On the strong pre-compactness property for entropy solutions of an ultra-parabolic equation with discontinuous flux

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Abstract

Under some non-degeneracy condition we show that sequences of entropy solutions of a nonlinear ultra-parabolic equation are strongly pre-compact in the general case of a Caratheodory flux vector. The proofs are based on localization principles for H-measures corresponding to sequences of measure-valued functions.

1. Introduction

We consider the equation

$$\operatorname{div}_x(\varphi(x,u) - A(x)\nabla g(u)) + \psi(x,u) = 0. \tag{1}$$

Here $\varphi(x,u) = (\varphi_1(x,u),\dots,\varphi_n(x,u))$, $u = u(x)$, $x = (x_1,\dots,x_n) \in \Omega$, where Ω is an open subset of $\mathbb{R}^n$; the flux vector $\varphi(x,u)$ is assumed to be a Caratheodory vector (i.e. it is continuous with respect to u and measurable with respect to x) such that the functions

$$\alpha_M(x) = \max_{|u|\le M} |\varphi(x,u)| \in L^2_{loc}(\Omega) \quad \forall M > 0 \tag{2}$$

(here and below $|\cdot|$ stands for the Euclidean norm of a finite-dimensional vector). We also assume that for any fixed $p \in \mathbb{R}$ the distribution

$$\operatorname{div}_x \varphi(x,p) = \gamma_p \in \mathrm{M}_{loc}(\Omega), \tag{3}$$

where $\mathrm{M}_{loc}(\Omega)$ is the space of locally finite Borel measures on Ω with the standard locally convex topology generated by semi-norms $p_\Phi(\mu) = \operatorname{Var}(\Phi\mu)$, $\Phi = \Phi(x) \in C_0(\Omega)$. In the parabolic term $A(x) = \{a_{ij}(x)\}_{i,j=1}^n$ is a non-negative matrix with coefficients $a_{ij}(x) \in C^1(\Omega)$, $g(u)$ is a continuous non-decreasing function on $\mathbb{R}$. The function $\psi(x,u)$ is assumed to be a Caratheodory function on $\Omega \times \mathbb{R}$ such that

$$\beta_M(x) = \max_{|u|\le M} |\psi(x,u)| \in L^1_{loc}(\Omega) \quad \forall M > 0. \tag{4}$$

Let $\gamma_p = \gamma_p^r + \gamma_p^s$ be the decomposition of the measure γ_p into the sum of the regular and the singular measures, so that $\gamma_p^r = \omega_p(x)dx$, $\omega_p(x) \in L^1_{loc}(\Omega)$, and γ_p^s is a singular measure (supported on a set of zero Lebesgue measure). We denote by $|\gamma_p^s|$ the variation of the measure γ_p^s, which is a non-negative locally finite Borel measure on Ω. Denote, as usual, $\operatorname{sign} u = \begin{cases} 1 & , u > 0, \\ -1 & , u < 0, \\ 0 & , u = 0. \end{cases}$

Now, we introduce the notion of entropy solution of (1).

Definition 1. A measurable function $u(x)$ on Ω is called an entropy solution of equation (1) if $\varphi(x,u(x)) \in L^1_{loc}(\Omega,\mathbb{R}^n)$, $g(u(x)), \psi(x,u(x)) \in L^1_{loc}(\Omega)$, and for all $p \in \mathbb{R}$ the Kruzhkov-type entropy inequality (see [8]) holds

$$\begin{aligned}\operatorname{div}_x [\operatorname{sign}(u(x)-p)(\varphi(x,u(x)) - \varphi(x,p)) - A(x)\nabla|g(u(x)) - g(p)|] +\\ \operatorname{sign}(u(x)-p)[\omega_p(x) + \psi(x,u(x))] - |\gamma_p^s| \le 0\end{aligned} \tag{5}$$

in the sense of distributions on Ω (in the space $\mathcal{D}'(\Omega)$); that is, for all non-negative functions $f(x) \in C_0^\infty(\Omega)$

$$\int_\Omega [\operatorname{sign}(u(x)-p)\,(\varphi(x,u(x)) - \varphi(x,p)) \cdot \nabla f(x) + |g(u(x)) - g(p)| \operatorname{div}(A(x)\nabla f(x)) -$$
$$\operatorname{sign}(u(x)-p)(\omega_p(x) + \psi(x,u(x)))f(x)]\,dx + \int_\Omega f(x)d|\gamma_p^s|(x) \ge 0$$

(here $u \cdot v$ denotes the scalar product of vectors $u, v \in \mathbb{R}^n$).

In the case when the second-order term is absent ($A(x) \equiv 0$) our definition extends the notion of an entropy solution for a first-order balance laws introduced for the case of one space variable in [6,7]. In general case our definition is a weaker form of the definition of entropy solution for ultra-parabolic equations used in [15].

Notice also that we do not require that $u(x)$ is a weak solution of (1). If $u(x) \in L^\infty(\Omega)$ and $\gamma_p^s = 0$ for all $p \in \mathbb{R}$ then any entropy solution $u(x)$ satisfies (1) in $\mathcal{D}'(\Omega)$, i.e. $u(x)$ is a weak solution of (1). Indeed, this follows from (5) with $p = \pm\|u\|_\infty$. But, generally, entropy solutions are not weak ones, even in the case when the singular measures γ_p^s are absent. For instance, as is easily verified, $u(x) = \operatorname{sign} x|x|^{-1/2}$ is an entropy solution of the first-order equation $(xu^2)_x = 0$ on the line $\Omega = \mathbb{R}$, but it does not satisfy this equation in $\mathcal{D}'(\mathbb{R})$.

We assume that equation (1) is non-degenerate in the sense of the following definition.

Definition 2. Equation (1) is said to be *non-degenerate* if for almost all $x \in \Omega$ for all $\xi \in \mathbb{R}^n$, $\xi \neq 0$ the functions $\lambda \to \xi \cdot \varphi(x,\lambda)$, $\lambda \to g(\lambda)A(x)\xi \cdot \xi$ are not constant simultaneously on non-degenerate intervals.

In this paper we shall establish the strong pre-compactness property for sequences of entropy solutions. This result generalizes the previous results of [9–12,14] to the case of ultra-parabolic equations. It also generalizes and improves results of S. Sazhenkov [15].

Theorem 1. *Suppose that u_k, $k \in \mathbb{N}$ is a sequence of entropy solutions of non-degenerate equation (1) such that $|\varphi(x,u_k(x))| + |\psi(x,u_k(x))| + |g(u_k(x))| + \rho(u_k(x))$ is bounded in $L^1_{loc}(\Omega)$, where $\rho(u)$ is a nonnegative super-linear function (i.e. $\rho(u)/u \to \infty$ as $u \to \infty$). Then there exists a subsequence of u_k, which converges in $L^1_{loc}(\Omega)$ to some entropy solution $u(x)$.*

Remark that the non-degeneracy condition is essential for the statement of Theorem 1. In the case of the equation $\operatorname{div}(\varphi(u) - A\nabla g(u)) = 0$ with constant matrix A this condition is necessary for strong pre-compactness property. For instance, if $\xi \cdot \varphi(u) = \mathrm{const}$, $g(u)A\xi \cdot \xi = \mathrm{const}$ on the segment $[a,b]$ with $\xi \in \mathbb{R}^n$, $\xi \neq 0$ then the sequence $u_k(x) = [a+b+(b-a)\sin(k\xi \cdot x)]/2$ of entropy solutions does not contain strongly convergent subsequences.

We also underline that for sequences of weak solutions (without additional entropy constraints) the statement of Theorem 1 does not hold. For example, the sequence $u_k = \operatorname{sign}\sin kx$ consists of weak solution for the Burgers equation $u_t + (u^2)_x = 0$ (as well as for the corresponding stationary equation $(u^2)_x = 0$) and converges only weakly, while the non-degeneracy condition is evidently satisfied.

In paper [15] another non-degeneracy condition was proposed for the evolutionary equation

$$u_t + \operatorname{div}_x(\varphi(t,x,u) - A(t,x)\nabla g(u)) = 0, \quad (t,x) \in \Pi = (0,T) \times \mathbb{R}^n,$$

namely the following condition

G) for a.e. $(t,x) \in \Pi$ $\forall(\tau,\xi) \in \mathbb{R}^{n+1}$ such that $(\tau,\xi) \neq 0$ and $A(t,x)\xi \cdot \xi = 0$, the set

$$\left\{ \lambda \mid \tau + \varphi'_\lambda(t,x,\lambda) \cdot \xi + (1/2)g'(\lambda) \sum_{i,j=1}^n (a_{ij})_{x_j}(t,x)\xi_i = 0 \right\}$$

has zero Lebesgue measure.

Here $a_{ij} = a_{ij}(t,x)$, ξ_i are components of the matrix A and the vector ξ, respectively, and it is assumed that $\varphi(t,x,u)$, $g(u)$ are C^1 with respect to the variable u, $g'(u) > 0$. In addition to condition G) it is required in [15] that the matrix A is either diagonal or has constant rank in Π.

Condition G) appeared in [15] due to some mistakes in calculations. Let us demonstrate that condition G) is wrong, i.e. the strong pre-compactness property may fail under this condition.

In the half-space $(t,x,y) \in \mathbb{R}_+ \times \mathbb{R}^2$ we consider the equation

$$u_t + \frac{\partial}{\partial x}(g(u)_x + yg(u)_y) + \frac{\partial}{\partial y}(yg(u)_x + y^2 g(u)_y) = 0$$

where $g(u) \in C^2(\mathbb{R})$, $g', g'' > 0$. This equation has the form

$$u_t + \operatorname{div} A\nabla g(u) = 0$$

with diffusion matrix $A = A(y) = \begin{pmatrix} 1 & y \\ y & y^2 \end{pmatrix}$. We see that the matrix $A \geq 0$ and has constant rank 1. We introduce the sequence $u_k = u_k(x, y) = \sin(kye^{-x})$. Since $g(u_k)_x + yg(u_k)_y = 0$ this sequence consists of stationary solutions of our equation while it does not contain strongly convergent subsequences.

Nevertheless, condition G) is satisfied. Indeed, let $y \neq 0$ and $(\tau, \xi) = (\tau, \xi_1, \xi_2) \neq 0$. Then the equalities

$$A\xi \cdot \xi = 0, \quad \tau + \frac{1}{2} g'(\lambda) \sum_{i=1,2} ((a_{i1})_x + (a_{i2})_y)\xi_i = 0$$

reduce to the relations

$$\xi_1 + y\xi_2 = 0, \quad \tau + \frac{1}{2} g'(\lambda)(\xi_1 + 2y\xi_2) = 0.$$

One can satisfy them simultaneously only for one value of λ. Otherwise, $\xi_1 + y\xi_2 = \xi_1 + 2y\xi_2 = 0$ and, therefore, $\xi = 0$. This in turn implies that also $\tau = 0$. Thus, $(\tau, \xi) = 0$, which contradicts to our assumptions. We see that for $y \neq 0$ and all $(\tau, \xi) \in \mathbb{R}^3 \setminus \{0\}$ such that $A\xi \cdot \xi = 0$ the set indicated in G) consists at most of one point and therefore has zero Lebesgue measure. We conclude that condition G) is satisfied. As is easy to see, the non-degeneracy condition in the sense of Definition 2 is violated here. Indeed, taking $\tau = 0$, $\xi = (-y, 1)$, we find that $\tau\lambda = g(\lambda)A(y)\xi \cdot \xi = 0$.

Theorem 1 will be proved in the last section. The proof is based on general localization properties for H-measures corresponding to bounded sequences of measure-valued functions. From these properties it follows also the strong convergence of various approximate solutions for equation (1). For example, in [14] we use approximations and the strong pre-compactness property in order to prove existence of entropy solution to the Cauchy problem for an evolutionary hyperbolic equation with discontinuous flux.

Now we consider approximations of ultra-parabolic equation (1). We assume for simplicity that $\psi(x, u) \equiv 0$, $g(u) \in C^1(\mathbb{R})$. As was shown in [14], there exists a sequence $\varphi_m(x, u) \in C^\infty(\Omega \times \mathbb{R})$ such that $\varphi_m(x, u) \underset{m\to\infty}{\to} \varphi(x, u)$ in $L^2_{loc}(\Omega, C(\mathbb{R}, \mathbb{R}^n))$ while $\operatorname{div}_x \varphi_m(x, p) = \gamma^m_{pr}(x) + \gamma^m_{ps}(x)$, where $\gamma^m_{pr}(x) \underset{m\to\infty}{\to} \omega_p(x)$ in $L^1_{loc}(\Omega)$, $|\gamma^m_{ps}(x)| \underset{m\to\infty}{\to} |\gamma^s_p|$ weakly in $M_{loc}(\Omega)$.

We can choose also the sequences of smooth symmetric matrices $A_m(x) = \{a^m_{ij}(x)\}^n_{i,j=1}$ such that $A_m \geq \varepsilon_m E$, $\varepsilon_m > 0$ (here E is the unite matrix), and a sequence $g_m(u) \in C^1(\mathbb{R})$ of strictly increasing functions such that $g'_m(u) \geq \varepsilon_m$ $a^m_{ij}(x) \underset{m\to\infty}{\to} a_{ij}(x)$ in $C^1(\Omega)$, $g_m(u) \underset{m\to\infty}{\to} g(u)$ in $C^1(\mathbb{R})$. We can always assume that

$$\varepsilon_m^{-1/2} \max_{x\in K} \|A_m(x) - A(x)\| \underset{m\to\infty}{\to} 0, \ \varepsilon_m^{-1/2} \max_{|u|\leq M} |g'_m(u) - g'(u)| \underset{m\to\infty}{\to} 0,$$

where $K \subset \Omega$ is an arbitrary compact, $M > 0$. Then we have the limit relations

$$(g'_m(u) - g'(u))/\sqrt{g'_m(u)} \to 0 \ \text{ in } C(\mathbb{R}), \ \|(A_m(x) - A(x))(A_m(x))^{-1/2}\| \to 0 \ \text{ in } C(\Omega).$$

Moreover, passing to subsequences of g_m, A_m if necessary, we may achieve that for each $M > 0$ and every compact $K \subset \Omega$

$$\max_{|u|\leq M} |g'_m(u) - g'(u)|/\sqrt{g'_m(u)} + \max_{x\in K} \|(A_m(x) - A(x))(A_m(x))^{-1/2}\| \underset{m\to\infty}{=} o\left(I_m(K, M+1)^{-1/2}\right), \tag{6}$$

where

$$I_m(K, M) = 1 + \int_K \int_{-M}^{M} |\operatorname{div}_x \varphi_m(x, p)| dp dx.$$

Generally, the sequence $I_m(K, M)$ may tends to infinity as $m \to \infty$. We consider the approximate equation

$$\operatorname{div}_x(\varphi_m(x, u) - A_m(x)\nabla g_m(u)) = 0 \tag{7}$$

and suppose that $u = u_m(x)$ is a bounded weak solution of (7) (for instance, we can take $u = u_m(x)$ being a weak solution to the Dirichlet problem with a bounded data at $\partial\Omega$). This means that $u \in$

$L^\infty(\Omega) \cap W^1_{2,loc}(\Omega)$, where $W^1_{2,loc}(\Omega)$ is the Sobolev space consisting of functions whose generalized derivatives lay in $L^2_{loc}(\Omega)$, and the following standard integral identity is satisfied: $\forall f = f(x) \in C^1_0(\Omega)$.

$$\int_\Omega [\varphi_m(x, u(x)) - A_m(x)\nabla g_m(u(x))] \cdot \nabla f(x) dx = 0. \tag{8}$$

We assume also that the sequence u_m is bounded in $L^\infty(\Omega)$.

Theorem 2. *Suppose that equation (1) is non-degenerate. Then the sequence* $u_m(x) \underset{m\to\infty}{\to} u(x)$ *in* $L^1_{loc}(\Omega)$*, where* $u = u(x)$ *is an entropy solution of (1).*

Remark that Theorem 2 allows to establish existence of entropy solutions of boundary value problems for equation (1) (as well as initial or initial boundary value problems for evolutionary equations of the kind (1)).

In next section 2 we describe the main concepts, in particular the concept of measure-valued functions. In sections 3,4 we introduce the notion of H-measure and prove the localization property. Finally, in the last section 5 these results are applied to prove Theorems 1, 2.

2. Main concepts

Recall (see [1,2,17]) that a *measure-valued* function on Ω is a weakly measurable map $x \to \nu_x$ of the set Ω into the space of probability Borel measures with compact support in $\mathbb{R}$. The weak measurability of ν_x means that for each continuous function $f(\lambda)$ the function $x \to \int f(\lambda) d\nu_x(\lambda)$ is Lebesgue-measurable on Ω.

Remark 1. If ν_x is a measure-valued function then, as was shown in [10], the functions $\int g(\lambda) d\nu_x(\lambda)$ are measurable in Ω for all bounded Borel functions $g(\lambda)$. More generally, if $f(x, \lambda)$ is a Caratheodory function and $g(\lambda)$ is a bounded Borel function then the function $\int f(x, \lambda) g(\lambda) d\nu_x(\lambda)$ is measurable. This follows from the fact that any Caratheodory function is strongly measurable as a map $x \to f(x, \cdot) \in C(\mathbb{R})$ (see [5], Chapter 2) and, therefore, is a pointwise limit of step functions $f_m(x, \lambda) = \sum_i g_{mi}(x) h_{mi}(\lambda)$ with measurable functions $g_{mi}(x)$ and continuous $h_{mi}(\lambda)$ so that for $x \in \Omega$ $f_m(x, \cdot) \underset{m\to\infty}{\to} f(x, \cdot)$ in $C(\mathbb{R})$.

A measure-valued function ν_x is said to be bounded if there exists $M > 0$ such that $\operatorname{supp} \nu_x \subset [-M, M]$ for almost all $x \in \Omega$. We denote the smallest value of M with this property by $\|\nu_x\|_\infty$.

Finally, measure-valued functions of the form $\nu_x(\lambda) = \delta(\lambda - u(x))$, where $\delta(\lambda - u)$ is the Dirac measure concentrated at u are said to be *regular*; we identify them with the corresponding functions $u(x)$. Thus, the set $MV(\Omega)$ of bounded measure-valued functions on Ω contains the space $L^\infty(\Omega)$. Note that for a regular measure-valued function $\nu_x(\lambda) = \delta(\lambda - u(x))$ the value $\|\nu_x\|_\infty = \|u\|_\infty$. Extending the concept of boundedness in $L^\infty(\Omega)$ to measure-valued functions we shall say that a subset A of $MV(\Omega)$ is *bounded* if $\sup_{\nu_x \in A} \|\nu_x\|_\infty < \infty$.

We define below the weak and the strong convergence of sequences of measure-valued functions.

Definition 3. Let $\nu^k_x \in MV(\Omega)$, $k \in \mathbb{N}$, and let $\nu_x \in MV(\Omega)$. Then
1) the sequence ν^k_x converges weakly to ν_x if for each $f(\lambda) \in C(\mathbb{R})$,

$$\int f(\lambda) d\nu^k_x(\lambda) \underset{k\to\infty}{\to} \int f(\lambda) d\nu_x(\lambda) \ \text{ in the weak-}* \text{ topology of } L^\infty(\Omega);$$

2) the sequence ν^k_x converges to ν_x *strongly* if for each $f(\lambda) \in C(\mathbb{R})$,

$$\int f(\lambda) d\nu^k_x(\lambda) \underset{k\to\infty}{\to} \int f(\lambda) d\nu_x(\lambda) \ \text{ in } L^1_{loc}(\Omega).$$

The next result was proved in [17] for regular functions ν^k_x. The proof can easily be extended to the general case, as was done in [10].

Theorem 3. *Let ν_x^k, $k \in \mathbb{N}$ be a bounded sequence of measure-valued functions. Then there exist a subsequence $\nu_x^r = \nu_x^k$, $k = k_r$, and a measure-valued function $\nu_x \in MV(\Omega)$ such that $\nu_x^r \to \nu_x$ weakly as $r \to \infty$.*

Theorem 3 shows that bounded sets of measure-valued functions are weak1y precompact. If $u_k(x) \in L^\infty(\Omega)$ is a bounded sequence, treated as a sequence of regular measure valued functions, and $u_k(x)$ weakly converges to a measure valued function ν_x then ν_x is regular, $\nu_x(\lambda) = \delta(\lambda - u(x))$, if and only if $u_k(x) \to u(x)$ in $L^1_{loc}(\Omega)$ (see [17]). Obviously, if $u_k(x)$ converges to ν_x strongly then $u_k(x) \to u(x) = \int \lambda d\nu_x(\lambda)$ in $L^1_{loc}(\Omega)$ and then $\nu_x(\lambda) = \delta(\lambda - u(x))$.

We shall study the strong pre-compactness property using Tartar's techniques of H-measures.

Let $F(u)(\xi)$, $\xi \in \mathbb{R}^n$, be the Fourier transform of a function $u(x) \in L^2(\mathbb{R}^n)$, $S = S^{n-1} = \{ \xi \in \mathbb{R} \mid |\xi| = 1 \}$ be the unit sphere in $\mathbb{R}^n$. Denote by $u \to \overline{u}$, $u \in \mathbb{C}$ the complex conjugation.

The concept of an H-measure corresponding to some sequence of vector-valued functions bounded in $L^2(\Omega)$ was introduced by Tartar [18] and Gerárd [4] on the basis of the following result. For $l \in \mathbb{N}$ let $U_k(x) = \left(U_k^1(x), \ldots, U_k^l(x)\right) \in L^2(\Omega, \mathbb{R}^l)$ be a sequence weakly convergent to the zero vector.

Proposition 1 (see [18], Theorem 1.1). *There exists a family of complex Borel measures $\mu = \left\{\mu^{ij}\right\}_{i,j=1}^l$ in $\Omega \times S$ and a subsequence $U_r(x) = U_k(x)$, $k = k_r$, such that*

$$\langle \mu^{ij}, \Phi_1(x)\overline{\Phi_2(x)}\psi(\xi)\rangle = \lim_{r\to\infty} \int_{\mathbb{R}^n} F(U_r^i\Phi_1)(\xi)\overline{F(U_r^j\Phi_2)(\xi)}\psi\left(\frac{\xi}{|\xi|}\right) d\xi \tag{9}$$

for all $\Phi_1(x), \Phi_2(x) \in C_0(\Omega)$ and $\psi(\xi) \in C(S)$.

The family $\mu = \left\{\mu^{ij}\right\}_{i,j=1}^l$ is called the H-measure corresponding to $U_r(x)$.

The concept of H-measure has been extended in [10] (see also [11,12]) to sequences of measure-valued functions. We study the properties of such H-measures in the next section.

3. H-measures corresponding to bounded sequences of measure-valued functions

Let $\nu_x^k \in MV(\Omega)$ be a bounded sequence of measure-valued functions weakly convergent to a measure-valued function $\nu_x^0 \in MV(\Omega)$. For $x \in \Omega$ and $p \in \mathbb{R}$ we introduce the distribution functions

$$u_k(x,p) = \nu_x^k((p,+\infty)), \quad u_0(x,p) = \nu_x^0((p,+\infty)).$$

Then, as mentioned in Remark 1, for $k \in \mathbb{N} \cup \{0\}$ and $p \in \mathbb{R}$ the functions $u_k(x,p)$ are measurable in $x \in \Omega$; thus, $u_k(x,p) \in L^\infty(\Omega)$ and $0 \le u_k(x,p) \le 1$. Let

$$E = E(\nu_x^0) = \left\{ p_0 \in \mathbb{R} \mid u_0(x,p) \underset{p\to p_0}{\to} u_0(x,p_0) \text{ in } L^1_{loc}(\Omega) \right\}.$$

We have the following result, the proof of which can be found in [10].

Lemma 1. *The complement $\bar{E} = \mathbb{R} \setminus E$ is at most countable and if $p \in E$ then $u_k(x,p) \underset{k\to\infty}{\to} u_0(x,p)$ weakly-$*$ in $L^\infty(\Omega)$.*

Let $U_k^p(x) = u_k(x,p) - u_0(x,p)$. Then, by Lemma 1, $U_k^p(x) \to 0$ as $k \to \infty$ weakly-$*$ in $L^\infty(\Omega)$ for $p \in E$. The next result, similar to Proposition 1, has also been established in [10].

Proposition 2. *1) There exists a family of locally finite complex Borel measures $\{\mu^{pq}\}_{p,q\in E}$ in $\Omega \times S$ and a subsequence $U_r(x) = \{U_r^p(x)\}_{p\in E}$, $U_r^p(x) = U_k^p(x)$, $k = k_r$ such that for all $\Phi_1(x), \Phi_2(x) \in C_0(\Omega)$ and $\psi(\xi) \in C(S)$*

$$\langle \mu^{pq}, \Phi_1(x)\overline{\Phi_2(x)}\psi(\xi)\rangle = \lim_{r\to\infty} \int_{\mathbb{R}^n} F(\Phi_1 U_r^p)(\xi)\overline{F(\Phi_2 U_r^q)(\xi)}\psi\left(\frac{\xi}{|\xi|}\right) d\xi. \tag{10}$$

2) The correspondence $(p,q) \to \mu^{pq}$ is a continuous map from $E \times E$ into the space $\mathrm{M}_{loc}(\Omega \times S)$.

Definition 4. We call the family of measures $\{\mu^{pq}\}_{p,q\in E}$ the H-measure corresponding to the subsequence $\nu_x^r = \nu_x^k$, $k = k_r$.

We point out the following important properties of an H-measure.

Lemma 2. *1) $\mu^{pp} \geq 0$ for each $p \in E$; 2) $\mu^{pq} = \overline{\mu^{qp}}$ for all $p, q \in E$; 3) for $p_1, \dots, p_l \in E$ and $g_1, \dots, g_l \in C_0(\Omega \times S)$ the matrix $A = a_{ij} = \langle \mu^{p_i p_j}, g_i \overline{g_j} \rangle$, $i, j = 1, \dots, l$ is Hermitian and positive-definite.*

Proof. We prove 3). First let the functions $g_i = g_i(x, \xi)$ be finite sums of functions of the form $\Phi(x)\psi(\xi)$, where $\Phi(x) \in C_0(\Omega)$ and $\psi(\xi) \in C(S)$. Then it follows from (10) that

$$a_{ij} = \lim_{r\to\infty} \int_{\mathbb{R}^n} H_r^i(\xi)\overline{H_r^j(\xi)} d\xi, \tag{11}$$

where $H_r^i(\xi) = F(g_i(\cdot, \xi/|\xi|)U_r^{p_i})(\xi)$. Hence setting $g_i(x, \xi) = g(x, \xi) = \sum\limits_{k=1}^{m} \Phi_k(x)\psi_k(\xi)$ we obtain

$$H_r^i(\xi) = \sum_{k=1}^{m} F(\Phi_k U_r^{p_i})(\xi)\psi_k\left(\frac{\xi}{|\xi|}\right).$$

It immediately follows from (11) that $a_{ji} = \overline{a_{ij}}$, $i, j = 1, \dots, l$, which shows that A is a Hermitian matrix. Further, for $\alpha_1, \dots, \alpha_l \in \mathbb{C}$ we have

$$\sum_{i,j=1}^{l} a_{ij}\alpha_i\overline{\alpha_j} = \lim_{r\to\infty} \int_{\mathbb{R}^n} |H_r(\xi)|^2 d\xi \geq 0, \quad H_r(\xi) = \sum_{i=1}^{l} H_r^i(\xi)\alpha_i,$$

which means that A is positive-definite.

In the general case of $g_i \in C_0(\Omega \times S)$ one carries out the proof of 3) by approximating the functions g_i, $i = 1, \dots, l$ in the uniform norm by finite sums of functions of the form $\Phi(x)\psi(\xi)$.

Assertions 1) and 2) are easy consequences of 3). For setting $l = 1$, $p_1 = p$ and $g_1 = g$ we obtain the relation $\langle \mu^{pp}, |g|^2 \rangle \geq 0$, which holds for all $g \in C_0(\Omega \times S)$, thus showing that μ^{pp} is real and non-negative. To prove 2) we represent an arbitrary function $g = g(x, \xi)$ with compact support in the form $g = g_1\overline{g_2}$. Let $l = 2$, $p_1 = p$ and $p_2 = q$. In view of 3),

$$\langle \mu^{pq}, g \rangle = \langle \mu^{pq}, g_1\overline{g_2} \rangle = \overline{\langle \mu^{qp}, g_2\overline{g_1} \rangle} = \overline{\langle \mu^{qp}, \overline{g} \rangle} = \langle \overline{\mu^{qp}}, g \rangle$$

and $\mu^{pq} = \overline{\mu^{qp}}$. The proof is complete. $\square$

We consider now a countable dense index subset $D \subset E$.

Proposition 3 (see [12]). *There exists a family of complex finite Borel measures μ_x^{pq} in the sphere S with $p, q \in D$, $x \in \Omega'$, where Ω' is a subset of Ω of full measure, such that $\mu^{pq} = \mu_x^{pq} dx$ that is, for all $\Phi(x, \xi) \in C_0(\Omega \times S)$ the function*

$$x \to \langle \mu_x^{pq}(\xi), \Phi(x, \xi) \rangle = \int_S \Phi(x, \xi) d\mu_x^{pq}(\xi)$$

is Lebesgue-measurable on Ω, bounded, and

$$\langle \mu^{pq}, \Phi(x, \xi) \rangle = \int_\Omega \langle \mu_x^{pq}(\xi), \Phi(x, \xi) \rangle dx.$$

Moreover, for $p, p', q \in D$, $p' > p$

$$\operatorname{Var} \mu_x^{pq} \leq 1 \quad \textit{and} \quad \operatorname{Var}(\mu_x^{p'q} - \mu_x^{pq}) \leq 2\left(\nu_x^0((p, p'))\right)^{1/2}. \tag{12}$$

Proof. We claim that $\mathrm{pr}_\Omega \mathrm{Var}\,\mu^{pq} \le \mathrm{meas}$ for $p,q\in E$, where meas is the Lebesgue measure on Ω. Assume first that $p=q$. By Lemma 2, the measure μ^{pp} is non-negative. Next, in view of relation (10) with $\Phi_1(x)=\Phi_2(x)=\Phi(x)\in C_0(\Omega)$ and $\psi(\xi)\equiv 1$,

$$\langle \mu^{pp}, |\Phi(x)|^2\rangle = \lim_{r\to\infty}\int_{\mathbb{R}^n} F(\Phi U_r^p)(\xi)\overline{F(\Phi U_r^p)(\xi)}d\xi =$$
$$\lim_{r\to\infty}\int_\Omega |U_r^p(x)|^2|\Phi(x)|^2dx \le \int_\Omega |\Phi(x)|^2dx$$

(we use here Plancherel's equality and the estimate $|U_r^p(x)|\le 1$). Thus, we see that that $\mathrm{pr}_\Omega\mu^{pp}\le \mathrm{meas}$.

Let $p,q\in E$, A be a bounded open subset of Ω, and $g=g(x,\xi)\in C_0(A\times S)$, $|g|\le 1$. We introduce functions $g_1=g/\sqrt{|g|}$ (we set $g_1=0$ for $g=0$) and $g_2=\sqrt{|g|}$. Then $g_1,g_2\in C_0(A\times S)$, $g=g_1\overline{g_2}$, $|g_1|^2=|g_2|^2=|g|$ and the matrix

$$\begin{pmatrix} \langle\mu^{pp},|g|\rangle & \langle\mu^{pq},g\rangle \\ \overline{\langle\mu^{pq},g\rangle} & \langle\mu^{qq},|g|\rangle \end{pmatrix}$$

is positive-definite by Lemma 2; in particular,

$$|\langle\mu^{pq},g\rangle| \le (\langle\mu^{pp},|g|\rangle\langle\mu^{qq},|g|\rangle)^{1/2} \le (\mu^{pp}(A\times S)\mu^{qq}(A\times S))^{1/2}\le \mathrm{meas}(A).$$

We take account of the inequalities $\mathrm{pr}_\Omega\mu^{pp}\le\mathrm{meas}$ and $\mathrm{pr}_\Omega\mu^{qq}\le\mathrm{meas}$ to obtain the last estimate. Since g can be an arbitrary function in $C_0(A\times S)$, $|g|\le 1$, we obtain the inequality $\mathrm{Var}\,\mu^{pq}(A\times S)\le \mathrm{meas}(A)$, The measure μ^{pq} is regular, therefore this estimate holds for all Borel subsets A of Ω and

$$\mathrm{pr}_\Omega \mathrm{Var}\,\mu^{pq}\le \mathrm{meas}\,. \tag{13}$$

It follows from (13) that for all $\psi(\xi)\in C(S)$ we have

$$\mathrm{Var}\,\mathrm{pr}_\Omega\,(\psi(\xi)\mu^{pq}(x,\xi))\le \|\psi\|_\infty\cdot \mathrm{pr}_\Omega\mathrm{Var}\,\mu^{pq}\le\|\psi\|_\infty\cdot\mathrm{meas}\,. \tag{14}$$

In view of (14) the measures $\mathrm{pr}_\Omega(\psi(\xi)\mu^{pq}(x,\xi))$ are absolutely continuous with respect to Lebesgue measure, and the Radon-Nikodym theorem shows that

$$\mathrm{pr}_\Omega\,(\psi(\xi)\mu^{pq}(x,\xi)) = h_\psi^{pq}(x)\cdot\mathrm{meas},$$

where the densities $h_\psi^{pq}(x)$ are measurable on Ω and, as seen from (14),

$$\|h_\psi^{pq}(x)\|_\infty\le\|\psi\|_\infty. \tag{15}$$

We now choose a non-negative function $K(x)\in C_0^\infty(\mathbb{R}^n)$ with support in the unit ball such that $\int K(x)dx=1$ and set $K_m(x)=m^nK(mx)$ for $m\in\mathbb{N}$. Clearly, the sequence of K_m converges in $\mathcal{D}'(\mathbb{R}^n)$ to the Dirac δ-function (that is, this sequence is an approximate unity).

Let $\operatorname*{B\lim}_{m\to\infty} c_m$ be a generalized Banach limit on the space l_∞ of bounded sequences $c=\{c_m\}_{m\in\mathbb{N}}$, i.e. $L(c)=\operatorname*{B\lim}_{m\to\infty} c_m$ is a linear functional on l_∞ with the property:

$$\varliminf_{m\to\infty} c_m\le L(c)\le\varlimsup_{m\to\infty} c_m$$

(in particular for convergent sequences $c=\{c_m\}$ $L(c)=\lim\limits_{m\to\infty} c_m$). For complex sequences $c_m=a_m+ib_m$ the Banach limits is defined by complexification: $\operatorname*{B\lim}_{m\to\infty} c_m=L(a)+iL(b)$, where $a=\{a_m\}$, $b=\{b_m\}$ are real and imaginary parts of the sequence $c=\{c_m\}$, respectively. Modifying the densities $h_\psi^{pq}(x)$ on subsets of measure zero, for instance, replacing them by the functions

$$\operatorname*{B\lim}_{m\to\infty}\int_\Omega h_\psi^{pq}(y)K_m(x-y)dy$$

(obviously, the value $h_\psi^{pq}(x)$ does not change for any Lebesgue point x of the function h_ψ^{pq}), we shall assume that for all $x\in\Omega$ we have

$$h_\psi^{pq}(x)=\operatorname*{B\lim}_{m\to\infty}\int_\Omega h_\psi^{pq}(y)K_m(x-y)dy. \tag{16}$$

Let Ω' be the set of common Lebesgue points of the functions $h_\psi^{pq}(x)$, $u_0(x,p) = \nu_x^0((p,+\infty))$, and $u_0^-(x,p) = \nu_x^0([p,+\infty)) = \lim\limits_{q\to p-} u_0(x,q)$, where $p,q \in D$ and ψ belongs to F, some countable dense subset of $C(S)$. The family of (p,q,ψ) is countable, therefore Ω' is of full measure.

The dependence of the h_ψ^{pq} on ψ, regarded as a map from $C(S)$ into $L^\infty(\Omega)$, is clearly linear and continuous (in view of (15)), therefore it follows from the density of F in $C(S)$ that $x \in \Omega'$ is a Lebesgue point of the functions $h_\psi^{pq}(x)$ for all $\psi(\xi) \in C(S)$ and $p,q \in D$. Here we also take account of (16), which implies that the functional $\psi \to h_\psi^{pq}(x)$ is continuous on $C(S)$ for all $x \in \Omega$.

For $p,q \in D$ and $x \in \Omega'$ (actually even for $x \in \Omega$) the equality $l(\psi) = h_\psi^{pq}(x)$ defines a continuous linear functional in $C(S)$; moreover, $\|l\| \le 1$ in view of (15). By the Riesz-Markov theorem this functional can be defined by integration with respect to some complex Borel measure $\mu_x^{pq}(\xi)$ in S and $\operatorname{Var}\mu_x^{pq} = \|l\| \le 1$. Hence

$$h_\psi^{pq}(x) = \langle \mu_x^{pq}(\xi), \psi\rangle = \int_S \psi(\xi) d\mu_x^{pq}(\xi) \tag{17}$$

for all $\psi(\xi) \in C(S)$.

Equality (17) shows that the functions $x \to \int_S \psi(\xi) d\mu_x^{pq}(\xi)$ are bounded and measurable for all $\psi(\xi) \in C(S)$. Next, for $\Phi(x) \in C_0(\Omega)$ and $\psi(\xi) \in C(S)$ we have

$$\begin{aligned}\int_\Omega \left(\int_S \Phi(x)\psi(\xi) d\mu_x^{pq}(\xi)\right) dx = \int_\Omega \Phi(x) h_\psi^{pq}(x) dx = \\ \int_\Omega \Phi(x) d\operatorname{pr}_\Omega\left(\psi(\xi)\mu^{pq}\right) = \int_{\Omega\times S} \Phi(x)\psi(\xi) d\mu^{pq}(x,\xi).\end{aligned} \tag{18}$$

Approximating an arbitrary function $\Phi(x,\xi) \in C_0(\Omega\times S)$ in the uniform norm by linear combinations of functions of the form $\Phi(x)\psi(\xi)$ we derive from (18) that the integral $\int_S \Phi(x,\xi) d\mu_x^{pq}(\xi)$ is Lebesgue-measurable with respect to $x \in \Omega$, bounded, and

$$\int_\Omega \left(\int_S \Phi(x,\xi) d\mu_x^{pq}(\xi)\right) dx = \int_{\Omega\times S} \Phi(x,\xi) d\mu^{pq}(x,\xi)$$

that is, $\mu^{pq} = \mu_x^{pq} dx$. Recall that $\operatorname{Var}\mu_x^{pq} \le 1$.

It remains to prove the last estimate in (12). Let $p,p',q \in D$, $p' > p$ and $x \in \Omega'$. We set $\Phi_m = \sqrt{K_m} \in C_0(\mathbb{R}^n)$, $m \in \mathbb{N}$, where the sequence of kernels K_m is as defined above. Starting from some index m the function $\Phi_m(x-y)$ (of the y-variable) belongs to $C_0(\Omega)$ and, in view of Proposition 2, for all $\psi(\xi) \in C(S)$ we have

$$\begin{aligned}\left|\int_\Omega K_m(x-y)\left(h_\psi^{p'q}(y) - h_\psi^{pq}(y)\right) dy\right| = \left|\langle (\mu^{p'q} - \mu^{pq})(y,\xi), K_m(x-y)\psi(\xi)\rangle\right| = \\ \lim_{r\to\infty}\left|\int_{\mathbb{R}^n} F(\Phi_m(U_r^{p'} - U_r^p))(\xi)\overline{F(\Phi_m U_r^q)(\xi)}\psi\left(\frac{\xi}{|\xi|}\right) d\xi\right| \le \\ \|\psi\|_\infty \varlimsup_{r\to\infty}\left[\left(\int_{\mathbb{R}^n} |F(\Phi_m(U_r^{p'} - U_r^p))(\xi)|^2 d\xi\right)^{1/2}\left(\int_{\mathbb{R}^n} |F(\Phi_m U_r^q)(\xi)|^2 d\xi\right)^{1/2}\right] = \\ \|\psi\|_\infty \varlimsup_{r\to\infty}\left[\left(\int_\Omega K_m(x-y)(U_r^{p'}(y) - U_r^p(y))^2 dy\right)^{1/2}\left(\int_\Omega K_m(x-y)(U_r^q(y))^2 dy\right)^{1/2}\right].\end{aligned} \tag{19}$$

Note that $|U_r^q| \le 1$, $\int_\Omega K_m(x-y)dy = 1$ and, therefore,

$$\int_\Omega K_m(x-y)(U_r^q(y))^2 dy \le 1. \tag{20}$$

Further,

$$\begin{aligned}\int_\Omega K_m(x-y)(U_r^{p'}(y) - U_r^p(y))^2 dy \le 2\int_\Omega K_m(x-y)|U_r^{p'}(y) - U_r^p(y)| dy \le \\ 2\int_\Omega K_m(x-y)(u_r(y,p) - u_r(y,p'))dy + 2\int_\Omega K_m(x-y)(u_0(y,p) - u_0(y,p'))dy\end{aligned} \tag{21}$$

(note that $u_r(y,p)-u_r(y,p')\ge 0$ for $r\in\mathbb{N}\cup\{0\}$). Since $p,p'\subset E$, it follows from Lemma 1 that $u_r(y,p)-u_r(y,p')\underset{r\to\infty}{\to} u_0(y,p)-u_0(y,p')$ in the weak-$*$ topology in $L^\infty(\Omega)$, therefore

$$\lim_{r\to\infty}\int_\Omega K_m(x-y)(u_r(y,p)-u_r(y,p'))dy=\int_\Omega K_m(x-y)(u_0(y,p)-u_0(y,p'))dy,$$

and by (21),

$$\overline{\lim_{r\to\infty}}\left(\int_\Omega K_m(x-y)(U_r^{p'}(y)-U_r^p(y))^2dy\right)^{1/2}\le$$
$$2\left(\int_\Omega K_m(x-y)(u_0(y,p)-u_0(y,p'))dy\right)^{1/2}. \tag{22}$$

From (19), in view of (20), (22), we obtain the estimate

$$\left|\int_\Omega K_m(x-y)(h_\psi^{p'q}(y)-h_\psi^{pq}(y))dy\right|\le 2\|\psi\|_\infty\left(\int_\Omega K_m(x-y)(u_0(y,p)-u_0(y,p'))dy\right)^{1/2}$$

and passing to the limit as $m\to\infty$, since $x\in\Omega'$ is a Lebesgue point of the functions $h_\psi^{p'q}$, h_ψ^{pq} and $u_0(\cdot,p)$, $u_0(\cdot,p')$, we obtain the inequality

$$\left|h_\psi^{p'q}(x)-h_\psi^{pq}(x)\right|\le 2\|\psi\|_\infty\left(u_0(x,p)-u_0(x,p')\right)^{1/2},$$

that is, for all $\psi(\xi)\in C(S)$ we have

$$\left|\langle\mu_x^{p'q}-\mu_x^{pq},\psi\rangle\right|\le 2\|\psi\|_\infty\left(u_0(x,p)-u_0(x,p')\right)^{1/2},$$

and therefore

$$\mathrm{Var}\,(\mu_x^{p'q}-\mu_x^{pq})\le 2\left(u_0(x,p)-u_0(x,p')\right)^{1/2}=2\left(\nu_x^0((p,p'])\right)^{1/2}. \tag{23}$$

Now we demonstrate that for $x\in\Omega'$ $\nu_x^0(\{p\})=0$ for each $p\in D$. Indeed, $\nu_x^0(\{p\})=u_0^-(x,p)-u_0(x,p)$ and since $p\in D\subset E$ is a continuity point of the map $p\to u_0(x,p)$ in $L^1_{loc}(\Omega)$ we conclude that $u_0^-(x,p)-u_0(x,p)=0$ a.e. in Ω. By the construction $x\in\Omega'$ is a common Lebesgue point of the functions $u_0(x,p)$, $u_0^-(x,p)$, therefore $\nu_x^0(\{p\})=u_0^-(x,p)-u_0(x,p)=0$, as required. In particular $\nu_x^0(\{p'\})=0$ and we can replace the segment $(p,p']$ in estimate (23) by the interval (p,p'). The proof is complete. □

Corollary 1. *The correspondence $(p,q)\to\mu_x^{pq}$ is a continuous map of the set $D\times D$ into the space* $\mathrm{M}(S)$ *of finite complex Borel measures in S (with norm* Var *).*

Proof. Suppose that $p,q,p'q'\in D$, and $p<p'$, $q<q'$ (the remaining cases are treated analogously). Using estimate (12) and the identity $\mu_x^{pq}=\overline{\mu_x^{qp}}$, which is an easy consequence of Lemma 2(2), we derive that

$$\mathrm{Var}\,(\mu_x^{p'q'}-\mu_x^{pq})\le\mathrm{Var}\,(\mu_x^{p'q'}-\mu_x^{pq'})+\mathrm{Var}\,(\mu_x^{pq'}-\mu_x^{pq})\le\nu_x^0((p,p'))+\nu_x^0((q,q')).$$

This estimate directly implies the continuity of the map $(p,q)\to\mu_x^{pq}$. □

Remark 2. a) Since the H-measure is absolutely continuous with respect to x-variables identity (10) is satisfied for $\Phi_1(x),\Phi_2(x)\in L^2(\Omega)$. Indeed, by Proposition 3 we can rewrite this identity in the form: $\forall\Phi_1(x),\Phi_2(x)\in C_0(\Omega)$, $\psi(\xi)\in C(S)$

$$\int_\Omega\Phi_1(x)\overline{\Phi_2(x)}\langle\psi(\xi),\mu_x^{pq}(\xi)\rangle dx=\lim_{r\to\infty}\int_{\mathbb{R}^n}F(\Phi_1U_r^p)(\xi)\overline{F(\Phi_2U_r^q)(\xi)}\psi\left(\frac{\xi}{|\xi|}\right)d\xi. \tag{24}$$

Both sides of this identity are continuous with respect to $(\Phi_1(x),\Phi_2(x))$ in $L^2(\Omega)\times L^2(\Omega)$ and since $C_0(\Omega)$ is dense in $L^2(\Omega)$ we conclude that (24) is satisfied for each $\Phi_1(x),\Phi_2(x)\in L^2(\Omega)$;

b) if $x\in\Omega'$ is a Lebesgue point of a function $\Phi(x)\in L^2(\Omega)$ then

$$\Phi(x)\langle\mu_x^{pq},\psi(\xi)\rangle=\lim_{m\to\infty}\lim_{r\to\infty}\int_{\mathbb{R}^n}F(\Phi\Phi_mU_r^p)(\xi)\overline{F(\Phi_mU_r^q)(\xi)}\psi\left(\frac{\xi}{|\xi|}\right)d\xi \tag{25}$$

for all $\psi(\xi) \in C(S)$, where $(\Phi\Phi_m U_r^p)(y) = \Phi(y)\Phi_m(x-y)U_r^p(y)$ and $(\Phi_m U_r^q)(y) = \Phi_m(x-y)U_r^q(y)$.

Indeed, it follows from (24) that

$$\lim_{r\to\infty} \int_{\mathbb{R}^n} F(\Phi\Phi_m U_r^p)(\xi)\overline{F(\Phi_m U_r^q)(\xi)}\psi\left(\frac{\xi}{|\xi|}\right)d\xi = \int_\Omega h_\psi^{pq}(y)\Phi(y)K_m(x-y)dy. \tag{26}$$

Now, since $x \in \Omega'$ is a Lebesgue point of the functions $h_\psi^{pq}(y)$ and $\Phi(y)$, and the function $h_\psi^{pq}(y)$ is bounded, x is also a Lebesgue point for the product of these functions. Therefore,

$$\lim_{m\to\infty} \int_\Omega h_\psi^{pq}(y)\Phi(y)K_m(x-y)dy = \Phi(x)h_\psi^{pq}(x) = \Phi(x)\langle\mu_x^{pq}, \psi(\xi)\rangle,$$

and (25) follows from (26) in the limit as $m \to \infty$;

c) for $x \in \Omega'$ and each families $p_i \in D$, $\psi_i(\xi) \in C(S)$, $i = 1, \dots, l$ the matrix $\langle\mu_x^{p_i p_j}, \psi_i\overline{\psi_j}\rangle$, $i, j = 1, \dots, l$ is positive definite. Indeed, as follows from Lemma 2(3), for $\alpha_1, \dots, \alpha_l \in \mathbb{C}$

$$\sum_{i,j=1}^{l} \langle\mu_x^{p_i p_j}, \psi_i\overline{\psi_j}\rangle\alpha_i\overline{\alpha_j} = \lim_{m\to\infty} \sum_{i,j=1}^{l} \langle\mu^{p_i p_j}(y,\xi), \Phi_m(x-y)\psi_i(\xi)\overline{\Phi_m(x-y)\psi_j(\xi)}\rangle\alpha_i\overline{\alpha_j} \ge 0.$$

Taking in the above property $l = 2$, $p_1 = p$, $p_2 = q$, $\psi_1(\xi) = \psi(\xi)/\sqrt{|\psi(\xi)|}$ ($\psi_1 = 0$ for $\psi = 0$) and $\psi_2(\xi) = \sqrt{|\psi(\xi)|}$, $\psi(\xi) \in C(S)$, we obtain, as in the proof of Proposition 3, that the matrix $\begin{pmatrix} \langle\mu_x^{pp}, |\psi|\rangle & \langle\mu_x^{pq}, \psi\rangle \\ \overline{\langle\mu_x^{pq}, \psi\rangle} & \langle\mu_x^{qq}, |\psi|\rangle \end{pmatrix}$ is positive definite. In particular,

$$|\langle\mu_x^{pq}, \psi\rangle| \le \left(\langle\mu_x^{pp}, |\psi|\rangle \cdot \langle\mu_x^{qq}, |\psi|\rangle\right)^{1/2}$$

and this easily implies that for any Borel set $A \subset S$

$$\operatorname{Var}\mu_x^{pq}(A) \le \left(\mu_x^{pp}(A)\mu_x^{qq}(A)\right)^{1/2}. \tag{27}$$

d) In view of c) $\mu_x^{pp} \ge 0$ for $x \in \Omega'$, $p \in D$. Then, by (25) with $\Phi(x) \equiv 1$ and the identity $F(u)(-\xi) = \overline{F(u)(\xi)}$ for real functions $u \in L^2$, we find

$$\langle\mu_x^{pp}, \psi(-\xi)\rangle = \lim_{m\to\infty}\lim_{r\to\infty} \int_{\mathbb{R}^n} F(\Phi_m U_r^p)(\xi)\overline{F(\Phi_m U_r^p)(\xi)}\psi\left(-\frac{\xi}{|\xi|}\right)d\xi =$$
$$\lim_{m\to\infty}\lim_{r\to\infty} \int_{\mathbb{R}^n} F(\Phi_m U_r^p)(-\xi)\overline{F(\Phi_m U_r^p)(-\xi)}\psi\left(\frac{\xi}{|\xi|}\right)d\xi =$$
$$\lim_{m\to\infty}\lim_{r\to\infty} \int_{\mathbb{R}^n} \overline{F(\Phi_m U_r^p)(\xi)}F(\Phi_m U_r^p)(\xi)\psi\left(\frac{\xi}{|\xi|}\right)d\xi = \langle\mu_x^{pp}, \psi(\xi)\rangle.$$

This means that the measure μ_x^{pp} is even, i.e. it is invariant under the map $\xi \to -\xi$.

Denote by $\theta(\lambda)$ the Heaviside function:

$$\theta(\lambda) = \begin{cases} 1, & \lambda > 0, \\ 0, & \lambda \le 0. \end{cases}$$

Below we shall frequently use the following simple estimate

Lemma 3. *Let* $p_0, p \in D$, $\chi(\lambda) = \theta(\lambda - p_0) - \theta(\lambda - p)$, $V_r(y) = \int |\chi(\lambda)|d(\nu_y^r(\lambda) + \nu_y^0(\lambda))$, $\Phi(y) \in L^2(\Omega)$, $x \in \Omega'$ *is a Lebesgue point of* $(\Phi(y))^2$. *Then*

$$\overline{\lim_{m\to\infty}}\ \overline{\lim_{r\to\infty}} \|\Phi_m(x-y)\Phi(y)V_r(y)\|_2 \le |\Phi(x)||u_0(x,p_0) - u_0(x,p)|^{1/2} \underset{p\to p_0}{\to} 0.$$

Proof. It is clear that

$$V_r(y) = |u_r(y,p_0) - u_r(y,p) + u_0(y,p_0) - u_0(y,p)| =$$
$$\operatorname{sign}(p-p_0)(u_r(y,p_0) - u_r(y,p) + u_0(y,p_0) - u_0(y,p)) \le 2$$

and in particular $(V_r(y))^2 \le 2V_r(y)$. Therefore,

$$\|\Phi_m(x-y)\Phi(y)V_r(y)\|_2^2 \le$$
$$2\operatorname{sign}(p-p_0)\int(\Phi(y))^2K_m(x-y)(u_r(y,p_0)-u_r(y,p)+u_0(y,p_0)-u_0(y,p))dy.$$

Since $p_0, p \in D \subset E$, $u_r(y,p_0) - u_r(y,p) \to u_0(y,p_0) - u_0(y,p)$ as $r \to \infty$ weakly-$*$ in $L^\infty(\Omega)$ and we derive from the above inequality that

$$\varlimsup_{r\to\infty}\|\Phi_m(x-y)\Phi(y)V_r(y)\|_2^2 \le 4\operatorname{sign}(p-p_0)\int(\Phi(y))^2K_m(x-y)(u_0(y,p_0)-u_0(y,p))dy.$$

Now, passing to the limit as $m \to \infty$ and taking into account that $x \in \Omega'$ is a Lebesgue point of the bounded function $u_0(y,p_0) - u_0(y,p)$ as well as the function $(\Phi(y))^2$ (therefore, x is a Lebesgue point of the product of these functions), we find

$$\varlimsup_{m\to\infty}\varlimsup_{r\to\infty}\|\Phi_m(x-y)\Phi(y)V_r(y)\|_2^2 \le 4(\Phi(x))^2|u_0(x,p_0)-u_0(x,p)|.$$

This implies the required relation

$$\varlimsup_{m\to\infty}\varlimsup_{r\to\infty}\|\Phi_m(x-y)\Phi(y)V_r(y)\|_2 \le 2|\Phi(x)||u_0(x,p_0)-u_0(x,p)|^{1/2}.$$

To complete the proof it only remains to observe that, as was demonstrated in the proof of Proposition 3, $\nu_x^0(\{p_0\}) = 0$ and therefore $u_0(x,p) \to u_0(x,p_0)$ as $p \to p_0$.

We now fix $x \in \Omega'$, $p_0, p \in D$. Let $\tilde{L}(p), L(p) \subset \mathbb{R}^n$ be the smallest linear subspaces containing $\operatorname{supp}\mu_x^{pp_0}$, $\operatorname{supp}\mu_x^{pp}$, respectively, and $L = L(p_0)$.

Lemma 4. *There exists positive δ such that $\tilde{L}(p) = L$ for each $p \in [p_0-\delta, p_0+\delta] \cap D$. If the space $L = L(p_0)$, $p_0 \in D$ has maximal dimension $l = \dim L$ among the spaces $L(p)$ then also $L(p) = L$ for each $p \in [p_0-\delta, p_0+\delta] \cap D$.*

Proof. First remark that, as it directly follows from (27), $\operatorname{supp}\mu_x^{pp_0} \subset \operatorname{supp}\mu_x^{p_0p_0} \subset L$ and, therefore $\tilde{L}(p) \subset L$. Similarly, $\operatorname{supp}\mu_x^{pp_0} \subset \operatorname{supp}\mu_x^{pp} \subset L(p)$, which implies the inclusion $\tilde{L}(p) \subset L(p)$. For positive r we denote $V_r = [p_0 - r, p_0 + r] \cap D$, $L_r = \bigcap_{p\in V_r}\tilde{L}(p)$. Clearly, $L_r \subset L$ is a decreasing (with respect to inclusion) family of linear subspaces of the finite-dimensional space L, therefore starting from some $r = \delta > 0$ for all $r \in (0,\delta]$ we have $L_r = L_0 \subset L$. To prove the lemma it suffices to show that $L_0 = L$. For in that case $L = L_0 \subset \tilde{L}(p) \subset L$ and the equality $\tilde{L}(p) = L$, $p \in V_\delta$ follows. Hence, $L = \tilde{L}(p) \subset L(p)$ and, in the case when $L = L(p_0)$ has maximal dimension, we conclude that $L(p) = L$.

We carry out the proof of the equality $L_0 = L$ by contradiction. Thus, we assume that $L_0 \ne L$. Then $m = \dim L_0 < l = \dim L$. We fix $\varepsilon > 0$. By Corollary 1 there exists $r \in (0,\delta]$ such that for $p \in V_r$ we have

$$\operatorname{Var}(\mu_x^{pp_0} - \mu_x^{p_0p_0}) < \varepsilon. \tag{28}$$

By the definition of the space L_r we can choose a strictly decreasing finite sequence of subspaces L_i', $i = 0,\dots,k$, such that $L_0' = L$, $L_k' = L_r = L_0$, and $L_i' = L_{i-1}' \cap \tilde{L}(p_i)$, where $p_i \in V_r$, $i = 1,\dots,k$. Clearly, $k \le \dim L - \dim\tilde{L} = l - m$. By the definition of the $\tilde{L}(p)$ we have $\operatorname{supp}\mu_x^{p_ip_0} \subset \tilde{L}(p_i)$. Hence $\operatorname{Var}(\mu_x^{p_ip_0}(C\tilde{L}(p_i)) = 0$, where CA for $A \subset \mathbb{R}^n$ is the difference $S \setminus A$. It now follows from (28) that

$$\mu_x^{p_0p_0}(C\tilde{L}(p_i)) < \varepsilon, \quad i = 1,\dots,k.$$

Since $L_0 = \bigcap_{i=1}^k\tilde{L}(p_i)$, it follows that $CL_0 = \bigcup_{i=1}^k C\tilde{L}(p_i)$ and

$$\mu_x^{p_0p_0}(CL_0) \le \sum_{i=1}^k\mu_x^{p_0p_0}(C\tilde{L}(p_i)) \le k\varepsilon.$$

Since ε is an arbitrary positive number, it follows that $\mu_x^{p_0p_0}(CL_0) = 0$ and $\operatorname{supp}\mu_x^{p_0p_0} \subset L_0$. Further, L is the smallest subspace such that $\operatorname{supp}\mu_x^{p_0p_0} \subset L$, therefore $L \subset L_0$, which is a contradiction. This completes the proof. $\square$

We consider now the complex linear subspace

$$R(p) = \left\{ \int \psi(\xi)\xi d\mu_x^{pp}(\xi) : \ \psi(\xi) \in C(S) \right\} \subset \mathbb{C}^n.$$

Lemma 5. *We have the equality $R(p) = \bar{L}(p)$, where $\bar{L}(p) = L(p) + iL(p) \subset \mathbb{C}^n$ is the complex linear subspace spanned by $L(p)$.*

Proof. The relation

$$v \cdot \int \psi(\xi)\xi d\mu_x^{pp}(\xi) = \int \psi(\xi) v \cdot \xi d\mu_x^{pp}(\xi), \quad v \in \mathbb{C}^n, \quad \psi(\xi) \in C(S)$$

(here and below we consider the scalar products of vectors in $\mathbb{C}^n$) shows us that the orthogonal complements $(R(p))^\perp = (L(p))^\perp$ are the same (in $\mathbb{C}^n$), which means that $R(p) = \bar{L}(p)$. The proof is complete. $\square$

Suppose that $f(y, \lambda)$ is a Caratheodory vector-function on $\Omega \times \mathbb{R}$. Assume also that the following estimate holds

$$\forall M > 0 \quad \|f(x, \cdot)\|_{M,\infty} = \max_{|\lambda| \le M} |f(x, \lambda)| \le \alpha_M(x) \in L^2_{loc}(\Omega). \tag{29}$$

Since the space $C(\mathbb{R}, \mathbb{R}^n)$ is separable with respect to the standard locally convex topology generated by seminorms $\|\cdot\|_{M,\infty}$, then, by the Pettis theorem (see [5], Chapter 3), the map $x \to F(x) = f(x, \cdot) \in C(\mathbb{R}, \mathbb{R}^n)$ is strongly measurable and in view of estimate (29) we see that $F(x) \in L^2_{loc}(\Omega, C(\mathbb{R}, \mathbb{R}^n))$, $|F(x)|^2 \in L^1_{loc}(\Omega, C(\mathbb{R}))$. In particular (see [5], Chapter 3), the set Ω_f of common Lebesgue points of the maps $F(x), |F(x)|^2$ has full measure. For $x \in \Omega_f$ we have

$$\forall M > 0 \ \lim_{m \to \infty} \int K_m(x - y)\|F(x) - F(y)\|_{M,\infty} dy = 0,$$
$$\lim_{m \to \infty} \int K_m(x - y)\||F(x)|^2 - |F(y)|^2\|_{M,\infty} dy = 0.$$

Since, evidently,

$$\|F(x) - F(y)\|^2_{M,\infty} \le 2\|F(x) - F(y)\|_{M,\infty}\|F(x)\|_{M,\infty} + \||F(x)|^2 - |F(y)|^2\|_{M,\infty},$$

from the above limit relations it follows that for $x \in \Omega_f$

$$\lim_{m \to \infty} \int K_m(x - y)\|F(x) - F(y)\|^2_{M,\infty} dy = 0 \quad \forall M > 0. \tag{30}$$

Clearly, each $x \in \Omega_f$ is a Lebesgue point of all functions $x \to f(x, \lambda)$, $\lambda \in \mathbb{R}$. Let $\Omega'' = \Omega' \cap \Omega_f$, $\gamma_x^r = \nu_x^r - \nu_x^0$. Suppose that $x \in \Omega''$, $p_0 \in D$, and the subspace L and the segment $V = V_\delta = [p_0 - \delta, p_0 + \delta] \cap D$ are determined as in Lemma 4. We suppose that $L = L(p_0)$ has maximal dimension. Let $\chi(\lambda) = \theta(\lambda - p_1) - \theta(\lambda - p_2)$, where $p_1, p_2 \in V$, $p_1 < p_2$. Assume also that $f(y, \lambda)$ takes its values in $L^\perp$. For a vector-function $h(y, \lambda)$ on $\Omega \times \mathbb{R}$, which is Borel and locally bounded with respect to the second variable, we denote $I_r(h)(y) = \int h(y, \lambda) d\gamma_y^r(\lambda)$. In view of the strong measurability of $F(x)$ and (29) we see that $I_r(f \cdot \chi)(y) \in L^2_{loc}(\Omega)$ (cf. Remark 1).

Proposition 4. *Under the above assumptions,*

$$\lim_{m \to \infty} \lim_{r \to \infty} \int_{\mathbb{R}^n} |\xi \cdot F(\Phi_m I_r(f \cdot \chi))(\xi)|^2 \, |\xi|^{-2} d\xi = 0.$$

Here $\Phi_m = \Phi_m(x - y) = \sqrt{K_m(x - y)}$ and $I_r(f \cdot \chi)$ are functions of the variable $y \in \Omega$.

Proof. Note that starting from some index m the supports of the $\Phi_m(x-y)$ lay in some compact subset B of Ω. Without loss of generality we can assume that $\operatorname{supp}\Phi_m \subset B$ for all $m \in \mathbb{N}$. Let $\tilde{f}(y,\lambda) = f(x,\lambda)$, $M = \sup \|\nu_y^r\|_\infty$. Then

$$|I_r((f-\tilde{f})\cdot\chi)(y)| \le \int |f(y,\lambda) - f(x,\lambda)| d\mathrm{Var}\,\gamma_y^r(\lambda) \le 2\|F(y) - F(x)\|_{M,\infty}.$$

Next, observe that

$$\int_{\mathbb{R}^n} |\xi \cdot F(\Phi_m I_r(f\cdot\chi))(\xi)|^2 \, |\xi|^{-2} d\xi = \|\xi \cdot F(\Phi_m I_r(f\cdot\chi))(\xi)|\xi|^{-1}\|_2^2,$$

where $\|\cdot\|$ is the norm in $L^2(\mathbb{R}^n)$, and with the help of Plancherel's identity we obtain that

$$\begin{gathered}\Big| \|\xi \cdot F(\Phi_m I_r(f\cdot\chi))(\xi)|\xi|^{-1}\|_2 - \|\xi \cdot F(\Phi_m I_r(\tilde{f}\cdot\chi))(\xi)|\xi|^{-1}\|_2 \Big| \le \\ \|\xi \cdot F(\Phi_m I_r((f-\tilde{f})\cdot\chi))(\xi)|\xi|^{-1}\|_2 \le \|F(\Phi_m I_r((f-\tilde{f})\cdot\chi))(\xi)\|_2 = \\ \|\Phi_m I_r((f-\tilde{f})\cdot\chi)\|_2 \le 2\left(\int K_m(x-y)\|F(y)-F(x)\|_{M,\infty}^2 dy\right)^{1/2}.\end{gathered}$$

From the above estimate and (30) it follows that

$$\lim_{m\to\infty}\lim_{r\to\infty}\Big| \|\xi \cdot F(\Phi_m I_r(f\cdot\chi))(\xi)|\xi|^{-1}\|_2 - \|\xi \cdot F(\Phi_m I_r(\tilde{f}\cdot\chi))(\xi)|\xi|^{-1}\|_2 \Big| = 0 \tag{31}$$

and it is sufficient to prove the proposition with f replaced by $\tilde{f}$. This function is continuous and does not depend on y. Therefore for any $\varepsilon > 0$ there exists a vector-valued function $g(\lambda)$ of the form $g(\lambda) = \sum_{i=1}^{k} v_i\theta(\lambda - p_i)$, where $v_i \in L^\perp$ and $p_i \in V$ such that $\|\tilde{f}\cdot\chi - g\|_\infty \le \varepsilon$ on $\mathbb{R}$.

Using again Plancherel's identity and the fact that

$$\left|\int (\tilde{f}\cdot\chi - g)(\lambda) d\gamma_y^r(\lambda)\right| \le \int |(\tilde{f}\cdot\chi - g)(\lambda)| d\mathrm{Var}\,(\gamma_y^r)(\lambda) \le 2\varepsilon,$$

we obtain

$$\begin{gathered}\Big| \|\xi \cdot F(\Phi_m I_r(\tilde{f}\cdot\chi))(\xi)|\xi|^{-1}\|_2 - \|\xi \cdot F(\Phi_m I_r(g))(\xi)|\xi|^{-1}\|_2 \Big| \le \\ \|\xi \cdot F(\Phi_m I_r(\tilde{f}\cdot\chi - g))(\xi)|\xi|^{-1}\|_2 \le \|\Phi_m I_r(\tilde{f}\cdot\chi - g)\|_2 \le 2\varepsilon\|\Phi_m\|_2 = 2\varepsilon.\end{gathered} \tag{32}$$

Since

$$I_r(g)(y) = \int \left(\sum_{i=1}^{k} v_i\theta(\lambda - p_i)\right) d\gamma_y^r(\lambda) = \sum_{i=1}^{k} v_i U_r^{p_i}(y),$$

from (25) it follows the limit relation

$$\begin{gathered}\lim_{m\to\infty}\lim_{r\to\infty} \|\xi \cdot F(\Phi_m I_r(g))(\xi)|\xi|^{-1}\|_2^2 = \\ \lim_{m\to\infty}\lim_{r\to\infty}\int_{\mathbb{R}^n} |\xi \cdot F(\Phi_m I_r(g))(\xi)|^2 \, |\xi|^{-2} d\xi = \sum_{i,j=1}^{k} \langle \mu_x^{p_i p_j}, (v_i\cdot\xi)(v_j\cdot\xi)\rangle = 0.\end{gathered} \tag{33}$$

The last equality is a consequence of the inclusion $\operatorname{supp}\mu_x^{p_i p_j} \subset L(p_i) = L$, which holds by Lemma 4 for all $i = 1,\dots,k$ (because $p_i \in V$), combined with the relation $v_i \perp L$. By (32) and (33),

$$\overline{\lim_{m\to\infty}}\,\overline{\lim_{r\to\infty}}\int_{\mathbb{R}^n} |\xi \cdot F(\Phi_m I_r(f\cdot\chi))(\xi)|^2 \, |\xi|^{-2} d\xi \le 4\varepsilon^2,$$

and it suffices to observe that $\varepsilon > 0$ can be arbitrary to complete the proof. $\square$

4. Localization principle and strong pre-compactness of bounded sequences of measure-valued functions

In this section we need some results concerning pseudo-differential operators (briefly p.d.o.) in $L^2(\mathbb{R}^n)$. Recall that a p.d.o. operator A with symbol $a(x,\xi)$, $x,\xi\in\mathbb{R}^n$ is defined as follows

$$F(Au)(\xi)=\int e^{-i\xi\cdot x}a(x,\xi)u(x)dx.$$

In particular, if $b(x)\in C_0(\mathbb{R}^n)$, $a(z)\in C(S)$ then the operators B,A with symbols $b(x)$, $a(\xi/|\xi|)$ are the multiplication operators $Bu(x)=b(x)u(x)$, $F(Au)(\xi)=a(\xi/|\xi|)F(u)(\xi)$. Obviously, these operators are well-defined and bounded in L^2. Moreover, the operator B is bounded in L^p, $1\le p\le\infty$. We shall use also the following statement known as the Hörmander-Mikhlin theorem on multipliers (see [16][Chapter 4]):

Lemma 6. *Let $a(z)\in C^l(S)$ for some $l>n/2$. Then the operator A with symbol $a(\xi/|\xi|)$ is a bounded operator $A:L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)$ for each $1<p<\infty$.*

We shall use operators represented, up to a compact operator, as finite sums $A=\sum A_kB_k$, where A_k,B_k are operators of the indicated above kind with symbols $a_k(\xi/|\xi|)$, $b_k(x)$ and call them admissible zero-order p.d.o. with symbols $\sum b_k(x)a_k(\xi/|\xi|)$. As was shown in [18], the commutator $[A_1,A_2]=A_1A_2-A_2A_1$ of admissible zero-order p.d.o. is a compact operator on L^2. Observe also that relation (9) in the definition of the Tartar H-measure can be written as follows

$$\langle\mu^{ij},p(x,\xi)\overline{q(x,\xi)}\rangle=\lim_{r\to\infty}(PU_r^i,QU_r^j)_2 \tag{34}$$

for each admissible zero-order p.d.o. P,Q with symbols $p(x,\xi)$, $q(x,\xi)$. Here $(\cdot,\cdot)_2$ is the scalar product in L^2. Indeed, suppose that $p(x,\xi)=\sum\limits_{k\in I_1}\Phi_k(x)\psi_k(\xi/|\xi|)$, $q(x,\xi)=\sum\limits_{l\in I_2}\Phi_l(x)\psi_l(\xi/|\xi|)$, where I_1,I_2 are finite sets, $\Phi_k(x),\Phi_l(x)\in C_0(\Omega)$, $\psi_k(\xi),\psi_l(\xi)\in C(S)$, $k\in I_1$, $l\in I_2$. Then

$$F(PU_r^i)(\xi)=\sum_{k\in I_1}F(\Phi_kU_r^i)(\xi)\psi_k(\xi/|\xi|)+F(E_1U_r^i)(\xi),$$
$$F(QU_r^j)(\xi)=\sum_{l\in I_2}F(\Phi_lU_r^j)(\xi)\psi_l(\xi/|\xi|)+F(E_2U_r^j)(\xi),$$

E_1,E_2 being compact operators in L^2. By compactness of E_1,E_2, we see that $E_1U_r^i\to0$, $E_2U_r^j\to0$ as $r\to\infty$ strongly in L^2. Therefore, with account of the Plancherel's identity and (9), we find

$$\lim_{r\to\infty}(PU_r^i,QU_r^j)_2=\lim_{r\to\infty}(F(PU_r^i),F(QU_r^j))_2=$$
$$\lim_{r\to\infty}\sum_{k\in I_1,l\in I_2}\int_{\mathbb{R}^n}F(\Phi_kU_r^i)(\xi)\overline{F(\Phi_lU_r^j)(\xi)}\psi_k(\xi/|\xi|)\overline{\psi_l(\xi/|\xi|)}d\xi=$$
$$\sum_{k\in I_1,l\in I_2}\langle\mu^{ij},\Phi_k(x)\psi_k(\xi)\overline{\Phi_l(x)\psi_l(\xi)}\rangle=\langle\mu^{ij},p(x,\xi)\overline{q(x,\xi)}\rangle$$

and (34) follows. Conversely, relation (9) follows from (34) with $p=\Phi_1(x)\psi(\xi/|\xi|)$, $q=\Phi_2(x)$.

The following lemma was also proved in [18][Lemma 3.2].

Lemma 7. *Let $b(x)\in C_0^1(\mathbb{R}^n)$, $a(z)\in C^l(S)$ with $l>(n+1)n/2$, A,B be operators in $L^2(\mathbb{R}^n)$ with symbols $a(\xi/|\xi|)$, $b(x)$. Then the commutator $[A,B]=AB-BA$ is a bounded operator from $L^2(\mathbb{R}^n)$ into $W_2^1(\mathbb{R}^n)$ and for $i=1,\dots,n$ $\partial_{x_i}[A,B]=C_i+E_i$, where C_i is an operator with symbol*

$$\xi_i\sum_{k=1}^n\frac{\partial a(\xi)}{\partial\xi_k}\frac{\partial b(x)}{\partial x_k}$$

while E_i is a compact operator in $L^2(\mathbb{R}^n)$.

Remark that the functions $\psi_{ik}(\xi)=\xi_i\dfrac{\partial a(\xi)}{\partial \xi_k}$ are homogeneous of zero order, i.e. $\psi_{ik}(\xi)=\psi_{ik}(\xi/|\xi|)$ with $\psi_{ik}(z)=z_i a_{\xi_k}(z)$, $z\in S$. In particular, $\partial_{x_i}[A,B]$ is an admissible zero-order pseudo-differential operator in L^2.

We define also operators (the Riesz potentials) J_α, $0<\alpha<n$ by the formula

$$F(J_\alpha u)(\xi)=|\xi|^{-\alpha}Fu(\xi).$$

It is known (see [16][Chapter 5]) that the operator J_α is a well-defined bounded operator from $L^p(\mathbb{R}^n)$, $p>1$ to the Sobolev space $W_p^\alpha(\mathbb{R}^n)$.

We denote by $R_j=\partial_{x_j}J_1=J_1\partial_{x_j}$ the Riesz transform, $j=1,\dots,n$. This is a zero-order p.d.o. with symbol $i\xi_j/|\xi|$. It is clear that the Riesz transforms commute with all admissible p.d.o. with symbols $\psi(\xi)$. Remark also that if $\overline{\psi(z)}=\psi(-z)=-\psi(z)$ then the p.d.o. A with symbols $\psi(\xi/|\xi|)$ acts in the space L^2 of real functions and A is an anti-selfadjoint operator:

$$\int_{\mathbb{R}^n} u\cdot Av dx=-\int_{\mathbb{R}^n} Au\cdot v dx \quad \forall u,v\in L^2.$$

Let $k(x)\in C_0^\infty(\mathbb{R}^n)$ be an even function with support in the unit ball such that $k(x)\ge 0$, $\int k(x)dx=1$. We define mollifiers $k_h(x)=h^{-n}k(x/h)$, $h>0$ and the corresponding averaging operators

$$u\to u^h=u*k_h(x)=\int_{\mathbb{R}^n} u(x-y)k_h(y)dy.$$

Here $u(x)\in X$ where $X=L^p_{loc}(\mathbb{R}^n)$ or $X=L^p(\mathbb{R}^n)$, $1\le p<\infty$. By the known properties of averaged functions $u^h(x)\in C^\infty(\mathbb{R}^n)$ and $u^h\to u$ as $h\to 0+$ in X. The averaging operator has symbol $F(k_h)(\xi)$ and therefore commutes with p.d.o. with symbols $\psi(\xi)$. Besides, since the kernel $k(x)$ is even, this operator is selfadjoint:

$$\int_{\mathbb{R}^n} u^h v dx=\int_{\mathbb{R}^n} uv^h dx \quad \forall u,v\in L^2. \tag{35}$$

We need in the sequel the following lemma

Lemma 8. *Suppose that $a(x)\in C^1(\mathbb{R}^n)$, $u(x)\in L^p_{loc}(\mathbb{R}^n)$, $1\le p<\infty$. Then $(au)^h-au^h \underset{h\to 0+}{\to} 0$ in the Sobolev space $W^1_{p,loc}(\mathbb{R}^n)$.*

The statement of this lemma follows from general result by R.J. DiPerna & P.L. Lions [3][Lemma II.1].

4.1. The first localization principle

We consider the bounded sequence of measure valued functions $\nu_x^k\in \mathrm{MV}(\Omega)$ and suppose that for some $p>1$ and each $a,b\in\mathbb{R}$, $a<b$ the sequence of distributions

$$\operatorname{div}_x\left(A(x)\nabla\int g(s_{a,b}(\lambda))d\nu_x^k(\lambda)\right) \text{ is pre-compact in } W^{-2}_{p,loc}(\Omega). \tag{36}$$

Here $s_{a,b}(u)=\max(a,\min(u,b))$ is the cut-off function and $W^{-s}_{p,loc}(\Omega)$ with $s>0$ denotes the locally convex space of distributions $u(x)$ such that $uf(x)$ belongs to the Sobolev space W_p^{-s} for all $f(x)\in C_0^\infty(\Omega)$. The topology in $W^{-s}_{p,loc}(\Omega)$ is generated by the family of semi-norms $u\to\|uf\|_{W_p^{-s}}$, $f(x)\in C_0^\infty(\Omega)$.

We choose the subsequence $\nu_x^r=\nu_x^k$, $k=k_r$ weakly convergent to a bounded measure-valued function ν_x^0 such that the H-measure $\mu^{pq}=\mu_x^{pq}dx$, $p,q\in D$ is well defined. Define the measures $\gamma_x^r=\nu_x^r-\nu_x^0$ and set of full measure Ω' as in the previous section.

The following Theorem shows that $\operatorname{supp}\mu_x^{pp}$ consists of $\xi\in S$ such that the function $\lambda\to A(x)\xi\cdot\xi g(\lambda)$ is constant in a vicinity of p.

Theorem 4. *Suppose that $x\in\Omega'$, $p_0\in D$ and $\xi\in L$, where L is a linear span of $\operatorname{supp}\mu_x^{p_0p_0}$. If $A(x)\xi\cdot\xi\ne 0$ then there exists $\delta>0$ such that $g(\lambda)=\mathrm{const}$ on the segment $\lambda\in[p_0,p_0+\delta]$.*

Proof. Throughout the proof we use the notation of section 3. Let $V = V_\delta = [p_0, p_0+\delta] \cap D$ be an interval, where $\delta > 0$ is chosen in accordance with Lemma 4, L be a linear span of $\operatorname{supp} \mu_x^{p_0 p_0}$, $p \in V$. As follows from (36) and the weak convergence $\nu_y^r \to \nu_y^0$,

$$\mathcal{L}_p^r(y) = \operatorname{div}_y \left(A(y)\nabla \int g(s_{p_0,p}(\lambda)) d\gamma_y^r(\lambda) \right) \underset{r\to\infty}{\to} 0 \text{ in } W_{p,loc}^{-2}(\Omega). \tag{37}$$

As is easy to compute,

$$g(s_{p_0,p}(\lambda)) = g(p_0) + (g(p) - g(p_0))\theta(\lambda - p_0) - (g(p) - g(\lambda))\chi(\lambda),$$

where $\chi(\lambda) = \theta(\lambda - p_0) - \theta(\lambda - p)$ is the indicator function of the interval $(p_0, p]$. Therefore, $\mathcal{L}_p^r = \operatorname{div}_y(A(y)\nabla Q_r^p(y))$ where the functions $Q_r^p(y)$ are as follows:

$$\begin{aligned} Q_r^p(y) = \int (g(p) - g(p_0))\theta(\lambda - p_0) d\gamma_y^r(\lambda) - \int (g(p) - g(\lambda))\chi(\lambda) d\gamma_y^r(\lambda) = \\ (g(p) - g(p_0)) U_r^{p_0}(y) - \int (g(p) - g(\lambda))\chi(\lambda) d\gamma_y^r(\lambda). \end{aligned} \tag{38}$$

For $\Phi(y) \in C_0^\infty(\Omega)$ we consider the sequence

$$\begin{aligned} L_r = \sum_{i,j=1}^n \frac{\partial^2}{\partial y_i \partial y_j} a_{ij}(y)\Phi(y) Q_r^p(y) = \sum_{i,j=1}^n \frac{\partial}{\partial y_i} \left(\Phi(y) a_{ij}(y) \frac{\partial Q_r^p(y)}{\partial y_j} \right) + \\ \sum_{i,j=1}^n \frac{\partial}{\partial y_i} \left(\frac{\partial a_{ij}(y)\Phi(y)}{\partial y_j} Q_r^p(y) \right) = \Phi(y) \operatorname{div}(A(y)\nabla Q_r^p(y)) + \\ \sum_{i,j=1}^n \left(\frac{\partial a_{ij}(y)\Phi(y)}{\partial y_j} Q_r^p(y) \right) + \sum_{i,j=1}^n a_{ij}(y) \frac{\partial \Phi(y)}{\partial y_i} \frac{\partial Q_r^p(y)}{\partial y_j}. \end{aligned}$$

Since the matrix $A(y)$ is symmetric we can transform the last term as follows

$$\begin{aligned} \sum_{i,j=1}^n a_{ij}(y) \frac{\partial \Phi(y)}{\partial y_i} \frac{\partial Q_r^p(y)}{\partial y_j} = \sum_{i,j=1}^n \frac{\partial}{\partial y_j} \left(Q_r^p(y) a_{ij}(y) \frac{\partial \Phi(y)}{\partial y_i} \right) - \\ Q_r^p(y) \sum_{i,j=1}^n \frac{\partial}{\partial y_j} \left(a_{ij}(y) \frac{\partial \Phi(y)}{\partial y_i} \right) = \operatorname{div}[Q_r^p(y) A(y) \nabla \Phi(y)] - Q_r^p(y) \operatorname{div}(A(y)\nabla\Phi(y)). \end{aligned}$$

Therefore,

$$\begin{aligned} L_r = \sum_{i,j=1}^n \frac{\partial^2}{\partial y_i \partial y_j} a_{ij}(y)\Phi(y) Q_r^p(y) = \Phi(y) \operatorname{div}(A(y)\nabla Q_r^p(y)) + \\ \sum_{i,j=1}^n \frac{\partial}{\partial y_i} \left(\frac{\partial a_{ij}(y)\Phi(y)}{\partial y_j} Q_r^p(y) \right) + \operatorname{div}[Q_r^p(y) A(y) \nabla \Phi(y)] - Q_r^p(y) \operatorname{div}(A(y)\nabla\Phi(y)) \end{aligned} \tag{39}$$

and, as follows from (37), the sequence L_r is pre-compact in W_p^{-2} since the term

$$\sum_{i,j=1}^n \frac{\partial}{\partial y_i} \left(\frac{\partial a_{ij}(y)\Phi(y)}{\partial y_j} Q_r^p(y) \right) + \operatorname{div}[Q_r^p(y) A(y) \nabla \Phi(y)] - Q_r^p(y) \operatorname{div}(A(y)\nabla\Phi(y))$$

lays in W_p^{-1} for all $p \ge 1$ and therefore is pre-compact in W_p^{-2}. Using the Fourier transformation and then applying the Riesz operator J_2, we obtain from (39) that

$$|\xi|^{-2} \sum_{i,j=1}^n \xi_i \xi_j \cdot F(a_{ij} Q_r^p \Phi)(\xi) = F(l_r), \quad l_r = J_2(L_r) \underset{r\to\infty}{\to} 0 \ \text{ in } L^p(\mathbb{R}^n). \tag{40}$$

Let $\psi(\xi) \in C^\infty(S)$. By (40), using the boundedness of the sequence $U_r^{p_0}\Phi(y)$ in L^q, $q^{-1}+p^{-1}=1$ and boundedness of the p.d.o. A with symbol $\overline{\psi(\xi/|\xi|)}$ (due to Lemma 6), we obtain

$$\int_{\mathbb{R}^n} |\xi|^{-2} \sum_{i,j=1}^n \xi_i\xi_j \cdot F(a_{ij}Q_r^p\Phi)(\xi)\overline{F(U_r^{p_0}\Phi)(\xi)}\psi\left(\frac{\xi}{|\xi|}\right)d\xi = \int_{\mathbb{R}^n} l_r(y)\overline{A(U_r^{p_0}\Phi(y))}dy \underset{r\to\infty}{\to} 0$$

(he we also use the fact that $l_r, A(U_r^{p_0}\Phi(y)) \in L^2(\mathbb{R}^n)$, which allows to apply the Plancherel's identity) or in view of (38),

$$(g(p)-g(p_0))\lim_{r\to\infty}\left\{\int_{\mathbb{R}^n} |\xi|^{-2} \sum_{i,j=1}^n \xi_i\xi_j \cdot F(a_{ij}U_r^{p_0}\Phi)(\xi)\overline{F(U_r^{p_0}\Phi)(\xi)}\psi\left(\frac{\xi}{|\xi|}\right)d\xi - \right.$$
$$\left.\int_{\mathbb{R}^n} |\xi|^{-2} \sum_{i,j=1}^n \xi_i\xi_j \cdot F(a_{ij}G_r^p\Phi)(\xi)\overline{F(U_r^{p_0}\Phi)(\xi)}\psi\left(\frac{\xi}{|\xi|}\right)d\xi\right\} = 0, \tag{41}$$

where

$$G_r^p(y) = \int (g(p)-g(\lambda))\chi(\lambda)d\gamma_y^r(\lambda).$$

We set in (41) $\Phi(y) = \Phi_m(x-y)$, where the functions Φ_m were defined in section 3 in the proof of Proposition 3, and pass to the limit as $m\to\infty$. By Remark 2 (see equality (25)) we obtain

$$\lim_{m\to\infty}\lim_{r\to\infty}\int_{\mathbb{R}^n} |\xi|^{-2} \sum_{i,j=1}^n \xi_i\xi_j \cdot F(a_{ij}U_r^{p_0}\Phi_m)(\xi)\overline{F(U_r^{p_0}\Phi_m)(\xi)}\psi\left(\frac{\xi}{|\xi|}\right)d\xi =$$
$$\langle \mu_x^{p_0p_0}, A(x)\xi\cdot\xi\psi(\xi)\rangle,$$

therefore

$$(g(p)-g(p_0))\cdot\langle \mu_x^{p_0p_0}, A(x)\xi\cdot\xi\psi(\xi)\rangle =$$
$$\lim_{m\to\infty}\lim_{r\to\infty}\int_{\mathbb{R}^n} |\xi|^{-2} \sum_{i,j=1}^n \xi_i\xi_j \cdot F(a_{ij}G_r^p\Phi_m)(\xi)\overline{F(U_r^{p_0}\Phi_m)(\xi)}\psi\left(\frac{\xi}{|\xi|}\right)d\xi. \tag{42}$$

This, by Bunyakovskii inequality and Plancherel's equality, gives us the estimate

$$|(g(p)-g(p_0))\langle \mu_x^{p_0p_0}, A(x)\xi\cdot\xi\psi(\xi)\rangle| \le$$
$$\overline{\lim_{m\to\infty}}\,\overline{\lim_{r\to\infty}}\sum_{i,j=1}^n \|a_{ij}G_r^p\Phi_m\|_2\cdot\|U_r^{p_0}\Phi_m\|_2\cdot\|\psi\|_\infty. \tag{43}$$

Since $g(\lambda)$ is a non-decreasing function $(g(p)-g(\lambda))\chi(\lambda) \le (g(p)-g(p_0))\chi(\lambda)$ and

$$|G_r^p(y)| \le (g(p)-g(p_0))\int \chi(\lambda)d\left(\nu_y^r(\lambda)+\nu_y^0(\lambda)\right)$$

and by Lemma 3 we obtain

$$\overline{\lim_{m\to\infty}}\,\overline{\lim_{r\to\infty}}\|a_{ij}(y)G_r^p\Phi_m\|_2 \le 2(g(p)-g(p_0))\cdot|a_{ij}(x)|\cdot(u_0(x,p_0)-u_0(x,p))^{1/2}. \tag{44}$$

Further, we have $|U_r^{p_0}| \le 1$, therefore $\|U_r^{p_0}\Phi_m\|_2 \le \|\Phi_m\|_2 = 1$ and, in view of (43) and (44),

$$(g(p)-g(p_0))|\langle \mu_x^{p_0p_0}, A(x)\xi\cdot\xi\psi(\xi)\rangle| \le 2n^2(g(p)-g(p_0))\max|a_{ij}(x)|\cdot\|\psi\|_\infty\omega(p), \tag{45}$$
$$\omega(p) = (u_0(x,p_0)-u_0(x,p))^{1/2} \underset{p\to p_0}{\to} 0.$$

Assume that $A(x)\xi\cdot\xi \ne 0$ for some $\xi \in L$. Then we can choose a function $\psi(\xi) \in C^\infty(S)$ such that $|\langle \mu_x^{p_0p_0}, A(x)\xi\cdot\xi\psi(\xi)\rangle| > 0$ (otherwise, $\operatorname{supp}\mu_x^{p_0p_0}$ lays in the kernel $\ker A(x)$ and therefore $L \subset \ker A(x)$). Then, from (45) it follows that for all $p \in V$

$$(g(p)-g(p_0)) \le c\omega(p)(g(p)-g(p_0)), \quad c = \text{const}. \tag{46}$$

Taking a smaller δ if necessary we can assume that $c\omega(p) < 1$ for all $p \in V$. Now, in view of (46), $g(p) = g(p_0)$ $\forall p \in [p_0, p_0 + \delta] \cap D$. Taking into account that D is dense in $\mathbb{R}$ and that $g(\lambda)$ is continuous, we obtain the required identity $g(p) = g(p_0)$ $\forall p \in [p_0, p_0 + \delta]$. The proof is complete. □

Corollary 2. *Assume that $p_0 \in E$ and $g(p)$ is not constant in each segment of $[p_0, p_0 + \delta]$, $\delta > 0$. Then $A(x)\xi \cdot \xi\mu^{p_0p_0} = 0$.*

Proof. Since D is arbitrary dense countable subset of E we may assume that $p_0 \in D$. Then, as readily follows from Theorem 4, $A(x)\xi \cdot \xi\mu_x^{p_0p_0} = 0$ for a.e. $x \in \Omega$ and since $\mu^{p_0p_0} = \mu_x^{p_0p_0}dx$ we conclude that $A(x)\xi \cdot \xi\mu^{p_0p_0} = 0$. □

Denote for $p_0, p \in E$, $p > p_0$

$$h(\lambda) = g(s_{p_0,p}(\lambda)) - g(p_0),\ Q_r(y) = \int h(\lambda)d\gamma_y^r(\lambda)$$

(remark that in the notations of Theorem 3 the sequence Q_r coincides with Q_r^p). Clearly, the sequence $Q_r(y)$ is bounded in $L^\infty(\Omega)$ and converges weakly-$*$ to zero as $r \to \infty$. After extraction of a subsequence (we keep the notation Q_r for it) we can assume that the Tartar H-measure $\bar{\mu} = \bar{\mu}^{00}$ is well-defined for this scalar sequence, i.e.

$$\langle \bar{\mu}, \Phi_1(x)\overline{\Phi_2(x)}\psi(\xi)\rangle = \lim_{r\to\infty} \int_{\mathbb{R}^n} F(Q_r\Phi_1)(\xi)\overline{F(Q_r\Phi_2)(\xi)}\psi\left(\frac{\xi}{|\xi|}\right)d\xi$$

for all $\Phi_1(x), \Phi_2(x) \in C_0(\Omega)$ and $\psi(\xi) \in C(S)$.

Proposition 5. *Under condition (36) we have $A(x)\xi \cdot \xi\bar{\mu} = 0$.*

Proof. If $g(p) = g(p_0)$ then $Q_r \equiv 0$ and there is nothing to prove. So we assume that $g(p) > g(p_0)$. Consider the set B consisting of values $v = g(u)$, $u \in [p_0, p]$ such that $g(\lambda) = \text{const}$ in some segment $[u, u + \delta]$, $\delta > 0$. For $v \in B$ the level set $g^{-1}(v) = \{\ u \in [p_0, p] \mid g(u) = v\ \}$ has positive Lebesgue measure: $\operatorname{meas} g^{-1}(v) > 0$. Since $\sum\limits_{v\in B} \operatorname{meas} g^{-1}(v) \le p - p_0$ we see that B is at most countable. The set $[p_0, p] \setminus E$ is at most countable as well. Therefore the union $B_1 = B \cup g([p_0, p] \setminus E)$ is at most countable and its complement $U = [g(p_0), g(p)] \setminus B_1$ is dense in $[g(p_0), g(p)]$. This allows us to find for every $\varepsilon > 0$ points $v_i \in U$, $i = 1, \ldots, k$ such that $g(p_0) \doteq v_0 < v_1 < \cdots < v_k < v_{k+1} \doteq g(p)$ and $v_i - v_{i-1} < \varepsilon$, $i = 1, \ldots, k + 1$. We choose $p_i \in [p_0, p]$ such that $g(p_i) = v_i$, $i = 1, \ldots, k$ and introduce the step function $s(\lambda) = \sum\limits_{i=1}^{k} c_i\theta(\lambda - p_i)$ with $c_i = v_i - v_{i-1}$. Then, as is easy to see, $\|h - s\|_\infty < \varepsilon$. From the assumption $v_i \in U$ it follows that $p_i \in E$ and $g(\lambda)$ is not constant in each segment $[p_i, p_i + \delta]$, $\delta > 0$. Let

$$\tilde{Q}_r(y) = \int s(\lambda)d\gamma_y^r(\lambda) = \sum_{i=1}^{k} c_i U_r^{p_i}(y), \tag{47}$$

A be the admissible zero-order p.d.e. with symbols $a(x, \xi) = \Phi(x)A(x)z \cdot z$, $z = \xi/|\xi|$, $\Phi(x) \in C_0(\Omega)$. It is clear that $\|Q_r - \tilde{Q}_r\|_\infty \le 2\|h - s\|_\infty \le 2\varepsilon$. Then, by the Bunyakovskii inequality

$$\begin{aligned} \left|(AQ_r, AQ_r)_2 - (A\tilde{Q}_r, A\tilde{Q}_r)_2\right| \le |(A(Q_r - \tilde{Q}_r), AQ_r)_2| + |(A\tilde{Q}_r, A(Q_r - \tilde{Q}_r))_2| \le \\ \text{const} \cdot \|(Q_r - \tilde{Q}_r)\Phi\|_2 \le C\varepsilon, \quad C = \text{const}. \end{aligned} \tag{48}$$

In view of representation (47) and relation (34) we find

$$\lim_{r\to\infty}(A\tilde{Q}_r, A\tilde{Q}_r)_2 = \lim_{r\to\infty}\sum_{i,j=1}^{k} c_ic_j(AU_r^{p_i}, AU_r^{p_j})_2 = \sum_{i,j=1}^{k} c_ic_j\langle\mu^{p_ip_j}, |a(x,\xi)|^2\rangle = 0. \tag{49}$$

Indeed, $p_i \in E$ for $i = 1, \dots, k$ and $g(\lambda)$ is not constant in each segment $[p_i, p_i + \delta]$, $\delta > 0$. Then by Corollary 2 $\langle \mu^{p_i p_i}, |a(x,\xi)|^2 \rangle = \langle \mu^{p_i p_i}, |\Phi(x)|^2 |A(x)\xi \cdot \xi|^2 \rangle = 0$ for all $i = 1, \dots, k$. By Lemma 2 the matrix $\langle \mu^{p_i p_j}, |a(x,\xi)|^2 \rangle$ is positive definite and in particular

$$|\langle \mu^{p_i p_j}, |a(x,\xi)|^2 \rangle|^2 \le \langle \mu^{p_i p_i}, |a(x,\xi)|^2 \rangle \cdot \langle \mu^{p_j p_j}, |a(x,\xi)|^2 \rangle = 0,$$

which evidently yields (49). From (48), (49) it follows that $\overline{\lim\limits_{r\to\infty}} (AQ_r, AQ_r)_2 \le C\varepsilon$ and since $\varepsilon > 0$ is arbitrary we obtain that $\lim\limits_{r\to\infty} (AQ_r, AQ_r)_2 = 0$. Therefore,

$$\langle \bar{\mu}^{00}, |\Phi(x)|^2 |A(x)\xi \cdot \xi|^2 \rangle = \langle \bar{\mu}^{00}, |a(x,\xi)|^2 \rangle = \lim_{r\to\infty} (AQ_r, AQ_r)_2 = 0$$

for all $\Phi(x) \in C_0(\Omega)$. Taking into account non-negativity of the measure $\bar{\mu} = \bar{\mu}^{00} \ge 0$, we conclude that $A(x)\xi \cdot \xi \bar{\mu} = 0$. □

Corollary 3. *For each $c(x,\xi) \in C_0^1(\Omega \times S)$ the vector $\langle \bar{\mu}, c(x,\xi)A(x)\xi \rangle = 0$.*

Proof. As follows from Proposition 5 and symmetricity of matrix $A(x)$

$$\operatorname{supp} \bar{\mu} \subset \{\ (x,\xi) \mid A(x)\xi = 0\ \}.$$

Hence, $c(x,\xi)A(x)\xi = 0$ on $\operatorname{supp} \bar{\mu}$ and $\langle \bar{\mu}, c(x,\xi)A(x)\xi \rangle = 0$. □

Corollary 4. *For each $j = 1, \dots, n$ $A_{x_j}(x)\xi \cdot \xi \bar{\mu} = \sum\limits_{k,l=1}^{n} (a_{kl}(x))_{x_j} \xi_k \xi_l \bar{\mu} = 0$.*

Proof. Let $c(x,\xi) \in C_0^1(\Omega \times S)$, $c(x,\xi) \ge 0$. We introduce the function

$$F(y) = \langle \bar{\mu}, c(x,\xi)A(x+y)\xi \cdot \xi \rangle.$$

This function is defined in a sufficiently small neighborhood V of zero, $F(y) \in C^1(V)$. By Proposition 5 $F(0) = 0$. Since $\bar{\mu} \ge 0$, $c(x,\xi)A(x+y)\xi \cdot \xi \ge 0$ we see that $F(y) \ge 0$. Hence $F(y)$ takes its minimal value at the point $y = 0$. Therefore $\nabla F(0) = 0$ and we claim that $\langle \bar{\mu}, c(x,\xi)A_{x_j}(x)\xi \cdot \xi \rangle = F_{y_j}(0) = 0$ for every $j = 1, \dots, n$. This relations hold for all $c(x,\xi) \in C_0^1(\Omega \times S)$ (because this function can be represented as a difference of two nonnegative function from $C_0^1(\Omega \times S)$) and we conclude that $A_{x_j}(x)\xi \cdot \xi \bar{\mu} = 0$. The proof is complete. □

4.2. The second localization principle.

Now we assume that a sequence of measure valued functions ν_x^k converges as $k \to \infty$ weakly to ν_x^0 and for each $a, b \in \mathbb{R}$, $a < b$ the sequence of distributions

$$\operatorname{div}_x \left(\int \varphi(x, s_{a,b}(\lambda)) d\nu_x^k(\lambda) - A(x)\nabla \int g(s_{a,b}(\lambda)) d\nu_x^k(\lambda) \right)$$
$$\text{is pre-compact in } W_{p,loc}^{-1}(\Omega) \qquad (50)$$

with some $p > 1$. We choose the subsequence $\nu_x^r = \nu_x^k$ such that the H-measure $\mu^{pq} = \mu_x^{pq} dx$, $p, q \in D$ is well defined. Define the measures $\gamma_x^r = \nu_x^r - \nu_x^0$ and set of full measure $\Omega'' = \Omega' \cap \Omega_\varphi$ as in section 3. Recall that Ω_φ consists of common Lebesgue points of the maps $x \to \varphi(x,\cdot) \in C(\mathbb{R}, \mathbb{R}^n)$ and $x \to |\varphi(x,\cdot)|^2 \in C(\mathbb{R})$. We also use notations of Lemma 4. In particular, $L(p)$ denotes a linear span of $\operatorname{supp} \mu_x^{pp}$. The main theorem of this paragraph is the following

Theorem 5. *Suppose that $x \in \Omega''$, $p_0 \in D$ and the space $L = L(p_0)$ has maximal dimension. Then there exists $\delta > 0$ such that $\varphi(x,\lambda) \cdot \xi = \text{const}$, $g(\lambda)A(x)\xi \cdot \xi = \text{const}$ on the segment $\lambda \in [p_0, p_0 + \delta]$ for all $\xi \in L$.*

To prove Theorem 5 we need some auxiliary results. Remark firstly that the sequence $\operatorname{div}_x \int \varphi(x, s_{a,b}(\lambda)) d\nu_x^k(\lambda)$ is bounded in $W_{2,loc}^{-1}(\Omega)$ and therefore it is pre-compact in $W_{2,loc}^{-2}(\Omega)$. By assumption (50) we see that relation (36) holds. This implies that the H-measure $\mu^{p_0p_0}$ satisfies the first localization principle and, in particular, $g(\lambda)A(x)\xi \cdot \xi = \text{const}$ on some segment $\lambda \in [p_0, p_0 + \delta]$ for all $\xi \in L$.

As follows from (50) and the weak convergence $\nu_y^r \to \nu_y^0$,

$$L_r = \operatorname{div}_y(P_r(y) - A(y)\nabla Q_r(y)) \underset{r\to\infty}{\to} 0 \text{ in } W_{p,loc}^{-1}(\Omega), \tag{51}$$

where we denote

$$P_r(y) = \int \varphi(y, s_{p_0,p}(\lambda)) d\gamma_y^r(\lambda) =$$
$$\int (\varphi(y,p) - \varphi(y,p_0))\theta(\lambda - p_0) d\gamma_y^r(\lambda) - \int (\varphi(y,p) - \varphi(y,\lambda))\chi(\lambda) d\gamma_y^r(\lambda) =$$
$$(\varphi(y,p) - \varphi(y,p_0)) U_r^{p_0}(y) - \int (\varphi(y,p) - \varphi(y,\lambda))\chi(\lambda) d\gamma_y^r(\lambda); \tag{52}$$
$$Q_r(y) = \int g(s_{p_0,p}(\lambda)) d\gamma_y^r(\lambda) =$$
$$(g(p) - g(p_0)) U_r^{p_0}(y) - \int (g(p) - g(\lambda))\chi(\lambda) d\gamma_y^r(\lambda), \tag{53}$$

$\chi(\lambda) = \theta(\lambda - p_0) - \theta(\lambda - p)$, $p > p_0$. Clearly, the sequences $P_r(y)$, $Q_r(y)$ are bounded in $L_{loc}^2(\Omega, \mathbb{R}^n)$, $L^\infty(\Omega)$ respectively, and converge weakly to zero as $r \to \infty$. Consider firstly the case

A) $g(\lambda) \equiv g(p_0)$ on some interval $[p_0, p_0 + \delta]$, $\delta > 0$.

In this case the statement of Theorem 5 follows from the following theorem.

Theorem 6. *Assume that condition A) is satisfied. Then there exists $\delta > 0$ such that $\xi \cdot \varphi(x, \lambda) = \xi \cdot \varphi(x, p_0)$ on $[p_0, p_0 + \delta]$ for all $\xi \in L$.*

Proof. Let $V = V_\delta = [p_0, p_0 + \delta] \cap D$, where $\delta > 0$ is chosen from Lemma 4, $L = L(p_0)$ be a linear span of $\operatorname{supp} \mu_x^{p_0p_0}$. Taking a smaller δ if necessary we can suppose that $g(\lambda) = \text{const}$ on $[p_0, p_0 + \delta]$. Let $p \in V_\delta$. Since $Q_r \equiv 0$ we derive from (51) that

$$\operatorname{div}_y P_r(y) \underset{r\to\infty}{\to} 0 \text{ in } W_{p,loc}^{-1}(\Omega)$$

and if $\Phi(y) \in C_0^\infty(\Omega)$ then

$$\operatorname{div}_y(P_r \Phi(y)) \underset{r\to\infty}{\to} 0 \ \text{ in } W_p^{-1}. \tag{54}$$

Using the Fourier transformation and the Riesz operator J_1, we obtain from (54) that

$$|\xi|^{-1}\xi \cdot F(P_r\Phi)(\xi) = F(l_r), \ l_r \to 0 \ \text{ in } L^p(\mathbb{R}^n) \tag{55}$$

as $r \to \infty$. Let $\psi(\xi) \in C^\infty(S)$. By (55), using the boundedness of the sequence $U_r^{p_0}\Phi(y)$ in L^q, $q^{-1} + p^{-1} = 1$ and Lemma 6, we obtain

$$\int_{\mathbb{R}^n} |\xi|^{-1}\xi \cdot F(P_r\Phi)(\xi) \overline{F(U_r^{p_0}\Phi)(\xi)} \psi\left(\frac{\xi}{|\xi|}\right) d\xi = \int_{\mathbb{R}^n} l_r(y) \overline{A(U_r^{p_0}\Phi)(y)} dy \to 0$$

as $r \to \infty$. Here A is a p.d.o. with symbols $\overline{\psi(\xi/|\xi|)}$. Thus, in view of (52),

$$\lim_{r\to\infty} \left\{ \int_{\mathbb{R}^n} |\xi|^{-1}\xi \cdot F(U_r^{p_0} f\Phi)(\xi) \overline{F(U_r^{p_0}\Phi)(\xi)} \psi\left(\frac{\xi}{|\xi|}\right) d\xi - \right.$$
$$\left. \int_{\mathbb{R}^n} |\xi|^{-1}\xi \cdot F(V_r^p\Phi)(\xi) \overline{F(U_r^{p_0}\Phi)(\xi)} \psi\left(\frac{\xi}{|\xi|}\right) d\xi \right\} = 0, \tag{56}$$

where

$$f(y) = \varphi(y,p) - \varphi(y,p_0) \ \text{ and } \ V_r^p(y) = \int (\varphi(y,p) - \varphi(y,\lambda))\chi(\lambda) d\gamma_y^r(\lambda).$$

Obviously, (56) holds for all $\psi(\xi) \in C(S)$. We set in (56) $\Phi(y) = \Phi_m(x-y)$, where the functions Φ_m were defined in section 3 in the proof of Proposition 3, and pass to the limit as $m \to \infty$. Observe that $x \in \Omega'' \subset \Omega_\varphi$ is a Lebesgue point of $\varphi(x,p)$ for every p. By Remark 2 (see equality (25)) we obtain

$$\lim_{m\to\infty}\lim_{r\to\infty}\int_{\mathbb{R}^n} |\xi|^{-1}\xi \cdot F(U_r^{p_0} f\Phi_m)(\xi)\overline{F(U_r^{p_0}\Phi_m)(\xi)}\psi\left(\frac{\xi}{|\xi|}\right)d\xi = (\varphi(x,p)-\varphi(x,p_0))\cdot \langle \mu_x^{p_0p_0}, \xi\psi(\xi)\rangle,$$

therefore

$$(\varphi(x,p)-\varphi(x,p_0))\cdot \langle \mu_x^{p_0p_0}, \xi\psi(\xi)\rangle = \lim_{m\to\infty}\lim_{r\to\infty}\int_{\mathbb{R}^n} |\xi|^{-1}\xi \cdot F(V_r^p\Phi_m)(\xi)\overline{F(U_r^{p_0}\Phi_m)(\xi)}\psi\left(\frac{\xi}{|\xi|}\right)d\xi. \tag{57}$$

Let π_1 and π_2 be the orthogonal projections of $\mathbb{R}^n$ onto the subspaces L and $L^\perp$ respectively; $\tilde\varphi(x,\lambda) = \pi_1(\varphi(x,\lambda))$, $\bar\varphi(x,\lambda) = \pi_2(\varphi(x,\lambda))$. Recall that L is the smallest subspace containing $\operatorname{supp}\mu_x^{p_0p_0}$. Hence

$$(\varphi(x,p)-\varphi(x,p_0))\cdot \langle \mu_x^{p_0p_0}, \xi\psi(\xi)\rangle = (\tilde\varphi(x,p)-\tilde\varphi(x,p_0))\cdot \langle \mu_x^{p_0p_0}, \xi\psi(\xi)\rangle. \tag{58}$$

Further, $V_r^p(y) = \pi_1(V_r^p(y)) + \pi_2(V_r^p(y))$ and

$$\pi_1(V_r^p(y)) = \int (\tilde\varphi(y,p) - \tilde\varphi(y,\lambda))\,\chi(\lambda)d\gamma_y^r(\lambda),$$
$$\pi_2(V_r^p(y)) = \int (\bar\varphi(y,p) - \bar\varphi(y,\lambda))\,\chi(\lambda)d\gamma_y^r(\lambda).$$

In the notation of Proposition 4,

$$\pi_2(V_r^p(y)) = I_r(h\cdot\chi),$$

where $h(y,\lambda) = \bar\varphi(y,p) - \bar\varphi(y,\lambda)$ is a Caratheodory vector taking its values in $L^\perp$. Now, by Proposition 4 we obtain

$$\lim_{m\to\infty}\lim_{r\to\infty}\int_{\mathbb{R}^n} |\xi|^{-1}\xi \cdot F(\pi_2(V_r^p)\Phi_m)(\xi)\overline{F(U_r^{p_0}\Phi_m)(\xi)}\psi\left(\frac{\xi}{|\xi|}\right)d\xi = 0. \tag{59}$$

Let $\tilde V_r^p(y) = \pi_1(V_r^p(y))$. From (57), in view of (58) and (59), we see that

$$(\tilde\varphi(x,p)-\tilde\varphi(x,p_0))\cdot \langle \mu_x^{p_0p_0}, \xi\psi(\xi)\rangle = \lim_{m\to\infty}\lim_{r\to\infty}\int_{\mathbb{R}^n} |\xi|^{-1}\xi \cdot F(\tilde V_r^p\Phi_m)(\xi)\overline{F(U_r^{p_0}\Phi_m)(\xi)}\psi\left(\frac{\xi}{|\xi|}\right)d\xi,$$

which in turn, by Bunyakovskii inequality and Plancherel's equality, gives us the estimate

$$|(\tilde\varphi(x,p)-\tilde\varphi(x,p_0))\cdot \langle \mu_x^{p_0p_0}, \xi\psi(\xi)\rangle| \le \varlimsup_{m\to\infty}\varlimsup_{r\to\infty}\|\tilde V_r^p\Phi_m\|_2\cdot\|U_r^{p_0}\Phi_m\|_2\cdot\|\psi\|_\infty. \tag{60}$$

Next, for $M_p(y) = \max\limits_{\lambda\in[p_0,p_0+\delta]}|\tilde\varphi(y,p)-\tilde\varphi(y,\lambda)|$

$$|\tilde V_r^p(y)| \le M_p(y)\int \chi(\lambda)d\left(\nu_y^r(\lambda)+\nu_y^0(\lambda)\right) = M_p(y)(u_r(y,p_0)-u_r(y,p)+u_0(y,p_0)-u_0(y,p))$$

and by Lemma 3

$$\varlimsup_{m\to\infty}\varlimsup_{r\to\infty}\|\tilde V_r^p\Phi_m\|_2 \le 2M_p(x)(u_0(x,p_0)-u_0(x,p)). \tag{61}$$

Here we bear in mind that x is a Lebesgue point of the function $(M_p(y))^2$ (the latter easily follows from the fact that $x\in\Omega_\varphi$ is a Lebesgue point of the maps $y\to\varphi(y,\cdot)$, $y\to|\varphi(y,\cdot)|^2$ into the spaces $C(\mathbb{R},\mathbb{R}^n)$,

$C(\mathbb{R})$, respectively). Further, we have $|U_r^{p_0}| \le 1$, therefore $\|U_r^{p_0}\Phi_m\|_2 \le \|\Phi_m\|_2 = 1$ and, in view of (59) and (61),

$$|(\tilde{\varphi}(x,p) - \tilde{\varphi}(x,p_0)) \cdot \langle \mu_x^{p_0p_0}, \xi\psi(\xi)\rangle| \le 2\|\psi\|_\infty M_p(x)\omega(p), \tag{62}$$
$$\omega(p) = (u_0(x,p_0) - u_0(x,p))^{1/2} \underset{p\to p_0}{\to} 0$$

(remind that $p_0 \in D$ is a continuity point of the function $p \to u_0(x,p)$ for $x \in \Omega'$). Next, by Lemma 5, the set of vectors of the form $\langle \mu_x^{p_0p_0}, \xi\psi(\xi)\rangle$, $\psi(\xi) \in C(S)$ spans the subspace $\overline{L} = L + iL$. Hence we can choose functions $\psi_i(\xi) \in C(S)$, $i = 1, \dots, l$ such that the vectors $v_i = \langle \mu_x^{p_0p_0}, \xi\psi_i(\xi)\rangle$ make up an algebraic basis in L. By (62), for $\psi(\xi) = \psi_i(\xi)$, $i = 1, \dots, l$, we obtain

$$|(\tilde{\varphi}(x,p) - \tilde{\varphi}(x,p_0)) \cdot v_i| \le c_i\omega(p)M_p(x), \quad c_i = \text{const},$$

and since v_i, $i = 1, \dots, l$ is a basis in L, these estimates show that for all $p \in V$

$$|\tilde{\varphi}(x,p) - \tilde{\varphi}(x,p_0)| \le c\omega(p)M_p(x) =$$
$$c\omega(p) \max_{\lambda\in[p_0,p_0+\delta]} |\tilde{\varphi}(x,p) - \tilde{\varphi}(x,\lambda)|, \quad c = \text{const}. \tag{63}$$

Taking a smaller δ if necessary we can assume that $2c\omega(p) \le \varepsilon < 1$ for all $p \in V$. Now, in view of (63),

$$|\tilde{\varphi}(x,p) - \tilde{\varphi}(x,p_0)| \le \frac{\varepsilon}{2} \max_{\lambda\in[p_0,p_0+\delta]} |\tilde{\varphi}(x,p) - \tilde{\varphi}(x,\lambda)|, \tag{64}$$

and since $\varphi(x,p)$ is continuous with respect to p and the set D is dense, the estimate (64) holds for all $p \in [p_0, p_0 + \delta]$.

We claim that now $\tilde{\varphi}(x,p) = \tilde{\varphi}(x,p_0)$ for $p \in [p_0, p_0 + \delta]$. Indeed, assume that for $p' \in [p_0, p_0 + \delta]$

$$|\tilde{\varphi}(x,p') - \tilde{\varphi}(x,p_0)| = \max_{\lambda\in[p_0,p_0+\delta]} |\tilde{\varphi}(x,\lambda) - \tilde{\varphi}(x,p_0)|.$$

Then for $\lambda \in [p_0, p']$ we have

$$|\tilde{\varphi}(x,p') - \tilde{\varphi}(x,\lambda)| \le |\tilde{\varphi}(x,\lambda) - \tilde{\varphi}(x,p_0)| +$$
$$|\tilde{\varphi}(x,p') - \tilde{\varphi}(x,p_0)| \le 2|\tilde{\varphi}(x,p') - \tilde{\varphi}(x,p_0)|$$

and

$$\max_{\lambda\in[p_0,p']} |\tilde{\varphi}(x,p') - \tilde{\varphi}(x,\lambda)| \le 2|\tilde{\varphi}(x,p') - \tilde{\varphi}(x,p_0)|.$$

We now derive from (64) with $p = p'$ that

$$|\tilde{\varphi}(x,p') - \tilde{\varphi}(x,p_0)| \le \varepsilon|\tilde{\varphi}(x,p') - \tilde{\varphi}(x,p_0)|,$$

and since $\varepsilon < 1$, this implies that

$$|\tilde{\varphi}(x,p') - \tilde{\varphi}(x,p_0)| = \max_{\lambda\in[p_0,p_0+\delta]} |\tilde{\varphi}(x,\lambda) - \tilde{\varphi}(x,p_0)| = 0.$$

We conclude that $\varphi(x,\lambda) - \varphi(x,p_0) \in L^\perp$ for all $\lambda \in [p_0, p_0 + \delta]$, i.e. $\varphi(x,\lambda) \cdot \xi = \varphi(x,p_0) \cdot \xi = \text{const}$ on the segment $\lambda \in [p_0, p_0 + \delta]$ for all $\xi \in L$. The proof is complete. $\square$

Now we consider the remaining case

B) $g(\lambda)$ is not constant on segments $[p_0, p_0 + \delta]$ or, in other words, $g(p) > g(p_0)$ $\forall p > p_0$.

To treat this case we shall mainly follow the ideas of S. Sazhenkov [15].

As was mentioned above, the sequence ν_x^k satisfies (36). So the first localization principle is satisfied. By Corollary 2 we find that $A(x)\xi \cdot \xi\mu^{p_0p_0} = 0$.

Let $\Phi_1(y), \Phi_2(y) \in C_0^\infty(\Omega)$, $\psi(z) \in C^\infty(S)$, $\overline{\psi(z)} = \psi(-z) = -\psi(z)$. Let A be the zero-order p.d.o. with symbol $\psi(\xi/|\xi|)$, $h > 0$. We apply relation (51) to the test function $f_{rh}(y) = \Phi_1(y)(J_1A[\Phi_2(Q_r)^h](y))^h \in C_0^\infty(\Omega)$. As a result, we arrive at the equality

$$\int_{\mathbb{R}^n} P_r(y)\cdot\nabla\Phi_1(y)(J_1A[\Phi_2(Q_r)^h])^h(y)dy +$$
$$\int_{\mathbb{R}^n} Q_r(y)\mathrm{div}A(y)\nabla\left(\Phi_1(y)(J_1A[\Phi_2(Q_r)^h])^h(y)\right)dy = -\langle l_r, f_{rh}\rangle, \tag{65}$$

where $l_r = \gamma(y)L_r$ and $\gamma(y) \in C_0^\infty(\Omega)$ is a function, which equals 1 in a vicinity of $\mathrm{supp}\,\Phi_1$. By our assumption $l_r \to 0$ in $W_p^{-1}(\mathbb{R}^n)$. We are going to pass to the limit in (65) firstly as $h \to 0+$ and then as $r \to \infty$. By the properties of the Riesz potential J_1 and Lemma 6 we see that

$$\Phi_1(y)(J_1A[\Phi_2(Q_r)^h](y))^h \underset{h\to 0+}{\to} \Phi_1(y)J_1A[\Phi_2Q_r](y) \text{ in } W_q^1,\ q^{-1}+p^{-1}=1.$$

Therefore,

$$\lim_{h\to 0+}\langle l_r, f_{rh}\rangle = \langle l_r, \Phi_1(y)J_1A[\Phi_2Q_r](y)\rangle; \tag{66}$$

$$\lim_{h\to 0+}\int_{\mathbb{R}^n} P_r(y)\cdot\nabla\left(\Phi_1(y)(J_1A[\Phi_2(Q_r)^h])^h(y)\right)dy =$$
$$\int_{\mathbb{R}^n} P_r(y)\cdot\nabla\left(\Phi_1(y)J_1A[\Phi_2Q_r](y)\right)dy =$$
$$\int_{\mathbb{R}^n}(P_r(y)\cdot\nabla\Phi_1(y))J_1A[\Phi_2Q_r](y)dy + \int_{\mathbb{R}^n}\Phi_1(y)P_{rj}(y)R_jA[\Phi_2Q_r](y)dy. \tag{67}$$

Here R_j, $j = 1,\dots,n$ are the Riesz operators and P_{rj} are the coordinates of P_r. In (67) and everywhere below we use the conventional rule of summation over repeated indices i,j,k,l in products. The passage to the limit in the second integral in (65) is more delicate and given in the following proposition.

Proposition 6. *In the limit as $h \to 0+$*

$$\int_{\mathbb{R}^n} Q_r(y)\mathrm{div}A(y)\nabla\left(\Phi_1(y)(J_1A[\Phi_2(Q_r)^h])^h(y)\right)dy \to$$
$$\int_{\mathbb{R}^n} Q_r(y)\{(\mathrm{div}A(y)\nabla\Phi_1(y))J_1A[\Phi_2Q_r](y) + a_{jk}(y)(\Phi_1(y))_{y_k}R_jA[\Phi_2Q_r](y)\}dy +$$
$$\frac{1}{2}\left\{\int_{\mathbb{R}^n}\gamma(y)Q_r(y)(D_{jk}+E_{jk})[a_{jk}\Phi_1Q_r](y)dy - \int_{\mathbb{R}^n}\gamma(y)Q_r(y)(C+E)[\Phi_2Q_r](y)dy +\right.$$
$$\left.\int_{\mathbb{R}^n} Q_r(y)(\Phi_2)_{y_j}(y)R_kA[a_{jk}\Phi_1Q_r](y)dy + \int_{\mathbb{R}^n} R_kA[\Phi_2Q_r](y)(a_{jk}(y)\Phi_1(y))_{y_j}Q_r(y)dy\right\}, \tag{68}$$

where C, D_{jk} are admissible zero-order p.d.o. with symbols $c(y,\xi)$, $d_{jk}(y,\xi)$ given by the expressions

$$c(y,\xi) = iz_j(a_{jk}(y)\Phi_1(y))_{y_l}\left(\psi(z)(\delta_{kl}-z_kz_l)+z_k\psi_{\xi_l}(z)\right),$$
$$d_{jk}(y,\xi) = iz_j(\Phi_2)_{y_l}(y)\left(\psi(z)(\delta_{kl}-z_kz_l)+z_k\psi_{\xi_l}(z)\right), \quad z=\xi/|\xi|\in S,$$

E, E_{jk} are compact operators on L^2, and $\gamma(y) \in C_0^\infty(\Omega)$ is an arbitrary function, which equals 1 *in a vicinity of* $\mathrm{supp}\,\Phi_1 \cup \mathrm{supp}\,\Phi_2$.

Proof. We have, with account of Lemma 8 and relation (35),

$$I(h) = \int_{\mathbb{R}^n} Q_r(y)\mathrm{div}A(y)\nabla\left(\Phi_1(y)(J_1A[\Phi_2(Q_r)^h])^h(y)\right)dy =$$
$$\int_{\mathbb{R}^n} Q_r(y)\{\mathrm{div}A(y)\nabla\Phi_1(y)\left(J_1A[\Phi_2(Q_r)^h]\right)^h(y) +$$
$$\mathrm{div}A(y)\Phi_1(y)\left(\nabla J_1A[\Phi_2(Q_r)^h]\right)^h(y)\}dy =$$
$$\int_{\mathbb{R}^n}(Q_r)^h(y)\{\mathrm{div}A(y)\nabla\Phi_1(y)J_1A[\Phi_2(Q_r)^h](y) +$$
$$\mathrm{div}A(y)\Phi_1(y)\nabla J_1A[\Phi_2(Q_r)^h](y)\}dy + \varepsilon_h,$$

where $\varepsilon_h \to 0$ as $h \to 0+$. Hence, up to a term vanishing as $h \to 0+$,

$$I(h) = \int_{\mathbb{R}^n} (Q_r)^h(y)\{(\mathrm{div} A(y)\nabla \Phi_1(y))J_1 A[\Phi_2 (Q_r)^h](y) + \\ a_{jk}(y)(\Phi_1(y))_{y_k} R_j A[\Phi_2(Q_r)^h](y)\}dy + \\ \int_{\mathbb{R}^n} (Q_r)^h(y)\partial_{y_j}\{a_{jk}(y)\Phi_1(y)R_k A[(\Phi_2(Q_r)^h)](y)\}dy. \tag{69}$$

The passage to the limit as $h \to 0+$ in the first integral on the right-hand side of (69) is plain since $(Q_r)^h \to Q_r$ in L^2_{loc}. The corresponding limit expression has the form

$$I = \int_{\mathbb{R}^n} Q_r(y)\{(\mathrm{div} A(y)\nabla \Phi_1(y))J_1 A[\Phi_2 Q_r](y) + \\ a_{jk}(y)(\Phi_1(y))_{y_k} R_j A[\Phi_2 Q_r](y)\}dy. \tag{70}$$

Concerning the second integral in (69), we transform it with the help of Lemma 7

$$I_1(h) = \int_{\mathbb{R}^n} (Q_r)^h(y)\partial_{y_j}\{a_{jk}(y)\Phi_1(y)R_k A[\Phi_2(Q_r)^h](y)\}dy = \\ \int_{\mathbb{R}^n} \gamma(y)(Q_r)^h(y)\partial_{y_j}\{R_k A[\Phi_2 a_{jk}\Phi_1(Q_r)^h](y)\}dy - I_2(h), \tag{71}$$

where $\gamma(y) \in C_0^\infty(\Omega)$ equals 1 in a vicinity of $\mathrm{supp}\,\Phi_1 \cup \mathrm{supp}\,\Phi_2$,

$$I_2(h) = \int_{\mathbb{R}^n} \gamma(y)(Q_r)^h(y)(C+E)[\Phi_2(Q_r)^h](y)dy,$$

C is a zero order p.d.o. with symbol $c(y,\xi)$,

$$c(y,\xi) = iz_j \frac{\partial a_{jk}(y)\Phi_1(y)}{\partial y_l}\frac{\partial z_k \psi(z)}{\partial \xi_l} = \\ iz_j(a_{jk}(y)\Phi_1(y))_{y_l}\left(\psi(z)(\delta_{kl} - z_k z_l) + z_k\psi_{\xi_l}(z)\right), \quad z = \xi/|\xi| \tag{72}$$

and E is a compact operator in L^2. By Lemma 7 again

$$\int_{\mathbb{R}^n} \gamma(y)(Q_r)^h(y)\partial_{y_j}\{R_k A[\Phi_2 a_{jk}\Phi_1(Q_r)^h](y)\}dy = \\ \int_{\mathbb{R}^n} (Q_r)^h(y)\partial_{y_j}\{\Phi_2(y)R_k A[a_{jk}\Phi_1(Q_r)^h](y)\}dy + I_3(h), \tag{73}$$

where

$$I_3(h) = \int_{\mathbb{R}^n} \gamma(y)(Q_r)^h(y)(D_{jk}+E_{jk})[a_{jk}\Phi_1(Q_r)^h](y)dy,$$

and D_{jk} is a zero order p.d.o. with symbol $d_{jk}(y,\xi)$,

$$d_{jk}(y,\xi) = iz_j(\Phi_2)_{y_l}(y)\left(\psi(z)(\delta_{kl} - z_k z_l) + z_k\psi_{\xi_l}(z)\right), \quad z = \xi/|\xi|, \tag{74}$$

E_{jk} is a compact operator in L^2. Now, we continue our transforms.

$$\int_{\mathbb{R}^n} (Q_r)^h(y)\partial_{y_j}\{\Phi_2(y)R_k A[a_{jk}\Phi_1(Q_r)^h](y)\}dy = \\ I_4(h) + \int_{\mathbb{R}^n} \Phi_2(y)(Q_r)^h(y)\partial_{y_j}\{R_k A[a_{jk}\Phi_1(Q_r)^h](y)\}dy, \tag{75}$$

$$I_4(h) = \int_{\mathbb{R}^n} (Q_r)^h(y)(\Phi_2)_{y_j}(y)R_k A[a_{jk}\Phi_1(Q_r)^h](y)dy.$$

Taking into account that $\partial_{y_j} R_k A = R_k A \partial_{y_j}$ and the operators $R_k A$ are self-adjoint, we obtain that

$$\int_{\mathbb{R}^n} \Phi_2(y)(Q_r)^h(y)\partial_{y_j}\{R_k A[a_{jk}\Phi_1(Q_r)^h](y)\}dy =$$
$$\int_{\mathbb{R}^n} R_k A[\Phi_2(Q_r)^h](y)\partial_{y_j}\{a_{jk}(y)\Phi_1(y)(Q_r)^h(y)\}dy =$$
$$\int_{\mathbb{R}^n} R_k A[\Phi_2(Q_r)^h](y)(a_{jk}(y)\Phi_1(y))_{y_j}(Q_r)^h(y)dy +$$
$$\int_{\mathbb{R}^n} a_{jk}(y)\Phi_1(y)R_k A[\Phi_2(Q_r)^h](y)((Q_r)^h)_{y_j}(y)dy =$$
$$\int_{\mathbb{R}^n} R_k A[\Phi_2(Q_r)^h](y)(a_{jk}(y)\Phi_1(y))_{y_j}(Q_r)^h(y)dy -$$
$$\int_{\mathbb{R}^n} \partial_{y_j}\{a_{jk}(y)\Phi_1(y)R_k A[\Phi_2(Q_r)^h](y)\}(Q_r)^h(y)dy = I_5(h) - I_1(h), \quad (76)$$

where

$$I_5(h) = \int_{\mathbb{R}^n} R_k A[\Phi_2(Q_r)^h](y)(a_{jk}(y)\Phi_1(y))_{y_j}(Q_r)^h(y)dy.$$

Hence, from (71), (73), (75), (76) it follows that

$$I_1(h) = (I_3(h) - I_2(h) + I_4(h) + I_5(h))/2.$$

Since expressions in the integrals $I_k(h)$, $k = 2, \dots, 5$ contain only bounded operators and $(Q_r)^h \to Q_r$ in L^2_{loc} as $h \to 0$ then in the limit as $h \to 0+$ these integrals converges respectively to

$$I_2 = \int_{\mathbb{R}^n} \gamma(y)Q_r(y)(C+E)[\Phi_2 Q_r](y)\}dy,$$
$$I_3 = \int_{\mathbb{R}^n} \gamma(y)Q_r(y)(D_{jk}+E_{jk})[a_{jk}\Phi_1 Q_r](y)dy,$$
$$I_4 = \int_{\mathbb{R}^n} Q_r(y)(\Phi_2)_{y_j}(y)R_k A[a_{jk}\Phi_1 Q_r](y)dy,$$
$$I_5 = \int_{\mathbb{R}^n} R_k A[\Phi_2 Q_r](y)(a_{jk}(y)\Phi_1(y))_{y_j} Q_r(y)dy.$$

Taking into account also (69), (70), we conclude that

$$\lim_{h\to 0+} I(h) = I + (I_3 - I_2 + I_4 + I_5)/2,$$

as was to be proved. □

Now we are ready to prove Theorem 5 in the case B).

Theorem 7. *Assume that condition B) is satisfied. Then there exists $\delta > 0$ such that $\xi \cdot \varphi(x,\lambda) = \xi \cdot \varphi(x,p_0)$ on $[p_0, p_0+\delta]$ for all $\xi \in L$.*

Proof. In view of (66), (67), (68), from (65) it follows that

$$-\langle l_r, \Phi_1(y)J_1 A[\Phi_2 Q_r](y)\rangle =$$
$$\int_{\mathbb{R}^n} (P_r(y)\cdot\nabla\Phi_1(y))J_1 A[\Phi_2 Q_r](y)dy + \int_{\mathbb{R}^n} \Phi_1(y)P_{rj}(y)R_j A[\Phi_2 Q_r](y)dy +$$
$$\int_{\mathbb{R}^n} Q_r(y)\{(\operatorname{div} A(y)\nabla\Phi_1(y))J_1 A[\Phi_2 Q_r](y) + a_{jk}(y)(\Phi_1(y))_{y_k} R_j A[\Phi_2 Q_r](y)\}dy +$$
$$\frac{1}{2}\left\{\int_{\mathbb{R}^n} \gamma(y)Q_r(y)(D_{jk}+E_{jk})[a_{jk}\Phi_1 Q_r](y)dy - \int_{\mathbb{R}^n} \gamma(y)Q_r(y)(C+E)[\Phi_2 Q_r](y)dy + \right.$$
$$\left.\int_{\mathbb{R}^n} Q_r(y)(\Phi_2)_{y_j}(y)R_k A[a_{jk}\Phi_1 Q_r](y)dy + \int_{\mathbb{R}^n} R_k A[\Phi_2 Q_r](y)(a_{jk}(y)\Phi_1(y))_{y_j} Q_r(y)dy\right\}. \quad (77)$$

Now we pass to the limit in (77) as $r\to\infty$. Firstly, observe that in view of Lemma 6 the sequence $\Phi_1(y)J_1A[\Phi_2Q_r](y)$ is bounded in W_q^1 while $l_r\to 0$ as $r\to\infty$ in W_p^{-1}, which is the dual space to W_q^1. Therefore,

$$\lim_{r\to\infty}\langle l_r,\Phi_1(y)J_1A[\Phi_2Q_r](y)\rangle=0. \tag{78}$$

Since $Q\to J_1A[\Phi_2Q]$ is a compact operator, the sequence $J_1A[\Phi_2Q_r]\to 0$ as $r\to\infty$ strongly in L^2. Therefore,

$$\lim_{r\to\infty}\int_{\mathbb{R}^n}(P_r(y)\cdot\nabla\Phi_1(y))J_1A[\Phi_2Q_r](y)dy=$$
$$\lim_{r\to\infty}\int_{\mathbb{R}^n}Q_r(y)(\mathrm{div}A(y)\nabla\Phi_1(y))J_1A[\Phi_2Q_r](y)dy=0. \tag{79}$$

By compactness of operators E, E_{jk} we have also

$$\lim_{r\to\infty}\int_{\mathbb{R}^n}\gamma(y)Q_r(y)E_{jk}[a_{jk}\Phi_1Q_r](y)dy=\lim_{r\to\infty}\int_{\mathbb{R}^n}\gamma(y)Q_r(y)E[\Phi_2Q_r](y)dy=0. \tag{80}$$

Extracting a subsequence if necessary we can suppose that the Tartar H-measure $\bar\mu=\{\bar\mu^{ij}\}_{i,j=0}^n$ corresponding to the sequence $(Q_r,P_r)\in\mathbb{R}^{n+1}$ is well-defined (here the zero index correspond to the component Q_r). By relation (9) we find

$$\lim_{r\to\infty}\left\{\int_{\mathbb{R}^n}\Phi_1(y)P_{rj}(y)R_jA[\Phi_2Q_r](y)dy+\int_{\mathbb{R}^n}Q_r(y)a_{jk}(y)(\Phi_1(y))_{y_k}R_jA[\Phi_2Q_r](y)dy+\right.$$
$$\frac{1}{2}\int_{\mathbb{R}^n}\gamma(y)Q_r(y)D_{jk}[a_{jk}\Phi_1Q_r](y)dy-\frac{1}{2}\int_{\mathbb{R}^n}\gamma(y)Q_r(y)C[\Phi_2Q_r](y)dy+$$
$$\frac{1}{2}\int_{\mathbb{R}^n}Q_r(y)(\Phi_2)_{y_j}(y)R_kA[a_{jk}\Phi_1Q_r](y)dy+$$
$$\left.\frac{1}{2}\int_{\mathbb{R}^n}R_kA[\Phi_2Q_r](y)(a_{jk}(y)\Phi_1(y))_{y_j}Q_r(y)dy\right\}=$$
$$\langle\bar\mu^{j0}(y,\xi),i\Phi_1(y)\Phi_2(y)\xi_j\psi(\xi)\rangle+\langle\bar\mu^{00}(y,\xi),H(y,\xi)\rangle, \tag{81}$$

where

$$H(y,\xi)=\Phi_2(y)a_{jk}(y)(\Phi_1)_{y_k}(y)i\xi_j\psi(\xi)+\frac{1}{2}\Phi_1(y)a_{jk}(y)d_{jk}(y,\xi)-$$
$$\frac{1}{2}\Phi_2(y)c(y,\xi)+\frac{1}{2}\Phi_1(y)a_{jk}(y)(\Phi_2)_{y_j}(y)i\xi_k\psi(\xi)+\frac{1}{2}\Phi_2(y)(a_{jk}(y)\Phi_1(y))_{y_j}i\xi_k\psi(\xi). \tag{82}$$

In view of (78)-(81) we derive from (77) that

$$\langle\bar\mu^{j0}(y,\xi),i\Phi_1(y)\Phi_2(y)\xi_j\psi(\xi)\rangle+\langle\bar\mu^{00}(y,\xi),H(y,\xi)\rangle=0 \tag{83}$$

for all real $\Phi_1(y),\Phi_2(y)\in C_0^1(\Omega)$, and all odd $\psi(\xi)=\psi(z)\in C(S)$, $z=\xi/|\xi|$ such that $i\psi(z)\in\mathbb{R}$. Here $H(y,\xi)$ depends on these test functions. Putting in (82) expressions for the symbols $c(y,\xi)$, $d_{jk}(y,\xi)$, we find after simple transforms that

$$H(y,\xi)=\Phi_2(y)a_{jk}(y)(\Phi_1)_{y_k}(y)i\xi_j\psi(\xi)+$$
$$\frac{1}{2}\Phi_1(y)a_{jk}(y)i\xi_j(\Phi_2)_{y_l}(y)\left(\psi(\xi)(\delta_{kl}-\xi_k\xi_l)+\xi_k\psi_{\xi_l}(\xi)\right)-$$
$$\frac{1}{2}\Phi_2(y)i\xi_j(a_{jk}(y)\Phi_1(y))_{y_l}\left(\psi(\xi)(\delta_{kl}-\xi_k\xi_l)+\xi_k\psi_{\xi_l}(\xi)\right)+$$
$$\frac{1}{2}\Phi_1(y)a_{jk}(y)(\Phi_2)_{y_j}(y)i\xi_k\psi(\xi)+\frac{1}{2}\Phi_2(y)(a_{jk}(y)\Phi_1(y))_{y_j}i\xi_k\psi(\xi).$$

To simplify the expression we set $\Phi_2 = \Phi \in C_0^1(\Omega)$ and suppose that $\Phi_1 = 1$ in a neighborhood of $\operatorname{supp}\Phi$. Then Φ_1 disappears in the above expression and we obtain that

$$\begin{aligned}
H(y,\xi) = \frac{1}{2}a_{jk}(y)i\xi_j\Phi_{y_l}(y)\left(\psi(\xi)(\delta_{kl}-\xi_k\xi_l)+\xi_k\psi_{\xi_l}(\xi)\right) - \\
\frac{1}{2}\Phi(y)i\xi_j(a_{jk}(y))_{y_l}\left(\psi(\xi)(\delta_{kl}-\xi_k\xi_l)+\xi_k\psi_{\xi_l}(\xi)\right) + \\
\frac{1}{2}a_{jk}(y)\Phi_{y_j}(y)i\xi_k\psi(\xi) + \frac{1}{2}\Phi(y)(a_{jk}(y))_{y_j}i\xi_k\psi(\xi) = \\
\frac{i}{2}\left(a_{jk}(y)\xi_j\Phi_{y_k}(y)+a_{jk}(y)\Phi_{y_j}(y)\xi_k\right)\psi(\xi) + \frac{i}{2}\left((a_{jk}(y))_{y_j}\xi_k-(a_{jk}(y))_{y_k}\xi_j\right)\Phi(y)\psi(\xi) + \\
\frac{i}{2}a_{jk}(y)\xi_j\xi_k(\psi_{\xi_l}(\xi)-\xi_l\psi(\xi))\Phi_{y_l}(y) - \frac{i}{2}(a_{jk}(y))_{y_l}\xi_j\xi_k(\psi_{\xi_l}(\xi)-\xi_l\psi(\xi))\Phi(y) = \\
ia_{jk}(y)\Phi_{y_j}(y)\xi_k\psi(\xi) + \frac{i}{2}a_{jk}(y)\xi_j\xi_k(\psi_{\xi_l}(\xi)-\xi_l\psi(\xi))\Phi_{y_l}(y) - \\
\frac{i}{2}(a_{jk}(y))_{y_l}\xi_j\xi_k(\psi_{\xi_l}(\xi)-\xi_l\psi(\xi))\Phi(y). \qquad (84)
\end{aligned}$$

Let us demonstrate that $\langle\bar{\mu}^{00}(y,\xi), H(y,\xi)\rangle = 0$. Indeed, in view of (84), $H(y,\xi)$ is a sum of functions of the kinds $b_j(y,\xi)a_{jk}(y)\xi_k$ and $c_l(y,\xi)(a_{jk}(y))_{y_l}\xi_j\xi_k$ with $b_j(y,\xi), c_l(y,\xi) \in C_0(\Omega\times S)$, and by Corollaries 3, 4 we have $\langle\bar{\mu}^{00}(y,\xi), H(y,\xi)\rangle = 0$ as required. In view of (83) we obtain that

$$\sum_{j=1}^{n}\langle\bar{\mu}^{j0}(y,\xi), \Phi(y)\xi_j\psi(\xi)\rangle = 0.$$

It is clear that this equality is satisfied for each $\Phi(y) \in C_0(\Omega)$ and odd $\psi(\xi) \in C(S)$. Replacing $\Phi(y)$ by $|\Phi(y)|^2$ and using the definition of H-measure, we derive that

$$\int_{\mathbb{R}^n}|\xi|^{-1}\xi\cdot F(P_r\Phi)(\xi)\overline{F(Q_r\Phi)(\xi)}\psi\left(\frac{\xi}{|\xi|}\right)d\xi \to 0$$

as $r\to\infty$. In view of (52), (53) we can rewrite this relation as follows

$$\begin{aligned}
\lim_{r\to\infty}\Bigg\{(g(p)-g(p_0))\int_{\mathbb{R}^n}|\xi|^{-1}\xi\cdot F(U_r^{p_0}f\Phi)(\xi)\overline{F(U_r^{p_0}\Phi)(\xi)}\psi\left(\frac{\xi}{|\xi|}\right)d\xi - \\
\int_{\mathbb{R}^n}|\xi|^{-1}\xi\cdot F(U_r^{p_0}f\Phi)(\xi)\overline{F(G_r^{p}\Phi)(\xi)}\psi\left(\frac{\xi}{|\xi|}\right)d\xi - \\
(g(p)-g(p_0))\int_{\mathbb{R}^n}|\xi|^{-1}\xi\cdot F(V_r^{p}\Phi)(\xi)\overline{F(U_r^{p_0}\Phi)(\xi)}\psi\left(\frac{\xi}{|\xi|}\right)d\xi + \\
\int_{\mathbb{R}^n}|\xi|^{-1}\xi\cdot F(V_r^{p}\Phi)(\xi)\overline{F(G_r^{p}\Phi)(\xi)}\psi\left(\frac{\xi}{|\xi|}\right)d\xi\Bigg\} = 0, \qquad (85)
\end{aligned}$$

where

$$f(y) = \varphi(y,p)-\varphi(y,p_0),\ V_r^p(y) = \int(\varphi(y,p)-\varphi(y,\lambda))\chi(\lambda)d\gamma_y^r(\lambda) \text{ and}$$
$$G_r^p(y) = \int(g(p)-g(\lambda))\chi(\lambda)d\gamma_y^r(\lambda).$$

We set in (85) $\Phi(y) = \Phi_m(x-y)$, where the functions Φ_m were defined in section 3 in the proof of Proposition 3, and pass to the limit as $m\to\infty$. By Remark 2 (see equality (25)) we obtain

$$\begin{aligned}
\lim_{m\to\infty}\lim_{r\to\infty}\int_{\mathbb{R}^n}|\xi|^{-1}\xi\cdot F(U_r^{p_0}f\Phi_m)(\xi)\overline{F(U_r^{p_0}\Phi_m)(\xi)}\psi\left(\frac{\xi}{|\xi|}\right)d\xi = \\
(\varphi(x,p)-\varphi(x,p_0))\cdot\langle\mu_x^{p_0p_0}, \xi\psi(\xi)\rangle,
\end{aligned}$$

therefore

$$\begin{aligned}
&(g(p)-g(p_0))(\varphi(x,p)-\varphi(x,p_0))\cdot\langle\mu_x^{p_0p_0},\xi\psi(\xi)\rangle = \\
\lim_{m\to\infty}\lim_{r\to\infty}&\left\{\int_{\mathbb{R}^n}|\xi|^{-1}\xi\cdot F(U_r^{p_0}f\Phi_m)(\xi)\overline{F(G_r^p\Phi_m)(\xi)}\psi\left(\frac{\xi}{|\xi|}\right)d\xi+\right. \\
(g(p)-g(p_0))&\int_{\mathbb{R}^n}|\xi|^{-1}\xi\cdot F(V_r^p\Phi_m)(\xi)\overline{F(U_r^{p_0}\Phi_m)(\xi)}\psi\left(\frac{\xi}{|\xi|}\right)d\xi- \\
&\left.\int_{\mathbb{R}^n}|\xi|^{-1}\xi\cdot F(V_r^p\Phi_m)(\xi)\overline{F(G_r^p\Phi_m)(\xi)}\psi\left(\frac{\xi}{|\xi|}\right)d\xi\right\}. \qquad (86)
\end{aligned}$$

Let π_1 and π_2 be the orthogonal projections of $\mathbb{R}^n$ onto the subspaces L and $L^\perp$ respectively; let $\tilde\varphi(y,\lambda)=\pi_1(\varphi(y,\lambda))$, $\bar\varphi(y,\lambda)=\pi_2(\varphi(y,\lambda))$, $\bar f(y)=\bar\varphi(y,p)-\bar\varphi(y,p_0)$, $\tilde f(y)=\tilde\varphi(y,p)-\tilde\varphi(y,p_0)$. Recall that L is the smallest subspace containing $\operatorname{supp}\mu_x^{p_0p_0}$. Hence $(\bar\varphi(x,p)-\bar\varphi(x,p_0))\cdot\xi=0$ on $\operatorname{supp}\mu_x^{p_0p_0}$ and

$$(\varphi(x,p)-\varphi(x,p_0))\cdot\langle\mu_x^{p_0p_0},\xi\psi(\xi)\rangle=(\tilde\varphi(x,p)-\tilde\varphi(x,p_0))\cdot\langle\mu_x^{p_0p_0},\xi\psi(\xi)\rangle; \qquad (87)$$

$$\lim_{m\to\infty}\lim_{r\to\infty}\int_{\mathbb{R}^n}||\xi|^{-1}\xi\cdot F(U_r^{p_0}\bar f\Phi_m)(\xi)|^2d\xi=\langle\mu_x^{p_0p_0},((\bar\varphi(x,p)-\bar\varphi(x,p_0))\cdot\xi)^2\rangle=0. \qquad (88)$$

From (88) it follows that

$$\lim_{m\to\infty}\lim_{r\to\infty}\||\xi|^{-1}\xi\cdot F(U_r^{p_0}f\Phi_m)-|\xi|^{-1}\xi\cdot F(U_r^{p_0}\tilde f\Phi_m)\|_2=0$$

and therefore

$$\begin{aligned}
\lim_{m\to\infty}\lim_{r\to\infty}&\left\{\int_{\mathbb{R}^n}|\xi|^{-1}\xi\cdot F(U_r^{p_0}f\Phi_m)(\xi)\overline{F(G_r^p\Phi_m)(\xi)}\psi\left(\frac{\xi}{|\xi|}\right)d\xi-\right. \\
&\left.\int_{\mathbb{R}^n}|\xi|^{-1}\xi\cdot F(U_r^{p_0}\tilde f\Phi_m)(\xi)\overline{F(G_r^p\Phi_m)(\xi)}\psi\left(\frac{\xi}{|\xi|}\right)d\xi\right\}=0. \qquad (89)
\end{aligned}$$

Further, $V_r^p(y)=\pi_1(V_r^p(y))+\pi_2(V_r^p(y))$ and

$$\pi_1(V_r^p(y))=\int(\tilde\varphi(y,p)-\tilde\varphi(y,\lambda))\,\chi(\lambda)d\gamma_y^r(\lambda),$$
$$\pi_2(V_r^p(y))=\int(\bar\varphi(y,p)-\bar\varphi(y,\lambda))\,\chi(\lambda)d\gamma_y^r(\lambda).$$

In the notation of Proposition 4,

$$\pi_2(V_r^p(y))=I_r(h\cdot\chi),$$

where $h(y,\lambda)=\bar\varphi(y,p)-\bar\varphi(y,\lambda)$ is a Caratheodory vector taking its values in $L^\perp$. Now, by Proposition 4 we obtain

$$\begin{aligned}
\lim_{m\to\infty}\lim_{r\to\infty}\int_{\mathbb{R}^n}|\xi|^{-1}\xi\cdot F(\pi_2(V_r^p)\Phi_m)(\xi)\overline{F(U_r^{p_0}\Phi_m)(\xi)}\psi\left(\frac{\xi}{|\xi|}\right)d\xi= \\
\lim_{m\to\infty}\lim_{r\to\infty}\int_{\mathbb{R}^n}|\xi|^{-1}\xi\cdot F(\pi_2(V_r^p)\Phi_m)(\xi)\overline{F(G_r^p\Phi_m)(\xi)}\psi\left(\frac{\xi}{|\xi|}\right)d\xi=0. \qquad (90)
\end{aligned}$$

Let $\tilde V_r^p(y)=\pi_1(V_r^p(y))$. From (86), in view of (87), (89) and (90), we see that

$$\begin{aligned}
&(g(p)-g(p_0))(\tilde\varphi(x,p)-\tilde\varphi(x,p_0))\cdot\langle\mu_x^{p_0p_0},\xi\psi(\xi)\rangle = \\
\lim_{m\to\infty}\lim_{r\to\infty}&\left\{\int_{\mathbb{R}^n}|\xi|^{-1}\xi\cdot F(U_r^{p_0}\tilde f\Phi_m)(\xi)\overline{F(G_r^p\Phi_m)(\xi)}\psi\left(\frac{\xi}{|\xi|}\right)d\xi+\right. \\
(g(p)-g(p_0))&\int_{\mathbb{R}^n}|\xi|^{-1}\xi\cdot F(\tilde V_r^p\Phi_m)(\xi)\overline{F(U_r^{p_0}\Phi_m)(\xi)}\psi\left(\frac{\xi}{|\xi|}\right)d\xi- \\
&\left.\int_{\mathbb{R}^n}|\xi|^{-1}\xi\cdot F(\tilde V_r^p\Phi_m)(\xi)\overline{F(G_r^p\Phi_m)(\xi)}\psi\left(\frac{\xi}{|\xi|}\right)d\xi\right\},
\end{aligned}$$

which in turn, by Bunyakovskii inequality and Plancherel's equality, gives us the estimate

$$(g(p)-g(p_0))\,|(\tilde\varphi(x,p)-\tilde\varphi(x,p_0))\cdot\langle\mu_x^{p_0p_0},\xi\psi(\xi)\rangle| \le \\ \varlimsup_{m\to\infty}\varlimsup_{r\to\infty}[\|U_r^{p_0}\tilde f\Phi_m\|_2\cdot\|G_r^p\Phi_m\|_2+\|\tilde V_r^p\Phi_m\|_2\times \\ ((g(p)-g(p_0))\|U_r^{p_0}\Phi_m\|_2+\|G_r^p\Phi_m\|_2)]\cdot\|\psi\|_\infty. \quad (91)$$

Next, for $M_p(y)=\max\limits_{\lambda\in[p_0,p]}|\tilde\varphi(y,p)-\tilde\varphi(y,\lambda)|$

$$|\tilde V_r^p(y)|\le M_p(y)\int\chi(\lambda)d\left(\nu_y^r(\lambda)+\nu_y^0(\lambda)\right)= \\ M_p(y)(u_r(y,p_0)-u_r(y,p)+u_0(y,p_0)-u_0(y,p))$$

so that, in view of Lemma 3

$$\varlimsup_{m\to\infty}\varlimsup_{r\to\infty}\|\tilde V_r^p\Phi_m\|_2\le 2M_p(x)\omega(p), \quad (92)$$

where $\omega(p)=(u_0(x,p_0)-u_0(x,p))^{1/2}$. Here we bear in mind that x is a Lebesgue point of the function $(M_p(y))^2$ (the latter easily follows from the fact that $x\in\Omega_\varphi$ is a Lebesgue point of the maps $y\to\varphi(y,\cdot)$, $y\to|\varphi(y,\cdot)|^2$ into the spaces $C(\mathbb{R},\mathbb{R}^n)$, $C(\mathbb{R})$, respectively).

Similarly, using the inequality $|g(p)-g(\lambda)|\le g(p)-g(p_0)$ for $\lambda\in[p_0,p]$, we derive the estimate

$$\varlimsup_{m\to\infty}\varlimsup_{r\to\infty}\|G_r^p\Phi_m\|_2\le 2(g(p)-g(p_0))\omega(p). \quad (93)$$

Further, we have $|U_r^{p_0}|\le 1$ and therefore

$$\|U_r^{p_0}\tilde f\Phi_m\|_2\le\|\tilde f\Phi_m\|_2=\left(\int(\tilde f(y))^2K_m(x-y)dy\right)^{1/2}\underset{m\to\infty}{\to}|\tilde f(x)|=|\tilde\varphi(x,p)-\tilde\varphi(x,p_0)|,$$

so that

$$\varlimsup_{m\to\infty}\varlimsup_{r\to\infty}\|U_r^{p_0}\tilde f\Phi_m\|_2\le|\tilde\varphi(x,p)-\tilde\varphi(x,p_0)|\le M_p(x). \quad (94)$$

Taking into account (92), (93), (94) and the obvious estimate $\|U_r^{p_0}\Phi_m\|_2\le 1$, we derive from (91) that for some constant C

$$(g(p)-g(p_0))|(\tilde\varphi(x,p)-\tilde\varphi(x,p_0))\cdot\langle\mu_x^{p_0p_0},\xi\psi(\xi)\rangle|\le C(g(p)-g(p_0))M_p(x)\omega(p)\|\psi\|_\infty,$$

By the condition B) $g(p)-g(p_0)>0$ and the above estimate implies that

$$|(\tilde\varphi(x,p)-\tilde\varphi(x,p_0))\cdot\langle\mu_x^{p_0p_0},\xi\psi(\xi)\rangle|\le CM_p(x)\omega(p)\|\psi\|_\infty. \quad (95)$$

Now observe that the measure $\mu_x^{p_0p_0}$ is even, i.e. it is invariant under the map $\xi\to-\xi$, see Remark 2. Therefore, $\langle\mu_x^{p_0p_0},a(\xi)\rangle=0$ for odd functions $a(\xi)$. Any continuous function $\psi(\xi)=\psi_e(\xi)+\psi_o(\xi)$, where $\psi_e(\xi)=(\psi(\xi)+\psi(-\xi))/2$, $\psi_o(\xi)=(\psi(\xi)-\psi(-\xi))/2$ are even and odd functions, respectively. By (95) we obtain that for each $\psi(\xi)\in C(S)$

$$|(\tilde\varphi(x,p)-\tilde\varphi(x,p_0))\cdot\langle\mu_x^{p_0p_0},\xi\psi(\xi)\rangle|=|(\tilde\varphi(x,p)-\tilde\varphi(x,p_0))\cdot\langle\mu_x^{p_0p_0},\xi\psi_o(\xi)\rangle|\le \\ CM_p(x)\omega(p)\|\psi_o\|_\infty=CM_p(x)\omega(p)\|\psi\|_\infty.$$

This relation coincides with (62), and it remains only to repeat the corresponding part in the proof of Theorem 6 to conclude that $\varphi(x,\lambda)\cdot\xi=\varphi(x,p_0)\cdot\xi=\mathrm{const}$ on the segment $\lambda\in[p_0,p_0+\delta]$ for all $\xi\in L$. The proof is complete. □

From statements of Theorem 3, 5, 6 it readily follows the assertion of Theorem 4. Under the non-degeneracy condition indicated in Definition 2 Theorem 4 yields the following result.

Theorem 8. *Suppose that the non-degeneracy condition is satisfied. Then any sequence ν_x^k weakly converging as $k\to\infty$ to ν_x^0 and satisfying (50) converges to ν_x^0 strongly.*

Proof. Let $\nu_x^r = \nu_x^k$, $k = k_r$, be a subsequence such that the H-measure $\{\mu^{pq}\}_{p,q\in E}$ is well defined. As directly follows from the assertion of Theorem 4 and non-degeneracy condition in Definition 2, $\mu_x^{pp} = 0$ for a.e. $x \in \Omega$ and $p \in D$. Indeed, in the notations of this Theorem $L = L(p_0) = \{0\}$ and since dimension of L is maximal we find that $L(p) = \{0\}$ for all $p \in D$. This means that $\mu_x^{pp} = 0$ for all $x \in \Omega''$, $p \in D$. Therefore, $\mu^{pp} = \mu_x^{pp} dx \equiv 0$ for $p \in D$. By Lemma 2,3) we see that $\mu^{pq} = 0$ for $p, q \in D$ and since D is dense and μ^{pq} is continuous in p, q (see Proposition 2) it follows that $\mu^{pq} \equiv 0$ for all $p, q \in E$. This implies that

$$u_r(x,p) \to u_0(x,p) \ \text{ in } \ L^2_{loc}(\Omega)$$

as $r \to \infty$. Indeed, it follows from the definition of an H-measure and Plancherel's equality that

$$\lim_{r\to\infty} \|U_r^p \Phi\|_2^2 = \langle \mu^{pp}, |\Phi(x)|^2 \rangle = 0$$

for all $\Phi(x) \in C_0(\Omega)$ and $p \in E$. Thus, for $p \in E$ we have

$$\int \theta(\lambda - p) d\nu_x^r(\lambda) \underset{r\to\infty}{\to} \int \theta(\lambda - p) d\nu_x^0(\lambda) \ \text{ in } L^2_{loc}(\Omega). \tag{96}$$

Any continuous function can be uniformly approximated on any compact subset by finite linear combinations of functions $\lambda \to \theta(\lambda - p)$, $p \in E$. Hence, it follows from (96) that for all $f(\lambda) \in C(\mathbb{R})$ we have

$$\int f(\lambda) d\nu_x^r(\lambda) \underset{r\to\infty}{\to} \int f(\lambda) d\nu_x^0(\lambda) \ \text{ in } L^2_{loc}(\Omega),$$

and therefore also in $L^1_{loc}(\Omega)$, that is, the subsequence ν_x^r converges to ν_x^0 strongly. Finally, for each admissible choice of the subsequence ν_x^r the limit measure-valued function is uniquely defined, therefore the original sequence ν_x^k is also strongly convergent to ν_x^0. The proof is now complete. $\square$

Taking account of Theorem 3 one can also give another formulation of Theorem 5: each bounded sequence of measure-valued functions satisfying (50) is pre-compact in the sense of strong convergence. Observe that in the regular case $\nu_x^k(\lambda) = \delta(\lambda - u_k(x))$ condition (50) has the form: $\forall a, b \in \mathbb{R}$, $a < b$

$$\begin{aligned} &\operatorname{div}_x\{\varphi(x, s_{a,b}(u_k(x))) - A(x)\nabla g(s_{a,b}(u_k(x)))\} \\ &\qquad\qquad\qquad \text{is pre-compact in } W^{-1}_{p,loc}(\Omega). \end{aligned} \tag{97}$$

In this case Theorem 8 yields the following

Corollary 5. *Under the non-degeneracy condition, each bounded sequence $u_k(x) \in L^\infty(\Omega)$ satisfying (97) contains a subsequence convergent in $L^1_{loc}(\Omega)$.*

Proof. It only need to note that if the sequence $u_k(x)$ converges to a measure-valued function ν_x^0 strongly in $MV(\Omega)$, then by the definition of strong convergence

$$u_k(x) \underset{k\to\infty}{\to} u_0(x) = \int \lambda d\nu_x^0(\lambda) \ \text{ in } L^1_{loc}(\Omega)$$

(which also shows that $\nu_x^0(\lambda) = \delta(\lambda - u_0(x))$ is regular in Ω). $\square$

The statements of Theorems 4 and 8 remains true also for sequences of unbounded measure-valued (or usual) functions. For the proof we should apply cut-off functions $s_{a,b}(u) = \max(a, \min(u, b))$, $a, b \in \mathbb{R}$ and derive that bounded sequences of measure-valued functions $s_{a,b}^* \nu_x^k$ (this is the image of ν_x^k under the map $s_{a,b}$) satisfy (50). Then, under the non-degeneracy condition, we obtain strong pre-compactness property for these sequences.

For instance, consider the sequence $u_k(x)$, $k \in \mathbb{N}$ of measurable functions on Ω. Suppose that condition (97) and the non-degeneracy condition hold. Let $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$, $v_k = s_{\alpha,\beta}(u_k) = \max(\alpha, \min(u_k, \beta))$. Then $v_k = v_k(x)$ is a bounded sequence in $L^\infty(\Omega)$ and for each $a, b \in \mathbb{R}$, $a < b$

$$\begin{aligned} &\operatorname{div}_x\{\varphi(x, s_{a,b}(v_k(x))) - A(x)\nabla g(s_{a,b}(v_k(x)))\} = \\ &\operatorname{div}_x\{\varphi(x, s_{a',b'}(u_k(x))) - A(x)\nabla g(s_{a',b'}(u_k(x)))\} \end{aligned}$$

where $a' = s_{a,b}(\alpha)$, $b' = s_{a,b}(\beta)$. From this identity and (97) it follows that the sequence $\operatorname{div}_x\{\varphi(x, s_{a,b}(v_k(x))) - A(x)\nabla g(s_{a,b}(v_k(x)))\}$ is pre-compact in $W^{-1}_{p,loc}(\Omega)$. By Corollary 5 the sequences $v_k(x) = s_{\alpha,\beta}(u_k)$ are pre-compact in $L^1_{loc}(\Omega)$ for every $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$. Using the standard diagonal extraction we can choose a subsequence $u_r(x) = u_{k_r}(x)$ such that for each $m \in \mathbb{N}$ the sequence $s_{-m,m}(u_r)$ converges as $r \to \infty$ to some function $w_m(x)$ in $L^1_{loc}(\Omega)$. Obviously, a.e. in Ω

$$|w_m(x)| \le m, \quad \text{and} \quad w_m(x) = s_{-m,m}(w_l(x)) \ \forall l > m.$$

This allows to define a unique (up to equality a.e.) measurable function $u(x) \in \mathbb{R} \cup \{\pm\infty\}$ such that $w_m(x) = s_{-m,m}(u(x))$ a.e. on Ω. If $a, b \in \mathbb{R}$, $a < b$ then for $m > \max(|a|, |b|)$

$$s_{a,b}(u_r) = s_{a,b}(s_{-m,m}(u_r)) \underset{r\to\infty}{\to} s_{a,b}(w_m) = s_{a,b}(s_{-m,m}(u)) = s_{a,b}(u) \quad \text{in } L^1_{loc}(\Omega).$$

In fact, we proved the following general statement.

Theorem 9. *Suppose that the sequence of measurable functions $u_k(x)$ satisfies (97) and the nondegeneracy condition holds. Then*

a) there exists a measurable function $u(x) \in \mathbb{R} \cup \{\pm\infty\}$ such that, after extraction of a subsequence u_r, $r \in \mathbb{N}$, $s_{a,b}(u_r) \to s_{a,b}(u)$ as $r \to \infty$ in $L^1_{loc}(\Omega)$ $\forall a, b \in \mathbb{R}$, $a < b$.

b) If in addition the following estimates are satisfied

$$\int_K \rho(u_k(x))dx \le C_K, \tag{98}$$

for each compact set $K \subset \Omega$, where $\rho(u)$ is a positive Borel function, such that $\rho(u)/u \underset{u\to\infty}{\to} \infty$, then $u(x) \in L^1_{loc}(\Omega)$ and $u_r \to u$ in $L^1_{loc}(\Omega)$ as $r \to \infty$.

Proof. We only need to prove b). Observe that, extracting a subsequence, if necessary, we can assume that $s_{-m,m}(u_r) \to s_{-m,m}(u)$ as $m \to \infty$ a.e. in Ω for every $m \in \mathbb{N}$. This implies that $u_r \to u$ a.e. in Ω and by Fatou lemma from (98) it follows that

$$\int_K \rho(u(x))dx \le C_K.$$

In particular, $u(x) \in L^1_{loc}(\Omega)$. Now, fix a compact $K \subset \Omega$ and $\varepsilon > 0$. By the assumption $\rho(u)/u \underset{u\to\infty}{\to} \infty$ we can choose $m \in \mathbb{N}$ such that $|u|/\rho(u) \le \varepsilon/(2C_K)$ for $|u| > m$. Then

$$\begin{aligned}\int_K |u_r(x) - u(x)|dx &\le \int_K |s_{-m,m}(u_r(x)) - s_{-m,m}(u(x))|dx + \\ &\int_K |u_r(x)|\theta(|u_r(x)| - m)dx + \int_K |u(x)|\theta(|u(x)| - m)dx \\ &\le \int_K |s_{-m,m}(u_r(x)) - s_{-m,m}(u(x))|dx + \\ &\frac{\varepsilon}{2C_K}\left(\int_K \rho(u_r(x))dx + \int_K \rho(u(x))dx\right) \le \\ &\int_K |s_{-m,m}(u_r(x)) - s_{-m,m}(u(x))|dx + \varepsilon.\end{aligned}$$

This implies that $\overline{\lim_{r\to\infty}} \int_K |u_r(x) - u(x)|dx \le \varepsilon$ and since $\varepsilon > 0$ is arbitrary we conclude that $\lim_{r\to\infty} \int_K |u_r(x) - u(x)|dx = 0$ for any compact $K \subset \Omega$, i.e. $u_r \to u$ in $L^1_{loc}(\Omega)$. The proof is complete. □

5. Proofs of Theorems 1,2

We need the following simple

Lemma 9. *Suppose $u = u(x)$ is an entropy solution of (1). Then for all $a, b \in \mathbb{R}$, $a < b$*

$$\operatorname{div}_x\{\varphi(x, s_{a,b}(u)) - A(x)\nabla g(s_{a,b}(u))\} = \zeta_{a,b} \quad \text{in } \mathcal{D}'(\Omega), \tag{99}$$

where $\zeta_{a,b} \in \mathrm{M}_{loc}(\Omega)$. Moreover, for each compact set $K \subset \Omega$ we have $\operatorname{Var}\zeta_{a,b}(K) \le C(K, a, b, I)$, where $I = I(x) = |\varphi(x, u(x))| + |\psi(x, u(x))| + |g(u(x))| \in L^1_{loc}(\Omega)$ and the map $I \to C(K, a, b, I)$ is bounded on bounded sets in $L^1_{loc}(\Omega)$.

Proof. By known representation property for non-negative distributions we derive from (5) that

$$\operatorname{div}_x [\operatorname{sign}(u(x) - p)(\varphi(x, u(x)) - \varphi(x, p)) - A(x)\nabla|g(u(x)) - g(p)|] + \\ \operatorname{sign}(u(x) - p)[\omega_p(x) + \psi(x, u(x))] - |\gamma_p^s| = -\kappa_p \quad \text{in } \mathcal{D}'(\Omega),$$

where $\kappa_p \in \mathrm{M}_{loc}(\Omega)$, $\kappa_p \ge 0$. Further, for a compact set $K \subset \Omega$ we choose a non-negative function $f_K(x) \in C_0^\infty(\Omega)$, which equals 1 on K. Then we have the estimate

$$\kappa_p(K) \le \int f_K(x) d\kappa_p(x) = \\ \int_\Omega [\operatorname{sign}(u(x) - p)(\varphi(x, u(x)) - \varphi(x, p)) \cdot \nabla f_K(x) + |g(u(x)) - g(p)| \operatorname{div}(A(x)\nabla f_K(x)) - \\ \operatorname{sign}(u(x) - p)(\omega_p(x) + \psi(x, u(x))) f_K(x)]\, dx + \int_\Omega f_K(x) d|\gamma_p^s|(x) \le \\ A(K, p, I) = \int_\Omega \big[I(x) \max(|f_K(x)|, |\nabla f_K(x)|, |\operatorname{div} A(x)\nabla f_K(x)|) + |\varphi(x, p)| \cdot |\nabla f_K(x)| + \\ |g(p)| \cdot |\operatorname{div} A(x)\nabla f_K(x)| + |\omega_p(x)| f_K(x)\big]\, dx + \int_\Omega f_K(x) d|\gamma_p^s|(x).$$

Hence,

$$\operatorname{div}_x [\operatorname{sign}(u(x) - p)(\varphi(x, u(x)) - \varphi(x, p)) - A(x)\nabla|g(u(x)) - g(p)|] = \zeta_p, \tag{100}$$

where

$$\zeta_p = |\gamma_p^s| - \kappa_p - \operatorname{sign}(u(x) - p)[\omega_p(x) + \psi(x, u(x))] \in \mathrm{M}_{loc}(\Pi).$$

In particular, taking into account the equality $|\gamma_p^s| + |\omega_p(x)|dx = |\gamma_p|$ we obtain the estimates for measures ζ_p: $|\zeta_p| \le \kappa_p + |\gamma_p| + |\psi(x, u(x))|dx$.

Further, notice that

$$\varphi(x, s_{a,b}(u)) = (\varphi(x, a) + \varphi(x, b))/2 + \\ \big(\operatorname{sign}(u - a)(\varphi(x, u) - \varphi(x, a)) - \operatorname{sign}(u - b)(\varphi(x, u) - \varphi(x, b))\big)/2; \\ g(s_{a,b}(u)) = (g(a) + g(b))/2 + \big(|g(u) - g(a)| - |g(u) - g(b)|\big)/2,$$

and it follows from (100) that relation (99) holds with $\zeta_{a,b} = (\zeta_a - \zeta_b + \gamma_a + \gamma_b)/2$. Moreover, we have

$$\operatorname{Var}\zeta_{a,b}(K) \le C(K, a, b, I) = (A(K, a, I) + A(K, b, I))/2 + \\ |\gamma_a|(K) + |\gamma_b|(K) + \int_K |\psi(x, u(x))| dx.$$

To complete the proof, it remains to note that for fixed K, a, b the constant $C(K, a, b, I)$ is bounded on bounded sets of $I(x) \in L^1_{loc}(\Omega)$. $\square$

5.1. Proof of Theorem 1

Taking into account that the sequence $I_k(x) = |\varphi(x, u_k(x))| + |\psi(x, u_k(x))| + |g(u_k(x))|$ is bounded in $L^1_{loc}(\Omega)$, we derive from Lemma 9 that for all $a, b \in \mathbb{R}$

$$\operatorname{div}\{\varphi(x, s_{a,b}(u_k)) - A(x)\nabla g(s_{a,b}(u_k))\} = \zeta^k_{a,b} \quad \text{in } \mathcal{D}'(\Omega),$$

where $\zeta^k_{a,b}$ is a bounded sequence in $\mathrm{M}_{loc}(\Omega)$. Since $\mathrm{M}_{loc}(\Omega)$ is compactly embedded in $W^{-1}_{p,loc}(\Omega)$ for each $p \in [1, n/(n-1))$ we see that condition (97) is satisfied. By our assumption condition (98) is also satisfied. By Theorem 9 we conclude that some subsequence u_r converges as $r \to \infty$ to a limit function u in $L^1_{loc}(\Omega)$. Extracting a subsequence if necessary we can assume that $u_r \underset{r\to\infty}{\to} u$ a.e. in Ω. Passing to the limit as $r \to \infty$ in relation (5) with $u = u_r$ we claim that the limit function $u = u(x)$ satisfies this relation for all p such that the level set $u^{-1}(p)$ has zero measure (then $\operatorname{sign}(u_r - p) \to \operatorname{sign}(u - p)$ as $r \to \infty$ a.e. in Ω). Since the set P of such p has full measure and, therefore, is dense, for an arbitrary $p \in \mathbb{R}$ we can choose sequences $p_r^- < p < p_r^+$, $p_r^\pm \in P$, $r \in \mathbb{N}$ convergent to p. Taking a sum of relations (5) with $p = p_r^-$ and $p = p_r^+$ and passing to the limit as $r \to \infty$, with account of the point-wise relation $\operatorname{sign}(u - p_r^-) + \operatorname{sign}(u - p_r^+) \underset{r\to\infty}{\to} 2\operatorname{sign}(u - p)$, we obtain that (5) holds for all $p \in \mathbb{R}$, i.e. $u(x)$ is an entropy solution of (1). $\square$

Remark 3. Based on relation (99), we can introduce the class of quasi-solutions, including, by Lemma 9, entropy solutions of (1) (as well as entropy sub- and super-solutions of this equation). As is seen from the proof of Theorem 1, the statement of this Theorem remains true for more general case when $u_k(x)$ are quasi-solutions of equation (1).

5.2. Proof of Theorem 2.

To simplify the notations, we temporarily drop the index m in equation (7), and underline that the flux $\varphi(x, u)$ in this equation is smooth.

First we show that a weak solution $u = u(x)$ of equation (7) is an entropy solution in the sense of Definition 1. For this observe that in relation (8) we can choose test functions $f(x) \in W^1_2(\Omega)$, which have compact support in Ω. In particular, for $\eta(u) \in C^2(\mathbb{R})$, $f = f(x) \in C_0^\infty(\Omega)$ the function $\eta'(u)f$, $u = u(x)$ is an admissible test function, and we derive from (8) that

$$\begin{aligned} 0 = -\int_\Omega [\varphi(x, u)\nabla\eta'(u)f - A(x)\nabla g(u) \cdot \nabla\eta'(u)f]\, dx = \\ \int_\Omega [(\operatorname{div}\varphi(x, u))\eta'(u)f + g'(u)\eta''(u)fA(x)\nabla u \cdot \nabla u + \\ A(x)\eta'(u)g'(u)\nabla u \cdot \nabla f]dx. \end{aligned} \tag{101}$$

Introduce the function $q(u)$ defined, up to an additive constant, by the identity $q'(u) = \eta'(u)g'(u)$. We also define the vector $\psi(x, u)$ such that $\psi'_u(x, u) = \eta'(u)\varphi'_u(x, u)$. This vector is determine by the above equality up to an additive constant $c = c(x)$. Now we transform the terms $\operatorname{div}\varphi(x, u)\eta'(u)f$, $\eta'(u)g'(u)\nabla u$ as follows

$$\begin{aligned} \operatorname{div}\varphi(x, u)\eta'(u)f = (\operatorname{div}_x\varphi(x, u) + \varphi'_u(x, u) \cdot \nabla u)\eta'(u)f = \\ (\eta'(u)\operatorname{div}_x\varphi(x, u))f + (\psi'_u(x, u) \cdot \nabla u)f = \\ f\operatorname{div}\psi(x, u) + (\eta'(u)\operatorname{div}_x\varphi(x, u) - \operatorname{div}_x\psi(x, u))f; \\ \eta'(u)g'(u)\nabla u = \nabla q(u). \end{aligned}$$

Putting these equalities into (101) and integrating by parts, we obtain that

$$\begin{aligned} \int_\Omega [\psi(x, u) \cdot \nabla f + (\operatorname{div}_x\psi(x, u) - \eta'(u)\operatorname{div}_x\varphi(x, u))f + \\ q(u)\operatorname{div}(A(x)\nabla f) - g'(u)\eta''(u)fA(x)\nabla u \cdot \nabla u]\, dx = 0. \end{aligned} \tag{102}$$

We shall assume that $\eta''(u) \geq 0$ and has compact support in $\mathbb{R}$. Let $R > 0$ be such that $\operatorname{supp} \eta''(u) \subset (-R, R)$ and $L = (\eta'(-R) + \eta'(R))/2$ (evidently, L does not depend on R). Then we can choose $\psi(x, u)$ in the following way

$$\psi(x,u) = \frac{1}{2} \int \operatorname{sign}(u-p)(\varphi(x,u) - \varphi(x,p))d\eta'(p) + L\varphi(x,u). \tag{103}$$

Indeed, taking $R > |u|$ and integrating by parts, we obtain the equality

$$\begin{aligned} \int \operatorname{sign}(u-p)(\varphi(x,u) - \varphi(x,p))d\eta'(p) = \\ \int_{-R}^{R} \operatorname{sign}(u-p)(\varphi(x,u) - \varphi(x,p))d\eta'(p) = \\ \int_{-R}^{u} (\varphi(x,u) - \varphi(x,p))d\eta'(p) - \int_{u}^{R} (\varphi(x,u) - \varphi(x,p))d\eta'(p) = \\ \int_{-R}^{u} \varphi'_u(x,p)\eta'(p)dp - \int_{u}^{R} \varphi'_u(x,p)\eta'(p)dp - \\ 2L\varphi(x,u) + \varphi(x,-R)\eta'(-R) + \varphi(x,R)\eta'(R) \end{aligned}$$

We see that, up to a function which does not depend on u,

$$\begin{aligned} \frac{1}{2} \int \operatorname{sign}(u-p)(\varphi(x,u) - \varphi(x,p))d\eta'(p) + L\varphi(x,u) = \\ \frac{1}{2} \left(\int_{-R}^{u} \varphi'_u(x,p)\eta'(p)dp - \int_{u}^{R} \varphi'_u(x,p)\eta'(p)dp \right) \end{aligned}$$

and therefore

$$\frac{\partial}{\partial u} \left(\frac{1}{2} \int \operatorname{sign}(u-p)(\varphi(x,u) - \varphi(x,p))d\eta'(p) + L\varphi(x,u) \right) = \eta'(u)\varphi'_u(x,u),$$

as required. In the similar way we find that, up to an additive constant,

$$q(u) = \frac{1}{2} \int |g(u) - g(p)| d\eta'(p) + Lg(u). \tag{104}$$

Concerning the function $\eta'(u)\operatorname{div}_x\varphi(x,u) - \operatorname{div}_x\psi(x,u)$, it admits the representation

$$\eta'(u)\operatorname{div}_x\varphi(x,u) - \operatorname{div}_x\psi(x,u) = \frac{1}{2} \int \operatorname{sign}(u-p)\operatorname{div}_x\varphi(x,p)d\eta'(p). \tag{105}$$

Indeed, in view of (103), we see that for sufficiently large R

$$\begin{aligned} 2\psi(x,u) = \int_{-R}^{u} (\varphi(x,u) - \varphi(x,p))d\eta'(p) - \int_{u}^{R} (\varphi(x,u) - \varphi(x,p))d\eta'(p) + 2L\varphi(x,u) = \\ \varphi(x,u)(\eta'(u) - \eta'(-R)) - \int_{-R}^{u} \varphi(x,p)d\eta'(p) - \varphi(x,u)(\eta'(R) - \eta'(u)) + \\ \int_{u}^{R} \varphi(x,p)d\eta'(p) + 2L\varphi(x,u) = 2\eta'(u)\varphi(x,u) - \int \operatorname{sign}(u-p)\varphi(x,p)d\eta'(p), \end{aligned}$$

where we use the equality $2L = \eta'(R) + \eta'(-R)$. Applying the operator div_x to the above equality, we arrive at (105).

Now, we transform (102), using equalities (103), (104), (105) and the identity

$$L \int_{\Omega} \{\varphi(x,u) \cdot \nabla f + q(u)\operatorname{div}(A(x)\nabla f)\}dx = 0.$$

We find that for each $f = f(x) \in C_0^\infty(\Omega)$, $f \geq 0$

$$\int\int_\Omega \{\operatorname{sign}(u-p)[(\varphi(x,u)-\varphi(x,p))\cdot\nabla f - f\operatorname{div}_x\varphi(x,p)] +$$
$$|g(u)-g(p)|\operatorname{div}(A(x)\nabla f)\}\eta''(p)dxdp = \int_\Omega g'(u)\eta''(u)fA(x)\nabla u\cdot\nabla u \geq 0$$

and since $\eta''(p)$ is arbitrary finite continuous function on $\mathbb{R}$ we arrive at

$$I(p) \doteq \int_\Omega \{\operatorname{sign}(u-p)[(\varphi(x,u)-\varphi(x,p))\cdot\nabla f - f\operatorname{div}_x\varphi(x,p)] +$$
$$|g(u)-g(p)|\operatorname{div}(A(x)\nabla f)\}dx \geq 0 \tag{106}$$

for all $p \in P$, where the set P consists of points p such that the level set $u^{-1}(p)$ has null Lebesgue measure. We use that the function $I(p)$ is continuous at any point of P. In view of (106) for all $p \in P$

$$\operatorname{div}[\operatorname{sign}(u-p)(\varphi(x,u)-\varphi(x,p))] + \operatorname{sign}(u-p)\operatorname{div}_x\varphi(x,p) - \operatorname{div}A(x)\nabla|g(u)-g(p)| \leq 0 \tag{107}$$

in $\mathcal{D}'(\Omega)$. Since the set P has full measure and, therefore, is dense, for an arbitrary $p \in \mathbb{R}$ we can choose sequences $p_r^- < p < p_r^+$, $p_r^\pm \in P$, $r \in \mathbb{N}$ convergent to p. Taking a sum of relations (107) with $p = p_r^-$ and $p = p_r^+$ and passing to the limit as $r \to \infty$, with account of the point-wise relation $\operatorname{sign}(u-p_r^-) + \operatorname{sign}(u-p_r^+) \underset{r\to\infty}{\to} 2\operatorname{sign}(u-p)$, we obtain that (107) holds for all $p \in \mathbb{R}$, i.e. $u(x)$ is an entropy solution of (7).

We need also a-priori estimate of ∇u. Choose $M \geq \|u\|_\infty$ and a function $\eta(u) \in C_0^2(\mathbb{R})$ such that $\eta(u) = u^2/2$ on the segment $[-M, M]$ and $\operatorname{supp}\eta(u) \in [-M-1, M+1]$. Then for $u = u(x)$ $\eta''(u) = 1$ a.e. in Ω and we derive from (102) that for each $f = f(x) \in C_0^\infty(\Omega)$, $f \geq 0$

$$\int_\Omega fg'(u)A(x)\nabla u\cdot\nabla u dx \leq$$
$$\left|\int_\Omega \left[\psi(x,u)\cdot\nabla f + (\operatorname{div}_x\psi(x,u) - \eta'(u)\operatorname{div}_x\varphi(x,u))f + q(u)\operatorname{div}(A(x)\nabla f)\right]dx\right|. \tag{108}$$

From (103), (104), (105) it follows that

$$|\psi(x,u)| \leq C\max_{|u|\leq M+1}|\varphi(x,u)|, \quad |q(u)| \leq C\max_{|u|\leq M+1}|g(u)|,$$
$$|\operatorname{div}_x\psi(x,u) - \eta'(u)\operatorname{div}_x\varphi(x,u)| \leq C\int_{-M-1}^{M+1}|\operatorname{div}_x\varphi(x,p)|dp,$$

where C is the constant depending only on the fixed function η. Putting these estimates into (108), we get

$$\int_\Omega fg'(u)A(x)\nabla u\cdot\nabla u dx \leq C\int_\Omega\{\max_{|u|\leq M+1}|\varphi(x,u)||\nabla f| +$$
$$\max_{|u|\leq M+1}|g(u)||\operatorname{div}A(x)\nabla f|\}dx + C\int_\Omega\int_{-M-1}^{M+1}|\operatorname{div}_x\varphi(x,p)|f(x)dpdx. \tag{109}$$

By our assumptions the sequences $\varphi_m(x,u)$, $g_m(u)$ converge as $m \to \infty$ in $L^2_{loc}(\Omega, C(\mathbb{R}))$ and in $C^1(\mathbb{R})$, respectively while $A_m(x) \to A(x)$ in C^1. Therefore, the sequence

$$\int_\Omega\{\max_{|u|\leq M+1}|\varphi_m(x,u)||\nabla f| + \max_{|u|\leq M+1}|g_m(u)||\operatorname{div}A_m(x)\nabla f|\}dx$$

is bounded by a constant depending only on f. Here we take $M \geq \sup_m\|u_m\|_\infty$. From estimate (109) it follows that

$$\int_\Omega fg_m'(u_m)A_m(x)\nabla u_m\cdot\nabla u_m dx \leq C_f I_m(K, M+1),$$

with $K = \operatorname{supp} f$, where the sequence

$$I_m(K,M) = 1 + \int_K \int_{-M}^{M} |\mathrm{div}_x \varphi_m(x,p)| dp dx$$

was indicated in Introduction. The obtained estimate can be written as follows

$$\int_\Omega |\sqrt{g'_m(u_m)}(A_m(x))^{1/2}\nabla u_m|^2 f(x)dx \le C_f I_m(K, M+1). \tag{110}$$

Now we take $a, b \in \mathbb{R}$, $a < b$. Let us demonstrate that the sequence

$$L_m = \mathrm{div}\,(\varphi(x, s_{a,b}(u_m)) - A(x)\nabla g(s_{a,b}(u_m)))$$

is pre-compact in $W^{-1}_{p,loc}$ with some $p > 1$. For that, recall that $u_m(x)$ is an e.s. of (7) and by Lemma 9

$$\mathrm{div}\,(\varphi_m(x, s_{a,b}(u_m)) - A_m(x)\nabla g_m(s_{a,b}(u_m))) = \xi_m$$

where ξ_m is a bounded sequence in the space $\mathrm{M}_{loc}(\Omega)$, which is compactly embedded in $W^{-1}_{p,loc}(\Omega)$ for each $p \in [1, n/(n-1))$. Further, we have $L_m = L_{1m} + L_{2m} + \xi_m$, where

$$L_{1m} = \mathrm{div}(\varphi(x, s_{a,b}(u_m)) - \varphi_m(x, s_{a,b}(u_m))),$$
$$L_{2m} = \mathrm{div}\,(A_m(x)\nabla g_m(s_{a,b}(u_m)) - A(x)\nabla g(s_{a,b}(u_m))).$$

In view of the estimate

$$|\varphi(x, s_{a,b}(u_m)) - \varphi_m(x, s_{a,b}(u_m))| \le \max_{|u|\le M} |\varphi_m(x,u) - \varphi(x,u)|$$

and the condition $\varphi_m(x,u) \underset{m\to\infty}{\to} \varphi(x,u)$ in $L^2_{loc}(\Omega, C(\mathbb{R}))$ we have $\varphi(x, s_{a,b}(u_m)) - \varphi_m(x, s_{a,b}(u_m)) \underset{m\to\infty}{\to} 0$ in $L^2_{loc}(\Omega)$. Hence $L_{1m} \to 0$ in $W^{-1}_{2,loc}(\Omega)$. Concerning the sequence L_{2m}, we first remark that by the chain rule a.e. in Ω

$$A_m(x)\nabla g_m(s_{a,b}(u_m)) = (g_m)'(u_m)\chi(u_m)A_m(x)\nabla u_m,$$
$$A(x)\nabla g(s_{a,b}(u_m)) = g'(u_m)\chi(u_m)A(x)\nabla u_m,$$

where $\chi(u)$ is the indicator function of the segment $[a,b]$. Therefore,

$$\begin{aligned}
&|A_m(x)\nabla g_m(s_{a,b}(u_m)) - A(x)\nabla g(s_{a,b}(u_m))| \le \\
&|(g_m)'(u_m)A_m(x)\nabla u_m - g'(u_m)A(x)\nabla u_m| \le \\
&|(g_m)'(u_m)(A_m(x) - A(x))\nabla u_m| + |((g_m)'(u_m) - g'(u_m))A(x)\nabla u_m| \le \\
&C\|(A_m(x) - A(x))(A_m(x))^{-1/2}\| \cdot |\sqrt{(g_m)'(u_m)}(A_m(x))^{1/2}\nabla u_m| + \\
&C|(g_m)'(u_m) - g'(u_m)|/\sqrt{(g_m)'(u_m)} \cdot \|(A(x))^{1/2}\| \cdot |\sqrt{(g_m)'(u_m)}(A_m(x))^{1/2}\nabla u_m|,
\end{aligned} \tag{111}$$

$C = \mathrm{const}$. Here we use the condition $A_m \ge A$, which implies that for any vector $v \in \mathbb{R}^n$

$$|Av| \le \|A^{1/2}\||A^{1/2}v| = \|A^{1/2}\|(Av\cdot v) \le \|A^{1/2}\|(A_m v\cdot v) = \|A^{1/2}\| \cdot |A_m^{1/2}v|.$$

From (111) it follows that for every $f = f(x) \in C_0^\infty(\Omega)$, $f \ge 0$

$$\begin{aligned}
&\left(\int_\Omega (A_m(x)\nabla g_m(s_{a,b}(u_m)) - A(x)\nabla g(s_{a,b}(u_m)))^2 f dx\right)^{1/2} \le \\
&C_1\left(\max_{x\in K}\|(A_m(x) - A(x))(A_m(x))^{-1/2}\| + \max_{|u|\le M}|(g_m)'(u) - g'(u)|/\sqrt{(g_m)'(u)}\right) \times \\
&\left(\int_\Omega |\sqrt{g'_m(u)}(A_m(x))^{1/2}\nabla u_m|^2 f(x)dx\right)^{1/2}.
\end{aligned} \tag{112}$$

where $K = \operatorname{supp} f$. Taking into account relation (6) and estimate (110) we derive that

$$\int_\Omega (A_m(x)\nabla g_m(s_{a,b}(u_m)) - A(x)\nabla g(s_{a,b}(u_m)))^2 f dx \underset{m\to\infty}{\to} 0,$$

i.e. $(A_m(x)\nabla g_m(s_{a,b}(u_m)) - A(x)\nabla g(s_{a,b}(u_m))) \to 0$ in $L^2_{loc}(\Omega)$. This implies that $L_{2m} \to 0$ in $W^{-1}_{2,loc}(\Omega)$. We conclude that $L_m = L_{1m} + L_{2m} + \xi_m$ is pre-compact in $W^{-1}_{p,loc}(\Omega)$ with some $p > 1$. Hence, assumption (97) is satisfied. By Corollary 5 we see that the sequence u_m converges in $L^1_{loc}(\Omega)$ to some function $u = u(x) \in L^\infty(\Omega)$. Obviously, $\|u\|_\infty \le M$. It only remains to demonstrate that u is an e.s. of (1). By relation (106) for each $p \in \mathbb{R}$, $f = f(x) \in C_0^\infty(\Omega)$, $f \ge 0$

$$\int_\Omega \{\operatorname{sign}(u_m - p)[(\varphi_m(x,u_m) - \varphi_m(x,p)) \cdot \nabla f - f \operatorname{div}_x \varphi_m(x,p)] + \\ |g_m(u_m) - g_m(p)| \operatorname{div}(A_m(x)\nabla f)\} dx \ge 0.$$

Since $\operatorname{div}_x \varphi_m(x,p) = \gamma^m_{pr}(x) + \gamma^m_{ps}(x)$ the above relation implies that

$$\int_\Omega \{\operatorname{sign}(u_m - p)[(\varphi_m(x,u_m) - \varphi_m(x,p)) \cdot \nabla f - f\gamma^m_{pr}(x)] + \\ f|\gamma^m_{ps}(x)| + |g_m(u_m) - g_m(p)| \operatorname{div}(A_m(x)\nabla f)\} dx \ge 0 \tag{113}$$

Passing to a subsequence, we may assume that $u_m(x) \to u(x)$ as $m \to \infty$ a.e. in Ω. Then

$$\operatorname{sign}(u_m - p)(\varphi_m(x,u_m) - \varphi_m(x,p)) \underset{m\to\infty}{\to} \operatorname{sign}(u - p)(\varphi(x,u) - \varphi(x,p)), \\ |g_m(u_m) - g_m(p)| \underset{m\to\infty}{\to} |g(u_m) - g(p)|, \\ \operatorname{sign}(u_m - p) \underset{m\to\infty}{\to} \operatorname{sign}(u - p)$$

a.e. in Ω and, as a consequence, in $L^1_{loc}(\Omega)$. The latter relation holds for such $p \in \mathbb{R}$ that the level set $u^{-1}(p)$ has zero Lebesgue measure. Besides, by our assumptions $\gamma^m_{pr}(x) \underset{m\to\infty}{\to} \omega_p(x)$ in $L^1_{loc}(\Omega)$, $|\gamma^m_{ps}(x)| \underset{m\to\infty}{\to} |\gamma^s_p|$ weakly in $M_{loc}(\Omega)$. Taking into account the indicated limit relations we can pass to the limit in (113) and obtain that

$$\int_\Omega \{\operatorname{sign}(u - p)[(\varphi(x,u_m) - \varphi(x,p)) \cdot \nabla f - f\omega_p(x)] + \\ |g(u) - g(p)| \operatorname{div}(A(x)\nabla f)\} dx + \int_\Omega f(x) d|\gamma^s_p|(x) \ge 0 \tag{114}$$

for all $p \in \mathbb{R}$ such that the level set $u^{-1}(p)$ has zero Lebesgue measure. Repeating the arguments concluding the proof of Theorem 1, we obtain that (114) holds for all $p \in \mathbb{R}$, i.e. $u(x)$ is an entropy solution of (1). This completes the proof of Theorem 2.

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