ON A STRONG PRECOMPACTNESS OF VELOCITY AVERAGES FOR A HETEROGENOUS TRANSPORT EQUATION WITH ROUGH COEFFICIENTS

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Abstract. We consider strong $L^1_{loc}(\mathbb{R}^d)$ precompactness of the sequence of averaged quantities $\int_{\mathbb{R}^m} h_n(x,\lambda)\rho(\lambda)d\lambda$, where $\rho \in C_0(\mathbb{R}^m)$, and $h_n \in L^p_{loc}(\mathbb{R}^d \times \mathbb{R}^m)$, $p > 1$, are solutions to the transport equations with flux explicitly depending on space:

$$\mathrm{div}_x(F(x,\lambda)h_n(x,\lambda)) = \sum_{i=1}^d \partial_{x_i}\partial_\lambda^{k_i} G_n^i(x,\lambda), \quad x \in \mathbb{R}^d, \quad \lambda \in \mathbb{R}^m,$$

where $F = (F_1, ..., F_d)$, and $F_i \in L^q_{loc}(\mathbb{R}^d \times \mathbb{R}^m)$, $i = 1, ..., d$, $1/p + 1/q < 1$, and $k_i = (k_1^i, ..., k_m^i) \in \mathbb{N}^m$, $i = 1, \dots d$, stands for multindex. For the sequences of functions $(G_n^i)_{n\in\mathbb{N}}$, $i = 1, \dots, d$, we assume that they strongly converge to zero as $n \to \infty$ in $L^{\tilde q}_{loc}(\mathbb{R}^d \times \mathbb{R}^m)$ for a $\tilde q > 1$.

In order to obtain the result we adapt H-measures [13, 35] and give positive (but partial) answer on the question whether it is possible to translate in an algebraic way the information "u_k is bounded in L^p, Pu_k is bounded in L^q for a $p, q > 1$", where P is a differential operator. This question was posed in [13]. The proof mostly involves the theory of multipliers.

1. Introduction

Consider a sequence $(h_n)_{n\in\mathbb{N}}$ of solutions to the first order transport equation:

$$\mathcal{L}(\nabla,\lambda)h_n = \mathrm{div}_x(a(\lambda)h_n(x,\lambda)) = \partial_\lambda g(x,\lambda), \quad \lambda \in \mathbb{R}^m,\ x \in \mathbb{R}^d, \tag{1}$$

where g is locally bounded Radon measure over $\mathbb{R}^d \times \mathbb{R}^m$.

It was firstly noticed by Agoshkov [1] that the family of averaged quantities

$$\int_{\mathbb{R}^m} \rho(\lambda)h_n(x,\lambda)d\lambda, \quad \rho \in C^1_0(\mathbb{R}^m) \tag{2}$$

demonstrates better properties then the family of solutions (h_n) itself. More precisely, in general one states that:

Lemma 1. *The sequence of averaged quantities* (2) *is strongly precompact in* $L^1_{loc}(\mathbb{R}^d)$.

Such kind of results are usually called velocity averaging lemmas.

In [27] one of the first and the most popular application of the velocity averaging is given. More precisely, it was shown that for an entropy admissible solution [25]

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of a homogeneous scalar conservation law, denote it by $u(t,x)$, $t \in I\!R^+$, $x \in I\!R^d$, so called kinetic function

$$h(t,x,\lambda) = \begin{cases} 1, & \text{for } \; 0 < \lambda \leq u(t,x) \\ -1, & \text{for } \; 0 > \lambda \geq u(t,x) \\ 0, & \text{otherwise} \end{cases}$$

satisfy a transport equation similar to (1). Then, for such transport equation the authors prove an averaging lemma which enables them to gain regularity results for the entropy admissible solution u to the corresponding conservation law.

From the beginning of 90s most of the results on velocity averaging were obtained, and most of them were restricted on the first order transport equation (deg$\mathcal{L}(i\xi,\cdot) = 1$). One can find most of the information and references on this issue in the introduction of [37] where the authors consider second order transport equation in general form:

$$\mathcal{L}(\nabla_x, \lambda) = \partial_\lambda g(x,\lambda),$$

(more precisely deg$\mathcal{L}(i\xi,\cdot) = 2$). They discovered that the symbol $\mathcal{L}(i\xi,\lambda)$ of such equation satisfies the "truncation property" which enabled them to prove apriori estimates which provided a regularity of solutions to various partial differential equations such as conservation laws, nonlinear degenerate parabolic equations, nonisotropic degenerate diffusion and other equations admitting kinetic formulation [27]. Also, we would like to mention that the theory of transport equations can be applied as well on nonlinear problems which does not admit kinetic formulation but are "linearizable" on some other way. For more information check [2, 8, 11].

It is important to notice that in all papers cited in [37] (except P.Gerard's work referenced as [13] in this paper), symbols $\mathcal{L}(i\xi,\lambda)$ corresponding to transport equations did not depend on space or time variables. Actually, this means that the equations describe processes occurring in homogeneous medias. On the other hand, most of natural phenomena take place in heterogeneous medias (flow in heterogeneous porous media, sedimentation processes, blood flow, gas flow in variable duct...).

But, it appears that it is much more complicated to work on heterogeneous transport equations, and that one can not apply techniques from the homogeneous case. This fact could be explained by the following simple observation. If we apply Fourier transform on equation (1), at least informally, we can separate solutions (h_n) and known coefficients. Thus, we are able to express the solution via known coefficients, and, consequently, "estimate" solutions via known coefficients. Still, it is far from being easy to formalize this observation (see e.g. [29, 37]).

In the heterogeneous case even such informal idea is not at our disposal. At the moment only possible approach is through a variant of defect measures [13, 35]. The defect measure is an object describing loss of compactness of a family of functions. Originally, the notion of defect measure was systematically studied for sequences satisfying elliptic estimates by P.L.Lions [26]. Since elliptic estimates automatically eliminate oscillations, the defect measures used in [26] were not appropriate enough for studying loss of compactness caused by oscillations, and which typically appear in the case of e.g. hyperbolic problems.

In order to control oscillations, natural idea was to introduce an object which distinguish oscillations of different frequencies. The idea is formalized by P. Gerard [13] and independently L. Tartar [35]. The first one named appropriate defect measure as microlocal defect measure and the second one H–measure. In the sequel we shall stick to Tartar's notion.

An H–measure is a Hermitian non-negative complex Radon measure on the co-spherical bundle over a domain in consideration (in general, the base space of the fibre bundle is a manifold, while the fibre is the unit sphere S^{d-1}). The following theorem is the corner stone of H-measures (for more information and other variants see also [3, 4, 5, 31, 32, 36]):

Theorem 2. [35] *If $(u_n) = ((u_n^1, \dots, u_n^r))$, $n \in \mathbf{N}$ is a sequence in $L^2(\mathbb{R}^d; \mathbb{R}^r)$ such that $u_n \rightharpoonup 0$ in $L^2(\mathbb{R}^d; \mathbb{R}^r)$, then there exists subsequence $(u_{n'}) \subset (u_n)$ and positive definite matrix of complex Radon measure $\mu = \{\mu^{ij}\}_{i,j=1,\dots,d}$ on $\mathbb{R}^d \times S^{d-1}$ such that for all $\varphi_1, \varphi_2 \in C_0(\mathbb{R}^d)$ and $\psi \in C(S^{d-1})$:*

$$
\begin{aligned}
\lim_{n'\to\infty} \int_{\mathbb{R}^d} \mathcal{F}(\varphi_1 u^i_{n'})(\xi)\overline{\mathcal{F}(\varphi_2 u^j_{n'})(\xi)}\psi(\frac{\xi}{|\xi|})d\xi &= \langle \mu^{ij}, \varphi_1 \bar{\varphi}_2 \psi\rangle \\
&= \int_{\mathbb{R}^d\times S^{d-1}} \varphi_1(x)\varphi_2(x)\psi(\xi)d\mu^{ij}(x,\xi), \quad i,j = 1,\dots,d,
\end{aligned}
\tag{3}
$$

where $\mathcal{F}$ is the Fourier transform.

Definition 3. The complex Radon measure $\{\mu^{ij}\}_{i,j=1,\dots r}$ defined in the previous theorem we call H-measure corresponding to the sequence $(u_n) \in L^2(\mathbb{R}^d; \mathbb{R}^r)$.

As we can see, the H–measure μ also depends on dual variable $\xi \in \mathbb{R}^d$ which actually describes frequency of an oscillation.

Although H–measures were in some sense step forward with respect to Lions's defect measures, they can only be applied on sequences belonging at least to L^2_{loc} (which means that only limited concentration effects are allowed). Such confinement forced P.Gerard's averaging lemma for heterogeneous transport equations [13, Theorem 2.5] to be proved only for sequences of solutions belonging to L^2_{loc} and with the righthand side being relatively precompact in H^{-1}_{loc}. If we compare that result with e.g. [29, Theorem 2] for homogeneous transport equation, we can see that Gerard's result has much stronger assumptions. In this paper, by adapting the notion of H–measure, we shall prove an averaging lemma for heterogeneous transport equations under the conditions analogous to the conditions given in [29, Theorem 2]. Actually, we have succeeded to translate in an algebraic way the information "u_k is bounded in L^p, Pu_k is bounded in L^q for a $p, q > 1$", where P is a differential operator of first order with homogeneous symbol. Thus, we gave partial answer on the question posed in [13].

Another shortcoming of H–measures is the fact that they can be applied only on transport equations with homogeneous symbol (since dual variable ξ belongs to the unit sphere), and in special cases of equations with non-homogeneous symbols [30, 33]. How to overcome this obstacle will be the subject of further investigations.

The paper is organized as follows.

In Section 2 we formulate the main result of the paper (Theorem 4 below), and introduce basic notions and notations.

In Section 3 we introduce the H–distributions – an extension of H–measures, which will be the basic tool for proving our main result. As in Theorem 2 above, we correspond an H–distribution to a sequence of finite dimension.

In Section 4 we give proof of the main result – Theorem 4.

In Section 5 we prove existence of solution to multidimensional scalar conservation law with flux discontinuous in the space variable and belonging to the Sobolev space $W^{1,p}(\mathbb{R})$ in the velocity variable. The same result can be found in [14, 31] but under different assumptions on the flux regularity.

2. The main result, notions and notations

We consider the following first order linear transport equation:

$$\mathcal{L}(\nabla_x, x, \lambda) = \operatorname{div}_x(F(x,\lambda)h_n(x,\lambda)) = \sum_{i=1}^{d} \partial_{x_j}\partial_\lambda^{k_i} G_n^i(x,\lambda), \quad x \in \mathbb{R}^d, \ \lambda \in \mathbb{R}^m, \tag{4}$$

where $k_i = (k_1^i, ..., k_d^i) \in \mathbf{N}^d$ and $\partial_\lambda^{k_i} = \partial_{\lambda_1}^{k_1^i} \dots \partial_{\lambda_d}^{k_d^i}$, and it is assumed that the flux F satisfies the following non-degeneracy condition [32]:

For almost every $x \in \mathbb{R}^d$ and every $\xi \in S^{d-1}$ the mapping:

$$\lambda \mapsto \sum_{k=1}^{d} F_k(x,\lambda)\xi_k \tag{5}$$

is not identically equal to zero on any set of positive Lebesgue measure.

The main result of the paper is the following theorem.

Theorem 4. *Assume that for the flux vector $F = (F_1, ..., F_d)$ appearing in (4) we have*

- *$F_i \in L^{1+\alpha}_{loc}(\mathbb{R}^d \times \mathbb{R}^m)$, $\alpha > 0$, $i = 1, .., d$.*
- *F satisfies nondegeneracy condition (5).*

Assume that the sequence $(h_n)_{n\in\mathbf{N}}$ is such that

$$h_n \rightharpoonup 0$$

weakly in $L^{1+\beta}_{loc}(\mathbb{R}^d \times \mathbb{R}^m)$ for a $\beta > 0$ such that:

$$\frac{1}{1+\alpha} + \frac{1}{1+\beta} < 1. \tag{6}$$

Assume that for every $i = 1, \dots, d$ we have

$$G_n^i \to 0, \quad n \to \infty$$

strongly in $L^{1+\gamma}_{loc}(\mathbb{R}^d \times \mathbb{R}^m)$, for a $\gamma > 0$.

Then for every $\rho \in C_0(\mathbb{R}^m)$, there exists a subsequence $(h_r) \subset (h_n)$ such that the sequence of averaged quantities

$$\int_{\mathbb{R}^m} h_r(x,\lambda)\rho(\lambda)d\lambda \to 0$$

strongly in $L^1_{loc}(\mathbb{R}^d \times \mathbb{R}^m)$ as $r \to \infty$.

Remark 5. Notice that the conditions

$$\begin{aligned} &h_n \rightharpoonup 0 \ \text{ in } \ L^{1+\beta}(\mathbb{R}^d \times \mathbb{R}^m), \\ &G_n^i \to 0 \ \text{ strongly in } \ L^{1+\gamma}(\mathbb{R}^d \times \mathbb{R}^m) \end{aligned}$$

are not essential. Namely, it is enough to assume that (h_n) is bounded in $L^{1+\beta}_{loc}(\mathbb{R}^d \times \mathbb{R}^m)$, and that $G_n^i \to G_i$ strongly in $L^{1+\gamma}_{loc}(\mathbb{R}^d \times \mathbb{R}^m)$ for a $G^i \in L^{1+\gamma}_{loc}(\mathbb{R}^d \times \mathbb{R}^m)$, $i = 1, \dots, d$.

Then, since every bounded set in $L^{1+\beta}_{loc}(\mathbb{R}^d \times \mathbb{R}^m)$ is weakly precompact, there exists a function $h \in L^p_{loc}(\mathbb{R}^d \times \mathbb{R}^m)$ such that along a subsequence as $n \to \infty$

$$h_n \rightharpoonup h.$$

It is clear that the subsequence (not relabeled) $(h_n - h)$ satisfies a transport equation fulfilling the conditions from Theorem 4. So, we can conclude that

$$\int_{\mathbb{R}^m} h_r(x,\lambda)\rho(\lambda)d\lambda \to \int_{\mathbb{R}^m} h(x,\lambda)\rho(\lambda)d\lambda,$$

strongly in $L^1_{loc}(\mathbb{R}^d)$ for a subsequence $(h_r) \subset (h_n)$.

Before we start the proof, we introduce necessary tools and notions.

By $\int_{x;loc}$ we imply $\int_K dx$ where $K \subset \mathbb{R}^d$ is a compact subset of $\mathbb{R}^d$ for an appropriate $d \in \mathbb{N}$.

Usually, it will be clear what is the value of d. Otherwise, we shall precise it.

By $B(0,l)$ we will denote the ball in an appropriate Euklid space centered in zero with the radius $l > 0$.

By $L^p_0(\mathbb{R}^d)$ we denote the space of functions belonging to $L^p(\mathbb{R}^d)$ being equal to zero out of some compact $V \subset\subset \mathbb{R}^d$. By $L^p_{loc}(\mathbb{R}^d)$ we denote the space of functions belonging to $L^p(V)$, for an arbitrary compact $V \subset\subset \mathbb{R}^d$.

By $(L^p_0(\mathbb{R}^d))^*$ we shall denote the set of bounded linear functionals defined on $L^p_0(\mathbb{R}^d)$. It is well known that $(L^p_0(\mathbb{R}^d))^* = L^q_{loc}(\mathbb{R}^d)$ for the q such that $1/p + 1/q = 1$. Still, to be as clear as possible in our considerations we will keep the notation $(L^p_0(\mathbb{R}^d))^*$.

By $W^{1,p}(\mathbb{R}^d)$ we denote the space of functions $f \in L^p(\mathbb{R}^d)$ having the Sobolev first derivative $\partial_{x_j} f$, $j = 1, \dots, d$, and such that $\partial_{x_j} f \in L^p(\mathbb{R}^d)$, $j = 1, \dots, d$.

For a multi-index $\alpha = (\alpha_1, \dots, \alpha_d) \in (\mathbb{N} \cup \{0\})^d$ we let $|\alpha| = \alpha_1 + \dots + \alpha_d$. Furthermore, for a function $u \in C^k(\Omega)$, $\Omega \subset \mathbb{R}^d$, and a multi-index $\alpha \in (\mathbb{N} \cup \{0\})^d$ we let $\partial^\alpha_x u = \partial^{\alpha_1}_{x_1} \dots \partial^{\alpha_d}_{x_d} u(x)$, $x \in \Omega$.

For a function $u \in C^\kappa(\Omega)$ we denote

$$\|u\|_{C^\kappa(\Omega)} = \max_{0 \le |\alpha| \le \kappa} \sup_{\xi \in \Omega} |\partial^\alpha_\xi u(\xi)|, \quad \alpha \in (\mathbb{N} \cup \{0\})^d.$$

By $C^1(\mathbb{R}^d; C^\kappa(S^{d-1}))$ we denote the space of functions $\phi(x,\xi)$, $x \in \mathbb{R}^d$, $\xi \in S^{d-1}$, such that

$$\max_{0 \le |\beta| \le 1} \sup_{x \in \mathbb{R}^d} \left(\max_{0 \le |\alpha| \le \kappa} \sup_{\xi \in S^{d-1}} |\partial^\beta_x \partial^\alpha_\xi \phi(x,\xi)| \right), \quad \alpha, \beta \in (\mathbb{N} \cup \{0\})^d.$$

By $L^p(\mathbb{R}^d; C^\kappa(S^{d-1}))$ we denote the space of functions $\phi(x,\xi)$, $x \in \mathbb{R}^d$, $\xi \in S^{d-1}$, such that

$$\|\phi\|_{L^p(\mathbb{R}^d; C^\kappa(S^{d-1}))} = \int_{\mathbb{R}^d} \|\phi(x,\cdot)\|^p_{C^\kappa(S^{d-1})} dx < \infty,$$

and additionally satisfying the following generalized Lebesgue point property:

Fix a positive smooth compactly supported real function ω with total mass one. For almost every $x \in \mathbb{R}^d$ we have:

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^d} \|\phi(x + \varepsilon z, \lambda + \varepsilon \nu, \tilde\lambda + \varepsilon \tilde\nu, \cdot) - \phi(x, \lambda, \tilde\lambda, \cdot)\|_{C^\kappa(S^{d-1})} \times \\ \times \prod_{i=1}^d \omega(z_i) \prod_{i=1}^r \omega(\eta_i) \prod_{i=1}^{r'} \omega(\tilde\nu_i) dz d\nu d\tilde\nu = 0. \tag{7}$$

We shall also need the following inequality for the means in an integral formulation:

Proposition 6. *Let $d, r \in I\!N$. Assume that $u \in L^p(I\!R^{d+r})$ for a $p \geq 1$. Then we have:*

$$\int_{I\!R^d} \|u(x,\cdot)\|_{L^p(I\!R^r)} dx \leq \|u\|_{L^p(I\!R^{d+r})}. \tag{8}$$

By $L^p_0(I\!R^d; C^k(S^{d-1}))$ we denote the subspace of $L^p(I\!R^d; C^k(S^{d-1}))$ such that for every $u \in L^p_0(I\!R^d; C^k(S^{d-1}))$ there exists a bounded set $V \subset I\!R^d$ such that $u(x,\xi) = 0$ if $x \in V$.

By $\mathcal{F}$ we denote the Fourier transform on $I\!R^d$, i.e. for a function u defined on $I\!R^d$ we put:

$$\hat{u}(\xi) := \mathcal{F}u(\xi) := \int_{I\!R^d} e^{-2\pi i \xi\cdot x} u(x) dx,$$

while its inverse $\bar{\mathcal{F}}$ is defined as:

$$u(x) := \bar{\mathcal{F}}\hat{u}(\xi) := \int_{I\!R^d} e^{2\pi i \xi\cdot x} \hat{u}(\xi) d\xi.$$

The main tool that we are going to use is the theory of multipliers [28, 34, 35]. We give a definition of a multiplier operator.

Definition 7. A multiplier operator $\mathcal{A}_\phi$ with a symbol $\phi \in C(I\!R^d)$ is defined by the formula:

$$\mathcal{F}[\mathcal{A}_\phi[f]](\xi) = \phi(\xi)\mathcal{F}[f](\xi),$$

where f is an integrable function.

Remark 8. Note that applying Plancharel's theorem we can rewrite (3) in the form:

$$\begin{aligned} &\lim_{n'\to\infty} \int_{I\!R^d} \phi_1 u^i_{n'}(t,x)\overline{\mathcal{A}_\psi[\phi_2 u^j_{n'}(\,\cdot\,,\,\cdot\,)](t,x)} dx dt \\ &= \int_{I\!R^d\times S^{d-1}} \phi_1(x,t)\overline{\phi_2(x,t)\psi(\xi)} d\mu^{pq}(x,\xi), \end{aligned} \tag{9}$$

where $\mathcal{A}_\psi$ is a multiplier operator on $I\!R^d$ with the symbol $\psi(\xi)$, $\xi \in S^{d-1}$.

The main theorem in the theory of multipliers is famous Hormander-Mikhlin theorem (see also Marcinkevich theorem [28, 34]):

Theorem 9. [28] *Let $d \geq 1$ be an integer, and let the function $\phi \in L^\infty(I\!R^d)$ has partial derivatives of order less then or equal to κ, where κ is the least integer greater then $d/2$. Given the q-tuple of integers denote $\alpha = (\alpha_1, \dots, \alpha_q)$, and let $n(\alpha) = \alpha_1 + \alpha_2 + \cdots + \alpha_q$.*

Suppose that for some constant $k > 0$ and for any real number $r > 0$ we have

$$\int_{\frac{r}{2}\leq\|\xi\|\leq r} |D^{n(\alpha)}\phi(\xi)|^2 d\xi \leq k^2 r^{n-2n(\alpha)}, \quad (n(\alpha) \leq \kappa). \tag{10}$$

Then for $1 < p < \infty$ and associated multiplier operator $\mathcal{A}_\phi$ there exists constant k_p such that

$$\|\mathcal{A}_\phi(f)\|_{L^p(I\!R^d)} \leq k_p \|f\|_{L^p(I\!R^d)}, \quad f \in L^p(I\!R^d). \tag{11}$$

Remark 10. It is very important to notice that carefully inspecting the proof of the Hormander-Mikhlin theorem we infer that the constant k_p from the Hormander-Mikhlin theorem has the form

$$k_p = C_p k, \tag{12}$$

where k is given in the Hormander-Mikhlin theorem, and C_p is a constant independent on the symbol ϕ. For the precise proof see [9].

Combining the former formulation of the Hormander-Mikhlin theorem and Remark 10 it is not difficult to prove the following corollary (see also [34, Sect. 3.2, Example 2]:

Corollary 11. *Assume that the symbol $\phi \in C^\kappa(I\!R^d)$, $\kappa \geq d/2$, $\kappa \in I\!N$ is defined on the unit sphere i.e. $\phi = \phi(\frac{\xi}{|\xi|})$. Then, the multiplier operator $\mathcal{A}_\phi$ with the symbol ϕ is continuous as a mapping $\mathcal{A}_\phi : L^p(I\!R^d) \to L^p(I\!R^d)$ satisfying:*

$$\|\mathcal{A}_\phi f\|_{L^p(I\!R^d)} \leq K_p \|\phi\|_{C^\kappa(S^{d-1})} \|f\|_{L^p(I\!R^d)}, \tag{13}$$

where K_p is a constant independent on ϕ, and $\|\phi\|_{C^\kappa} = \max\limits_{0\leq j\leq\kappa} \sup\limits_{z\in I\!R^d} |\phi^{(j)}(z)|$.

Proof: As in Theorem 9 for a given q-tuple of integers denote $\alpha = (\alpha_1, \ldots, \alpha_q)$, and let $n(\alpha) = \alpha_1 + \alpha_2 + \cdots + \alpha_q \leq \kappa$. Then notice that

$$\left| D^{n(\alpha)} \phi \left(\frac{\xi}{|\xi|} \right) \right| \leq \|\phi\|_{C^\kappa(S^{d-1})} \frac{K}{|\xi|^{n(\alpha)}},$$

for a constant K independent on ϕ. From here we have for an arbitrary $r > 0$:

$$\begin{aligned}\int_{\frac{r}{2}<\|\xi\|<r} \left| D^{n(\alpha)} \phi \left(\frac{\xi}{|\xi|} \right) \right|^2 d\xi &\leq \frac{\|\phi\|^2_{C^\kappa}(S^{d-1})}{r^{2n(\alpha)}} r^d \left(1 - \frac{1}{2^d}\right) \\ &\leq \|\phi\|^2_{C^\kappa(S^{d-1})} r^{d-2n(\alpha)} \left(1 - \frac{1}{2^d}\right)\end{aligned}$$

implying that condition (10) is satisfied. From the Hormander-Mikhlin theorem, more precisely Remark 10, we conclude that $\mathcal{A}_\phi$ is bounded as mapping from $L^p(I\!R^d)$ to $L^p(I\!R^d)$ satisfying (13).

❑

In the following definition we introduce two very important multipliers. We are going to use them substantially in the proof Theorem 4.

Definition 12. For every $f \in L^p(I\!R^d)$, $p > 1$, the Riesz potential $\mathcal{I}_\alpha$, $0 < \alpha < d$, is defined by the formula

$$\mathcal{F}[\mathcal{I}_\alpha[f]](\xi) = (2\pi|\xi|)^{-\alpha} \mathcal{F}[f](\xi).$$

The zero order multiplier $\mathcal{R}_j$, $j = 1, ..., d$, with the symbol $i\xi_j/|\xi|$ is called the Riesz transform.

We provide basic properties of the Riesz transform and Riesz potential.

We have:

$$\begin{aligned}(\mathcal{I}_\alpha \circ \mathcal{I}_\beta)[f] &= \mathcal{I}_{\alpha+\beta}[f] \\ \partial_{x_j} \mathcal{I}_1[f] &= \mathcal{I}_1[\partial_{x_j} f] = \mathcal{R}_j[f], \quad j = 1, ..., d, \\ \|\mathcal{R}_j[f]\|_{L^p} &\leq C_p \|f\|_{L^p}, \quad j = 1, ..., d.\end{aligned} \tag{14}$$

The Riesz potentials $\mathcal{I}_1$ are characterized by the following important lemma:

Lemma 13. [34] *If $p > d$ then the Riesz potential $\mathcal{I}_1$ is a compact operator from $L^p(I\!R^d)$ to $C(I\!R^d)$. If $1 < p \leq d$ then the Riesz potential $\mathcal{I}_1$ is compact operator from $L^p(I\!R^d)$ to $L^q(I\!R^d)$ for an arbitrary $q \in [1, pd(d-p)^{-1}]$.*

3. H-distributions - beyond H-measures

In order to describe loss of compactness for a sequence $(u_n) \in L^p(\Omega)$, $p > 1$, we shall use a notion similar to H-measures; we call it H-distribution. But, since we decreased regularity of the sequence (u_n), our H-distribution will not have so nice properties as H-measures.

The main theorem of the section is the following one:

Theorem 14. *Assume that (u_n), $n \in \mathbf{N}$, is a sequence in $L^{1+\beta}_{loc}(I\!R^{d+r})$ such that $u_n \rightharpoonup 0$ in $L^{1+\beta}_{loc}(I\!R^{d+r})$, $\beta > 0$. Assume that (v_n), $n \in \mathbf{N}$, is a sequence bounded in $L^\infty_{loc}(I\!R^{d+r'})$.*

Then, there exist subsequences $(u_{n'}) \subset (u_n)$ and $(v_{n'}) \subset (v_n)$ such that there exists a complex valued functional μ such that

$$\mu \in \left(L^{\beta^*}_0(I\!R^{d+r+r'}; C^k(S^{d-1})) \right)^*,$$

for a β^* satisfying $\frac{1}{1+\beta} + \frac{1}{\beta^*} < 1$, where $\left(L^{\beta^*}_0(I\!R^d; C^k(S^{d-1})) \right)^*$ is the set of bounded linear functionals over $L^{\beta^*}_0(I\!R^{d+r+r'}; C^k(S^{d-1}))$, such that for every $\varphi_1 \in L^{\beta^*}_0(I\!R^{d+r})$, $\varphi_2 \in C_0(I\!R^d)$, $\rho_2 \in L^{\beta^*}(I\!R^{r'})$, and $\psi \in C^\kappa(S^{d-1})$, $\kappa > d/2$:

$$\begin{aligned} \lim_{n' \to \infty} \int_{I\!R^{d+r+r'}} (\varphi_1 u^i_{n'})(x,\lambda) \mathcal{A}_\psi[\rho_2(\tilde{\lambda})\varphi_2(\cdot) v^j_{n'}(\cdot,\tilde{\lambda})](x) dx d\lambda d\tilde{\lambda} \\ = \langle \mu, \varphi_1 \varphi_2 \rho_2 \psi \rangle. \end{aligned} \tag{15}$$

where $\mathcal{A}_\psi : L^p(I\!R^d_x) \to L^p(I\!R^d_x)$ is a multiplier operator with the symbol $\psi(\xi)$, $\xi \in S^{d-1}$.

The functional μ we call H–distribution corresponding to (u_n) and (v_n).

Remark 15. Notice that in the case of our H-distributions we can not formulate the theorem via the Fourier transform since we cannot estimate $\mathcal{F}(u)$ if $u \in L^p$ for $p > 2$ (Hausdorff-Young inequality [28] holds only for $1 \leq p \leq 2$).

The proof of the theorem follows the steps from the proof of [35, Theorem 1.1]. Thus, we shall need a variant of Tartar's first commutation lemma. To formulate it, we need the following operators:

Let $a \in C^\kappa(S^{d-1})$, $\kappa > d/2$, and b continuous function with compact support defined on $I\!R^d$. We associate to a and b linear continuous operators $\mathcal{A}$ and B on $L^p(I\!R^d)$, $p > 1$ arbitrary, by the formulae:

$$\mathcal{F}(\mathcal{A}u)(\xi) = a(\frac{\xi}{|\xi|})\mathcal{F}(u)(\xi) \quad a.e. \ \ \xi \in I\!R^d, \tag{16}$$

$$Bu(x) = b(x)u(x) \quad a.e. \ \ x \in I\!R^d, \tag{17}$$

where $\chi_{B(0,l)}$ is characteristic function of the ball $B(0,l)$. These operators are bounded operators on $L^p(I\!R)$, $p > 1$ (see (13)). We have the following lemma:

Lemma 16. *(First commutation lemma) $C = \mathcal{A}B - B\mathcal{A}$ is a compact operator from $L^\infty_0(I\!R^d)$ into $L^{p_0}_{loc}(I\!R^d)$ for every $p_0 \geq 2$.*

Proof: First, notice that we can assume $b \in C_0^1(I\!R^d)$. Indeed, if we assume merely $b \in C_0(I\!R^d)$ then we can uniformly approach the function b by a sequence $(b_n) \in C_0^1(I\!R^d)$. The corresponding sequence of commutators $C_n = \mathcal{A}B_n - B_n\mathcal{A}$, where $B_n(u) = b_n u$, converges in norm toward C. So, if we prove that C_n are compact for each n the same will hold for C as well.

On the first step notice that according to (13) we have:

$$\|C\| \leq 2const\|a\|_{C^\kappa(S^{d-1})}\|b\|_{L^\infty(I\!R^d)},$$

where *const* is a constant independent on the symbol a and the function b.

Then, fix a real non-negative function ω with compact support and total mass one. Take the characteristic function $\chi_{B(0,2)}$ of the ball $B(0,2) \subset I\!R^d$ and denote:

$$\chi^\varepsilon_{B(0,2)} = \chi_{B(0,2)} \star \frac{1}{\varepsilon^d}\,\Pi_{i=1}^d\,\omega(\frac{x_i}{\varepsilon}), \quad \varepsilon > 0.$$

Choose an $\varepsilon > 0$ small enough so that we have $\chi^\varepsilon_{B(0,2)}(x) = 1$ for $x \in B(0,1)$, and $(1 - \chi^\varepsilon_{B(0,2)}) \equiv 1$ out of the ball $B(0,2)$.

Next, notice that

$$\mathcal{A} = \mathcal{A}_{a\chi^\varepsilon_{B(0,2)}} + \mathcal{A}_{a(1-\chi^\varepsilon_{B(0,2)})},$$

where $\mathcal{A}_{a\chi^\varepsilon_{B(0,2)}}$ is the multiplier operator with the symbol $a\chi^\varepsilon_{B(0,2)}$, and $\mathcal{A}_{a(1-\chi^\varepsilon_{B(0,2)})}$ is the multiplier operator with the symbol $a(1 - \chi^\varepsilon_{B(0,2)})$.

Accordingly,

$$\begin{aligned} C = \mathcal{A}B - B\mathcal{A} &= \mathcal{A}_{a\chi^\varepsilon_{B(0,2)}}B - B\mathcal{A}_{a\chi^\varepsilon_{B(0,2)}} \\ &\quad + \mathcal{A}_{a(1-\chi^\varepsilon_{B(0,2)})}B - B\mathcal{A}_{a(1-\chi^\varepsilon_{B(0,2)})} = C_{a\chi^\varepsilon_{B(0,2)}} + C_{a(1-\chi^\varepsilon_{B(0,2)})}, \end{aligned}$$

where

$$\begin{aligned} C_{a\chi^\varepsilon_{B(0,2)}} &= \mathcal{A}_{a\chi^\varepsilon_{B(0,2)}}B - B\mathcal{A}_{a\chi^\varepsilon_{B(0,2)}}, \\ C_{a(1-\chi^\varepsilon_{B(0,2)})} &= \mathcal{A}_{a(1-\chi^\varepsilon_{B(0,2)})}B - B\mathcal{A}_{a(1-\chi^\varepsilon_{B(0,2)})}. \end{aligned}$$

We shall consider separately the commutators $C_{a(1-\chi^\varepsilon_{B(0,2)})}$ and $C_{a\chi^\varepsilon_{B(0,2)}}$.

First, notice that since $a(1-\chi^\varepsilon_{B(0,2)})$ has compact support the multiplier $\mathcal{A}_{a(1-\chi^\varepsilon_{B(0,2)})}$ is actually the convolution operator with the kernel $\psi_\varepsilon(x) = \bar{\mathcal{F}}(a(1-\chi^\varepsilon_{B(0,2)}))(x) \in L^2(I\!R^d)$:

$$\mathcal{A}_{a(1-\chi^\varepsilon_{B(0,2)})}(u) = \psi_\varepsilon \star u, \quad u \in L^p(I\!R^d). \tag{18}$$

Therefore, we can state that

$$C_{a(1-\chi^\varepsilon_{B(0,2)})}u(x) = \int_{I\!R^d} (b(x) - b(y))\,\psi(x-y)u(y)dy, \tag{19}$$

is compact operator from $(L_0^\infty(I\!R^d), \|\cdot\|_{L^\infty})$ into $L^{p_0}_{loc}(I\!R^d)$, $p_0 \geq 2$.

Indeed, take an arbitrary sequence $(u_n) \subset L_0^\infty(I\!R^d)$ such that $u_n \rightharpoonup 0$ weak-$\star$ in $L^\infty(I\!R^d)$ and $\mathrm{supp} u_n \subset \hat{V} \subset\subset I\!R^d$, for a compact set $\hat{V}$. In order to prove that $C_{a(1-\chi^\varepsilon_{B(0,2)})}$ is compact, it is enough to prove that $C_{a(1-\chi^\varepsilon_{B(0,2)})}u_n$ strongly converges to zero in $L^{p_0}_{loc}(I\!R^d)$.

Since $\psi_\varepsilon \in L^2(I\!R^d)$ we also have $\psi \in L^1_{loc}(I\!R^d)$. Thus, for every fixed $x \in I\!R^d$ we have

$$C_{a(1-\chi^\varepsilon_{B(0,2)})}u_n(x) = \int_{\hat{V}} (b(x) - b(y))\,\psi(x-y)u_n(y)dy \to 0, \quad n \to 0. \tag{20}$$

Next, since the sequence (u_n) has compact support we also have:

$$|C_{a(1-\chi^{\varepsilon}_{B(0,2)})}u_n(x)| \leq \hat{C}, \tag{21}$$

for a constant $\hat{C}$ depending on the support of the sequence (u_n) as well as L^2 norm of the kernel ψ.

Combining (20) and (21) with Lebesgue dominated convergence theorem we have for an arbitrary compact $V \subset\subset I\!R^d$ and every $p_0 > 0$:

$$\int_V |C_{a(1-\chi^{\varepsilon}_{B(0,2)})}u_n(x)|^{p_0} dx \to 0, \quad n \to \infty, \tag{22}$$

proving (19).

In order to prove that $C_{a\chi^{\varepsilon}_{B(0,2)}}$ is compact we need more subtle arguments basically involving techniques from the proof of the Hormander-Mikhlin theorem from e.g. [28].

So, let Θ be a non-negative function with support in the set $\{\xi \in R^n : \frac{1}{2} \leq \|\xi\| \leq 2\}$, which is infinitely differentiable and is such that $\Theta(\xi) > 0$ when $2^{-\frac{1}{2}} \leq \|\xi\| \leq 2^{\frac{1}{2}}$. Also let

$$\theta(\xi) = \Theta(\xi) \Big/ \sum_{j=-\infty}^{\infty} \Theta(2^{-j}\xi).$$

Then, θ is non-negative, has support in the set $\{\xi \in R^n : \frac{1}{2} \leq \|\xi\| \leq 2\}$, is infinitely differentiable and is such that if $\xi \neq 0$, then

$$\sum_{j=-\infty}^{\infty} \theta(2^{-j}\xi) = 1.$$

Now, let $a_j(\xi) = a(\xi/|\xi|)(1-\chi^{\varepsilon}_{B(0,2)}(\xi))\theta(2^{-j}\xi)$, $j > 0$. Then, a_j has support in the set

$$\{\xi \in R^n : 2^{j-1} \leq \|\xi\| \leq 2^{j+1}\}, \quad j > 0,$$

and

$$a(\xi/|\xi|)(1-\chi^{\varepsilon}_{B(0,2)}(\xi)) = \sum_{j=0}^{\infty} a_j(\xi).$$

By $\bar{a}_j$ we denote the inverse Fourier transform of the function a_j:

$$\bar{a}_j(x) = \bar{\mathcal{F}}(a_j)(x).$$

Notice that $a \in C^{\kappa}(I\!R^d)$ satisfies condition (10) (see also Corollary 11). From here, the Cauchy-Schwartz inequality, Plancharel's theorem and the well known properties of the Fourier transform, for every $s > 0$ we have (see also the proof of

[28, Theorem 7.5.13] and [9, Theorem 8]):

$$\begin{aligned}\int_{\|x\|>s}|\bar a_j(x)|dx &\leq \Big(\int_{\|x\|\geq s}\|x\|^{-2\kappa}dx\Big)^{1/2}\Big(\int_{\|x\|\geq s}\|x\|^{2\kappa}|\bar a_j(x)|^2 d\xi\Big)^{1/2} \\ &\leq \Big(\int_{\|x\|\geq s}\|x\|^{-2\kappa}dx\Big)^{1/2}\Big(\int_{I\!R^d}\|x\|^{2\kappa}|\bar a_j(x)|^2 dx\Big)^{1/2} \\ &\leq \Big(\frac{2\pi^{d-1}s^{d-2\kappa}}{2\kappa-d}\Big)^{1/2}\Big(2\kappa\sum_{i=1}^d\int_{I\!R^d}|x_i|^{2\kappa}|\bar a_j(x)|^2dx\Big)^{1/2} \\ &= \Big(\frac{2\pi^{d-1}s^{d-2\kappa}}{2\kappa-d}\Big)^{1/2}\Big(2\kappa\sum_{i=1}^d\int_{I\!R^d}|D^\kappa_{\xi_i}a_j(\xi)|^2d\xi\Big)^{1/2} \\ &\leq C_1\|a\|_{C^k(S^{d-1})}(2^js)^{(\frac12 d-\kappa)},\end{aligned} \tag{23}$$

for a constant C_1 depending on the functions θ and $\chi^\varepsilon_{B(0,2)}$.

Next, consider the functional series

$$\bar A_n(x)=\sum_{j=0}^n \bar a_j(x).$$

For an arbitrary fixed $\varepsilon>0$, the series $\sum_{j=-n}^n \bar a_j(x)$ is absolutely convergent in $L^1(I\!R^d\backslash B(0,\varepsilon))$. Indeed, we have

$$\begin{aligned}\|\bar A_n(x)\|_{L^1(I\!R^d\backslash B(0,\varepsilon))} &\leq \sum_{j=0}^n\|\bar a_j\|_{L^1(I\!R^d\backslash B(0,\varepsilon))} \\ &\leq C_1\|a\|_{C^k(S^{d-1})}s^{(\frac12 d-\kappa)}\sum_{j=0}^n 2^{j(\frac12 d-\kappa)}\leq C_3<\infty,\end{aligned} \tag{24}$$

for a constant $C_3>0$, since $\frac12 d-\kappa\leq 0$.

Thus, for every $\varepsilon>0$ there exists $\bar A_\varepsilon\in L^1(I\!R^d\backslash B(0,\varepsilon))$ such that

$$\sum_{j=0}^\infty \bar a_j(x)=\bar A_\varepsilon(x),\quad x\in I\!R^d\backslash B(0,\varepsilon). \tag{25}$$

Similarly, from (23) we have

$$\int_{\|x\|<s}\|x\|\cdot|\bar A_n(x)|dx\to 0,\quad s\to 0, \tag{26}$$

since $\frac12 d-\kappa\geq -1$.

Now, take the convolution operator:

$$\mathcal{A}_n(u)=\bar A_n\star u,\quad u\in L^\infty_0(I\!R^d).$$

and consider the commutator

$$C_n=\mathcal{A}_nB-B\mathcal{A}_n.$$

It holds:

$$C_n(u)(x)=\int_{I\!R^d}\bar A_n(x-y)(b(x)-b(y))u(y)dy.$$

Given a fixed $\varepsilon > 0$ rewrite $C_n(u)$ in the following way:

$$
\begin{aligned}
C_n(u)(x) &= \int_{\mathbb{R}^d} \bar{A}_n(x-y)(b(x)-b(y))u(y)dy \\
&= \int_{\|x\|>\varepsilon} \bar{A}_n(x-y)(b(x)-b(y))u(y)dy + \int_{\|x\|\leq\varepsilon} \bar{A}_n(x-y)(b(x)-b(y))u(y)dy.
\end{aligned}
$$

From here, using the fact that $b \in C_0^1(\mathbb{R}^d)$ together with (25) and (26) we conclude

$$
\limsup_{n\to\infty} \|C_n(u)(x)\|_{L^{p_0}_{loc}(\mathbb{R}^d)} \leq \| \int_{\|x\|>\varepsilon} \bar{A}^\varepsilon(x-y)(b(x)-b(y))u(y)dy\|_{L^{p_0}_{loc}(\mathbb{R}^d)} + o_\varepsilon(1), \tag{27}
$$

where

$$
\begin{aligned}
o_\varepsilon(1) &= \int_{\|x\|\leq\varepsilon} \bar{A}_n(x-y)(b(x)-b(y))u(y)dy \\
&= \int_{\|x\|\leq\varepsilon} \|x-y\|\bar{A}_n(x-y)\frac{(b(x)-b(y))}{\|x-y\|}u(y)dy \to^{(26)} 0, \quad \varepsilon \to 0.
\end{aligned}
$$

Furthermore, notice that the operator

$$
u \mapsto \int_{\|x\|>\varepsilon} \bar{A}^\varepsilon(x-y)(b(x)-b(y))u(y)dy
$$

is a compact operator from $L^\infty_0(\mathbb{R}^d)$ to $L^{p_0}_{loc}(\mathbb{R}^d)$ for an arbitrary $p_0 > 1$.

Then, as in the final steps of the proof of [28, Theorem 7.5.13], from the Fatou's lemma one can conclude that for any $p_0 > 1$ there exists a subsequence $(C_{n_k}) \subset (C_n)$ such that:

$$
\begin{aligned}
\|C_{a(1-\chi^\varepsilon_{B(0,2)})}u\|_{L^{p_0}_{loc}(\mathbb{R}^d)} &\leq \limsup_{k\to\infty} \|C_{n_k}u\|_{L^{p_0}_{loc}(\mathbb{R}^d)} \\
&\leq \| \int_{\|x\|>\varepsilon} A^\varepsilon(x-y)(b(x)-b(y))u(y)dy\|_{L^{p_0}_{loc}(\mathbb{R}^d)} + o_\varepsilon(1).
\end{aligned}
$$

This actually means that the commutator $C_{a(1-\chi^\varepsilon_{B(0,2)})}$ can be bounded by the sum of a compact operator and a bounded operator whose norm is arbitrarily small. This implies that $C_{a(1-\chi^\varepsilon_{B(0,2)})}$ is the compact operator from $L^\infty_0(\mathbb{R}^d), \|\cdot\|_\infty$ to $L^{p_0}_{loc}(\mathbb{R}^d)$ for an arbitrary $p_0 > 1$.

Thus we see that C can be represented as sum of two compact operators $C_{a\chi^\varepsilon_{B(0,2)}}$ and $C_{a(1-\chi^\varepsilon_{B(0,2)})}$, which means that $\mathcal{A}$ is compact operator itself.

This concludes the proof. ❑

Now, we are able to prove Theorem 14.

Proof of Theorem 14:

Since $(v_n) \in L^\infty_{loc}(\mathbb{R}^{d+r'})$ it follows that $(v_n) \in L^{\beta^*}_{loc}(\mathbb{R}^{d+r'})$ for every $\beta^* \geq 1$. Therefore, there exists a subsequence $(v_{n'})$ such that we have

$$
v_{n'} \rightharpoonup v \in L^\infty_{loc}(\mathbb{R}^{d+r'}) \ \text{ in } L^{\beta^*}_{loc}(\mathbb{R}^{d+r'}). \tag{28}
$$

From here, since $u_{n'} \rightharpoonup 0$, $n' \to \infty$, in $L^{1+\beta}_{loc}(I\!\!R^{d+r})$ and $\varphi_1 \mathcal{A}_\psi[\varphi_2\rho_2 v] \in L_0^{\beta^*}(I\!\!R^{d+r+r'})$ for any $\varphi_1 \in L_0^{\beta^*}(I\!\!R^{d+r})$, $\varphi_2 \in C_0(I\!\!R^d)$, $\tilde\rho \in L^{\beta^*}(I\!\!R^{r'})$, we have:

$$\begin{aligned}
&\lim_{n'\to\infty} \int_{I\!\!R^{d+r+r'}} \varphi_1(x,\lambda) u_{n'}(x,\lambda) \overline{\mathcal{A}_\psi[\varphi_2(\cdot)\tilde\rho(\tilde\lambda) v_{n'}(\cdot,\tilde\lambda)](x)} dx d\lambda d\tilde\lambda \qquad (29)\\
&= \lim_{n'\to\infty} \int_{I\!\!R^{d+r+r'}} \varphi_1(x,\lambda) u_{n'}(x,\lambda) \mathcal{A}_\psi[\varphi_2(\cdot)\tilde\rho(\tilde\lambda)(v_{n'}-v)(\cdot,\tilde\lambda)](x) dx d\lambda \tilde\lambda,
\end{aligned}$$

where $\mathcal{A}_\psi$ is the multiplier operator with the symbol $\psi(\xi/|\xi|)$, $\psi \in C^\kappa(I\!\!R^d)$.

Assume that we have supp$\varphi_2 \subset B(0,l) \subset I\!\!R^{d+r'}$. Then, from (29) and Lemma 16 we have:

$$\begin{aligned}
&\lim_{n'\to\infty} \int_{I\!\!R^{d+r+r'}} \varphi_1(x,\lambda) u_{n'}(x,\lambda) \mathcal{A}_\psi[\varphi_2(\cdot)\tilde\rho(\tilde\lambda) v_{n'}(\cdot,\tilde\lambda)](x) dx d\lambda d\tilde\lambda \qquad (30)\\
&= \lim_{n'\to\infty} \int_{I\!\!R^{d+r+r'}} \varphi_1(x,\lambda) u_{n'}(x,\lambda) \mathcal{A}_\psi[\varphi_2(\cdot)\tilde\rho(\tilde\lambda)\chi_{B(0,l)}(\cdot)(v_{n'}-v)(\cdot,\tilde\lambda)](x) dx d\lambda d\tilde\lambda\\
&= \lim_{n'\to\infty} \int_{I\!\!R^{d+r+r'}} \varphi_1(x,\lambda)\varphi_2(x)\tilde\rho(\tilde\lambda) u_{n'}(x,\lambda) \mathcal{A}_\psi[\chi_{B(0,l)}(\cdot)(v_{n'}-v)(\cdot,\tilde\lambda)](x) dx d\lambda d\tilde\lambda\\
&= \lim_{n\to\infty} \int_{I\!\!R^{d+r+r'}} \varphi_1(x,\lambda)\varphi_2(x)\tilde\rho(\tilde\lambda) u_{n'}(x,\lambda) \mathcal{A}_\psi[\chi_{B(0,l)}(\cdot) v_{n'}(\cdot,\tilde\lambda)](x) dx,
\end{aligned}$$

where we implicitly used the Lebesgue dominated convergence theorem.

From here, denoting $\varphi(x,\lambda) = \varphi_1(x,\lambda)\varphi_2(x)$, and Corollary 11 we see that the functional

$$\mu_{n,l}(\varphi,\tilde\rho,\psi) = \lim_{n'\to\infty} \int_{I\!\!R^{d+r+r'}} \varphi(x,\lambda)\tilde\rho(\tilde\lambda) u_{n'}(x,\lambda) \mathcal{A}_\psi[(\chi_{B(0,l)} v_{n'})(\cdot,\tilde\lambda)](x) dx d\lambda d\tilde\lambda \qquad (31)$$

is linear with respect to $\varphi \in L_0^{\beta*}(I\!\!R^{d+r})$, $\tilde\rho \in L^{\beta^*}(I\!\!R^{r'})$ and $\psi \in C^\kappa(S^{d-1})$, and bounded with $\|\varphi\|_{L^{\beta^*}(I\!\!R^{d+r'})} \|\tilde\rho\|_{L^{\beta^*}(I\!\!R^{r})} \|\psi\|_{C^\kappa(S^{d-1})}$. Indeed, using the Holder inequality, inequality between means (8), and Corollary 11:

$$\begin{aligned}
&\Big| \int_{I\!\!R^{d+r+r'}} \varphi(x,\lambda)\tilde\rho(\tilde\lambda) u_{n'}(x,\lambda) \mathcal{A}_\psi[(\chi_{B(0,l)} v_{n'})(\cdot,\tilde\lambda)](x) dx d\lambda d\tilde\lambda \Big|\\
&\leq C \|\psi\|_{C^\kappa(S^{d-1})} \|\varphi\|_{L^{\beta^*}(I\!\!R^d)} \|u_{n'}\|_{L^{1+\beta}(\text{supp}\varphi)} \|v_{n'}\|_{L^\theta(B(0,l))}\\
&\leq \tilde C \|\psi\|_{C^\kappa(S^{d-1})} \|\varphi\|_{L^{\beta^*}(I\!\!R^{d+r})} \|\tilde\rho\|_{L^{\beta^*}(I\!\!R^{r'})},
\end{aligned}$$

where $\frac{1}{1+\beta} + \frac{1}{\beta^*} + \frac{1}{\theta} = 1$, and C and $\hat C$ are appropriate constants.

Using the weak precompactness property of the space $\left(L_0^{\beta^*}(I\!\!R^{d+r}) \times L_0^{\beta^*}(I\!\!R^{r'}) \times C^\kappa(S^{d-1})\right)^*$ (Banach-Alaoglu theorem) we conclude that there exists a $\mu_l \in \left(L_0^{\beta^*}(I\!\!R^{d+r}) \times L_0^{\beta^*}(I\!\!R^{r'}) \times C^\kappa(S^{d-1})\right)^*$ such that along subsequences (not relabeled) $(u_{n'}) \subset (u_n)$ and $(v_{n'}) \subset (v_n)$ we have for $\varphi_1 \in L_0^{\beta^*}(I\!\!R^{d+r})$, $\tilde\rho \in L_0^{\beta^*}(I\!\!R^{r'})$, and $\varphi_2 \in C_0(I\!\!R^d)$, supp$\varphi_2\tilde\rho \subset B(0,l)$ (compare (31) when $\varphi = \varphi_1\varphi_2$ with (30)):

$$\lim_{n'\to\infty} \int_{I\!\!R^{d+r+r'}} (\varphi_1 u_{n'})(x,\lambda) \mathcal{A}[\varphi_2(\cdot)\tilde\rho(\tilde\lambda) v_{n'}(\cdot,\tilde\lambda)](x) dx d\lambda d\tilde\lambda = \langle \mu_l, \varphi_1\varphi_2\tilde\rho\psi\rangle.$$

Choosing $l \in I\!\!N$ (or some other countable set) we can assume that the same subsequences $(u_{n'})$ and $(v_{n'})$ define the distributions μ_l for any $l \in I\!\!N$.

Standard arguments involving e.g. Cantor diagonalization procedure show that there exists μ so that for every $\psi \in C^{\kappa}(S^{d-1})$, $\tilde{\rho} \in L_0^{\beta^*}(I\!R^{r'})$, and $\varphi \in L_0^{\beta^*}(I\!R^{d+r})$ such that for every $p \in I\!R^r$ we have supp$\varphi \subset B(0,l)$:

$$\langle \mu, \varphi\tilde{\rho}\psi\rangle = \langle \mu_l, \varphi\tilde{\rho}\psi\rangle, \quad l \in \mathbf{N}.$$

Clearly, the latter μ satisfies (15).

Now, using the Schwartz kernel theorem we can conclude that μ is a distribution in variables $(x,\lambda,\tilde{\lambda}) \in I\!R^{d+r+r'}$ and $\xi \in S^{d-1}$. Still, we assert that the functional μ is much more than that, and we need to explicitly extend the functional $\mu \in \Big(L_0^{\beta^*}(I\!R^{d+r+r'}) \times C^{\kappa}(S^{d-1})\Big)^*$ on a functional $\mu \in \Big(L_0^{\beta^*}(I\!R^{d+r+r'}; C^{\kappa}(S^{d-1}))\Big)^*$.

First, fix once and for all the partition of the space $I\!R^{d+r}$ on disjoint cubes K_i^n, $i \in Z\!\!\!Z$, with the edge length $\frac{1}{2^{n(r+d)}}$, $n \in I\!N$. Assume that the partition K_i^{n+1}, $i \in Z\!\!\!Z$, is obtained by partitioning the cubes K_i^n, $i \in I\!N$, on $\frac{1}{2^{r+d}}$ equal cubes. In the completely same manner we fix a partitioning of $I\!R^{r'}$ on disjoint cubes $\tilde{K}_i^n$, $i \in Z\!\!\!Z$, with the edge length $\frac{1}{2^{r'd}}$.

By χ_i^n we denote the characteristic function of the cube K_i^n, and by $\tilde{\chi}_j^n$ we denote the characteristic function of the cube $\tilde{K}_j^n$.

Then, take an arbitrary $\phi \in L_0^{\beta^*}(I\!R^{d+r+r'}; C^{\kappa}(S^{d-1})) \cap C^1(I\!R^{d+r+r'}; C^{\kappa}(S^{d-1}))$, $\phi = \phi(x,\lambda,\tilde{\lambda},\xi)$, $(x,\lambda,\tilde{\lambda}) \in I\!R^{d+r+r'}$, $\xi \in S^{d-1}$. Furthermore, assume that for every $n \in I\!N$, every $\xi \in S^{d-1}$, and some $N(n) \in I\!N$:

$$\text{supp}\phi(\cdot,\cdot,\cdot,\xi) \subset V \subset \bigcup_{i,j=-N(n)}^{N(n)} K_i^n \times \tilde{K}_j^n. \tag{32}$$

Denote by $\Phi_1 \in C_0^1(I\!R^d)$ the function such that $\Phi_1(x) \equiv 1$, $x \in \text{proj}_x V$, where $\text{proj}_x V$ is projection of the set V on the x-subspace $I\!R^d \times \{0\} \subset I\!R^{d+r+r'}$.

Denote by $\Phi_2 \in C_0^1(I\!R^r)$ the function such that $\Phi_2(\lambda) = 1$, $\lambda \in \text{proj}_\lambda V$ is projection of the set V on the λ-subspace $I\!R^r \times \{0\} \subset I\!R^{d+r+r'}$.

Denote by $\Phi_3 \in C_0^1(I\!R^{r'})$ the function such that $\Phi_3(\tilde{\lambda}) = 1$, $\tilde{\lambda} \in \text{proj}_{\tilde{\lambda}} V$ is projection of the set V on the $\tilde{\lambda}$-subspace $I\!R^{r'} \times \{0\} \subset I\!R^{d+r+r'}$.

Denote by

$$\phi_n(x,\lambda,\tilde{\lambda},\xi) = \sum_{i,j=-N}^{N} \phi(x_i^n,\lambda_i^n,\tilde{\lambda}_j^n,\xi)\chi_i^n(x,\lambda)\tilde{\chi}_j^n(\tilde{\lambda}),$$

where $(x_i^n,\lambda_i^n) \in K_i^n$ is center of the cube $\tilde{\lambda}_j^n \in \tilde{K}_i^n$, and $N = N(n) \in I\!N$ is such that (32) is satisfied.

Now, we define:

$$\langle \mu, \phi(x,\lambda,\tilde{\lambda},\xi)\rangle := \lim_{N\to\infty} \sum_{i,j=-N}^{N} \langle \mu, \Phi_1^2(x)\Phi_2(\lambda)\chi_i^n(x,\lambda)\Phi_3(\tilde{\lambda})\tilde{\chi}_j^n(\tilde{\lambda})\phi(x_i^n,\lambda_i^n,\tilde{\lambda}_j^n,\xi)\rangle. \tag{33}$$

To prove that (33) is well defined extension, we need to prove that the sequence $\Big(\sum\limits_{i,j=-N}^{N} \langle \mu, \Phi_1^2(x)\Phi_2(\lambda)\chi_i^n(x,\lambda)\Phi_3(\tilde{\lambda})\tilde{\chi}_j^n(\tilde{\lambda})\phi(x_i^n,\lambda_i^n,\tilde{\lambda}_j^n,\xi)\rangle\Big)_{n\in I\!N}$ is convergent.

We shall use the definition of the functional $\mu \in \left(L_0^{\beta^*}(\mathbb{R}^{d+r+r'}) \times C^\kappa(S^{d-1})\right)^*$, and the Cauchy criterion. Accordingly, assume that $n \geq m$, and consider:

$$
\begin{aligned}
&\Big|\sum_{i,j=-N}^{N} \langle \mu, \Phi_1^2(x)\Phi_2(\lambda)\chi_i^n(x,\lambda)\Phi_3(\tilde\lambda)\tilde\chi_j^n(\tilde\lambda)\phi(x_i^n,\lambda_i^n,\tilde\lambda_j^n,\xi)\rangle \\
&-\sum_{i,j=-M}^{M} \langle \mu, \Phi_1^2(x)\Phi_2(\lambda)\chi_i^m(x,\lambda)\Phi_3(\tilde\lambda)\tilde\chi_j^m(\tilde\lambda)\phi(x_i^m,\lambda_i^m,\tilde\lambda_j^m,\xi)\rangle\Big| \\
&=\Big|\sum_{i,j=-N}^{N} \langle \mu, \Phi_1^2(x)\Phi_2(\lambda)\chi_i^n(x,\lambda)\Phi_3(\tilde\lambda)\tilde\chi_j^n(\tilde\lambda)\left(\phi(x_i^n,\lambda_i^n,\tilde\lambda_j^n,\xi)-\phi(x_{i(n)}^m,\lambda_{i(n)}^m,\tilde\lambda_{j(n)}^m,\xi)\right)\rangle\Big|,
\end{aligned}
\tag{34}
$$

where $M = M(m)$ and $N = N(n)$ are such that (32) is satisfied. The point $(x_{i(n)}^m, \lambda_{i(n)}^m)$ is such that $(x_{i(n)}^m, \lambda_{i(n)}^m) = (x_i^m, \lambda_i^m)$ as long as $K_i^n \subset K_i^m$, and the point $\tilde\lambda_{j(n)}^m$ is such that $\tilde\lambda_{j(n)}^m = \tilde\lambda_j^m$ as long as $\tilde K_j^n \subset \tilde K_j^m$.

Let $\frac{1}{1+\beta} + \frac{1}{\tilde\beta} = 1$ for $\tilde\beta > 1$.

By (34) and definition of the functional μ we have

$$
\begin{aligned}
&\Big|\sum_{i,j=-N}^{N} \langle \mu, \Phi_1^2(x)\Phi_2(\lambda)\chi_i^n(x,\lambda)\Phi_3(\tilde\lambda)\tilde\chi_j^n(\tilde\lambda)\phi(x_i^n,\lambda_i^n,\tilde\lambda_j^n,\xi)\rangle \\
&-\sum_{i,j=-M}^{M} \langle \mu, \Phi_1^2(x)\Phi_2(\lambda)\chi_i^m(x,\lambda)\Phi_3(\tilde\lambda)\tilde\chi_j^m(\tilde\lambda)\phi(x_i^m,\lambda_i^m,\tilde\lambda_j^m,\xi)\rangle\Big| \\
&=\Big|\sum_{i,j=-N}^{N} \lim_{k\to\infty}\int_{\mathbb{R}^{d+r+r'}} \Phi_1(x)\Phi_2(\lambda)(u_k\chi_i^n)(x,\lambda)\times \\
&\qquad \times \mathcal{A}_{\left(\phi(x_i^n,\lambda_i^n,\tilde\lambda_j^n,\frac{\xi}{|\xi|})-\phi(x_{i(n)}^m,\lambda_{i(n)}^m,\tilde\lambda_{j(n)}^m,\frac{\xi}{|\xi|})\right)}[\Phi_1(\cdot)\Phi_2(\tilde\lambda)\tilde\chi_j^m(\tilde\lambda)v_k(\cdot,\tilde\lambda)](x)\,dx\,d\lambda\,d\tilde\lambda\Big| \\
&\leq \lim_{k\to\infty}\sum_{i,j=-N}^{N} \|\Phi_1\Phi_2 u_k^i\chi_i^n\|_{L^{1+\beta}(\mathbb{R}^{d+r})}\|\Phi_1\Phi_3 v_k\tilde\chi_j^n(\tilde\lambda)\|_{L^{\tilde\beta}(\mathbb{R}^{d+r'})}\times \\
&\qquad \times \|\phi(x_i^n,\lambda_i^n,\tilde\lambda_j^m,\xi)-\phi(x_{i(n)}^m,\lambda_{i(n)}^m,\tilde\lambda_{j(n)}^m,\xi)\|_{C^\kappa(S^{d-1})} \\
&\leq \lim_{k\to\infty}\|\Phi_1\Phi_2 u_k\|_{L^{1+\beta}(\mathbb{R}^{d+r})}\|\Phi_1\Phi_3 v_k\|_{L^{\tilde\beta}(\mathbb{R}^{d+r'})}\times \\
&\qquad \times \sum_{i,j=-N}^{N}\|(x_i^n,\lambda_i^n,\tilde\lambda_j^n)-(x_{i(n)}^m,\lambda_{i(n)}^m,\tilde\lambda_{j(n)}^m)\|\sum_{\alpha=1}^{d}\|\partial_{x_\alpha}\phi(x,\lambda,\tilde\lambda,\xi)\|_{C^0(\mathbb{R}^d;C^\kappa(S^{d-1}))} \\
&= C\frac{\sum_{\alpha=1}^{d}\|\partial_{x_\alpha}\phi(x,\xi)\|_{C^0(\mathbb{R}^d;C^\kappa(S^{d-1}))}}{2^M},
\end{aligned}
$$

since $\|(x_i^n,\lambda_i^n,\tilde\lambda_j^n)-(x_{i(n)}^m,\lambda_{i(n)}^m,\tilde\lambda_{i(n)}^m)\|\leq \frac{1}{2^M}$ (according to the definition of the cubes K_i^n and $\tilde K_j^n$). Above we have

$$\begin{aligned}&\sum_{i,j=-N}^{N}\|\Phi_1\Phi_2u_k^i\chi_i^n\|_{L^{1+\beta}(I\!R^{d+r})}\|\Phi_1\Phi_3v_k\tilde\chi_j^n(\tilde\lambda)\|_{L^{\tilde\beta}(I\!R^{d+r'})}\\&=\|\Phi_1\Phi_2u_k\|_{L^{1+\beta}(I\!R^{d+r})}\|\Phi_1\Phi_3v_k\|_{L^{\tilde\beta}(I\!R^{d+r'})}\leq\hat C,\end{aligned}$$

for a constant $\hat C$.

Thus, we see that $(\sum\limits_{i=-N}^{N}\langle\mu,\Phi_1^2(x)\Phi_2(\lambda)\chi_i^n(x,\lambda)\Phi_3(\tilde\lambda)\tilde\chi_j^n(\tilde\lambda)\phi(x_i^n,\lambda_i^n,\tilde\lambda_j^n,\xi)\rangle)$ is Cauchy sequence implying that extension (33) is well defined on the space $\Big(C^1(I\!R^d;C^\kappa(S^{d-1})),\|\cdot\|_{L_0^{\beta^*}(I\!R^d;C^\kappa(S^{d-1}))}\Big)$.

In order to prove that the functional μ, can be extended on $L_0^{\beta^*}(I\!R^{d+r+r'};C^\kappa(S^{d-1}))$, it is enough to prove that $\Big(C^1(I\!R^{d+r+r'};C^\kappa(S^{d-1})),\|\cdot\|_{L_0^{\beta^*}(I\!R^{d+r+r'};C^\kappa(S^{d-1}))}\Big)$ is dense in $L_0^{\beta^*}(I\!R^{d+r+r'};C^\kappa(S^{d-1}))$.

Take an arbitrary $\phi\in L_0^{\beta^*}(I\!R^{d+r+r'};C^\kappa(S^{d-1}))$ and consider the family of convolutions

$$\begin{aligned}\phi_\varepsilon(x,\lambda,\tilde\lambda,\xi)=\int_{I\!R^{d+r+r'}}&\phi(y,\eta,\tilde\eta,\xi)\frac{1}{\varepsilon^{d+r+r'}}\,\Pi_{i=1}^d\,\omega(\frac{x_i-y_i}{\varepsilon})\times\\&\times\Pi_{i=1}^r\,\omega(\frac{\lambda_i-\eta_i}{\varepsilon})\,\Pi_{i=1}^{r'}\,\omega(\frac{\tilde\lambda_i-\tilde\eta_i}{\varepsilon})dyd\eta d\tilde\eta,\end{aligned}$$

where ω is real non-negative smooth function with total mass one.

We shall prove that along a subsequence we have:

$$\lim_{\varepsilon\to0}\|\phi_\varepsilon(x,\lambda,\tilde\lambda,\xi)-\phi(x,\lambda,\tilde\lambda,\xi)\|_{L_0^{\beta^*}(I\!R^{d+r+r'};C^\kappa(S^{d-1}))}=0. \tag{35}$$

We have:

$$\begin{aligned}&\|\phi_\varepsilon(x,\lambda,\tilde\lambda,\xi)-\phi(x,\lambda,\tilde\lambda,\xi)\|^{\beta^*}_{L_0^{\beta^*}(I\!R^d;C^\kappa(S^{d-1}))} \qquad (36)\\&=\int_{I\!R^{d+r+r'}}\|\phi_\varepsilon(x,\lambda,\tilde\lambda,\xi)-\phi(x,\lambda,\tilde\lambda,\xi)\|^{\beta^*}_{C^\kappa(S^{d-1})}dxd\lambda d\tilde\lambda\\&\leq\int_{I\!R^{d+r+r'}}\int_{I\!R^{d+r+r'}}\|\phi((x,\lambda,\tilde\lambda)+\varepsilon(z,\nu,\tilde\nu),\xi)-\phi(x,\lambda,\tilde\lambda,\xi)\|^{\beta^*}_{C^\kappa(S^{d-1})}\times\\&\qquad\times\Pi_{i=1}^d\,\omega(z_i)\,\Pi_{i=1}^r\,\omega(\eta_i)\,\Pi_{i=1}^{r'}\,\omega(\tilde\eta_i)dzd\nu d\tilde\nu dxd\lambda d\tilde\lambda\end{aligned}$$

where on the last step we used standard change of variables $z_i=\frac{x_i-y_i}{\varepsilon}$, $\nu=\frac{\lambda_i-\eta_i}{\varepsilon}$, $\tilde\nu=\frac{\tilde\lambda_i-\tilde\eta_i}{\varepsilon}$, and the well known inequality $\|\int f(x,y)dy\|\leq\int\|f(x,y)\|dy$.

Next, using generalized Lebesgue point property (7) we have for almost every $(x,\lambda,\tilde\lambda)\in I\!R^{d+r+r'}$

$$\begin{aligned}\lim_{\varepsilon\to0}\int_{I\!R^{d+r+r'}}&\|\phi((x,\lambda,\tilde\lambda)+\varepsilon(z,\nu,\tilde\nu),\xi)-\phi(x,\lambda,\tilde\lambda,\xi)\|^{\beta^*}_{C^\kappa(S^{d-1})}\times\\&\times\Pi_{i=1}^d\,\omega(z_i)\,\Pi_{i=1}^r\,\omega(\eta_i)\,\Pi_{i=1}^{r'}\,\omega(\tilde\eta_i)dzd\nu d\tilde\nu=0.\end{aligned}$$

From here, applying the Lebesgue dominated convergence theorem on (36) we conclude that (35) holds.

This concludes the proof.

□

Simple corollary of Theorem 14 is the following proposition:

Proposition 17. *The H-distribution* $\mu \in \left(L_0^{\beta^*}(\mathbb{R}^{d+r+r'}; C^\kappa(S^{d-1}))\right)^*$ *defined in Theorem 14 is a family* $\mu(x,\lambda,\tilde{\lambda},\xi) \in (C^\kappa(S^{d-1}))^*$, $(x,\lambda,\tilde{\lambda}) \in \mathbb{R}^{d+r+r'}$, *of complex functionals defined on* $C^\kappa(S^{d-1})$ *such that for all* $\phi \in L_0^{\beta^*}(\mathbb{R}^{d+r+r'}; C^\kappa(S^{d-1}))$ *the mapping*

$$\mathbb{R}^{d+r+r'} \ni (x,\lambda,\tilde{\lambda}) \mapsto \langle \mu(x,\lambda,\tilde{\lambda},\xi), \phi\rangle$$

belongs to $L_0^{\beta^*}(\mathbb{R}^{d+r+r'})$.

The latter proposition actually means that we can write (15) as:

$$\lim_{n'\to\infty} \int_{\mathbb{R}^{d+r+r'}} (\varphi_1 u^i_{n'})(x,\lambda)\mathcal{A}[\rho_2(\tilde{\lambda})\varphi_2(\cdot)v^j_{n'}(\cdot,\tilde{\lambda})](x)dxd\lambda d\tilde{\lambda} \tag{37}$$
$$= \int_{\mathbb{R}^{d+r+r'}} \varphi_1(x,\lambda)\varphi_2(x)\rho_2(\tilde{\lambda})\langle \mu(x,\lambda,\tilde{\lambda},\cdot),\psi\rangle dxd\lambda d\tilde{\lambda}.$$

4. The proof of Theorem 4

We shall need the operator $T_l : \mathbb{R} \to \mathbb{R}$, $l \in \mathbb{N}$, reminding on the truncation operator (see e.g. [7, 12]) which is often used for controlling concentration effects:

$$T_l(v) = \begin{cases} 0, & |v| > l \\ v, & |v| \leq l. \end{cases} \tag{38}$$

In order to use the results from the previous section we take, for a fixed l and fixed $\rho \in C_0^{|k|}(\mathbb{R}^m)$, where $|k|$ is given in Theorem 4:

$$u_n(x,\lambda) = h_n(x,\lambda), \quad \text{and}$$
$$v_n(x) = T_l(\int_{\mathbb{R}^m} \rho(q)h_n(x,q)dq).$$

Thus, with the notation from the previous section, we have $r = m$ and $r' = 0$.

The proof starts with the choice of special test function. For a fixed $p \in \mathbb{R}^d$ and $l \in \mathbb{N}$ we take (compare with [33, (35)]):

$$\varphi(x,\lambda) = \zeta_1(x)\zeta_2(\lambda)(\mathcal{I}_1 \circ \mathcal{A}_\psi)[\zeta_3(\cdot)T_l(\int_{\mathbb{R}^m} \rho(p)h_n(\cdot,p)dp)](x), \tag{39}$$

where $\zeta_1 \in C_0^1(\mathbb{R}^d)$, $\zeta_2 \in C_0^{|k|}(\mathbb{R}^m)$ for the multindex k given in (4), and $\zeta_3 \in C_0^1(\mathbb{R}^d)$, and $\mathcal{A}_\psi$ is multiplier with a symbol $\psi \in C^\kappa(S^{d-1})$. Since $h_n \rightharpoonup 0$ in $L^{1+\beta}(\mathbb{R}^d \times \mathbb{R}^m)$ we can assume that $T_l(\int_{\mathbb{R}^m} \rho(p)h_n(\cdot,p)dp) \rightharpoonup 0$ weakly-$\star$ in $L^\infty(\mathbb{R}^d)$ as well. Otherwise, we can take L^∞ weak-$\star$ limit h^l of $T_l(\int_{\mathbb{R}^m} \rho(p)h_n(\cdot,p)dp$ and put $T_l(\int_{\mathbb{R}^m} \rho(p)h_n(\cdot,p)dp) - h_l$ in the place of $T_l(\int_{\mathbb{R}^m} \rho(p)h_n(\cdot,p)dp)$ in (39).

Then, substitute the function φ in (4). We get after integrating over $\mathbb{R}^m$ and using (14):

$$
\begin{aligned}
&\int_{\mathbb{R}^{d+m}}\sum_{i=1}^d F_i(x,\lambda)h_n(x,\lambda)\Big(\partial_{x_i}\zeta_1\zeta_2(\mathcal{I}_1\circ\mathcal{A}_\psi)[\zeta_3(\cdot)T_l(\int_{\mathbb{R}^m}\rho(p)h_n(\cdot,p)dp)]\\
&\qquad+\zeta_1\zeta_2(\mathcal{R}_i\circ\mathcal{A}_\psi)[\zeta_3(\cdot)T_l(\int_{\mathbb{R}^m}\rho(p)h_n(\cdot,p)dp)]\Big)dxd\lambda\\
&=\sum_{i=1}^d\int_{\mathbb{R}^d}G^i_n(x,\lambda)\Big(\partial_{x_i}\zeta_1\partial^k_\lambda\zeta_2(\mathcal{I}_1\circ\mathcal{A}_\psi)[\zeta_3(\cdot)T_l(\int_{\mathbb{R}^m}\rho(p)h_n(\cdot,p)dp)]\\
&\qquad+\zeta_1\partial^k_\lambda\zeta_2(\mathcal{R}_i\circ\mathcal{A}_\psi)[\zeta_3(\cdot)T_l(\int_{\mathbb{R}^m}\rho(p)h_n(\cdot,p)dp)]\Big)dxd\lambda.
\end{aligned}
\tag{40}
$$

To proceed, note that from Lemma 13 it follows that the multiplier $\mathcal{A}_\psi\circ\mathcal{I}_1$ is compact operator from $L^p(\mathbb{R}^d)$ to $C(\mathbb{R}^d)$ for any $p>d$. Therefore, it transforms weakly convergent sequence $(T_l(\int_{\mathbb{R}^m}\rho(p)h_n(\cdot,p)dp))$ into the strongly convergent one.

Thus, every term containing as a subintegral expression $\mathcal{A}_\psi\circ\mathcal{I}_1$ multiplied by a function/sequence which is bounded in L^q for any $q>1$ will converge to zero. Similarly, from (14) it follows that every term containing as a subintegral expression $\partial_{x_i}\mathcal{A}_\psi\circ\mathcal{I}_1=\mathcal{A}_\psi\circ\mathcal{R}_i$ multiplied by a sequence which is strongly convergent to zero in L^q for any $q>1$ will converge to zero as well.

So, combining the Holder inequality, Hormander-Mikhlin theorem, and Theorem 14 (more precisely (37)), we get after letting $r\to\infty$ in (40) along a subsequence $(h_r)\subset(h_n)$ such that:

$$
\sum_{i=1}^d\int_{\mathbb{R}^{d+m}}\langle F_i(\lambda,x)\xi_i\zeta_1(x)\zeta_2(\lambda)\zeta_3(x)\psi(\xi),\mu_l(x,\lambda,\xi)\rangle dxd\lambda=0 \tag{41}
$$

Clearly, since l belongs to the countable set $\mathbb{N}$, we can assume that the same subsequence (h_r) for every $l\in\mathbb{N}$ defines the distribution $\mu_l(x,q,\xi)$ from (41).

According to Theorem 14 instead of (41) we can write:

$$
\int_{\mathbb{R}^{d+m}}\langle\sum_{i=1}^d F_i(x,\lambda)\xi_i\zeta_4(x,\lambda,\xi),\mu_l(x,\lambda,\xi)\rangle dxd\lambda=0, \tag{42}
$$

for an arbitrary $\zeta_4\in L_0^{\beta^*}(\mathbb{R}^{d+m};C^\kappa(S^{d-1})))$.

Now, we take in (42):

$$
\zeta_4(x,\lambda,\xi)=\frac{\left(\sum\limits_{i=1}^d F_i(x,\lambda)\xi_i\right)\zeta(x,\lambda,\xi)}{\left(\sum\limits_{i=1}^d F_i(x,\lambda)\xi_i\right)^2+\varepsilon},
$$

where $\zeta\in L_0^{\beta^*}(\mathbb{R}^{d+m};C^\kappa(S^{d-1}))$ is arbitrary. We obtain

$$
\int_{\mathbb{R}^{d+m}}\langle\frac{\left(\sum\limits_{i=1}^d F_i(x,\lambda)\xi_i\right)^2\zeta(x,\lambda,\xi)}{\left(\sum\limits_{i=1}^d F_i(x,\lambda)\xi_i\right)^2+\varepsilon},\mu_l(x,\lambda,\xi)\rangle d\lambda dx=0,
$$

Rewrite the latter expression in the form:

$$\int_{\mathbb{R}^{d+m}} \langle \frac{\left(\sum\limits_{i=1}^{d} F_i(x,\lambda)\xi_i\right)^2 \zeta(x,\lambda,\xi)}{\left(\sum\limits_{i=1}^{d} F_i(x,\lambda)\xi_i\right)^2 + \varepsilon}, \mu_l(x,\lambda,\xi)\rangle dx d\lambda = 0. \tag{43}$$

From condition (5) we know that for every $(x,\xi) \in D \times S^{d-1}$, where $D \subset \mathbb{R}^d$ is of full Lebesgue measure with respect to $\mathbb{R}^d$, the set $\{\lambda \in \mathbb{R}^m : \sum\limits_{k=1}^{d} F_k(x,\lambda)\xi_k = 0\}$ has zero Lebesgue measure. Therefore, for almost every $\lambda \in \mathbb{R}^m$ we have:

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}^d} \langle \frac{\left(\sum\limits_{i=1}^{d} F_i(x,\lambda)\xi_i\right)^2 \zeta(x,\lambda,\xi)}{\left(\sum\limits_{i=1}^{d} F_i(x,\lambda)\xi_i\right)^2 + \varepsilon}, \mu_l(x,\lambda,\xi)\rangle dx$$
$$= \int_{\mathbb{R}^d} \langle \zeta(x,\lambda,\xi), \mu_l(x,\lambda,\xi)\rangle dx.$$

Then, using the Lebesgue dominated convergence theorem we obtain from (43) after letting $\varepsilon \to 0$:

$$\int_{\mathbb{R}^{d+m}} \langle \zeta(x,\lambda,\xi), \mu_l(x,\lambda,\xi)\rangle dx d\lambda = 0.$$

Due to arbitrariness of ζ we conclude that

$$\mu_l(x,\lambda,\xi) \equiv 0.$$

From here, applying (15) with $\psi \equiv 1$, $\varphi_1(x,\lambda) = \varphi_2(x)\rho(\lambda)$ for $\varphi_2 \in C_0(\mathbb{R}^d)$, $\rho \in C_0(\mathbb{R}^m)$ we get (recall that $\tilde\lambda$ does not exists in (15) since $r' = 0$):

$$\begin{aligned} 0 &= \int_{\mathbb{R}^{d+m}} \langle \rho(\lambda)\varphi^2(x), \mu_l(x,\lambda,\xi)\rangle dx d\lambda \\ &= \lim_{r\to\infty} \int_{\mathbb{R}^{d+m}} \rho(\lambda)\varphi^2(x) h_r(x,\lambda) T_l(\int_{\mathbb{R}^m} \rho(p) h_r(x,p)dp) dx d\lambda \\ &= \lim_{r\to\infty} \int_{\mathbb{R}^d} \varphi^2(x)[T_l(\int_{\mathbb{R}^m} \rho(p)h_r(x,p)dp)]^2 dx \end{aligned}$$

From here, it immediately follows that for every fixed $l \in \mathbb{N}$:

$$\lim_{r\to 0} \|T_l(\int_{\mathbb{R}^m} \rho(\lambda)h_r(x,\lambda)d\lambda)\|_{L^1_{loc}(\mathbb{R}^d)} = 0.$$

Using the fact that (h_n) is bounded in L^p_{loc} for the $p>1$ given in Theorem 4, it is not difficult to prove that (see also [12])

$$\lim_{r\to 0} \| \int_{\mathbb{R}^m} \rho(\lambda)h_r(x,\lambda)d\lambda\|_{L^1_{loc}(\mathbb{R}^d)} = 0.$$

Indeed, denote by $\chi_{|h_r|>l}$ characteristic function of the set $\{(x,\lambda) : |h_r(x,\lambda)| > l\}$ and by $\chi_{|h_r|\leq l}$ characteristic function of the set $\{(x,\lambda) : |h_r(x,\lambda)| \leq l\}$. Then, consider

$$
\begin{aligned}
&\int_{x,loc} |\int_{\mathbb{R}^m} \rho(\lambda) h_r(x,\lambda)d\lambda|dx \qquad (44)\\
&= \int_{x,loc} |\int_{\mathbb{R}^m} \rho(\lambda)\left(\chi_{|h_r|\leq\lambda l}(x,\lambda)+\chi_{|h_r|>l}(x,\lambda)\right) h_r(x,\lambda)d\lambda|dx\\
&= \int_{x,loc} |\int_{\mathbb{R}^m} \rho(\lambda)T_l(h_r)(x,\lambda)d\lambda|dx + \int_{x,loc} |\int_{\mathbb{R}^m} \rho(\lambda)\chi_{|h_r|>l}(x,\lambda)h_r(x,\lambda)d\lambda|dx\\
&\leq \int_{x,loc} |\int_{\mathbb{R}^m} \rho(\lambda)T_l(h_r)(x,\lambda)d\lambda|dx + \|\chi_{|h_r|>l}\|_{L^{\kappa}_{loc}(\mathbb{R}^d\times\mathbb{R}^m)}\|h_r\|_{L^p_{loc}(\mathbb{R}^d\times\mathbb{R}^m)}\\
&= \int_{x,loc} |\int_{\mathbb{R}^m} \rho(\lambda)T_l(h_r)(x,\lambda)d\lambda|dx + o(1), \quad l\to\infty,
\end{aligned}
$$

uniformly in r since $h_r \in L^p_{loc}$, $p>1$ implies:

$$
\|\chi_{|h_r|>l}\|_{L^{\kappa}_{loc}(\mathbb{R}^d\times\mathbb{R}^m)} = \|\chi_{|h_r|>l}\|^{\kappa}_{L^1_{loc}(\mathbb{R}^d\times\mathbb{R}^m)} \to 0 \ \text{ as } \ l\to\infty
$$

uniformly in r. Thus, letting $r\to\infty$ in (44) we get:

$$
\lim_{r\to\infty}\int_{x,loc} |\int_{\mathbb{R}^m} \rho(\lambda)h_r(x,\lambda)d\lambda|dx = o(1), \quad l\to\infty.
$$

Finally, letting $l\to\infty$ here, we conclude the theorem.

5. Scalar conservation law with discontinuous flux

In this section we shall show how to apply the previous results in order to prove existence of solution to a Cauchy problem for multidimensional scalar conservation law with discontinuous flux:

$$
\begin{aligned}
&\partial_t u + \mathrm{div} f(t,x,u) = 0, \quad (t,x)\in\mathbb{R}^+\times\mathbb{R}^d\\
&u|_{t=0} = u_0(x) \in L^\infty(\mathbb{R}^d).
\end{aligned} \tag{45}
$$

We assume that the flux $f=f(t,x,u)$ is a Caratheodory vector on $\mathbb{R}^+\times\mathbb{R}^d\times\mathbb{R}$ such that $f(t,x,\cdot)\in W^{1,p}(\mathbb{R};\mathbb{R}^d)$, $p>1$, for fixed (t,x). Furthermore, we suppose that

$$
\begin{aligned}
&\text{for some } a,b\in\mathbb{R},\ a<b, \text{ we have}\\
&0 = f(\cdot,\cdot,a) = f(\cdot,\cdot,b) \ \text{ and } \ a<u_0(x)<b.
\end{aligned} \tag{46}
$$

Also, we suppose that

$$
\begin{aligned}
&\max_{a\leq u\leq b}|f(\cdot,\cdot,u)| \in L^q(\mathbb{R}^+\times\mathbb{R}^d), \quad q>1,\\
&\mathrm{div}_x f(\cdot,\cdot,p) = \gamma_p \in \mathcal{M}_{loc}(\mathbb{R}^+\times\mathbb{R}^d),
\end{aligned} \tag{47}
$$

where $\mathcal{M}_{loc}(\mathbb{R}^+\times\mathbb{R}^d)$ is a set of locally bounded measures.

Finally, we need to assume that the flux $f=(f_1,\dots,f_d)$ satisfies the nondegeneracy condition analogous to (5):

We assume that for almost every $(t,x)\in\mathbb{R}^+\times\mathbb{R}^d$ and every $\xi\in S^d\subset\mathbb{R}^{d+1}$ the mapping

$$
\mathbb{R}\ni\lambda\mapsto\sum_{i=1}^d \xi_i\partial_\lambda f_i(t,x,\lambda) \tag{48}
$$

is not identically equal to zero on any set $A\subset\mathbb{R}$ of positive Lebesgue measure.

Since the beginning of the 80s, problems such as (45) have been the subject of intensive investigations. The reason for such interest is in applicability of the problem – it models many natural phenomena many of which are in connection with the oil industry. Therefore, it is not surprising that most of the efforts in the field was made by a Norwegian group of mathematicians [16, 17, 18, 19, 20, 21, 22, 23, 24] (rather incomplete list). Still, almost all of the mentioned papers dealt with one dimensional variant of the problem.

On the other hand, the question of existence of solution for multidimensional scalar conservation with Caratheodory flux that we are considering here was open for a relatively long time.

The question was settled for the first time in [31] for the case when $\max\limits_{a\leq u\leq b}|f(\cdot,\cdot,u)|\in L^q(I\!R^+\times I\!R^d)$, for a $q>2$. Thus, we shall improve the result from [31] since we demand less regularity on the flux (see (47)). Still, notice that using the method from [31] one can prove existence of solution to Cauchy problem (45) merely assuming that the flux $f=f(t,x,\lambda)$ is continuous in λ (see [14] for more general situation – diffusion-dispersion limits for scalar conservation law with discontinuous flux).

As usual, initially we take a C^1 approximation f_ε of the flux f satisfying

$$\lim_{\varepsilon\to 0}\max_{\lambda\in[a,b]}\|\partial_\lambda f_\varepsilon(t,x,\lambda)-\partial_\lambda f(t,x,\lambda)\|=0 \ \text{ in } \ L^q_{loc}(I\!R^+\times I\!R^d). \tag{49}$$

For instance, we can choose $f_\varepsilon(t,x,\lambda)=f(\cdot,\cdot,\lambda)\star\frac{1}{\varepsilon^{d+1}}\omega(\frac{t}{\varepsilon})\,\Pi_{i=1}^d\,\omega(\frac{x_i}{\varepsilon})$, where ω is a smooth positive compactly supported real function with total mass one (see e.g. [31, (56)]).

Then, we consider usual approximation of problem (45):

$$\begin{aligned}&\partial_t u_\varepsilon+\mathrm{div} f_\varepsilon(t,x,u_\varepsilon)=0,\quad (t,x)\in I\!R^+\times I\!R^d,\\&u|_{t=0}=u_0(x)\in L^\infty(I\!R^d).\end{aligned}\tag{50}$$

Problem (50) has unique entropy admissible solution u_ε for every fixed $\varepsilon>0$. It is also well known that, due to (46), the family of solutions (u_ε), $\varepsilon>0$, remains uniformly bounded, i.e. $a\leq u_\varepsilon\leq b$, $\varepsilon>0$.

In order to prove that a solution to Cauchy problem (45) exists it is enough to prove that the family of solutions (u_ε) is strongly precompact in $L^1_{loc}(I\!R^+\times I\!R^d)$. Notice that if we put $u_\varepsilon(t,x)\equiv 0$, $t<0$, we can assume that the family (u_ε) is defined on entire $I\!R^{d+1}$.

To prove this, we shall reduce the equation from (50) to a transport equation having form (4). Such approach for this kind of problem is initially proposed in [6]. Actually, what we are essentially do is rewriting Cauchy problem (45) in the kinetic formulation. For more information and strict consideration of heterogeneous scalar conservation law (with smooth flux) in the kinetic framework one should check [10].

So, for a fixed $\lambda\in I\!R$ take Kružkov semi entropies

$$\eta_+(u)=|u-\lambda|^+=\max\{0,u-\lambda\},\quad \eta_-(u)=|u-\lambda|^-=\max\{0,-u+\lambda\},$$

and apply them in the definition of an entropy solution (see [25]) of (50) . We have for every fixed λ in distributional sense, respectively:

$$\partial_t|u_\varepsilon-\lambda|^+ + \mathrm{div}_x \mathrm{sgn}_+(u_\varepsilon-\lambda)(f_\varepsilon(t,x,u_\varepsilon)-f_\varepsilon(t,x,\lambda)) \tag{51}$$
$$\leq -\mathrm{sgn}_+(u_\varepsilon-\lambda)\mathrm{div}_x f_\varepsilon(t,x,\lambda) \leq |\mathrm{div}_x f_\varepsilon(t,x,\lambda)|,$$
$$\partial_t|u_\varepsilon-\lambda|^- + \mathrm{div}_x \mathrm{sgn}_-(u_\varepsilon-\lambda)(f_\varepsilon(t,x,u_\varepsilon)-f_\varepsilon(t,x,\lambda)) \tag{52}$$
$$\leq -\mathrm{sgn}_-(u_\varepsilon-\lambda)\mathrm{div}_x f_\varepsilon(t,x,\lambda) \leq |\mathrm{div}_x f_\varepsilon(t,x,\lambda)|,$$

where $\mathrm{sgn}_+(\lambda)=H(\lambda)$ and $\mathrm{sgn}_-(\lambda)=-H(-\lambda)$ for the Heaviside function H. Using the Schwartz lemma for non-negative distributions we can rewrite (51) and (52) as follows:

$$\partial_t|u_\varepsilon-\lambda|^+ + \mathrm{div}_x \mathrm{sgn}_+(u_\varepsilon-\lambda)(f_\varepsilon(t,x,u_\varepsilon)-f_\varepsilon(t,x,\lambda)) \tag{53}$$
$$= |\mathrm{div}_x f_\varepsilon(t,x,\lambda)| + \mu_+^\varepsilon(t,x,\lambda),$$
$$\partial_t|u_\varepsilon-\lambda|^- + \mathrm{div}_x \mathrm{sgn}_-(u_\varepsilon-\lambda)(f_\varepsilon(t,x,u_\varepsilon)-f_\varepsilon(t,x,\lambda)) \tag{54}$$
$$= |\mathrm{div}_x f_\varepsilon(t,x,\lambda)| + \mu_-^\varepsilon(t,x,\lambda),$$

where the distributions $\mu_\pm^\varepsilon$ are locally bounded negative Radon measures over $I\!R^+\times I\!R^d\times I\!R$. Next, we apply partial derivative in λ on (53) and (54) to obtain:

$$\begin{aligned}
&-\partial_t \mathrm{sgn}_+(u_\varepsilon-\lambda) - \mathrm{div}_x \mathrm{sgn}_+(u_\varepsilon-\lambda)\partial_\lambda f_\varepsilon(t,x,\lambda)\\
&\quad= \partial_\lambda\left(|\mathrm{div}_x f_\varepsilon(t,x,\lambda)| + \mu_+^\varepsilon(t,x,\lambda)\right),\\
&-\partial_t \mathrm{sgn}_-(u_\varepsilon-\lambda) - \mathrm{div}_x \mathrm{sgn}_-(u_\varepsilon-\lambda)\partial_\lambda f_\varepsilon(t,x,\lambda)\\
&\quad= \partial_\lambda(|\mathrm{div}_x f_\varepsilon(t,x,\lambda)| + \mu_-^\varepsilon(t,x,\lambda)),
\end{aligned}$$

Adding the latter two equalities we get:

$$\begin{aligned}
&-\partial_t h_\varepsilon(t,x,\lambda) - \mathrm{div}_x(\partial_\lambda f_\varepsilon(t,x,\lambda)h_\varepsilon(t,x,\lambda))\\
&= \partial_\lambda\left(2|\mathrm{div}_x f_\varepsilon(t,x,\lambda)| + \mu_+^\varepsilon(t,x,\lambda) + \mu_-^\varepsilon(t,x,\lambda)\right),
\end{aligned} \tag{55}$$

where

$$h_\varepsilon(t,x,\lambda) = \mathrm{sgn}_+(u_\varepsilon-\lambda) + \mathrm{sgn}_-(u_\varepsilon-\lambda) = \begin{cases} 1, & 0\leq\lambda\leq u_\varepsilon(t,x),\\ -1, & u_\varepsilon(t,x)\leq\lambda\leq 0,\\ 0, & else. \end{cases}$$

Then, we rewrite (55) as:

$$\begin{aligned}
&-\partial_t h_\varepsilon(t,x,\lambda) - \mathrm{div}_x(\partial_\lambda f(t,x,\lambda)h_\varepsilon(t,x,\lambda))\\
&=\mathrm{div}_x[(\partial_\lambda f_\varepsilon(t,x,\lambda) - \partial_\lambda f(t,x,\lambda))h_\varepsilon(t,x,\lambda)]\\
&+\partial_\lambda\left(2|\mathrm{div}_x f_\varepsilon(t,x,\lambda)| + \mu_+^\varepsilon(t,x,\lambda) + \mu_-^\varepsilon(t,x,\lambda)\right).
\end{aligned} \tag{56}$$

From (47) and (49), it is clear that equation (56) and appropriate family of solutions (h_ε) satisfy the transport equation from Theorem 4. Using nondegeneracy condition (48) and having in mind Remark 5, from Theorem 4 we conclude that the family of averaged quantities

$$\left(\int_\lambda h_\varepsilon(t,x,\lambda)\rho(\lambda)d\lambda\right), \quad \rho\in C_0(I\!R),$$

is strongly precompact in $L^1_{loc}(I\!R^+\times I\!R^d)$. Now, standard arguments show that (u_ε) is strongly precompact in $L^1_{loc}(I\!R^+\times I\!R^d)$ (see e.g. [15]).

It is clear that an aggregation point $u\in L^1_{loc}(I\!R^+\times I\!R^d)$ of the family (u_ε) represents a weak solution to (45).

This concludes the paper.

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