Symmetric Euler and Navier-Stokes shocks in stationary barotropic flow on a bounded domain

Erik Endres[*] Helge Kristian Jenssen [†] Mark Williams[‡]

Revised: April 22, 2008

Abstract

We construct stationary solutions to the barotropic, compressible Euler and Navier-Stokes equations in several space dimensions with spherical or cylindrical symmetry. For given Dirichlet data on a sphere or a cylinder we first construct smooth and radially symmetric solutions to the Euler equations in the exterior domain. On the other hand, stationary smooth solutions in the interior domain necessarily become sonic and can not be continued beyond a critical inner radius. We then use these solutions to construct entropy-satisfying shocks for the Euler equations in the region between two concentric spheres or cylinders. Next we construct smooth Navier-Stokes solutions converging to the previously constructed Euler shocks in the small viscosity limit. In the process we introduce a new technique for constructing smooth solutions, which exhibit a fast transition in the interior, to a class of two-point boundary problems.

Contents

[*]Department of Mathematics, Penn State University, University Park, State College, PA 16802, USA (endres@math.psu.edu)

[†]Department of Mathematics, Penn State University, University Park, State College, PA 16802, USA (jenssen@math.psu.edu). Partially supported by NSF grant DMS-0539549

[‡]Department of Mathematics, University of North Carolina, Chapel Hill, NC 27599, USA (williams@email.unc.edu). Partially supported by NSF grant DMS-0701201

1 Introduction

Consider the domain between two concentric spheres $r = a$ and $r = b$, where $a < b$, and imagine that a compressible fluid is injected with a prescribed constant density ρ_a and constant radial velocity u_a at the inner boundary $r = a$. Depending on how fast the fluid is allowed to exit at the outer boundary, fluid may or may not pile up in the interior and a shock may or may not form. Similarly, one can consider the case where fluid is injected radially at the outer boundary, or the cases where spheres are replaced by cylinders.

As a first step in constructing stationary shock solutions of this type to the Euler equations, we first construct *inner solutions*, that is, smooth solutions defined everywhere in the exterior $r \geq a$ of a sphere $r = a$ with data (ρ_a, u_a) prescribed at the inner boundary, and *outer solutions*, which are smooth and defined inside $r = b$ when data (ρ_b, u_b) is prescribed at the outer boundary. Note that the stationary equations reduce to ODEs under the symmetry assumption. We find that inner solutions remain subsonic (resp., supersonic) everywhere if they are subsonic (resp., supersonic) at $r = a$. A similar result holds for outer solutions, with the interesting difference that there is a critical inner radius at which the flow becomes sonic and beyond which the stationary solution cannot be extended. In the cylindrically symmetric (CS) case we allow swirling flows with nonzero angular (v) and axial (w) components, but we find that it is only the *radial* Mach number that is relevant for classifying solutions (and for determining the critical radius in the case of outer solutions). The main results on inner and outer solutions are summarized in Propositions 2.2, 2.4, 2.6, and 2.7.

In section 3 we show how to build symmetric, entropy-satisfying shock solutions to the Euler equations by using either inner or outer solutions. In each case, since we consider only stationary solutions and due to the compressibility of admissible shocks, there is only a single shock in these solutions. The main results are summarized in Theorems 3.1 and 3.2.

Section 4 addresses the following question: Taking a, b, and data at $r = a$ as fixed, can one formulate necessary and sufficient conditions on the flow variables at $r = b$ which guarantee the existence of a stationary, weak solution of the barotropic Euler equations with these boundary values, and which contains a single shock at *some* location $\bar{r} \in (a, b)$. The answer is provided, for cylindrically symmetric flow with or without swirl, in Theorem 4.1.

Remark 1.1. *The inviscid solutions we build have been studied, for isentropic flow, also by Chen and Glimm [CG1] - [CG2] in their analysis of the initial value problem on exterior domains. In these works the shock solutions serve as building blocks in a Godunov type scheme and their analysis requires detailed local L^∞ estimates. The inviscid part of the present work applies to more general barotropic flows and we are interested in properties in the large.*

The goal of the second part of the paper is to construct smooth Navier-Stokes solutions converging to the previously constructed Euler shocks in the small viscosity limit. We focus on the spherically symmetric case with prescribed supersonic inflow at $r = a$. (The same arguments treat the cylindrically symmetric case when both $v = 0$ and $w = 0$.) We assume we are given an inviscid shock taking values (ρ_a, u_a) at $r = a$ and (ρ_b, u_b) at $r = b$, and we seek solutions to the second-order viscous equations on $[a, b]$, which assume these boundary values for each fixed viscosity ϵ, and which converge (in an appropriate sense) to the given inviscid shock as $\epsilon \to 0$.

The density equation can be used to eliminate an unknown, say ρ, and one can attempt to apply classical two-point boundary theory to the second-order ODE for u that remains. This problem has the form

$$\begin{aligned} &u_{rr} = \frac{1}{\epsilon} f(r, u, u_r, \epsilon) \text{ on } [a, b], \\ &u(a) = u_a, u(b) = u_b. \end{aligned} \tag{1.1}$$

The approach based on fixed point theorems (e.g., [H], Chapter 12) gives existence and uniqueness for *large* epsilon, but provides no information for small ϵ. In fact, as $\epsilon \to 0$ the length of the interval on which one can solve two-point problems of the form (1.1) generally shrinks to zero. The methods based on comparison theorems, upper and lower solutions, and shooting methods [BSW, DH, K] also appear unsuitable for constructing solutions involving fast interior transitions like the shock layers in our viscous solutions.

In section 5 after the change of variables $s = r - \bar{r}$, where $r = \bar{r}$ $(a < \bar{r} < b)$ is the inviscid shock location, we reformulate the stationary Navier-Stokes equations as a second-order, 2×2 *transmission problem* on the bounded interval $[a - \bar{r}, b - \bar{r}]$. The unknowns are $(\rho, u) = (\rho_\pm(s), u_\pm(s))$ in $\pm s \geq 0$ and transmission conditions at $s = 0$ are given by

$$[\rho] = 0, [u] = 0, [u_s] = 0, \tag{1.2}$$

where, for example, $[u] := u_+(0) - u_-(0)$. Boundary conditions are now imposed at $s = a - \bar{r}$ and $s = b - \bar{r}$.

Writing $w_\pm = (w^1, w^2) := (\rho, u)$ (and suppressing some $\pm$), in section 6 we show how to construct high order *approximate* solutions to the transmission problem,

$$\tilde{w}^\epsilon(s) = \left(\mathcal{U}^0(s, z) + \epsilon \mathcal{U}^1(s, z) + \cdots + \epsilon^M \mathcal{U}^M(s, z)\right)|_{z = \frac{s}{\epsilon}}, \tag{1.3}$$

where $\mathcal{U}^j(s,z) = U^j(s) + V^j(z)$, $V^j(z) \to 0$ exponentially fast as $z \to \pm\infty$, and $U^0(s)$ is the given inviscid shock. Thus, for small $\delta > 0$,

$$\begin{aligned} &\tilde{w}^\epsilon(s) \to U^0(s) \text{ in } L^\infty(|s| \geq \delta), \text{ while} \\ &\tilde{w}^\epsilon(s) \to U^0(s) \text{ in } L^p(|s| \leq \delta), 1 \leq p < \infty. \end{aligned} \tag{1.4}$$

Observe that the terms $V^j(\frac{s}{\epsilon})$ describe the fast transition in the viscous solutions that occurs near the inviscid shock front at $s = 0$. The construction of approximate solutions is summarized in Proposition 6.6. The method used has much in common with the construction in [GW, GMWZ3], but the fact that the transmission problem here is an ODE rather than a PDE allows for some significant simplifications.

The last steps of the analysis are carried out in section 7. There we prove the existence of an exact solution $w^\epsilon(s)$ to the transmission problem that is close to the approximate solution. We look for w^ϵ in the form

$$w^\epsilon(s) = \tilde{w}^\epsilon(s) + \epsilon^L v^\epsilon(s), \; 1 \leq L < M, \tag{1.5}$$

where the v^ϵ satisfy an appropriate error problem and turn out to be uniformly bounded in $L^\infty[a - \overline{r}, b - \overline{r}]$ as $\epsilon \to 0$. The second-order 2×2 problem for $v^\epsilon = (v^1, v^2)$ is written as a 3×3 first-order system for $V = (v^1, v^2, \epsilon v^2_s)$ (see (7.8)):

$$\begin{aligned} &V_s = \frac{1}{\epsilon} GV + F \text{ on } [a - \overline{r}, b - \overline{r}] \\ &[V] = 0 \text{ on } s = 0 \\ &(v^1, v^2) = \overline{v} \text{ at } s = a - \overline{r}. \end{aligned} \tag{1.6}$$

There are two main obstacles to obtaining uniformly bounded solutions to (1.6) as $\epsilon \to 0$. The first is that the entries of the matrix $G = G(\tilde{w}^\epsilon + \epsilon^L v)$ are functions $g^{ij}(\frac{s}{\epsilon}, q^\epsilon(s))$ that undergo fast transitions near $s = 0$. The eigenvalues of G therefore exhibit similar behavior. If all the eigenvalues of G had a favorable sign and remained bounded away from 0, the factor of $\frac{1}{\epsilon}$ in front of G would not pose a serious problem. In fact, fast transitions make the eigenvalues of G hard to analyze, and we know that at least one changes sign near $s = 0$, so the factor $\frac{1}{\epsilon}$ is a serious problem. The second obstacle is the need to smoothly piece together the part of the solution in $|s| \geq \delta > 0$ that changes slowly and takes on prescribed boundary values at $s = a - \overline{r}$, with the part of the solution in $|s| \leq \delta$ that undergoes a fast transition.

The matrix G in (1.6) can be written

$$G = G(z, q)|_{z=\frac{s}{\epsilon}, q=q^\epsilon(s)}, \tag{1.7}$$

where, roughly speaking, the first argument describes fast behavior, and the second argument slow behavior. The exponential decay of $V^0(z)$ to 0 as $z \to \pm\infty$ implies that there exist limiting matrices $G(\pm\infty, q)$ to which $G(z,q)$ converges exponentially fast as $z \to \pm\infty$:

$$|G(z,q) - G(\pm\infty, q)| \leq Ce^{-\kappa|z|}, \text{ for some } C > 0, \kappa > 0. \tag{1.8}$$

We deal with the first of the obstacles described above by using a conjugation argument first introduced in [MZ], and also used in later papers such as [GMWZ2, GMWZ3], that effectively allows us to replace the matrix $G(z, q)$ by $G(\pm\infty, q)$ when analyzing (1.6) on the fast transition subinterval $|s| \leq \delta$ (see Lemma 7.7). The removal of the fast scale in G greatly simplifies the analysis of eigenvalues. One observes readily that two of the eigenvalues of $G(\pm\infty, q^\epsilon(s))$ are $O(\epsilon)$, while the third is $O(1)$ and changes sign at $s = 0$. This change of sign reflects the transition from supersonic to subsonic flow across the inviscid shock. A second and more straightforward conjugation can then be used to reduce G to the block forms

$$G_{B\pm}(q^\epsilon(s)) = \begin{pmatrix} O(\epsilon) & O(\epsilon) & 0 \\ O(\epsilon) & O(\epsilon) & 0 \\ 0 & 0 & g^{33}_\pm(q^\epsilon(s)) + O(\epsilon) \end{pmatrix} \text{ on } \{|s| \leq \delta\} \cap \{\pm s \geq 0\}, \tag{1.9}$$

as in Proposition 7.8. Observe that on $|s| \geq \delta$, $V^0(\frac{s}{\epsilon})$ is already negligible for ϵ small, so in that region the G matrix in (1.6) can be conjugated directly to the form (1.9) without a preliminary conjugation to remove the fast scale.

We deal with the second obstacle by splitting the transmission problem (1.6) into four separate boundary problems labelled I, II, III, and IV on the subintervals $[a - \overline{r}, -\delta]$, $[-\delta, 0]$, $[0, \delta]$, and $[\delta, b - \overline{r}]$ respectively. The sign of g^{33} in (1.9) is positive in $s \leq 0$, and this means that boundary data for the third scalar unknown must be prescribed at the *right* endpoint in problems I and II. (The factor of $\frac{1}{\epsilon}$ in (1.6) causes the third unknown to blow up exponentially as $\epsilon \to 0$ if data is prescribed at the left endpoint.) This complicates the smooth patching together of solutions at the joining point $s = -\delta$. We accomplish this by first allowing the boundary data in problems I and II to depend on several unknown scalar parameters, $p_1, \ldots, p_4$, and then showing that parameter choices exist for which the solutions to problems I and II match up smoothly at $s = -\delta$. The matching for Problems III and IV is easier to handle, because g^{33} is negative in $s \geq 0$, so all boundary data can in each case be prescribed at the left endpoint. Our main result for the small viscosity limit is summarized in Theorem 7.15.

Remark 1.2. *1. We wish to explain the relation of this work to the paper [GMWZ3], in which smooth solutions of the Navier-Stokes equations are constructed which converge to a given Euler shock (about which no symmetry assumptions are made) in the small viscosity limit. There are three main differences. First, the viscous solutions in [GMWZ3] are time-dependent solutions of PDEs and exist on a* finite *time interval independent of ϵ. The solutions in this paper are solutions of the same PDEs which are symmetric and* stationary*, and can therefore be viewed as existing for all time. Their construction is based on the solution of an appropriate ODE.*

The second difference concerns the estimates and the techniques used for passing from local solutions to global solutions. In [GMWZ3] global L^2 and Sobolev space estimates are proved under an appropriate Evans function hypothesis by localization via a smooth partition of unity. The errors introduced by the cutoffs are easily absorbed by adjusting certain weights in the estimates. For the ODEs considered here, direct L^∞ estimates are most natural, no Evans hypothesis is needed, and partitions of unity are no longer feasible (the errors introduced are too large). Rather, we are forced to accomplish the passage from local to global solutions, and in particular the patching together of "slow" and "fast" pieces of solutions, by careful matching arguments involving the introduction of several unknown parameters.

A third difference is that the analysis in [GMWZ3] takes place in an unbounded domain without boundary. The only boundary-type conditions in that paper are the transmission conditions which are imposed on a (free) boundary located roughly in the middle of the fast transition region. In this paper, in addition to transmission conditions at $s = 0$ we have boundary conditions at $s = a - \overline{r}$ and $r = b - \overline{r}$.

2. A result demonstrating convergence of smooth viscous solutions to inviscid 1D shocks was given in [GX]. The viscous shocks constructed there, as in [GMWZ3], were time-dependent and existed on a finite time interval independent of ϵ.

3. Geometric singular perturbation theory (or "Fenichel theory") has been used by several authors (e.g., [J, GS]) to construct solutions which exhibit both slow and fast interior behavior and connect equilibrium points *of ODEs. Our viscous solutions exhibit both slow and fast behavior, but since, for example, the endstates (ρ_a, u_a), (ρ_b, u_b) are not equilibria, we do not see how to apply Fenichel theory to construct the viscous solutions being sought here. Even if that were possible, we believe the direct and self-contained approach presented here has much to recommend it.*

There is by now a well-developed long-time stability theory for multidimensional planar viscous shocks (see, for example, [Z, GMWZ1]). Under an appropriate Evans hypothesis, it is known that small perturbations introduced at time zero of a planar viscous shock will eventually dissipate and disappear in the limit as $t \to \infty$. One of the motivations of the present work is to set the stage for the study of such questions for non-planar viscous shocks. For the long-time stability question to make sense, one must perturb a curved viscous shock that is already known to exist for all time. Thus, it is natural to work with *stationary* curved viscous shocks.

In the sequel to this paper [EJW] we construct stationary shock solutions for the full, nonbarotropic Euler and Navier-Stokes equations.

1.1 Equations

The barotropic Navier-Stokes equations express the conservation of mass and the balance of momentum. In Eulerian coordinates the equations in $\mathbb{R}^3$ take the form

$$\rho_t + \operatorname{div}(\rho \mathbf{U}) = 0 \tag{1.10}$$

$$(\rho \mathbf{U}^i)_t + \operatorname{div}(\rho \mathbf{U}^i \mathbf{U}) + P(\rho)_{x_i} = \mu \Delta \mathbf{U}^i + (\lambda + \mu) \operatorname{div} \mathbf{U}_{x_i} + 0 \quad i = 1, 2, 3\,. \tag{1.11}$$

Here $x = (x_1, x_2, x_3)$ and $\rho, \mathbf{U} = (\mathbf{U}^1, \mathbf{U}^2, \mathbf{U}^3)$, and $P(\rho)$ are the density, velocity, and pressure, respectively. μ and λ are positive viscosity coefficients (constants).

In the case of spherical or cylindrical symmetry the density and velocities at a point depend only on time and the radial distance to either the origin or to the x_3-axis. We refer to these as the spherically symmetric (SS) and the cylindrically symmetric (CS) cases, respectively. We let (u, v, w) be the velocity components in either spherical or cylindrical coordinates. We set $r = |x|$ in the SS case, while $r = \sqrt{x_1^2 + x_2^2}$ in the CS case. In either case, with a slight abuse of notation we write $\rho(x, t) = \rho(r, t)$, *etc.* Thus

$$\mathbf{U}(x, t) = u(r, t) \frac{x}{r}\,, \qquad v = w \equiv 0$$

in the SS case, while

$$\mathbf{U}(x,t) = u(r,t)\frac{(x_1,x_2,0)}{r} + v(r,t)\frac{(-x_2,x_1,0)}{r} + w(r,t)(0,0,1)\,,$$

in the CS case. The equations (1.10)-(1.11) then take the form

$$\rho_t + (\rho u)_\xi = 0 \tag{1.12}$$

$$(\rho u)_t + (\rho u^2)_\xi - \frac{\rho v^2}{r} + P(\rho)_r - \nu u_{\xi r} = 0 \tag{1.13}$$

$$(\rho v)_t + (\rho u v)_\xi + \frac{\rho u v}{r} - \mu v_{\xi r} = 0 \tag{1.14}$$

$$(\rho w)_t + (\rho u w)_\xi - \mu w_{r\xi} = 0 \tag{1.15}$$

where $\nu := \lambda + 2\mu$ and $\partial_\xi \equiv \partial_r + m/r$, and $m = 1$ (CS case) or $m = 2$ (SS case). Finally, the inviscid compressible Euler equations are obtained by setting $\mu = \lambda = \nu = 0$. The compressible and Euler and Navier-Stokes equations are discussed, for example, in [RJ].

1.2 Setup and assumptions

Our first task is to construct stationary profiles for the barotropic Euler equations We treat both the SS and CS cases in domains which are bounded by concentric and fixed spheres or cylinders with radii $b > a > 0$. The solutions are constructed to take on given values at the inner or outer boundaries $\{r = a\}$ and $\{r = b\}$. To analyze stationary solutions we will make the following assumptions about the pressure $P(\rho)$:

(A1) The function $\rho \mapsto P(\rho)$ is a twice differentiable on $(0, +\infty)$ with

$$P'(\rho) > 0 \quad \text{for all } \rho > 0\,. \tag{1.16}$$

(A2)

$$P''(\rho) \geq 0 \quad \text{for all } \rho > 0\,. \tag{1.17}$$

(A3)

$$\limsup_{\rho \to +\infty} \int_1^\rho \frac{P'(\sigma)}{\sigma}\, d\sigma = +\infty\,. \tag{1.18}$$

(A4)

$$\lim_{\rho \downarrow 0} P'(\rho) = 0\,. \tag{1.19}$$

Remark 1.3. *We note that the pressure function of a polytropic ideal gas for isentropic flow, i.e.* $P(\rho) = K\rho^\gamma$ *with* $\gamma > 1$*, satisfies all of (A1)-(A4).*

We denote by c the local sound speed,

$$c = c(\rho) := \sqrt{P'(\rho)}\,.$$

2 Stationary solutions of the barotropic Euler equations

ODE system for spherically symmetric flow In the SS case the barotropic Euler equations reduce to the ODE system

$$\begin{aligned} \frac{d(\rho u r^2)}{dr} &= 0 && (2.1)\\ u\frac{du}{dr} + \frac{1}{\rho}\frac{d(P(\rho))}{dr} &= 0\,, && (2.2) \end{aligned}$$

and the Rankine-Hugoniot conditions reduce to

$$[\rho u] = 0\,, \qquad \left[P(\rho) + \rho u^2\right] = 0\,.$$

ODE system for cylindrically symmetric flow In the CS case the barotropic Euler equations reduce to the ODE system

$$\begin{aligned} \frac{d(\rho u r)}{dr} &= 0 && (2.3)\\ \frac{d(\rho u^2 r)}{dr} - \rho v^2 + r\frac{d(P(\rho))}{dr} &= 0 && (2.4)\\ \frac{d(\rho u r v)}{dr} + \rho u v &= 0 && (2.5)\\ \frac{d(\rho u r w)}{dr} &= 0\,, && (2.6) \end{aligned}$$

and the Rankine-Hugoniot conditions reduce to

$$[\rho u] = 0 \qquad \left[P(\rho) + \rho u^2\right] = 0\,, \tag{2.7}$$

together with

$$[v] = 0\,, \qquad [w] = 0\,. \tag{2.8}$$

Inner and outer solutions We are seeking stationary solutions defined in regions between fixed, concentric spheres or cylinders with radii $b > a > 0$. Different situations occur according to whether the Dirichlet data for ρ and u, v, w are prescribed at $r = a$ or at $r = b$. Solution with data prescribed at the inner (outer) boundary will be referred to as inner (outer) solutions.

Remark 2.1. *For barotropic flow the problem of solving the stationary equations will be reduced a single algebraic equation. One could also analyze the ODEs more directly. Indeed, this approach leads to a separable ODE for the velocity in the case of the full Euler system for an ideal gas (see [EJW]). While this is not true for the barotropic case, we note that the ODEs for the radial velocity are*

$$u_r = \frac{2uc^2}{r(u^2 - c^2)}\,, \qquad u_r = \frac{u(v^2 + c^2)}{r(u^2 - c^2)} \tag{2.9}$$

in the SS and CS cases, respectively. Thus, sonic points are singular points for the ODEs.

2.1 Inner solutions in the spherically symmetric case

We now consider the case with given Dirichlet data ρ_a, u_a at the inner boundary $r = a$, and we seek a smooth, stationary solution to the barotropic Euler equations (2.1)-(2.2) in the region $r \geq a$. Note that we are not (yet) saying anything about the sign or size of u_a. The ODE (2.1) yields

$$u(r) = \frac{C_a}{\rho(r)r^2} \qquad \text{for } r \geq a, \text{ where} \qquad C_a := \rho_a u_a a^2 . \tag{2.10}$$

Notice that this implies that if $u_a \gtrless 0$, then $u(r) \gtrless 0$ for all $r \geq a$. We will only consider the case where $\rho_a > 0$ and $u_a \neq 0$. To analyze the ODE (2.2) it is convenient to introduce the function $\Pi : \mathbb{R}^2_+ \to \mathbb{R}$ defined by

$$\Pi(\rho_2, \rho_1) := \int_{\rho_1}^{\rho_2} \frac{2P'(\sigma)}{\sigma}\, d\sigma . \tag{2.11}$$

We note that (A3) amounts to $\Pi(\rho, \rho_1) \to +\infty$ as $\rho \to \infty$. From (2.2) we get

$$\frac{d}{dr}\big[u(r)^2 + \Pi(\rho(r), \rho_a)\big] = 0 .$$

Since $\Pi(\rho_a, \rho_a) = 0$, it follows that

$$u(r)^2 + \Pi(\rho(r), \rho_a) \equiv u_a^2, \qquad \text{for } r \geq a . \tag{2.12}$$

Using (2.10) to eliminate $u(r)$ we thus get that $\rho = \rho(r)$ satisfies

$$\frac{C_a^2}{\rho(r)^2 r^4} + \Pi(\rho(r), \rho_a) \equiv u_a^2 \qquad \text{for } r \geq a . \tag{2.13}$$

We need to show that equation (2.13) can be solved for $\rho(r)$ in terms of r when $r \geq a$. Let's define the function

$$\phi(\rho, \rho_a, u_a) := \rho^2\big[u_a^2 - \Pi(\rho, \rho_a)\big] , \tag{2.14}$$

such that (2.13) takes the form

$$\phi(\rho, \rho_a, u_a) = \frac{C_a^2}{r^4} . \tag{2.15}$$

We now consider ρ_a and u_a as fixed and let $' = \frac{d}{d\rho}$. With $\phi(\rho) \equiv \phi(\rho, \rho_a, u_a)$ and $\Pi(\rho) \equiv \Pi(\rho, \rho_a)$ we thus have

$$\phi'(\rho) = 2\rho\big[u_a^2 - \big(\Pi(\rho) + P'(\rho)\big)\big] . \tag{2.16}$$

From (A1) and (A2) it follows that the map $\rho \mapsto \Pi(\rho) + P'(\rho)$ is strictly increasing on $(0, \infty)$, and from (A3) it follows that it tends to $+\infty$ as $\rho \uparrow \infty$. Note that (A4) together with the definition of Π, and the fact that $\rho_a > 0$, implies $u_a^2 > 0 > (\Pi(\rho) + P'(\rho))|_{\rho=0}$. Thus, by (2.16), $\phi'(\rho) > 0$ for $\rho > 0$ sufficiently small, while $\phi'(\rho) \downarrow -\infty$ as $\rho \uparrow \infty$. It follows that there are unique ρ-values $0 < \rho_* < \rho_0$ such that

$$\phi'(\rho_*) = 0 , \qquad \phi(\rho_0) = 0 .$$

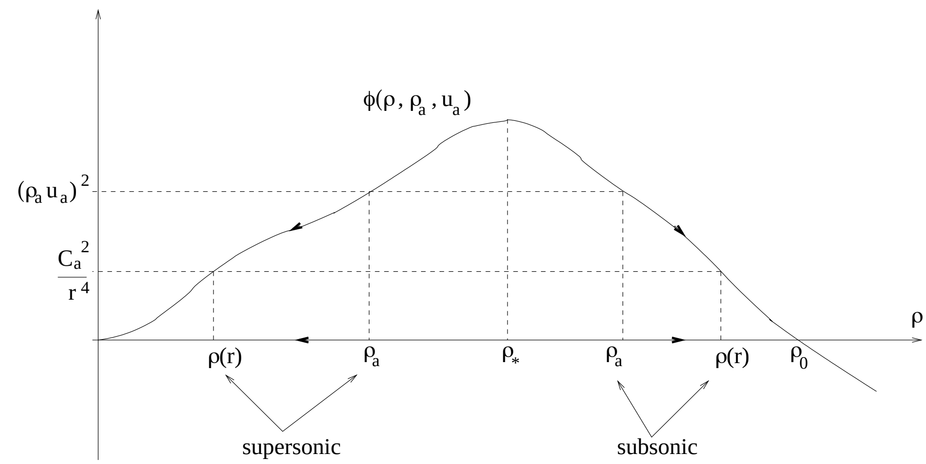

Figure 1: Inner solutions. The function $\phi(\rho, \rho_a, u_a)$. Arrows indicate direction as r increases from $r = a$. The function $\psi(\rho, \rho_a, u_a, v_a)$ in Section 2.2 has the same form.

Using (A4) it also follows that both $\phi(\rho)$ and $\phi'(\rho)$ vanish as $\rho \downarrow 0$. The graph of ϕ thus looks like in Figure 1. As $\phi(\rho_a) = \rho_a^2 u_a^2 > 0$ it follows that $\rho_a < \rho_0$. Note that, by construction, we have $\phi(\rho_a) < \phi(\rho_*)$. We observe that ρ_* and ρ_0 depend on a, ρ_a, u_a, and that ρ_* is implicitly given by

$$\partial_\rho \phi(\rho, \rho_a, u_a)\Big|_{\rho=\rho^*} = 0\,.$$

For given a, ρ_a, and u_a there are thus two possibilities: $\rho_* < \rho_a$ or $\rho_* > \rho_a$. From the figure it's clear that $\rho_a \gtrless \rho_*$ if and only if $\phi'(\rho_a) \lesseqgtr 0$. By (2.16) and the fact that $\Pi(\rho_a) = 0$, we see that $\phi'(\rho_a) > 0$ ($\phi'(\rho_a) < 0$) if and only if the flow is supersonic (subsonic) at $r = a$.

Returning to equation (2.15) we see that its right-hand side is a strictly decreasing function of r. The two cases are thus given as follows ($c_a^2 := c(\rho_a)^2 = P'(\rho_a)$):

- Subsonic case: $|u_a| < c_a$. In this case we find a unique, smooth solution $\rho(r)$ of (2.15) for all $r > a$, with $\frac{d\rho(r)}{dr} > 0$;
- Supersonic case: $|u_a| > c_a$. In this case we find a unique, smooth solution $\rho(r)$ of (2.15) for all $r > a$, with $\frac{d\rho(r)}{dr} < 0$.

It is clear from Figure 1 that these smooth solutions are defined for all $r \geq a$.

Consider the subsonic case and recall the convexity assumption (A2). This condition implies that the sound speed along the profile, $c(\rho(r))$, is an increasing function of r in this case. On the other hand, since $\Pi(\rho, \rho_a)$ is increasing with respect to ρ, it follows from (2.12) that $|u(r)|$ is a strictly decreasing function of r in this case. Thus, the Mach number $M(r) := |u(r)|/c(\rho(r))$ decreases as r increases: if the flow is subsonic at $r = a$, then the same is true for all $r \geq a$. A similar argument applies in the supersonic case: if the flow is supersonic at $r = a$, then the same is true for all $r \geq a$. Note that these conclusions are independent of the direction of flow, i.e. we get a solution for both inflow ($u_a > 0$) and outflow ($u_a < 0$) boundary data. Summing up we have:

Proposition 2.2. (Existence of spherically symmetric stationary inner solutions) *Consider the stationary barotropic Euler equations with spherical symmetry (2.1)-(2.2) in the exterior of a sphere with radius $a > 0$, and with prescribed Dirichlet data $\rho_a > 0$, $u_a \neq 0$ at $r = a$. Assume that the pressure P satisfies the assumptions (A1)-(A4) and that the data are non-sonic (i.e. $u_a^2 \neq P'(\rho_a)$).*

Then (2.1)-(2.2) have a unique smooth solution defined for all $r \geq a$. The resulting flow is strictly subsonic/supersonic for all $r \geq a$ if and only if it is strictly subsonic/supersonic at the inner boundary $r = a$.

Remark 2.3. *If the data at $r = a$ are sonic, then there are two solutions to (2.1)-(2.2) defined for $r \geq a$ - one supersonic and one subsonic.*

2.2 Inner solutions the cylindrically symmetric case

Next we construct inner solutions for CS flow: given Dirichlet data $\rho_a > 0$, $u_a \neq 0$, v_a, w_a at the inner boundary $r = a$, we seek a smooth, stationary solution to the barotropic Euler equations (2.3)-(2.6) in the region $r \geq a$. From (2.3) we see that (2.6) is satisfied with $w \equiv w_a$. By using (2.3) in (2.4) and (2.5) we reduce the remaining equations to

$$\rho u r \equiv C_a \,, \tag{2.17}$$

$$r v \equiv D_a \,, \tag{2.18}$$

$$u\frac{du}{dr} - \frac{D_a^2}{r^3} + \frac{1}{\rho}\frac{d(P(\rho))}{dr} = 0 \,, \tag{2.19}$$

where

$$C_a = a\rho_a u_a \,, \qquad D_a = a v_a \,. \tag{2.20}$$

Defining $\Pi(\rho_2, \rho_1)$ as in (2.11) and integrating (2.19) once, we get that the density $\rho(r)$ satisfies the algebraic equation

$$\frac{1}{r^2} = \frac{\rho(r)^2}{C_a^2 + D_a^2\,\rho(r)^2}\left[V_a^2 - \Pi(\rho(r), \rho_a)\right] , \qquad \text{where } V_a^2 := u_a^2 + v_a^2. \tag{2.21}$$

To analyze this equation we define the function

$$\psi(\rho) \equiv \psi(\rho, \rho_a, u_a, v_a) := \frac{\rho^2}{C_a^2 + D_a^2\,\rho^2}\left[V_a^2 - \Pi(\rho, \rho_a)\right] ,$$

where the dependence on u_a and v_a are given by (2.20). When $D_a = 0$ this reduces to the function ϕ defined in (2.14). We get that

$$\psi'(\rho) = \frac{2\rho}{\left(C_a^2 + D_a^2\,\rho^2\right)^2}\left\{C_a^2 V_a^2 - \left[C_a^2\big(\Pi(\rho, \rho_a) + P'(\rho)\big) + D_a^2\,\rho^2 P'(\rho)\right]\right\} . \tag{2.22}$$

As in the SS case (see the argument above for $\phi(\rho)$) we have that the map

$$\rho \mapsto C_a^2\big(\Pi(\rho, \rho_a) + P'(\rho)\big) + D_a^2\,\rho^2 P'(\rho) \tag{2.23}$$

is strictly increasing, tends to $+\infty$ as $\rho \to +\infty$, and tends to a strictly negative value as $\rho \downarrow 0$. (Recall that we assume $\rho_a > 0$). Also, from (A3), it follows that $\psi(\rho) < 0$ for ρ

sufficiently large. Hence, just as for $\phi(\rho)$, we have that $\psi(\rho)$ is positive for small positive ρ, tends to 0 as $\rho \downarrow 0$, and that there are unique values $0 < \rho_* < \rho_0$ for which

$$\psi'(\rho_*) = 0\,, \qquad \psi(\rho_0) = 0\,.$$

As $\psi(\rho_a) > 0$ it follows that $\rho_0 > \rho_a$. The situation is thus the same as for the case without a tangential velocity component, and $\psi(\rho) \equiv \psi(\rho, \rho_a, u_a, v_a)$ has the same shape as $\phi(\rho, \rho_a, u_a)$ in Figure 1.

Returning to the algebraic equation (2.21) for $\rho(r)$ we observe that equality holds at $r = a$ by definition. As r increases from $r = a$ we see that the properties of ψ guarantees a solution $\rho(r)$, defined for all $r \geq a$. Just as in the case with no tangential velocity there are two cases: $\rho_a \gtrless \rho_*$, which is the case if and only if $\psi'(\rho_a) \lesseqgtr 0$, which holds if and only if $u_a^2 \lesseqgtr P'(\rho_a) \equiv c_a^2$. Note that the tangential velocity is irrelevant at this point. We refer to these cases as *radially* super/sub-sonic. To analyze the sonicity we define the

$$\textit{Radial Mach number} = M_{rad}(r) := \frac{|u(r)|}{c(r)}\,,$$

as well as the

$$\textit{(proper) Mach number} = M(r) := \frac{V(r)}{c(r)}\,,$$

where $V(r) := \sqrt{u(r)^2 + v(r)^2}$. We thus have two cases:

- Radially subsonic case: $|u_a| < c(a)$. In this case we find a unique, smooth solution $\rho(r)$ of (2.21) for all $r > a$, with $\frac{d\rho(r)}{dr} > 0$;
- Radially supersonic case: $|u_a| > c(a)$. In this case we find a unique, smooth solution $\rho(r)$ of (2.21) for all $r > a$, with $\frac{d\rho(r)}{dr} < 0$.

Consider the radially subsonic case where $\frac{d\rho(r)}{dr} > 0$. From (2.17), (2.18), and (2.21) we have

$$V(r)^2 + \Pi(\rho(r), \rho_a) \equiv V_a^2\,. \tag{2.24}$$

Since $\Pi(\rho, \rho_a)$ is increasing with respect to ρ, it follows that the speed $V(r)$ is a strictly decreasing function of r in this case. As $\rho ur \equiv C_a$ it follows that $|u(r)|$ is decreasing in this case, while the sound speed $c(r)$ increases with r. Thus, in the radially subsonic case we have that *both* the radial and the proper Mach numbers are decreasing as r increases from $r = a$.

Next, consider the radially supersonic case where $\frac{d\rho(r)}{dr} < 0$. Again, (2.24) holds and we conclude that the speed $V(r)$ is strictly increasing in this case. Also, since $\rho(r)$ decreases, so does the sound speed $c(r)$. The flow, which is supersonic at $r = a$ (as it is radially supersonic there) therefore becomes more supersonic as r increases from $r = a$. We proceed to show that the flow remains also *radially* supersonic as r increases. Multiplying (2.21) by $C_a^2/\rho(r)^2$, and using (2.24) we get that

$$u(r)^2 = \frac{C_a^2 V(r)^2}{C_a^2 + D_a^2 \rho(r)^2}\,.$$

As $V(r)$ increases with r, in the the present case, while $\rho(r)$ decreases, it follows that $|u(r)|$, and thus $M_{rad}(r)$, increases as r increases.

Thus, stationary, cylindrically symmetric flow (possibly with swirl) which is *radially* super- or sub-sonic at the inner boundary $r = a$, becomes increasingly so as r increases. Summing up we have:

Proposition 2.4. (Existence of cylindrically symmetric stationary inner solutions) *Consider the stationary barotropic Euler equations with cylindrical symmetry (2.3)-(2.6) in the exterior of a cylinder with radius $a > 0$, and with prescribed Dirichlet data $\rho_a > 0, u_a \neq 0, v_a, w_a$ at $r = a$. Assume that the pressure satisfies assumptions (A1)-(A4) and that the data are radially non-sonic ($u_a^2 \neq P'(\rho_a)$).*

Then (2.3)-(2.6) have a unique, smooth solution defined for all $r \geq a$. The flow is subsonic/supersonic throughout $r > a$ if and only if it is subsonic/supersonic at $r = a$.

Remark 2.5. *As in the SS case, if the data are sonic at $r = a$ then there are two smooth, stationary solutions defined in $r > a$ - one supersonic and one subsonic.*

2.3 Outer solutions in the spherically symmetric case

We now give Dirichlet data $\rho_b > 0, u_b \neq 0$ at an *outer* boundary $r = b$, and we seek a smooth, stationary solution to the Euler equations. Differently from inner solutions which were defined everywhere outside the inner boundary, outer solutions are not defined for all $r < b$: there is a critical radius $r^* = r^*(b, \rho_b, u_b) \in (0, b]$ where the flow becomes sonic and beyond which a stationary solution cannot be extended. In order to construct a solution in a nontrivial interval we will assume that the data are strictly super- or subsonic at $r = b$.

We define the functions Π and ϕ as in Section 2.1. An entirely similar analysis shows that the density profile $\rho(r)$ in the present case is given as the solution to the algebraic equation

$$\frac{C_b^2}{r^4} = \phi(\rho(r), \rho_b, u_b) \qquad \text{where} \quad C_b := \rho_b u_b b^2 \,. \tag{2.25}$$

Again as in Section 2.1: setting $\phi(\rho) \equiv \phi(\rho, \rho_b, u_b)$ we get that there are unique ρ-values $0 < \rho^* < \rho^0$ (each depending on b, ρ_b, and u_b) such that $\phi'(\rho^*) = 0\,, \qquad \phi(\rho^0) = 0$. Note that, by construction, $\phi(\rho_b) < \phi(\rho^*)$. As $\phi(\rho_b) = \rho_b^2 u_b^2 > 0$ it follows that $\rho^0 > \rho_b$. The situation thus looks like in Figure 2, and we get the following two possibilities:

- Subsonic case: $|u_b| < c_b$. In this case we find a unique, smooth solution $\rho(r)$ to (2.25) with $\frac{d\rho(r)}{dr} > 0$;
- Supersonic case: $|u_b| > c_b$. In this case we find a unique, smooth solution $\rho(r)$ to (2.25) with $\frac{d\rho(r)}{dr} < 0$.

So far the analysis is similar to the analysis of inner solutions. However, as r decreases from $r = b$, there is a limiting value of the radius $r = r^* = r^*(b, \rho_b, u_b)$ below which (2.25) does not have a solution $\rho(r)$. Observe that r^* is given by $\rho(r^*) = \rho^*$, and that ρ^* is implicitly given by $\phi'(\rho^*) = 0$. Using (2.16) (with u_b instead of u_a and with ϕ and Π as in this section)

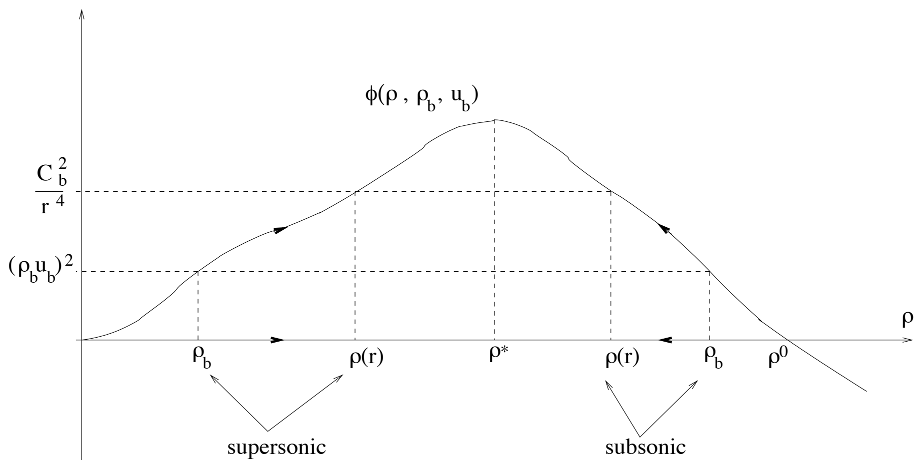

Figure 2: Outer solutions. The function $\phi(\rho, \rho_b, u_b)$. Arrows indicate direction as r decreases from $r = b$. The function $\psi(\rho, \rho_b, u_b, v_b)$ in Section 2.4 has the same form.

we have $u_b^2 - \Pi(\rho^*, \rho_b) = P'(\rho^*)$, and it follows that $\phi(\rho^*) = \rho^{*2}(u_b^2 - \Pi(\rho^*, b)) = \rho^{*2} P'(\rho^*)$. By (2.25) the limiting value $r = r^*$ is thus given implicitly by

$$\rho^{*2} P'(\rho^*) = \frac{C_b^2}{r^{*4}} = \rho^{*2} u^{*2} .$$

That is, as we let r decrease from $r = b$ we reach the limiting radius at the sonic point.

To analyze the sonicity of flow in outer SS solutions we can argue in a similar manner as for inner solutions to reach the following conclusions: if the flow is subsonic (supersonic) at $r = b$, then the flow becomes less subsonic (supersonic) as r decreases. We summarize our findings for outer SS solutions:

Proposition 2.6. (Existence of spherically symmetric stationary outer solutions) *Consider the stationary barotropic Euler equations with spherical symmetry (2.1)-(2.2) inside a sphere with radius $b > 0$, and with prescribed Dirichlet data $\rho_b > 0$, $u_b \neq 0$ at $r = b$. Assume that the pressure satisfies the assumptions (A1)-(A4) and that the data are non-sonic ($u_b^2 \neq P'(\rho_b)$).*

Then there is a critical inner radius $r^* = r^*(b, \rho_b, u_b) > 0$ where the flow becomes sonic, and below which there is no solution of the equations. For each fixed $\bar{r} > r^*$ the equations have a unique smooth solution defined for $\bar{r} \leq r \leq b$. The resulting flow is strictly subsonic (supersonic) throughout $[\bar{r}, b]$ if and only if it is strictly subsonic (supersonic) at the outer boundary $r = b$.

2.4 Outer solutions in the cylindrically symmetric case

We proceed to analyze solutions of the system (2.3)-(2.5) which takes on given Dirichlet data $\rho_b > 0$, u_b, v_b at the outer boundary $r = b$. (We have now set $w \equiv w_b$). The analysis follows the same steps as in the earlier sections.

We define Π as in Section 2.1 and observe that the same analysis as in Section 2.2 shows that the density profile $\rho(r)$ in the present case satisfies the algebraic equation

$$\frac{1}{r^2} = \psi(\rho) \equiv \psi(\rho, \rho_b, u_b, v_b) \tag{2.26}$$

where

$$\psi(\rho) \equiv \psi(\rho, \rho_b, u_b, v_b) := \frac{\rho^2}{C_b^2 + D_b^2\,\rho^2}\left[V_b^2 - \Pi(\rho, \rho_b)\right],$$

and with $V_b^2 := u_b^2 + v_b^2$, $C_b = b\rho_b u_b$, and $D_b = bv_b$. An argument entirely similar to the one in Section 2.2 gives unique values $0 < \rho^* < \rho^0$ (now depending on ρ_b, u_b, v_b) such that

$$\psi'(\rho^*) = 0\,, \qquad \psi(\rho^0) = 0\,.$$

As $\psi(\rho_b) > 0$ it follows that $\rho^0 > \rho_b$. The situation is thus the same as for the case without a tangential velocity component, and $\psi(\rho) \equiv \psi(\rho, \rho_b, u_b, v_b)$ has the same shape as $\phi(\rho, \rho_b, u_b)$ in Figure 2. We have the two cases: $\rho_b \gtrless \rho^*$, which holds if and only if $M_{rad}(r) \lessgtr 1$, where

$$M_{rad}(r) = \textit{Radial Mach number} := \frac{|u(r)|}{c(r)}\,.$$

We thus have the two cases for any $\bar{r} > r^*$:

- Radially subsonic case: $|u_b| < c(b)$. In this case we find a unique, smooth solution $\rho(r)$ of (2.26) on $[\bar{r}, b]$, with $\frac{d\rho(r)}{dr} > 0$;
- Radially supersonic case: $|u_b| > c(b)$. In this case we find a unique, smooth solution $\rho(r)$ of (2.26) on $[\bar{r}, b]$, with $\frac{d\rho(r)}{dr} < 0$.

Consider the radially subsonic case where $\frac{d\rho(r)}{dr} > 0$. Combining the ODEs with data given at $r = b$ we get that

$$V(r)^2 + \Pi(\rho(r), \rho_b) \equiv V_b^2\,, \tag{2.27}$$

where $V(r) := \sqrt{u(r)^2 + v(r)^2}$. Using the convexity of the pressure we get that the sound speed $c(r)$ increases with r. Thus, in the radially subsonic case we have that *both* the radial and the proper Mach numbers increases as r decreases from $r = b$. In particular, the flow becomes *less* radially subsonic as the particles flow towards the origin. A similar argument shows that supersonic flow becomes less radially supersonic as r decreases from $r = b$.

Thus, stationary, cylindrically symmetric flow (possibly with swirl) which is *radially* super- or sub-sonic at the outer boundary $r = b$, becomes less so as r decreases from $r = b$.

As in the case without a tangential velocity component we see from Figure 2 that, as r decreases from $r = b$, there is a limiting value of the radius $r = r^* = r^*(b, \rho_b, u_b)$ below which (2.26) does not have a solution $\rho(r)$. We proceed to show that this occurs exactly where the flow becomes *radially* sonic (i.e. $M_{rad}(r^*) = 1$). First observe that r^* is given by $\rho(r^*) = \rho^*$, and that ρ^* is implicitly given by $\psi'(\rho^*) = 0$. Using (2.22) (with C_b, D_b instead of C_a, D_a and with $\Pi = \Pi(\cdot, \rho_b)$) we get that

$$C_b^2 V_b^2 = C_b^2\left[\Pi(\rho^*, \rho_b) + P'(\rho^*)\right] + D_b^2 \rho^{*2} P'(\rho^*)\,.$$

Using the relations $\Pi(\rho^*, \rho_b) = V_b^2 - V^{*2}$, $C_b = u^* \rho^* r^*$, $D_b = v^* r^*$, it follows that

$$u^{*2} = P'(\rho^*) =: c^{*2} .$$

Summarizing we have:

Proposition 2.7. (Existence of spherically symmetric stationary outer solutions) *Consider the stationary barotropic Euler equations with cylindrical symmetry (2.3)-(2.6) inside a sphere with radius* $b > 0$, *and with prescribed Dirichlet data* $\rho_b > 0$, $u_b \neq 0$, v_b, w_b *at* $r = b$. *Assume that the pressure satisfies the assumptions (A1)-(A4) and that the data are radially non-sonic* ($u_b^2 \neq P'(\rho_b)$)*.*

Then there is a critical inner radius $r^* = r^*(b, \rho_b, u_b, v_b) > 0$ *where the flow becomes radially sonic, and below which there is no solution of the equations. For each fixed* $\bar{r} > r^*$ *the equations have a unique smooth solution defined for* $\bar{r} \leq r \leq b$. *The resulting flow is radially subsonic (supersonic) throughout* $[\bar{r}, b]$ *if and only if it is radially subsonic (supersonic) at the outer boundary* $r = b$.

Remark 2.8. *A calculation shows that* r^* *is an increasing function of* $|v_b|$ *for fixed values of* b, ρ_b, u_b. *That is, faster rotation of the fluid decreases the interval of existence of a stationary solution.*

3 Stationary solutions with shocks

We next use the inner and outer solutions to construct *weak* solutions with spherical or cylindrical symmetry, and with a single stationary, entropy admissible shock. As with the inner and outer solutions in the previous section the construction will depend on whether we work outward with data given $r = a$, or inward with data given at $r = b$. The two cases are treated separately in the two next subsections. In the last subsection we consider the problem of deciding existence and possible location of a shock when data are provided at both $r = a$ and at $r = b$.

We note that the Rankine-Hugoniot relations for density and momentum are identical for SS and CS flow. It follows from (2.8) that only the radial part of the velocity changes across a discontinuity in the CS case. We thus treat the two cases as one case, bearing in mind that $v = w \equiv 0$ in the SC case.

3.1 Shock solution built from inner solutions

We assume we are given Dirichlet data $\rho_a > 0, u_a, v_a, w_a$ at the inner boundary $r = a$ (a being fixed from now on) and we assume that the flow is either strictly *radially* supersonic or strictly *radially* subsonic at $r = a$. That is, we assume $u_a^2 \gtrless c(a)^2$, where $c(a)^2 = P'(\rho_a)$.

Next we fix any radius $b > a$ together with any intermediate radius $\bar{r} \in (a, b)$. According to the proposeitions above we can solve (2.1)-(2.2) in the SS case, or (2.3)-(2.6) in the CS case, for $r \in (a, \bar{r})$ with the given values at $r = a$ as initial data. This provides the values $\rho(\bar{r}-)$, $u(\bar{r}-)$, $v(\bar{r}-)$, $w(\bar{r}-)$. The Rankine-Hugoniot conditions then give $\rho(\bar{r}+)$, $u(\bar{r}+)$, $v(\bar{r}+)$, $w(\bar{r}+)$ (see below), which we use as initial data for (2.1)-(2.2) (or (2.3)-(2.6)) in the outer region $r \in (\bar{r}, b)$. Appealing once more to the earlier discussion we obtain a stationary solution defined for all $r \in [a, b]$ and with a single discontinuity at any intermediate location.

It remains to verify that $\rho(\bar{r}+)$, $u(\bar{r}+)$, $v(\bar{r}+)$, $w(\bar{r}+)$ are uniquely determined by the Rankine-Hugoniot relations, and to analyze the admissibility of the resulting solutions. As selection criteria we impose that the flow should be *compressive*: a fluid particle suffers an increase in density as it crosses a shock. Let $\bar{\rho} = \rho(\bar{r}_-)$, $\bar{u} = u(\bar{r}_-)$, and let $\hat{\rho} = \rho(\bar{r}_+)$, $\hat{u} = u(\bar{r}_+)$. Defining $F(\rho) := P(\rho) - P(\bar{\rho})$ and $G(\rho) := \bar{\rho}\bar{u}^2(1 - \bar{\rho}/\rho)$, the Rankine-Hugoniot conditions are

$$F(\hat{\rho}) = G(\hat{\rho}), \qquad \hat{u} = \frac{\bar{\rho}\bar{u}}{\hat{\rho}}. \tag{3.1}$$

We have

$$F'(\rho) = P'(\rho), \qquad \text{and} \qquad G'(\rho) = \left(\frac{\bar{\rho}\bar{u}}{\rho}\right)^2.$$

It follows that the flow at $\bar{r}_-$ is radially supersonic if and only if $G'(\bar{\rho}) > F'(\bar{\rho})$, and in this case the Rankine-Hugoniot conditions have a unique nontrivial solution $\hat{\rho} > \bar{\rho}$. On the other hand, the flow at $\bar{r}_-$ is radially subsonic if and only if $G'(\bar{\rho}) < F'(\bar{\rho})$, and in this case the Rankine-Hugonoiot conditions have a unique nontrivial solution $\hat{\rho} < \bar{\rho}$. (These conclusions are consequences of the convexity assumption (A2)).

Recall that $\bar{u} \gtrless 0$ if and only if $u_a \gtrless 0$ and that the radial sonicity is conserved as we move away from the origin (for an inner solution, which is what we consider here). It follows that if the flow is supersonic at the inner boundary $r = a$, then the flow there must be into the domain. Similarly, if the flow is subsonic at the inner boundary $r = a$, then the flow there must be out of the domain.

Having determined the flow in $(a, \bar{r})$, as well as the values of the flow variables at $r = r_+$, we can now find a unique stationary and smooth solution in the outer region $\bar{r} < r < b$.

Proposition 3.1. (Stationary symmetric shocks built from inner solutions) *Consider the barotropic Euler equations with spherical (or cylindrical) symmetry in the domain between two concentric spheres (cylinders) with radii $a < b$, and with prescribed density $\rho_a > 0$ and velocities $u_a \neq 0$, v_a, w_a at $r = a$. Assume that flow at $r = a$ is (radially) non-sonic and that the pressure satisfies the assumptions (A1)-(A4). Given any radius $\bar{r} \in (a, b)$.*

Then there is a unique weak admissible solution with a single shock located at $\bar{r}$ if and only if, either, the flow is radially supersonic at $r = a$ and directed into the domain (i.e. $u_a > 0$), or the flow is radially subsonic at $r = a$ and directed out of the domain (i.e. $u_a < 0$). In the former case the flow is (radially) supersonic in $(a, \bar{r})$ and (radially) subsonic in $(\bar{r}, b)$, while the opposite holds in the latter case.

3.2 Shock solution built from outer solution

The procedure for constructing stationary, symmetric solutions with an admissible shock from data given at the outer boundary, is similar to, but slightly more involved than, the procedure in the previous subsection. To formulate the result we need to identify the two critical radii involved. For concreteness let's consider the SS case. First, from the data at $r = b$ we calculate $\rho_1^* = \rho_1^*(\rho_b, u_b)$ from the equation

$$\partial_\rho \phi(\rho, \rho_b, u_b)\Big|_{\rho=\rho_1^*} = 0, \tag{3.2}$$

where ϕ is defined in (2.14). We then calculate $r_1^* = r_1^*(\rho_b, u_b)$ from

$$\frac{C_b^2}{r_1^{*4}} = \phi(\rho_1^*, \rho_b, u_b) \qquad \text{where} \quad C_b = \rho_b u_b b^2\,. \tag{3.3}$$

Now, given any intermediate radius $\bar{r} \in (r_1^*, b)$ we know that we can solve the flow equations on $(\bar{r}, b)$ to find $(\hat{\rho},\, \hat{u}) := (\rho(\bar{r}_+),\, u(\bar{r}_+)$. The earlier analysis shows that the flow will be strictly super- or subsonic at $\bar{r}_+$ if and only if it is so at $r = b$. As in the case of inner solutions the Rankine-Hugoniot relations now determine a unique state $(\bar{\rho},\, \bar{u})$ at $\bar{r}_-$ which connects to $(\hat{\rho},\, \hat{u})$. The flow at $\bar{r}_-$ is supersonic (subsonic) if and only if the flow at $\bar{r}_+$ is subsonic (supersonic). Repeating the above analysis we find that the flow can be extended inwards until a second critical radius $r_2^* < r_1^*$ which is determined as above (with with data $\bar{r}$, $\bar{\rho}$, $\bar{u}$ instead of b, ρ_b, u_b). (For the CS case there are similarly two critical radii r_1^* and r_2^* determined in the same way from the data at $r = b$.) Finally, the admissibility condition dictates the direction of the flow in the same way as in Section 3.1. Summarizing we have:

Proposition 3.2. (Stationary symmetric shocks built from outer solutions) *Consider the barotropic Euler equations with spherical (or cylindrical) symmetry in the domain between two concentric spheres (cylinders) with radii $a < b$, and with prescribed density $\rho_b > 0$ and velocities $u_b \neq 0$, v_b, w_b at $r = b$. Assume that flow at $r = b$ is (radially) non-sonic and that the pressure satisfies the assumptions (A1)-(A4). Given any radius $\bar{r} \in (r_1^*, b)$ and assume that $a \in (r_2^*, r_1^*)$, where r_1^*, r_2^* are determined as above.*

Then there is a unique weak admissible solution, defined on (a, b) and with a single shock located at $\bar{r}$ if and only if, either, the flow is radially supersonic at $r = b$ and directed into the domain (i.e. $u_b < 0$), or the flow is radially subsonic at $r = b$ and directed out of the domain (i.e. $u_b < 0$). In the former case the flow is (radially) supersonic in $(\bar{r}, b)$ and (radially) subsonic in $(a, \bar{r})$, while the opposite holds in the latter case.

4 When can a shock solution be found?

We next consider the possibility of finding shock solutions for given boundary data. For concreteness we treat supersonic CS flow and ask: Given data $(\rho_a,\, u_a,\, v_a,\, w_a)$ at $r = a$ together with data $(\rho_b,\, u_b,\, v_b,\, w_b)$ at $r = b$; does there exist a solution of the stationary barotropic Euler equations with these boundary data and with an admissible shock located at some intermediate radius $\bar{r} \in (a, b)$? To analyze this question we fix a, b, and $(\rho_a,\, u_a,\, v_a,\, w_a)$, and then formulate necessary and sufficient conditions on ρ_b, u_b, v_b, w_b which guarantee the existence of a solution with a shock in (a, b).

From the earlier analysis we know that $\rho(r)u(r)r^2 \equiv C_a := \rho_a u_a a^2$, $rv(r) \equiv D_a := av_a$, and $w(r) \equiv w_a$ along any stationary solution (smooth or not). Thus, a necessary condition for the existence of a shock solution with "final" data ρ_b, u_b, v_b, w_b at $r = b$ is that

$$\rho_b u_b = \frac{C_a}{b^m}\,, \qquad v_b = \frac{D_a}{b}\,, \qquad w_b = w_a\,. \tag{4.1}$$

We choose to work with the density as the primary unknown so that the issue becomes: what final densities ρ_b can be attained for a solution with a shock at $\bar{r} \in (a, b)$. For

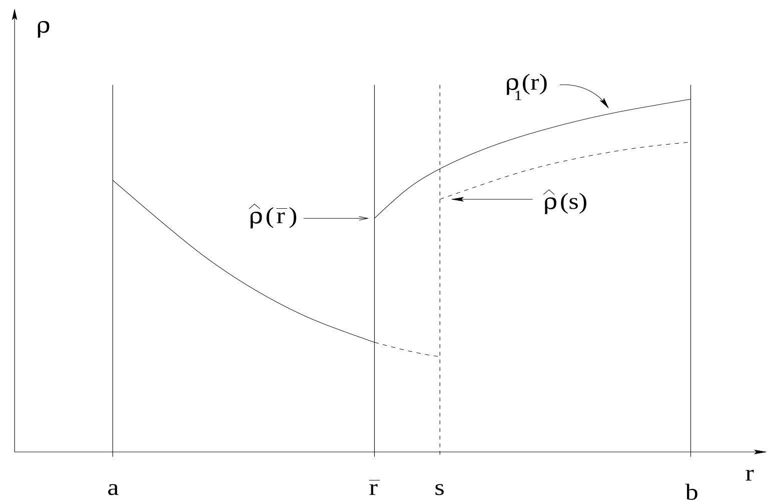

Figure 3: Configuration in the case of supersonic inflow at $r = a$.

concreteness we consider the case with (radially) supersonic inflow at $r = a$, that is, $u_a > 0$ and $u_a^2 > c_a^2 = P'(\rho_a)$.

To see how the final density ρ_b depends on the shock location $\bar{r}$, we first observe that the ODE for the density takes the same form in the two intervals $(a, \bar{r})$ and $(\bar{r}, b)$, and it is independent of $\bar{r}$. Indeed, from (2.19) it follows that

$$\frac{d\rho}{dr} = \frac{\rho(v^2 + u^2)}{r(c^2 - u^2)} . \tag{4.2}$$

From the earlier analysis we know that the flow remains (radially) subsonic for all $r > \bar{r}$, whence (4.2) is a well-behaved ODE with unique solutions. Thus, if $\rho_1(r)$, $\rho_2(r)$ are two smooth solutions with $\rho_1(s) > \rho_2(s)$ for some s, then necessarily $\rho_1(r) > \rho_2(r)$ for all $r > s$.

We can use this to infer how ρ_b varies with the shock location $\bar{r}$. Specifically we will show that an increase in $\bar{r}$ implies a lower ending value for the density at $r = b$, see Figure 3. Let $\rho_1(r)$ denote the solution to (4.2) for $r > \bar{r}$, and let $\hat{\rho}(r)$ denote the density immediately on the outside of the shock. It follows from the uniqueness of solutions to (4.2) that an increase in $\bar{r}$ implies a lower ending value for the density at $r = b$ if and only if

$$\hat{\rho}'(\bar{r}) < \rho_1'(\bar{r}+) , \tag{4.3}$$

which we will show to hold. From (4.2) we have

$$(\hat{c}^2 - \hat{u}^2)\rho_1'(\bar{r}+) = \frac{\hat{\rho}(\bar{v}^2 + \hat{u}^2)}{\bar{r}} , \tag{4.4}$$

where a bars (hats) denote evaluation immediately on the inside (outside) of the shock. To express $\hat{\rho}'(\bar{r})$ we use the Rankine-Hugoniot relations. As $\rho u = C_a/r$ throughout we get from $(2.7)_2$ that

$$P(\hat{\rho}) + \frac{C_a^2}{\hat{\rho}\,\bar{r}^2} = P(\bar{\rho}) + \frac{C_a^2}{\bar{\rho}\,\bar{r}^2} .$$

Taking the derivative with respect to $\bar{r}$ and rearranging gives

$$(\hat{c}^2 - \hat{u}^2)\hat{\rho}' \;=\; \frac{2C_a^2}{\bar{r}^3}\Big(\frac{1}{\hat{\rho}} - \frac{1}{\bar{\rho}}\Big) + (\bar{c}^2 - \bar{u}^2)\bar{\rho}' \;=\; \frac{2C_a^2}{\bar{r}^3}\Big(\frac{1}{\hat{\rho}} - \frac{1}{\bar{\rho}}\Big) + \frac{\bar{\rho}(\bar{v}^2 + \bar{u}^2)}{\bar{r}}\,, \tag{4.5}$$

where we have used that (4.2) holds throughout (a, b). From (4.4) and (4.5), and recalling that $\hat{c}^2 - \hat{u}^2 > 0$, we get that (4.3) holds if and only if $C_a^2(\bar{\rho} - \hat{\rho}) < D_a^2\bar{\rho}\hat{\rho}(\hat{\rho} - \bar{\rho})$, which holds since the shock is compressive ($\hat{\rho} > \bar{\rho}$).

It follows that the minimal value α for ρ_b is attained by placing the shock at $\bar{r} = b-$, while the maximal value for ρ_b is attained by placing the shock at $\bar{r} = a+$. We summarize our findings in:

Theorem 4.1. (Possible shocks in cylindrically symmetric flow) *Consider the stationary, cylindrically symmetric, barotropic Euler equations. Given radii $a < b$ and data ρ_a, u_a, v_a, w_a which corresponds to supersonic inflow at $r = a$ (i.e. $u_a^2 > c_a^2$, $u_a > 0$).*

Then there is a finite interval (α, β) of ρ_b-values that can be reached from the data at $r = a$ through a stationary, compressive shock located at some location $\bar{r} \in (a, b)$. α, β depend only on a, ρ_a, u_a, v_a, and b, and there is a one-to-one correspondence between ρ_b values in (α, β) and shock locations in (a, b).

Remark 4.2. *A similar analysis applies to the case of CS flow with (radially) subsonic data given at $r = a$, as well as to the case of SS flows.*

5 Exact Navier-Stokes solutions converging to Euler shocks

In this second part of the paper we construct smooth stationary solutions to the Navier-Stokes equations which converge to the previously constructed inviscid shocks in the small viscosity limit. We focus on the spherically symmetric case with prescribed supersonic inflow at $r = a$. The same arguments treat the cylindrically symmetric case when both $v = 0$ and $w = 0$. The small viscosity limit for the CS case when either $v \neq 0$ or $w \neq 0$ entails additional difficulties similar to those which appear in the nonbarotropic SS case. These are treated in [EJW].

In the following sections we sometimes denote derivatives with respect to r or s by $d_r f$ or $d_s f$, and we denote viscosity by ϵ.

5.1 Formulation as a transmission problem

The stationary viscous equations for the spherically symmetric case are given by the 2×2 second order system

$$\begin{aligned} &d_r(\rho u) + \frac{2\rho u}{r} = 0 \\ &d_r(\rho u^2) + \frac{2\rho u^2}{r} + d_r P(\rho) = \epsilon\left(u_{rr} + \frac{2u_r}{r} - \frac{2u}{r^2}\right), \ \ \epsilon > 0. \end{aligned} \tag{5.1}$$

We suppose that we are given a stationary inviscid shock solution $U^0(r) = (\rho^0(r), u^0(r))$ (as constructed earlier) with supersonic inflow at $r = a$, shock surface $r = \overline{r} \in (a, b)$, and taking the values (ρ_a, u_a) at $r = a$ and (ρ_b, u_b) at $r = b$. Setting $s = r - \overline{r}$, $\tilde{\rho}(s) := \rho(s + \overline{r})$, $\tilde{u}(s) := u(s + \overline{r})$ and dropping tildes, we obtain an equivalent problem on $[a - \overline{r}, b - \overline{r}]$ with shock surface at $s = 0$ now:

$$d_s f(\rho, u) + g(\rho, u, s) = \epsilon h(u, u_s, u_{ss}, s) \tag{5.2}$$

where

$$f(\rho, u) = \begin{pmatrix} \rho u \\ \rho u^2 + P(\rho) \end{pmatrix}, \qquad g(\rho, u, s) = \begin{pmatrix} \frac{2\rho u}{s+\overline{r}} \\ \frac{2\rho u^2}{s+\overline{r}} \end{pmatrix}, \quad \text{and} \tag{5.3}$$

$$h(u, u_s, u_{ss}, s) = \begin{pmatrix} 0 \\ u_{ss} + \frac{2u_s}{s+\overline{r}} - \frac{2u}{(s+\overline{r})^2} \end{pmatrix}. \tag{5.4}$$

For viscosity $\epsilon > 0$ sufficiently small, we shall construct exact smooth solutions to (5.2), which assume the values (ρ_a, u_a) at $s = a - \overline{r}$, the values $(\rho_b, u_b) + O(\epsilon)$ at $s = b - \overline{r}$, and which "converge" to the inviscid shock $\tilde{U}^0(s) = U^0(s + \overline{r})$ as $\epsilon \to 0$ (e.g., in L^2 near $s = 0$, in L^∞ for $|s| \geq \delta > 0$). The tilde on U^0 is suppressed below.

To obtain exact viscous solutions converging to $U^0(s)$ as $\epsilon \to 0$, first we replace (5.2) with an equivalent *transmission* problem on $[a - \overline{r}, b - \overline{r}]$:

$$\begin{aligned} &(a)\ d_s f(\rho, u) + g(\rho, u, s) - \epsilon h(u, u_s, u_{ss}, s) = 0 \text{ on } [a - \overline{r}, b - \overline{r}] \cap \{\pm s \geq 0\} \\ &(b)\ [\rho] = 0, [u] = 0, [u_s] = 0 \text{ on } s = 0, \end{aligned} \tag{5.5}$$

where now $(\rho, u) = (\rho^\epsilon_\pm, u^\epsilon_\pm)$ in $\pm s \geq 0$. Using the obvious correspondence between C^1 functions on $[a-\overline{r}, b-\overline{r}]$ and piecewise C^1 functions satisfying transmission conditions as in (5.5), we see that the problems (5.2) and (5.5) are equivalent in the sense that (ρ, u) solves (5.2) if and only if $(\rho_\pm, u_\pm)$ solves (5.5). Here and often in what follows, we suppress the $\pm$ and ϵ on $(\rho^\epsilon_\pm, u^\epsilon_\pm)$.

Remark 5.1. *1. For a given interval length $p = b - a$, if ϵ is* large *enough, one can use the fact that*

$$\rho(r) u(r) r^2 = \rho_a u_a a^2 \tag{5.6}$$

to replace (5.1) *with an equivalent equation for u, and apply classical two-point boundary theory for second-order problems (e.g., [H], Chapter 12, Thm. 4.1 and Cor. 4.1) to obtain a unique solution taking the values u_a, u_b at a, b respectively. However, for small ϵ the standard theory yields two-point existence results only on intervals whose lengths shrink to zero with ϵ. Furthermore it gives no information about the proximity of these large ϵ solutions to the given inviscid shock. Formulating the viscous problem as a transmission problem at the location of the inviscid shock effectively resolves the issue of locating the transition region.*

6 Approximate viscous solutions

We seek an exact solution $w^\epsilon(s) = (\rho(s), u(s))$ of (5.5). We first construct approximate viscous solutions of the form

$$\tilde{w}^\epsilon(s) = \left(\mathcal{U}^0(s,z) + \epsilon\mathcal{U}^1(s,z) + \cdots + \epsilon^M\mathcal{U}^M(s,z)\right)|_{z=\frac{s}{\epsilon}}. \tag{6.1}$$

where we shall need to take $M \geq 2$. Here

$$\mathcal{U}^j(s,z) = U^j(s) + V^j(z), \tag{6.2}$$

with $U^0_\pm(s)$ the given inviscid solution, and the $V^j_\pm$ are transmission layer profiles which (turn out to be) exponentially decreasing as $z \to \pm\infty$. The following construction is an adaptation, to ODEs on a bounded interval with fast and slow scales, of the construction of approximate solutions of the Navier-Stokes equations on $\mathbb{R}^{d+1}$ in [GMWZ3], section 5 (see also [GG]).

6.1 Interior profile equations

Substitute (6.1) into (5.5) and write the result as

$$\sum_{j=-1}^{M} \epsilon^j \mathcal{F}^j(s,z)|_{z=\frac{s}{\epsilon}} + \epsilon^M R^{M,\epsilon}(s), \tag{6.3}$$

where

$$\mathcal{F}^j(s,z) = F^j(s) + G^j(z). \tag{6.4}$$

Here the G^j decrease exponentially to 0 as $z \to \pm\infty$, since the same is true of the $V^j(z)$ in (6.2).

Remark 6.1. *1. When substituting $\tilde{w}^\epsilon$ into a nonlinear function $f(w)$, we Taylor expand as follows:*

$$\begin{aligned} &f(\tilde{w}^\epsilon) = f(\mathcal{U}^0 + O(\epsilon)) = f(U^0 + V^0) + O(\epsilon) = \\ &f(U^0) + \left(f(U^0 + V^0) - f(U^0)\right) + O(\epsilon) = f(U^0) + \left(f(U^0(0) + V^0) - f(U^0(0))\right) + O(\epsilon). \end{aligned} \tag{6.5}$$

The $O(s)$ error introduced by replacing $U^0(s)$ by $U^0(0)$ can be viewed as an $O(\epsilon)$ error by writing $s = \epsilon z|_{z=\frac{s}{\epsilon}}$ and using the exponential decay of $V^0(z)$. Higher order terms are obtained similarly.

2. The terms $F^j(s)$, $G^j(z)$ appearing in (6.3) are unique for $j \leq M-1$. For example, to see that $F^{-1}(s)$ is uniquely determined, multiply (6.3) by ϵ, fix $s \neq 0$, let $\epsilon \downarrow 0$, and note $G^{-1}(\frac{s}{\epsilon}) \to 0$. For fixed $z_0 \neq 0$ this in turn implies uniqueness of $G^{-1}(z_0)$, by evaluating at $s^\epsilon = z_0\epsilon$ and letting $\epsilon \downarrow 0$.

The interior profile equations are obtained by setting the F^j, G^j equal to zero. In the following expressions for $G^j(z)$, the functions $U^j_\pm$ and their derivatives are evaluated at $s=0$. With $f(w)$ as in (5.3), define 2×2 matrices

$$A(w)=d_w f(w)=\begin{pmatrix}A^{11} & A^{12}\\ A^{21} & A^{22}\end{pmatrix}=\begin{pmatrix}u & \rho\\ u^2+P'(\rho) & 2\rho u\end{pmatrix},\qquad B=\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}. \tag{6.6}$$

The F^j, G^j are given by

$$\begin{aligned}&F^{-1}(s)=0\\ &G^{-1}(z)=-B\partial_z^2\mathcal{U}^0+\partial_z f(\mathcal{U}^0),\end{aligned} \tag{6.7}$$

$$\begin{aligned}&F^0(s)=A(U^0)\partial_s U^0+g(U^0,s)\\ &G^0(z)=-B\partial_z^2 V^1+\partial_z\left(A(\mathcal{U}^0)(U^1+V^1)\right)+Q^0(U^0,V^0),\end{aligned} \tag{6.8}$$

where $Q^0=Q^0(z)$ (for short) decays exponentially to zero as $z\to\pm\infty$. For $j\ge 1$,

$$\begin{aligned}&F^j(s)=A(U^0)\partial_s U^j+d_w g(U^0,s)U^j-P^{j-1}(s)\\ &G^j(z)=-B\partial_z^2 V^{j+1}+\partial_z\left(A(\mathcal{U}^0)(U^{j+1}+V^{j+1})\right)+Q^j(z),\end{aligned} \tag{6.9}$$

where P^j, Q^j depend only on (U^k,V^k) and its derivatives for $k\le j$, and Q^j decays exponentially to 0 as $z\to\pm\infty$. The $P^j(s)$ have some dependence on s coming from the third argument of $g(\rho,u,s)$.

6.2 Profile transmission conditions

The following equations are obtained by substituting the expansion (6.1) into the transmission conditions (5.5)(b), and setting coefficients of the different powers of ϵ equal to 0. Here $U^j_\pm$, $V^j_\pm$ denote limits as s, respectively z approaches $0^\pm$. Below we use the notation $U^j=(U^{j,1},U^{j,2})$ and similarly for V^j.

We obtain at $s=0$ the conditions:

$$\begin{aligned}&(a)\ \epsilon^0: U^{0,1}_+ + V^{0,1}_+ = U^{0,1}_- + V^{0,1}_-,\\ &(b)\ \epsilon^0: U^{0,2}_+ + V^{0,2}_+ = U^{0,2}_- + V^{0,2}_-,\\ &(c)\ \epsilon^{-1}: \partial_z V^{0,2}_+ = \partial_z V^{0,2}_-,\end{aligned} \tag{6.10}$$

and for $1\le j\le M$,

$$\begin{aligned}&(a)\ \epsilon^j: U^{j,1}_+ + V^{j,1}_+ = U^{j,1}_- + V^{j,1}_-,\\ &(b)\ \epsilon^j: U^{j,2}_+ + V^{j,2}_+ = U^{j,2}_- + V^{j,2}_-,\\ &(c)\ \epsilon^{j-1}: \partial_s U^{j-1,2}_+ + \partial_z V^{j,2}_+ = \partial_s U^{j-1,2}_- + \partial_z V^{j,2}_-.\end{aligned} \tag{6.11}$$

Remark 6.2. *Observe that the boundary conditions* (6.10), (6.11) *imply that* $\tilde w$ *as in* (6.1) *satisfies*

$$d_s\tilde w^2_+ = d_s\tilde w^2_- + \epsilon^M K \ \text{ at } s=0, \tag{6.12}$$

for some constant K. *By adding* $-s\phi(s)\epsilon^M K$ *to* $\tilde w^2_+$, *where* ϕ *is a smooth cutoff supported near* $s=0$ *and identically one near* $s=0$, *we obtain an approximate solution satisfying the transmission conditions exactly.*

6.3 Solution of the profile equations.

1. Note that $F^0_\pm = 0$ already by our assumption that U^0 is a shock.

2. V^0 **and the reduced profile equation.** Recall that $G^{-1} = 0$ represents one equation on $z \geq 0$ and one on $z \leq 0$. Continuing to suppress the $\pm$ subscript, we define

$$\mathbb{G}^j(z) = \begin{cases} \int_{+\infty}^z G^j(\zeta)d\zeta \text{ for } z \geq 0 \\ \int_{-\infty}^z G^j(\zeta)d\zeta \text{ for } z \leq 0 \end{cases}. \tag{6.13}$$

Anticipating $\mathcal{U}^0_\pm(0,z) \to U^0_\pm(0)$ as $z \to \pm\infty$, we find that the equations $\mathbb{G}^{-1}(z) = 0$ are, with $f = (f^1, f^2)$,

$$\begin{aligned} 0 &= f^1(\mathcal{U}^0) - f^1(U^0), \\ \partial_z \mathcal{U}^{0,2} &= f^2(\mathcal{U}^0) - f^2(U^0). \end{aligned} \tag{6.14}$$

Set $W(z) = \mathcal{U}^0(0,z) = U^0(0) + V^0(z)$, where V^0 is unknown. The existence of a *smooth* profile W on $\mathbb{R}_z$ satisfying (6.14) with

$$W(z) \to U^0_\pm(0) \text{ as } z \to \pm\infty \tag{6.15}$$

is classical. For example, in [Gi] Gilbarg shows that for a convex pressure law, such profiles exist for shocks $U^0_\pm(0)$ of any strength. (This is easy in the barotropic SS case, where the phase space for the reduced profile equation (6.16) is one dimensional). Taking $V^0_\pm(z) := W(z) - U^0_\pm(0)$, we observe that $G^{-1} = 0$ and the transmission conditions (6.10) hold.

For later reference, we obtain the *reduced profile equation* by solving the first equation of (6.14) for $\mathcal{U}^{0,1}$ in terms of $\mathcal{U}^{0,2}$, thereby obtaining $\mathcal{U}^{0,1} = k(\mathcal{U}^{0,2})$ and:

$$\partial_z \mathcal{U}^{0,2} = f^2(k(\mathcal{U}^{0,2}), \mathcal{U}^{0,2}) - f^2(k(U^{0,2}), U^{0,2})) := f^2_r(\mathcal{U}^{0,2}) - f^2_r(U^{0,2}). \tag{6.16}$$

Observe

$$\partial_z \mathcal{U}^{0,1} = \mathcal{A}\partial_z \mathcal{U}^{0,2}, \quad \text{where } \mathcal{A} := -(A^{11})^{-1}A^{12}. \tag{6.17}$$

3. Determining the jump $[U^1]$. Define $\mathbb{Q}^0(z)$ from $Q^0(z)$ in the same way (6.13) that $\mathbb{G}^j$ was defined from G^j. The equations $\mathbb{G}^0(z) = 0$ can be written

$$\begin{aligned} &(a)\ 0 = A^{11}(U^{1,1} + V^{1,1}) + A^{12}(U^{1,2} + V^{1,2}) - \big(A(U^0)U^1\big)^1 + \mathbb{Q}^{0,1} \\ &(b)\ \partial_z V^{1,2} = A^{21}(U^{1,1} + V^{1,1}) + A^{22}(U^{1,2} + V^{1,2}) - \big(A(U^0)U^1\big)^2 + \mathbb{Q}^{0,2}. \end{aligned} \tag{6.18}$$

We show now that the jump $[U^1]$ is determined by the requirement that the equations (6.18) be compatible with the transmission conditions (6.11).

Suppose for a moment that $[\mathcal{U}^{1,2}] = [U^{1,2} + V^{1,2}] = 0$. Then (6.18)(a) shows that $[\mathcal{U}^{1,1}] = 0$ if and only if

$$[A(U^0)U^1]^1 = [\mathbb{Q}^{0,1}]. \tag{6.19}$$

We seek a condition on $[A(U^0)U^1]^2$ that will imply (6.11)(c) for $j = 1$ assuming that (6.18) and (6.11)(a),(b) hold. Using (6.18)(b) and

$$[\mathcal{U}^0] = 0, \ \ [\partial_z V^0] = 0, \ \ [U^1 + V^1] = 0, \tag{6.20}$$

we compute

$$[\partial_z V^{1,2}] = -[A(U^0)U^1]^2 + [\mathbb{Q}^0]^2. \tag{6.21}$$

Now (6.11)(c) for $j=1$ means $[\partial_z V^{1,2}] = -[\partial_s U^{0,2}]$. This holds if and only if

$$[A(U^0)U^1]^2 = [\mathbb{Q}^{0,2}] + [\partial_s U^{0,2}]. \tag{6.22}$$

Equations (6.19) and (6.22) give the transmission conditions for the problem satisfied by U^1. Since $A(U^0_\pm(0))$ is invertible, these conditions determine the jump $[U^1]$.

For later use we use (6.17) and (6.18)(a) to write

$$\begin{aligned}&(a)\ U^{1,1} + V^{1,1} = \mathcal{A}(U^{1,2}+V^{1,2}) + H(z),\ \text{where}\\ &(b)\ H(z) := -(A^{11})^{-1}\left(-(A(U^0)U^1)^1 + \mathbb{Q}^{0,1}\right).\end{aligned} \tag{6.23}$$

4. Solve for U^1. First solve $F^1(s)=0$,

$$A(U^0)\partial_s U^1 + d_w g(U^0,s)U^1 = P^0(s), \tag{6.24}$$

on $[a-\overline{r},0]$ with boundary conditions

$$U^1 = 0 \text{ at } s = a-\overline{r}. \tag{6.25}$$

This gives $U^1_-(0)$. Knowing $[U^1]$ we obtain $U^1_+(0)$, and then use that as initial data for solving (6.24) on $[0,b-\overline{r}]$. Observe that we have no reason to expect $U^1(b-\overline{r})=0$.

5. Stable and unstable manifolds. Let $W^s \subset \mathbb{R}^1$ (resp. $W^u \subset \mathbb{R}^1$) denote the stable (resp. unstable) manifold of the reduced profile equation (6.16) for the rest point $U^{0,2}_+(0)$ (resp. $U^{0,2}_-(0)$). The properties of $U^0(s)$ (in particular, supersonic flow to the right in $[a-\overline{r},0]$, subsonic flow to the right in $[0,b-\overline{r}]$) imply that these are 1 dimensional manifolds which intersect (transversally of course) along the image of the profile $\mathcal{U}^{0,2}(0,z)$. The tangent space to W^s (resp. W^u) at $\mathcal{U}^{0,2}(0,0)$ ($\mathbb{R}$ in both cases) is the space of initial data at $z=0$ of solutions to the linearized equation

$$\partial_z V^{1,2} = A_r(\mathcal{U}^{0,2})V^{1,2}, \tag{6.26}$$

on $z\geq 0$ (resp. $z\leq 0$) which decay to 0 as $z\to+\infty$ (resp. $z\to-\infty$). Here

$$A_r(\mathcal{U}^{0,2}) := d_{\mathcal{U}^{0,2}} f_r = \left(A^{22} - A^{21}(A^{11})^{-1}A^{12}\right)(k(\mathcal{U}^{0,2}),\mathcal{U}^{0,2}). \tag{6.27}$$

Remark 6.3. *1. A direct computation using the given properties of $U^0(s)$ shows that in the barotropic SS case, for some $\alpha>0$,*

$$A_r(U^{0,2}_-(s)) \geq \alpha \text{ in } [a-\overline{r},0],\ \text{while } A_r(U^{0,2}_+(s)) \leq -\alpha \text{ in } [0,b-\overline{r}]. \tag{6.28}$$

From (6.28) we see that the sign of $A_r(\mathcal{U}^{0,2}(0,z))$ changes as z varies from $-\infty$ to $+\infty$.

2. In the full SS case $A_r(U^{0,2}(s))$ is 2×2 with two positive eigenvalues in $[a-\overline{r},0]$ and with eigenvalues of opposite signs in $[0,b-\overline{r}]$. Thus, the sign of one of the eigenvalues of $A_r(\mathcal{U}^{0,2}(0,z))$ changes along the profile. The manifolds W^s, W^u are submanifolds of $\mathbb{R}^2$ of dimensions 2 and 1, respectively, and intersect tranversally along the trace of $\mathcal{U}^{0,2}(0,z)$.

6. Solve for V^1. We will first obtain $V^{1,2}$ exponentially decaying to 0 as $z \to \pm\infty$, and then use (6.23) to solve for $V^{1,1}$. From (6.23) and the decay of $V^{1,2}$ it will then be clear that $\partial_z V^{1,1}$ must decay exponentially to 0. From (6.23)(b) we see that

$$H(\pm\infty) = U^{1,1}_\pm(0) - (\mathcal{A}|_{z=\pm\infty})U^{1,2}_\pm(0), \tag{6.29}$$

so $V^{1,1}$ itself decays exponentially to 0. The equation for $V^{1,2}$ is (6.18)(b), where now $U^{1,1} + V^{1,1}$ is given by (6.23)(a). In view of the compatibility conditions that have been arranged by the choice of U^1, in order to obtain V^1 satisfying (6.18) and the transmission conditions (6.11)(a),(b),(c) for $j = 1$, it suffices now to find an exponentially decaying solution to (6.18)(b) such that (6.11)(b) holds: $[U^{1,2} + V^{1,2}] = 0$.

Using (6.23) we observe that (6.18)(b) has the form

$$\partial_z V^{1,2} = A_r(\mathcal{U}^{0,2})V^{1,2} + \mathcal{F}(z), \tag{6.30}$$

where $\mathcal{F}$ is expressible in terms of already determined profiles and exponentially decreasing to 0 as $z \to \pm\infty$. Let $\mathcal{W}^s$ and $\mathcal{W}^u$ be the linear submanifolds of $\mathbb{R}$ consisting of initial data at $z = 0$ of solutions to (6.30) that decay to 0 as $z \to \pm\infty$. Both $\mathcal{W}^s$ and $\mathcal{W}^u$ equal $\mathbb{R}$ in the barotropic case. (In the full SS case, they are translates of the tangent spaces to W^s and W^u, respectively, at $\mathcal{U}^{0,2}(0,0)$.)

Clearly, we should choose initial data

$$(V^{1,2}_+(0), V^{1,2}_-(0)) \in (\mathcal{W}^s \times \mathcal{W}^u) \cap \{(v_1, v_2) \in \mathbb{R}^2 : v_1 - v_2 = U^{1,2}_-(0) - U^{1,2}_+(0)\}. \tag{6.31}$$

We call this line $\mathcal{L}^1$, the *line of connection initial data for* $V^{1,2}_\pm(z)$. Any point on $\mathcal{L}^1$ gives a choice of initial data for (6.30) corresponding to a decaying solution that satisfies (6.11)(b). Thus, we now have an exponentially decaying $V^1(z)$ satisfying (6.18) and (6.11) for $j = 1$.

Remark 6.4. *1. In the full SS case* (6.31) *is a transversal intersection of linear submanifolds of* $\mathbb{R}^4$ *of dimensions* 3 *and* 2 *respectively. Again, we obtain a line* $\mathcal{L}^1$ *with direction*

$$\mathbb{U}^{0,2}(0) := (\partial_z \mathcal{U}^{0,2}(0,0), \partial_z \mathcal{U}^{0,2}(0,0)). \tag{6.32}$$

2. Different choices of points along $\mathcal{L}^1$ *lead to an indeterminacy in* $V^1_+(\frac{b-\overline{r}}{\epsilon})$ *or* $V^1_-(\frac{a-\overline{r}}{\epsilon})$ *of size* $e^{-\frac{C}{\epsilon}}$*, for some* $C > 0$.

7. Repeat. The remaining profiles are solved for according to the same pattern:

$$U^1 \to V^1 \to U^2 \to V^2 \to \cdots . \tag{6.33}$$

For each $j \geq 1$ take $U^j(a - \overline{r}) = 0$. The jump condition at $s = 0$ for the problem satisfied by U^j is always the compatibility condition for V^j. The line $\mathcal{L}^j$ of connection initial data for $V^j_\pm$ always has direction $\mathbb{U}^{0,2}(0)$.

We summarize the result of this construction using the following spaces to keep track of regularity:

Definition 6.5 (Spaces)**.** *1. For* $k \in \mathbb{N}$ *let* C^k_p *(the subscript indicates "piecewise") be the set of functions* $U(s)$ *on* $[a - \overline{r}, b - \overline{r}]$ *such that the restrictions* $U_\pm$ *belong to* $C^k([a - \overline{r}, b - \overline{r}] \cap \{\pm s \geq 0\})$.

2. Let $\tilde{C}^k_p$ be the set of functions $V(z)$ on $\mathbb{R}$ such that the restrictions $V_\pm$ belong to $C^k(\pm z \geq 0)$ and satisfy, for some $\beta > 0$,

$$\left|\left(\frac{d}{dz}\right)^j V(z)\right| \leq C_j e^{-\beta|z|} \text{ for } j \leq k. \tag{6.34}$$

The inviscid construction shows that if the functions $f(\rho, u)$, $g(\rho, u, s)$ appearing in (5.5) are C^k functions of their arguments, then $U^0(s) \in C^k_p$.

Proposition 6.6 (Approximate solutions)**.** *Let k and $M \geq 1$ be integers with $k \geq M + 2$. Assume that the functions $f(\rho, u)$, $g(\rho, u, s)$ appearing in (5.5) are C^k functions of their arguments. Let $U^0(s) \in C^k_p$ be a stationary inviscid shock on $[a - \overline{r}, b - \overline{r}]$ with supersonic inflow at $a - \overline{r}$, shock surface at $s = 0$, and taking the values (ρ_a, u_a), (ρ_b, u_b) at $s = a - \overline{r}$ and $s = b - \overline{r}$ respectively. With $w = (w^1, w^2) := (\rho, u)$, write the interior equation (5.5)(a) as $\mathcal{E}(w) = 0$. Then one can construct an approximate solution*

$$\tilde{w}^\epsilon(s) = \left(\mathcal{U}^0(s, z) + \epsilon\mathcal{U}^1(s, z) + \cdots + \epsilon^M\mathcal{U}^M(s, z)\right)|_{z=\frac{s}{\epsilon}} \tag{6.35}$$

satisfying

$$\begin{aligned}
&\mathcal{E}(\tilde{w}^\epsilon) = \epsilon^M R^{M,\epsilon}(s) \text{ on } [a - \overline{r}, b - \overline{r}] \\
&[\tilde{w}^\epsilon] = 0,\ [d_s\tilde{w}^{\epsilon,2}] = 0 \text{ on } s = 0 \\
&\tilde{w}^\epsilon(a - \overline{r}) = (\rho_a, u_a) + O\left(e^{-\frac{\beta}{\epsilon}}\right) \text{ for some } \beta > 0,\ \ \tilde{w}^\epsilon(b - \overline{r}) = (\rho_b, u_b) + O(\epsilon).
\end{aligned} \tag{6.36}$$

Here $\mathcal{U}^j(s, z) = U^j(s) + V^j(z)$, with $U^0(s)$ the given inviscid shock, and

$$\begin{aligned}
&U^j \in C^{k-j}_p \\
&V^0 \in \tilde{C}^k_p,\ \ V^j \in \tilde{C}^{k-1}_p \text{ for } j \geq 1,
\end{aligned} \tag{6.37}$$

and there exist constants M_j such that

$$|(\epsilon\partial_s)^j R^{M,\epsilon}|_{C^0_p} \leq M_j,\ \text{for } j \leq k - M - 2. \tag{6.38}$$

Proof. It just remains to discuss the regularity of the profiles. Since $f(\rho, u) \in C^k$, the solution $\mathcal{U}^0(0, z)$ of (6.14) is C^k on $\mathbb{R}$, and thus $V^0(z) \in C^k_p$ (in fact, $\mathcal{U}^{0,2} \in C^{k+1}$ on $\mathbb{R}$). The term $Q^0(z)$ in (6.8) involves two derivatives of V^0. Thus, the terms on the right in the equation for $V^{1,2}$ obtained from (6.18)(b) are in C^{k-1}_p. So $V^{1,2} \in C^k_p$. We get $V^{1,1} \in C^{k-1}_p$, since $A^{ij}(\mathcal{U}^0) \in C^{k-1}$. The term $Q^1(z)$ in (6.9) involves two derivatives of V^1, hence $\mathbb{Q}^1(z) \in C^{k-2}_p$, and therefore $V^2 \in C^{k-1}_p$ again. Similarly, $V^j \in C^{k-1}_p$ for $j \geq 1$.

In (6.24) $P^0(s)$ involves terms in which $U^0(s)$ is differentiated twice, so $U^1(s) \in C^{k-1}_p$, and similarly $U^j \in C^{k-j}_p$.

The remainder $R^{M,\epsilon}$ has a contribution that involves two derivatives of $U^M \in C^{k-M}_p$. Hence we obtain (6.38). □

7 The error problem

We now look for an exact solution to the transmission problem (5.5) in the form

$$w^\epsilon = \tilde{w}^\epsilon + \epsilon^L v^\epsilon, \ 1 \le L < M, \tag{7.1}$$

where $\tilde{w}^\epsilon$ is the approximate solution given by (6.35). It would suffice, for example, to choose $L = 1$ and $M = 2$, but the arguments below work for any L, M satisfying $1 \le L < M$.

We will often suppress the superscript ϵ. By subtracting $\mathcal{E}(w) - \mathcal{E}(\tilde{w})$ and cancelling ϵ^L we obtain the *error problem* for $v^\epsilon = (v^1, v^2)$:

$$\begin{aligned}
&(a)\ A(\tilde{w} + \epsilon^L v) d_s v + \left(v \cdot \int_0^1 \partial_w A(\tilde{w} + \sigma\epsilon^L v) d\sigma \right) d_s \tilde{w} + v \cdot \int_0^1 \partial_w g(\tilde{w} + \sigma\epsilon^L v, s) d\sigma \\
&\qquad - \epsilon h(v^2, v^2_s, v^2_{ss}, s) = -\epsilon^{M-L} R^{M,\epsilon} \qquad \text{on } [a - \overline{r}, b - \overline{r}] \cap \{\pm s \ge 0\} \\
&(b)\ [v] = 0,\ [d_s v^2] = 0 \text{ on } s = 0 \\
&(c)\ v(a - \overline{r}) = \epsilon^{-L} \left((\rho_a, u_a) - \tilde{w}(a - \overline{r}) \right) = O(e^{-\frac{\beta}{\epsilon}}) \text{ for some } \beta > 0.
\end{aligned} \tag{7.2}$$

7.1 First-order system

Let us begin by rewriting (7.2)(a) as

$$A d_s v + \frac{1}{\epsilon} \mathcal{B} v + C v - \epsilon h = -\epsilon^{M-L} R^M, \tag{7.3}$$

where

$$\begin{aligned}
\mathcal{B} v &:= \epsilon \left(v \cdot \int_0^1 \partial_w A(\tilde{w} + \sigma\epsilon^L v) d\sigma \right) d_s \tilde{w}, \\
C v &:= v \cdot \int_0^1 \partial_w g(\tilde{w} + \sigma\epsilon^L v, s) d\sigma.
\end{aligned} \tag{7.4}$$

Recalling the definition of h, (5.4), this becomes

$$\mathcal{A} d_s v + \frac{1}{\epsilon} \mathcal{B} v + \mathcal{C} v - \epsilon \begin{pmatrix} 0 \\ v^2_{ss} \end{pmatrix} = -\epsilon^{M-L} R^M, \tag{7.5}$$

where now

$$\mathcal{A} := A - \begin{pmatrix} 0 & 0 \\ 0 & \frac{2\epsilon}{s+\overline{r}} \end{pmatrix}, \qquad \mathcal{C} := C + \begin{pmatrix} 0 & 0 \\ 0 & \frac{2\epsilon}{(s+\overline{r})^2} \end{pmatrix}. \tag{7.6}$$

Setting $E := \mathcal{B} + \epsilon \mathcal{C}$, we next split the matrix equation into components:

$$\begin{aligned}
&\mathcal{A}^{11} d_s v^1 + \mathcal{A}^{12} d_s v^2 + \frac{1}{\epsilon} E^{11} v^1 + \frac{1}{\epsilon} E^{12} v^2 = -\epsilon^{M-L} R^{M,1} \\
&\mathcal{A}^{21} d_s v^1 + \mathcal{A}^{22} d_s v^2 + \frac{1}{\epsilon} E^{21} v^1 + \frac{1}{\epsilon} E^{22} v^2 - \epsilon v^2_{ss} = -\epsilon^{M-L} R^{M,2}.
\end{aligned} \tag{7.7}$$

Define $V = (v^1, v^2, v^3)^t$, where $v^3 = \epsilon v^2_s$, and rewrite (7.2) as a 3×3 first-order transmission problem on $[a-\overline{r}, b-\overline{r}]$:

$$\begin{aligned}
&d_s V = \frac{1}{\epsilon} GV + F,\\
&[V] = 0 \text{ on } s = 0,\\
&\begin{pmatrix} v^1 \\ v^2 \end{pmatrix}(a-\overline{r}) = \epsilon^{-L}\left((\rho_a, u_a) - \tilde{w}(a-\overline{r})\right) := \overline{v}.
\end{aligned} \tag{7.8}$$

where

$$G = \begin{pmatrix} g^{11} & g^{12} & g^{13} \\ 0 & 0 & 1 \\ g^{31} & g^{32} & g^{33} \end{pmatrix}, \qquad F = \begin{pmatrix} -(\mathcal{A}^{11})^{-1}\epsilon^{M-L}R^{M,1} \\ 0 \\ \epsilon^{M-L}R^{M,2} - \mathcal{A}^{21}(\mathcal{A}^{11})^{-1}\epsilon^{M-L}R^{M,1} \end{pmatrix}, \tag{7.9}$$

with

$$\begin{aligned}
&g^{11} = -(\mathcal{A}^{11})^{-1}E^{11}, \qquad g^{12} = -(\mathcal{A}^{11})^{-1}E^{12}, \qquad g^{13} = -(\mathcal{A}^{11})^{-1}\mathcal{A}^{12}\\
&g^{31} = E^{21} + \mathcal{A}^{21}g^{11}, \qquad g^{32} = E^{22} + \mathcal{A}^{21}g^{12}, \qquad g^{33} = \mathcal{A}^{22} - \mathcal{A}^{21}(\mathcal{A}^{11})^{-1}\mathcal{A}^{12}.
\end{aligned} \tag{7.10}$$

Notation 7.1. *We introduce the notation $q^\epsilon(s) = (q^{\epsilon,1}, \dots, q^{\epsilon,6})$ by*

1. $q^{\epsilon,1}(s) = U^0(0)$,
2. $q^{\epsilon,2}(s) = (U^0(s) - U^0(0)) + \epsilon\left(U^1(s) + V^1(\frac{s}{\epsilon})\right) + \cdots + \epsilon^M\left(U^M(s) + V^M(\frac{s}{\epsilon})\right)$,
3. $q^{\epsilon,3}(s) = \epsilon^L v^\epsilon$, *where we suppose that $v^\epsilon(s)$ is bounded in C^j_p for some $j \geq 1$,*
4. $d_s\tilde{w} = \frac{1}{\epsilon}\left(d_z V^0(z)|_{z=\frac{s}{\epsilon}} + q^{\epsilon,4}(s)\right)$ *(hence, $q^{\epsilon,4}(s) = O(\epsilon)$ uniformly on $[a-\overline{r}, b-\overline{r}]$),*
5. $q^{\epsilon,5}(s) = \epsilon\mathcal{C}$,
6. $q^{\epsilon,6}(s) = \frac{2\epsilon}{s+\overline{r}}$. *This occurs only in the second term in the definition of $\mathcal{A}$, (7.6).*
7. *For any function $f^\epsilon(s)$, the expression $f^\epsilon = O(\epsilon^k)$ means $|f^\epsilon(s)| \leq C\epsilon^k$ uniformly on $[a-\overline{r}, b-\overline{r}]$ for $0 < \epsilon \leq \epsilon_0$,*
8. *ϵ_0 will always denote some sufficiently small positive number,*
9. *When $\epsilon \downarrow 0$ observe that $q^\epsilon(s) \to q^0(s) := (U^0(0), U^0(s) - U^0(0), 0)$ in C^0_p.*

Suppressing some epsilons, we have

$$\begin{aligned}
&(a)\ \tilde{w}(s) = V^0(z)|_{z=\frac{s}{\epsilon}} + q^1 + q^2\\
&(b)\ w(s) = V^0(z)|_{z=\frac{s}{\epsilon}} + q^1 + q^2 + q^3\\
&(c) d_s\tilde{w}(s) = \frac{1}{\epsilon}\left(d_z V^0(z)|_{z=\frac{s}{\epsilon}} + q^4\right).
\end{aligned} \tag{7.11}$$

In the obvious way, we may now regard the matrix coefficients appearing in (7.3)-(7.10) as defining corresponding functions of $(z,q) \in \mathbb{R}_z \times \Omega$, for some $\Omega \subset \mathbb{R}^{13}$. For example, we write with slight abuse,

$$\begin{aligned} &A(\tilde{w} + \epsilon^L v) = A(z,q)|_{z=\frac{s}{\epsilon}, q=q^\epsilon(s)}\,, \qquad E = E(z,q)|_{z=\frac{s}{\epsilon}, q=q^\epsilon(s)}, \\ &G = G(z,q)|_{z=\frac{s}{\epsilon}, q=q^\epsilon(s)}. \end{aligned} \tag{7.12}$$

Note that z-dependence in the above functions of (z,q) enters *only* through V^0 or $d_z V^0$.

Recalling the definitions of $\mathcal{A}$ and E, the properties of $U^0(s)$, and using (7.11), we see that for v valued in a bounded subset of $\mathbb{R}^2$ and for $0 < \epsilon \le \epsilon_0$ with ϵ_0 sufficiently small,

$$\begin{aligned} &(a)\; E\left(\frac{s}{\epsilon}, q^\epsilon(s)\right) = O\left(|d_z V^0|_{z=\frac{s}{\epsilon}}|\right) + \mathcal{E}^\epsilon(s), \text{ where } \mathcal{E}^\epsilon(s) = O(\epsilon)\,, \\ &(b)\; \mathcal{A}\left(\frac{s}{\epsilon}, q^\epsilon(s)\right) = O(1)\,, \qquad (\mathcal{A}^{11})^{-1}\left(\frac{s}{\epsilon}, q^\epsilon(s)\right) = O(1)\,. \end{aligned} \tag{7.13}$$

7.2 Strategy

We solve the error problem (7.8) on $[a - \overline{r}, b - \overline{r}]$ by choosing a sufficiently small $\delta > 0$ (in a manner explained below) and solving subproblems on the s-intervals

$$[a - \overline{r}, -\delta],\ [-\delta, 0],\ [0, \delta],\ [\delta, b - \overline{r}]. \tag{7.14}$$

Let us refer to these subproblems as problems I, II, III, IV respectively.

In problems I and IV observe that the $d_z V^0$ term in (7.13)(a) is uniformly $O(e^{-\frac{\beta}{\epsilon}})$ for some $\beta > 0$, so using (7.10) we have

$$G = \begin{pmatrix} O(\epsilon) & O(\epsilon) & g^{13} \\ 0 & 0 & 1 \\ O(\epsilon) & O(\epsilon) & g^{33} \end{pmatrix} \text{ for problems } I, IV. \tag{7.15}$$

Thus, in $|s| \ge \delta$, G has two eigenvalues $\lambda_1^\epsilon(s)$, $\lambda_2^\epsilon(s)$ that are $O(\epsilon)$ and a third, $\lambda_3^\epsilon(s)$ that is $g^{33} + O(\epsilon)$. From Remark 6.3 and the definition of g^{33} we see that for $0 < \epsilon \le \epsilon_0$ and some $\alpha > 0$,

$$\lambda_3(s) \ge \alpha \text{ in problem } I;\ \ \lambda_3(s) \le -\alpha \text{ in problem } IV. \tag{7.16}$$

Moreover, g^{33} changes sign on $[-\delta, \delta]$.

Remark 7.2. *1. In order to determine a unique solution to the transmission problem (7.8), an additional scalar condition must be imposed on V. The fact that λ_3 is positive in problem I implies that if we try to prescribe $v^3 = \epsilon d_s v^2$ at $a - \overline{r}$, "most" solutions will blow up exponentially as $\epsilon \to 0$. This is one difficulty. On the other hand the sign of λ_3 in problem IV allows us to prescribe initial data for V at $s = \delta$. Problems I and IV describe the slow variation of v away from the transmission layer near $s = 0$.*

2. Problems II and III describe the fast variation of v in the transmission layer. Here there are two difficulties. On the one hand $g^{33}(\frac{s}{\epsilon}, q^\epsilon(s))$ varies rapidly and changes sign on $[-\delta, \delta]$. In addition, the $d_z V^0$ term in (7.13)(a) is $O(1)$ on $[-\delta, \delta]$, so we no longer have (7.15). All nonzero terms in G are now $O(1)$.

3. Consider the restrictions to $|s| \geq \delta > 0$ of the functions of (s,ϵ) given by $q^\epsilon(s)$, $\mathcal{A}\left(\frac{s}{\epsilon}, q^\epsilon(s)\right)$, etc., as in (7.12), (7.13). The exponential decay of $V^0(z)$ implies that these restrictions extend to $\{|s| \geq \delta\} \times [0,\epsilon_0]$ with the same regularity in (s,ϵ) that they have on $\{|s| \geq \delta\} \times (0,\epsilon_0]$.

Each problem will be reduced to a simpler form by conjugation. The conjugations for problems I and IV are straightforward conjugations to block form, where one 2×2 block corresponds to the $O(\epsilon)$ eigenvalues λ_1, λ_2 and the other 1×1 block corresponds to λ_3. Two conjugations are used for each of problems II and III. The first conjugation is of the sort introduced as a key tool in [MZ, GMWZ2, GMWZ3], and is designed to replace $G(z,q)$ by $G(-\infty,q)$ and $G(+\infty,q)$ in problems II and III respectively. This has the effect of setting $V^0(z)$ or $d_zV^0(z)$ equal to zero in all coefficients. The second conjugation is a conjugation to block form as in problems I, IV.

We'll produce a family of solutions to problem I depending on a single scalar parameter p_1, and a family of solutions to problem II depending on 3 scalar parameters (p_2,p_3,p_4). For example, we take

$$\begin{pmatrix} v^1 \\ v^2 \end{pmatrix}(a-\overline{r}) = \epsilon^{-L}\big((\rho_a,u_a) - \tilde{w}(a-\overline{r})\big) = \overline{v}\,, \qquad v^3(-\delta) = p_1 \quad \text{in problem } I. \tag{7.17}$$

The requirement that the solutions to problems I and II agree at $s=-\delta$ will allow us to determine (p_1,p_2,p_3) in terms of p_4, which at the moment remains free. The favorable sign of $g^{33}(U^0_+(s))$ allows us to prescribe three scalar initial conditions at $s=0$ for problem III and at $s=\delta$ for problem IV. Hence, additional parameters are not needed for those problems.

7.3 Problem I on $[a-\overline{r},-\delta]$

For the moment we allow $-\delta$ to be any fixed number in $(a-\overline{r},0)$, and we assume the unknown $v^\epsilon(s)$ is uniformly bounded with respect to ϵ in C^0_p.

Proposition 7.3. *There is an invertible 3×3 matrix $S(\frac{s}{\epsilon}, q^\epsilon(s))$ defined on $[a-\overline{r},-\delta] \times (0,\epsilon_0]$, with the same regularity in (s,ϵ) as G in (7.12), such that*

$$S^{-1}GS = \begin{pmatrix} H & 0 \\ 0 & P \end{pmatrix} := G_B \tag{7.18}$$

where H is 2×2, P is 1×1, and

$$H\left(\frac{s}{\epsilon}, q^\epsilon(s)\right) = O(\epsilon), \quad P\left(\frac{s}{\epsilon}, q^\epsilon(s)\right) = \lambda_3^\epsilon(s) = g^{33}(U^0_-(s)) + O(\epsilon). \tag{7.19}$$

As a function of (s,ϵ), S extends with no loss of regularity as an invertible matrix on $[a-\overline{r},-\delta]\times[0,\epsilon_0]$. At $\epsilon = 0$ S has the form

$$\begin{pmatrix} S^{11} & S^{12} \\ 0 & S^{22} \end{pmatrix}, \tag{7.20}$$

with S^{11} and S^{22} invertible on $[a-\overline{r},-\delta]$, and of size 2×2 and 1×1 respectively.

Proof. It is clear from (7.15) that when $\epsilon = 0$, 0 is an eigenvalue of G with a two dimensional eigenspace. Together with the spectral separation described before Remark 7.2, this implies that there is an invertible matrix S such that (7.18) holds with H and P as in (7.19). In view of part (3) of Remark 7.2, S has the stated regularity.

The span of the first two columns of S, span S_I, is an invariant subspace for G, and at $\epsilon = 0$ this space is

$$\ker G = \mathbb{R}^2 \times \{0\}. \tag{7.21}$$

Thus, we must have

$$S_I = \begin{pmatrix} S^{11} \\ S^{21} \end{pmatrix} \tag{7.22}$$

with S^{11} is 2×2 and invertible, and S^{21} vanishes at $\epsilon = 0$. The last column of S, S_{II} is an eigenvector associated to λ_3. Since S is invertible, at $\epsilon = 0$ we must have

$$S_{II} = \begin{pmatrix} S^{12} \\ S^{22} \end{pmatrix} \tag{7.23}$$

with $S^{22} \neq 0$.

□

We shall solve

$$\begin{aligned} &d_s V = \frac{1}{\epsilon} G V + F \\ &\begin{pmatrix} v^1 \\ v^2 \end{pmatrix} (a - \overline{r}) = \overline{v} = O(e^{-\frac{\beta}{\epsilon}}),\ \beta > 0,\ \text{(as in (7.8))}\ , \qquad v^3(-\delta) = p_1 \in \mathbb{R}\,, \end{aligned} \tag{7.24}$$

by first studying the conjugated problem for $\mathcal{V} = (\nu^1, \nu^2, \nu^3)^t$ defined by $V = S\mathcal{V}$:

$$\begin{aligned} &d_s \mathcal{V} = \frac{1}{\epsilon} G_B \mathcal{V} + S^{-1} F - (S^{-1} \partial_s S) \mathcal{V} \qquad \text{on } [a - \overline{r}, -\delta] \\ &\begin{pmatrix} \nu^1 \\ \nu^2 \end{pmatrix} (a - \overline{r}) = \zeta \in \mathbb{R}^2\,, \qquad \nu^3(-\delta) = \eta \in \mathbb{R}. \end{aligned} \tag{7.25}$$

Remark 7.4. *Recall that* G_B *and* S *both depend on* $q^{\epsilon,3} = \epsilon^L v$ *and hence on the unknown* V*. The form of the functional dependence of* S *on* q^3 *is known from Proposition 7.3. A simple fixed point argument shows that, for* $\mathcal{V}$ *in a bounded set of* $\mathbb{R}^3$*, the equation*

$$V - S(..., \epsilon^L v, ...)\mathcal{V} = 0 \tag{7.26}$$

uniquely determines $V = V(s, \epsilon, \mathcal{V})$*. The regularity of this map may be determined from the known regularity of* S *by the implicit function theorem. Under the assumptions of Proposition 6.6, the regularity is at least* C^{k-M-1} *(*$d_s U^M$ *occurs in the* E^{ij}*). Thus, (7.25) is a well-defined nonlinear equation for* $\mathcal{V}$*.*

Proposition 7.5. *We make the same regularity assumptions as in Proposition 6.6. For fixed $R > 0$ and $|\zeta, \eta| \le R$, there exists an $\epsilon_0 > 0$ such that for $0 < \epsilon \le \epsilon_0$, the problem (7.25) has a solution on $[a - \overline{r}, -\delta]$, $\mathcal{V}(s, \epsilon, \zeta, \eta)$, that is uniformly C^1 in all its arguments. In addition we have*

$$\nu^3(a - \overline{r}) = O(\epsilon). \tag{7.27}$$

Proof. **1. Rewrite equations.** Setting $\mathcal{V} = (\nu^1, \nu^2, \nu^3) = (\nu^*, \nu^3)$ and using the properties of G_B, we rewrite (7.25) as

$$\begin{aligned} &d_s\nu^* = B_1(\epsilon^L\mathcal{V})\nu^* + B_2(\epsilon^L\mathcal{V})\nu^3 + \epsilon^{M-L}H^*(\epsilon^L\mathcal{V}) \\ &d_s\nu^3 = \frac{1}{\epsilon}P(\epsilon^L\mathcal{V})\nu^3 + B_3(\epsilon^L\mathcal{V})\nu^* + \epsilon^{M-L}H^3(\epsilon^L\mathcal{V}), \\ &\nu^*(a - \overline{r}) = \zeta, \;\; \nu^3(-\delta) = \eta. \end{aligned} \tag{7.28}$$

where P is as in Prop. 7.3, $H = (H^*, H^3) := S^{-1}F$, the matrices B_j are uniformly bounded with respect to ϵ, and we've suppressed the dependence of the coefficients on all arguments except $\epsilon^L\mathcal{V}$. With slight abuse we sometimes write

$$V = S(\epsilon^L\mathcal{V})\mathcal{V}. \tag{7.29}$$

2. Iteration scheme. The scheme is not quite standard so we write it explicitly:

$$\begin{aligned} &(a) \qquad d_s\nu^*_{n+1} = B_1(\epsilon^L\mathcal{V}_n)\nu^*_{n+1} + B_2(\epsilon^L\mathcal{V}_n)\nu^3_{n+1} + \epsilon^{M-L}H^*(\epsilon^L\mathcal{V}_n) \\ &(b) \qquad d_s\nu^3_{n+1} = \frac{1}{\epsilon}P(\epsilon^L\mathcal{V}_n)\nu^3_{n+1} + B_3(\epsilon^L\mathcal{V}_n)\nu^*_n + \epsilon^{M-L}H^3(\epsilon^L\mathcal{V}_n), \\ &(c) \qquad \nu^*_{n+1}(a - \overline{r}) = \zeta, \;\; \nu^3_{n+1}(-\delta) = \eta. \end{aligned} \tag{7.30}$$

Observe that ν^3_{n+1} occurs in (7.30)(a), but ν^*_{n+1} does not occur in (7.30)(b).

3. Estimates. For the moment we assume $|\mathcal{V}_n|_\infty \le K$ for all n. Denoting $L^\infty([a - \overline{r}, -\delta])$ norms by $|\cdot|_\infty$ we have

$$\begin{aligned} &(a) \qquad |\nu^*_{n+1}|_\infty \le C_1(|\nu^3_{n+1}|_\infty + \epsilon^{M-L}) + |\zeta| \\ &(b) \qquad |\nu^3_{n+1}|_\infty \le C_2\epsilon(|\nu^*_n|_\infty + \epsilon^{M-L}) + |\eta| \end{aligned} \tag{7.31}$$

where the constants C_1, C_2 may be chosen independently of K for $0 < \epsilon \le \epsilon_0$ provided $\epsilon_0 = \epsilon_0(K)$ is small enough. Since the coefficients of (7.30)(a) are uniformly bounded with respect to ϵ, the first estimate is standard ([CL], Chpt. 1, Thm. 2.1). To prove (7.31)(b) for $s < -\delta$ set

$$P(\epsilon^{M-L}\mathcal{V}_n) := b(s)\,, \qquad p(s) := \int_{-\delta}^{s} b(\sigma)d\sigma\,, \tag{7.32}$$

$$f(s) := B_3(\epsilon^{M-L}\mathcal{V}_n)\nu^*_n + \epsilon^{M-L}H^3. \tag{7.33}$$

We have

$$\nu^3_{n+1}(s) = \int_{-\delta}^{s} e^{\frac{p(s)-p(t)}{\epsilon}} f(t)dt + e^{\frac{p(s)}{\epsilon}}\eta \;=:\; A + B. \tag{7.34}$$

Since $b(s) \geq \alpha > 0$ on $[a - \overline{r}, -\delta]$ (by (7.16)), we obtain

$$\begin{aligned} &(a)\ |A| \leq |f|_\infty \int_s^{-\delta} e^{\frac{\alpha}{\epsilon}(s-t)} dt \leq |f|_\infty \frac{\epsilon}{\beta} \\ &(b)\ |B| \leq e^{\int_{-\delta}^s \frac{\alpha}{\epsilon} d\sigma} \eta = e^{\frac{\alpha}{\epsilon}(s+\delta)} \eta. \end{aligned} \tag{7.35}$$

This gives (7.31).

4. Induction step. To initialize take $\nu_0^* = \zeta$, $\nu_0^3 = \eta$. With C_1 as in (7.31) we will show that for ϵ small enough,

$$\begin{aligned} &|\nu_n^*|_\infty \leq C_1(|\eta| + 1) + |\zeta| + 1 \\ &|\nu_n^3|_\infty \leq |\eta| + 1 \end{aligned} \tag{7.36}$$

for all n. Indeed, assuming (7.36) for a given n, the estimate (7.31)(b) implies

$$|\nu_{n+1}^3|_\infty \leq C_2 \epsilon \left((C_1(|\eta| + 1) + |\zeta| + 1) + \epsilon^{M-L} \right) + |\eta| \leq |\eta| + 1 \tag{7.37}$$

for ϵ small. Estimate (7.31)(a) then gives

$$|\nu_{n+1}^*|_\infty \leq C_1(|\eta| + 1 + \epsilon^{M-L}) + |\zeta| \leq C_1(|\eta| + 1) + |\zeta| + 1 \tag{7.38}$$

for ϵ small.

5. Contraction. Set $y_n := \nu_{n+1}^* - \nu_n^*$ and $z_n := \nu_{n+1}^3 - \nu_n^3$. We have

$$\begin{aligned} d_s y_n = {} & B_1(\epsilon^L \mathcal{V}_n) y_n + \left(B_1(\epsilon^L \mathcal{V}_n) - B_1(\epsilon^L \mathcal{V}_{n-1}) \right) \nu_n^* + \\ & B_2(\epsilon^L \mathcal{V}_n) z_n + \left(B_2(\epsilon^L \mathcal{V}_n) - B_2(\epsilon^L \mathcal{V}_{n-1}) \right) \nu_n^3 + \epsilon^{M-L} \left(H^*(\epsilon^L \mathcal{V}_n) - H^*(\epsilon^L \mathcal{V}_{n-1}) \right) \\ d_s z_n = {} & \frac{1}{\epsilon} P(\epsilon^L \mathcal{V}_n) z_n + \frac{1}{\epsilon} \left(P(\epsilon^L \mathcal{V}_n) - P(\epsilon^L \mathcal{V}_{n-1}) \right) \nu_n^3 + \\ & B_3(\epsilon^L \mathcal{V}_n) y_{n-1} + \left(B_3(\epsilon^L \mathcal{V}_n) - B_3(\epsilon^L \mathcal{V}_{n-1}) \right) \nu_{n-1}^* + \epsilon^{M-L} \left(H^3(\epsilon^L \mathcal{V}_n) - H^3(\epsilon^L \mathcal{V}_{n-1}) \right) \end{aligned} \tag{7.39}$$

with $y_n(a - \overline{r}) = 0$ and $z_n(-\delta) = 0$. The estimates (7.31) and (7.36) give, for some new constants C_3, C_4:

$$\begin{aligned} &(a)\ |z_n|_\infty \leq C_4 \left(\epsilon^L |y_{n-1}, z_{n-1}|_\infty + \epsilon |y_{n-1}|_\infty \right) \\ &(b)\ |y_n|_\infty \leq C_3 \left(\epsilon^L |y_{n-1}, z_{n-1}|_\infty + |z_n|_\infty \right). \end{aligned} \tag{7.40}$$

Note that the z_n term on the right in (7.40)(b) can be replaced by the right side of (7.40)(a), so we obtain

$$|y_n, z_n|_\infty \leq \frac{1}{2} |y_{n-1}, z_{n-1}|_\infty \tag{7.41}$$

for ϵ small.

Observe that $\nu^3(a - \overline{r}) = O(\epsilon)$ now follows from (7.34) and (7.35).

6. **C^1 dependence on parameters.** Consider for example the scheme satisfied by $\dot{\mathcal{V}}_n := \partial_\eta \mathcal{V}_n$, which is obtained by differentiating (7.30) with respect to η:

$$\begin{aligned} d_s\dot{\nu}^*_{n+1} &= B_1(\epsilon^L\mathcal{V}_n)\dot{\nu}^*_{n+1} + B_2(\epsilon^L\mathcal{V}_n)\dot{\nu}^3_{n+1} + (\partial_w B_1\cdot\epsilon^L\dot{\mathcal{V}}_n)\nu^*_{n+1} + (\partial_w B_2\cdot\epsilon^L\dot{\mathcal{V}}_n)\nu^3_{n+1}+ \\ &\quad \epsilon^{M-L}\left(\partial_w H^*\cdot\epsilon^L\dot{\mathcal{V}}_n\right) \\ d_s\dot{\nu}^3_{n+1} &= \frac{1}{\epsilon}P(\epsilon^L\mathcal{V}_n)\dot{\nu}^3_{n+1} + \frac{1}{\epsilon}(\partial_w P\cdot\epsilon^L\dot{\mathcal{V}}_n)\nu^3_{n+1} + B_3(\epsilon^L\mathcal{V}_n)\dot{\nu}^*_n + (\partial_w B_3\cdot\epsilon^L\dot{\mathcal{V}}_n)\nu^*_n+ \\ &\quad \epsilon^{M-L}\left(\partial_w H^3\cdot\epsilon^L\dot{\mathcal{V}}_n\right), \end{aligned} \tag{7.42}$$

where $\dot{\nu}^*_n(a-\overline{r}) = 0$, $\dot{\nu}^3_n(-\delta) = 1$. The iterates $\dot{\mathcal{V}}_n$ satisfy estimates similar to (7.31). Thus, using the known convergence of the $\mathcal{V}_n$, we obtain uniform convergence of the $\dot{\mathcal{V}}_n$ on $[a-\overline{r},-\delta]$ by the same analysis as above.

□

Since the problem (7.24) on $[a-\overline{r},-\delta]$ involves boundary data at both $a-\overline{r}$ and $-\delta$, it is not yet clear that we can solve (7.24) for arbitrary data $\overline{v}$ and p_1 by using Proposition 7.5 and the transformation S. The next Proposition shows we can.

Proposition 7.6. *We make the same regularity assumptions as in Proposition 6.6. For fixed $R>0$ and $|\overline{v},p_1|\le R$, there exists an $\epsilon_0>0$ such that for $0<\epsilon\le\epsilon_0$, the problem (7.24) has a solution on $[a-\overline{r},-\delta]$, $V_I(s,\epsilon,\overline{v},p_1)$, that is uniformly C^1 in all its arguments.*

Proof. Writing $V_I=(v^*,v^3)$, we have a C^1 map

$$(\zeta,\eta)\to(v^*(a-\overline{r},\epsilon,\zeta,\eta),v^3(-\delta,\epsilon,\zeta,\eta)) = (\overline{v},p_1) \tag{7.43}$$

given by the composition

$$(\zeta,\eta)\to\mathcal{V}(s,\epsilon,\zeta,\eta)\to\begin{pmatrix}\mathcal{V}(a-\overline{r},\epsilon,\zeta,\eta)\\ \mathcal{V}(-\delta,\epsilon,\zeta,\eta)\end{pmatrix}\to\begin{pmatrix}V(a-\overline{r},\epsilon,\zeta,\eta)\\ V(-\delta,\epsilon,\zeta,\eta)\end{pmatrix}\to\begin{pmatrix}v^*(a-\overline{r},\epsilon,\zeta,\eta)\\ v^3(-\delta,\epsilon,\zeta,\eta)\end{pmatrix}, \tag{7.44}$$

where the map represented by the third arrow is defined using S as in (7.29). From Proposition 7.3 we have

$$S=\begin{pmatrix}S^{11} & S^{12}\\ O(\epsilon) & S^{22}\end{pmatrix} \tag{7.45}$$

where S^{11} and S^{22} are uniformly invertible on $[a-\overline{r},-\delta]$ for ϵ small. Thus we have

$$\begin{aligned} v^*(a-\overline{r}) &= S^{11}\zeta + S^{12}\nu^3(a-\overline{r}) \\ v^3(-\delta) &= O(\epsilon)\nu^*(-\delta) + S^{22}\eta. \end{aligned} \tag{7.46}$$

Since $\nu^3(a-\overline{r}) = O(\epsilon)$ (recall (7.27)), we see that the map (7.43) has an inverse with C^1 dependence on $(\epsilon,\overline{v},p_1)$ defined for $|\overline{v},p_1|\le R$ and for $0<\epsilon\le\epsilon_0(R)$ small enough.

Thus, given boundary data $(\overline{v},p_1)$ as in (7.24), we can choose data $(\zeta(\epsilon,\overline{v},p_1),\eta(\epsilon,\overline{v},p_1))$ for $\mathcal{V}$ satisfying (7.25) so that $V_I=S\mathcal{V}$ is a solution to (7.24). □

7.4 Problem II on $[-\delta, 0]$

In solving problems II and III we will deal with the rapid variation and change of sign of g^{33} (recall (7.10)) on $[-\delta, \delta]$ by performing conjugations that replace the matrix $G(z, q)$ by the limiting matrices

$$G(\pm\infty, q) = \lim_{z \to \pm\infty} G(z, q). \tag{7.47}$$

The proof of the following Lemma takes advantage of the fact that for $\beta > 0$ as in (6.34):

$$|G(z, q) - G(\pm\infty, q)| = O(e^{-\beta|z|}) \text{ on } \pm z \geq 0. \tag{7.48}$$

Lemma 7.7 (See [MZ], Lemma 2.6)**.** *Let $U^0_\pm(0)$ be the endstates of the given inviscid shock. There are neighborhoods $\mathcal{Q}_\pm$ of $(U^0_\pm(0), 0, \ldots, 0)$ in $\mathbb{R}^{13}$ and matrices $T_\pm(z, q)$ defined and C^1 on $\{\pm z \geq 0\} \times \mathcal{Q}_\pm$ satisfying:*

(a) $T_\pm$ and $(T_\pm)^{-1}$ are uniformly bounded and there is a $\beta > 0$ such that for $q \in \mathcal{Q}_\pm$ and $|\alpha| \leq 1$,

$$|\partial^\alpha_{z,q} \left(T_\pm(z, q) - Id\right)| = O(e^{-\beta|z|}) \text{ on } \pm z \geq 0; \tag{7.49}$$

(b) $T_\pm$ satisfies the matrix differential equation on $\pm z \geq 0$

$$\partial_z T_\pm(z, q) = G(z, q)T_\pm(z, q) - T_\pm(z, q)G(\pm\infty, q). \tag{7.50}$$

$T_\pm(z, q)$ can be chosen to have the same regularity as $G(z, q)$.

An immediate corollary is that $V_\pm(z)$ satisfies

$$d_z V_\pm = G(z, q)V + f_\pm \text{ on } \pm z \geq 0 \tag{7.51}$$

if and only if $\mathcal{V}_\pm := (T_\pm)^{-1} V_\pm$ satisfies

$$d_z \mathcal{V}_\pm = G(\pm\infty, q)\mathcal{V}_\pm + (T_\pm)^{-1} f_\pm \text{ on } \pm z \geq 0. \tag{7.52}$$

For the moment we assume that $V^\epsilon(s)$ as in (7.8) is uniformly bounded in $C^0([-\delta, 0])$ with respect to ϵ. Observe that for small enough positive constants ϵ_0 and δ and for $q^\epsilon(s)$ as defined in Notation 7.1, we have $q^\epsilon(s) \in \mathcal{Q}$ when $0 < \epsilon \leq \epsilon_0$ and $|s| \leq \delta$. Parallel to Proposition 7.3 we now have

Proposition 7.8. *There are invertible C^1 matrices $S_\pm(q)$ defined on $\mathcal{Q}_\pm$ such that*

$$S^{-1}_\pm G(\pm\infty, q)S_\pm = \begin{pmatrix} H_\pm(q) & 0 \\ 0 & P_\pm(q) \end{pmatrix} := G_{B\pm}(q) \tag{7.53}$$

where $H_\pm$ is 2×2 and $P_\pm$ is 1×1. For small enough positive constants ϵ_0 and δ we have

$$H_\pm\left(q^\epsilon(s)\right) = O(\epsilon), \quad P_\pm\left(q^\epsilon(s)\right) = g^{33}(U^0_\pm(s)) + O(\epsilon) \tag{7.54}$$

for $0 < \epsilon \leq \epsilon_0$ and $|s| \leq \delta$.

$S_\pm(q)$ can be chosen with the same regularity as $G(\pm\infty, q)$ and such that

$$S_\pm(q^\epsilon(s)) = \begin{pmatrix} S^{11}_\pm & S^{12}_\pm \\ O(\epsilon) & S^{22}_\pm \end{pmatrix}, \tag{7.55}$$

with $S^{11}_\pm$ and $S^{22}_\pm$ invertible on $\{\pm s \geq 0, |s| \leq \delta\}$ with inverses bounded uniformly with respect to $\epsilon \in (0, \epsilon_0]$.

Proof. Observe that $G(\pm\infty, q)$ can be obtained from $G(z,q)$ by setting $V^0(z)$ and $d_z V^0(z)$ equal to zero in the entries that define $G(z,q)$. Thus, $w(s)$ is replaced by $q^{\epsilon,1}(s) + q^{\epsilon,2}(s) + q^{\epsilon,3}(s)$ (recall (7.11)). The proof is now a repetition of that of Proposition 7.3. □

Remark 7.9. *Setting* $\mathbb{S}_\pm(z,q) := T_\pm(z,q)S_\pm(q)$ *and using* (7.50) *and* (7.53)*, we note that* $V_\pm(z)$ *satisfies* (7.51) *on* $\pm z \geq 0$ *if and only if* $\mathcal{V}_\pm := (\mathbb{S}_\pm)^{-1}V_\pm$ *satisfies*

$$d_z\mathcal{V}_\pm = G_{B\pm}(q)\mathcal{V}_\pm + (\mathbb{S}_\pm)^{-1}f \text{ on } \pm z \geq 0. \tag{7.56}$$

We return now to problem II for $V^\epsilon(s)$ on $[-\delta, 0]$:

$$\begin{aligned} d_z V &= \frac{1}{\epsilon}GV + F \\ v^*(-\delta) &= (p_2, p_3) \in \mathbb{R}^2 \end{aligned} \tag{7.57}$$

for G and F as in (7.9). Note that $v^*(-\delta)$ is prescribed, but $v^3(0)$ is left unspecified for the moment. Consider also the conjugated problem for $\mathcal{V}$ defined by $V = \mathbb{S}\left(\frac{s}{\epsilon}, q^\epsilon(s)\right)\mathcal{V}$ (we suppress the minus subscript):

$$\begin{aligned} d_s\mathcal{V} &= \frac{1}{\epsilon}G_B\mathcal{V} + (\partial_q\mathbb{S}\cdot\partial_s q^\epsilon)\mathcal{V} + (\mathbb{S})^{-1}F \text{ on } [-\delta, 0] \\ \nu^*(-\delta) &= \zeta \in \mathbb{R}^2,\ \nu^3(0) = p_4 \in \mathbb{R}. \end{aligned} \tag{7.58}$$

As before $\mathbb{S}$ and G_B depend on the unknown V through $q^{\epsilon,3} = \epsilon^L v$, but arguing as in Remark 7.4, we see that (7.58) is a well-defined nonlinear problem for $\mathcal{V}$.

Proposition 7.10. *We make the same regularity assumptions as in Proposition 6.6. For fixed* $R > 0$ *and* $|\zeta, \eta| \leq R$*, there exists an* $\epsilon_0 > 0$ *such that for* $0 < \epsilon \leq \epsilon_0$*, the problem* (7.58) *has a solution on* $[-\delta, 0]$*,* $\mathcal{V}(s, \epsilon, \zeta, p_4)$*, that is* C^0 *in* s *and* C^1 *in* (ζ, p_4) *uniformly with respect to* ϵ*. In addition we have*

$$\nu^3(-\delta) = O(\epsilon) \tag{7.59}$$

Proof. We have $\partial_q\mathbb{S}\cdot\partial_s q^\epsilon = O(1)$ and by Remark 6.3

$$g^{33}(U^0_-(s)) \geq \alpha > 0 \text{ on } [-\delta, 0]. \tag{7.60}$$

Together with (7.53) and (7.54), this means that (7.58) is the same type of problem as (7.25). Thus, we can just repeat the proof of Proposition 7.5. The property described in Remark 7.2, part 3, no longer holds, however, so we obtain only uniform C^0 regularity in s on $[-\delta, 0]$. (The equation then implies that $\epsilon\mathcal{V}$ is C^1 in s uniformly with respect to $\epsilon \in (0, \epsilon_0]$.)

□

Next, given (p_2, p_3), we produce a family of solutions parametrized by p_4 to the problem (7.57).

Proposition 7.11. *For* $|p_2, p_3| \leq R_1$ *and* $|p_4| \leq R_2$*, there exists an* $\epsilon_0 = \epsilon_0(R_1, R_2)$ *such that for* $0 < \epsilon \leq \epsilon_0$*, there is a solution* $V_{II}(s, \epsilon, p_2, p_3, p_4)$ *satisfying* (7.57) *on* $[-\delta, 0]$*. The function* V_{II} *is* C^0 *in* s *and* C^1 *in* (p_2, p_3, p_4) *uniformly with respect to* $\epsilon \in (0, \epsilon_0]$*.*

Proof. We can set $V_{II} = \mathbb{S}\mathcal{V}$ for $\mathcal{V}(s, \epsilon, \zeta, p_4)$, provided ζ can be chosen so that $v^*_{II}(-\delta) = (p_2, p_3)$. Using the property (7.49) of $T(z, q)$ and (7.55), we see that for ϵ small enough

$$\mathbb{S}\left(\frac{-\delta}{\epsilon}, q^\epsilon(s)\right) = \begin{pmatrix} \mathbb{S}^{11} & \mathbb{S}^{12} \\ O(\epsilon) & \mathbb{S}^{22} \end{pmatrix} \tag{7.61}$$

with $\mathbb{S}^{11}$ and $\mathbb{S}^{22}$ uniformly invertible. Thus,

$$v^*_{II}(-\delta) = \mathbb{S}^{11}\zeta + \mathbb{S}^{12}\nu^3(-\delta). \tag{7.62}$$

Since $\nu^3(-\delta) = O(\epsilon)$ (recall (7.59)), for ϵ small enough we can find $\zeta = \zeta(\epsilon, p_2, p_3)$ with C^1 dependence on (p_2, p_3) satisfying

$$(p_2, p_3) = \mathbb{S}^{11}\zeta + \mathbb{S}^{12}\nu^3(-\delta). \tag{7.63}$$

□

7.5 Matching exact solutions at $s = -\delta$

We have now constructed a solution to problem I, $V_I(s, \epsilon, \overline{v}, p_1)$ satisfying

$$\begin{aligned} d_z V_I &= \frac{1}{\epsilon} G V_I + F \\ v^*_I(a - \overline{r}) &= \overline{v},\ v^3_I(-\delta) = p_1, \end{aligned} \tag{7.64}$$

and a family, parametrized by p_4, of solutions to problem II, $V_{II}(s, \epsilon, p_2, p_3, p_4)$, satisfying

$$\begin{aligned} d_z V_{II} &= \frac{1}{\epsilon} G V_{II} + F \\ v^*_{II}(-\delta) &= (p_2, p_3). \end{aligned} \tag{7.65}$$

We now show that for a given p_4, the parameters (p_1, p_2, p_3) can be chosen so that for small enough ϵ

$$V_I(-\delta, \epsilon, \overline{v}, p_1) = V_{II}(-\delta, \epsilon, p_2, p_3, p_4). \tag{7.66}$$

Hence, for such a choice of parameters we obtain a smooth solution to the error equation on $[a - \overline{r}, 0]$.

Observe that the map

$$p_1 \to V_I(-\delta, \epsilon, \overline{v}, p_1) = (v^*_I(-\delta, \epsilon, \overline{v}, p_1), p_1) \tag{7.67}$$

defines a C^1 curve in $\mathbb{R}^3$, while the map

$$(p_2, p_3) \to V_{II}(-\delta, \epsilon, p_2, p_3, p_4) = (p_2, p_3, v^3_{II}(-\delta, \epsilon, p_2, p_3, p_4)) \tag{7.68}$$

defines a family, parametrized by p_4, of C^1 surfaces in $\mathbb{R}^3$. For a given p_4, the next Proposition shows that for ϵ small enough, we can always find a point where the curve intersects the surface.

Proposition 7.12. *Fix positive constants R_1 and R_2 and suppose $|p_4| \leq R_1$. There exists $\epsilon_0(R_1, R_2) > 0$ such that for $0 < \epsilon \leq \epsilon_0$, there is a C^1 function $p_4 \to (p_1, p_2, p_3)(p_4)$ such that*

$$V_I(-\delta, \epsilon, \overline{v}, p_1(p_4)) = V_{II}(-\delta, \epsilon, (p_2, p_3)(p_4), p_4) \tag{7.69}$$

with $|p_1(p_4)| < R_2$.

Proof. For ϵ_0 chosen as in Proposition 7.6, there exists $R_3 > 0$ such that

$$|v_I^*(-\delta, \epsilon, \overline{v}, p_1)| \leq R_3 \text{ for } 0 < \epsilon \leq \epsilon_0,\ |p_1| \leq R_2. \tag{7.70}$$

Suppose now that $|p_4| \leq R_1$ and $|p_2, p_3| \leq R_3 + 1$. Since $V_{II} = \mathbb{S}\mathcal{V}$ as in the proof of Proposition 7.11, we see using (7.61) that

$$v_{II}^3(-\delta, \epsilon, p_2, p_3, p_4) = O(\epsilon)\nu^*(-\delta) + \mathbb{S}^{22}\nu^3(-\delta). \tag{7.71}$$

But $\nu^3(-\delta) = O(\epsilon)$ (recall (7.59)), so by shrinking ϵ_0 if necessary we can insure

$$|v_{II}^3(-\delta, \epsilon, p_2, p_3, p_4)| < R_2 \text{ for } 0 < \epsilon \leq \epsilon_0,\ |p_2, p_3| \leq R_3 + 1,\ |p_4| \leq R_1. \tag{7.72}$$

Now (7.70) and (7.72) imply that the curve (7.67) and the surface (7.68) have at least one point of intersection with $|p_1| < R_2$. The C^1 dependence of (p_1, p_2, p_3) on p_4 follows by applying the implicit function theorem to

$$F(p_1, p_2, p_3, p_4) := \begin{pmatrix} (p_2, p_3) - v_I^*(-\delta, \epsilon, \overline{v}, p_1) \\ p_1 - v_{II}^3(-\delta, \epsilon, p_2, p_3, p_4) \end{pmatrix} = 0, \tag{7.73}$$

and using

$$\partial_{p_2, p_3} v_{II}^3(-\delta, \epsilon, p_2, p_3, p_4) = O(\epsilon). \tag{7.74}$$

□

Corollary 7.13. *Let R_1, R_2, $\epsilon_0(R_1, R_2)$ and $(p_1, p_2, p_3)(p_4)$ be as in Proposition 7.12. Then the function defined for $|p_4| \leq R_1$ by*

$$V(s, \epsilon, (p_1, p_2, p_3)(p_4), p_4) = \begin{cases} V_I(s, \epsilon, \overline{v}, p_1(p_4)), s \in [a - \overline{r}, -\delta] \\ V_{II}(s, \epsilon, (p_2, p_3)(p_4), p_4), s \in [-\delta, 0] \end{cases} \tag{7.75}$$

is an exact solution to the error problem (7.8) on $[a - \overline{r}, 0]$. V is C^0 in s and C^1 in p_4 uniformly with respect to $0 < \epsilon \leq \epsilon_0$. The function ϵV is C^1 in s uniformly with respect to $0 < \epsilon \leq \epsilon_0$.

7.6 Problems III and IV

Let $V(s,\epsilon,(p_1,p_2,p_3)(p_4),p_4)$ be the solution to the error equation on $[a-\overline{r},0]$ defined in Corollary 7.13. Problem III is

$$\begin{aligned} &d_z V_{III} = \frac{1}{\epsilon} G V_{III} + F \text{ on } [0,\delta] \\ &V_{III}(0) = V(0,\epsilon,(p_1,p_2,p_3)(p_4),p_4), \end{aligned} \tag{7.76}$$

and Problem IV is

$$\begin{aligned} &d_z V_{IV} = \frac{1}{\epsilon} G V_{IV} + F \text{ on } [\delta, b-\overline{r}] \\ &V_{IV}(\delta) = V_{III}(\delta). \end{aligned} \tag{7.77}$$

Clearly, the boundary condition in (7.76) is chosen so that the transmission condition $[V] = 0$ in (7.8) holds, and the boundary condition in (7.77) is chosen so that V_{III} and V_{IV} match smoothly at $s=\delta$.

The solution of problems III and IV is quite similar to that of problems II and I respectively, but now the argument is simpler because $g^{33}(U_+^0(s))$ has a favorable sign that allows one to prescribe data at the left boundary point in the equations for v^3. By Remark 6.3 we have

$$g^{33}(U_+^0(s)) \leq -\alpha < 0 \text{ for } s \in [0, b-\overline{r}], \tag{7.78}$$

so there is no need to split the boundary conditions or to introduce extra parameters as in problems I and II. The rapid variation and possible change of sign of $g^{33}(\frac{s}{\epsilon}, q^\epsilon(s))$ in (7.8) presents the same difficulty in problem III as in problem II, but we handle that by using $\mathbb{S}_+$ to conjugate G to G_{B+} (recall Remark 7.9) just as in (7.58). Problem IV is conjugated to a block form similar to (7.25) using a conjugator S just like the one constructed in Proposition 7.3. After conjugation problems III and IV are both solved by iteration schemes like the one used earlier, where the iterates satisfy estimates like (7.31), except that now (ζ,η) specifies data only at the left boundary point.

Summarizing, we have proved

Proposition 7.14. *Let $V_{I,II}(s,\epsilon,p_4)$ be the function constructed in Corollary 7.13, and let $V_{III}(s,\epsilon)$, $V_{IV}(s,\epsilon)$ be the solutions to problems III and IV above. Then the function defined for $|p_4| \leq R_1$ and ϵ_0 small enough by*

$$V(s,\epsilon,p_4) = \begin{cases} V_{I,II}(s,\epsilon,p_4), s \in [a-\overline{r},0] \\ V_{III}(s,\epsilon), s \in [0,\delta] \\ V_{IV}(s,\epsilon), s \in [\delta, b-\overline{r}] \end{cases} \tag{7.79}$$

is an exact solution to the error problem (7.8) on $[a-\overline{r}, b-\overline{r}]$. V is C^0 in s and C^1 in p_4 uniformly with respect to $0 < \epsilon \leq \epsilon_0$. The function ϵV is C^1 in s uniformly with respect to $0 < \epsilon \leq \epsilon_0$.

This finishes the proof of the main result of this section:

Theorem 7.15. *Let $V(s,\epsilon,p_4) = \big(v^*(s,\epsilon), v^3(s,\epsilon)\big)$ be the function defined in Proposition 7.14, and set $w^\epsilon(s) = \tilde{w} + \epsilon^L v^*$, where $\tilde{w}$ is the approximate solution constructed in Proposition 6.6. For ϵ_0 small enough, w^ϵ is an exact solution to the transmission problem* (5.5) *for $0 < \epsilon \le \epsilon_0$ with $w^\epsilon(a - \overline{r}) = (\rho_a, u_a)$, the inflow data at $r = a$ for the original inviscid shock. In particular, we have for any $\beta > 0$:*

$$\begin{aligned} &\lim_{\epsilon\to 0} w^\epsilon(s) = U^0(s) \ \text{ in } L^p\left([a-\overline{r}, b-\overline{r}]\right), 1 \le p < \infty \\ &\lim_{\epsilon\to 0} w^\epsilon(s) = U^0(s) \ \text{ in } L^\infty\left([a-\overline{r}, b-\overline{r}] \cap \{|s| \ge \beta\}\right), \end{aligned} \tag{7.80}$$

where $U^0_\pm(s)$ is the original inviscid shock with discontinuity at $s = 0$.

Remark 7.16. *We have stated Theorem 7.15 for barotropic SS shocks with supersonic inflow at $r = a$. The same result holds by exactly the same arguments for the barotropic CS case with supersonic inflow at $r = a$ in the case when angular (v) and axial (w) velocity components are zero. The small viscosity result in the CS case when either $v \neq 0$ or $w \neq 0$ involves additional difficulties due to the fact that now G_{B+} has positive eigenvalues of size O(1) in $s \ge 0$ in addition to a single negative eigenvalue of size O(1). This complicates the matching arguments and leads to difficulties which are identical to those encountered in the full, nonbarotropic SS case. These matters are treated in [EJW].*

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