Sound traveling-waves in wind instruments as solutions to non linear non homogeneous gas dynamics equations

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Abstract

The sound propagation is usually described by a linear homogeneous wave equation, though the air flow in a duct is described by the gas dynamics equations, using a variable cross section, which corresponds to a non linear non homogneous system. The aim of this paper is to exhibit a common periodic solution to both models, with several free parameters such as frequency or amplitude, able to represent any sound. By taking in account a friction term linked to the material (wood or brass for instance) of the duct, it is possible to build an analytic such solution when the cross section fullfills some condition which corresponds exactly to the general shape of the wind instruments. The conclusion is that in the wind intruments, the shape yields linearity.

1 The gas dynamics models with source terms

The duct is supposed to have a cylindrical symetry along its axis which can be a straight line (as for a clarinet) or a curve line (as for a horn) and the abscissa along this axis is denoted by $x > 0$, with $x = 0$ corresponding to the narrowest side of the duct. The variable cross section is denoted by $a(x) > 0$, the air density and velocity are ρ and u, both depending on the position x and time t. Since people blows at $x = 0$, the velocity u is expected to be positive. The mass conservation is described by

$$a(x)\ \rho_t\ +\ (a(x)\ \rho\ u)_x\ =\ 0\ ,$$

which becomes the transport equation

$$q_t\ +\ m_x\ =\ 0\ , \tag{1.1}$$

by using the transported quantity $q = a(x)\rho$ and the flux $m = a(x)\rho u$. This flux is ruled by the general dynamic equation

$$m_t\ +\ 2\ u\ m_x\ +\ \left(c^2 - u^2\right)\ q_x\ +\ S(q, m)\ =\ 0\ , \tag{1.2}$$

where $c = c(q)$ is the sound velocity and S is a source term to be detailed later. The system (1.1), (1.2) is an inhomogeneous form of the isentropic Euler equations, with the usual wave velocities $u - c$ and $u + c$.

A traveling-wave is expected to have the form $q(x,t) = q(x - At)$, $m = m(x - At)$, with some constant $A > 0$ corresponding to a velocity, so that both q and m are solution to the single wave equation: $q_t + A\, q_x = 0$, $m_t + A\, m_x = 0$.

Following the notion of *source waves*, such that Roll waves in hydraulics (see [1] or [3]), such solutions are exhibited by writing $m = m(q)$ in (1.1) and (1.2). One gets

$$\left((m'(q) - u)^2 - c^2 \right) q_x \;=\; S(q, m(q)) \;.$$

Next, by introducing a function $\psi(q)$ such that

$$\psi'(q) \;=\; \frac{(m'(q) - u)^2 \;-\; c^2}{S(q, m(q))} \;,$$

we obtain the two equations $\psi'(q)\; q_x \;=\; 1$ and $\psi'(q)\; q_t \;=\; -m'(q)\; \psi'(q)\; q_x \;=\; -m'(q)$.
By integrating the first equation with respect to x we get $\psi(q) \;=\; x \;-\; K(t)$, where $K(t)$ does not depend on x. Next, a derivation with respect to t , using $\psi'(q)\; q_t = -m'(q)$, leads to $m'(q) = K'(t)$. Another derivation, with respect to x this time gives $m''(q)\; q_x = 0$. Since the expected wave is not flat, we select $m''(q) = 0$, which leads to the two expressions $m(q) \;=\; A\, q \;-\; B$, $K'(t) \;=\; A$, with some constants A and B , and the relation

$$\psi(q) \;=\; x - At \;, \tag{1.3}$$

with the integration constant put inside the expression of $\psi(q)$. From (1.3) it is clear that q and then $m = Aq - B$ are both functions of $x - At$, and therefore solutions of a linear wave equation with the constant velocity A (traveling-waves). The expression of $\psi'(q)$ gets simpler by noticing that $u = \frac{m}{q} \;=\; A - \frac{B}{q}$ so that $m'(q) - u = \frac{B}{q}$, which leads to

$$\psi'(q) \;=\; \frac{B^2 \;-\; q^2 c^2}{q^2\; S(q, Aq - B)} \;. \tag{1.4}$$

It is important to notice that the hypothesis $S(q,m) \not\equiv 0$ is fondamental for this result, which does not apply for homogeneous systems. Another fondamental hypothesis is that $\psi'(q)$ is a function of *q only*. The result does not apply to the case of a source term depending on q, m and x for instance or to a sound speed depending on q and x. In the next sections we look for state laws and duct shapes for which these hypotheses are verified, and conclude that it is genuinely the case for the usual shapes of wind instruments, as flutes, clarinets or horns.

2 Application to an air flow in a duct

The cross section $a(x)$ is supposed to be increasing with x ($a'(x) > 0$). We look for expressions of $c(q)$ and $S(q,m)$ compatible with the previous hypotheses, that is independant of x. The expression of $c(q)$ comes from the pressure P using $c^2 = \frac{\gamma P}{\rho}$, where $\gamma = 1.4$ is the adiabatic constant, and the pressure is linked to the density ρ and the temperature T (in $^\circ$Kelvin) by a state law. A well known state law is given by the Boyle-Mariotte law $P = K_0 \rho T$, with $K_0 = 287.06$ in M.K.S. units, with P expressed in *Pascals*, which provides the following sound velocity

$$c \;=\; \sqrt{\gamma\; K_0\; T} \;.$$

Since the usual temperatures are greater than 270 $^\circ$ K, the variations of the profile of c are negligible, and we can fix the temperature to a reference value T_0 and take $c = c_0 \equiv \sqrt{\gamma\ K_0\ T_0}$. This corresponds to the "*isothermal*" hypothesis.

Another state law is given by the "*isentropic*" case $P = K\ \rho^\gamma$, with $K = 69259.5$ in M.K.S. units. The expression of the sound speed reads $c = \sqrt{\gamma K}\ \rho^{\frac{\gamma-1}{2}}$, which is not a function of q only. Since $c^2 = P'(\rho)$, we can derive the expression of the source term at rest, that is when the velocity is zero and the pressure is constant: $u = 0$, $\frac{\partial P}{\partial x} = 0$. The dynamical equation (1.2) reduces to $c^2 q_x + S(q,0) = 0$. Since $q_x = a(x)\ \rho_x\ +\ a'(x)\ \rho$, we obtain $c^2\ a'(x)\ \rho\ +\ S(q,0)\ =\ 0$, which leads to two possible of the source term:

$$S(q,0) \ = \ \begin{cases} -\ c_0^2\ \frac{a'(x)}{a(x)}\ q & \textit{in the isothermal case}\ , \\ -\gamma\ K\ \ \frac{a'(x)}{a(x)^\gamma}\ q^\gamma & \textit{in the isentropic case}\ . \end{cases}$$

We introduce a friction term, of the Strickler type, to select the following source term

$$S(q,m) \ = \ S(q,0) \ + \ k\ |u|\, u\ ,$$

where k is the Strickler friction (dimensionless) coefficient.

The retained model is a combination between the isothermal and the isentropic cases. The sound velocity is a constant $c = c_0 \equiv \sqrt{\gamma K_0 T_0}$, with a fixed temperature T_0. For instance with $T_0 = 300K$ we get $c_0 = 347.225\ ms^{-1}$.

The selected source term is $S(q,m) = k\,|u|\,u - \gamma K q^\gamma\ \frac{a'(x)}{a(x)^\gamma}$,which becomes independent on x if we impose a profile such that

$$\frac{a'(x)}{a(x)^\gamma} \ = \ Constant \ = \ k\ \frac{D^2}{\gamma K}\ , \tag{2.1}$$

where D is a constant. Expression (2.1) is a differential equation whose solutions are

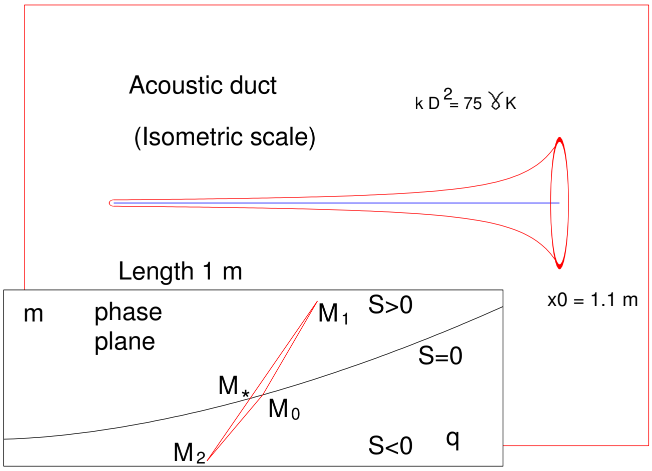
$$a(x) \ = \ \left(\frac{\gamma\ K}{(\gamma - 1)\ \ k\ D^2\ \ (x_0 - x)} \right)^{\frac{1}{\gamma - 1}}$$

where x_0 is a constant to be taken larger than L the length of the wind intrument. The radius of the duct is given by $r(x) = \sqrt{\frac{a(x)}{\pi}}$, and we recognize the usual shape of the wind instrument as seen on the figure below, where $kD^2 = 75\ \gamma K$. The value of x_0 corresponds to an infinite radius, or to a maximal length for a wind instrument. By the way, using the isothermal case leads to another expression of the cross section which reads $a(x) = a_0\ exp\,(Cx)$ where C and a_0 are positive constants, and allows an infinite length.

The dynamic equation (1.2) becomes

$$m_t + 2um_x + \left(c_0^2 - u^2\right) q_x \ + \ k\left(u^2 - D^2 q^\gamma\right) \ = \ 0\ , \tag{2.2}$$

where the source term is independent on x which allows waves of the form $m = Aq - B$.



3 The wave profile

The updated form of $\psi'(q)$ reads

$$\psi'(q) \;=\; \frac{B^2 - c_0^2\; q^2}{k\left((Aq-B)^2 \;-\; D^2\; q^{\gamma+2}\right)} \tag{3.1}$$

and we consider a so-called *Reference state* $M_* = (q_*, m_*)$, with $q_* > 0,\ m_* > 0$ and set $u_* = \frac{m_*}{q_*}$ and take $A = u_* + c_0$ and $B = q_*\ c_0$. The value q_* is a possible realistic value of$q$ which is a root of the numerator in (3.1). Since $\psi'(q)q_x = 1$, either q_x becomes infinite when $q = q_*$, or q_* is also a root of the denaminator in (3.1) and in this case $u_* = D\ q_*^{\frac{\gamma}{2}}$, and M_* belongs to the set $S_0 = \{(q, m)\ \mid\ S(q, m) = 0\ \}$ in the phase plane (q, m). The line $m = (u_* + c_0)\, q - q_* c_0$ cuts S_0 in M_*, coming from the set $\{S < 0\}$ for $q < q_*$ and going towards $\{S > 0\}$ for $q > q_*$. Since

$$\psi'(q) \;=\; \frac{(q_*^2 - q^2)\ \ c_0^2}{q^2\ S\left(q, (u_* + c_0)\, q - q_* c_0\right)}\ ,$$

we see that for $\psi'(q) < 0$ for $q < q_*,\ S < 0$ and for $q > q_*,\ S > 0$. Since $\psi'(q)q_x = 1$, we get that $q_x < 0$, which means that the wave profile corresponding to the line $m = (u_* + c_0)\, q - q_* c_0$ is decreasing.

Now let us consider in $\{S > 0\}$ a point $M_1 = (q_1, m_1)$ on the line $m = (u_* + c_0)\, q - q_* c_0$,and then the line whose reference point is M_1, that is the line of equation $m = (u_1 + c_0)\, q - q_1 c_0$. Along this line, we have

$$\psi'(q) = \frac{\left(q_1^2 - q^2\right)\ c_0^2}{q^2\ S\left(q, \left(u_1 + c_0\right) q - q_1 c_0\right)}\ ,$$

which is positive for $q < q_1$. This means that the corresponding profile is increasing. By the same way, let us consider in $\{S < 0\}$ a point $M_2 = (q_2, m_2)$ on the line $m = (u_* + c_0)\, q - q_* c_0$,and then the line whose reference point is M_2, of equation $m = (u_2 + c_0)\, q - q_2 c_0$. Along this line, we have

$$\psi'(q) = \frac{\left(q_2^2 - q^2\right)\ c_0^2}{q^2\ S\left(q, \left(u_2 + c_0\right) q - q_2 c_0\right)}\ ,$$

which is positive for $q > q_2$. This means that the corresponding profile is increasing too. This is reprensented in a window on both figure, above and below. Up to now, the two points M_1 and M_2 have been chosen without any restriction, and it is possible to get the two lines $m = (u_1 + c_0)\, q - q_1 c_0$ and $m = (u_2 + c_0)\, q - q_2 c_0$ cutting S_0 at the same point $M_0 \in S_0$. This can be done by choicing first $M_0 \in S_0$ with $q_0 > q_*$, then searching a point $M = (q, (u_* + c_0)q - q_* c_0)$ such that M_0 belongs to the lines of slope $u_j - c_0$ passing through $M_j,\ j = 1, 2$. This corresponds to the equation

$$q + \frac{q_* q_0}{q} = \left(u_* + 2c_0 - D q_0^{\frac{\gamma}{2}} \right) \frac{q_0}{c_0}\ , \tag{3.2}$$

which have two solutions which are actually q_1 and q_2 with $q_2 < q_* < q_0 < q_1$ as shown in the figure below.

The profiles are computed by integrating $\psi'(q)\ q_x = 1$ for each part of the profile, using a numerical integration method, with

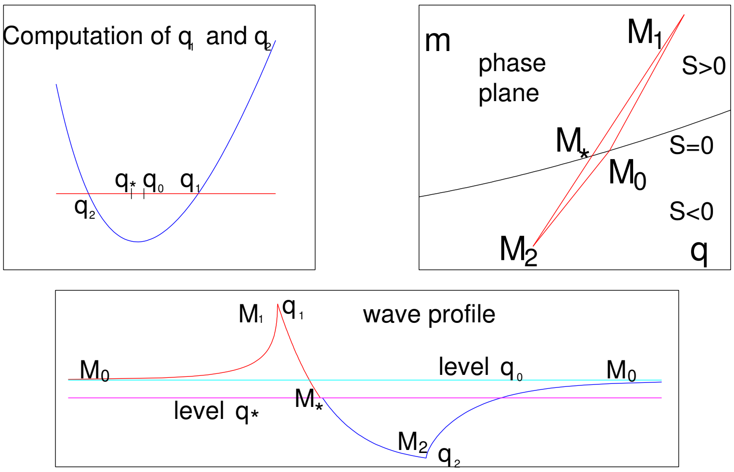
$$\psi'(q_*) = \frac{2 q_*\ c_0^2}{k\left(2\left(u_* + c_0\right) u_* q_*^2 - D^2\left(\gamma + 2\right) q_*^{\gamma+1}\right)}$$

for $q = q_*$. We first compute M_1 and M_2 from (3.2) and starting from M_1 we compute the two branches $M_1 M_0$ and $M_1 M_*$, next starting from M_2 we compute the two branches $M_2 M_0$ and $M_2 M_*$.

4 The wave propagation

The profile is made of a sequence of 3 travelling waves: a rear part from the state M_0 to M_1 with the constant velocity $A_1 = u_1 + c_0 = u_* + 2c_0 - c_0 \frac{q_*}{q_1}$, a central part from the state M_1 to M_* and M_2 with the constant velocity $A_* = u_* + c_0$ and a front part from the state M_2 back to M_0 with the velocity $A_2 = u_2 + c_0 = u_* + 2c_0 - c_0 \frac{q_*}{q_2}$. Since $q_2 < q_* < q_0 < q_1$, we have $A_1 > A_* > A_2$. When q_* and q_0 are near, which is the case in reality, these velocities are also very near and we can consider that the whole wave travels at the velocity $u_* + c_0$. The figure above corresponds to a choice of the parameters such that q_* and q_0 are not too near and then the three lines in the phase plane do not seem superimposed for the reader.

The effect of the different velocities together with the mass conservation principle is the emergence of two small shock waves, one on the top of the rear wave, in M_1 and another at the lowest



part of the front wave in M_2. The amplitude of these two shocks increases during the propagation but stay imperceptible since the travel time in the duct is very short, of the order of 10^{-3} s. However, such mass effects can be observed in the real world, with similar equations, sometimes at a very different scale in space and time (see for instance [4], on the modelling of the Rogue Waves using a small difference of the velocities of two long waves on the ocean).

The sound production is produced by the concatenation of such waves with the same parameters, involving a constant wavelength and therefore a constant frequency.

5 Some remarks and conclusion

The wave profile presents a decreasing part, from M_1 to M_2, which never degenerates into a shock as often expected in hydrodynamics (as for the Roll waves, see [1] or [3]). This comes obviously from the source term. However, the occurrence of a shock may be analysed. Since the sound speed is a constant, the pressure law corresponds to $P(q) = c_0^2 q$ and the shock condition derived from the well known Rankine-Hugoniot relations for a shock wave linking M_1 to M_2 read

$$\frac{u_2 - u_1}{q_2 - q_1} = \sqrt{\frac{P(q_2) - P(q_1)}{q_2 - q_2}} = \frac{c_0}{\sqrt{q_1 q_2}} \,.$$

We shall compute the rate $\frac{u_2-u_1}{q_2-q_1}$ by two other ways. First we write that M_0 belongs to the lines issued from M_1 and M_2, with the respective slopes u_1+c_0 and u_2+c_0, that is

$$(m_0 \ =) \ \ (u_1+c_0)\, q_0 - c_0 q_1 \ = \ (u_2+c_0)\, q_0 - c_0 q_2 \ ,$$

which gives

$$\frac{u_2-u_1}{q_2-q_1} \ = \ \frac{c_0}{q_0} \ .$$

Next we write that M_*, M_1 ans M_2 are aligned or

$$u_1 = u_* + c_0 - c_0 \frac{q_*}{q_1} \ \ , \ u_2 = u_* + c_0 - c_0 \frac{q_*}{q_2} \ ,$$

which gives

$$\frac{u_2-u_1}{q_2-q_1} \ = \ \frac{c_0 q_*}{q_1 q_2} \ .$$

We have obtained

$$\left(\frac{u_2-u_1}{q_2-q_1} \ = \right) \ c_0 \ \frac{1}{\sqrt{q_1 q_2}} \ = \ c_0 \ \frac{1}{q_0} \ = \ c_0 \ \frac{q_*}{q_1 q_2} \ , \text{ hence } q_* = \sqrt{q_1 q_2} = q_0 \ .$$

Thus $u_* = u_0$ and $(u_* =) u_1 + c_0 - c_0 \frac{q_1}{q_*} = u_1 - c_0 + c_0 \frac{q_*}{q_1}$, which gives $2q_* q_1 - q_1^2 - q_*^2 = 0$, that is $(q_* - q_1)^2 = 0$, or $q_1 = q_*$. The same arguments give $q_2 = q_*$. Our hypothetical shock is reduced to one point, which means obviously that no shock occurs.

The wave velocity A is close to c_0, but a priori different since $A - c_0 = u_2$ for the front part of the wave for instance. Depending on the (still realistic) values of the parameters of the model, u_2 can be negative or positive, and a possible constraint on the parameters may consist into setting $u_2 = 0$. The arguments for this approach are settled in the reference [2].

This construction of a wave should also feed the discussion between linear or non linear models (see for instance [5]). It seems, as a conclusion, that the difference is not so important when a source term is present. By taking in account a non linear diffusion term instead of a friction term, other travelling waves may be expected. The main conclusion is that the linearity is naturally induced by the shape in a wind instrument.

References

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