On existence and uniqueness of entropy solutions to the Cauchy problem for a conservation law with discontinuous flux.

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Abstract

We study the Cauchy problem for a conservation law with space discontinuous flux of generalized Audusse-Perthame form. It is shown that, after a change of unknown function, entropy solutions in the sense of Audusse-Perthame correspond to Kruzhkov's generalized entropy solutions for the transformed equation. This observation allows to use the Kruzhkov method of doubling variable (instead of rather complicated variant of this method invented by Audusse & Perthame). Applying this method for measure valued solutions, we establish the uniqueness and the existence of entropy solutions to the problem under consideration.

Introduction.

In a half-plane $\Pi = \mathbb{R}_+ \times \mathbb{R}$ we study the Cauchy problem for a conservation law

$$u_t + \varphi(x,u)_x = 0, \tag{1}$$

with initial condition

$$u(0,x) = u_0(x) \in L^\infty(\mathbb{R}). \tag{2}$$

We assume that the flux $\varphi(x,u) = g(\beta(x,u))$ where $g(\beta) \in C(\mathbb{R})$ and $\beta(x,u)$ is a Caratheodory function (i.e. this function is measurable with respect to x and continuous with respect to u). We also suppose that $\beta(x,u)$ strictly increases with respect to u, and there exist functions $h_-(u), h_+(u) \in C(\mathbb{R})$ such that

$$h_-(u) \le |\beta(x,u)| \le h_+(u) \quad \forall x \in \mathbb{R}, \tag{3}$$

with $h_-(u) \to +\infty$ as $u \to \infty$.

More generally, we can study the multi-dimensional problem (1), (2) when $x \in \mathbb{R}^m$, $m \in \mathbb{N}$ and $g(u) \in C(\mathbb{R}, \mathbb{R}^m)$. All the results remain true also in this case (see concluding Remark 2) but, to simplify the proofs, we will be occupied only by the case $m = 1$.

Observe that flux functions indicated in paper of Audusse-Perthame [1] satisfy the above assumptions. For instance, if $\varphi(x,u)$ strictly decreases for $u \le u_*(x)$, strictly increases for $u \ge u_*(x)$, and it's minimal value $\varphi(x, u_*(x)) = M$ does not depend on x then we can take $\beta(x,u) = \text{sign}(u - u_*(x))(\varphi(x,u) - M)$, $g(\beta) = M + |\beta|$. To prove that this function $\beta(x,u)$ is measurable with respect to x one could take into account that $u_*(x)$ is a measurable function. The latter follows from the representation: $\forall \lambda \in \mathbb{R}$

$$\{ \ x \in \mathbb{R} \mid u_*(x) \ge \lambda \ \} = \bigcap_{u \in \mathbb{Q}, u < \lambda} \{ \ x \in \mathbb{R} \mid \varphi(x,u) > \varphi(x,\lambda) \ \}.$$

(here $\mathbb{Q}$ denotes the set of rational numbers). Hence, the set $\{ \ x \in \mathbb{R} \mid u_*(x) \ge \lambda \ \}$ is measurable as a countable intersection of measurable sets, and this yields the measurability of $u_*(x)$.

Finally, the fact that $\alpha(x,u)$ satisfies conditions (3) directly follows from the predicted in [1] two-sided estimates of $|\varphi(x,u)|$.

Let us consider the more general equation

$$\varphi_0(x,u)_t + \varphi(x,u)_x = 0, \tag{1'}$$

where $\varphi_0(x,u)$, $\varphi(x,u)$ are Caratheodory functions such that

$$\max_{|u| \le R}(|\varphi_0(x,u)| + |\varphi(x,u)|) \in L^\infty(\mathbb{R}) \ \forall R > 0,$$

and $\varphi_0(x,u)$ strictly increases with respect to u.

Consider firstly the case when the flux $\varphi(\cdot, u) \in C^1(\mathbb{R})$ for every $u \in \mathbb{R}$. In this case we can introduce the generalized entropy solutions of (1'), (2) in the sense of S.N. Kruzhkov [5]. Let us recall the definition.

Definition 1. A function $u = u(t,x) \in L^\infty(\Pi)$ is called a generalized entropy solutions (g.e.s. for short) of (1'), (2) if for all $k \in \mathbb{R}$

$$|\varphi_0(x,u) - \varphi_0(x,k)|_t + [\text{sign}(u-k)(\varphi(x,u) - \varphi(x,k))]_x + \text{sign}(u-k)\varphi_x(x,k) \le 0 \tag{4}$$

in the sense of distribution on Π (in $\mathcal{D}'(\Pi)$), and

$$\operatorname*{ess\,lim}_{t \to 0+} u(t,\cdot) = u_0 \ \text{ in } L^1_{loc}(\mathbb{R}). \tag{5}$$

It is rather well-known (cf. Proposition 1 below) that conditions (4), (5) can be written in the form of the single integral inequality: for each $k \in \mathbb{R}$ and all non-negative test functions $f = f(t,x) \in C_0^1(\bar{\Pi})$, with $\bar{\Pi} = [0,+\infty) \times \mathbb{R}$ being a closure of Π,

$$\int_\Pi [|\varphi_0(x,u) - \varphi_0(x,k)|f_t + \operatorname{sign}(u-k)(\varphi(x,u) - \varphi(x,k))f_x -$$
$$\operatorname{sign}(u-k)\varphi_x(x,k)f]dtdx + \int_{\mathbb{R}} |\varphi_0(x,u_0(x)) - \varphi_0(x,k)|f(0,x)dx \ge 0. \qquad (6)$$

Observe that by the doubling variable method developed in [5] one can derive from (4) the important relation

$$|\varphi_0(x,u) - \varphi_0(x,v)|_t + [\operatorname{sign}(u-v)(\varphi(x,u) - \varphi(x,v))]_x \le 0 \ \text{ in } \mathcal{D}'(\Pi), \qquad (7)$$

which holds for a pair $u = u(t,x)$, $v = v(t,x)$ of g.e.s. This relation is a keystone in the proof of the uniqueness for g.e.s.

Returning to the case of equation (1), we claim that Definition 1 is not valid because the term $\operatorname{sign}(u-k)\varphi_x(x,k)$ is not well-defined in $\mathcal{D}'(\Pi)$. However, as was firstly observed in [1], this obstacle can be removed if to take in (4) some family of stationary solutions instead of constants k. Then we arrived at relations like (7) being correctly defined in $\mathcal{D}'(\Pi)$ since they do not contain terms $\operatorname{sign}(u-k)\varphi_x(x,k)$ anymore. The stationary solutions of equation (1) are defined by the relations $\beta(x,u_k(x)) = k \in \mathbb{R}$, in other words $u_k(x,u) = \alpha(x,k)$, where for fixed $x \in \mathbb{R}$ $\alpha(x,u)$ is an inverse function to $\beta(x,\cdot)$. Obviously, $\alpha(x,u)$ is a Caratheodory function strictly increasing with respect to u. Besides, as follows from (3), $\max\limits_{|u|\le R} |\alpha(x,u)| \in L^\infty(\mathbb{R})$ for every $R > 0$. Indeed, $|\alpha(x,u)| \le C_R$ for all $x \in \mathbb{R}$, $u \in [-R,R]$, where the constant C_R is chosen from the condition $h_-(u) \ge R$ for $|u| > C_R$. In particular, $\alpha(x,k) \in L^\infty(\mathbb{R})$ for every $k \in \mathbb{R}$. Since $\varphi(x,\alpha(x,k)) = g(k) = \text{const}$ then $\alpha(x,k)$ are weak solutions of (1).

Now let us introduce the definition of entropy solution in the sense of Audusse-Perthame [1].

Definition 2. A function $u = u(t,x) \in L^\infty(\Pi)$ is called an entropy solution (briefly - e.s.) of problem (1), (2) if $\forall k \in \mathbb{R}$, $f = f(t,x) \in C_0^1(\bar{\Pi})$, $f \ge 0$

$$\int_\Pi [|u - \alpha(x,k)|f_t + \operatorname{sign}(u - \alpha(x,k))(\varphi(x,u) - g(k))f_x]dtdx$$
$$+ \int_{\mathbb{R}} |u_0(x) - \alpha(x,k)|f(0,x)dx \ge 0. \qquad (8)$$

Now we discuss another natural approach to the problem (1), (2). We make in (1) the change $v = \beta(x, u)$. Then this equation reduces to the equation

$$\alpha(x, v)_t + g(v)_x = 0, \tag{9}$$

which is particular case of (1'). The initial condition (2) is transformed to the following one

$$v(0, x) = v_0(x) = \beta(x, u_0(x)). \tag{10}$$

Since in (9) $\varphi(x, u) = g(u)$ does not depend on x the notion of the Kruzhkov's g.e.s. of (9), (10) is well-defined. Hence, we can introduce the entropy solution $u = u(t, x)$ of the original problem required that $u = \alpha(x, v)$, $v = v(t, x)$ being a g.e.s. of (9), (10) in the sense of Definition 1. Here the entropy relation (6) acquires the form: for each $k \in \mathbb{R}$ and all $f = f(t, x) \in C_0^1(\bar{\Pi})$, $f \geq 0$

$$\int_\Pi [|\alpha(x, v) - \alpha(x, k)|f_t + \operatorname{sign}(v - k)(g(v) - g(k))f_x]dtdx +$$
$$\int_{\mathbb{R}} |\alpha(x, v_0(x)) - \alpha(x, k)|f(0, x)dx \geq 0. \tag{11}$$

Since $u = \alpha(x, v)$, $\operatorname{sign}(v - k)(g(v) - g(k)) = \operatorname{sign}(\alpha(x, v) - \alpha(x, k))(g(\beta(x, u)) - g(k)) = \operatorname{sign}(u - \alpha(x, k))(\varphi(x, u) - g(k))$ relations (11) and (8) are equivalent. Thus, we have proven the following result.

Theorem 1. *A function $u = u(t, x)$ is an e.s. of (1), (2) in the sense of Audusse-Perthame if and only if $v = \beta(x, u(t, x))$ is a g.e.s. of (9), (10) in the sense of Kruzhkov.*

One should also take into account that $u \in L^\infty(\Pi)$ if and only if $v \in L^\infty(\Pi)$. The latter easily follows from estimates (3).

In some sense the second approach, based on the reduction to problem (9), (10), is more convenient. In particular, it allows to avoid the rather complicated variant of the doubling variable method invented in [1] (instead, the "usual" Kruzhkov's method can be applied).

In the sequel we need the more general class of measure valued solutions. Recall (see [2, 11]) that a measure valued function on Π is a weakly measurable map $(t, x) \mapsto \nu_{t,x}$ of Π into the space $\operatorname{Prob}_0(\mathbb{R})$ of probability Borel measures with compact support in $\mathbb{R}$.

The weak measurability of $\nu_{t,x}$ means that for each continuous function $g(u)$ the function $(t, x) \to \langle g(u), \nu_{t,x}(u)\rangle = \int g(u)d\nu_{t,x}(u)$ is measurable on Π.

We say that a measure valued function $\nu_{t,x}$ is *bounded* if there exists $R > 0$ such that $\operatorname{supp} \nu_{t,x} \subset [-R, R]$ for almost all $x \in \Pi$. We shall denote by $\|\nu_{t,x}\|_\infty$ the smallest of such R.

Finally, we say that measure valued functions of the kind $\nu_{t,x}(u) = \delta(u - u(t,x))$, where $u(t,x) \in L^\infty(\Pi)$ and $\delta(u - u^*)$ is the Dirac measure at $u^* \in \mathbb{R}$, are *regular*. We identify these measure valued functions and the corresponding functions $u(t,x)$, so that there is a natural embedding $L^\infty(\Pi) \subset MV(\Pi)$, where $MV(\Pi)$ is the set of bounded measure valued functions on Π.

Measure valued functions naturally arise as weak limits of bounded sequences in $L^\infty(\Pi)$ in the sense of the following theorem of Tartar (see [11]).

Theorem T. *Let $u_m(t,x) \in L^\infty(\Pi)$, $m \in \mathbb{N}$ be a bounded sequence. Then there exist a subsequence $u_r(t,x)$ and a measure valued function $\nu_{t,x} \in \mathrm{MV}(\Pi)$ such that*

$$\forall g(u) \in C(\mathbb{R}) \quad g(u_r) \underset{r\to\infty}{\to} \langle g(u), \nu_{t,x}(u)\rangle \quad \textit{weakly-}* \textit{ in } L^\infty(\Pi). \tag{12}$$

Besides, $\nu_{t,x}$ is regular, i.e. $\nu_{t,x} = \delta(u - u(t,x))$ if and only if $u_r(t,x) \underset{r\to\infty}{\to} u(t,x)$ in $L^1_{loc}(\Pi)$.

More generally, if $g(t,x,u)$ is a Caratheodory function bounded on the sets $\Pi \times [-R, R]$, $R > 0$ then for each $r \in \mathbb{N}$ the functions $g(t,x,u_r(t,x)) \in L^\infty(\Pi)$, $\int g(t,x,u) d\nu_{t,x}(u) \in L^\infty(\Pi)$, and

$$g(t,x,u_r(t,x)) \underset{r\to\infty}{\to} \langle g(t,x,u), \nu_{t,x}(u)\rangle = \int g(t,x,u) d\nu_{t,x}(u) \quad \text{weakly-}* \text{ in } L^\infty(\Pi). \tag{13}$$

This follows from the fact that any Caratheodory function is strongly measurable as a map $x \to g(x,\cdot) \in C(\mathbb{R})$ (see [4], Chapter 2) and, therefore, is a pointwise limit of step functions $g_m(t,x,u) = \sum_i l_{mi}(t,x) h_{mi}(u)$ so that for $(t,x) \in \Pi$ $g_m(t,x,\cdot) \underset{m\to\infty}{\to} g(t,x,\cdot)$ in $C(\mathbb{R})$.

As was shown in [8] (see also [9]), for a measure valued function $\nu_{t,x}$ we can introduce the function

$$u(t,x,s) = \inf\{\, v \,|\, \nu_{t,x}((v, +\infty)) \le s \,\}$$

such that the measures $\nu_{t,x}$ is an image of the Lebesgue measure ds on $I = (0,1)$ with respect to the map $s \to u(t,x,s)$: $\nu_{t,x} = u(t,x,\cdot)^* ds$. Moreover, the function $s \to u(t,x,s)$ is a unique non-increasing and right-continuous function with the property $\nu_{t,x} = u(t,x,\cdot)^* ds$. As is easy to verify (see [8, 9]) $u(t,x,s)$ is measurable on $\Pi \times I$, $u(t,x,s) \in L^\infty(\Pi \times I)$, and $\|u\|_\infty = \|\nu_{t,x}\|_\infty$. Observe also that $u(t,x,s) = u(t,x)$ for a regular function $\nu_{t,x} \sim u(t,x)$. By the identity $\nu_{t,x} = u(t,x,\cdot)^* ds$ we have $\displaystyle\int g(t,x,u)d\nu_{t,x}(u) = \int_I g(t,x,u(t,x,s))ds$ for each Caratheodory function $g(t,x,u)$. Therefore, the limit relation (13) can be rewritten as follows

$$g(t,x,u_r(x)) \underset{r\to\infty}{\to} \int_I g(t,x,u(t,x,s))ds \quad \text{weakly-}* \text{ in } L^\infty(\Pi). \tag{14}$$

Remark that the function $u(t,x,s)$ was used in [8, 9] in the definition of a *strong measure valued solution* for a scalar conservation law. This function was called later in [3] *a bounded measurable process* on Π (if to be exact, the non-decreasing version of u was used in [3] instead). We will use a shorter name *a process* in the sequel. Hence, a process on Π is a function $u(t,x,s) \in L^\infty(\Pi \times I)$, which is non-increasing and continuous from the right with respect to s. Obviously, correspondence $\nu_{t,x} = u(t,x,\cdot)^* ds$ between processes and measure valued functions on Π is one to one.

Now we introduce the notions of a process entropy solution to problem (1), (2).

Definition 3. A process $u = u(t,x,s)$ on Π is called *a process entropy solution* (process e.s. for short) of (1), (2) if $\forall k \in \mathbb{R}$, $\forall f = f(t,x) \in C_0^1(\bar{\Pi})$, $f \geq 0$

$$\int_{\Pi\times I} [|u - \alpha(x,k)|f_t + \operatorname{sign}(u - \alpha(x,k))(\varphi(x,u) - g(k))f_x]dtdxds \\ + \int_{\mathbb{R}} |u_0(x) - \alpha(x,k)|f(0,x)dx \geq 0. \tag{15}$$

It is clear that in the case $u(t,x,s) = u(t,x)$ the notion of process e.s. reduces to the notion of e.s. introduced in Definition 2. In the same way as in Theorem 1, setting $v = v(t,x,s) = \beta(x,u(t,x,s))$, we obtain the process g.e.s. of (9), (10) in the sense of the relation: $\forall k \in \mathbb{R}$, $\forall f = f(t,x) \in C_0^1(\bar{\Pi})$, $f \geq 0$

$$\int_{\Pi\times I} [|\alpha(x,v) - \alpha(x,k)|f_t + \operatorname{sign}(v - k)(g(v) - g(k))f_x]dtdxds + \\ \int_{\mathbb{R}} |\alpha(x,v_0(x)) - \alpha(x,k)|f(0,x)dx \geq 0 \tag{16}$$

similar to (11).

We underline that condition (15) is equivalent to the requirement: $\forall k \in \mathbb{R}$

$$\frac{\partial}{\partial t}\int_I |u-\alpha(x,k)|ds + \frac{\partial}{\partial x}\int_I \operatorname{sign}(u-\alpha(x,k))(\varphi(x,u)-g(k))ds \le 0 \qquad (17)$$

in $\mathcal{D}'(\Pi)$ and the initial condition

$$\operatorname*{ess\,lim}_{t\to 0+}\int_I |u(t,x,s)-u_0(x)|ds = 0 \ \text{ in } L^1_{loc}(\mathbb{R}) \qquad (18)$$

(and in similar way one can also reformulate condition (16)).

It is rather well-known but for completeness we put below the proof.

Proposition 1. *Condition (15) is equivalent to (17), (18).*

Proof. Suppose that relation (15) is satisfied. Let $\omega(s) \in C_0^\infty(\mathbb{R})$ be a function such that $\operatorname{supp}\omega(s) \subset (0,1)$, $\omega(s) \ge 0$, $\int \omega(s)ds = 1$. We set for $r \in \mathbb{N}$ $\omega_r(s) = r\omega(rs)$, $\theta_r(s) = \int_{-\infty}^s \omega_r(x)dx$. It is clear that $\omega_r(s)$ converges weakly in $\mathcal{D}'(\mathbb{R})$ to the Dirac δ-function as $r\to\infty$ while $\theta_r(s)$ converges point-wise to the Heaviside function $\theta(s) = \begin{cases} 1, & s>0, \\ 0, & s\le 0 \end{cases}$. For $h(x) \in C_0^1(\mathbb{R})$, $t_0 > 0$, $r \in \mathbb{N}$ we set $f = f(t,x) = h(x)\theta_r(t_0-t)$. It is clear that $f \in C_0^1(\bar\Pi)$, $f \ge 0$ and by (15) for each $k \in \mathbb{R}$ and $r > 1/t_0$

$$-\int_{\Pi\times I} |u(t,x,s)-\alpha(x,k)|h(x)\omega_r(t_0-t)dtdxds +$$
$$\int_{\Pi\times I} \operatorname{sign}(u-\alpha(x,k))(\varphi(x,u)-g(k))h'(x)\theta_r(t_0-t)dtdxds +$$
$$\int_{\mathbb{R}} |u_0(x)-\alpha(x,k)|h(x)dx \ge 0. \qquad (19)$$

Assume that $t_0 \in S$, where the set S consists of $t>0$ being Lebesgue points of the function $t \to u(t,x,s)$ for a.e. $(x,s) \in \mathbb{R}\times I$. Obviously, $S \subset \mathbb{R}_+$ is a set of full Lebesgue measure. Since $t_0 \in S$ then

$$\int |u(t,x,s)-\alpha(x,k)|\omega_r(t_0-t)dt \underset{r\to\infty}{\to} |u(t_0,x,s)-\alpha(x,k)|$$

for a.e. $(x,s) \in \mathbb{R}\times I$, and by the Lebesgue dominated convergence theorem

$$\int_{\Pi\times I} |u(t,x,s)-\alpha(x,k)|h(x)\omega_r(t_0-t)dtdxds \underset{r\to\infty}{\to}$$
$$\int_{\mathbb{R}\times I} |u(t_0,x,s)-\alpha(x,k)|h(x)dxds$$

while

$$\int_{\Pi\times I} \operatorname{sign}(u-\alpha(x,k))(\varphi(x,u)-g(k))h'(x)\theta_r(t_0-t)dtdxds \underset{r\to\infty}{\to}$$
$$\int_{(0,t_0)\times\mathbb{R}\times I} \operatorname{sign}(u-\alpha(x,k))(\varphi(x,u)-g(k))h'(x)dtdxds$$

because the sequence $\theta_r(t_0-t)$ is uniformly bounded and convergent to $\theta(t_0-t)$. Taking into account the above limit relations we derive from (19) that

$$\int_{\mathbb{R}\times I} |u(t_0,x,s)-\alpha(x,k)|h(x)dxds \le \int_{\mathbb{R}} |u_0(x)-\alpha(x,k)|h(x)dx +$$
$$\int_{(0,t_0)\times\mathbb{R}\times I} \operatorname{sign}(u-\alpha(x,k))(\varphi(x,u)-g(k))h'(x)dtdxds \le$$
$$\int_{\mathbb{R}} |u_0(x)-\alpha(x,k)|h(x)dx + C_h t_0, \quad C_h = \text{const.} \tag{20}$$

From (20) it follows that

$$\limsup_{t_0\to 0, t_0\in S} \int_{\mathbb{R}\times I} |u(t_0,x,s)-\alpha(x,k)|h(x)dxds \le \int_{\mathbb{R}} |u_0(x)-\alpha(x,k)|h(x)dx. \tag{21}$$

Since a map $t_0 \to u(t_0,\cdot)$, $t_0\in S$ is bounded in $L^\infty(\mathbb{R}\times I)$ and C_0^1 is dense in $L^1(\mathbb{R})$ we see that the limit relation (21) holds for all nonnegative $h(x)\in L^1(\mathbb{R})$. We fix such a function $h(x)\in L^1(\mathbb{R})$, $h(x)\ge 0$ and remark that the function $v_0(x)=\beta(x,u_0(x))$ can be approximated in $L^1(\mathbb{R},h(x)dx)$ by a bounded in $L^\infty(\mathbb{R})$ sequence of step functions $v_n(x)=\sum_{i=1}^{m_n} k_{in}\chi_{A_{in}}(x)$, where k_{in} are constants, and $\chi_{A_{in}}(x)$ are indicator functions of measurable sets $A_{in}\subset\mathbb{R}$ such that $\{A_{in}\}_{i=1}^{m_n}$ is a partition of $\mathbb{R}$. Moreover, after extraction a subsequence we can assume that $v_n(x)\to v_0(x)$ a.e. on $\mathbb{R}$ as $n\to\infty$. Since $\alpha(x,u)$ is a Caratheodory function then $u_n(x)\doteq\alpha(x,v_n(x))\to\alpha(x,v_0(x))=u_0(x)$ as $n\to\infty$ a.e. on $\mathbb{R}$ and, by the Lebesgue dominated convergence theorem, $u_n(x)\to u_0(x)$ in $L^1(\mathbb{R},h(x)dx)$ as well. Observe that $u_n(x)=\sum_{i=1}^{m_n}\alpha(x,k_{in})\chi_{A_{in}}(x)$. Thus, for every $\varepsilon>0$ there exists a function $\tilde u(x)=\sum_{i=1}^{m}\alpha(x,k_i)\chi_{A_i}(x)$ such that $\int |u_0(x)-\tilde u(x)|h(x)dx<\varepsilon$. Then

$$\int_{\mathbb{R}\times I} |u(t_0,x,s)-u_0(x)|h(x)dxds < \varepsilon + \int_{\mathbb{R}\times I} |u(t_0,x,s)-\tilde u(x)|h(x)dxds =$$

$$\varepsilon + \sum_{i=1}^{m} \int_{\mathbb{R}\times I} |u(t_0,x,s) - \alpha(x,k_i)| h(x)\chi_{A_i}(x)dxds$$

and, in view of (21),

$$\limsup_{t_0\to 0, t_0\in S} \int_{\mathbb{R}\times I} |u(t_0,x,s) - u_0(x)| h(x)dxds \le$$
$$\varepsilon + \sum_{i=1}^{m} \int_{\mathbb{R}} |u_0(x) - \alpha(x,k_i)| h(x)\chi_{A_i}(x)dx$$
$$= \varepsilon + \int_{\mathbb{R}} |u_0(x) - \tilde{u}(x)| h(x)dx < 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we obtain that $\forall h(x) \in L^1(\mathbb{R})$, $h(x) \ge 0$

$$\lim_{t_0\to 0, t_0\in S} \int_{\mathbb{R}\times I} |u(t_0,x,s) - u_0(x)| h(x)dxds = 0$$

and (18) follows.

Conversely, assume that both conditions (17), (18) are satisfied, and $f = f(t,x) \in C_0^1(\bar{\Pi})$, $f \ge 0$. Then the nonnegative function $f(t,x)\theta_r(t) \in C_0^1(\Pi)$ for each $r \in \mathbb{N}$ and applying (17) to this test function, we arrive at

$$\int_{\Pi\times I} |u(t,x,s) - \alpha(x,k)| f\omega_r(t)dtdxds + \int_{\Pi\times I} [|u(t,x,s) - \alpha(x,k)| f_t +$$
$$\operatorname{sign}(u - \alpha(x,k))(\varphi(x,u) - g(k))f_x]\theta_r(t)dtdxds \ge 0. \qquad (22)$$

As readily follows from (18)

$$\int_{\Pi\times I} |u(t,x,s) - \alpha(x,k)| f\omega_r(t)dtdxds \underset{r\to\infty}{\to} \int_{\mathbb{R}} |u_0(x) - \alpha(x,k)| f(0,x)dx.$$

Further, since $\theta_r(t)$ point-wise converges as $r \to \infty$ to the Heaviside function $\theta(t)$ then, by the Lebesgue dominated convergence theorem, the second integral in (22) converges as $r \to \infty$ to the integral

$$\int_{\Pi\times I} [|u(t,x,s) - \alpha(x,k)| f_t + \operatorname{sign}(u - \alpha(x,k))(\varphi(x,u) - g(k))f_x]dtdxds.$$

Due to above limit relations, (15) follows from (22) in the limit as $r \to \infty$. The proof is complete.

§ 1. The uniqueness of en e.s.

Relation (16) allows to apply the Kruzhkov's doubling variable method and establish the following result.

Theorem 2. *Let $u_1 = u_1(t,x,s)$, $u_2 = u_2(t,x,s)$ be two process e.s. of (1), (2). Then*

$$\frac{\partial}{\partial t}\int_{I^2} |u_1(\cdot,p) - u_2(\cdot,q)|dpdq + \frac{\partial}{\partial x}\int_{I^2} \operatorname{sign}(u_1(\cdot,p) - u_2(\cdot,q)) \\ \times(\varphi(x,u_1(\cdot,p)) - \varphi(x,u_2(\cdot,q)))dpdq \le 0 \text{ in } \mathcal{D}'(\Pi). \tag{23}$$

Proof. Making the change $v_1 = v_1(t,x,p) = \beta(x,u_1(t,x,p))$, $v_2 = v_2(t,x,q) = \beta(x,u_2(t,x,q))$, we reduce (23) to the relation

$$\frac{\partial}{\partial t}\int_{I^2} |\alpha(x,v_1(\cdot,p)) - \alpha(x,v_2(\cdot,q))|dpdq + \\ \frac{\partial}{\partial x}\int_{I^2} \operatorname{sign}(v_1(\cdot,p) - v_2(\cdot,q))(g(v_1(\cdot,p)) - g(v_2(\cdot,q)))dpdq \le 0 \text{ in } \mathcal{D}'(\Pi). \tag{24}$$

From (16) it follows that for each $k\in\mathbb{R}$ $\forall f = f(t,x) \in C_0^1(\Pi)$, $f \ge 0$

$$\int_{\Pi\times I} [|\alpha(x,v_1(\cdot,p)) - \alpha(x,k)|f_t + \operatorname{sign}(v_1(\cdot,p)-k)(g(v_1(\cdot,p)) - g(k))f_x]dtdxdp \ge 0,$$

that is

$$\frac{\partial}{\partial t}\int_{I} |\alpha(x,v_1(\cdot,p)) - \alpha(x,k))|dp + \\ \frac{\partial}{\partial x}\int_{I} \operatorname{sign}(v_1(\cdot,p) - k)(g(v_1(\cdot,p)) - g(k))dp \le 0 \ \text{ in } \mathcal{D}'(\Pi).$$

Taking in this relation $k = v_2(\tau,y,q)$, with $(\tau,y)\in\Pi$, $q\in I$ and integrating over $q\in I$, we arrive at

$$\frac{\partial}{\partial t}\int_{I^2} |\alpha(x,v_1(t,x,p)) - \alpha(x,v_2(\tau,y,q))|dpdq + \\ \frac{\partial}{\partial x}\int_{I^2} \operatorname{sign}(v_1(t,x,p) - v_2(\tau,y,q))(g(v_1(t,x,p)) - g(v_2(\tau,y,q)))dpdq \le 0 \tag{25}$$

in $\mathcal{D}'(\Pi\times\Pi)$. Analogously, changing the roles of the variables (t,x) and (τ,y) and the process g.e.s. v_1 and v_2, we obtain the relation

$$\frac{\partial}{\partial \tau}\int_{I^2} |\alpha(y,v_1(t,x,p)) - \alpha(y,v_2(\tau,y,q))|dpdq + \\ \frac{\partial}{\partial y}\int_{I^2} \operatorname{sign}(v_1(t,x,p) - v_2(\tau,y,q))(g(v_1(t,x,p)) - g(v_2(\tau,y,q)))dpdq \le 0 \tag{26}$$

in $\mathcal{D}'(\Pi\times\Pi)$. Putting (25), (26) together, we derive that

$$\frac{\partial}{\partial t}P_1(t,x;\tau,y)+\frac{\partial}{\partial \tau}P_2(t,x;\tau,y)+$$
$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)Q(t,x;\tau,y)\le 0 \ \text{ in } \mathcal{D}'(\Pi\times\Pi), \tag{27}$$

where we denote

$$P_1(t,x;\tau,y)=\int_{I^2}|\alpha(x,v_1(t,x,p))-\alpha(x,v_2(\tau,y,q))|dpdq,$$
$$P_2(t,x;\tau,y)=\int_{I^2}|\alpha(y,v_1(t,x,p))-\alpha(y,v_2(\tau,y,q))|dpdq,$$
$$Q(t,x;\tau,y)=\int_{I^2}\operatorname{sign}(v_1(t,x,p)-v_2(\tau,y,q))(g(v_1(t,x,p))-g(v_2(\tau,y,q)))dpdq.$$

Let $f(t,x)\in C_0^1(\Pi)$, $h(t,x;\tau,y)=f(t,x)\omega_l(\tau-t)\omega_r(y-x)$, where $r,l\in\mathbb{N}$ and the sequence $\omega_r(s)$ was defined in the proof of Proposition 1. Then $h=h(t,x;\tau,y)\in C_0^1(\Pi\times\Pi)$, $h\ge 0$ Applying (27) to the test function h, we obtain after simple transforms that

$$\int_{\Pi\times\Pi}\Big[(P_1(t,x;\tau,y)(f(t,x)\omega_l(\tau-t))_t+$$
$$P_2(t,x;\tau,y)(f(t,x)\omega_l(\tau-t))_\tau)\omega_r(y-x)$$
$$+Q(t,x;\tau,y)f_x(t,x)\omega_l(\tau-t)\omega_r(y-x)\Big]dtdxd\tau dy\ge 0. \tag{28}$$

We are going to pass in (28) to the limit as $r,l\to\infty$. Let $R=\max(\|v_1\|_\infty,\|v_2\|_\infty)$, $\rho(\delta)=\max\limits_{u,v\in[-R,R],|u-v|\le\delta}|g(u)-g(v)|$ be a continuity modulus of g on the segment $[-R,R]$. Then, as is easily verified, for a.e. $(t,x,\tau,y)\in\Pi\times\Pi$

$$|Q(t,x;\tau,y)-Q(t,x;\tau,x)|\le 2\int_I\rho(|v_2(\tau,y,q)-v_2(\tau,x,q)|)dq,$$
$$|Q(t,x;\tau,x)-Q(t,x;t,x)|\le 2\int_I\rho(|v_2(\tau,x,q)-v_2(t,x,q)|)dq. \tag{29}$$

We introduce the set

$$E_1=\{\ x\in\mathbb{R}\mid x \text{ is a Lebesgue point of } v_2(t,\cdot,s) \text{ for a.e. } (t,s)\in\mathbb{R}_+\times I\ \}.$$

It is clear that E_1 is a measurable set of full Lebesgue measure. Further, since $\alpha(x,u)$ is a Caratheodory function and the space $C([-R,R])$ is separable then, by the Pettis theorem (see [4], Chapter 3), the map $x \to A(x) \doteq \alpha(x,\cdot) \in C([-R,R])$ is strongly measurable and in view of estimate (3) we see that $A(x) \in L^\infty(\mathbb{R}, C([-R,R]))$. In particular (see [4], Chapter 3), the set E_2 of Lebesgue points of the map $A(x)$ has full measure. For $x \in E_2$ we have

$$\lim_{r\to\infty} \int \omega_r(x-y) \max_{|u|\le R} |\alpha(x,u)-\alpha(y,u)|dy = \lim_{r\to\infty} \int \omega_r(x-y)\|A(x)-A(y)\|_\infty dy = 0. \tag{30}$$

If $x \in E_1$ then, taking into account the first estimate in (29), we obtain that for a.e. $(t,\tau) \in \mathbb{R}_+ \times \mathbb{R}_+$

$$\left|\int Q(t,x;\tau,y)\omega_r(y-x)dy - Q(t,x;\tau,x)\right| \le \int |Q(t,x;\tau,y)-Q(t,x;\tau,x)|\omega_r(y-x)dy \le 2\int_{\mathbb{R}\times I} \rho(|v_2(\tau,y,q)-v_2(\tau,x,q)|)\omega_r(y-x)dydq \underset{r\to\infty}{\to} 0 \tag{31}$$

since x is a Lebesgue point of $v_2(\tau,\cdot,q) \in L^\infty(\mathbb{R})$ for a.e. (τ,q). Observe that the sequence $J_r(t,x,\tau) = \int Q(t,x;\tau,y)\omega_r(y-x)dy$ is bounded in $L^\infty(\Pi\times\mathbb{R}_+)$ and by (31) it converges as $r \to \infty$ to $Q(t,x;\tau,x)$ for a.e. $(t,x,\tau) \in \Pi \times \mathbb{R}_+$. Therefore, by the Lebesgue dominated convergence theorem,

$$\lim_{r\to\infty} \int_{\Pi\times\Pi} Q(t,x;\tau,y)f_x(t,x)\omega_l(\tau-t)\omega_r(y-x)dtdxd\tau dy = \int_{\Pi\times\mathbb{R}_+} Q(t,x;\tau,x)f_x(t,x)\omega_l(\tau-t)dtdxd\tau. \tag{32}$$

Let S be a set of $t \in \mathbb{R}_+$ being Lebesgue points of the functions $t \to v_2(t,x,q)$ for a.e. $(x,q) \in \mathbb{R}\times I$. Then $S \subset \mathbb{R}_+$ is a set of full measure, and for $t \in S$ for a.e. $x \in \mathbb{R}$ we have the relation similar to (31)

$$\left|\int Q(t,x;\tau,x)\omega_l(\tau-t)d\tau - Q(t,x;t,x)\right| \le \int |Q(t,x;\tau,x)-Q(t,x;t,x)|\omega_l(\tau-t)d\tau \le 2\int_{\mathbb{R}_+\times I} \rho(|v_2(\tau,x,q)-v_2(t,x,q)|)\omega_l(\tau-t)d\tau dq \underset{l\to\infty}{\to} 0. \tag{33}$$

Here we used the second estimate in (29). Using again the Lebesgue dominated convergence theorem we derive from (33) the relation

$$\lim_{l\to\infty} \int_{\Pi\times\mathbb{R}_+} Q(t,x;\tau,x) f_x(t,x)\omega_l(\tau-t)dtdxd\tau = \int_{\Pi} Q(t,x;t,x)f_x(t,x)dtdx. \quad (34)$$

Now, suppose that $x\in E_1\cap E_2$. Evidently,

$$|P_1(t,x;\tau,y)-P_1(t,x;\tau,x)| \le \int_I |\alpha(x,v_2(\tau,y,q))-\alpha(x,v_2(\tau,x,q))|dq.$$

Since x is a Lebesgue point of bounded function $v_2(\tau,\cdot,q)$ for a.e. (τ,q) while $\alpha(x,\cdot)$ is continuous, x is also a Lebesgue point of the composition $\alpha(x,v_2(\tau,\cdot,q))$ for a.e. (τ,q). Therefore, for a.e. (t,τ)

$$\begin{aligned} \left|\int P_1(t,x;\tau,y)\omega_r(y-x)dy - P_1(t,x;\tau,x)\right| \le \\ \int |P_1(t,x;\tau,y)-P_1(t,x;\tau,x)|\omega_r(y-x)dy \le \\ \int_{\mathbb{R}\times I} |\alpha(x,v_2(\tau,y,q))-\alpha(x,v_2(\tau,x,q))|\omega_r(y-x)dydq \underset{r\to\infty}{\to} 0. \end{aligned} \quad (35)$$

Further,

$$\begin{aligned} |P_2(t,x;\tau,y)-P_2(t,x;\tau,x)| \le \\ \int_I |\alpha(x,v_2(\tau,y,q))-\alpha(x,v_2(\tau,x,q))|dq + 2\|A(y)-A(x)\|_\infty. \end{aligned}$$

Hence, for a.e. (t,τ)

$$\begin{aligned} \left|\int P_2(t,x;\tau,y)\omega_r(y-x)dy - P_2(t,x;\tau,x)\right| \le \\ \int |P_2(t,x;\tau,y)-P_2(t,x;\tau,x)|\omega_r(y-x)dy \le \\ \int_{\mathbb{R}\times I} |\alpha(x,v_2(\tau,y,q))-\alpha(x,v_2(\tau,x,q))|\omega_r(y-x)dydq + \\ 2\int_{\mathbb{R}} \|A(y)-A(x)\|\omega_r(y-x)dy \underset{r\to\infty}{\to} 0. \end{aligned} \quad (36)$$

Here we take into account (35) and (30). Observe that $P_1(t,x;\tau,x) = P_2(t,x;\tau,x)$. We see that limit relations (35), (36) hold for a.e. $(t,x,\tau)\in\Pi\times\mathbb{R}_+$.

By the Lebesgue dominated convergence theorem, from (35), (36) it follows the limit relation

$$\lim_{r\to\infty}\int_{\Pi\times\Pi}\Big(P_1(t,x;\tau,\xi)(f(t,x)\omega_l(\tau-t))_t+$$
$$P_2(t,x;\tau,y)(f(t,x)\omega_l(\tau-t))_\tau\Big)\omega_r(y-x)dtdxd\tau dy=$$
$$\int_{\Pi\times\mathbb{R}_+}P_1(t,x;\tau,x)f_t(t,x)\omega_l(\tau-t)dtdxd\tau, \qquad (37)$$

where we also use the fact that $(\partial/\partial t+\partial/\partial\tau)\omega_l(\tau-t)=0$. Now, we pass to the limit in (37) as $l\to\infty$. For this, we observe firstly that

$$|P_1(t,x;\tau,x)-P_1(t,x;t,x)|\le\int_I|\alpha(x,v_2(\tau,x,q))-\alpha(x,v_2(t,x,q))|dq,$$

which implies, in the same way as in the derivation of (33) the relation

$$\left|\int P_1(t,x;\tau,x)\omega_l(\tau-t)d\tau-P_1(t,x;t,x)\right|\le$$
$$\int|P_1(t,x;\tau,x)-P_1(t,x;t,x)|\omega_l(\tau-t)d\tau\le$$
$$\int_{\mathbb{R}_+\times I}|\alpha(x,v_2(\tau,x,q))-\alpha(x,v_2(t,x,q))|\omega_l(\tau-t)d\tau dq\underset{l\to\infty}{\to}0 \qquad (38)$$

for a.e. $(t,x)\in\Pi$. This in turn implies that

$$\lim_{l\to\infty}\int_{\Pi\times\mathbb{R}_+}P_1(t,x;\tau,x)f_t(t,x)\omega_l(\tau-t)dtdxd\tau=\int_\Pi P_1(t,x;t,x)f_t(t,x)dtdx. \quad (39)$$

Passing in (28) to the limit firstly as $r\to\infty$ and then as $l\to\infty$ with account of relations (32), (34), (37), (39), we arrive at

$$\int_\Pi[P_1(t,x;t,x)f_t(t,x)+Q(t,x;t,x)f_x(t,x)]dtdx\ge0\;\forall f(t,x)\in C_0^1(\Pi),\,f(t,x)\ge0,$$

i.e.

$$\frac{\partial}{\partial t}P_1(t,x;t,x)+\frac{\partial}{\partial x}Q(t,x;t,x)\le0 \text{ in } \mathcal{D}'(\Pi).$$

This is exactly (24) because

$$P_1(t,x;t,x)=\int_{I^2}|\alpha(x,v_1(t,x,p))-\alpha(x,v_2(t,x,q))|dpdq,$$

$$Q(t,x;t,x)=\int_{I^2}\operatorname{sign}(v_1(t,x,p)-v_2(t,x,q))(g(v_1(t,x,p))-g(v_2(t,x,q)))dpdq.$$

The proof is complete.

The statement of Theorem 2 is a key-stone in the proof of the following uniqueness result.

Theorem 3. *Suppose that the flux $\varphi(x,u)$ is uniformly continuous with respect to $u \in [-R,R]$ for every $R>0$. Then a process e.s. $u(t,x,s)$ of the problem (1), (2) is unique. Moreover, $u(t,x,s)=u(t,x)$, where $u(t,x)$ is a unique e.s. of (1), (2).*

Proof. Let $u_1=u_1(t,x,s)$, $u_2=u_2(t,x,s)$ be two process e.s. of (1), (2), and $R=\max(\|u_1\|_\infty, \|u_2\|_\infty)$. By the uniform continuity of $\varphi(x,u)$ there exists a non-decreasing sub-additive function $\rho(\delta)$ on $[0,+\infty)$ (the modulus of continuity) such that $0=\rho(0)=\lim\limits_{\delta\to 0+}\rho(\delta)$, and $|\varphi(x,u)-\varphi(x,v)| \le \rho(|u-v|)$ for all $u,v\in[-R,R]$ and a.e. $x\in\mathbb{R}$. Observe that for each positive ε

$$\frac{\rho(\delta)}{\delta+\varepsilon} \le \frac{\rho(\varepsilon)}{\varepsilon} \quad \forall \delta \ge 0. \tag{40}$$

Indeed, we can choose $k\in\mathbb{N}$ such that $\delta\in[(k-1)\varepsilon, k\varepsilon)$. Then, since $\rho(\delta)$ is non-decreasing and sub-additive, $\rho(\delta)\le\rho(k\varepsilon)\le k\rho(\varepsilon)$ while $\delta+\varepsilon\ge k\varepsilon$, and (40) follows. By (23) we see that for each $\varepsilon>0$

$$\frac{\partial}{\partial t}(P(t,x)+\varepsilon)+\frac{\partial}{\partial x}Q(t,x)\le 0 \ \text{ in } \mathcal{D}'(\Pi), \tag{41}$$

where we denote

$$P(t,x)=\int_{I^2}|u_1(t,x,p)-u_2(t,x,q)|dpdq,$$

$$Q(t,x)=\int_{I^2}\operatorname{sign}(u_1(t,x,p)-u_2(t,x,q))(\varphi(x,u_1(t,x,p))-\varphi(x,u_2(t,x,q)))dpdq.$$

Suppose that $f(t,x)\in C_0^1(\bar\Pi)$, $f(t,x)\ge 0$; $\theta_r(t)=\int_{-\infty}^t \omega_r(s)ds$ (see the proof of Proposition 1). Then for $r\in\mathbb{N}$ the nonnegative function $f(t,x)\theta_r(t)\in C_0^1(\Pi)$. Applying (41) to this test function, we obtain the relation

$$\int_\Pi [(P(t,x)+\varepsilon)f_t(t,x)+Q(t,x)f_x(t,x)]\theta_r(t)dtdx + \int_\Pi (P(t,x)+\varepsilon)\omega_r(t)f(t,x)dtdx \ge 0. \tag{42}$$

Now, we observe that, by Proposition 1, the process e.s. $u = u_1, u_2$ satisfy limit relation (18). Therefore,

$$0 \le P(t,x) = \int_{I^2} |u_1(t,x,p) - u_2(t,x,q)| dpdq \le$$
$$\int_{I^2} |u_1(t,x,p) - u_0(x)| dpdq + \int_{I^2} |u_0(x) - u_2(t,x,q)| dpdq =$$
$$\int_I |u_1(t,x,s) - u_0(x)| ds + \int_I |u_2(t,x,s) - u_0(x)| ds \to 0 \ \text{ in } L^1_{loc}(\mathbb{R})$$

as $t \to 0$ running over some set $S \subset \mathbb{R}_+$ of full measure. This easily implies that, as $r \to \infty$,

$$\int_\Pi (P(t,x) + \varepsilon)\omega_r(t) f(t,x) dtdx \to \varepsilon \int_{\mathbb{R}} f(0,x) dx. \tag{43}$$

Since $\theta_r(t) \to \theta(t)$ as $r \to \infty$ and $0 \le \theta_r(t) \le 1$ then, by the Lebesgue dominated convergence theorem,

$$\int_\Pi [(P(t,x) + \varepsilon) f_t(t,x) + Q(t,x) f_x(t,x)]\theta_r(t) d_t dx \underset{r\to\infty}{\longrightarrow}$$
$$\int_\Pi [(P(t,x) + \varepsilon) f_t(t,x) + Q(t,x) f_x(t,x)] dtdx. \tag{44}$$

Taking into account (43), (44) we derive from (42) in the limit as $r \to \infty$ that

$$\int_\Pi [(P(t,x) + \varepsilon) f_t(t,x) + Q(t,x) f_x(t,x)] dtdx + \varepsilon \int_{\mathbb{R}} f(0,x) dx \ge 0. \tag{45}$$

Denote $N(\varepsilon) = \rho(\varepsilon)/\varepsilon$ and set for $0 < t_0 < T$, $C > 1$, $r \in \mathbb{N}$

$$f(t,x) = \theta_1(C + N(\varepsilon)(T - t) - |x|)\theta_r(t_0 - t).$$

Evidently, $f(t,x) \in C^1_0(\bar{\Pi})$, $f(t,x) \ge 0$. Applying (45) to the test function $f(t,x)$, we obtain that for $r > 1/t_0$

$$-\int_\Pi (P(t,x) + \varepsilon)\theta_1(C + N(\varepsilon)(T - t) - |x|)\omega_r(t_0 - t) dtdx -$$
$$\int_\Pi [N(\varepsilon)(P(t,x) + \varepsilon) + Q(t,x) \operatorname{sign} x]\omega_1(C + N(\varepsilon)(T - t) - |x|)\theta_r(t_0 - t) dtdx$$
$$+\varepsilon \int_R \theta_1(C + N(\varepsilon)T - |x|) dx \ge 0. \tag{46}$$

Since, with account of (40), for a.e. $(t,x)\in\Pi$

$$|Q(t,x)|\le\int_{I^2}|\varphi(x,u_1(t,x,p))-\varphi(x,u_2(t,x,q))|dpdq\le$$
$$\int_{I^2}\rho(|u_1(t,x,p)-u_2(t,x,q)|)dpdq\le$$
$$N(\varepsilon)\int_{I^2}(|u_1(t,x,p)-u_2(t,x,q)|+\varepsilon)dpdq=N(\varepsilon)(P(t,x)+\varepsilon)$$

we see that the second integral in (46) is nonnegative and therefore

$$-\int_\Pi(P(t,x)+\varepsilon)\theta_1(C+N(\varepsilon)(T-t)-|x|)\omega_r(t_0-t)dtdx+$$
$$\varepsilon\int_R\theta_1(C+N(\varepsilon)T-|x|)dx\ge 0. \tag{47}$$

Assuming that t_0 is a Lebesgue point of the function

$$t\to\int_\Pi P(t,x)\theta_1(C+N(\varepsilon)(T-t)-|x|)dx,$$

we can pass to the limit in (46) as $r\to\infty$ and arrive at

$$-\int_{\mathbb{R}}(P(t_0,x)+\varepsilon)\theta_1(C+N(\varepsilon)(T-t_0)-|x|)dx+\varepsilon\int_R\theta_1(C+N(\varepsilon)T-|x|)dx\ge 0.$$

Hence, for a.e. $t_0\in(0,T)$

$$\int_{\mathbb{R}}P(t_0,x)\theta_1(C-|x|)dx\le\int_{\mathbb{R}}(P(t_0,x)+\varepsilon)\theta_1(C+N(\varepsilon)(T-t_0)-|x|)dx\le$$
$$\varepsilon\int_R\theta_1(C+N(\varepsilon)T-|x|)dx\le\varepsilon\int_{|x|\le C+N(\varepsilon)T}dx=2C\varepsilon+2T\rho(\varepsilon).$$

This implies that

$$\int_{(0,T)\times\mathbb{R}}P(t,x)\theta_1(C-|x|)dtdx\le 2T(C\varepsilon+T\rho(\varepsilon))\underset{\varepsilon\to0+}{\to}0.$$

Therefore,

$$\int_{(0,T)\times\mathbb{R}}P(t,x)\theta_1(C-|x|)dtdx=0$$

and, since $T>0$, $C>1$ are arbitrary, we conclude that

$$P(t,x)=\int_{I^2}|u_1(t,x,p)-u_2(t,x,q)|dpdq=0\ \text{ a.e. on }\Pi.$$

This readily implies that $u_1(t,x,p) = u_2(t,x,q) = u(t,x)$, where $u(t,x) = \int_I u_1(t,x,s)ds$. Hence, process e.s. $u(t,x,s)$ is unique and necessarily has the form $u(t,x,s) = u(t,x)$. We conclude that $u(t,x)$ is a unique e.s. of (1), (2). The prove is complete.

Remark 1. In the same way as in paper [1] we can adapt the above proof to establish the more general comparison principle for process entropy sub- and super-solutions of (1), (2).

The condition of uniform continuity of $\varphi(x,u)$ is essential for the uniqueness, even in the case when $\varphi(x,u)$ is a continuous function of both variables. We confirm this by the following simple example.

Example 1. We consider equation (1) with

$$\varphi(x,u) = \begin{cases} (1+x^2)u, & |u| \le 1/(1+x^2), \\ u + \dfrac{x^2 \operatorname{sign} u}{1+x^2}, & |u| > 1/(1+x^2). \end{cases}$$

Then $\varphi(x,u)$ is continuous on $\mathbb{R}^2$, $u \to \varphi(x,u)$ is increasing and Lipschitz with constant $(1+x^2)$, $|u| \le |\varphi(x,u)| \le 1+|u|$. Thus, our assumptions (and even ones of [1]) are satisfied. But evidently the function

$$u(t,x) = \begin{cases} 0, & x > \tan(t-\pi/2), t \in (0,\pi), \\ c/(1+x^2), & \text{otherwise} \end{cases}$$

is an e.s. of (1) with zero initial data for each constant $c \in [-1,1]$. Hence, even the zero solution is not unique.

§ 2. The existence of an e.s.

Now, we are going to prove the existence of an e.s. For that we use an approximation of the flux. We define $\tilde{\beta}_n(x,u) = \beta(s_n(x),u)$, where $s_n(x) = \max(-n, \min(x,n))$ is a truncation function, $n \in \mathbb{N}$; $\gamma_n(x,u) = (\tilde{\beta}_n(\cdot,u) * \omega_n)(x) = \int \tilde{\beta}_n(x-y,u)\omega_n(y)dy$. Then $\gamma_n(\cdot,u) \in C^\infty(\mathbb{R}, C(\mathbb{R}))$, $(\gamma_n)_x(x,u) = 0$ for $|x| > n+1$; Besides, $\gamma_n(x,u)$ is continuous, strictly increasing with respect to u, and satisfies the uniform estimates like (3):

$$\tilde{h}_-(u) \doteq h_-(u) - 4h_+(0) \le |\gamma_n(x,u)| \le \tilde{h}_+(u) \doteq h_+(u). \tag{48}$$

Indeed, the upper bound in (49) readily follows from (3). To establish the low bound, remark that for $u \ge 0$

$$\beta(x,u) = \beta(x,u) - \beta(x,0) + \beta(x,0) = |\beta(x,u) - \beta(x,0)| + \beta(x,0) \ge$$
$$|\beta(x,u)| - |\beta(x,0)| + \beta(x,0) \ge |\beta(x,u)| - 2|\beta(x,0)| \ge h_-(u) - 2h_+(0)$$

while for $u < 0$

$$-\beta(x,u) = \beta(x,0) - \beta(x,u) - \beta(x,0) = |\beta(x,u) - \beta(x,0)| - \beta(x,0) \geq$$
$$|\beta(x,u)| - |\beta(x,0)| - \beta(x,0) \geq |\beta(x,u)| - 2|\beta(x,0)| \geq h_-(u) - 2h_+(0).$$

After the convolution, we derive from the above estimates that $\pm\gamma_n(x,u) \geq h_-(u) - 2h_+(0)$ for $\pm u \geq 0$. This implies that for $\pm u \geq 0$

$$|\gamma_n(x,u)| \geq |\gamma_n(x,u) - \gamma_n(x,0)| - |\gamma_n(x,0)| =$$
$$\pm(\gamma_n(x,u) - \gamma_n(x,0)) - |\gamma_n(x,0)| \geq \pm\gamma_n(x,u) - 2|\gamma_n(x,0)| \geq$$
$$\pm\gamma_n(x,u) - 2h_+(0) \geq h_-(u) - 4h_+(0) = \tilde{h}_-(u),$$

as was to be proved.

By the property of averaged functions, $\gamma_n(x,u) \to \beta(x,u)$ as $n \to \infty$ in $L^1_{loc}(\mathbb{R}, C(\mathbb{R}))$. Now we average $\gamma_n(x,u)$ with respect to the second variable, setting $\gamma_{nm}(x,u) = (\gamma_n(x,\cdot) * \omega_m)(u) = \int \gamma_n(x,u-v)\omega_m(v)dv$. Since $\gamma_n(x,u)$ is uniformly continuous on the sets $\mathbb{R} \times [-R,R]$ for each $R > 0$ (because it does not depend on x for $|x| > n+1$) the sequence $\gamma_{nm}(x,u) \to \gamma_n(x,u)$ as $m \to \infty$ uniformly on the sets $\mathbb{R} \times [-R,R]$, $R > 0$. Obviously, the functions $\gamma_{nm}(x,u) \in C^\infty(\mathbb{R}^2)$, $(\gamma_{nm})_u(x,u) > 0$, and they satisfy the estimates $\bar{h}_-(u) \leq |\gamma_{nm}(x,u)| \leq \bar{h}_+(u)$ with $\bar{h}_-(u) = \min\limits_{|v-u|\leq 1} \tilde{h}_-(v)$, $\bar{h}_+(u) = \max\limits_{|v-u|\leq 1} \tilde{h}_+(v)$. We underline that $\bar{h}_\pm(u) \in C(\mathbb{R})$, and $\bar{h}_-(u) \to +\infty$ as $u \to \infty$. Using the standard diagonal extraction, we can define a sequence $\beta_n(x,u) = \gamma_{nm_n}(x,u) \in C^\infty(\mathbb{R}^2)$, which converges to $\beta(x,u)$ in $L^1_{loc}(\mathbb{R}, C(\mathbb{R}))$. By the construction $\beta_n(x,u)$ satisfy the uniform estimate

$$\bar{h}_-(u) \leq |\beta_n(x,u)| \leq \bar{h}_+(u) \tag{49}$$

like (3). Let $g_n(\beta) \in C_0^\infty(\mathbb{R})$ be a sequence convergent to $g(u)$ uniformly on compact sets, and $\varphi_n(x,u) = g_n(\beta_n(x,u))$. Then $\varphi_n(x,u) \in C^\infty(\mathbb{R}^2)$, the derivatives $(\varphi_n)_x(x,u)$, $(\varphi_n)_u(x,u)$, $(\varphi_n)_{xu}(x,u)$ are bounded because $\varphi_n(x,u)$ does not depend on x for sufficiently large $|x|$ and $\varphi_n(x,u) = 0$ for sufficiently large $|u|$ (recall that $g_n(\beta)$ has a compact support). Hence, $\varphi(x,u)$ satisfies the Kruzhkov's assumptions, which ensure the existence of g.e.s. $u_n(t,x)$ to the Cauchy problem for the approximate equation

$$u_t + \varphi_n(x,u)_x = 0 \tag{50}$$

with initial data (2). Let $\alpha_n(x,u)$ be the inverse function to $\beta_n(x,\cdot)$. Since $\varphi_n(x,\alpha_n(x,k)) = g_n(k)$, we see that $\alpha_n(x,k)$ is a smooth and bounded solution of (50) for each $k \in \mathbb{R}$. Therefore, it is a g.e.s. of (50) as well. If $R = \|u_0\|_\infty$ then $\alpha_n(x, k_-) \le u_0(x) \le \alpha_n(x, k_+)$ a.e. on $\mathbb{R}$, where $k_- = \inf_{x,n} \beta_n(x, -R)$, $k_+ = \sup_{x,n} \beta_n(x, R)$ (as follows from (49), these values are finite), and by the comparison result from [5] we conclude that $\alpha_n(x, k_-) \le u_n(t, x) \le \alpha_n(x, k_+)$ a.e. on Π. Since the functions $\alpha_n(x, k)$ are uniformly bounded the sequence $u_n(t, x)$ is bounded in $L^\infty(\Pi)$. By Theorem T there exists a process $u(t, x, s)$ such that after extraction of a subsequence, if necessary (we keep the notation u_n for it), $u_n(t, x)$ converges to $u(t, x, s)$ in the sense of relation (14).

Proposition 2. *$u(t, x, s)$ is a process e.s. of (1), (2).*

Proof. Since $u_n(t, x)$ are g.e.s. of (50), (2), and $\alpha_n(x, k)$ is a stationary g.e.s. of this problem for every $k \in \mathbb{R}$ then the Kruzhkov's entropy relation like (7) holds

$$|u_n - \alpha_n(\cdot, k)|_t + [\operatorname{sign}(u_n - \alpha_n(x, k))(\varphi_n(x, u_n) - g_n(k))]_x \le 0 \text{ in } \mathcal{D}'(\Pi). \quad (51)$$

Recall that $\operatorname*{ess\,lim}_{t \to 0+} u_n(t, \cdot) = u_0$ in $L^1_{loc}(\mathbb{R})$. As in the proof of Proposition 1 we derive from this relation and (51) the integral inequality: $\forall f = f(t, x) \in C^1_0(\bar{\Pi})$, $f \ge 0$

$$\int_\Pi [|u_n - \alpha_n(\cdot, k)| f_t + \operatorname{sign}(u_n - \alpha_n(x, k))(\varphi_n(x, u_n) - g_n(k)) f_x] dt dx + \int_{\mathbb{R}} |u_0(x) - \alpha_n(x, k)| f(0, x) dx \ge 0. \quad (52)$$

Further, in view of (14)

$$|u_n(t, x) - \alpha(x, k)| \underset{n \to \infty}{\to} \int_I |u(t, x, s) - \alpha(x, k)| ds, \quad (53)$$

$$\operatorname{sign}(u_n(t, x) - \alpha(x, k))(\varphi(x, u_n(t, x)) - g(k)) \underset{n \to \infty}{\to} \int_I \operatorname{sign}(u(t, x, s) - \alpha(x, k))(\varphi(x, u(t, x, s)) - g(k)) ds \text{ weakly-}* \text{ in } L^\infty(\Pi). \quad (54)$$

We take sufficiently large $R > 0$ such that $\|\alpha_n(x, k)\|_\infty \le R$, $\|u_n\|_\infty \le R \; \forall n \in \mathbb{N}$ and denote by $\rho(x, \delta)$ a continuity modulus of $\varphi(x, \cdot)$ on the segment $[-R, R]$. Evidently, as $n \to \infty$ $\alpha_n(x, k) \to \alpha(x, k)$ in $L^1_{loc}(\Pi)$, $\varphi_n(x, \cdot) \to \varphi(x, \cdot)$ in $L^1_{loc}(\Pi, C([-R, R]))$ and therefore

$$||u_n - \alpha_n(x, k)| - |u_n - \alpha(x, k)|| \le |\alpha_n(x, k) - \alpha(x, k)| \to 0,$$

$$| \operatorname{sign}(u_n - \alpha_n(x,k))(\varphi_n(x,u_n) - g_n(k)) - \operatorname{sign}(u_n - \alpha(x,k))(\varphi(x,u_n) - g(k))| \le$$
$$|\varphi_n(x,u_n) - \varphi(x,u_n)| + 2\rho(x, |\alpha_n(x,k) - \alpha(x,k)|) \to 0$$

in $L^1_{loc}(\Pi)$. This together with (53), (54) yields the limit relations

$$|u_n(t,x) - \alpha_n(x,k)| \underset{n\to\infty}{\rightarrow} \int_I |u(t,x,s) - \alpha(x,k)| ds,$$
$$\operatorname{sign}(u_n(t,x) - \alpha_n(x,k))(\varphi_n(x,u_n(t,x)) - g_n(k)) \underset{n\to\infty}{\rightarrow}$$
$$\int_I \operatorname{sign}(u(t,x,s) - \alpha(x,k))(\varphi(x,u(t,x,s)) - g(k)) ds \text{ weakly-}* \text{ in } L^\infty(\Pi).$$

They allow to pass to the limit in (52) as $n \to \infty$ and obtain that for each $k \in \mathbb{R}$, $f = f(t,x) \in C^1_0(\bar{\Pi})$, $f \ge 0$

$$\int_{\Pi\times I} [|u - \alpha(x,k)| f_t + \operatorname{sign}(u - \alpha(x,k))(\varphi(x,u) - g(k)) f_x] dt dx ds$$
$$+ \int |u_0(x) - \alpha(x,k)| f(0,x) dx \ge 0$$

with $u = u(t,x,s)$, which is exactly (15). Hence, $u(t,x,s)$ is a process e.s. of (1), (2), as was to be proved.

From Proposition 2 and Theorem 3 it readily follows the existence of e.s. Moreover, we have the following statement.

Theorem 4. *Suppose that $\varphi(x,u)$ is uniformly continuous with respect to $u \in [-R, R]$ for every $R > 0$. Then the sequence $u_n(t,x)$ converges in $L^1_{loc}(\Pi)$ to unique e.s. $u(t,x)$ of (1), (2).*

Proof. By Theorem 3 the process e.s. $u(t,x,s) = u(t,x)$ where $u(t,x)$ is a unique e.s. of (1), (2). Thus the limit measure-valued function corresponding to this process is regular and by Theorem T $u_n(t,x) \to u(t,x)$ as $n \to \infty$ in $L^1_{loc}(\Pi)$. Finally, since the limit function $u(t,x)$ does not depend on the prescribed above choice of the subsequence, we conclude that the original sequence converges to $u(t,x)$ in $L^1_{loc}(\Pi)$. The proof is complete.

§ 3. Concluding remarks.

Remark 2. All the result of this paper remain valid for a multi-dimensional equation

$$u_t + \operatorname{div}_x \varphi(x,u) = 0, \quad (t,x) \in \Pi = \mathbb{R}_+ \times \mathbb{R}^m \tag{55}$$

where the vector $\varphi(x,u) = g(\beta(x,u))$, $g(\beta) \in C(\mathbb{R},\mathbb{R}^m)$, $\beta(x,u)$ being a Caratheodory function on $\mathbb{R}^m \times \mathbb{R}$, strictly increasing with respect to u and satisfying estimates (3). Recall that our flux is supposed to be only continuous with respect to u and in order to prove the analog of Theorem 3, we need to require some additional conditions on character of continuity of the flux vector, similar to ones in [6, 7]. For instance, it is sufficient to suppose that for each $R > 0$ there exist non-decreasing sub-additive functions $\rho_i(r)$ on $[0,+\infty)$, $i = 1,\dots,m$ such that $\rho_i(r) > 0$ for $r > 0$, and

$$|\varphi_i(x,u) - \varphi_i(x,v)| \le \rho_i(|u-v|)\ \forall u,v \in [-R,R], x \in \mathbb{R}^m,\ i = 1,\dots,n, \quad (56)$$

$$\liminf_{r\to 0+} r^{1-m} \prod_{i=1}^{m} \rho_i(r) < \infty. \quad (57)$$

Clearly, in the case when $\varphi(x,u)$ is Lipschitz on any segment $u \in \mathbb{R}$ (with the Lipschitz constant independent of x) above conditions (56), (57) are trivially satisfied. Observe also that (57) always fulfils in one-dimensional case $m = 1$.

Remark 3. The existence of entropy solution to the Cauchy problem for equation (55) can be proved without assumptions (56), (57) if the following non-degeneracy condition is satisfied: for a.e. $x \in \mathbb{R}^m$ $\forall \xi = (\xi_1,\dots,\xi_m) \in \mathbb{R}^m$, $\xi \ne 0$ the function $u \to \xi\cdot\varphi(x,u) = \sum_{i=1}^{m} \xi_i\varphi_i(x,u)$ is not affine on non-degenerate intervals.

Indeed, as follows from results of [10], the approximated sequence $u_n(t,x)$ (which is constructed in the same way as in the one-dimensional case) is strongly pre-compact and therefore, after extraction of a subsequence it converges to a function $u(t,x)$ in $L^1_{loc}(\Pi)$. Obviously, this limit function $u(t,x)$ is an e.s. to the original Cauchy problem.

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