A CONVERGENCE RESULT FOR FINITE VOLUME SCHEMES ON 2-DIMENSIONAL RIEMANNIAN MANIFOLDS

JAN GIESSELMANN

Abstract. This paper studies a family of finite volume schemes for the hyperbolic scalar conservation law $u_t + \nabla_g \cdot f(x,u) = 0$ on a closed Riemannian manifold. For an initial value in $\mathrm{BV}(M)$ and an at most 2-dimensional manifold we will show that these schemes converge with a $h^{\frac{1}{4}}$ convergence rate towards the entropy solution. When M is 1-dimensional the schemes are TVD and we will show that this improves the convergence rate to $h^{\frac{1}{2}}$.

1. INTRODUCTION

Hyperbolic partial differential equations on curved manifolds occur in many applications. These include shallow water models for the atmosphere or ocean [4], [13], [16], the propagation of sound waves on curved surfaces [21] and passive tracer advection in the atmosphere. Further examples are the propagation of magneto-gravity waves in the solar tachocline [20], [5], [10] and relativistic matter flows near compact objects like black holes [9], [14].

For the numerics of these problems finite difference [9], finite volume [14], discontinuous Galerkin [12] and wave propagation methods [19] have been used. For convergence analysis of finite volume schemes, we will consider the following scalar model problem for non-linear hyperbolic conservation laws:

$$u_t + \nabla_g \cdot (\tilde{f}(u)v(x)) = 0 \text{ in } M \times \mathbb{R}_+ \tag{1}$$

$$u(x,0) = u_0(x) \text{ on } M. \tag{2}$$

Here (M,g) is a 1- or 2-dimensional closed oriented Riemannian manifold, v is a smooth vector-field on M and g is a fixed Riemannian metric on M. By $\nabla_g \cdot$ we denote the divergence operator on M induced by g. The aim of this paper is to prove a convergence rate for finite volume schemes for this model problem.

For this problem one has the notion of entropy solution, analogous to the Kruzkov definition in Euclidean space.

Definition 1. *A function $u \in L^\infty(M \times \mathbb{R}_+)$ is called an* **entropy solution** *of (1),(2) if*

$$\int_{M\times\mathbb{R}_+} \left[|u-\kappa|\varphi_t + g(x)\big((\tilde{f}(u\top\kappa) - \tilde{f}(u\bot\kappa))v(x), \nabla_g\varphi\big)\right] dv_g\, dt \tag{3}$$
$$+ \int_M |u_0-\kappa|\varphi(\cdot,0)\, dv_g \geq 0 \quad \forall \kappa\in\mathbb{R}, \forall\varphi\in C_0^\infty(M\times\mathbb{R}_+,\mathbb{R}_+).$$

The well-posedness of this problem was investigated by Ben-Artzi and LeFloch in [2]. They show that given $u_0 \in L^\infty(M) \cap L^1(M)$ and a geometry compatible flux, i.e. $\nabla_g \cdot v = 0$, the problem (1),(2) has a unique entropy solution u. Furthermore for $u_0 \in L^\infty(M) \cap \mathrm{BV}(M)$ the total variation of the entropy solution is bounded for every time $t \geq 0$ in the

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Universität Stuttgart (IANS), Pfaffenwaldring 57, 70569 Stuttgart, Germany
Email: jan.giesselmann@mathematik.uni-stuttgart.de.

sense that there exists $C_1 \geq 0$ depending only on $\|u_0\|_{L^\infty(M)}$ and the geometry of M such that

$$\mathrm{TV}_M(u(\cdot,t)) \leq e^{C_1 t}(1 + \mathrm{TV}_M(u_0)) \text{ for all } t \geq 0.$$

In [1] it is shown that for $\nabla_g \cdot v = 0$ we have

$$TV_X(u(\cdot,t)) \leq TV_X(u_0)$$

for every vector-field X with $[X,v] = 0$, where

$$TV_X(u) := \sup_{\phi \in C^\infty(M): \|\phi\|_{L^\infty}} \int_M u \nabla_g \cdot (\phi X) \, \mathrm{dv}_g(x).$$

This implies that for $d = 1$ the entropy solution is total variation diminishing, i.e. $\mathrm{TV}_M(u(\cdot,t)) \leq TV_M(u_0)$. Furthermore they prove convergence for a class of finite volume schemes for the Cauchy-problem (1),(2). In this paper we will prove convergence rates for these schemes. We will follow the ideas of Eymard et. al. in [8] for the proof of convergence rates for finite volume schemes in Euclidean space. As in the Euclidean case we are able to prove convergence of order $\frac{1}{2}$ in one space dimension and convergence of order $\frac{1}{4}$ for two space dimensions. The new problems in the convergence analysis, which are caused by the differential geometric properties of the problem, mostly occur in the proofs of Lemmas 13 and 15. We refer to [19] and [18] for a treatment of the wave propagation method on curved manifolds. In [3] different approaches to construct grids on spheres are treated and we refer to [11], [17] for geodesic grids on a sphere.

We make the following hypotheses on the data:

$$\left\{ \begin{array}{c} u_0 \in L^\infty(M) \cap \mathrm{BV}(M), U_m, U_M \in \mathbb{R} : U_m \leq u_0 \leq U_M \ a.e., \\ \nabla_g \cdot v = 0, \\ \tilde{f} \in C^1(\mathbb{R}, \mathbb{R}). \end{array} \right. \tag{4}$$

The hypothesis $\nabla_g \cdot v = 0$ is used to ensure the well-posedness of the problem and to avoid technical problems. Like in the Euclidean case it should not be necessary for the convergence rate.

The outline of this paper is as follows: In section 2 we will recall some helpful definitions and notations from differential geometry and give some results, which are necessary for the proof of the main theorem, Theorem 16. In sections 3, 4 we will present the notion of triangulation and the construction of finite volume schemes on Riemannian manifolds respectively. In section 5 we will state the main Theorem and prove it.

2. Differential geometry

2.1. **Notation and definitions.** We will consider a connected, closed, oriented d-dimensional Riemannian manifold (M,g), with $d = 1,2$, i.e. M is a compact, smooth, oriented manifold without boundary and g is a fixed Riemannian metric on M. This means $g(x)$ is a scalar product on the tangent space T_xM of M at x. In local coordinates $(x^j)_{1 \leq j \leq d}$ the partial derivatives $\partial_j = \frac{\partial}{\partial x^j}$ form a basis of the tangent space T_xM and we have the metric tensor $g_{ij}(x) := g(x)(\partial_i, \partial_j)$ with inverse g^{ij}. This enables us to define the divergence operator $\nabla_g \cdot$ by

$$\nabla_g \cdot f(x) := \frac{1}{\sqrt{|g(x)|}} \partial_j \left(\sqrt{|g(x)|} f^j(x) \right)$$

where $|g(x)| := |\det(g_{ij}(x))|$, for every smooth vector-field f on M with local representation $f = f^j \partial_j$. This is only well-defined in the local coordinate system, but in fact the definition is independent of the choice of the local coordinates and so the divergence is well-defined all over M. Similarly for every smooth function u on M the gradient of u is defined by

$$(\nabla_g u)^i = g^{ij}\frac{\partial u}{\partial x^j}.$$

The Riemannian metric also defines a volume form dv_g on the manifold, a volume form dv_N on every submanifold N and a metric d_g on M. Spaces of functions of bounded variation are defined similar to the definition in Euclidean space

Definition 2.

$$\begin{aligned}
\mathrm{TV}_M(u) &:= \sup_{X\in\Gamma(TM):\|X\|_\infty\leq 1}\int_M u\nabla_g\cdot X\,dv_g,\\
\mathrm{BV}(M) &:= \left\{u\in L^1(M):\mathrm{TV}_M(u)<\infty\right\},
\end{aligned}$$

where $\Gamma(TM)$ denotes the smooth vector-fields on M, i.e. the smooth sections of the tangent bundle TM.

Definition 3. *An open subset $U\subset M$ is called* **convex***, if for every pair of points $x,y\in U$ there exists a unique minimising geodesic from x to y lying in U.*

2.2. **Geodesic polar coordinates.** We will now define geodesic polar coordinates. They are helpful for the definition of cut-off functions, which we need in a doubling of variables argument in Lemma 15. We consider a point $x\in M$ and local geodesic polar coordinates (ρ,θ) around x (cf. [6] for example). The metric tensor has the form

$$g_{(\rho,\theta)} = \begin{pmatrix} 1 & 0\\ 0 & G(\rho,\theta)\end{pmatrix}.$$

The function G fulfils

$$\lim_{\rho\to 0} G = 0,\quad \lim_{\rho\to 0}(\sqrt{G})_\rho = 1,\quad (\sqrt{G})_{\rho\rho}+K\sqrt{G}=0,$$

where K is the Gaussian curvature of M. Because M is compact K, $|\nabla_g K|$ and $|\nabla_g^2 K|$ are bounded on M. We recall that there exists a $R>0$ such that for every $y\in M$ the mapping $\exp_y : B_R(0)\subset T_yM\longrightarrow M$ is a diffeomorphism on its image (cf. [7] for example). Let

$$A := \left\{(y,v)\in TM : \|v\|_g\leq \frac{R}{2}\right\}$$

then A is compact and there exists some $C>0$ such that

$$\|(T\exp_y)_v\|\leq C\,\forall y\in M,\ v\in B_{\frac{R}{2}}(0)\subset T_yM.$$

Therefore

$$|G(\exp_y(v))| = \left\|(T\exp_y)_v(\frac{\partial}{\partial\theta})\right\|^2\leq \|(T\exp_y)_v\|^2\left\|\frac{\partial}{\partial\theta}\right\|^2\leq C^2\frac{R^2}{4}.$$

So $\sqrt{G}$ is bounded and for fixed θ the function $u(\rho):=\sqrt{G(\rho,\theta)}$ satisfies

$$u''(\rho)+K(\rho,\theta)u(\rho)=0.$$

Multiplying this equation by u' and integrating with respect to ρ yields

$$\begin{aligned}
&\int_0^r u''(\rho)u'(\rho)\,d\rho+\int_0^r K(\rho,\theta)u(\rho)u'(\rho)\,d\rho = 0\\
\Longrightarrow\quad & u'(r)^2-u'(0)^2+K(r,\theta)u(r)^2-K(0,\theta)u(0)^2\\
&-\int_0^r K_\rho(\rho,\theta)u(\rho)^2\,d\rho = 0\\
\Longrightarrow\quad & u'(r)^2\leq 1-K(r,\theta)u(r)^2+\int_0^r\|\nabla_g K(\rho,\theta)\|_g u(\rho)^2\,d\rho.
\end{aligned}$$

So for $r < \frac{R}{2}$ the function $(\sqrt{G})_\rho$ is bounded independently of the choice of x. Because $u'' = -Ku$ the same is true for $(\sqrt{G})_{\rho\rho}$. We can easily calculate

$$\frac{\partial \sqrt{G}}{\partial \rho^4} = +K^2\sqrt{G} - K_{\rho\rho}\sqrt{G} - 2K_\rho(\sqrt{G})_\rho.$$

Thus for $r < \frac{R}{2}$ the function $\frac{\partial\sqrt{G}}{\partial\rho^4}$ is bounded on M independently of x by a constant C. So the Taylor formula implies

$$\sqrt{G}(\rho,\theta) = \rho - \frac{\rho^3}{6}K(x) + R \tag{5}$$

where $|R| \leq C\rho^4$.

2.3. **Parallel transport.** In the proof of Lemma 15 we will have to use parallel transport to extend vectors to local vector-fields. For $x, y \in M$ with $0 < d_g(x,y) < R$ there exists a unique minimising geodesic γ_{xy} from x to y parametrised by arc-length. So we get a well defined mapping

$$P_{xy} : T_xM \longrightarrow T_yM$$

defined by parallel transport along this geodesic. By definition of geodesic we know that $P_{xy}(\gamma'_{xy}(0)) = \gamma'_{xy}(d_g(x,y))$. Obviously we have for $0 < d_g(x,y) < R$ the identities $\nabla_{g,x}d_g(x,y) = -\gamma'_{xy}(0)$ and $\nabla_{g,y}d_g(x,y) = \gamma'_{xy}(d_g(x,y))$. Let v be a smooth vector-field on M then

$$\frac{d}{dt}g(\gamma_{xy}(t))\left(P_{x\gamma_{xy}(t)}(v(x)), \gamma'_{xy}(t)\right) = 0$$

and therefore

$$g(x)\left(v(x), \nabla_{g,x}d_g(x,y)\right) = -g(y)\left(P_{xy}(v(x)), \nabla_{g,y}d_g(x,y)\right). \tag{6}$$

Furthermore we have the following lemma whose proof is given in the appendix.

Lemma 4. *Let M be a smooth closed Riemannian manifold and $R > 0$ such that $\exp_x^{-1} : B_R(x) \longrightarrow T_xM$ is a smooth chart for every $x \in M$ and for all $x, y \in M$ with $d_g(x,y) < R$ there exists a unique minimising geodesic from x to y. Let $\bar{v} \in \Gamma(TM)$ and $v(x,\xi)$ for $d_g(x,\xi)$ small enough given by parallel transport of $\bar{v}(\xi)$ along the unique minimising geodesic from ξ to x. Then*

$$\tilde{v} : \left\{(\xi, x) \in M^2 : d_g(\xi,x) < \frac{R}{4}\right\} \longrightarrow TM$$

is C^2 and there exists $C > 0$ such that

$$|\nabla_{g,x}\tilde{v}(x,\xi)| < C_l d_g(x,\xi) \text{ for } \xi \in M, d_g(x,\xi) < \frac{R}{8}. \tag{7}$$

2.4. **Cut-off functions.** These are necessary for a doubling of variables argument in the proof of Lemma 15. In this paragraph we will assume $d = 2$. Let $\psi : \mathbb{R} \longrightarrow \mathbb{R}_+$ be a smooth function with $\operatorname{supp}\psi \subset [-1, 0]$ such that

$$\int_{\mathbb{R}} \psi(x)\,dx = 1.$$

Let $\psi_\varepsilon(x) := \frac{1}{\varepsilon}\psi\left(\frac{x}{\varepsilon}\right)$. Let $\chi : \mathbb{R} \longrightarrow \mathbb{R}_+$ be a smooth function with support in $[-1, 1]$, which is even, decreasing on $[0, 1]$ and fulfils

$$\int_{\mathbb{R}^2} \chi(|x|) = 1.$$

We define

$$\chi_\varepsilon : M \times M \longrightarrow \mathbb{R}, \quad (x,y) \mapsto \frac{1}{\varepsilon^2}\chi\left(\frac{d_g(x,y)}{\varepsilon}\right). \tag{8}$$

By (5) we have for any fixed $y \in M$ and $\varepsilon < \frac{R}{2}$

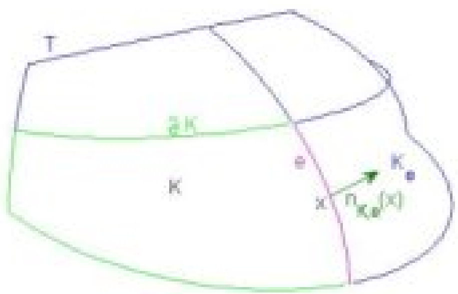
$$\begin{aligned}(9) \qquad \int_M \chi_\varepsilon(x,y)\,\mathrm{dv}_g(x) &= \int_0^\infty \int_0^{2\pi} \frac{1}{\varepsilon^2}\chi\left(\frac{\rho}{\varepsilon}\right)\left(\rho - \frac{\rho^3}{6}K(y) + R\right) d\theta d\rho \\ &= 1 + \tilde{R}\end{aligned}$$

where $|\tilde{R}| \leq \frac{\varepsilon^2}{6}|K(y)| + C\varepsilon^3$.

3. Triangulation

Definition 5. *A* **triangulation** *on (M,g) is a set $\mathcal{T}$ of curved polyhedra K on M such that $M = \cup_{\mathcal{T}} \bar{K}$ and the interior of each polyhedron is convex. When M is 1-dimensional we impose that for distinct "polyhedra" $K_1, K_2 \in \mathcal{T}$ the section $K_1 \cap K_2$ is a common face of K_1, K_2, this means a single point. When M is 2-dimensional we impose $K_1 \cap K_2$ is a common face of K_1, K_2, a single point or empty. Furthermore we assume that the faces are geodesic lines. This is not only necessary for the convergence analysis, but also sensible for numerical calculations. On the sphere it ensures that the normal vectors are constant along the faces.*

The set of the faces e of a polyhedron K is denoted by ∂K and the unique polyhedron sharing the face e with K is denoted by K_e. By $n_{K,e}(x) \in T_xM$ we denote the unit outer normal to a polyhedron K in a point $x \in e$. Finally $|K|, |e|$ denote the d- and $(d-1)$-dimensional Hausdorff measures of K, e respectively.



We will need the following assumption on the triangulation: There exist $\beta, h > 0$ and $k \in \mathbb{N}$ such that for every $K \in \mathcal{T}$ and $e \in \partial K$ the following conditions are fulfilled

$$\beta h^d \leq |K|, \tag{10}$$

$$|e| \leq h^{d-1} \tag{11}$$

$$\#\partial K \leq k \tag{12}$$

$$\delta(K) \leq h, \tag{13}$$

where $\delta(K) := \sup\{d_g(x,y) : x, y \in K\}$ and $\#\partial K$ denotes the number of elements of ∂K, i.e. the number of faces of K. Now we can state the following approximation result, which is proven in the appendix.

Lemma 6. *For h small enough and every $u \in \mathrm{BV}(M)$ there is a constant $C > 0$ depending on M but not on the triangulation $\mathcal{T}$ such that*

$$\|u - \bar{u}\|_{L^1(M)} \leq Ch$$

where

$$\bar{u}(x) := \frac{1}{|K|}\int_K u(x)\,\mathrm{dv}_g(x) \text{ for } x \in K$$

so $\bar{u}$ is well-defined allmost everywhere on the manifold.

4. The scheme

We define

$$f(x,u) := \tilde{f}(u)v(x). \tag{14}$$

For every polyhedron $K \in \mathcal{T}$ and face $e \in \partial K$ we consider a numerical flux function $f_{K,e} : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ such that the following properties are satisfied:

$$\text{Conservation:} \quad f_{K,e}(a,b) = -f_{K_e,e}(b,a), \tag{15}$$

$$\text{Consistency:} \quad f_{K,e}(a,a) = \frac{1}{|e|}\int_e f(x,a) n_{K,e}(x)\,\mathrm{dv}_e(x), \tag{16}$$

$$\text{Monotonicity:} \quad f_{K,e} \quad \text{is nondecreasing in the first and nonincreasing in the second variable.} \tag{17}$$

Furthermore we impose that the $f_{K,e}$ are uniformly locally Lipschitz continuous. We will consider the following **semi-discrete scheme**:

$$(u_K^h)_t = -\frac{1}{|K|}\sum_{e \in \partial K} |e| f_{K,e}(u_K^h, u_{K_e}^h) \tag{18}$$

$$u_K^h(0) = \frac{1}{|K|}\int_K u_0(x)\,\mathrm{dv}_g(x) \tag{19}$$

$$u^h(x,t) = u_K^h(t) \text{ for } x \in K. \tag{20}$$

5. Proof of convergence rates

We first show that a solution of (18)-(20) exists and that it is bounded.

Lemma 7. *Assume the local existence of a solution of* (18)-(20) *and let* $u_0(x) \in [U_m, U_M]$ *for almost every* $x \in M$, *then* $u_K^h(t) \in [U_m, U_M]$ *for every* $t \geq 0$ *and* $K \in \mathcal{T}$.

Proof. It is obvious that $u_K^h(0) \in [U_m, U_M]$ for every K. First observe that for fixed K and $u_{K_e}^h \leq u_K^h$ for all $e \in \partial K$ we have

$$\begin{aligned}(u_K^h)_t &= -\frac{1}{|K|}\sum_{e \in \partial K} |e| f_{K,e}(u_K^h, u_{K_e}^h) \\ &\overset{monotonicity}{\leq} -\frac{1}{|K|}\sum_{e \in \partial K} |e| f_{K,e}(u_K^h, u_K^h) \\ &\overset{consistency}{\leq} -\frac{1}{|K|}\sum_{e \in \partial K} \int_e f(x, u_K^h) n_{K,e}\,\mathrm{dv}_g(x) = 0.\end{aligned}$$

Now we will prove that $u_K^h \leq U_M$ for all t, the proof for $u_K^h \geq U_m$ is analogous. Let

$$s := \sup\{T \geq 0 : u_K^h(t) \in [U_m, U_M] \forall t \in [0,T] \text{ and } K \in \mathcal{T}\}.$$

We have $s \geq 0$. Let $E := \max |e|$. Assume $s < \infty$. Due to continuity we have $u_K^h(s) \in [U_m, U_M]\,\forall K$. Because the $f_{K,e}$ are locally Lipschitz continuous, it exists $\delta > 0$ such that a solution $\{u_K^h\}_{K \in \mathcal{T}}$ of (1),(2) exists in $[0, s+\delta)$. Let $A := \sup\{u_K^h(t) : t \leq s + \frac{\delta}{2}\}$ and L the uniform Lipschitz constant of the $f_{K,e}$ on $[-A, A]$. Because $s < \infty$ there are $a_1, \varepsilon > 0$ and $K_1 \in \mathcal{T}$ such that $a_1 < \min\left(\frac{\delta}{2}, \frac{1}{kLE}\right)$ and

$$u_{K_1}^h(s + a_1) = U_M + \varepsilon. \tag{21}$$

Now we will prove by induction that there exist $0 < a_n \leq a_1$ and $K_n \in \mathcal{T}$ such that

$$u_{K_n}^h(s + a_n) \geq U_M + \frac{\varepsilon}{(a_1 kLE)^{n-1}}. \tag{22}$$

The induction starts with (21). If (22) is fulfilled there has to be an $a_{n+1} < a_n$ such that

$$u^h_{K_n}(s + a_{n+1}) \geq U_M \text{ and}$$
$$(u^h_{K_n})_t(s + a_{n+1}) \geq \frac{\varepsilon}{a_n(a_1kLE)^{n-1}} \geq \frac{\varepsilon}{a_1(a_1kLE)^{n-1}}.$$

Thus due to the monotonicity and Lipschitz property of the $f_{K,e}$ there must be a $K_{n+1} \in \mathcal{T}$ such that

$$u^h_{K_{n+1}}(s + a_{n+1}) \geq U_M + \frac{\varepsilon}{(a_1kLE)^n}.$$

There are only finitely many $K \in \mathcal{T}$ so there is a subsequence a_{k_l} and some $K \in \mathcal{T}$ such that

$$u^h_K(s + a_{k_l}) \overset{l\longrightarrow\infty}{\longrightarrow} \infty,$$

because all a_{k_l} are smaller than a_1 this is a contradiction to the continuity of u^h_K on $[0, s + \delta)$. So $s = \infty$. □

As an immediate consequence of Lemma 7 and the local Lipschitz continuity of the numerical fluxes we have:

Corollary 8. *There exists a global solution of the system* (18)-(20).

The next step is to prove a TVD estimate in the $d = 1$ and a weak BV -estimate in the $d = 2$ case. For brevity we introduce the following notation: for real numbers a, b we define

$$C(a, b) := \{(c, d) \in [a\perp b, a\top b]^2 : (b - a)(d - c) \geq 0, \}$$

where $a\top b$ and $a\perp b$ denote the maximum and minimum of a and b respectively. For every $t \geq 0$ we define

$$E(t) := \{(K, e) : K \in \mathcal{T}, e \in \partial K, u^h_K(t) > u^h_{K_e}(t)\}.$$

Lemma 9 (TVD property)**.** *Let M be 1 -dimensional then the scheme (18)-*(20) *is TVD, i.e.*

$$\mathrm{TV}_M(u^h(\cdot, t)) \leq \mathrm{TV}_M(u_0) \textit{ for all } t > 0.$$

This implies that for every $T > 0$ there exists a $C > 0$ depending only on f, u_0, M, $f_{K,e}, T$ such that

$$\int_0^T \sum_{K\in\mathcal{T}} \sum_{e\in\partial K} |e| \max_{(c,d)\in C(u^h_K, u^h_{K_e})} |f_{K,e}(c, d) - f_{K,e}(c, c)| \; dt \leq C.$$

Proof. We will consider times t where $\frac{d}{dt}|u^h_K - u^h_{K_e}|$ exists for all $K \in \mathcal{T}$ and $e \in \partial K$. These derivatives exist for almost every $t \geq 0$ and we have

$$\frac{d}{dt}\mathrm{TV}_M(u^h(\cdot, t)) = \frac{1}{2} \sum_{K\in\mathcal{T}} \sum_{e\in\partial K} \frac{d}{dt}|u^h_K(t) - u^h_{K_e}(t)|.$$

Now we fix one $K \in \mathcal{T}$ and observe that K has exactly two neighbours K_1, K_2.

- If $u^h_{K_1}(t) \leq u^h_K(t) \leq u^h_{K_2}(t)$ or $u^h_{K_2}(t) \leq u^h_K(t) \leq u^h_{K_1}(t)$ then $(u^h_K)_t$ occurs exactly twice with different signs in the sum and therefore vanishes.
- If $u^h_K(t) > u^h_{K_1}(t), u^h_{K_2}(t)$ the term

$$(u^h_K)_t(t) = -\sum_{e\in\partial K} \frac{|e|}{|K|} f_{K,e}(u^h_K(t), u^h_{K_e}(t)) \leq -\sum_{e\in\partial K} \frac{|e|}{|K|} f_{K,e}(u^h_K(t), u^h_K(t)) = 0$$

occurs twice in the sum.

- If $u^h_K(t) < u^h_{K_1}(t), u^h_{K_2}(t)$ the term
$$-(u^h_K)_t(t) = \sum_{e\in\partial K} \frac{|e|}{|K|} f_{K,e}(u^h_K(t), u^h_{K_e}(t)) \le \sum_{e\in\partial K} \frac{|e|}{|K|} f_{K,e}(u^h_K(t), u^h_K(t)) = 0$$
occurs twice in the sum.

So we know $\mathrm{TV}_M(u^h(\cdot,t))$ is nonincreasing in time. For every $K\in\mathcal{T}$ there exist $x_K, y_K \in K$ such that
$$u_0(x_K) \ge u^h_K(0) \ge u_0(y_K).$$
Let K_1, K_2 be the neighboring elements for some $K\in\mathcal{T}$, then we define
$$\zeta_K = \begin{cases} x_K & : \quad u^h_K > u^h_{K_1}, u^h_{K_2} \\ y_K & : \quad \text{else.} \end{cases}$$
We have
$$\begin{aligned} 2\,\mathrm{TV}_M(u^h(\cdot,0)) &= \sum_{K\in\mathcal{T}}\sum_{e\in\partial K} |u^h_K(0) - u^h_{K_e}(0)| \\ &\le \sum_{K\in\mathcal{T}}\sum_{e\in\partial K} |u_0(\zeta_K) - u_0(\zeta_{K_e})| \le 2\,\mathrm{TV}_M(u_0). \end{aligned}$$
This proves the TVD property. For $(c,d)\in C(u^h_K, u^h_{K_e})$ we have
$$|f_{K,e}(c,d) - f_{K,e}(c,c)| \le L|c-d| \le |u^h_K - u^h_{K_e}|,$$
where L is the uniform Lipschitz constant for all $f_{K,e}$ on $[U_m, U_M]$. Using $|e|=1$ we get
$$\begin{aligned} &\int_0^T \sum_{K\in\mathcal{T}}\sum_{e\in\partial K} |e| \max_{(c,d)\in C(u^h_K, u^h_{K_e})} |f_{K,e}(c,d) - f_{K,e}(c,c)|\, dt \\ \le\ & \int_0^T L \sum_{K\in\mathcal{T}}\sum_{e\in\partial K} |u^h_K - u^h_{K_e}|\, dt \\ \le\ & 2L \int_0^T \mathrm{TV}_M(u^h(\cdot,t))\, dt \le 2LT\,\mathrm{TV}_M(u_0). \end{aligned}$$
□

In the 2-dimensional case there is no TVD estimate, but we can prove a weak BV estimate which will play a similar role in the convergence proof.

Lemma 10 (weak BV-estimate)**.** *Let M be 2-dimensional. For every $T>0$ there exists a $C>0$ depending only on f, u_0, M, β, $\{f_{K,e}\}, T, k$ such that*
$$\int_0^T \sum_{K\in\mathcal{T}}\sum_{e\in\partial K} |e| \max_{(c,d)\in C(u^h_K, u^h_{K_e})} |f_{K,e}(c,d) - f_{K,e}(c,c)|\, dt \le \frac{C}{\sqrt{h}}.$$

Proof. We have
$$\begin{aligned} \int_0^T \sum_{K\in\mathcal{T}} |K| u^h_K (u^h_K)_t\, dt &= \frac{1}{2}\int_0^T \sum_{K\in\mathcal{T}} |K| \left((u^h_K)^2\right)_t dt \\ &= \frac{1}{2}\sum_{K\in\mathcal{T}} |K| \left((u^h_K)^2(T) - (u^h_K)^2(0)\right) \qquad (23) \\ &\ge -\frac{1}{2}\sum_{K\in\mathcal{T}} |K| (u^h_K)^2(0) \\ &= -\frac{1}{2}\|u^h(0)\|^2_{L^2(M)} \ge -\frac{1}{2}\|u_0\|^2_{L^2(M)}. \end{aligned}$$

Now we multiply (18) by $|K|u_K^h(t)$ and sum over all $K \in \mathcal{T}$

$$\begin{aligned}
\int_0^T \sum_{K\in\mathcal{T}} |K| u_K^h (u_K^h)_t \, dt &= -\int_0^T \sum_{K\in\mathcal{T}} \sum_{e\in\partial K} |e| f_{K,e}(u_K^h, u_{K_e}^h) u_K^h \, dt \\
&\overset{(16)(4)}{=} \int_0^T \sum_{K\in\mathcal{T}} \sum_{e\in\partial K} |e| \left(f_{K,e}(u_K^h, u_K^h) - f_{K,e}(u_K^h, u_{K_e}^h) \right) u_K^h \, dt \\
&= \int_0^T \sum_{(K,e)\in E(t)} |e| \Big[\left(f_{K,e}(u_K^h, u_K^h) - f_{K,e}(u_K^h, u_{K_e}^h) \right) u_K^h \\
&\quad + \left(f_{K_e,e}(u_{K_e}^h, u_{K_e}^h) - f_{K_e,e}(u_{K_e}^h, u_K^h) \right) u_{K_e}^h \Big] \, dt \\
&\overset{(15)}{=} \int_0^T \sum_{(K,e)\in E(t)} |e| \Big[\left(f_{K,e}(u_K^h, u_K^h) - f_{K,e}(u_K^h, u_{K_e}^h) \right) u_K^h \\
&\quad - \left(f_{K,e}(u_{K_e}^h, u_{K_e}^h) - f_{K,e}(u_K^h, u_{K_e}^h) \right) u_{K_e}^h \Big] \, dt.
\end{aligned} \tag{24}$$

Now we define $F_{K,e}(a) := f_{K,e}(a,a)$ and let $\Phi_{K,e}$ be a primitive of $a \mapsto aF'_{K,e}(a)$ satisfying $\Phi_{K,e}(0) = 0$. Let $a = u_K^h, b = u_{K_e}^h$ then every single summand has the form

$$|e| \left[a(F_{K,e}(a) - f_{K,e}(a,b)) - b(F_{K,e}(b) - f_{K,e}(a,b)) \right].$$

Integration by parts yields

$$\begin{aligned}
\Phi_{K,e}(b) - \Phi_{K,e}(a) &= \int_a^b u F'_{K,e}(u) \, du \\
&= b(F_{K,e}(b) - f_{K,e}(a,b)) - a(F_{K,e}(a) - f_{K,e}(a,b)) \\
&\quad - \int_a^b (F_{K,e}(u) - f_{K,e}(a,b)) \, du.
\end{aligned}$$

Due to the conservation property (15) of the numerical fluxes we have $F_{K,e} = -F_{K_e,e}$ and therefore $\Phi_{K,e} = -\Phi_{K_e,e}$. Because the flux is geometry compatible (4) we have

$$\sum_{e\in\partial K} |e| f_{K,e}(a,a) = 0 \Longrightarrow \sum_{e\in\partial K} |e| F'_{K,e}(a) = 0 \Longrightarrow \sum_{e\in\partial K} |e| \Phi_{K,e}(a)$$

for every $K \in \mathcal{T}$ and $a \in \mathbb{R}$. Thus we have

$$\begin{aligned}
\sum_{(K,e)\in E(t)} |e| \left(\Phi_{K,e}(u_K^h) - \Phi_{K,e}(u_{K_e}^h) \right) &= \sum_{(K,e)\in E(t)} |e| \left(\Phi_{K,e}(u_K^h) + \Phi_{K_e,e}(u_{K_e}^h) \right) \\
&= \sum_{(K,e)\in E(t)} |e| \left(\Phi_{K,e}(u_K^h) + \Phi_{K_e,e}(u_{K_e}^h) \right) \\
&\quad + \underbrace{\sum_{\{(K,e): u_K^h = u_{K_e}^h\}} |e| \left(\Phi_{K,e}(u_K^h) \right)}_{0} \\
&= \sum_{K\in\mathcal{T}} \sum_{e\in\partial K} |e| \Phi_{K,e}(u_K^h) = 0.
\end{aligned}$$

Using this in (24) implies

$$\begin{aligned}\int_0^T \sum_{K\in\mathcal{T}} |K| u_K^h (u_K^h)_t\,dt &= -\int_0^T \sum_{K\in\mathcal{T}} \sum_{e\in\partial K} |e| f_{K,e}(u_K^h, u_{K_e}^h) u_K^h\,dt \\ &= -\int_0^T \sum_{(K,e)\in E(t)} |e| \int_{u_K^h}^{u_{K_e}^h} \Big(f_{K,e}(u,u) - f_{K,e}(u_K^h, u_{K_e}^h)\Big)\,du\,dt \qquad (25)\\ &= \int_0^T \sum_{(K,e)\in E(t)} |e| \int_{u_{K_e}^h}^{u_K^h} \Big(f_{K,e}(u,u) - f_{K,e}(u_K^h, u_{K_e}^h)\Big)\,du\,dt.\end{aligned}$$

For $u_{K_e}^h \leq c \leq d \leq u_K^h$ we have due to (17)

$$\begin{aligned}\int_{u_{K_e}^h}^{u_K^h} \underbrace{\Big(f_{K,e}(u_K^h, u_{K_e}^h) - f_{K,e}(u,u)\Big)}_{\geq 0}\,du &\geq \int_c^d \Big(f_{K,e}(u_K^h, u_{K_e}^h) - f_{K,e}(u,u)\Big)\,du \\ &\geq \int_c^d \left(f_{K,e}(d,c) - f_{K,e}(u,u)\right)\,du.\end{aligned}$$

We will now use the following fact which can be found in [8]:

Lemma 11. *Let* $g : \mathbb{R} \longrightarrow \mathbb{R}$ *be a monotonic Lipschitz continuous function, with Lipschitz constant* $G > 0$. *Then*

$$\left| \int_c^d (g(u) - g(c))\,du \right| \geq \frac{1}{2G}(g(d) - g(c))^2, \;\; \forall c, d \in \mathbb{R}.$$

Thus (17) and the Lipschitz continuity of the $f_{K,e}$ imply

$$\begin{aligned}\int_c^d \left(f_{K,e}(d,c) - f_{K,e}(u,u)\right)du &\geq \int_c^d \left(f_{K,e}(d,c) - f_{K,e}(d,u)\right)du \\ &\geq \frac{1}{2L}\left(f_{K,e}(d,c) - f_{K,e}(d,d)\right)^2\end{aligned}$$

and

$$\begin{aligned}\int_c^d \left(f_{K,e}(d,c) - f_{K,e}(u,u)\right)du &\geq \int_c^d \left(f_{K,e}(d,c) - f_{K,e}(u,c)\right)du \\ &\geq \frac{1}{2L}\left(f_{K,e}(d,c) - f_{K,e}(c,c)\right)^2,\end{aligned}$$

where L is the uniform Lipschitz constant of the $f_{K,e}$ on $[U_m, U_M]$. Multiplying both inequalities with $\frac{1}{2}$ and adding them yields with (23) and(25)

$$\begin{aligned}\frac{1}{2}\|u_0\|^2_{L^2(M)} &\geq \int_0^T \sum_{(K,e)\in E(t)} \int_{u_{K_e}^h}^{u_K^h} |e| \Big(f_{K,e}(u_K^h, u_{K_e}^h) - f_{K,e}(u,u)\Big)\,du\,dt \\ &\geq \int_0^T \sum_{(K,e)\in E(t)} \frac{|e|}{2L} \Bigg(\max_{u_{K_e}^h \leq c\leq d\leq u_K^h} \left(f_{K,e}(d,c) - f_{K,e}(d,d)\right)^2 \\ &\qquad + \max_{u_{K_e}^h \leq c\leq d\leq u_K^h} \left(f_{K,e}(d,c) - f_{K,e}(c,c)\right)^2 \Bigg)\,dt \qquad (26)\\ &\geq \int_0^T \sum_{K\in\mathcal{T}} \sum_{e\in\partial K} \frac{|e|}{2L} \max_{(c,d)\in C(u_K^h, u_{K_e}^h)} |f_{K,e}(c,d) - f_{K,e}(c,c)|^2\,dt.\end{aligned}$$

Now by Cauchy Schwartz inequality we get

$$\begin{aligned}
&\int_0^T \sum_{K\in\mathcal{T}} \sum_{e\in\partial K} |e| \max_{(c,d)\in C(u_K^h, u_{K_e}^h)} |f_{K,e}(c,d) - f_{K,e}(c,c)|\, dt \\
\leq\ & \left(\int_0^T \sum_{K\in\mathcal{T}} \sum_{e\in\partial K} |e|^2 \max_{(c,d)\in C(u_K^h, u_{K_e}^h)} |f_{K,e}(c,d) - f_{K,e}(c,c)|^2\ dt\right)^{\frac{1}{2}} \\
&\cdot \left(\sum_{K\in\mathcal{T}} \sum_{e\in\partial K} 1\right)^{\frac{1}{2}} \\
\leq\ & CL^{\frac{1}{2}} \|u_0\|_{L^2(M)} h^{\frac{d-1}{2}} h^{-\frac{d}{2}} \frac{1}{\beta^{\frac{1}{2}}} k^{\frac{1}{2}}.
\end{aligned}$$

the last line follows from (26) and the assumptions on the grid (10)-(12).

□

Next we prove a weak discrete entropy inequality for the approximate solution, which is an auxiliary result to prove a continuous entropy inequality for the approximate solution. This continuous entropy inequality is important for the main convergence proof and has a similar importance for the proof like the entropy inequality for the exact solution.

Lemma 12 (Weak discrete entropy inequality). *For every $\kappa \in [U_m, U_M]$, every polyhedron $K \in \mathcal{T}$ and every test function $\varphi \in C_0^\infty(\mathbb{R}_+, \mathbb{R}_+)$ the following inequality holds*

$$\begin{aligned}
&\int_{\mathbb{R}_+} |K||u_K^h(t) - \kappa|\varphi_t\, dt + |K||u_K^h(0) - \kappa|\varphi(0) \\
&\quad - \int_{\mathbb{R}_+} \sum_{e\in\partial K} |e| \left(f_{K,e}(u_K^h \top \kappa, u_{K_e}^h \top \kappa) - f_{K,e}(u_K^h \bot \kappa, u_{K_e}^h \bot \kappa)\right) \varphi\, dt \geq 0.
\end{aligned}$$

Proof. Let $B = \sup\{t > 0 : \varphi(t) \neq 0\}$. Consider disjoint intervals $\{I_j = (a_j, b_j) : j \in \mathcal{J}\}$, where $\mathcal{J}$ is some countable index set, such that

$$A := \bigcup_{j\in\mathcal{J}} I_j = \{t \in (0,B) : u_K^h(t) > \kappa\}.$$

For all b_j we have $u_K^h(b_j) = \kappa$ or $\varphi(b_j) = 0$. For all but at most one a_j we have $u_K^h(a_j) = \kappa$. If there is an $a_* \in \{a_j : j \in \mathcal{J}\}$ with $u_K^h(a_*) \neq \kappa$ we have $a_* = 0$. To make the proof shorter we nevertheless denote one a_j by a_* satisfying $a_* = 0$ or $u_K^h(a_*) = \kappa$. Using this notation we have

$$\begin{aligned}
&|K| \int_{\mathbb{R}_+} (u_K^h(t) \top \kappa)\varphi_t\, dt = |K| \sum_j \int_{I_j} u_K^h \varphi_t\, dt + |K| \int_{\mathbb{R}_+ \setminus A} \kappa\varphi_t\, dt \\
=\ & |K| \sum_j \int_{I_j} (u_K^h - \kappa)\varphi_t\, dt + |K| \int_{\mathbb{R}_+} \kappa\varphi_t\, dt \\
=\ & |K| \left(\sum_j \left[(u_K^h - \kappa)(b_j)\varphi(b_j) - (u_K^h - \kappa)(a_j)\varphi(a_j) \int_{I_j} (u_K^h)_t \varphi\, dt\right] - \kappa\varphi(0)\right) \\
=\ & -|K| u_K^h(a_*)\varphi(a_*) + \kappa|K|(\varphi(a_*) - \varphi(0)) - \int_A |K|(u_K^h)_t \varphi\, dt \\
\geq\ & -(u_K^h(0) \top \kappa)\varphi(0)|K| - \int_A |K|(u_K^h)_t \varphi\, dt.
\end{aligned}$$

For $t \in A$ we have by (17) and (18)

$$\begin{aligned} |K|(u_K^h)_t\varphi &= -\sum_{e\in\partial K} |e| f_{K,e}(u_K^h \top \kappa, u_{K_e}^h)\varphi \\ &\leq -\sum_{e\in\partial K} |e| f_{K,e}(u_K^h \top \kappa, u_{K_e}^h \top \kappa)\varphi \end{aligned}$$

while for $t \in \mathbb{R}\backslash A$ we have by (4),(16) and (17)

$$\begin{aligned} 0 &= \varphi \sum_{e\in\partial K} |e| f_{K,e}(u_K^h \top \kappa, \kappa) \\ &\leq -\sum_{e\in\partial K} |e| f_{K,e}(u_K^h \top \kappa, u_{K_e}^h \top \kappa)\varphi. \end{aligned}$$

Thus we get

$$\begin{aligned} &|K| \int_{\mathbb{R}_+} (u_K^h \top \kappa)\varphi_t \, dt + (u_K^h(0) \top \kappa)\varphi(0)|K| \\ - &\int_{\mathbb{R}_+} \sum_{e\in\partial K} |e| f_{K,e}(u_K^h \top \kappa, u_{K_e}^h \top \kappa)\varphi \, dt \geq 0. \end{aligned}$$

In a similar way we can prove

$$\begin{aligned} &|K| \int_{\mathbb{R}_+} (u_K^h \perp \kappa)\varphi_t \, dt + (u_K^h(0) \perp \kappa)\varphi(0)|K| \\ - &\int_{\mathbb{R}_+} \sum_{e\in\partial K} |e| f_{K,e}(u_K^h \perp \kappa, u_{K_e}^h \perp \kappa)\varphi \, dt \leq 0. \end{aligned}$$

The Lemma follows from $|u_K^h(t) - \kappa| = (u_K^h(t) \top \kappa) - (u_K^h(t) \perp \kappa)$. □

We observe that because M is compact the norms $\|f\|_{L^\infty(M)}$ and $\|\nabla f\|_g$ (which denotes the operator norm of the covariant derivative $\nabla f \in \Gamma(T^*M \otimes TM)$) are bounded by a constant C_2. This means particularly for every unit vector t tangent to M the following estimate for the covariant derivative in direction t holds: $\|\nabla_t f\|_g \leq C_2$ on $M \times [U_m, U_M]$.

Lemma 13 (Continuous entropy inequality)**.** *Provided the assumptions (10)-(13) on the grid with h small enough and (15)-(17) on the numerical fluxes, there is a constant $C > 0$ such that for every $\varphi \in C_0^\infty(M \times \mathbb{R}_+, \mathbb{R}_+)$ and $\kappa \in [U_m, U_M]$ we have*

$$\begin{aligned} &\int_0^T \int_M |u^h(x,t) - \kappa|\varphi_t(x,t) \, \mathrm{dv}_g(x)dt + \int_M |u_0(x) - \kappa|\varphi(x,0) \, \mathrm{dv}_g(x) \\ &+ \int_0^T \int_M \Big(f(x, u^h(x,t) \top \kappa) - f(x, u^h(x,t) \perp \kappa) \Big) \cdot \nabla_g \varphi(x,t) \, \mathrm{dv}_g(x) \, dt \\ &\geq -\int_M |u^h(x,0) - u_0(x)|\varphi(x,0) \, \mathrm{dv}_g(x) \\ &- 2\int_0^T \sum_{K\in\mathcal{T}} \sum_{e\in\partial K} |e| \left[\max_{(c,d)\in C(u_K^h, u_{K_e}^h)} |f_{K,e}(c,d) - f_{K,e}(c,c)| + C\delta(K) \right] r_{K,e}(t) \, dt \end{aligned}$$

with

$$r_{K,e}(t) := \frac{1}{|K||e|} \int_e \int_K \int_0^{d_g(x,y)} \|\nabla_g \varphi(\gamma_{xy}(\theta), t)\|_g \, d\theta \, \mathrm{dv}_e(y) \, \mathrm{dv}_g(x). \tag{27}$$

Proof. We start by using $\psi(t) := \frac{1}{|K|}\int_K \varphi(x,t)\ \mathrm{dv}_g(x)$ as test function in the weak discrete entropy inequality (Lemma 12) and summing over all $K \in \mathcal{T}$. Using that f is geometry compatible (4) and the consistency property of the numerical fluxes (16) we get $T_1+T_2 \leq 0$ with

$$\begin{aligned}
T_1 &:= -\int_0^T \sum_{K\in\mathcal{T}} |u_K^h(t)-\kappa| \left(\int_K \varphi(x,t)\ \mathrm{dv}_g(x)\right)_t dt \\
&\quad - \sum_{K\in\mathcal{T}} |u_K^h(0)-\kappa| \int_K \varphi(x,0)\ \mathrm{dv}_g(x) \\
&= -\int_0^T \int_M |u^h(x,t)-\kappa|\varphi_t(x,t)\ \mathrm{dv}_g(x)dt \\
&\quad - \int_M |u^h(x,0)-\kappa|\varphi(x,0)\,\mathrm{dv}_g(x) \\
T_2 &:= \int_0^T \sum_{K\in\mathcal{T}}\sum_{e\in\partial K} \frac{|e|}{|K|}\Big(f_{K,e}(u_K^h(t)\top\kappa, u_{K_e}^h(t)\top\kappa) \\
&\quad - f_{K,e}(u_K^h(t)\top\kappa, u_K^h(t)\top\kappa) - f_{K,e}(u_K^h(t)\bot\kappa, u_{K_e}^h(t)\bot\kappa) \\
&\quad + f_{K,e}(u_K^h(t)\bot\kappa, u_K^h(t)\bot\kappa)\Big) \int_K \varphi(x,t)\ \mathrm{dv}_g(x)dt.
\end{aligned}$$

Now let

$$\begin{aligned}
T_{10} &:= -\int_0^T \int_M |u^h(x,t)-\kappa|\varphi_t(x,t)\ \mathrm{dv}_g(x)dt \\
&\quad - \int_M |u_0(x)-\kappa|\varphi(x,0)\ \mathrm{dv}_g(x) \\
T_{20} &:= -\int_0^T \int_M \Big(f(x,u^h(x,t)\top\kappa) \\
&\quad - f(x,u^h(x,t)\bot\kappa)\Big)\nabla_g\varphi(x,t)\ \mathrm{dv}_g(x)dt.
\end{aligned}$$

We are going to estimate $|T_1-T_{10}|$ and $|T_2-T_{20}|$. Obviously we have

$$\begin{aligned}
|T_1-T_{10}| &\leq \int_M \Big||u^h(x,0)-\kappa| - |u_0(x)-\kappa|\Big|\,\varphi(x,0)\ \mathrm{dv}_g(x) \\
&\leq \int_M |u^h(x,0)-u_0(x)|\varphi(x,0)\ \mathrm{dv}_g(x).
\end{aligned}$$

Due to the geometry compatibility of the numerical fluxes (4) we have

$$\begin{aligned}
T_{20} &= -\int_0^T \sum_{K\in\mathcal{T}} \int_K \nabla_g\cdot\Big[\Big(f(x,u_K^h(t)\top\kappa) - f(x,u_K^h(t)\bot\kappa)\Big)\,\varphi(x,t)\Big]\ \mathrm{dv}_g(x) \\
&= -\int_0^T \sum_{K\in\mathcal{T}}\sum_{e\in\partial K} \int_e \Big(f(x,u_K^h(t)\top\kappa) - f(x,u_K^h(t)\bot\kappa)\Big)\,n_{K,e}(x)\varphi(x,t)\ \mathrm{dv}_e(x) \\
&\quad + \int_0^T \sum_{K\in\mathcal{T}}\sum_{e\in\partial K} |e|\,\Big(f_{K,e}(u_K^h(t)\top\kappa, u_{K_e}^h(t)\top\kappa) \\
&\quad - f_{K,e}(u_K^h(t)\bot\kappa, u_{K_e}^h(t)\bot\kappa)\Big)\,\frac{1}{|e|}\int_e \varphi(x,t)\ \mathrm{dv}_e(x)dt
\end{aligned}$$

because the last summand is zero due to the fact that each face e is a face of exactly two polyhedra and the conservation property (15) of the numerical fluxes. Therefore we get

$$
\begin{aligned}
|T_2 - T_{20}| \leq \int_0^T \Big| \sum_{K\in\mathcal{T}} \sum_{e\in\partial K} \int_e \Big(f(y, u_K^h(t)\top\kappa) - f(y, u_K^h(t)\bot\kappa)\Big) \\
\cdot n_{K,e}(y) \left(\varphi(y,t) - \frac{1}{|K|}\int_K \varphi(x,t)\ \mathrm{dv}_g(x)\right) \mathrm{dv}_e(y) \\
+ \quad |e| \Big(f_{K,e}(u_K^h(t)\top\kappa, u_{K_e}^h(t)\top\kappa) - f_{K,e}(u_K^h(t)\bot\kappa, u_{K_e}^h(t)\bot\kappa)\Big) \\
\left(\frac{1}{|K|}\int_K \varphi(x,t)\ \mathrm{dv}_g(x) - \frac{1}{|e|}\int_e \varphi(y,t)\ \mathrm{dv}_e(y)\right) \Big|\, dt.
\end{aligned} \tag{28}
$$

To estimate this further we need an estimate for

$$\Big| f(x, u_K^h(t)\top\kappa)\cdot n_{K,e}(x) - f_{K,e}(u_K^h\top\kappa, u_K^h\top\kappa)\Big|$$

for every $x\in e$. The fact that $f\cdot n_{K,e}$ is continuous with respect to the space variable implies due to (16)yjg1707

$$
\begin{aligned}
f_{K,e}(u_K^h\top\kappa, u_K^h\top\kappa) &= \frac{1}{|e|}\int_e f(x, u_K^h(t)\top\kappa) n_{K,e}(x)\ \mathrm{dv}_e(x) \\
&= f(\xi, u_K^h(t)\top\kappa) n_{K,e}(\xi)
\end{aligned}
$$

for some $\xi\in e$. Due to Lemma 14 below we have for every unit tangent vector $t\in T_x e$

$$
\begin{aligned}
& t\langle f(x,(u_K^h(t)\top\kappa)), n_{K,e}(x)\rangle_g \\
&= \langle \nabla_t f(x,(u_K^h(t)\top\kappa)), n_{K,e}(x)\rangle_g + \langle f(x,(u_K^h(t)\top\kappa)), \nabla_t n_{K,e}(x)\rangle_g \\
&\leq C_2.
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
& \Big| f(x,(u_K^h(t)\top\kappa)) n_{K,e}(x) - f_{K,e}(u_K^h\top\kappa, u_K^h\top\kappa)\Big| \\
&= \Big| f(x,(u_K^h(t)\top\kappa)) n_{K,e}(x) - f(\xi,(u_K^h(t)\top\kappa)) n_{K,e}(\xi)\Big| \\
&\leq \delta(e) C_2 \leq \delta(K) C_2.
\end{aligned}
$$

Using a similar estimate for the $\bot$ case we get from (28)

$$
\begin{aligned}
|T_2 - T_{20}| \leq & \int_0^T \sum_{K\in\mathcal{T}} \sum_{e\in\partial K} \Big[|e| \Big| - f_{K,e}(u_K^h(t)\top\kappa, u_K^h(t)\top\kappa) \\
& + f_{K,e}(u_K^h(t)\bot\kappa, u_K^h(t)\bot\kappa) \\
& + f_{K,e}(u_K^h(t)\top\kappa, u_{K_e}^h(t)\top\kappa) - f_{K,e}(u_K^h(t)\bot\kappa, u_{K_e}^h(t)\bot\kappa)\Big| \\
& \frac{1}{|e||K|}\int_e\int_K |\varphi(x,t)-\varphi(y,t)|\,\mathrm{dv}_e(y)\,\mathrm{dv}_g(x) \\
+ \quad & \delta(K) C \int_e \Big| \varphi(y,t) - \frac{1}{|K|}\int_K \varphi(x,t)\,\mathrm{dv}_g(x)\Big| \mathrm{dv}_e(y)\Big].
\end{aligned}
$$

For h small enough and $x \in K$ and $y \in e \in \partial K$ let γ_{xy} denote the unique minimising geodesic from x to y parametrised by arc length. Then we have

$$\begin{aligned} & \frac{1}{|K||e|}\int_e\int_K |\varphi(x,t)-\varphi(y,t)|\,\mathrm{dv}_e(y)\,\mathrm{dv}_g(x) \\ = \; & \frac{1}{|K||e|}\int_e\int_K \left|\int_0^{d_g(x,y)} \langle \nabla_g\varphi(\gamma_{xy}(s),t),\gamma'_{xy}(s)\rangle_g\,ds\right| \mathrm{dv}_g(x)\,\mathrm{dv}_e(y) \\ = \; & \frac{1}{|K||e|}\int_e\int_K \int_0^{d_g(x,y)} \|\nabla_g\varphi(\gamma_{xy}(\theta),t)\|_g\,d\theta\,\mathrm{dv}_e(y)\,\mathrm{dv}_g(x). \end{aligned}$$

This finally yields

$$\begin{aligned} |T_2-T_{20}| \le \int_0^T \sum_{K\in\mathcal{T}}\sum_{e\in\partial K} |e| \Big[\Big| & -f_{K,e}(u_K^h(t)\top\kappa, u_K^h(t)\top\kappa) \\ & + f_{K,e}(u_K^h(t)\bot\kappa, u_K^h(t)\bot\kappa) + f_{K,e}(u_K^h(t)\top\kappa, u_{K_e}^h(t)\top\kappa) \\ & - f_{K,e}(u_K^h(t)\bot\kappa, u_{K_e}^h(t)\bot\kappa)\Big| + C\delta(K)\Big] r_{K,e} \end{aligned} \tag{29}$$

with $r_{K,e}$ given in (27).

Now we want to estimate the right hand side of the above inequality (29). Due to the monotonicity (17) of the numerical fluxes we observe for $u_K^h \ge u_{K_e}^h$

$$\begin{aligned} 0 & \le -f_{K,e}(u_K^h\top\kappa, u_K^h\top\kappa) + f_{K,e}(u_K^h\top\kappa, u_{K_e}^h\top\kappa) \\ & \le \max_{u_{K_e}^h \le c\le d\le u_K^h} (-f_{K,e}(d,d)+f_{K,e}(d,c)) \\ & = \max_{(c,d)\in C(u_K^h,u_{K_e}^h)} |f_{K,e}(c,d)-f_{K,e}(c,c)| \end{aligned}$$

and for $u_K^h \le u_{K_e}^h$

$$\begin{aligned} 0 & \le f_{K,e}(u_K^h\top\kappa, u_K^h\top\kappa) - f_{K,e}(u_K^h\top\kappa, u_{K_e}^h\top\kappa) \\ & \le \max_{u_K^h \le c\le d\le u_{K_e}^h} (+f_{K,e}(c,c)-f_{K,e}(c,d)) \\ & = \max_{(c,d)\in C(u_K^h,u_{K_e}^h)} |f_{K,e}(c,d)-f_{K,e}(c,c)|. \end{aligned}$$

There are similar estimates for $\bot$ instead of $\top$ which show that

$$\begin{aligned} & \sum_{K\in\mathcal{T}}\sum_{e\in\partial K} |e| \Big| f_{K,e}(u_K^h(t)\top\kappa, u_K^h(t)\top\kappa) - f_{K,e}(u_K^h(t)\top\kappa, u_{K_e}^h(t)\top\kappa) \\ - \; & f_{K,e}(u_K^h(t)\bot\kappa, u_K^h(t)\bot\kappa) + f_{K,e}(u_K^h(t)\bot\kappa, u_{K_e}^h(t)\bot\kappa)\Big| + C\delta(K)\Big] \\ \le \; & 2\sum_{K\in\mathcal{T}}\sum_{e\in\partial K} |e| \left[\max_{(c,d)\in C(u_K^h,u_{K_e}^h)} |f_{K,e}(c,d)-f_{K,e}(c,c)| + C\delta(K)\right]. \end{aligned}$$

This implies together with (29)

$$\begin{aligned} |T_2-T_{20}| \; & \le \; \int_0^T 2\sum_{K\in\mathcal{T}}\sum_{e\in\partial K} |e| \Big[\max_{(c,d)\in C(u_K^h,u_{K_e}^h)} |f_{K,e}(c,d)-f_{K,e}(c,c)| \\ & + \; C\delta(K)\Big] r_{K,e}(t) \end{aligned}$$

which implies the Lemma. □

To finish the proof of Lemma 13 we have to prove Lemma 14.

Lemma 14. *For every vector* $t \in T_x e$ *, i.e.* t *is tangent to* e *in a point* $x \in e$ *we have*

$$\nabla_t n_{K,e} = 0.$$

Proof. We assumed that e is a geodesic line segment, so there exists a geodesic $\gamma : [a,b] \longrightarrow M$ parametrised by arc length such that $\gamma([a,b]) = e$. Then we have

$$\begin{aligned} \|\nabla_t n_{K,e}\|_g^2 &= (g(x)(\nabla_t n_{K,e}, n_{K,e}))^2 + \left(g(x)(\nabla_t n_{K,e}, \gamma')\right)^2 \\ &= \frac{1}{2} t \underbrace{(g(x)(n_{K,e}, n_{K,e}))}_{1} + t \underbrace{\left(g(x)(n_{K,e}, \gamma')\right)}_{0} - g(x)(n_{K,e}, \underbrace{\nabla_t \gamma'}_{0}) \\ &= 0. \end{aligned}$$

□

The next Lemma is a very important step in the convergence proof. There will be different estimates for $d = 1, 2$. This is due to the fact that while we have the TVD property (Lemma 9) in the $d = 1$ case, we only have the weak BV estimate (Lemma 10) in the $d = 2$ case. The proof will be done for $d = 2$ only. The proof for $d = 1$ follows from the same arguments using Lemma 9 instead of Lemma 10.

Lemma 15. *Provided the assumptions from Lemma 13 there exists a constant* $C > 0$ *depending only on* $M, g, u_0, \{f_{K,e}\}, f, \beta, k$ *such that for small enough* h *and every test function* $\alpha \in C_0^\infty(M \times \mathbb{R}_+, \mathbb{R}_+)$ *the following inequality holds*

$$\begin{aligned} &\int_{M\times\mathbb{R}_+} |u^h(x,t) - u(x,t)| \alpha_t(x,t)\, \mathrm{dv}_g(x)\, dt \\ + &\int_{M\times\mathbb{R}_+} |\tilde{f}(u(x,t) \top u^h(x,t)) - \tilde{f}(u(x,t) \bot u^h(x,t))| v(x) \cdot \nabla_g \alpha(x,t)\, \mathrm{dv}_g(x)\, dt \\ \geq &\begin{cases} -Ch^{\frac{1}{2}} & : \ d = 1 \\ -Ch^{\frac{1}{4}} & : \ d = 2 \end{cases}. \end{aligned}$$

Proof. The proof is based on a doubling of variables argument. We recall the entropy inequality (3) fulfilled by the entropy solution u of (1),(2)

$$\begin{aligned} &\int_{M\times\mathbb{R}_+} |u(y,s) - \kappa| \varphi_s(y,s) \\ + &\ [f(y, u(y,s) \top \kappa) - f(y, u(y,s) \bot \kappa)] \cdot \nabla_g \varphi(y,s)\ \mathrm{dv}_g(y) ds \\ + &\int_M |u_0(y) - \kappa| \varphi(y,0)\ \mathrm{dv}_g(y) \geq 0 \end{aligned}$$

for all $\kappa \in \mathbb{R}$ and $\varphi \in C_0^\infty(M \times \mathbb{R}_+, \mathbb{R}_+)$. In (3) we set $\kappa = u^h(x,t)$ and $\varphi(y,s) = \alpha(x,t)\chi_\varepsilon(x,y)\psi_\varepsilon(t-s)$, where χ_ε and ψ_ε are cut-off functions as defined in subsection 2.4. Now we integrate this equation with respect to x and t. In the continuous entropy inequality from Lemma 13 we set $\kappa = u(y,s)$ and $\varphi(x,t) = \alpha(x,t)\chi_\varepsilon(x,y)\psi_\varepsilon(t-s)$ and integrate with respect to y and s. Adding both equations yields

$$
\begin{aligned}
& \int_{M^2\times\mathbb{R}_+^2} |u^h(x,t)-u(y,s)| \\
& \quad \alpha_t(x,t)\chi_\varepsilon(x,y)\psi_\varepsilon(t-s)\,\mathrm{dv}_g(x)\,\mathrm{dv}_g(y)dtds \\
+ & \int_{M^2\times\mathbb{R}_+^2} \Big[f(y,u(y,s)\top u^h(x,t)) - f(y,u(y,s)\bot u^h(x,t))\Big] \\
& \quad \cdot\alpha(x,t)\nabla_{g,y}\chi_\varepsilon(x,y)\psi_\varepsilon(t-s)\,\mathrm{dv}_g(x)\,\mathrm{dv}_g(y)dtds \\
+ & \int_{M^2\times\mathbb{R}_+^2} \Big[f(x,u(y,s)\top u^h(x,t)) - f(x,u(y,s)\bot u^h(x,t))\Big] \\
& \quad \cdot\nabla_g\alpha(x,t)\chi_\varepsilon(x,y)\psi_\varepsilon(t-s)\,\mathrm{dv}_g(x)\,\mathrm{dv}_g(y)dtds \\
+ & \int_{M^2\times\mathbb{R}_+^2} \Big[f(x,u(y,s)\top u^h(x,t)) - f(x,u(y,s)\bot u^h(x,t))\Big] \\
& \quad \cdot\alpha(x,t)\nabla_{g,x}\chi_\varepsilon(x,y)\psi_\varepsilon(t-s)\,\mathrm{dv}_g(x)\,\mathrm{dv}_g(y)dtds \\
+ & \int_{M^2\times\mathbb{R}_+} |u_0(x)-u(y,s)|\alpha(x,0)\chi_\varepsilon(x,y)\psi_\varepsilon(-s)\,\mathrm{dv}_g(x)\,\mathrm{dv}_g(y)ds \\
\geq & -\int_{M^2\times\mathbb{R}_+} |u^h(x,t)-u_0(x)|\alpha(x,0)\chi_\varepsilon(x,y)\psi_\varepsilon(-s)\,ds\,\mathrm{dv}_g(y)\,\mathrm{dv}_g(x) \\
- & \; 2\int_{M\times\mathbb{R}_+}\int_0^T \sum_{K\in\mathcal{T}}\sum_{e\in\partial K} |e| \Bigg[\max_{(c,d)\in C(u_K^h,u_{K_e}^h)} |f_{K,e}(c,d)-f_{K,e}(c,c)| \\
& \qquad +C\delta(K)\Bigg] r_{K,e}(t)\,dt\,ds\,\mathrm{dv}_g(y).
\end{aligned}
\tag{30}
$$

Now we estimate term by term and use the decomposition (14) $f = \tilde{f}v$. We will start with the most difficult summand. Let E_2 be the sum of the second and fourth summand, i.e.

$$
\begin{aligned}
E_2 \;:=\; & \int_{M^2\times\mathbb{R}_+^2} \Big[\tilde{f}(u(y,s)\top u^h(x,t)) - \tilde{f}(u(y,s)\bot u^h(x,t))\Big]\,\alpha(x,t) \\
& \psi_\varepsilon(t-s)\,(v(y)\cdot\nabla_{g,y}\chi_\varepsilon(x,y) + v(x)\cdot\nabla_{g,x}\chi_\varepsilon(x,y))\,\mathrm{dv}_g(x)\,\mathrm{dv}_g(y)dtds.
\end{aligned}
$$

We also define

$$
\begin{aligned}
E_{2b} \;=\; & \int_{M^2\times\mathbb{R}_+^2} \Big[\tilde{f}(u(x,t)\top u^h(x,t)) - \tilde{f}(u(x,t)\bot u^h(x,t))\Big]\,\alpha(x,t) \\
& \psi_\varepsilon(t-s)\,(v(y)\cdot\nabla_{g,y}\chi_\varepsilon(x,y) + v(x)\cdot\nabla_{g,x}\chi_\varepsilon(x,y))\,\mathrm{dv}_g(x)\,\mathrm{dv}_g(y)dtds.
\end{aligned}
$$

Using integration by parts we get

$$
\begin{aligned}
|E_{2b}| \stackrel{(6)}{=} & \Bigg|\int_{M^2\times\mathbb{R}_+^2} \Big[\tilde{f}(u(x,t)\top u^h(x,t)) - \tilde{f}(u(x,t)\bot u^h(x,t))\Big]\,\alpha(x,t) \\
& \psi_\varepsilon(t-s)\big(v(y)\cdot\nabla_{g,y}\chi_\varepsilon(x,y) - P_{xy}v(x)\cdot\nabla_{g,y}\chi_\varepsilon(x,y)\big)\,\mathrm{dv}_g(x)\,\mathrm{dv}_g(y)dtds\Bigg| \\
= & \Bigg|\int_{M^2\times\mathbb{R}_+^2} \Big[\tilde{f}(u(x,t)\top u^h(x,t)) - \tilde{f}(u(x,t)\bot u^h(x,t))\Big]\,\alpha(x,t)\;\psi_\varepsilon(t-s) \\
& \big(-\nabla_{g,y}\cdot v(y)\chi_\varepsilon(x,y) + \nabla_{g,y}\cdot(P_{xy}v(x))\chi_\varepsilon(x,y)\big)\,\mathrm{dv}_g(x)\,\mathrm{dv}_g(y)dtds\Bigg| \\
\stackrel{(4),(7)}{\leq} & \; C\varepsilon.
\end{aligned}
$$

Furthermore the Lipschitz continuity of f with respect to u (4) and the L^∞-estimates for u_0 in (4) and u^h in Lemma 7 imply

$$\begin{aligned}|E_2 - E_{2b}| \;&\leq\; C\int_{M^2\times\mathbb{R}^2_+} |u(x,t)-u(y,s)|\alpha(x,t)\psi_\varepsilon(t-s)\\ &\quad |v(y)\cdot\nabla_{g,y}\chi_\varepsilon(x,y)+v(x)\cdot\nabla_{g,x}\chi_\varepsilon(x,y)|\,\mathrm{dv}_g(x)\,\mathrm{dv}_g(y)dtds.\end{aligned}$$

Now we cover M with finitely many geodesic balls $B_{r_1}(x_1),\dots,B_{r_N}(x_N)$ where $r_i \leq \frac{R}{3}$ and R is chosen such that for every $x\in M$ the mapping $\exp_x^{-1}|_{B_R(x)}$ is a chart. Furthermore we restrict to the $\varepsilon < \min r_i$ case. Let $B_i := B_{r_i}(x_i)$ and $\tilde{B}_i = B_{2r_i}(x_i)$.

$$\begin{aligned}|E_2 - E_{2b}| \;&\leq\; C\sum_i\int_{\mathbb{R}^2_+}\int_{B_i}\int_{\tilde{B}_i} |u(x,t)-u(y,s)|\alpha(x,t)\psi_\varepsilon(t-s)\\ &\quad |v(y)\cdot\nabla_{g,y}\chi_\varepsilon(x,y)+v(x)\cdot\nabla_{g,x}\chi_\varepsilon(x,y)|\,\mathrm{dv}_g(x)\,\mathrm{dv}_g(y)dtds\\ &\leq\; C\sum_i\int_{\mathbb{R}^2_+}\int_{B_{r_i}(0)}\int_{B_{2r_i}(0)} |u(\exp_{x_i}(a),t)-u(\exp_{x_i}(b),s)|\\ &\quad \alpha(\exp_{x_i}(a),t)\psi_\varepsilon(t-s)\,\big|v(\exp_{x_i}(b))\cdot\nabla_{g,y}\chi_\varepsilon(\exp_{x_i}(a),\exp_{x_i}(b))\\ &\quad +v(\exp_{x_i}(a))\cdot\nabla_{g,x}\chi_\varepsilon(\exp_{x_i}(a),\exp_{x_i}(b))\big|\,da\,db\,dt\,ds.\end{aligned}\tag{31}$$

Here we use that the metric tensors are bounded.

Let γ_{xy} be the minimising geodesic from x to y parametrised by arc-length. Then we have

$$\begin{aligned}&v(x)\cdot\nabla_{g,x}\chi_\varepsilon(x,y)+v(y)\cdot\nabla_{g,y}\chi_\varepsilon(x,y)\\ =\;&\frac{1}{\varepsilon^3}\chi'\left(\frac{d_g(x,y)}{\varepsilon}\right)\big(v(x)\cdot\gamma'_{xy}(0)-v(y)\cdot\gamma'_{xy}(d_g(x,y))\big).\end{aligned}$$

Because v is smooth we have

$$\gamma'_{xy}(\langle v,\gamma'_{xy}\rangle_g) = \langle\nabla_{\gamma'_{xy}}v,\gamma'_{xy}\rangle_g + \langle v,\nabla_{\gamma'_{xy}}\gamma'_{xy}\rangle_g \leq \|\nabla v\|_g.\tag{32}$$

Thus

$$\begin{aligned}v(x)\cdot\gamma'_{xy}(0)-v(y)\cdot\gamma'_{xy}(d_g(x,y)) \;&=\; \int_0^{d_g(x,y)}\gamma'_{xy}(\langle v(s),\gamma'_{xy}(s)\rangle_g)\,ds\\ &\overset{(32)}{\leq}\; \|\nabla v\|_{L^\infty(M)}d_g(x,y).\end{aligned}$$

Then we have by definition of χ_ε in (8) for all $a,b\in B_R(0)$

$$\begin{aligned}&\big|v(\exp_{x_i}(b))\cdot\nabla_{g,y}\chi_\varepsilon(\exp_{x_i}(a),\exp_{x_i}(b))+v(\exp_{x_i}(a))\cdot\nabla_{g,x}\chi_\varepsilon(\exp_{x_i}(a),\exp_{x_i}(b))\big|\\ &\qquad\leq C\varepsilon^{-2}\mathbb{I}_{\{d_g(\exp_{x_i}(a),\exp_{x_i}(b))<\varepsilon\}}.\end{aligned}\tag{33}$$

Because the derivative of $\exp_{x_i}^{-1}$ is bounded there exists a constant $C_i > 0$ such that

$$d_g(\exp_{x_i}(a),\exp_{x_i}(b)) > C_i\|a-b\|,\ \forall a\in B_{r_i}(0), b\in B_{2r_i}(0)\tag{34}$$

which implies

$$\mathbb{I}_{\{d_g(\exp_{x_i}(a),\exp_{x_i}(b))<\varepsilon\}} \leq \mathbb{I}_{\{\|b-a\|<\frac{\varepsilon}{C_i}\}}.$$

Hence we have by (31) and (33)

$$\begin{aligned}|E_2 - E_{2b}| \;&\leq\; C\sum_i\int_{\mathbb{R}^2_+}\int_{B_{r_i}(0)}\int_{B_{2r_i}(0)} |u(\exp_{x_i}(a),t)-u(\exp_{x_i}(b),s)|\\ &\quad \varepsilon^{-2}\mathbb{I}_{\{\|b-a\|<\frac{\varepsilon}{C_i}\}}\,da\,db\,dt\,ds\\ &\leq\; C\varepsilon\end{aligned}$$

because each $u \circ \exp_{x_i}$ has bounded variation. Finally we have

$$|E_2| \leq C\varepsilon.$$

Let

$$\begin{aligned} E_1 &:= \int_{M^2\times\mathbb{R}_+^2} |u^h(x,t)-u(y,s)|\alpha_t(x,t)\chi_\varepsilon(x,y)\psi_\varepsilon(t-s)\,\mathrm{dv}_g(x)\,\mathrm{dv}_g(y)dtds, \\ E_{1b} &:= \int_{M\times\mathbb{R}_+} |u^h(x,t)-u(x,t)|\alpha_t(x,t)\,\mathrm{dv}_g(x)\,dt. \end{aligned}$$

Due to (9) we have

$$\begin{aligned} |E_1-E_{1b}| &\leq \int_{M^2\times\mathbb{R}_+^2} |u(x,t)-u(y,s)|\alpha_t(x,t)\chi_\varepsilon(x,y) \\ &\qquad \psi_\varepsilon(t-s)\,\mathrm{dv}_g(x)\,\mathrm{dv}_g(y)\,dt\,ds + C\varepsilon^2. \end{aligned}$$

To estimate the first part of the right hand side we again cover M with balls B_i like in the estimate for E_2. From the definition of χ_ε in (8) we know that χ is decreasing for positive x , this yields the following inequality

$$\begin{aligned} |E_1-E_{1b}| &\leq \sum_i \int_{\mathbb{R}_+^2}\int_{B_{r_i}(0)}\int_{B_{r_i}(0)} |u(\exp_{x_i}(a),t)-u(\exp_{x_i}(b),s)| \\ &\qquad |\alpha_t(\exp_{x_i}(a),t)|\frac{1}{\varepsilon^2}\chi\left(\frac{C_i\|b-a\|}{\varepsilon}\right)\psi_\varepsilon(t-s)\,da\,db\,dt\,ds \\ &\qquad +C\varepsilon^2 \\ &\leq C\varepsilon, \end{aligned}$$

because the L^1 -norms of the $\frac{1}{\varepsilon^2}\chi\left(\frac{C_i\|b-a\|}{\varepsilon}\right)$ are uniformly bounded with respect to ε. The constants C_i were chosen like in (34).

Let

$$\begin{aligned} E_3 &:= \int_{M^2\times\mathbb{R}_+^2} |\tilde f(u(y,s)\top u^h(x,t))-\tilde f(u(y,s)\bot u^h(x,t))| \\ &\qquad v(x)\cdot\nabla_g\alpha(x,t)\chi_\varepsilon(x,y)\psi_\varepsilon(t-s)\,\mathrm{dv}_g(x)\,\mathrm{dv}_g(y)\,dt\,ds \\ E_{3b} &:= \int_{M\times\mathbb{R}_+} |\tilde f(u(x,t)\top u^h(x,t))-\tilde f(u(x,t)\bot u^h(x,t))| \\ &\qquad v(x)\cdot\nabla_g\alpha(x,t)\,\mathrm{dv}_g(x)\,dt. \end{aligned}$$

Then we have

$$\begin{aligned} |E_3-E_{3b}| &\leq C\int_{M^2\times\mathbb{R}_+^2} |u(y,s)-u(x,t)|v(x)\cdot\nabla_g\alpha(x,t) \\ &\qquad \chi_\varepsilon(x,y)\psi_\varepsilon(t-s)\,\mathrm{dv}_g(x)\,\mathrm{dv}_g(y)\,dt\,ds + C\varepsilon^2 \\ &\leq C\varepsilon, \end{aligned}$$

like in the estimate for E_1. To estimate the fifth summand on the left hand side of (30), deonted E_4, we consider the entropy inequality (3) fulfilled by u. For fixed $x\in M$ we define

$$\varphi(x,y,s) := \alpha(x,0)\chi_\varepsilon(x,y)\int_s^\infty \psi_\varepsilon(-\tau)\,d\tau.$$

and $\kappa = u_0(x)$. Then integration with respect to x yields

$$\begin{aligned}
&- \int_{M^2\times\mathbb{R}_+} |u(y,s)-u_0(x)|\alpha(x,0)\chi_\varepsilon(x,y)\psi_\varepsilon(-s)\,\mathrm{dv}_g(x)\,\mathrm{dv}_g(y)\,ds\\
&+ \int_{M^2\times\mathbb{R}_+} \Big(\tilde f(u(y,s)\top u_0(x)) - \tilde f(u(y,s)\bot u_0(x))\Big)\, v(y)\cdot\nabla_{g,y}\chi_\varepsilon(x,y)\\
&\quad \alpha(x,0)\left(\int_s^\infty \psi_\varepsilon(-\tau)d\tau\right)\mathrm{dv}_g(x)\,\mathrm{dv}_g(y)\,ds\\
&+ \int_{M^2} |u_0(y)-u_0(x)|\alpha(x,0)\chi_\varepsilon(x,y)\left(\int_0^\infty \psi_\varepsilon(-\tau)d\tau\right)\mathrm{dv}_g(x)\,\mathrm{dv}_g(y) \geq 0.
\end{aligned}$$

We note that the first summand here is exactly $-E_4$, thus we deonte the summands by $-E_4, E_5, E_6$ respectively. To estimate E_5 we define E_{5b} by

$$\begin{aligned}
E_{5b} &:= \int_{M^2\times\mathbb{R}_+}\int_s^\infty \Big(\tilde f(u(y,s)\top u_0(y)) - \tilde f(u(y,s)\bot u_0(y))\Big)\\
&\qquad v(y)\cdot\nabla_{g,y}\chi_\varepsilon(x,y)\alpha(x,0)\psi_\varepsilon(-\tau)\,d\tau\,ds\ \mathrm{dv}_g(x)\,\mathrm{dv}_g(y)\\
&\overset{\varepsilon\leq R}{=} -\int_{M^2\times\mathbb{R}_+}\int_s^\infty \Big(\tilde f(u(y,s)\top u_0(y)) - \tilde f(u(y,s)\bot u_0(y))\Big)\\
&\qquad (P_{yx}v(y))\cdot\nabla_{g,x}\chi_\varepsilon(x,y)\alpha(x,0)\psi_\varepsilon(-\tau)\,d\tau\,ds\ \mathrm{dv}_g(x)\,\mathrm{dv}_g(y)\\
&= \int_{M^2\times\mathbb{R}_+}\int_s^\infty \Big(\tilde f(u(y,s)\top u_0(y)) - \tilde f(u(y,s)\bot u_0(y))\Big)\\
&\qquad [\nabla_{g,x}\cdot P_{yx}(v(y))]\,\chi_\varepsilon(x,y)\alpha(x,0)\psi_\varepsilon(-\tau)\,d\tau\,ds\ \mathrm{dv}_g(x)\,\mathrm{dv}_g(y)\\
&+ \int_{M^2\times\mathbb{R}_+}\int_s^\infty \Big(\tilde f(u(y,s)\top u_0(y)) - \tilde f(u(y,s)\bot u_0(y))\Big)\\
&\qquad \chi_\varepsilon(x,y)(P_{yx}v(y))\cdot\nabla_g\alpha(x,0)\psi_\varepsilon(-\tau)\,d\tau\,ds\ \mathrm{dv}_g(x)\,\mathrm{dv}_g(y).
\end{aligned}$$

From the arguments for the estimate of E_2 we know that $\nabla_{g,x}\cdot P_{yx}(v(y))$ is well-defined for x in an ε-neighbourhood of y. We use the fact that the Levi-Civita connection is compatible with the metric, which implies $\|P_{yx}(v(y))\|_g = \|v(y)\|_g$, and the divergence of $P_{yx}(v(y))$ with respect to x is bounded. So the integrands of the above integrals are bounded and the supports with respect to s lie in $[0,\varepsilon]$, so $E_{5b}\leq C\varepsilon$. Furthermore we have

$$\begin{aligned}
|E_5 - E_{5b}| &\leq C\int_{M^2\times\mathbb{R}_+}\int_s^\infty |u_0(x)-u_0(y)||v(y)\cdot\nabla_{g,y}\chi_\varepsilon(x,y)|\psi_\varepsilon(-\tau)\\
&\qquad d\tau\,ds\,\mathrm{dv}_g(y)\,\mathrm{dv}_g(x).
\end{aligned}$$

Integrating with respect to τ and s yields

$$|E_5 - E_{5b}| \leq C\int_{M^2} |u_0(x)-u_0(y)||v(y)\cdot\nabla_{g,y}\chi_\varepsilon(x,y)|\varepsilon\,\mathrm{dv}_g(y)\,\mathrm{dv}_g(x)$$

because the integral over τ is bounded by 1 and the support with respect to s lies in $[0,\varepsilon]$. Then we use the fact

$$\varepsilon|v(y)\cdot\nabla_{g,y}\chi_\varepsilon(x,y)| \leq C\varepsilon^{-2}\mathbb{I}_{\{d_g(x,y)<\varepsilon\}}.$$

We cover M with balls like in the estimate for E_2 again and a similar argument yields

$$|E_5 - E_{5b}| \leq C\varepsilon.$$

Another version of this argument implies

$$|E_6| \leq C\varepsilon.$$

So we finally have

$$0 \leq E_4 \leq C\varepsilon.$$

Now we have to find an estimate for the right hand side of (30): Keeping in mind the weak BV-estimate (10), the essential part of this estimate is an estimate for

$$\left| \int_{M \times \mathbb{R}_+} r_{K,e}(t)\, ds\, \mathrm{dv}_g(y) \right|,$$

where $r_{K,e}$ was defined in (27). Because the test function φ in the definition of $r_{K,e}$ now has the form

$$\varphi(x,t) = \alpha(x,t)\chi_\varepsilon(x,y)\psi_\varepsilon(t-s)$$

we have

$$\begin{aligned}
& \left| \int_{M \times \mathbb{R}_+} r_{K,e}(t)\ \mathrm{dv}_g(x) ds \right| \\
= \quad & \frac{C}{|K||e|} \int_M \int_{\mathbb{R}_+} \int_K \int_e \int_0^{d_g(x,z)} \|\nabla_g \alpha(\gamma_{xz}(\theta),t)\|_g \\
& \chi_\varepsilon(\gamma_{xz}(\theta),y)\psi_\varepsilon(t-s) d\theta\, \mathrm{dv}_e(z)\, \mathrm{dv}_g(x)\, ds\ \mathrm{dv}_g(y) \\
+ \quad & \frac{C}{|K||e|} \int_M \int_{\mathbb{R}_+} \int_K \int_e \int_0^{d_g(x,z)} \|\nabla_1 \chi_\varepsilon(\gamma_{xz}(\theta),y)\|_g \\
& \alpha(\gamma_{xz}(\theta),t)\psi_\varepsilon(t-s) d\theta\, \mathrm{dv}_e(z)\, \mathrm{dv}_g(x)\, ds\ \mathrm{dv}_g(y).
\end{aligned} \tag{35}$$

Now integration over s, y yields that the first summand in (35) can be estimated by

$$\begin{aligned}
& \frac{C}{|K||e|} \int_K \int_e \int_0^{d_g(x,z)} \underbrace{\|\nabla_g \alpha(\gamma_{xz}(\theta),t)\|_g}_{\leq \|\nabla_g \alpha\|_{L^\infty(M)}} (1 + C\varepsilon^2) d\theta\, \mathrm{dv}_e(z)\, \mathrm{dv}_g(x) \\
& \leq C(1+\varepsilon^2)\delta(K).
\end{aligned}$$

We know that

$$\int_{B_\varepsilon(x)} \mathrm{dv}_g(x) = \int_{B_\varepsilon(0) \subset T_xM} \underbrace{|\det((T\exp_x)_v)|}_{\text{bounded for } |v| < \frac{R}{2}}\, dv \leq C\varepsilon^2.$$

To estimate the second summand in (35) we observe

$$\|\nabla_1 \chi_\varepsilon(\gamma_{xz}(\theta),y)\|_g \quad \leq \quad C\varepsilon^{-3}\mathbb{I}_{\{d_g(\gamma_{xz}(\theta),y) \leq \varepsilon\}}.$$

Then integration over s and y yields that the second summand in (35) is smaller than

$$\frac{C}{|K||e|} \int_K \int_e \int_0^{d_g(x,y)} C\varepsilon^{-1}\, d\theta\, \mathrm{dv}_e(z)\, \mathrm{dv}_g(x) \leq C\frac{\delta(K)}{\varepsilon}.$$

So we have due to the weak BV estimate Lemma 10

$$\begin{aligned}
& \int_{M \times \mathbb{R}_+} \int_0^T \sum_{K \in \mathcal{T}} \sum_{e \in \partial K} \left[\max_{(c,d) \in C(u_K^h, u_{K_e}^h)} |e||f_{K,e}(c,d) - f_{K,e}(c,c)| \right] r_{K,e}\, \mathrm{dv}_g(x) dt ds \\
& \qquad \leq \frac{C}{\sqrt{h}} \left(\frac{h}{\varepsilon} + h + h\varepsilon^2 \right) = C\sqrt{h} \left(\frac{1}{\varepsilon} + 1 + \varepsilon^2 \right).
\end{aligned} \tag{36}$$

We observe that due to (10)-(13) we have

$$\int_0^T \sum_{K \in \mathcal{T}} \sum_{e \in \partial K} C\delta(K)|e| \leq 3C\beta^{-1}k.$$

Now it remains to estimate

$$\int_{M^2\times\mathbb{R}_+} |u^h(x,t)-u_0(x)|\alpha(x,0)\chi_\varepsilon(x,y)\psi_\varepsilon(-s)\,ds\,\mathrm{dv}_g(y)\,\mathrm{dv}_g(x).$$

Integrating with respect to y,s yields that this term is smaller than

$$C(1+C\varepsilon^2)\int_M |u^h(x,0)-u_0(x)|\,\mathrm{dv}_g(x)\leq C(1+\varepsilon^2)h$$

for ε,h small enough by Lemma 6. This finally implies

$$\begin{aligned}
&\int_{M\times\mathbb{R}_+} |u^h(x,t)-u(x,t)|\alpha_t(x,t)\,\mathrm{dv}_g(x)\,dt\\
+\ &\int_{M\times\mathbb{R}_+} |\tilde f(u(x,t)\top u^h(x,t))-\tilde f(u(x,t)\bot u^h(x,t))|v(x)\cdot\nabla_g\alpha(x,t)\\
&\mathrm{dv}_g(x)\,dt\\
\geq\ & -C(\varepsilon+Ch+h\varepsilon^2+\frac{\sqrt{h}}{\varepsilon}+\sqrt{h}+2\sqrt{h}\varepsilon^2+h\varepsilon^2)\\
=\ & -C(h^{\frac14}+h+h^{\frac32}+h^{\frac14}+h^{\frac12}+h+h^{\frac32})
\end{aligned}$$

where we set $\varepsilon=h^{\frac14}$ for the last equality. □

Now the convergence proof is quite easy and only consists of choosing a sensible test function α in Lemma 15.

Theorem 16. *Provided the assumptions of Lemma 15 hold, then we have for every time $T>0$ a constant $C>0$ depending only on $f,u_0,M,g,T,\{f_{K,e}\},\beta,k$ such that*

$$\int_0^T\int_M |u^h(x,t)-u(x,t)|\,\mathrm{dv}_g(x)\,dt\leq\begin{cases} Ch^{\frac12} & : \quad d=1\\ Ch^{\frac14} & : \quad d=2,\end{cases}$$

Proof. For $t\geq 0$ we define

$$\rho(t):=\begin{cases}(1-t)\exp\left(\frac{1}{t^2-1}\right) & : \quad t\leq 1\\ 0 & : \quad t\geq 1\end{cases}$$

and $\rho_T(t):=\rho\left(\frac{t}{2T}\right)$. We have $\rho_T(t)\in[0,e]$ and there exists $\varepsilon>0$ such that $\rho'(t)<-\varepsilon\ \forall t\in[0,\frac12]$ and therefore $\rho_T'(t)<\frac{-\varepsilon}{2T}\ \forall t\in[0,T]$. Now we define $\alpha(x,t)=\rho_T(t)$, then we have $\alpha\in C_0^\infty(M\times\mathbb{R}_+,\mathbb{R}_+)$ and $\nabla_g\alpha=0$ which implies

$$\begin{aligned}
&\frac{-\varepsilon}{2T}\int_M\int_0^T |u^h(x,t)-u(x,t)|\,\mathrm{dv}_g(x)\,dt\\
\geq\ &\int_M\int_0^T \alpha_t(x,t)|u^h(x,t)-u(x,t)|\,\mathrm{dv}_g(x)\,dt\\
\geq\ &\begin{cases}-Ch^{\frac12} & : \quad d=1\\ -Ch^{\frac14} & : \quad d=2,\end{cases}
\end{aligned}$$

which proves the Theorem. □

6. Appendix

6.1. **Proof of Lemma 4.** We cover M with balls $B_{\frac{R}{4}}(x_1),\dots,B_{\frac{R}{4}}(x_N)$. Then we fix one $l\in\{1,\dots,N\}$ and identify $B_{\frac{R}{2}}(x_l)$ with $B_{\frac{R}{2}}(0)\subset\mathbb{R}^2$ by $\exp_{x_l}$, with the induced metric tensor. Let the tangent spaces be trivialised by an orthonormal frame over $B_{\frac{R}{2}}(0)$. Because $\exp_{x_l}$ is an Diffeomorphism on $B_R(x_l)$, the g_{ij} and all their derivatives are bounded on $B_{\frac{R}{2}}(0)$. Especially the Christoffel symbols Γ_{ij}^k are bounded in this domain and $\sqrt{|g_{ij}|} > \varepsilon$ for some $\varepsilon > 0$. Due to the same cause there exists $C > 0$ such that $\|(D\exp_x)_v\| < C$ and $\|(D\exp_x^{-1})_y\| < C$ for all $x, y, v \in B_{\frac{R}{3}}(0)$. Let $\tilde{v}(x,\xi) = P_{\xi x}\bar{v}(\xi)$, i.e. the vector achieved from $\bar{v}(\xi)$ by parallel transport along the unique minimising geodesic from ξ to x. We want to define $\tilde{v}$ in this way on $\tilde{B} := \{(x,\xi) \in \mathbb{R}^2 : \xi \in B_{\frac{R}{4}}(0), d_g(\exp_{x_l}^{-1}(x), \exp_{x_l}^{-1}(\xi)) < \frac{R}{4}\}$. This means $\tilde{v}$ fulfils the differential equation

$$D_1\tilde{v}(x,\xi)\cdot w(x,\xi) + \tilde{v}^T(x,\xi)\Gamma(x)\cdot w(x,\xi) = 0 \text{ in } \tilde{B} \tag{37}$$

$$\tilde{v}(\xi,\xi) = \bar{v}(\xi) \text{ in } B_{\frac{R}{4}}(0) \tag{38}$$

where $w(x,y) := \exp_x^{-1}(y)$ and D_1 denotes the Jacobi-matrix with respect to the first variable. Then we see that (37),(38) is equivalent to

$$\begin{aligned} &D_1\tilde{v}(x,\xi)(D\exp_\xi)_{w(\xi,x)}w(\xi,x) \\ &+\tilde{v}^T(x,\xi)\Gamma(x)(D\exp_\xi)_{w(\xi,x)}w(\xi,x) = 0 \text{ in } \tilde{B} \end{aligned} \tag{39}$$

$$\tilde{v}(\xi,\xi) = \bar{v}(\xi) \text{ in } B_{\frac{R}{4}}(0) \tag{40}$$

because $(D\exp_\xi)_{w(\xi,x)}\cdot w(\xi,x) = -w(x,\xi)$. Now we consider the following diffeomorphism

$$\begin{aligned} \Phi : \tilde{B} &\longrightarrow B_{\frac{R}{4}}(0) \times B_{\frac{R}{4}}(0) \\ (x,\xi) &\mapsto (\exp_\xi^{-1}(x), \xi) \\ (\exp_\xi(a), \xi) &\leftarrow\!\shortmid (a, \xi). \end{aligned}$$

We define $v = \tilde{v}\circ\Phi^{-1}$ and an easy calculation yields (39),(40) is equivalent to

$$\begin{aligned} &D_1 v(x,\xi)\cdot x \\ &+v^T(x,\xi)\Gamma(\exp_\xi(x))(D\exp_\xi)_x\cdot x = 0 \text{ in } B_{\frac{R}{4}}(0)\times B_{\frac{R}{4}}(0) \end{aligned} \tag{41}$$

$$\tilde{v}(0,\xi) = \bar{v}(\xi) \text{ in } B_{\frac{R}{4}}(0). \tag{42}$$

Let us recall that

$$A(x,\xi) := \Gamma(\exp_\xi(x))(D\exp_\xi)_x : B_{\frac{R}{3}}(0)\times B_{\frac{R}{3}}(0) \longrightarrow M_{2\times 2}(\mathbb{R})\otimes\mathbb{R}^2$$

is smooth. We will express x in polar coordinates (r,θ) such that (41) becomes

$$\partial_r v(r,\theta,\xi) + v^T(r,\theta,\xi)A(r\cos(\theta), r\sin(\theta),\xi)\begin{pmatrix}\cos(\theta)\\ \sin(\theta)\end{pmatrix} = 0 \tag{43}$$

$$\text{for } r < \frac{R}{4}, \theta\in[0,2\pi], \xi\in B_{\frac{R}{4}}(0)$$

$$v(0,\theta,\xi) = \bar{v}(\xi) \text{ for } \theta\in[0,2\pi], \xi\in B_{\frac{R}{4}}(0). \tag{44}$$

Because $A(r\cos(\theta), r\sin(\theta),\xi)\begin{pmatrix}\cos(\theta)\\ \sin(\theta)\end{pmatrix}$ is smooth and bounded as a function of r,θ,ξ, we know by ODE theory that $v(r,\theta,\xi) \in C^2((0,\frac{1}{4})\times[0,2\pi]\times B_{\frac{R}{4}}(0), \mathbb{R}^2)$ and hence $v(x,y,\xi) \in C^2(B_{\frac{R}{4}}(0)\setminus\{0\}\times B_{\frac{R}{4}}(0), \mathbb{R}^2)$. We know $v|_{\{0\}\times B_{\frac{R}{4}}(0)} = \bar{v}$ is smooth. Furthermore we can show that for $f = v, v_x, v_y, v_{xx}, v_{xy}, v_{yy}$ the limit

$$\lim_{r\to 0} f(r,\theta,\xi)$$

exists, is independent of θ, depends smooth on ξ and the convergence is uniform w.r.t. ξ and θ. Given a sequence (r_n,θ_n,ξ_n) with $\lim_{n\to\infty} r_n = 0$ and $\lim_{n\to\infty}\xi_n = \xi_*$ we have

$$\begin{array}{rl} & |f(r_n,\theta_n,\xi_n)-f(0,\xi_*)| \\ \leq & \underbrace{|f(r_n,\theta_n,\xi_n)-f(0,\xi_n)|}_{\to 0 \text{ because conv. uniform in } \theta,\xi} + \underbrace{|f(0,\xi_n)-f(0,\xi_*)|}_{\to 0 \text{ because the limit depends smooth on } \xi}. \end{array}$$

Hence f is continuous on $B_{\frac{R}{4}}(0)\times B_{\frac{R}{4}}(0)$ and $v\in C^2(B_{\frac{R}{4}}(0)\times B_{\frac{R}{4}}(0),\mathbb{R}^2)$. The transition from $\tilde v$ to v was done by a diffeomorphism and so $\tilde v\in C^2(\tilde B,\mathbb{R}^2)$. Particularly $\nabla_x \mathrm{div}_x\tilde v(x,\xi)$ is continuous. The divergence of a vectorfield is the trace of its covariant derivative so by construction $\mathrm{div}_x\tilde v(x,\xi)$ vanishes for $x=0$. So there exists a constant C_l such that

$$|\mathrm{div}_x\tilde v(x,\xi)|<C_l d_g(x,\xi) \text{ for } \xi\in B_{\frac{R}{4}}(x_l), d_g(x,\xi)<\frac{R}{8}.$$

The Lemma follows because there are only finitely many l.

6.2. **Proof of Lemma 6.**

Proof. Miranda et. al. showed in [15] that there exists a sequence $(f_j)_j\in C^\infty(M)$ such that

$$\|f_j-u\|_{L^1(M)}\leq\frac{1}{j} \quad\text{and}\quad \lim_{j\to\infty}\int_M\|\nabla_g f_j\|_g\,\mathrm{dv}_g(x)=\mathrm{TV}_M(u)<\infty. \tag{45}$$

For every j we have

$$\begin{aligned} \|u-\bar u\|_{L^1(M)} &\leq \|u-f_j\|_{L^1(M)}+\|f_j-\bar f_j\|_{L^1(M)}+\|\bar f_j-\bar u\|_{L^1(M)}\\ \|\bar f_j-\bar u\|_{L^1(M)} &= \sum_K\|\bar f_j-\bar u\|_{L^1(K)}\leq\sum_K\|f_j-u\|_{L^1(K)}=\|f_j-u\|_{L^1(M)}\\ \Rightarrow \|u-\bar u\|_{L^1(M)} &\leq 2\|u-f_j\|_{L^1(M)}+\|f_j-\bar f_j\|_{L^1(M)}. \end{aligned}$$

Furthermore we have for every $K\in\mathcal{T}$

$$\begin{aligned} \|f_j-\bar f_j\|_{L^1(K)} &\leq \int_K\left|f_j(x)-\frac{1}{|K|}\int_K f_j(y)\,\mathrm{dv}_g(y)\right|\mathrm{dv}_g(x)\\ &\leq \frac{1}{|K|}\int_{K^2}|f_j(x)-f_j(y)|\,\mathrm{dv}_g(y)\,\mathrm{dv}_g(x). \end{aligned}$$

Because K is convex for every pair of points $x,y\in K$ there is a unique minimising geodesic from x to y. It can be written as

$$\gamma:[0,1]\longrightarrow M\quad \theta\mapsto\begin{cases}\exp_y((1-\theta)\exp_y^{-1}(x)) & \text{for}\quad 0\leq\theta\leq\frac12\\ \exp_x(\theta\exp_x^{-1}(y)) & \text{for}\quad \frac12\leq\theta\leq 1.\end{cases}$$

This implies

$$\begin{aligned} \|f_j-\bar f_j\|_{L^1(K)} \leq\ & \frac{1}{|K|}\int_{K^2}\int_0^{\frac12}\Big|\nabla_g f_j(\exp_y((1-\theta)\exp_y^{-1}(x)))\\ &\cdot(T\exp_y)_{(1-\theta)\exp_y^{-1}(x)}(\exp_y^{-1}(x))\Big|\,d\theta\,\mathrm{dv}_g(x)\,\mathrm{dv}_g(y)\\ +\ & \frac{1}{|K|}\int_{K^2}\int_{\frac12}^1\Big|\nabla_g f_j(\exp_x(\theta\exp_x^{-1}(y)))\\ &\cdot(T\exp_x)_{\theta\exp_x^{-1}(y)}(\exp_x^{-1}(y))\Big|\,d\theta\,\mathrm{dv}_g(x)\,\mathrm{dv}_g(y). \end{aligned}\tag{46}$$

We have

$$\|(T\exp_y)_{(1-\theta)\exp_y^{-1}(x)}(\exp_y^{-1}(x))\|_g\leq C\delta(K), \tag{47}$$

because $\exp$ is smooth on the compact set

$$\mathcal{K} := \left\{ (x,v) : x \in M, v \in T_x M, \|v\|_g \leq \frac{R}{2} \right\},$$

the operator norm is continuous and

$$\| \exp_y^{-1}(x)\|_g = \| \exp_x^{-1}(y)\|_g \leq \delta(K).$$

Inserting (47) in (46) implies

$$\begin{aligned}
\|f_j - \bar{f}_j\|_{L^1(K)} &\leq \frac{C\delta(K)}{|K|} \int_{K^2} \int_0^{\frac{1}{2}} \|\nabla_g f_j(\exp_y((1-\theta)\exp_y^{-1}(x)))\|_g \\
&\quad d\theta \, \mathrm{dv}_g(x) \, \mathrm{dv}_g(y) \\
&+ \frac{C\delta(K)}{|K|} \int_{K^2} \int_{\frac{1}{2}}^1 \|\nabla_g f_j(\exp_x(\theta \exp_x^{-1}(y)))\|_g \\
&\quad d\theta \, \mathrm{dv}_g(x) \, \mathrm{dv}_g(y) \\
&= \frac{C\delta(K)}{|K|} \int_K \int_0^{\frac{1}{2}} \int_{(1-\theta)\exp_y^{-1}(K)} \|\nabla_g f_j(\exp_y(w))\|_g \\
&\quad \frac{1}{(1-\theta)^d} |\det(T\exp_y)_w| \, d\theta \, dw \, \mathrm{dv}_g(y) \\
&+ \frac{C\delta(K)}{|K|} \int_K \int_{\frac{1}{2}}^1 \int_{\theta \exp_x^{-1}(K)} \|\nabla_g f_j(\exp_x(v))\|_g \\
&\quad \frac{1}{\theta^d} |\det(T\exp_x)_v| \, d\theta \, dv \, \mathrm{dv}_g(x)
\end{aligned}$$

where the determinants are computed with respect to orthonormal bases of the respective tangent spaces. The determinant of $(T\exp_x)_v$ is continuous and positive on $\mathcal{K}$ so there exists $C > 0$ such that

$$\frac{1}{C} < |\det(T\exp_x)_v| < C \;\forall (x,v) \in \mathcal{K}. \tag{48}$$

We have

$$\begin{aligned}
\|f_j - \bar{f}_j\|_{L^1(K)} &\leq \frac{C\delta(K)}{|K|} \int_K \int_{\exp_y((1-\theta)\exp_y^{-1}(K))} \|\nabla_g f_j(z)\|_g \\
&\quad |\det((T\exp_y^{-1})_z)| \, \mathrm{dv}_g(z) \, \mathrm{dv}_g(y) \\
&+ \frac{C\delta(K)}{|K|} \int_K \int_{\exp_x(\theta \exp_x^{-1}(K))} \|\nabla_g f_j(z)\|_g \\
&\quad |\det((T\exp_x^{-1})_z)| \, \mathrm{dv}_g(z) \, \mathrm{dv}_g(x).
\end{aligned}$$

Because the interior of K is convex we have

$$\begin{aligned}
\exp_y((1-\theta)\exp_y^{-1}(K)) &\subset K \text{ for } 0 \leq \theta \leq 1 \text{ and } x,y \in K \\
\exp_x(\theta \exp_x^{-1}(K)) &\subset K \text{ for } 0 \leq \theta \leq 1 \text{ and } x,y \in K.
\end{aligned}$$

This implies by (48)

$$\begin{aligned}\|f_j - \bar f_j\|_{L^1(K)} &\leq \frac{C\delta(K)}{|K|}\int_{K^2}\|\nabla_g f_j(z)\|_g \,\mathrm{dv}_g(z)\,\mathrm{dv}_g(y)\\ &\quad + \frac{C\delta(K)}{|K|}\int_{K^2}\|\nabla_g f_j(z)\|_g \,\mathrm{dv}_g(z)\,\mathrm{dv}_g(x)\\ &\leq C\delta(K)\|\nabla_g f_j\|_{L^1(K)}.\end{aligned}$$

Finally we have due to (45)

$$\begin{aligned}\|u - \bar u\|_{L^1(M)} &\leq \lim_{j\to\infty} 2\|u - f_j\|_{L^1(M)} + \lim_{j\to\infty}\|f_j - \bar f_j\|_{L^1(M)}\\ &= \lim_{j\to\infty}\sum_K \|f_j - \bar f_j\|_{L^1(K)}\\ &\leq Ch\lim_{j\to\infty}\sum_K \|\nabla_g f_j\|_{L^1(K)}\\ &\leq ChTV_M(u).\end{aligned}$$

□

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