|_{L^\infty} |W - W_+|_{L^\infty} |w|^2_{H^1}$$

Thanks to (6), we get

$$\mathrm{Re} \, \langle w', K \, \mathrm{d}Q \, w \rangle \leq C \left(\eta^{-1} \, |\Pi w|^2_{L^2} + \eta |w'|^2_{L^2} \right) + C |W - W_-|_{L^\infty} |W - W_+|_{L^\infty} |w|^2_{L^2}$$

where $\eta > 0$ is a positive constant to be chosen later on (small enough). Hence, the following estimate holds

$$\mathrm{Re} \, \langle w', K \, \mathrm{d}F \, w' \rangle = C \left\{ (\mathrm{Re} \, \lambda + |W'|_{L^\infty} + |W - W_-|_{L^\infty} |W - W_+|_{L^\infty}) |w|^2_{H^1} \right.$$
$$\left. + \, \eta^{-1} \, |\Pi w|^2_{L^2} + \eta |w'|^2_{L^2} \right\}$$

By (20), Re$\lambda \leq C\,\varepsilon^2$, hence, in term of ε, we get (22). This concludes the proof of Lemma 2.1. □

Assuming, in addition, hypothesis A2, we prove estimates (12) and (13).

Proof of Proposition 1.5. Thanks to the small amplitude assumption, it is possible to choose $K = K(W_+)$ constant in the Shizuta–Kawashima condition (7), since this is an open condition so persists under small perturbations. Hence, summing estimates (20)-(21) with (22), we obtain

$$\text{Re}\,\lambda\,|w|^2_{L^2} + |\Pi\,w|^2_{L^2} + |w'|^2_{L^2} \leq C\,\varepsilon^2\,|w|^2_{H^1} + C\,(1+\eta^{-1})\varepsilon^2|w|^2_{L^2} + C(\varepsilon^2+\eta)|w'|^2_{L^2}$$

which yields

$$\text{Re}\,\lambda\,|w|^2_{L^2} + |\Pi\,w|^2_{L^2} + |w'|^2_{L^2} \leq C\,\varepsilon^2\,|w|^2_{L^2}$$

for ε and η sufficiently small. Since $|w|^2_{L^2} \leq C\big(|u|^2_{L^2} + |\Pi\,w|^2_{L^2}\big)$ we get the estimate (12) for ε small.

Estimate (20) implies the bound on the real part of the eigenvalue λ. Taking the imaginary part of (23),

$$\text{Im}\,\big(\lambda\,\langle A_0\,w, w\rangle\big) = -\text{Im}\,\langle A_0\,w, \text{d}^2F\,W'\,w\rangle - \text{Im}\,\langle A_0\,w, \text{d}F\,w'\rangle + \text{Im}\,\langle A_0\,w, \text{d}Q\,w\rangle.$$

Hence, for $\eta > 0$ to be chosen,

$$|\text{Im}\,\lambda|\,|w|^2_{L^2} \leq C\left(|W'|_{L^\infty}\,|w|^2_{L^2} + \eta\,|w|^2_{L^2} + \eta^{-1}|w'|^2_{L^2} + \eta^{-1}|\Pi w|^2_{L^2}\right).$$

Thanks to (10) and (12), we get

$$|\text{Im}\,\lambda|\,|w|^2_{L^2} \leq C\left(\varepsilon^2 + \eta + \eta^{-1}\varepsilon^2\right)\,|w|^2_{L^2}.$$

Thus, choosing $\eta = \varepsilon$, we obtain the result for ε small enough. □

3. The reduced system for the conserved integrated variables

As stressed in the Introduction, the next step consists in estimating the conserved densities u in term of an appropriate weighted L^2-norm of the conserved quantities z, defined by

$$z(x) := \int_{-\infty}^{x} u(y)\,dy. \tag{28}$$

The first step is to the deduce a balance law satisfied by the variable z with source terms depending on Πw and w'.

Setting

$$\mathrm{d}F := \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \quad \text{and} \quad \mathrm{d}Q := \begin{pmatrix} 0 & 0 \\ q_1 & q_2 \end{pmatrix},$$

equation (5) can be rewritten as

$$\begin{cases} \lambda\, u + (A_{11}\, u + A_{12}\, v)' = 0, \\ \lambda\, v + (A_{21}\, u + A_{22}\, v)' - q_1 u - q_2 v = 0. \end{cases}$$

Let $\hat{v} := q_2^{-1}\, q_1\, u + v$. In particular, $c_1|\hat{v}|_{L^2} \leq |\Pi\, w|_{L^2} \leq c_2|\hat{v}|_{L^2}$ for some $c_1, c_2 > 0$. Hence, in the following, we consider the variable $\hat{v}$ in place of Πw.

Then the couple $(u, \hat{v})$ satisfies

$$\begin{cases} \lambda\, u + \big(a\, u + A_{12}\, \hat{v}\big)' = 0, \\ (\lambda I_r - q_2)\, \hat{v} + \big(c\, u + A_{22}\, \hat{v}\big)' + q_2^{-1} q_1 \big(a\, u + A_{12}\, \hat{v}\big)' = 0 \end{cases}$$

where

$$a := A_{11} - A_{12} q_2^{-1} q_1, \qquad c := A_{21} - A_{22} q_2^{-1} q_1.$$

With z defined in (28), we can write the above system as

$$\begin{cases} \lambda\, z + a\, z' + A_{12}\, \hat{v} = 0, \\ (\lambda I_r - q_2)\, \hat{v} + \big(c\, z' + A_{22}\, \hat{v}\big)' + q_2^{-1} q_1 \big(a\, z' + A_{12}\, \hat{v}\big)' = 0 \end{cases}$$

Next the idea is to obtain an expression for $\hat{v}$ from the second equation and inserting it in the first one, in order to obtained a reduced system of viscous conservation laws with source terms. Since we want to derive energy estimates, it is useful to change variables in the first equation in order to symmetrize the term containing the first order derivative.

Let a_0 be a symmetric and positive definite matrix such that $a_0 a$ is symmetric, as in assumption B1. Let $\tilde{z} := a_0^{1/2} z$. The new variable $\tilde{z}$ and the variable $\hat{v}$ satisfy

$$\begin{cases} \lambda\, \tilde{z} + a_0^{1/2}\, a\, (a_0^{-1/2}\, \tilde{z})' + a_0^{1/2}\, A_{12}\, \hat{v} = 0, \\ (\lambda I_r - q_2)\, \hat{v} + \big(c\, (a_0^{-1/2}\, \tilde{z})' + A_{22}\, \hat{v}\big)' + q_2^{-1} q_1 \big(a\, (a_0^{-1/2}\, \tilde{z})' + A_{12}\, \hat{v}\big)' = 0 \end{cases}$$

Hence

$$\begin{cases} \lambda\, \tilde{z} + \tilde{a}\, \tilde{z}' + a_0^{1/2}\, A_{12}\, \hat{v} = O(1)\, W'\, \tilde{z}', \\ (\lambda I_r - q_2)\, \hat{v} + \big(c\, a_0^{-1/2}\, \tilde{z}' + O(1) W' \tilde{z} + O(1) \hat{v}\big)' \\ \qquad\qquad + q_2^{-1} q_1 \big(a\, a_0^{-1/2}\, \tilde{z}' + O(1) W' \tilde{z} + O(1) \hat{v}\big)' = 0 \end{cases}$$

where the matrix $\tilde{a} := a_0^{1/2}\, a\, a_0^{-1/2}$ is symmetric. From the second equation, using (10), we get

$$\hat{v} = -(\lambda I_r - q_2)^{-1} \big(c + q_2^{-1} q_1 a\big) a_0^{-1/2}\, \tilde{z}'' + O(1) W' \big(\tilde{z}' + \varepsilon \tilde{z} + \hat{v}\big) + O(1) \hat{v}', \tag{29}$$

or, equivalently, using the $O(1)$ notation,

$$\hat{v} = O(1)W'\big(\tilde{z}' + \varepsilon\tilde{z} + \hat{v}\big) + O(1)w'. \tag{30}$$

Plugging (29) in the equation satisfied by $\tilde{z}$, we get

$$\lambda\,\tilde{z} + \tilde{a}\,\tilde{z}' - \tilde{b}\tilde{z}'' = O(1)W'\big(\tilde{z}' + \varepsilon\tilde{z} + \hat{v}\big) + O(1)\hat{v}' \tag{31}$$

where

$$b := A_{12}(\lambda I_r - q_2)^{-1}\big(c + q_2^{-1}q_1 a\big) \qquad \text{and} \qquad \tilde{b} := a_0^{1/2}\,b\,a_0^{-1/2}.$$

Estimate on $\tilde{z}$ will be obtained by mutliplying (31) by z and integrating. With the present form, we would obtain a "bad" term $\langle\tilde{z}, O(1)\tilde{v}'\rangle$. For this reason, it is useful to use the relation (30) to obtain the following new version of (31)

$$\lambda\,\tilde{z} + \tilde{a}\,\tilde{z}' - \tilde{b}\tilde{z}'' = O(1)W'\big(\tilde{z}' + \varepsilon\tilde{z} + \hat{v}\big) + O(1)\big(O(1)W'\big(\tilde{z}' + \varepsilon\tilde{z} + \hat{v}\big) + O(1)w'\big)'.$$

For $\lambda = 0$, the term b represents the viscosity term given by the Chapman–Enskog expansion for the relaxation system. Hence, it is significant to decompose b as follows

$$b = b_0 + \lambda b_1$$

where matrices b_0 and b_1 are given by

$$b_0 := -A_{12}q_2^{-1}(c + q_2^{-1}q_1 a), \quad b_1 := A_{12}(\lambda - q_2)^{-1}q_2^{-1}(c + q_2^{-1}q_1 a).$$

Hence we get the following equation satisfied by the variable $\tilde{z}$

$$\lambda\,\tilde{z} + \tilde{a}\,\tilde{z}' - \tilde{b}_0\,\tilde{z}'' = \lambda\tilde{b}_1\tilde{z}'' + \Theta.$$

where $\tilde{b}_i := a_0^{1/2}\,b_i\,a_0^{-1/2}$ for $i = 0, 1$ and Θ is appropriately defined. By assumptions B1. on the reduced system, the matrix $\tilde{b}_0$ is symmetric and positive semidefinite.

From now on, we drop the tildas for shortness and consider the following equation

$$\lambda\,z + a\,z' - b_0\,z'' = \lambda b_1 z'' + \Theta_1 + \Theta_2'. \tag{32}$$

where a is a symmetric matrix, b_0 is a symmetric, positive semidefinite matrix, and

$$\begin{cases} \Theta_1 := O(1)W'\big(z' + \varepsilon z + \hat{v} + w'\big) \\ \Theta_2 := O(1)W'\big(z' + \varepsilon z + \hat{v}\big) + O(1)w' \end{cases} \tag{33}$$

Lemma 3.1. *Assume hypothesis* C. *Let $\lambda \in \mathbb{C}$ satisfying (13), let z be a solution (32) with a symmetric, b_0 symmetric, positive semidefinite and Θ_1, Θ_2 given in (33) and let k be a smooth function from $\mathbb{R}^n$ to the set of real skew-symmetric matrices.*

Then, for $\varepsilon, \eta > 0$ sufficiently small, the following estimates hold:

$$\mathrm{Re}\,\lambda\,|z|^2_{L^2} + \langle z', b_0\, z'\rangle \le C\big(|z|^2_{W'} + (\varepsilon+\eta)|z'|^2_{L^2} + \varepsilon^2|\hat v|^2_{L^2} + \eta^{-1}|w'|^2_{L^2}\big)$$
$$\mathrm{Re}\,\langle z', k\, a\, z'\rangle \le C\big(\eta|\mathrm{Re}\,\lambda||z|^2_{L^2} + \varepsilon|z|^2_{W'} + (\eta^{-1}\varepsilon^2+\eta)|z'|^2_{L^2} + \varepsilon^2|\hat v|^2_{L^2} + \eta^{-1}\,|w'|^2_{L^2}\big)$$

Proof. Taking the real part of the scalar product of z against (32), we get

$$\mathrm{Re}\,\lambda\,|z|^2_{L^2} - \frac{1}{2}\langle z, \mathrm{d}a\, W' z\rangle - \mathrm{Re}\,\langle z, b_0\, z''\rangle = \mathrm{Re}\,\langle z, \lambda\, b_1\, z''\rangle + \mathrm{Re}\,\langle z, \Theta\rangle.$$

having used the symmetry of a. Since

$$\begin{aligned}\mathrm{Re}\,\langle z, b_0\, z''\rangle &= -\mathrm{Re}\,\langle z', b_0\, z'\rangle - \mathrm{Re}\,\langle z, \mathrm{d}b_0\, W' z'\rangle\\ &\le -\mathrm{Re}\,\langle z', b_0\, z'\rangle + C(|z|^2_{W'} + |W'|_{L^\infty}|z'|^2_{L^2}).\end{aligned}$$

we obtain, thanks to (10),

$$\mathrm{Re}\,\lambda\,|z|^2_{L^2} + \mathrm{Re}\,\langle z', b_0\, z'\rangle \le C\big(|z|^2_{W'} + \varepsilon^2|z'|^2_{L^2} + \mathrm{Re}\,\langle z, \lambda\, b_1\, z''\rangle + \mathrm{Re}\,\langle z, \Theta\rangle\big).$$

The term containing b_1 can be easily estimated by

$$\begin{aligned}|\mathrm{Re}\,\langle z, \lambda b_1\, z''\rangle| &\le |\lambda|\,(|\langle z', b_1\, z'\rangle| + |\langle z, \mathrm{d}b_0\, W' z'\rangle|)\\ &\le C|\lambda|\left(|z'|^2_{L^2} + |z|^2_{W'} + |W'|_{L^\infty}|z'|^2_{L^2}\right).\end{aligned}$$

Taking in account (10) and (13), we get

$$|\mathrm{Re}\,\langle z, \lambda b_1\, z''\rangle| \le C\,\varepsilon\left(|z'|^2_{L^2} + |z|^2_{W'}\right).$$

Therefore, we obtain

$$\mathrm{Re}\,\lambda\,|z|^2_{L^2} + \mathrm{Re}\,\langle z', b_0\, z'\rangle \le C\big(|z|^2_{W'} + \varepsilon|z'|^2_{L^2} + \mathrm{Re}\,\langle z, \Theta\rangle\big). \tag{34}$$

It remains to deal with the term with Θ. For what concerns Θ_1, using (19) (with $\eta = 1$) and (10), we have

$$|\mathrm{Re}\,\langle z, \Theta_1\rangle| \le C|z|^2_{W'} + C\varepsilon^2\big(|z'|^2_{L^2} + |\hat v|^2_{L^2} + |w'|^2_{L^2}\big)$$

The term with Θ_2 can be dealt with integrating by parts

$$|\mathrm{Re}\,\langle z, \Theta_2'\rangle| = |\mathrm{Re}\,\langle z', \Theta_2\rangle| \le C\varepsilon|z|^2_{W'} + C(\varepsilon^2+\eta)|z'|^2_{L^2} + C\varepsilon^2|\hat v|^2_{L^2} + C\eta^{-1}|w'|^2_{L^2}$$

where η is any positive constant and C is independent on η. Inserting the last three estimates in (34), we get i. in Lemma 3.1.

Applying k to (32), k as defined in B2, and taking the L^2 scalar product against z', we get

$$\mathrm{Re}\,\langle z', k\, a\, z'\rangle = -\mathrm{Re}\,\lambda\,\langle z', kz\rangle + \mathrm{Re}\,\langle z', k\, b\, z''\rangle + \mathrm{Re}\,\langle z', k\Theta\rangle. \tag{35}$$

Using the eigenvalue estimate on $\mathrm{Re}\,\lambda$, stated in Proposition 1.5, we obtain

$$\mathrm{Re}\,\langle z', k\,a\,z'\rangle \le C\left(\eta|\mathrm{Re}\,\lambda||z|^2_{L^2} + (\eta^{-1}\varepsilon^2+\eta)\,|z'|^2_{L^2} + \eta^{-1}\,|z''|^2_{L^2}\right) + \mathrm{Re}\,\langle z', k\Theta\rangle$$

Finally, using once more (10), for the term with Θ there hold

$$\begin{aligned}|\mathrm{Re}\,\langle z', k\Theta_1\rangle| &\le C\varepsilon|z|^2_{W'} + C\varepsilon^2\left(|z'|^2_{L^2} + |\hat v|^2_{L^2} + |w'|^2_{L^2}\right)\\ |\mathrm{Re}\,\langle z', k\Theta_2'\rangle| &\le |\mathrm{Re}\,\langle z'', k\Theta_2\rangle| + |\mathrm{Re}\,\langle z', O(1)W'\Theta_2\rangle|\\ &\le C\varepsilon|z|^2_{W'} + C\varepsilon^2\big(|z'|^2_{L^2} + |\hat v|^2_{L^2}\big) + C|w'|^2_{L^2}.\end{aligned}$$

Collecting all of these estimates, we complete the proof of Lemma 3.1. □

Proof of Proposition 1.6. Choosing $\eta = \varepsilon$ and summing up the estimates in Lemma 3.1, we obtain, for ε small enough,

$$\mathrm{Re}\,\lambda\,|z|^2_{L^2} + |z'|^2_{L^2} \le C\big(|z|^2_{W'} + \varepsilon^2|\hat v|^2_{L^2} + \varepsilon^{-1}\,|w'|^2_{L^2}\big)$$

Using (in place of a first-order Friedrichs estimate) the bound

$$|\hat v|^2_{L^2} + |w'|^2_{L^2} \le C\left(|\Pi v|^2_{L^2} + |w'|^2_{L^2}\right) \le C\varepsilon^2|u|^2_{L^2}$$

obtained in Proposition 1.5, we get

$$\mathrm{Re}\,\lambda\,|z|^2_{L^2} + |u|^2_{L^2} \le C|z|^2_{W'} + C\varepsilon|u|^2_{L^2}.$$

Hence estimate (14) holds for ε small. □

The reduced Kawashima estimate (14) shows that it is possible to bound the L^2 estimate of u in term of $|z|_{W'}$. If we are able to prove a Poincaré–like inequality and bound the weighted norm $|z|_{W'}$ by small multiples of the L^2 norm of u and higher derivatives, we are done. This we can accomplish by changing variables in an appropriate way and applying a weighted energy method in the spirit of Goodman [4, 5, 6].

Lemma 3.2 ([5]). *Let $a = a(W)$ and $b = b(W)$ be symmetric matrices, $b(W) \ge 0$, with one eigenvalue α_0 of a close to zero and the others strictly negative or positive (and uniformly separated from α_0). Then, there exist smooth, real matrix-valued functions $r = r(x)$, $\ell = \ell(x)$, $\ell(x)\,r(x) = I$ for any x, satisfying, for some $C, c > 0$,*

$$(\ell\,r')_{pp} = 0, \qquad |\ell'|, |r'| \le C|W'|;$$

$$\ell\,a\,r = \mathrm{diag}\,(\alpha_-, \alpha_p, \alpha_+) = \begin{pmatrix}\alpha_- & 0 & 0\\ 0 & \alpha_p & 0\\ 0 & 0 & \alpha_+\end{pmatrix}; \tag{36}$$

with α_p scalar, α_-, α_+ symmetric square matrices (with dimensions $p-1$ and $n-p$ respectively), $\alpha_- \leq -c < 0 < c \leq \alpha_+$; and

$$\operatorname{Re} \ell\, b\, r \geq -C\varepsilon. \tag{37}$$

Proof. Since a is symmetric, it is possible to find an othonormal transformation $\omega = \omega(W)$ such that $\omega^t\, a\, \omega$ is (block-)diagonal with the decomposition given in the righthand side of (36). The spectral separation assumption guarantees the positivity/negativity of α_+/α_-. Moreover, the matrix $\omega^t\, b\, \omega$ is positive semidefinite, $\operatorname{Re} b \geq 0$.

Let ω_p denote the pth column of ω and $\gamma = \gamma(x)$ be the solution of the first order linear differential equation

$$\gamma' = -(\omega_p \cdot \mathrm{d}\omega_p\, W')\, \gamma, \quad \gamma(0) = 1, \tag{38}$$

or, equivalently, set

$$\gamma(x) := \exp\left(\int_0^x \omega_p^t(W)\, \mathrm{d}\omega_p(W)\, W'\, dy \right). \tag{39}$$

Define the matrix r and ℓ as

$$r(x) := \omega(W)\, \mathrm{diag}\,(I_{p-1}, \gamma(x), I_{n-p}), \qquad \ell(x) := r^{-1}(x). \tag{40}$$

Clearly estimates on $|r'|$ and $|\ell'|$ hold and

$$\begin{aligned} \ell\, a\, r &= \mathrm{diag}\,(I_{p-1}, \gamma^{-1}, I_{n-p})\mathrm{diag}\,(\alpha_-, \alpha_p, \alpha_+)\mathrm{diag}\,(I_{p-1}, \gamma, I_{n-p}) \\ &= \mathrm{diag}\,(\alpha_-, \alpha_p, \alpha_+), \end{aligned}$$

hence ℓ and r still block-diagonalize a in the manner claimed. Moreover

$$(\ell r')_{pp} = \gamma^{-1}\omega_p \cdot (\gamma\omega_p)' = \omega_p \cdot \big(\mathrm{d}\omega_p W' - (\omega_p \cdot \mathrm{d}\omega_p\, W')\omega_p\big) = 0,$$

since ω_p has norm equal to 1.

By (39), it follows that $\gamma = 1 + O(\varepsilon)$; hence bound (37) follows from assumptions on b and continuity. □

Proof of Proposition 1.7. To prove (15), it is sufficient to establish the corresponding result for $\zeta := \ell\, z$ with ℓ given in Lemma 3.2. Left multiplying (32) by ℓ, we get the eigenvalue equation for ζ

$$\lambda\, \zeta + \alpha\, \ell\, r' \zeta + \alpha\, \zeta' - \beta\, \zeta'' = \Xi, \tag{41}$$

where

$$\alpha := \ell\, a\, r, \qquad \beta := \ell\, b_0\, r, \qquad \Xi := \Xi_1 + \Xi_2'$$

and

$$\begin{cases} \Xi_1 := O(1)|W'|\big(\varepsilon\,\zeta + \zeta' + \hat{v}' + w'\big), \\ \Xi_2 := O(1)|W'|\big(\zeta + \hat{v}\big) + O(1)\varepsilon\,\zeta' + O(1)\,w'. \end{cases}$$

Following [4], set $\rho_0(x) := 1$ for any x, and define the two weights $\rho_\pm$ as the solutions to the Cauchy problem

$$\rho_\pm' = \mp M\,|W'|\,c^{-1}\,\rho_\pm, \qquad \rho_\pm(0) := 1, \tag{42}$$

where c is given in Lemma 3.2 and M is a constant to be chosen later. Therefore, for ε so small that $O(M\varepsilon) < 1$,

$$\rho_\pm(x) = \exp\left(\pm\int_0^x M|W'(\xi)|c^{-1}\,d\xi\right) = 1 + O\left(M\int_{\mathbb{R}}|W'(\xi)|\,d\xi\right) = O(1), \tag{43}$$

and

$$\rho_j'(x) = O(1)\,|W'(x)|, \qquad j \in \{-,0,+\}. \tag{44}$$

Let $\rho = \rho(x)$ be the block diagonal matrix defined by

$$\rho(x) := \mathrm{diag}\,(\rho_-(x)\,I_h, \rho_0(x), \rho_+(x)\,I_k)$$

where I_n denotes the identity $n\times n$ matrix. Taking the real part of the L^2-scalar product of $\rho\,\zeta$ against (41), we get

$$\mathrm{Re}\,\lambda\,\langle\rho\,\zeta,\zeta\rangle + \mathrm{Re}\,\langle\rho\,\zeta,\alpha\,\ell\,r'\zeta\rangle + \mathrm{Re}\,\langle\rho\,\zeta,\alpha\,\zeta'\rangle - \mathrm{Re}\,\langle\rho\,\zeta,\beta\,\zeta''\rangle = \mathrm{Re}\,\langle\rho\,\zeta,\Xi\rangle. \tag{45}$$

The weights $\rho_0, \rho_\pm$ are positive and $O(1)$, hence $\langle\rho\,\zeta,\zeta\rangle^{1/2}$ is equivalent to $|\zeta|_{_{L^2}}$.

Both ρ and $\rho\,\alpha$ are symmetric, hence

$$\mathrm{Re}\,\langle\rho\,\zeta,\alpha\,\zeta'\rangle = \mathrm{Re}\,\langle\zeta,\rho\,\alpha\,\zeta'\rangle = -\frac{1}{2}\mathrm{Re}\,\langle\zeta,(\rho'\,\alpha + \rho\,\mathrm{d}\alpha\,W')\,\zeta\rangle.$$

By (9) and (11), we have the key fact[3]

$$\mathrm{d}\alpha_0\,W' \le -C|W'|$$

for some $C > 0$. By definition of $\rho_\pm$, we have also

$$\rho_\pm'\,\alpha_\pm + \rho_\pm\,\mathrm{d}\alpha_\pm\,W' = \mp\rho_\pm\,(M\,|W'|c^{-1}\alpha_\pm - \mathrm{d}\alpha_\pm\,W').$$

Thus, for M sufficiently large, there exists $C > 0$, independent on ε, such that

$$\begin{aligned}\rho'\,\alpha + \rho\,\mathrm{d}\alpha\,W' &= \mathrm{diag}\,(\rho_-'\,\alpha_- + \rho_-\,\mathrm{d}\alpha_-\,W', \mathrm{d}\alpha_0\,W', \rho_+'\,\alpha_+ + \rho_+\,\mathrm{d}\alpha_+\,W') \\ &\le -C\,|W'|\,\mathrm{diag}\,(M,1,M).\end{aligned}$$

[3]Indeed, this is what drives the Goodman estimate; see [3, 4].

Decomposing ζ as $(\zeta_-, \zeta_0, \zeta_+)$ and setting $\hat\zeta := (\zeta_-, \zeta_+)$, we get the "good" term

$$\mathrm{Re}\,\langle \rho\zeta, \alpha\,\zeta'\rangle \geq C \int_{\mathbb{R}} (M|\hat\zeta|^2 + |\zeta_0|^2)\,|W'|\,dx.$$

Next, let us deal with the "bad" term $\langle \rho\,\zeta, \alpha\,\ell\,r'\zeta\rangle$. Since $(\ell\,r')_{pp} = 0$, there holds

$$|\mathrm{Re}\,\langle \rho\,\zeta, \alpha\,\ell\,r'\zeta\rangle| \leq C \int_{\mathbb{R}} |\hat\zeta|^2 |W'|\,dx$$

Hence, by choosing M large enough, we get from (45)

$$\mathrm{Re}\,\lambda|\zeta|^2_{L^2} + |\zeta|^2_{W'} - \mathrm{Re}\,\langle \rho\,\zeta, \beta\,\zeta''\rangle \leq C\,\big|\mathrm{Re}\,\langle \rho\,\zeta, \Xi\rangle\big|$$

Since

$$\mathrm{Re}\,\langle \rho\,\zeta, \beta\,\zeta''\rangle = -\mathrm{Re}\,\langle \rho\,\zeta', \beta\,\zeta'\rangle - \mathrm{Re}\,\langle \rho\,\zeta, \beta'\,\zeta'\rangle - \mathrm{Re}\,\langle \rho'\,\zeta, \beta\,\zeta'\rangle$$

the term with β can be estimated by

$$\mathrm{Re}\,\langle \rho\,\zeta, \beta\,\zeta''\rangle \leq C\varepsilon\big(|\zeta|^2_{W'} + |\zeta'|^2_{L^2}\big)$$

having used (37). Hence, we obtain

$$\mathrm{Re}\,\lambda|\zeta|^2_{L^2} + |\zeta|^2_{W'} \leq C\varepsilon\big(|\zeta|^2_{W'} + |\zeta'|^2_{L^2}\big) + C\,\big|\mathrm{Re}\,\langle \rho\,\zeta, \Xi\rangle\big| \tag{46}$$

Given $\eta > 0$, recalling (10), we deduce

$$\big|\mathrm{Re}\,\langle \rho\,\zeta, \Xi_1\rangle\big| \leq C(\varepsilon + \eta)|\zeta|^2_{W'} + C\eta^{-1}\varepsilon^2(|\zeta'|^2_{L^2} + |\hat v|^2_{L^2} + |w'|^2_{L^2})$$

with C independent on η. For what concerns the term with Ξ_2, integrating by parts and using (44), there holds

$$\big|\mathrm{Re}\,\langle \rho\,\zeta, \Xi_2'\rangle\big| = \big|\mathrm{Re}\,\langle \rho'\,\zeta, \Xi_2\rangle\big| + \big|\mathrm{Re}\,\langle \rho\,\zeta', \Xi_2\rangle\big| \leq \big|\mathrm{Re}\,\langle O(1)|W'|\,\zeta, \Xi_2\rangle\big| + \big|\mathrm{Re}\,\langle \rho\,\zeta', \Xi_2\rangle\big|$$

For any $\eta > 0$, estimating one by one the terms in Ξ_2, we obtain

$$\big|\mathrm{Re}\,\langle O(1)|W'|\,\zeta, \Xi_2\rangle\big| \leq C(\varepsilon + \eta)|\zeta|^2_{W'} + C\varepsilon^2(|\zeta'|^2_{L^2} + |\hat v|^2_{L^2}) + C\eta^{-1}\varepsilon^2|w'|^2_{L^2},$$

$$\big|\mathrm{Re}\,\langle \rho\,\zeta', \Xi_2\rangle\big| \leq C\eta|\zeta|^2_{W'} + C(\varepsilon + \eta^{-1}\varepsilon^2 + \eta)|\zeta'|^2_{L^2} + C\eta^{-1}\varepsilon^2|\hat v|^2_{L^2} + C\eta^{-1}|w'|^2_{L^2}.$$

Choosing $\eta = \varepsilon$ and summing up, we get

$$\big|\mathrm{Re}\,\langle \rho\,\zeta, \Xi_2'\rangle\big| \leq C\varepsilon\big(|\zeta|^2_{W'} + |\zeta'|^2_{L^2} + |\hat v|^2_{L^2}\big) + C\varepsilon^{-1}|w'|^2_{L^2}.$$

Inserting these estimates in (46), we get, for ε sufficiently small,

$$\mathrm{Re}\,\lambda|\zeta|^2_{L^2} + |\zeta|^2_{W'} \leq C\big(|\zeta'|^2_{L^2} + \varepsilon|\hat v|^2_{L^2} + \varepsilon^{-1}|w'|^2_{L^2}\big).$$

Since $\zeta = \ell\,z$, from the above estimate we deduce

$$\mathrm{Re}\,\lambda|z|^2_{L^2} + |z|^2_{W'} \leq C\big(\varepsilon|z'|^2_{L^2} + \varepsilon|\hat v|^2_{L^2} + \varepsilon^{-1}|w'|^2_{L^2}\big).$$

for ε sufficiently small. Recalling that $z' = u$, estimate (15) is proved. $\square$

Appendix A. Structural hypotheses

In this Appendix, we briefly discuss the structural hypotheses of the introduction, verifying the assertions of Remark 1.2 that A1–A2 and B1–B2 follow from conditions (i)–(ii) of the remark (i.e., partial simultaneous symmetrizability plus genuine coupling) together with the assumed structure $Q = (0_n, q)$.

Lemma A.1. *Let $Q = (0_n, q)$. Then, (i)–(ii) of Rmk.1.2 imply* A1–A2 *and* B1–B2.

Proof. These follow by more general results of Yong [10].[4] We give a proof for completeness. As all properties are coordinate-independent properties of the linearization about constant states, we may without loss of generality take A^0 block-diagonal. For, TA^0 is block-lower triangular for T block-upper triangular, whence TA^0T^* is symmetric block-diagonal, and a left symmetrizer for the system obtained by the change of coordinates $w \to (T^*)^{-1}w$, $A \to (T^*)^{-1}AT^*$, $Q \to (T^*)^{-1}QT^*$.

Observing that $\tilde A^0 := (A^0)^{-1}$ is a right symmetrizer if A^0 is a left symmetrizer, we obtain

$$\tilde A^0 w_t + \tilde A w_x = \tilde Q w,$$

where $\tilde A^0$ is symmetric positive definite and block-diagonal, $\tilde A$ is symmetric, and $\mathrm{d}\tilde Q = \begin{pmatrix} 0 & 0 \\ 0 & \tilde q \end{pmatrix}$ symmetric with $q < 0$. (Note: the latter key fact follows by

$$\mathrm{d}\tilde Q = (T^*)^{-1}\mathrm{d}Q\, T^* \tilde A^0,$$

the fact that T^* is block-lower triangular, and that the first block row of $\mathrm{d}Q$ by assumption vanishes.) Rewriting, we have

$$w_t + \bar A w_x = \bar Q w,$$

where

$$\bar A = \begin{pmatrix} (\tilde A^0_{11})^{-1}\tilde A_{11} & (\tilde A^0_{11})^{-1}\tilde A_{12} \\ (\tilde A^0_{22})^{-1}\tilde A_{21} & (\tilde A^0_{22})^{-1}\tilde A_{22} \end{pmatrix}, \qquad \mathrm{d}\bar Q = \begin{pmatrix} 0 & 0 \\ 0 & (\tilde A^0_{22})^{-1}\tilde q \end{pmatrix}.$$

In these coordinates, one readily computes that

$$a_0 = (\tilde A^0_{11})^{-1}\tilde A_{11}, \qquad b_0 = -(\tilde A^0_{11})^{-1}\tilde A^*_{12}\tilde q \tilde A^*_{12},$$

hence $\ker b_0 = \ker \tilde A_{12}$, and genuine coupling for the reduced system is the condition that no eigenvector of $a_0 = (\tilde A^0_{11})^{-1}\tilde A_{11}$ lie in $\ker \tilde A_{12}$, the same condition as for

[4]Symmetrizability is not explicitly stated in [10], but is clear from the development.

genuine coupling of the full system, and $\tilde{A}^0_{11}$ is a left symmetrizer for the reduced system with $\tilde{A}^0_{11} b_0 = -\tilde{A}^*_{12}\tilde{q}\tilde{A}^*_{12}$ symmetric positive semidefinite since $\tilde{q}$ is symmetric negative definite. □

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