

A mathematical model for Tsunami generation using a conservative velocity-pressure hyperbolic system

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May 26, 2008

Abstract

By using the Hugoniot curve in detonics as a Riemann invariant of a velocity-pressure model, we get a conservative hyperbolic system similar to the Euler equations. The only differences are the larger value of the adiabatic constant ($\gamma = 8.678$ instead of 1.4 for gas dynamics) and the mass density replaced by a strain density depending on the pressure. The model is not homogeneous since it involves a gravity and a friction term. After the seismic wave reaches up the bottom of the ocean, one gets a pressure wave propagating toward the surface, which is made of a frontal shock wave followed by a regular decreasing profile. Since this regular profile propagates faster than the frontal shock waves, the amplitude of the pressure wave is strongly reduced when reaching the surface. Only in the case of a strong earth tremor the residual pressure wave is still sufficient to generate a water elevation with a sufficient wavelengths enable to propagate as a Saint Venant water wave and to become a tsunami when reaching the shore. We describe the construction of the model and the computation of the wave profile and discuss about the formation or not of a wave.

We propose a model using a constant mass density because variable mass density models are often unstable, since a tiny variation of the density allways causes a large variation of the pressure. In the model presented here, the transported variable is not the mass density but a new variable, called the strain density, which has the same properties of conservation without this drawback of numerical instability.

The first section deals with the construction of the velocity-pressure model and the new conservative variable of strain density is designed in the second section. The new non homogeneous Euler-like model is then studied in the third section and one dimension numerical computations of the profile of the wave are reported in the fourth section, for different values of the friction coefficient. We conclude by some discussions about the emergence of a tsunami wave or not.

1 The velocity-pressure model

The Hugoniot curves (see [6]) correspond to the linkage between the velocity w and the pressure p in a shock wave travelling through a material after an impact. By starting from a position at rest,

the algebraic equation of such curves has the form

$$p = \rho_0 (c_0 w + S_0 w^2) ,$$

where ρ_0 is the mass density of the material, c_0 is its speed of sound, and S_0 is a dimensionless constant related to this material, obtained from experiments. For water, $\rho_0 = 1000 \text{ kg m}^{-3}$, $c_0 = 1647 \text{ m s}^{-1}$ and $S_0 = 1.921$. By solving with respect to w , we get

$$w(p) = \pm \alpha_0 \left(\sqrt{1 + \beta_0 p} - 1 \right) ,$$

where $\alpha_0 = \frac{c_0}{2 S_0} = 429 \text{ m s}^{-1}$ and $\beta_0 = \frac{4 S_0}{\rho_0 c_0^2} = 2.84 \cdot 10^{-9}$ whose unit is the inverse of an energy. We notice the relation $\alpha_0 \beta_0 \rho_0 c_0 = 2$. The values of these parameters come from [5].

A general one dimension velocity-pressure model, with a constant mass density ρ_0 , is made of a dynamics equation of the form

$$\rho_0 (w_t + w w_z) + p_z = 0 \tag{1.1}$$

where t is the time and z is the position along a vertical upwards oriented axis (with $z = 0$ at sea level), and a Hooke law of the form

$$p_t + w p_z + \rho_0 c(p)^2 w_z = 0 \tag{1.2}$$

where $c(p) > 0$ stands for the pressure depending wave velocity, to be identified. These two equations compose a hyperbolic system whose Riemann invariants have the form

$$w'(p) = \pm \frac{1}{\rho_0 c(p)} .$$

By derivating the expression of $w(p)$ from the Hugoniot curves, one gets

$$w'(p) = \pm \frac{\alpha_0 \beta_0}{2 \sqrt{1 + \beta_0 p}} .$$

Identifying the two expressions and using $\alpha_0 \beta_0 \rho_0 c_0 = 2$ lead to the formula

$$c(p) = c_0 \sqrt{1 + \beta_0 p} .$$

This reads like a state law for our velocity pressure model.

Up to now we were only concerned with homogeneous equations since Riemann invariants only exists in this case. We also have to take in account the gravity effects, since the pressure is increasing with the depth of water, and some friction effect since stillness is a stable configuration. We use a friction term of the Strickler type as usual in hydraulics, but other choices are possible and will lead to similar results. That way, the dynamics equation is replaced by

$$\rho_0 (w_t + w w_z) + p_z + \rho_0 g + k |w| w = 0 , \tag{1.3}$$

where g is the gravity constant and k the friction parameter. The Hooke law and the state law are unchanged, and we take $g = 9.8 \text{ m s}^{-2}$ in the numerical experiments. The size of the friction parameter is a priori unknown and will be discussed later.

2 The conservative strain density

We look for a quantity $q = q(p)$ satisfying the transport equation

$$q_t + (qw)_z = 0 . \quad (2.1)$$

Since q depends on p we get

$$q'(p) (p_t + w p_z) + q(p) w_z = 0 ,$$

to be compared with the Hooke law. We get

$$\frac{q'(p)}{q(p)} = \frac{1}{\rho_0 c(p)^2} = \frac{1}{\rho_0 c_0^2 (1 + \beta_0 p)} ,$$

which is easily solved and gives

$$q(p) = (1 + \beta_0 p)^{\frac{1}{\beta_0 \rho_0 c_0^2}} = (1 + \beta_0 p)^{\frac{\alpha_0}{2 c_0}} ,$$

for the choice $q(0) = 1$. The wave velocity can be written as a function of the strain density q as

$$c(q) = c_0 q^{\frac{c_0}{\alpha_0}} .$$

We check now the conservation of the momentum $m = qw$. We get

$$m_t + (mw)_z + \frac{q}{\rho_0} p_z + gq + \frac{k}{\rho_0} q |w| w = 0 ,$$

where

$$\frac{q}{\rho_0} p_z = \frac{2 c_0}{\alpha_0 \beta_0 \rho_0} q^{2 \frac{c_0}{\alpha_0}} q_z = c_0^2 q^{2 \frac{c_0}{\alpha_0}} q_z = c(q)^2 q_z .$$

We introduce a pressure term, standing as a strain pressure,

$$P(q) = c_0^2 \frac{q^{2 \frac{c_0}{\alpha_0} + 1}}{2 \frac{c_0}{\alpha_0} + 1} = \frac{c_0^2}{\gamma_0} q^{\gamma_0} , \quad \text{with } \gamma_0 = 2 \frac{c_0}{\alpha_0} + 1 (= 8.678)$$

and we get the conservative equation for the momentum

$$m_t + (mw + P(q))_z + gq + \frac{k}{\rho_0} q |w| w = 0 . \quad (2.2)$$

The system made of (2.1) (2.2) has the form of the well known Euler equations for gas dynamics with a larger adiabatic coefficient $\gamma_0 = 8.678$ instead of the usual value 1.4 for gases, and can be handled in the same way, especially for the shock waves. The same Rankine Hugoniot condition is valid, connecting the velocity of a shock wave to the two states (q_1, w_1) and (q_2, w_2) by

$$z'(t) = \frac{w_1 + w_2}{2} + \frac{q_1 + q_2}{2} \sqrt{\frac{1}{q_1 q_2} \frac{P(q_2) - P(q_1)}{q_2 - q_1}} . \quad (2.3)$$

3 The profile of the strain wave

We first compute the state at rest. In case of stillness the equations (2.1) and (2.2) reduce to

$$q_t = 0 \quad , \quad c_0^2 q^{2\frac{c_0}{\alpha_0}-1} q_z + g = 0 \quad ,$$

since $q \neq 0$. By denoting $q = q_0(z)$ the strain density at rest, the integration gives

$$\frac{\alpha_0 c_0}{2} q_0(z)^{2\frac{c_0}{\alpha_0}} + g z = Constant \quad .$$

Recalling that $q^{2\frac{c_0}{\alpha_0}} = 1 + \beta_0 p$, we can use the atmospheric pressure p_a at the surface ($z = 0$) and get

$$\frac{\alpha_0 \beta_0 c_0}{2} (p - p_a) + g z = 0 \quad , \quad \text{that is} \quad p = p_a - \rho_0 g z \quad \text{or} \quad q_0(z) = (1 + \beta_0 (p_a - \rho_0 g z))^{\frac{\alpha_0}{2c_0}}$$

which corresponds to the geostrophic equilibrium state.

To compute the strain density profile we use the deviation variable $\eta = q - q_0$ and look for linkage of the form $m = A \eta - B$, as in any $q - m$ -system with a source term (see [4], or annex below in Section 5), where A and B are constant. Since stillness is reached for $\eta = 0$, we have $B = 0$. Besides, since q_0 does not depend on t , we have

$$\eta_t + A \eta_z = 0 \quad ,$$

which means that A corresponds to the wave velocity, that we name the reference velocity, corresponding to a reference state $(q_{ref}, q_{ref} w_{ref})$ such that $A = w_{ref} + c(q_{ref})$. We compute, for $q > q_0$ which is always expected,

$$w = \frac{m}{q} = A \frac{q - q_0}{q} \quad , \quad m_t = A \eta_t = -A^2 \eta_z \quad , \quad m_z = A \eta_z \quad , \quad q_z = \eta_z - \frac{g q_0}{c(q_0)^2}$$

to be introduced into (2.2) which becomes

$$\left[-A^2 + 2A^2 \frac{q - q_0}{q} + c^2 - A^2 \left(\frac{q - q_0}{q} \right)^2 \right] \eta_z + g q - \frac{g q_0}{c(q_0)^2} \left(c^2 - A^2 \left(\frac{q - q_0}{q} \right)^2 \right) + \frac{k q A^2}{\rho_0 q^2} (q - q_0)^2 = 0 \quad .$$

This equation reduces to

$$\left[c^2 - A^2 \frac{q_0^2}{q^2} \right] \eta_z + g q + \frac{g q_0}{c(q_0)^2} \left[A^2 \left(\frac{q - q_0}{q} \right)^2 - c^2 \right] + \frac{k A^2}{\rho_0 q} (q - q_0)^2 = 0 \quad .$$

By multiplying by q^2 and using $c = c_0 q^{\frac{c_0}{\alpha_0}}$ we get

$$(q^2 c^2 - q_0^2 A^2) \eta_z + g q^3 \left(1 - \left(\frac{q}{q_0} \right)^{2\frac{c_0}{\alpha_0}} - 1 \right) + \frac{g A^2 q_0}{c(q_0)^2} + \frac{k q A^2}{\rho_0} (q - q_0)^2 = 0 \quad . \quad (3.1)$$

Since $q = \eta + q_0$, this is a differential equation which can be integrated by using standard numerical methods. An increasing profile is expected. Since the reference velocity A is far larger than $c(q)$ the

coefficient $q^2 c^2 - q_0^2 A^2$ of η_z is always negative in practice. The two last terms are always positive, and the friction term is the predominant one. The term $gq^3 \left(1 - \left(\frac{q}{q_0}\right)^{2\frac{c_0}{\alpha_0} - 1}\right)$ is always negative and is always balanced by the friction term when the friction coefficient k is not too small.

The value of A is determined by the strength of the seismic wave at the bottom of the ocean, whose depth is denoted z_f . This corresponds to a reference state (q_{ref}, m_{ref}) . We have

$$A = w_{ref} + c_{ref} \quad , \quad m_{ref} = q_{ref} w_{ref} = A (q_{ref} - q_0(z_f)) \quad , \quad c_{ref} = c_0 \frac{c_0}{\alpha_0} q_{ref} .$$

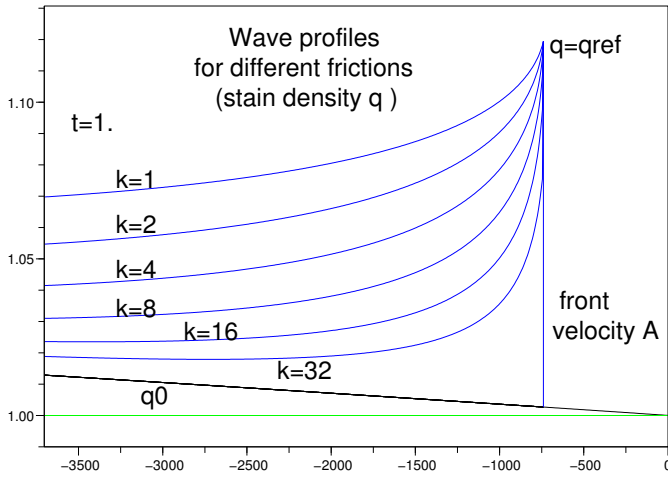
We get

$$A = A \left(1 - \frac{q_0(z_f)}{q_{ref}}\right) + c_{ref} \quad ,$$

which gives $A = c_{ref} \frac{q_{ref}}{q_0(z_f)}$ and $w_{ref} = c_{ref} \left(\frac{q_{ref}}{q_0(z_f)} - 1\right)$.

Now we can compute the profile of the strain wave as the solution of (3.1).

We present a series of numerical computation tests using the reference value $q_{ref} = 1.1296$, for a depth $z_f = 3700$ meters, that is an increasing of about 1500% above the natural pressure on the bottom of the ocean.



The velocity field w_{ref} increases from a few meters per second near the bottom to more than 300 meters per second near the front shock wave drawn here, which is here an hypothetic one. The tests performed with too small friction coefficients (observed here for $k < 0.15$) yield decreasing profiles, as expected from the remark above about the size of the friction term. The real front shock wave will progress more slowly than the strain wave, and its amplitude will decrease rapidly, and the effective values of the velocity field w_{ref} will be strongly reduced near the front shock wave.

The front shock wave connects the geostrophic equilibrium state q_0 , with the velocity $w = 0$, to a value q on the strain wave, with the velocity $w = A \left(1 - \frac{q_0}{q}\right)$. The Rankine Hugoniot condition

(2.3) gives the velocity of this shock wave, which reads here:

$$z'(t) = \frac{A}{2} \left(1 - \frac{q_0}{q}\right) + \frac{q_0 + q}{2} \sqrt{\frac{1}{q q_0} \frac{P(q) - P(q_0)}{q - q_0}}, \quad \text{with } P(q) = \frac{c_0^2}{\gamma_0} q^{\gamma_0}. \quad (3.2)$$

We have the following result:

Proposition 3.1 *The shock wave propagates slower than the stain wave, and faster than the local wave speed $c(q_0(z))$, that is*

$$c_0 q_0(z(t))^{\frac{c_0}{\alpha_0}} < z'(t) < A.$$

Proof: We fix t and set $q_s(t)$ and $q_0 = q_0(z(t))$, as the left and right values of the shock wave. Then the shock velocity reads

$$z'(t) = \frac{A}{2} \frac{q_s - q_0}{q_s} + \frac{q_s + q_0}{2} \sqrt{\frac{1}{q_s q_0} \frac{P(q_s) - P(q_0)}{q_s - q_0}}.$$

We shall use some $\xi \in]q_0, q_s[$ such that

$$\frac{P(q_s) - P(q_0)}{q_s - q_0} = P'(\xi) = c_0^2 \xi^{\frac{2c_0}{\alpha_0}}.$$

We have

$$z'(t) < A \iff \frac{q_s + q_0}{2} \sqrt{\frac{1}{q_s q_0} \frac{P(q_s) - P(q_0)}{q_s - q_0}} < A \left(1 - \frac{1}{2} + \frac{q_0}{2q_s}\right),$$

where $\frac{q_s + q_0}{2} = q_s \left(\frac{1}{2} + \frac{q_0}{2q_s}\right)$. We get

$$z'(t) < A \iff q_s \left(\frac{1}{2} + \frac{q_0}{2q_s}\right) \sqrt{\frac{1}{q_s q_0} c_0^2 \xi^{\frac{2c_0}{\alpha_0}}} < A \left(\frac{1}{2} + \frac{q_0}{2q_s}\right),$$

which reduces to

$$c_0 \sqrt{\frac{q_s}{q_0}} \xi^{\frac{c_0}{\alpha_0}} < A = c_0 q_{ref}^{\frac{c_0}{\alpha_0}} \frac{q_{ref}}{q_0(z_f)},$$

and is equivalent to

$$1 < \left(\frac{q_{ref}}{\xi}\right)^{\frac{c_0}{\alpha_0}} \frac{q_{ref}}{q_0(z_f)} \sqrt{\frac{q_s}{q_0}},$$

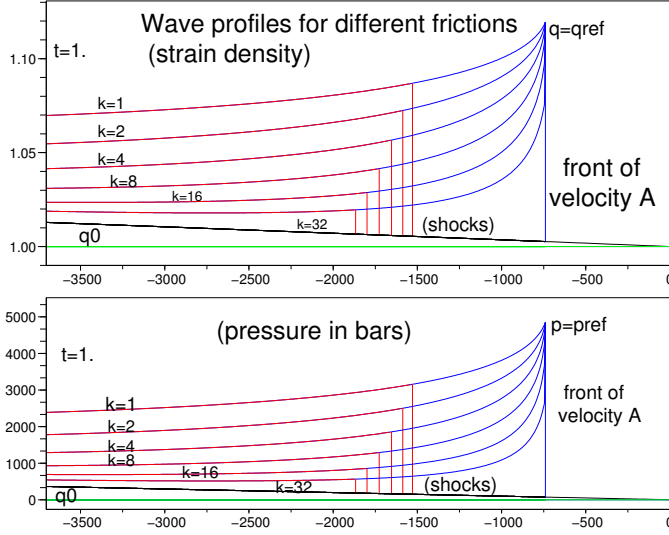
which is true since $q_{ref} > q_0(z_f)$ and $q_{ref} > q_s > \xi > q_0$.

On the other hand, since the expression

$$\frac{A}{2} \frac{q_s - q_0}{q_s} + \frac{q_s + q_0}{2} \sqrt{\frac{1}{q_s q_0} \frac{P(q_s) - P(q_0)}{q_s - q_0}}$$

is an increasing function of q_s for $q_s \geq q_0$, we get obviously the other inequality.(End of proof)

Now we can construct the whole wave, made of a regular part corresponding to a part of the strain wave and a front shock whose position is determined by the Rankine Hugoniot condition (3.2) interpreted as a differential equation whose solution $z(t)$ gives the position of the shock. The next picture shows the different positions of the shock wave for different values of the friction coefficient.



The velocity A is equal to 2932.5 m/s that is $A = 1.78 c_0$. The shock on the bottom of the ocean, at the depth $z_f = -3700\text{m}$, corresponds to 15 times the value of the usual geostrophic pressure, that is

$$p_{bottom} = 363.6 \cdot 10^5 \text{ pascal} = 363.6 \text{ bars} \quad , \quad p_{ref} = 5.454 \cdot 10^8 \text{ pascal} = 5.454 \text{ kbars} .$$

This corresponds to strain densities

$$q_{fond} = 1.01288 \quad , \quad q_{ref} = 1.1296 .$$

We notice that the variation from $q_{bottom} = q_0(z_f)$ to q_{ref} corresponds to an increasing of 11.5 % which is an increasing of about 1400 % of the pressure. The velocity of the wave is computed from the values

$$c_{ref} = c_0 \frac{c_0}{q_{ref}} = 2629.5 \text{ m/s} \quad , \quad w_{ref} = c_{ref} \frac{q_{ref}}{q_f} = 303 \text{ m/s} .$$

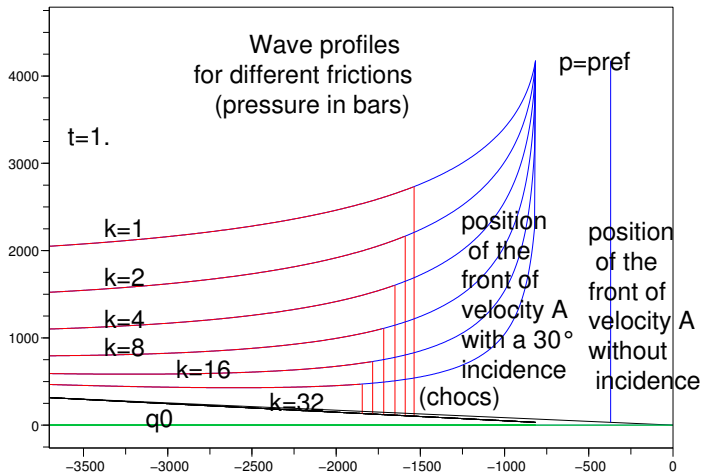
We notice that the velocities c_{ref} and $A = w_{ref} + c_{ref}$ are for more important than c_0 . The real profiles are drawn in red, and are shaped as a part of the strain wave profile cut by the slower shock wave. The eliminated parts are drawn in blue. We observe an important difference between the velocities of the different fronts, for several values of the friction coefficient.

4 A wave or not ?

We denote by H ($= -z_f$ here) the mean depth of the ocean. The wavelength λ of the Saint-Venant waves must satisfy a condition of the form

$$\lambda \geq 2 N \frac{H \sqrt{gH}}{c_s} , \quad (4.1)$$

where N the number of sonic interactions (back-and-forth) between the bottom and the surface of the ocean. This condition means that along a horizontal distance of a wavelength λ , there are at least N such sonic interactions . The use of the Saint-Venant model is as more appropriate as N is great. As in [2] for Rogue waves, we propose to require $N \geq 25$, which implies for example a wavelength greater than $21400 m$ for an ocean depth of $3700m$. A tsunami wave is expected when the condition (4.1) is fulfilled. A linear model for the surface elevation propagation was proposed in the historical paper [1] by K.Kajiura.



The numerical tests in section 3 show the effect of the friction, whose effect is to shape up the amplitude of the wave near the front shock. At the same time, this front shock magnitude is eroded progressively as the wave propagates upwards to the surface. Since the amplitude of the front shock decreases, the velocity of this front shock decreases too. By following the propagation of a wave with an incidence of angle ϕ (that is only changing g into $g \cos\phi$ and z into $z/\cos\phi$), we get a longer path to travel with a weaker gravity constant) and a delayed wave compare to the case without incidence. This is show in the next figure.

The value of the initial amplitude has a capital effect. It must be large enough to get, after erosion by the friction, a remaining wave near the surface which is sufficient to raise up the sea surface and provoke a wave. The physical wave starts as a sperical wave, and propagates according to the incidences. The part with a small incidence will reach the surface later and will help the formation of the water wave. The part with a larger incidence will disappear because of the friction

effect. The question of the value of the friction coefficient stays open, since the use of the strain density was never done before. It seems from the numerical tests that the correct values lay between 1 and 10. Too large values provoke a sharp front wave which erodes rapidly and will never reach the surface with a sufficient amplitude to make a wave, which is not expected, since sometimes, tsunamis really occur.

5 Annex: the source wave linearity

We consider a general 2x2 hyperbolic system whose first equation has the form

$$q_t + m_x = 0 \quad . \quad (5.1)$$

We denote by λ_1 and $\lambda_2 \geq \lambda_1$ the eigenvalues of the flux matrix, which depend on q and m only. Then the general form of the second equation is

$$m_t + (\lambda_1 + \lambda_2) m_x - \lambda_1 \lambda_2 q_x = S(q, m) ,$$

or

$$m_t + 2 u m_x + (c^2 - u^2) q_x = S(q, m) , \quad (5.2)$$

by using the notations

$$u = \frac{\lambda_1 + \lambda_2}{2} , \quad c = \frac{\lambda_2 - \lambda_1}{2} .$$

and $S(q, m)$ is a source term, assumed to be not identically zero. We have the following result:

Theorem 5.1 *The nonlinear non homogeneous system (5.1), (5.2) admits non constant local solutions which are also solutions to the linear homogeneous system*

$$q_t + A q_x = 0 , \quad m_t + A m_x = 0 , \quad (5.3)$$

where A is a real constant, with the linkage $Aq - m = B$, another real constant.

Proof: We look for local solution with a linkage of the form $m = m(q)$. Then the system becomes

$$q_t + m'(q)q_x = 0 , \quad m'(q) (q_t + 2uq_x) + (c^2 - u^2) q_x = S(q, m(q)) .$$

Since $q_t = -m'(q) q_x$, and $S(q, m(q)) \neq 0$, the second equation becomes

$$\frac{(u(q, m(q)) - m'(q))^2 - c(q, m(q))^2}{S(q, m(q))} q_x = 1 , \quad (5.4)$$

which has the form

$$\psi'(q) q_x = 1 ,$$

by introducing a real function $\psi(q)$ whose derivative is $\psi'(q) = \frac{(u(q, m(q)) - m'(q))^2 - c(q, m(q))^2}{S(q, m(q))}$.

Now, integrating with respect to x gives $\psi(q) = x - K(t)$, where $K(t)$ is an integrating constant

which may depend on t . Next, derivating with respect to t gives $\psi'(q) q_t = -K'(t)$, where we set $q_t = -m'(q) q_x$. Hence we get $m'(q) \psi'(q) q_x = K'(t)$, and recalling that $\psi'(q) q_x = 1$, it remains $m'(q) = K'(t)$. A new derivation with respect to x leads to $m''(q) q_x = 0$, and since the solution is not constant, we get $m''(q) = 0$, that is $m(q) = Aq - B$, with some constants A and B . Next, $K'(t) = m'(q) = A$, and we have got $\psi(q) = x - At - x_0$, for some constant x_0 , or, locally, $q = \psi^{-1}(x - x_0 - At)$, which satisfies to (5.3), since $m_t + Am_x = m'(q) (q_t + Aq_x) = 0$. (end of proof)

This result is a very general one, since no special hypotheses were needed on the second equation (5.2). Such waves are very common in the nature: water waves such as roll waves, rogue waves, tidal bore waves or also many other waves as reported in [2] or [4]. For example the double property of being either a solution to a non linear, non homogeneous systems and a linear homogeneous system provides the linkage between acoustics and gas dynamics in a wind instrument (see [3]).

In the case of a conservative system, invariant by Galilean transform, the only choice of the function u is reduced to $u(q, m) = m/q$. We easily construct this way the usual Saint Venant system in hydraulics or the Euler equations in gas dynamics.

In Section 3, the state at rest is not $q = 0$, $m = 0$, but $q = q_0$, $m = 0$, so we look for a solution with the linkage $A\eta - m = B$ since $\eta = q - q_0 = 0$ at rest. We find a differential equation which is more complex than (5.4).

6 Bibliography

We have used the idea of source term linearization effects on waves as in [2],[3] and [4]. Other models are developed in [1], [5] and [7]. The data [6] were used to valuate the parameters.

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