On weak convergence of entropy solutions to scalar conservation laws.

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Abstract

We prove that weak limits of entropy solutions to a one-dimensional scalar conservation law are entropy solutions as well.

We consider a scalar conservation law

$$u_t + f(u)_x = 0, \quad (t,x) \in \Pi = (0,+\infty) \times \mathbb{R}. \tag{1}$$

The flux function $f(u)$ is supposed to be only continuous: $f(u) \in C(\mathbb{R})$. Recall the notion of an entropy solution of (1) in the sense of Kruzhkov [6].

Definition 1. A bounded measurable function $u = u(t,x) \in L^\infty(\Pi)$ is called an entropy solution (e.s. for short) of (1) if $\forall k \in \mathbb{R}$

$$\frac{\partial}{\partial t}|u-k| + \frac{\partial}{\partial x}[\operatorname{sign}(u-k)(f(u)-f(k))] \le 0 \tag{2}$$

in the sense of distributions on Π (in $\mathcal{D}'(\Pi)$).

Here $\operatorname{sign} u = \begin{cases} 1 & , \quad u > 0, \\ -1 & , \quad u \le 0. \end{cases}$, and relation (2) means that for each test function $h = h(t,x) \in C_0^1(\Pi)$, $h \ge 0$

$$\int_\Pi [|u-k|h_t + \operatorname{sign}(u-k)(f(u)-f(k))h_x]dtdx \ge 0.$$

Taking in (2) $k = \pm R$, $R \ge \|u\|_\infty$, we derive that $u_t + f(u)_x = 0$ in $\mathcal{D}'(\Pi)$, i.e. an e.s. $u = u(t,x)$ is a weak solution of (1). We recall also that $u = u(t,x)$ is an e.s. of the Cauchy problem for (1) with initial data

$$u(0,x) = u_0(x) \tag{3}$$

if in addition to (2) the following initial condition holds:

$$\operatorname*{ess\,lim}_{t\to 0+} u(t,\cdot) = u_0 \ \text{ in } L^1_{loc}(\mathbb{R}). \tag{4}$$

As was shown in [10] (see also [7, 8] for more details), for every $u_0(x) \in L^\infty(\mathbb{R})$ there exists a unique e.s. to problem (1), (3). We underline that $f(u)$ is assumed to be only continuous and it is essential for the uniqueness that we have only one space variable.

Now we consider a bounded sequence $u_n = u_n(t,x)$ of e.s. weakly convergent to $u = u(t,x) \in L^\infty(\Pi)$. In the case $f(u) \in C^1(\mathbb{R})$ it is rather well-known that $u = u(t,x)$ is a weak solution of (1), i.e. $u_t + f(u)_x = 0$ in $\mathcal{D}'(\Pi)$. This is a simple application of compensated compactness theory, for the proof we refer to [2, 16], see also books [3, 15]. We establish here the stronger version of this result, namely that u is an e.s. of (1), in the case of only continuous $f(u)$. Actually, we shall prove even the more general statement concerning measure valued e.s. (see the main Theorem 1 below).

Recall (see [4, 16]) that a measure valued function on Π is a weakly measurable map $(t,x) \mapsto \nu_{t,x}$ of Π into the space $\mathrm{Prob}_0(\mathbb{R})$ of probability Borel measures with compact support in $\mathbb{R}$.

The weak measurability of $\nu_{t,x}$ means that for each continuous function $g(\lambda)$ the function $(t,x) \to \langle \nu_{t,x}, g(\lambda) \rangle = \int g(\lambda) d\nu_{t,x}(\lambda)$ is measurable on Π.

We say that a measure valued function $\nu_{t,x}$ is *bounded* if there exists $R > 0$ such that $\operatorname{supp} \nu_{t,x} \subset [-R, R]$ for almost all $(t,x) \in \Pi$. We shall denote by $\|\nu_{t,x}\|_\infty$ the smallest of such R.

Finally, we say that measure valued functions of the kind $\nu_{t,x}(\lambda) = \delta(\lambda - u(t,x))$, where $u(t,x) \in L^\infty(\Pi)$ and $\delta(\lambda - u^*)$ is the Dirac measure at $u^* \in \mathbb{R}$, are *regular*. We identify these measure valued functions and the corresponding functions $u(t,x)$, so that there is a natural embedding $L^\infty(\Pi) \subset MV(\Pi)$, where $MV(\Pi)$ is the set of bounded measure valued functions on Π.

Measure valued functions naturally arise as weak limits of bounded sequences in $L^\infty(\Pi)$ in the sense of the following theorem of Tartar (see [16]).

Theorem T. *Let $u_m(t,x) \in L^\infty(\Pi)$, $m \in \mathbb{N}$ be a bounded sequence. Then there exist a subsequence $u_r(t,x)$ and a measure valued function $\nu_{t,x} \in \mathrm{MV}(\Pi)$ such that*

$$\forall g(\lambda) \in C(\mathbb{R}) \quad g(u_r) \underset{r\to\infty}{\to} \langle \nu_{t,x}, g(\lambda) \rangle \quad \text{weakly-}* \text{ in } L^\infty(\Pi). \tag{5}$$

Besides, $\nu_{t,x}$ is regular, i.e. $\nu_{t,x}(\lambda) = \delta(\lambda - u(t,x))$ if and only if $u_r(t,x) \underset{r\to\infty}{\to} u(t,x)$ in $L^1_{loc}(\Pi)$.

Now we recall the notion of a measure valued e.s. of (1) in the sense of [4].

Definition 2. A bounded measure valued function $\nu_{t,x} \in \mathrm{MV}(\Pi)$ is called *a measure valued e.s.* of (1) if $\forall k \in \mathbb{R}$

$$\frac{\partial}{\partial t}\langle \nu_{t,x}, |\lambda - k|\rangle + \frac{\partial}{\partial x}\langle \nu_{t,x}, \operatorname{sign}(\lambda - k)(f(\lambda) - f(k))\rangle \le 0 \ \text{ in } \mathcal{D}'(\Pi). \tag{6}$$

It is clear that the regular measure valued function $\nu_{t,x}(\lambda) = \delta(\lambda - u(t,x))$ is a measure valued e.s. of (1) if and only if $u(t,x)$ is an e.s. of this equation. Applying the measure valued analogue of the Kruzhkov doubling variable method, we derive the following statement

Proposition 1. *Let $\nu_{t,x}$, $\tilde{\nu}_{t,x}$ be two measure valued e.s. of (1). Then*

$$\begin{gathered}\frac{\partial}{\partial t}\langle \tilde{\nu}_{t,x}(\mu), \langle \nu_{t,x}(\lambda), |\lambda - \mu|\rangle\rangle + \\ \frac{\partial}{\partial x}\langle \tilde{\nu}_{t,x}(\mu), \langle \nu_{t,x}(\lambda), \operatorname{sign}(\lambda - \mu)(f(\lambda) - f(\mu))\rangle\rangle = \\ \frac{\partial}{\partial t}\iint |\lambda - \mu| d\nu_{t,x}(\lambda) d\tilde{\nu}_{t,x}(\mu) + \\ \frac{\partial}{\partial x}\iint \operatorname{sign}(\lambda - \mu)(f(\lambda) - f(\mu)) d\nu_{t,x}(\lambda) d\tilde{\nu}_{t,x}(\mu) \le 0 \quad \textit{in } \mathcal{D}'(\Pi).\end{gathered} \tag{7}$$

For the proof of Proposition 1 we refer to [10, 11]. In the case when one of the measure valued e.s. is regular the statement of Proposition 1 was proved earlier in [4]. Our main result is the following

Theorem 1. *Suppose that $\nu_{t,x}$ is a measure valued e.s. to equation (1) such that for a.e. $(t,x) \in \Pi$ $f(u)$ is affine on the closed convex hull $\overline{\mathrm{co}}\,\operatorname{supp}\nu_{t,x}$ of* $\operatorname{supp}\nu_{t,x}$. *Let $u(t,x) = \langle \nu_{t,x}, \lambda\rangle = \int \lambda d\nu_{t,x}(\lambda)$. Then $u = u(t,x)$ is an e.s. of the Cauchy problem (1), (3) with some initial data $u_0(x)$.*

Proof. Let $R = \|\nu_{t,x}\|_\infty$. As follows from (6) with $k = \pm R$

$$\frac{\partial}{\partial t}\langle \nu_{t,x}, \lambda\rangle + \frac{\partial}{\partial x}\langle \nu_{t,x}, f(\lambda)\rangle = 0 \ \text{ in } D'(\Pi).$$

Since $f(u)$ is affine on $\operatorname{supp}\nu_{t,x}$ for a.e. $(t,x) \in \Pi$ then $\langle \nu_{t,x}, f(\lambda)\rangle = f(\langle \nu_{t,x}, \lambda\rangle) = f(u(t,x))$ and the above relation acquires the form $u_t + f(u)_x = 0$ in $\mathcal{D}'(\Pi)$. Hence $u = u(t,x)$ is a weak solution of (1). We see that $(u, f(u))$ is a divergence-free vector field on Π. By the known results on existence of weak normal traces (see, for instance, [1]) there exists a weak trace $u_0(x) \in L^\infty(\mathbb{R})$ of $u(t,x)$ that is $\operatorname*{ess\,lim}_{t\to 0+} u(t,\cdot) = u_0$ in the weak-$*$ topology of $L^\infty(\mathbb{R})$. It is clear that $\|u_0\|_\infty \le R$. Let $v(t,x)$ be a unique e.s. to the Cauchy problem (1), (3) with initial data $u_0(x)$. By the maximum principle (see [10, 11]) $\|v\|_\infty = \|u_0\|_\infty \le R$. We shall prove that $u = v$ a.e. on Π. Observe that $(u-v)_t + (f(u) - f(v))_x = 0$ in $\mathcal{D}'(\Pi)$. Therefore, there exists a Lipschitz function $P(t,x)$ (a potential), such that $P_x = u - v$, $P_t = f(v) - f(u)$ in $\mathcal{D}'(\Pi)$. This function is extended by continuity to the closure $\bar{\Pi} = [0,+\infty) \times \mathbb{R}$ of Π. Subtracting the constant if necessary, we can assume that $P(0,0) = 0$. Let us demonstrate that $P(0,x) \equiv 0$. By the construction $P_x(t,\cdot) = (u-v)(t,\cdot) \to 0$ weakly-$*$ in $L^\infty(\mathbb{R})$ as $t \to 0$ running over some set $E \subset (0,+\infty)$ of full Lebesgue measure. On the other hand, evidently $P_x(t,\cdot) \underset{t\to 0+}{\to} P_x(0,x)$ in $\mathcal{D}'(\mathbb{R})$, and we conclude that $P_x(0,x) = 0$ in $\mathcal{D}'(\mathbb{R})$. Since $P(0,x)$ is continuous, the latter means that $P(0,x) \equiv P(0,0) = 0$, as was announced.

Now, we observe that for $u = u(t,x)$, $v = v(t,x)$

$$\langle \nu_{t,x}(\lambda), |\lambda - v|\rangle P_t + \langle \nu_{t,x}(\lambda), \operatorname{sign}(\lambda - v)(f(\lambda) - f(v))\rangle P_x =$$
$$\langle \nu_{t,x}(\lambda), \operatorname{sign}(\lambda - v)(f(\lambda) - f(v))\rangle (u-v) - \langle \nu_{t,x}(\lambda), |\lambda - v|\rangle (f(u) - f(v)) = 0 \quad (8)$$

a.e. on Π. Indeed, if $\operatorname{supp}\nu_{t,x} = \{u(t,x)\}$ then (8) reduces to the trivial identity

$$\operatorname{sign}(u-v)(f(u)-f(v))(u-v) - |u-v|(f(u)-f(v)) = 0, \quad u = u(t,x), v = v(t,x)$$

while in the case when $\overline{\operatorname{co}}\operatorname{supp}\nu_{t,x} = [a,b]$, $a < b$ we have $f(u) = \alpha u + \beta$ on $[a,b]$ with some constants $\alpha, \beta \in \mathbb{R}$. Here there are two possibilities: $v(t,x) \in [a,b]$ and $v(t,x) \notin [a,b]$. In the first case

$$\langle \nu_{t,x}(\lambda), \operatorname{sign}(\lambda - v)(f(\lambda) - f(v))\rangle = \alpha \langle \nu_{t,x}(\lambda), |\lambda - v|\rangle,$$
$$f(u) - f(v) = \alpha(u - v),$$

and (8) follows. In the second case $v(t,x) \notin [a,b]$ we have

$$\langle \nu_{t,x}(\lambda), \operatorname{sign}(\lambda - v)(f(\lambda) - f(v))\rangle = \operatorname{sign}(u-v)(\langle \nu_{t,x}, f(\lambda)\rangle - f(v)) =$$
$$\operatorname{sign}(u-v)(f(u) - f(v)), \quad \langle \nu_{t,x}(\lambda), |\lambda - v|\rangle = |u - v|,$$

and (8) is again satisfied.

Let $Q(t,x) = g(P(t,x))$ where $g(u) = u^2/(1+u^2)$. Then $Q(t,x)$ is a Lipschitz function, $0 \le Q(t,x) < 1$, $Q_t = g'(P)P_t$, $Q_x = g'(P)P_x$ in $\mathcal{D}'(\Pi)$. From (8) it follows that

$$\langle \nu_{t,x}(\lambda), |\lambda - v|\rangle Q_t + \langle \nu_{t,x}(\lambda), \mathrm{sign}(\lambda - v)(f(\lambda) - f(v))\rangle Q_x = 0 \tag{9}$$

a.e. on Π. In view of (9) and inequality (7) with $\tilde{\nu}_{t,x}(\mu) = \delta(\mu - v(t,x))$

$$\frac{\partial}{\partial t}(\langle \nu_{t,x}(\lambda), |\lambda - v|\rangle Q + \varepsilon) + \frac{\partial}{\partial x}\langle \nu_{t,x}(\lambda), \mathrm{sign}(\lambda - v)(f(\lambda) - f(v))\rangle Q \le 0 \tag{10}$$

in $\mathcal{D}'(\Pi)$ for all $\varepsilon > 0$. Let

$$\rho(\delta) = \sup\{\ |f(u) - f(v)|\ \mid u, v \in [-R, R], |u - v| \le \delta\ \}$$

be the modulus of continuity of $f(u)$ on the segment $[-R, R]$. Then $\rho(\delta)$ is a non-decreasing sub-additive function on $[0, +\infty)$ such that $0 = \rho(0) = \lim\limits_{\delta \to 0+} \rho(\delta)$, and $|f(u) - f(v)| \le \rho(|u - v|)$ for all $u, v \in [-R, R]$. Observe that for each positive ε

$$\frac{\rho(\delta)}{\delta + \varepsilon} \le \frac{\rho(\varepsilon)}{\varepsilon} \quad \forall \delta \ge 0. \tag{11}$$

Indeed, we can choose $k \in \mathbb{N}$ such that $\delta \in [(k-1)\varepsilon, k\varepsilon)$. Then, since $\rho(\delta)$ is non-decreasing and sub-additive, $\rho(\delta) \le \rho(k\varepsilon) \le k\rho(\varepsilon)$ while $\delta + \varepsilon \ge k\varepsilon$, and (11) follows.

We denote $N(\varepsilon) = \rho(\varepsilon)/\varepsilon$ and set $g = g(t,x) = \theta(R + N(\varepsilon)(T - t) - |x|)$, where $\theta(s) \in C^1(\mathbb{R})$, $\theta'(s) \ge 0$, $\theta(s) = 0$ for $s \le 0$, $\theta(s) = 1$ for $s \ge 1$; $R > 1$, $T > 0$. Observe that $g = g(t,x) \in C^1((0,T) \times \mathbb{R})$, $g \ge 0$ and $g_t = -N(\varepsilon)|g_x|$. Therefore, for a.e. $(t,x) \in (0,T) \times \mathbb{R}$

$$\begin{aligned}(\langle \nu_{t,x}(\lambda), |\lambda - v|\rangle Q + \varepsilon) g_t + \langle \nu_{t,x}(\lambda), \mathrm{sign}(\lambda - v)(f(\lambda) - f(v))\rangle Q g_x \le \\ \{\langle \nu_{t,x}(\lambda), |f(\lambda) - f(v)|\rangle - N(\varepsilon)\langle \nu_{t,x}(\lambda), |\lambda - v| + \varepsilon\rangle\} Q |g_x| \le 0,\end{aligned} \tag{12}$$

because $Q < 1$ and, in view of (11), for a.e. $(t,x) \in \Pi$ such that $\operatorname{supp} \nu_{t,x} \subset [-R, R]$, $v = v(t,x) \in [-R, R]$

$$\langle \nu_{t,x}(\lambda), |f(\lambda) - f(v)|\rangle \le \langle \nu_{t,x}(\lambda), \rho(|\lambda - v|)\rangle \le N(\varepsilon)\langle \nu_{t,x}(\lambda), |\lambda - v| + \varepsilon\rangle.$$

Let $\gamma(t) \in C_0^1((0,T))$, $\gamma(t) \ge 0$. Applying (10) to the test function $h = g(t,x)\gamma(t)$ and taking into account (12), we obtain that

$$\int_{\Pi} (\langle \nu_{t,x}(\lambda), |\lambda - v|\rangle Q + \varepsilon) g(t,x) \gamma'(t) dt dx \ge 0.$$

Since $\gamma(t)$ is an arbitrary smooth non-negative function from $C_0^1((0,T))$ this inequality means that

$$\frac{\partial}{\partial t}\int_{\mathbb{R}}(\langle\nu_{t,x}(\lambda),|\lambda-v(t,x)|\rangle Q(t,x)+\varepsilon)g(t,x)dx\le 0 \ \text{ in } \mathcal{D}'((0,T)).$$

This readily implies that for a.e. $t\in(0,T)$

$$\int_R\langle\nu_{t,x}(\lambda),|\lambda-v(t,x)|\rangle Q(t,x)\theta(R-|x|)dx\le$$
$$\int_{\mathbb{R}}(\langle\nu_{t,x}(\lambda),|\lambda-v(t,x)|\rangle Q(t,x)+\varepsilon)\theta(R+N(\varepsilon)(T-t)-|x|)dx\le$$
$$\operatorname*{ess\,lim}_{t\to0+}\int_{\mathbb{R}}(\langle\nu_{t,x}(\lambda),|\lambda-v(t,x)|\rangle Q(t,x)+\varepsilon)\theta(R+N(\varepsilon)(T-t)-|x|)dx=$$
$$\varepsilon\int_R\theta(R+N(\varepsilon)T-|x|)dx\le 2\varepsilon(R+N(\varepsilon)T)=2R\varepsilon+2T\rho(\varepsilon),$$

where we use the identity $Q(0,x)\equiv0$. This implies that

$$\int_{(0,T)\times\mathbb{R}}\langle\nu_{t,x}(\lambda),|\lambda-v|\rangle Q(t,x)\theta(R-|x|)dtdx\le 2T(R\varepsilon+T\rho(\varepsilon))\underset{\varepsilon\to0}{\to}0.$$

Therefore, in the limit as $\varepsilon\to0$ we derive the identity

$$\int_{(0,T)\times\mathbb{R}}\langle\nu_{t,x}(\lambda),|\lambda-v(t,x)|\rangle Q(t,x)\theta(R-|x|)dxdt=0.$$

Since $R>1$, $T>0$ are arbitrary and the integrand is nonnegative we obtain that

$$\langle\nu_{t,x}(\lambda),|\lambda-v(t,x)|\rangle Q(t,x)=0 \ \text{ a.e. on } \Pi. \tag{13}$$

By (13) we see that $\langle\nu_{t,x}(u),|u-v(t,x)|\rangle=0$ a.e. on the open set $\{P(t,x)\neq0\}$. This implies that $v(t,x)=u(t,x)=\langle\nu_{t,x}(u),u\rangle$ a.e. on this set. On the other hand, almost everywhere on the set $\{P(t,x)=0\}$ we have $u(t,x)-v(t,x)=P_x(t,x)=0$, by known properties of Lipschitz functions (see, for example, [5]). Thus, $u(t,x)=v(t,x)$ a.e. on Π, and u is an e.s. of (1), (3). The proof is complete.

Corollary 1. *Any e.s. $u(t,x)$ of (1) admits a strong trace at $t=0$ in the sense of relation (4).*

Proof. Applying Theorem 1 to a regular measure valued e.s. $u(t,x)$, we derive that u must be an e.s. to the Cauchy problem (1), (3) with initial function

$u_0(x)$ being a weak trace of u. By condition (4), we conclude that this trace is actually strong. The proof is complete.

Remark that the general results on the existence of the strong normal traces for solutions of multidimensional conservation laws was established in recent papers [13, 14].

As another consequence of Theorem 1, we obtain the already announced statement that weak limits of a sequence $u_r(t,x)$ of entropy solutions to (1) are actually e.s. of this equation. For the proof, we establish that the limit measure valued function of the sequence $u_r(t,x)$ satisfies the assumptions of Theorem 1. In the case $f(u) \in C^1(\mathbb{R})$ this is well-known, see [2, 16], one could find the proof also in books [3, 15]. For the extension to the case $f(u) \in C(\mathbb{R})$ we need the following simple technical result.

Lemma 1. *Assume that ν is a finite non-negative Borel measure on $\mathbb{R}$, $[a,b] = \overline{\mathrm{co}}\,\mathrm{supp}\,\nu$, $\bar{u} \in [a,b]$; $H(u) \in C(\mathbb{R})$ and for each $k \in (\bar{u}, b)$*

$$\int (H(\lambda) - H(k))\,\mathrm{sign}^+(\lambda - k)d\nu(\lambda) = 0, \tag{14}$$

where $\mathrm{sign}^+(\lambda) = (1 + \mathrm{sign}\,\lambda)/2$ is the Heaviside function. Then $H(u) = \mathrm{const}$ on $[\bar{u}, b]$.

Proof. Assuming the contrary, we can find a point $c \in (\bar{u}, b)$ such that $H(u)$ is not constant on any interval $(c,d) \subset (a,b)$ (otherwise, $H(u)$ takes at most countable set of values on $[\bar{u}, b]$ and therefore must be constant on this segment). Hence, there exists a sequence c_r, $r \in \mathbb{N}$ such that $c < c_{r+1} < c_r < b$ $\forall r \in \mathbb{N}$, $c_r \to c$ as $r \to \infty$, and

$$|H(c_r) - H(c)| = \max_{u \in [c, c_r]} |H(u) - H(c)| > 0. \tag{15}$$

Denote $H_k(u) = (H(u) - H(k))\,\mathrm{sign}^+(u - k)$, $k \in \mathbb{R}$ and set for $h_r = H(c_r) - H(c)$

$$\psi_r(u) = (H_c(u) - H_{c_r}(u))/h_r = \begin{cases} 0 & , \quad u \le c, \\ (H(u) - H(c))/h_r & , \quad c < u \le c_r, \\ 1 & , \quad u > c_r. \end{cases} \tag{16}$$

By (14) we have

$$\int \psi_r(\lambda) d\nu(\lambda) = 0 \quad \forall r \in \mathbb{N}. \tag{17}$$

As follows from (15), $|\psi_r(\lambda)| \le 1$, and obviously the sequence $\psi_r(\lambda)$ converges point-wise to the Heaviside function $\psi(\lambda) = \operatorname{sign}^+(\lambda - c)$. By the Lebesgue dominated convergence theorem, we can pass to the limit in (17) as $r \to \infty$ and conclude that $\nu((c,b]) = \int \operatorname{sign}^+(\lambda - c) d\nu(\lambda) = 0$. But this contradicts to the fact that $[a,b]$ is the minimal segment containing $\operatorname{supp}\nu$ and therefore $\nu((c,b]) > 0$. The proof is complete.

Corollary 2. *Suppose that the following identity*

$$\int (H(\lambda) - H(k)) \operatorname{sign}^-(\lambda - k) d\nu(\lambda) = 0 \quad \forall k \in (a, \bar{u}) \tag{18}$$

is satisfied instead of (14), where $\operatorname{sign}^-(\lambda) = -\operatorname{sign}^+(-\lambda)$. *Then* $H(u) = \mathrm{const}$ *on* $[a, \bar{u}]$.

Proof. Making the change $\lambda \to -\lambda$, $k \to -k$ in (18), we obtain

$$\int (H(-\lambda) - H(-k)) \operatorname{sign}^+(\lambda - k) d\tilde{\nu}(\lambda) = 0 \quad \forall k \in (-\bar{u}, -a)$$

where $\tilde{\nu}$ is the image of ν under the map $\lambda \to -\lambda$. It is clear that $\overline{\mathrm{co}} \operatorname{supp} \tilde{\nu} = [-b, -a]$ and the above relation coincides with (14) applied to the function $H(-u)$. By Lemma 1 we conclude that this function must be constant on $[-\bar{u}, -a]$, which is equivalent to our statement $H(u) = \mathrm{const}$ on $[a, \bar{u}]$.

Now we are ready to prove our second main theorem.

Theorem 2. *Suppose that a bounded sequence* $u_r = u_r(t,x)$, $r \in \mathbb{N}$ *of e.s. of (1) converges as* $r \to \infty$ *to a function* $u = u(t,x)$ *weakly-$*$ in* $L^\infty(\Pi)$. *Then* u *is an e.s. of (1).*

Proof. Extracting a subsequence if necessary we can assume that u_r converges as $r \to \infty$ to a measure valued function $\nu_{t,x} \in \mathrm{MV}(\Pi)$ in the sense of relation (5). Since u_r is an e.s. of (1) then for each $k \in \mathbb{R}$

$$\frac{\partial}{\partial t}|u_r - k| + \frac{\partial}{\partial x}[\operatorname{sign}(u_r - k)(f(u_r) - f(k))] \le 0 \ \text{ in } \mathcal{D}'(\Pi). \tag{19}$$

By (5) as $r \to \infty$

$$|u_r - k| \to \langle \nu_{t,x}(\lambda), |\lambda - k| \rangle, \quad \operatorname{sign}(u_r - k)(f(u_r) - f(k)) \to \\ \langle \nu_{t,x}(\lambda), \operatorname{sign}(\lambda - k)(f(\lambda) - f(k)) \rangle \ \text{ weakly-}* \text{ in } L^\infty(\Pi), \tag{20}$$

$$u_r \to \langle \nu_{t,x}, \lambda \rangle \ \text{ weakly-}* \text{ in } L^\infty(\Pi). \tag{21}$$

From (21) it follows that $u(t,x) = \langle \nu_{t,x}, \lambda \rangle$. Further, in view of (20) relations (19) in the limit as $r \to \infty$ yield (6). Therefore, $\nu_{t,x}$ is a measure valued e.s. of (1). We denote

$$\eta_k^+(u) = (u-k)^+ = \max(u-k,0),\ \psi_k^+(u) = (f(u)-f(k))\operatorname{sign}^+(u-k);$$
$$\eta_k^-(u) = (u-k)^- = \max(k-u,0),\ \psi_k^-(u) = (f(u)-f(k))\operatorname{sign}^-(u-k),$$

where the functions $\operatorname{sign}^{\pm}(u)$ were defined above in the formulations of Lemma 1 and Corollary 2. As we know, u_r are weak solutions of (1) and this yields

$$\frac{\partial}{\partial t}(u_r - k) + \frac{\partial}{\partial x}(f(u_r) - f(k)) = 0 \ \text{ in } \mathcal{D}'(\Pi) \tag{22}$$

for all $k \in \mathbb{R}$. Putting (19) together with (22) multiplied by ± 1, we derive that for each $k \in \mathbb{R}$

$$\frac{\partial}{\partial t}\eta_k^{\pm}(u_r) + \frac{\partial}{\partial x}\psi_k^{\pm}(u_r) \le 0 \ \text{ in } \mathcal{D}'(\Pi). \tag{23}$$

Remark that by our assumption the sequence $u_r = u_r(t,x)$ is bounded in $L^\infty(\Pi)$. Then, as was demonstrated in [16], from (23) it follows that the sequences of distributions

$$(l_k^{\pm})_r = \frac{\partial}{\partial t}\eta_k^{\pm}(u_r) + \frac{\partial}{\partial x}\psi_k^{\pm}(u_r)$$

are pre-compact in $H_{loc}^{-1}(\Pi)$, which is a locally convex space of distributions $l = l(t,x)$ such that lh belongs to the Sobolev space H_2^{-1} for all $h = h(t,x) \in C_0^\infty(\Pi)$. The topology in $H_{loc}^{-1}(\Pi)$ is generated by the family of semi-norms $l \to \|lh\|_{H_2^{-1}}$, $h(t,x) \in C_0^\infty(\Pi)$. We see also that the sequences

$$\frac{\partial}{\partial t}|u_r - k| + \frac{\partial}{\partial x}[\operatorname{sign}(u_r - k)(f(u_r) - f(k))] = (l_k^-)_r + (l_k^+)_r$$

are pre-compact in $H_{loc}^{-1}(\Pi)$. By Tartar-Murat commutation relations (see [9, 16]) there exists a set $E \subset \Pi$ of full measure such that for all $(t,x) \in E$

$$\begin{gathered}\langle \nu_{t,x}, p_1(\lambda)q_2(\lambda) - p_2(\lambda)q_1(\lambda)\rangle = \\ \langle \nu_{t,x}, p_1(\lambda)\rangle\langle \nu_{t,x}, q_2(\lambda)\rangle - \langle \nu_{t,x}, p_2(\lambda)\rangle\langle \nu_{t,x}, q_1(\lambda)\rangle\end{gathered} \tag{24}$$

for every pairs of continuous functions (p_1,q_1), (p_2,q_2) such that the sequences $(p_i(u_r))_t + (q_i(u_r))_x$, $i = 1,2$ are pre-compact in $H_{loc}^{-1}(\Pi)$. We fix $(t,x) \in E$ and denote $\nu = \nu_{t,x}$, $\bar{u} = u(t,x) = \langle \nu, \lambda \rangle$. We are going to show that $f(u)$ is linear on $[a,b] = \overline{\operatorname{co}}\operatorname{supp}\nu$. If $a = b$ then there is nothing to prove. So, we assume that $a < b$. Applying (24) to $p_1(\lambda) = \lambda - k$, $q_1(\lambda) = f(\lambda) - f(k)$; $p_2(\lambda) = |\lambda - k|$, $q_2(\lambda) = (f(\lambda) - f(k)) \operatorname{sign}(\lambda - k)$ an taking into account that $p_1(\lambda)q_2(\lambda) - p_2(\lambda)q_1(\lambda) \equiv 0$, we obtain that $\forall k \in \mathbb{R}$

$$\langle \nu, \lambda - k \rangle \langle \nu, (f(\lambda) - f(k)) \operatorname{sign}(\lambda - k) \rangle = \langle \nu, |\lambda - k| \rangle \langle \nu, f(\lambda) - f(k) \rangle. \tag{25}$$

We take in (25) $k = \bar{u}$. Then $\langle \nu, \lambda - k \rangle = 0$ and therefore $\langle \nu, |\lambda - \bar{u}| \rangle \langle \nu, f(\lambda) - f(\bar{u}) \rangle = 0$. Since $a < b$ (i.e. ν is not a Dirac measure) then $\langle \nu, |\lambda - \bar{u}| \rangle > 0$ and from the above equality it follows that $\langle \nu, f(\lambda) \rangle = f(\bar{u})$. Applying (24) to the pairs $p_1(\lambda) = \eta^+_{\bar{u}}(\lambda)$, $q_1(\lambda) = \psi^+_{\bar{u}}(\lambda)$; $p_2(\lambda) = \eta^+_k(\lambda)$, $q_2(\lambda) = \psi^+_k(\lambda)$ with $k > \bar{u}$, we obtain

$$\langle \nu, \eta^+_{\bar{u}}(\lambda) \rangle \langle \nu, \psi^+_k(u) \rangle - \langle \nu, \eta^+_k(\lambda) \rangle \langle \nu, \psi^+_{\bar{u}}(\lambda) \rangle = \langle \nu, Q(\lambda) \rangle, \tag{26}$$

where

$$\begin{aligned} Q(\lambda) = \eta^+_{\bar{u}}(\lambda)\psi^+_k(\lambda) - \eta^+_k(\lambda)\psi^+_{\bar{u}}(\lambda) = \\ \operatorname{sign}^+(\lambda - \bar{u}) \operatorname{sign}^+(\lambda - k)[(\lambda - \bar{u})(f(\lambda) - f(k)) - (\lambda - k)(f(\lambda) - f(\bar{u}))] = \\ \operatorname{sign}^+(\lambda - k)[(\lambda - \bar{u})(f(\lambda) - f(k)) - (\lambda - k)(f(\lambda) - f(\bar{u}))] = \\ (\lambda - \bar{u})\psi^+_k(\lambda) - \eta^+_k(\lambda)(f(\lambda) - f(\bar{u})), \end{aligned}$$

where we use the condition $k > \bar{u}$. By (24) again, we find that

$$\begin{aligned} \langle \nu, Q(\lambda) \rangle = \langle \nu, (\lambda - \bar{u})\psi^+_k(\lambda) - \eta^+_k(\lambda)(f(\lambda) - f(\bar{u})) \rangle = \\ \langle \nu, \lambda - \bar{u} \rangle \langle \nu, \psi^+_k(\lambda) \rangle - \langle \nu, \eta^+_k(\lambda) \rangle \langle \nu, f(\lambda) - f(\bar{u}) \rangle = 0 \end{aligned}$$

because $\langle \nu, \lambda - \bar{u} \rangle = \langle \nu, f(\lambda) - f(\bar{u}) \rangle = 0$. Then from (26) it follows that

$$\langle \nu, \eta^+_{\bar{u}}(\lambda) \rangle \langle \nu, \psi^+_k(\lambda) \rangle = \langle \nu, \eta^+_k(\lambda) \rangle \langle \nu, \psi^+_{\bar{u}}(\lambda) \rangle.$$

This implies that $\forall k \in \mathbb{R}$, $k > \bar{u}$

$$\langle \nu, \psi^+_k(\lambda) - c\eta^+_k(\lambda) \rangle = \langle \nu, \psi^+_k(\lambda) \rangle - c\langle \nu, \eta^+_k(\lambda) \rangle = 0 \tag{27}$$

with $c = \langle \nu(u), \psi^+_{\bar{u}}(u) \rangle / \langle \nu(u), \eta^+_{\bar{u}}(\lambda) \rangle$ (observe that $\langle \nu, \eta^+_{\bar{u}}(\lambda) \rangle > 0$). Denote $H(u) = f(u) - cu$. Then (27) acquires the form

$$\int (H(\lambda) - H(k)) \operatorname{sign}^+(\lambda - k) d\nu(\lambda) = 0 \quad \forall k > \bar{u}$$

and by Lemma 1 $H(u) \equiv d_1 = \text{const}$ on $[\bar{u}, b]$. Analogously, from (24) with $p_1(\lambda) = \eta_{\bar{u}}^-(\lambda)$, $q_1(\lambda) = \psi_{\bar{u}}^-(\lambda)$; $p_2(\lambda) = \eta_k^-(\lambda)$, $q_2(\lambda) = \psi_k^-(\lambda)$, $k < \bar{u}$ it follows the relation similar to (27)

$$\langle \nu, \psi_k^-(\lambda) - c\eta_k^-(\lambda) \rangle = \langle \nu, \psi_k^-(\lambda) \rangle - c\langle \nu, \eta_k^-(\lambda) \rangle = 0, \qquad (28)$$

where the constant $c = \langle \nu, \psi_{\bar{u}}^-(\lambda) \rangle / \langle \nu, \eta_{\bar{u}}^-(\lambda) \rangle$ coincides with one in (27) since $\langle \nu, \psi_{\bar{u}}^-(\lambda) \rangle = \langle \nu, \psi_{\bar{u}}^+(\lambda) \rangle$, $\langle \nu, \eta_{\bar{u}}^-(\lambda) \rangle = \langle \nu, \eta_{\bar{u}}^+(\lambda) \rangle$. Indeed,

$$\langle \nu, \psi_{\bar{u}}^+(\lambda) \rangle - \langle \nu, \psi_{\bar{u}}^-(\lambda) \rangle = \langle \nu, \psi_{\bar{u}}^+(\lambda) - \psi_{\bar{u}}^-(\lambda) \rangle = \langle \nu, f(\lambda) - f(\bar{u}) \rangle = 0,$$
$$\langle \nu, \eta_{\bar{u}}^+(\lambda) \rangle - \langle \nu, \eta_{\bar{u}}^-(\lambda) \rangle = \langle \nu, \eta_{\bar{u}}^+(\lambda) - \eta_{\bar{u}}^-(\lambda) \rangle = \langle \nu, \lambda - \bar{u} \rangle = 0.$$

Rewriting (28) in the form $\int (H(\lambda) - H(k)) \operatorname{sign}^-(\lambda - k) d\nu(\lambda) = 0$ for each $k < \bar{u}$ and applying Corollary 2 we derive that $H(u) \equiv d_2 = \text{const}$ on $[a, \bar{u}]$. By continuity of $H(u)$, $d_1 = d_2 = d$ and we conclude that $f(u) - cu = d$ on $[a, b]$. Thus, $f(u)$ is affine on $\overline{\text{co}}\,\operatorname{supp} \nu_{t,x}$ for a.e. $(t, x) \in \Pi$, as was announced.

We see that $\nu_{t,x}$ satisfies the assumptions of Theorem 1. By this Theorem $u(t, x) = \langle \nu_{t,x}, \lambda \rangle$ is an e.s. of (1), as was to be proved.

Corollary 3. *Suppose that the function $f(u)$ is not affine on non-degenerate intervals. Then the sequence $u_r \underset{r\to\infty}{\to} u$ in $L^1_{loc}(\Pi)$.*

Proof. Let $\nu_{t,x}$ be the limit measure valued function, which corresponds to some subsequence of u_r. As was demonstrated in the proof of Theorem 2, $f(u)$ must be affine on $\overline{\text{co}}\,\operatorname{supp} \nu_{t,x}$ for a.e. $(t, x) \in \Pi$, and by our assumption we conclude that $\overline{\text{co}}\,\operatorname{supp} \nu_{t,x} = \{u(t, x)\}$. Hence, $\nu_{t,x} = \delta(\lambda - u(t, x))$ and by Theorem T $u_r \underset{r\to\infty}{\to} u$ in $L^1_{loc}(\Pi)$. Finally, since the limit function u does not depend on the choice of a subsequence, this limit relation holds for the original sequence.

As was shown in [12], the strong pre-compactness property for sequences of e.s. remains valid for multidimensional equations $u_t + \operatorname{div}_x f(u) = 0$, $f(u) \in C(\mathbb{R}, \mathbb{R}^n)$ under the condition that linear combinations $\xi \cdot f(u)$, $\xi \in \mathbb{R}^n$, $\xi \neq 0$ of flux functions are not affine on non-degenerate intervals.

Remark 1. The statements of Theorems 1,2 remain valid (with the same proofs) for the more general equation

$$u_t + f(t, x, u)_x = 0, \ (t, x) \in \Pi_T = (0, T) \times \mathbb{R}.$$

It is sufficient here to require that $f(t, x, u)$ satisfies the Kruzhkov's assumptions [6].

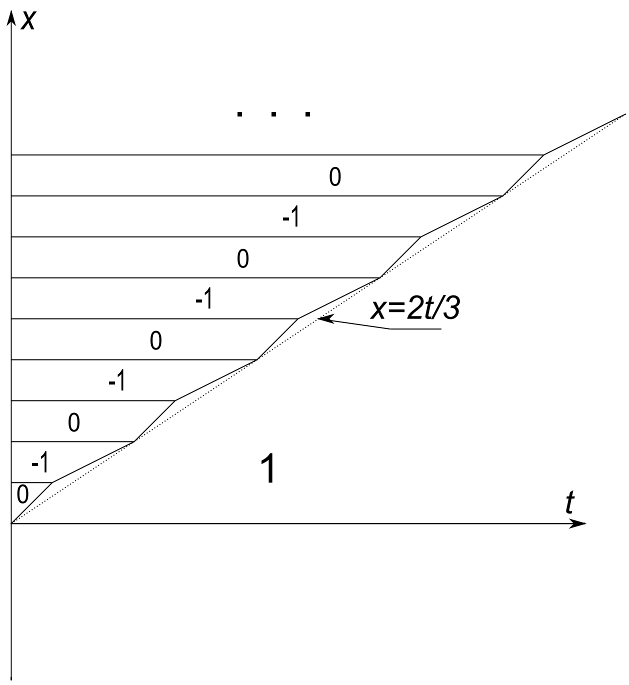

Figure 1:

It is clear that the assertion of Theorem 2 is fulfilled also for a bounded sequence u_r of approximate solutions of (1). We only need that for each $k \in \mathbb{R}$

$$\frac{\partial}{\partial t}|u_r - k| + \frac{\partial}{\partial x}[\mathrm{sign}(u_r - k)(f(u_r) - f(k))] \underset{r\to\infty}{\to} l_k \text{ in } H^{-1}_{loc}(\Pi),$$

where the distributions $l_k \le 0$ in $\mathcal{D}'(\Pi)$.

Remark 2. Let us consider the more general equation

$$g(u)_t + f(u)_x = 0, \tag{29}$$

where $f(u), g(u) \in C(\mathbb{R})$ and $g(u)$ is an increasing invertible function. E.s. $u = u(t,x)$ of (29) is defined by the relation similar to (2): $\forall k \in \mathbb{R}$

$$\frac{\partial}{\partial t}|g(u) - g(k)| + \frac{\partial}{\partial x}[\mathrm{sign}(u - k)(f(u) - f(k))] \le 0 \ \text{ in } \mathcal{D}'(\Pi).$$

For equation (29) Theorems 1,2 are not generally true. Moreover, it may occur that a weak limit of a sequence of e.s. of (29) is not even a weak solution of this equation. We confirm this fact by the following simple example. Let us choose $g(u) = u^3$, $f(u) = \max(0, u^3)$ and define $u(t,x) = \begin{cases} w(x) & , \quad x > x(t), \\ 1 & , \quad x \le x(t) \end{cases}$ (see Fig. 1) where $w(x) = ((-1)^{[x]} - 1)/2$ ($[x]$ denotes the integer part of x), and $x(t)$ is a piece-wise affine function on $\mathbb{R}_+$ such that $x(0) = 0$ and $x'(t) = \begin{cases} 1 & , \quad t \in [3k, 3k+1), \\ 1/2 & , \quad t \in [3k+1, 3k+3), \end{cases}$ $k = 0, 1, \cdots$ so that on the line $x = x(t)$ the Rankine-Hugoniot condition $x'(t) = (f(w(x)) - f(1))/(g(w(x)) - g(1))$ is satisfied. Since $f(g^{-1}(v)) = \max(0, v)$ is a convex function and $1 > w(x)$, the shock line $x = x(t)$ is admissible and $u(t,x)$ is an e.s. of (29). Consider the sequence $u_r = u(rt, rx)$, $r \in \mathbb{N}$. Evidently this sequence consists of e.s. of (29) and converges to $\bar{u}(t,x) = \begin{cases} -1/2 & , \quad x > 2t/3, \\ 1 & , \quad x \le 2t/3 \end{cases}$ weakly-$*$ in $L^\infty(\Pi)$. As is easy to verify, on the line $x = 2t/3$ the Rankine-Hugoniot relation is violated. Indeed, $3(f(1) - f(-1/2)) - 2(g(1) - g(-1/2)) = 3/4 \neq 0$. Therefore, $\bar{u}(t,x)$ is not even a weak solution of (29).

Remark that $u = u(t,x)$ is an e.s. of (29) if and only if $v = g(u)$ is an e.s. of the equation $v_t + f(g^{-1}(v))_x = 0$. Using this observation, we can revise the assertion of Theorem 2 for equation (29):

Theorem 2'. *Suppose that $u_r(t,x)$ is a bounded sequence of e.s. of (29) such that $v_r = g(u_r(t,x))$ converges to a function $v(t,x)$ weakly-$*$ in $L^\infty(\Pi)$. Then $u = g^{-1}(v(t,x))$ is an e.s. of (29). Moreover, u satisfies initial condition (3) with some function $u_0(x)$ in the sense of relation (4).*

Remark 3. Certainly, our results are purely one-dimensional. For instance, the statement of Theorem 2 is not true for multidimensional equations even with only one non-linear flux component. Indeed, if $f(u)$ is not affine then we can always find e.s. $u_1 = u_1(t,x)$, $u_2 = u_2(t,x)$ of (1) such that $u = (u_1 + u_2)/2$ is not a weak solution of (1). Consider the sequence $u_r(t,x,y) = (1 - \alpha(ry))u_1(t,x) + \alpha(ry)u_2(t,x)$, $r \in \mathbb{N}$, where $\alpha(y) = (1 + (-1)^{[y]})/2$. Obviously, this sequence consists of e.s. of equation

$$u_t + f(u)_x = 0, \quad u = u(t,x,y) \tag{30}$$

in the half-space $t > 0$, $(x,y) \in \mathbb{R}^2$. As is easy to see, $u_r(t,x,y)$ converges to $\bar{u}(t,x,y) = u(t,x) = (u_1 + u_2)/2$ weakly-$*$ in L^∞ while $u(t,x)$ is not even a weak solution of (30).

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