GLOBAL WAVE INTERACTIONS IN ISENTROPIC GAS DYNAMICS

ROBIN YOUNG

Abstract. We give a complete description of nonlinear waves and their pairwise interactions in isentropic gas dynamics. Our analysis includes rarefactions, compressions and shock waves. Because the waves are arbitrarily large, we describe the change of states across the wave exactly, without resolving the characteristic patterns. We similarly describe the states between nonlinear waves in any pairwise interaction. When the strengths and reflected waves are described correctly, we show that whenever two (arbitrary) nonlinear waves of the same family interact, their strengths simply add. Also, if a wave crosses a simple wave (rarefaction or compression) of the opposite family, its strength is unchanged, and the change in the opposite simple wave is explicitly given by exact formulae. In addition, we analyze the crossing of two arbitrary shocks. We obtain bounds for the outgoing middle state, and use this to estimate the outgoing wave strengths in terms of the incident strengths. Our estimates are global in that they apply to waves of arbitrary strength, and they are uniform in the incoming middle state. In particular, the estimates continue to hold as the middle state approaches vacuum.

1. Introduction

One of the major open questions in hyperbolic conservation laws is the global existence in BV of solutions having large L^∞ data. There are several examples which show that global solutions do not exist in general systems, although all such examples are from nonphysical systems which do not possess convex entropies [12, 6, 14, 16]. The celebrated Glimm-Lax theory states that solutions to many 2×2 systems of conservation laws decay in the total variation norm, provided the supnorm of the data is small [5]. These systems include the fundamental system of isentropic gas dynamics, commonly known as the p-system. The restriction to small supnorm is a serious one, because Glimm's theory relies fundamentally on asymptotic expansions of waves, and the analysis is thus restricted to a small neighborhood [4, 11].

In order to obtain BV existence results for data that includes large waves, it is necessary to derive estimates of interactions of strong nonlinear waves, uniform in the base state. In this paper, we treat waves *exactly* at the level of states, without resolving the characteristic patterns inside the interaction region. We then derive estimates on the interactions of these waves which

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are uniform in the base state and applicable to waves of arbitrary strength. This is a part of the author's long-term project to obtain long-time BV estimates for solutions to the p-system for arbitrary BV_{loc} data.

We are interested in the p-system, modeling isentropic flow in one dimension in a Lagrangian frame of reference. The system is

$$\begin{pmatrix} v \\ u \end{pmatrix}_t + \begin{pmatrix} -u \\ p(v) \end{pmatrix}_x = 0, \tag{1.1}$$

where v is the specific volume of the fluid, u is the fluid velocity, and p the pressure, which is assumed to be a convex function of the volume v.

We treat the case of a polytropic ideal gas, for which

$$p(v) = A_0\, v^{-\gamma}, \tag{1.2}$$

with positive constant A_0 and ideal gas constant $\gamma > 1$. The *Lagrangian sound speed* $c(v)$, given by $c^2(v) = -p'(v)$, so that

$$c(v) = \sqrt{-p'(v)} = \sqrt{A_0\,\gamma}\; v^{-(\gamma+1)/2}, \tag{1.3}$$

is real and decreasing for $v > 0$, which means that the system is strictly hyperbolic and genuinely nonlinear (except at vacuum).

The vacuum state corresponds to $v = \infty$, and the pressure, density $1/v$ and sound speed c vanish there,

$$p(v) \to 0 \quad \text{and} \quad c(v) \to 0 \quad \text{as} \quad v \to \infty.$$

For $\gamma > 1$, the integral

$$\int_1^\infty c(v)\, dv = \int_1^\infty \sqrt{-p'(v)}\, dv < \infty, \tag{1.4}$$

converges, which is a necessary condition for the possible occurrence of a vacuum in the solution [2, 11]. This last condition fails for the isothermal case $\gamma = 1$ treated in [9].

First, we make a global nonlinear change of coordinates which is tuned to our needs. Noting that the equations are linear in velocity u, we always use that as one variable, but we have some freedom in choosing the thermodynamic variable. Instead of the specific volume v, density $\rho = 1/v$ or pressure p, we use Riemann's coordinate, which is the the integrated sound speed

$$h(v) = \int_v^\infty c(v)\, dv = \frac{2\sqrt{A_0\,\gamma}}{\gamma - 1}\; v^{-(\gamma-1)/2}, \tag{1.5}$$

and which is clearly a monotone decreasing function of specific volume. The limits of the integral are chosen so that also $h = 0$ at vacuum, and make sense because (1.4) holds.

Using h as the thermodynamic variable, it is easy to see that

$$v'(h) = \frac{-1}{c(h)} \quad \text{and} \quad p'(h) = c(h). \tag{1.6}$$

It is convenient to make this change of variables explicit, by rewriting the system (1.1) as

$$\begin{pmatrix} v(h) \\ u \end{pmatrix}_t + \begin{pmatrix} -u \\ p(h) \end{pmatrix}_x = 0, \tag{1.7}$$

where the constitutive relation is determined by prescribing the Lagrangian sound speed $c(h)$. Using (1.6), we then define the pressure and specific volume by

$$p(h) = \int c(h)\,dh \quad \text{and} \quad v(h) = \int \frac{-1}{c(h)}\,dh. \tag{1.8}$$

In these coordinates, $u \pm h$ are Riemann invariants, and so the simple wave curves become straight lines, and there is a reflection symmetry between the forward and backward waves. Moreover, for a polytropic gas (1.2), homogeneity of $p(h)$ and $v(h)$ implies an important scaling property. Consequently, all (forward and backward) wave curves can be described as scalings of a single curve, namely

$$u_r - u_l = h_a\, g(\frac{h_b}{h_a}), \tag{1.9}$$

where the subscripts indicate the right, left, ahead and behind states, respectively. Thus in order to understand the waves and their interactions, we need to analyze $g(z)$ and related functions of a single variable. This makes the finding of estimates significantly easier and suggests the possibility of uniform estimates.

Having described all waves by the single equation (1.9), which in particular is linear in u, we can easily solve the Riemann problem by eliminating u and solving one nonlinear equation for a single unknown h_*. The interaction problem is treated similarly, and for a pairwise Glimm interaction we also get a single nonlinear equation of one unknown. The "Glimm interaction problem" is the problem of resolving the states in the interaction of two or more waves, while avoiding the complex characteristic patterns which develop. This interaction is the basis of difference approximations such as the Random Choice and Front Tracking methods, in which states are resolved accurately while their characteristic curves are approximated; Glimm's convergence argument implies that in the limit of such approximations, the exact characteristic pattern is attained [4, 11, 1].

We define both scaled and unscaled strengths of a wave; the unscaled strength turns out to be the usual change of appropriate Riemann coordinate across the wave. In order to accurately describe the interactions of strong waves, we extend the class of waves we use to include compression waves. This is in contrast to the usual practice of approximating compressions by many small shocks [17, 1]. Our reason for doing so is, since the waves are strong, rarefactions will continue to expand until they interact, so the interaction will have finite width. Thus any reflected wave also has finite width, and so cannot be a shock. That said, it is clear that the compression could collapse into a shock within a short time, but we treat this as a separate interaction. We remark that this more accurately captures the internal structure of the interaction, while the simpler case of using shocks instead of compressions underestimates the waves actually in the interaction, essentially just giving the asymptotic states from the corresponding Riemann problem.

Although adding compressions leads to a larger class of interactions, the advantage gained is that several interactions are simplified, and we can get accurate descriptions of interactions for waves of arbitrary strength. Our first result here is that except for the crossing of two shocks, wave strengths combine *linearly* even for strong nonlinear waves, and the reflected waves are described *exactly*.

Theorem 1. *Whenever two waves of the same family interact, their unscaled strengths add* exactly*, and the reflected wave of the opposite family is given exactly and bounded in terms of wave strengths. That is, if waves of strengths $\mathcal{A}$ and $\mathcal{B}$ interact, the resulting wave has strength $\mathcal{A}+\mathcal{B}$, and the reflected wave has strength given explicitly in terms of $\mathcal{A}$, $\mathcal{B}$ and the base state h. If the interacting waves are compressive ($\mathcal{A}$, $\mathcal{B} < 0$), then the reflected wave is a rarefaction, while if a shock meets a rarefaction ($\mathcal{A} > 0$ or $\mathcal{B} > 0$), the reflected wave is a compression.*

If any wave crosses a simple wave of the other family, its unscaled strength doesn't change, and if a simple wave crosses a shock, its unscaled strength increases by an amount which is exactly given.

Since a wave crossing simple waves doesn't undergo any change in wave strength, and since the entropy condition implies that any wave adjacent to a vacuum state must be simple, our theorem implies that a shock approaching vacuum will meet the vacuum with known strength, and thus affect the vacuum in an explicitly given way [8, 13]. This enables us to consider solutions containing vacuums, and indeed, the author has found exact solutions to the p-system that clearly demonstrate how the vacuum evolves in time [15].

The exact expressions for reflected waves can be used to build a nonincreasing interaction potential which should lead to global BV bounds on solutions: this is the subject of the author's ongoing research. Indeed, we have constructed such a potential for monotonic data, and this will appear in a forthcoming paper.

The above theorem excludes what is arguably the most interesting case, namely the interaction of strong shocks of opposite families. Our method for treating this case is easily described: namely, we approximate the shock curves by power functions, and use these in the interaction equation to bound the outgoing middle state. We then obtain bounds on the outgoing wave strengths. Consider the interaction of two shocks of strengths $\mathcal{A}$ and $\mathcal{B}$ sharing the common forward state h_s, and resulting in outgoing shocks $\mathcal{A}'$ and $\mathcal{B}'$.

Theorem 2. *If $\mathcal{A} \le \mathcal{B} < 0$, then the unscaled wave strength $\mathcal{A}'$ satisfies*

$$K_7\,|\mathcal{A}| \le |\mathcal{A}'| \le K_6\,|\mathcal{A}|\,,$$

while for $\mathcal{B}'$ we have three cases: if $|\mathcal{B}| \ge |\gamma_\#|\,h_s$,

$$K_8\,|\mathcal{A}|^{\frac{2}{d+1}}\,|\mathcal{B}|^{\frac{d-1}{d+1}} \le |\mathcal{B}'| \le K_9\,|\mathcal{A}|^{\frac{2}{d+1}}\,|\mathcal{B}|^{\frac{d-1}{d+1}}\,; \tag{1.10}$$

next, if $|\mathcal{B}| \le |\gamma_\#|\,h_s \le |\mathcal{A}|$,

$$\widehat{K}_8\,\frac{|\mathcal{A}|^{\frac{2}{d+1}}|\mathcal{B}|}{h_s^{\frac{2}{d+1}}} \le |\mathcal{B}'| \le \widehat{K}_9\,\frac{|\mathcal{A}|^{\frac{2}{d+1}}|\mathcal{B}|}{h_s^{\frac{2}{d+1}}} \le \overline{K}_9\,|\mathcal{A}|^{\frac{2}{d+1}}\,|\mathcal{B}|^{\frac{d-1}{d+1}}\,;$$

and finally, if $|\mathcal{B}| \le |\mathcal{A}| \le |\gamma_\#|\,h_s$, then

$$\widetilde{K}_8\,|\mathcal{B}| \le |\mathcal{B}'| \le \widetilde{K}_9\,|\mathcal{B}|\,.$$

Similarly, if $|\mathcal{B}| \ge |\mathcal{A}|$, then by symmetry we get the same estimates with the positions of $\mathcal{A}$ and $\mathcal{B}$ reversed.

Here $\gamma_\#$ and K_i are constants depending only on the gas constant γ. The (small) constant $\gamma_\# < 0$ is regarded as a threshold for scaled shock strength: that is, if a wave has scaled strength $\alpha \equiv \frac{\mathcal{A}}{h_s} \le \gamma_\#$, we regard it as fully nonlinear and express it as a power; on the other hand, if $\gamma_\# \le \alpha < 0$ then we say the shock is weakly nonlinear and use linear estimates. We expect that Glimm's quadratic interaction estimates will hold for these weakly nonlinear waves. We note that our estimates are uniform in the base state h_s, and as this approaches vacuum, the nonlinear estimate (1.10) applies.

The paper is laid out as follows. In Section 2, we recall the hyperbolic waves that appear in the system and describe the wave curves and their properties, including an important scaling property which is a consequence of the ideal gas law (1.2). In Section 3, we recall the solution of the Riemann problem, which is classical, and introduce the "extended Riemann problem," which is designed to handle embedded vacuums of finite width. In Section 4, we state and solve the pairwise Glimm interaction problem for waves of arbitrary strength. In Section 5, we give bounds on the reflected waves that emanate as a result of the nonlinear interaction, in terms of the incident wave strengths. Finally, in Section 6, we fully analyze the interaction of two shocks of arbitrary strength, giving bounds for the shock strengths of the outgoing shocks in terms of the incoming shock strengths.

2. Elementary Waves and Wave Curves

We consider sound speeds of the form

$$c(h) = B_0\,h^d, \tag{2.1}$$

with constant $d > 1$, so that (1.8) yields

$$p(h) = \frac{B_0\,h^{d+1}}{d+1} \quad \text{and} \quad v(h) = \frac{h^{1-d}}{B_0\,(d-1)}. \tag{2.2}$$

Since $d > 1$, $c(h)$, $p(h)$ and $v(h)$ are monotone convex functions, and p and c can be extended up to the vacuum, at which $p = c = h = 0$. Eliminating h and writing $p = p(v)$, it is routine to check that (2.1) describes (1.2); indeed, the gas constant γ is given by

$$\gamma = \frac{d+1}{d-1} \quad \text{or} \quad d = \frac{\gamma+1}{\gamma-1}\,, \tag{2.3}$$

and scaling constant

$$A_0 = \frac{(\gamma-1)^2}{4\,\gamma}\, C_0^{\gamma-1} \quad \text{with} \quad C_0 = \frac{1}{(d-1)\; B_0}.$$

Note that $\gamma \to 1$ as $d \to \infty$ above, which is the isothermal case studied by Nishida [9]. To recover this case, we set

$$c(h) = \alpha\; e^{h/\alpha}, \tag{2.4}$$

and easily check that this yields $p(v) = \alpha^2/v$.

For smooth solutions the quasilinear form of equations (1.7) is

$$h_t + c(h)\; u_x = 0, \qquad u_t + c(h)\; h_x = 0, \tag{2.5}$$

which is clearly a nonlinear wave equation, and from which the Riemann invariant form of the equations are easily obtained. Moreover, changing the role of the dependent and independent variables in on regions where

$$u_t\; h_x - u_x\; h_t \neq 0,$$

we get the *linear* system

$$t_u + \frac{1}{c(h)}\; x_h = 0, \qquad x_u + c(h)\; t_h = 0,$$

which was known to Riemann [10]; this is just the 1-D hodograph transform [2].

It is clear from (2.5) that the system is hyperbolic, and when it is written as a quasilinear system

$$\mathbf{u}_t + \mathbf{A}\; \mathbf{u}_x = \mathbf{0},$$

with $\mathbf{u} = (h\;\, u)^t$, then the flux matrix $\mathbf{A} = \mathbf{A}(h)$ has eigenvalues $\pm c(h)$ and eigenvectors

$$r_- = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \text{and} \quad r_+ = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \tag{2.6}$$

corresponding to the backward ($-c < 0$) and forward ($c > 0$) waves and wavespeeds, respectively.

2.1. **Simple Waves.** The simple wave curves consist of those states that can be connected to a given fixed state by a one-D rarefaction or compression, and are calculated as the integrals of the eigenvectors. In other words, they solve the equation

$$\frac{d}{d\epsilon}\begin{pmatrix} h \\ u \end{pmatrix} = r_\mp = \begin{pmatrix} 1 \\ \mp 1 \end{pmatrix}.$$

Using h as the parameter, we get

$$u - u_0 = \pm(h_0 - h), \tag{2.7}$$

where the $+$ ($-$) corresponds to backward (resp. forward) waves. For 2×2 systems, these are also the level curves of the opposite Riemann invariants $u \pm h$ [7, 11].

The simple wave is a rarefaction if it is expanding, so that the wavespeed is increasing across the wave from left to right. Thus a backward rarefaction with left state $(h_0\ u_0)^t$ satisfies

$$-c(h_0) = \lambda_-(v_0) \le \lambda_-(v) = -c(h),$$

which, since c increases with h, gives $h \le h_0$. Similarly, a forward rarefaction with left state $(h_0\ u_0)^t$ satisfies (2.7) for $h \ge h_0$.

On the other hand, compressions satisfy the same equation (2.7), but the wavespeed decreases from left to right across the wave. Thus $h \ge h_0$ for a backward compression, and $h \le h_0$ for a forward compression.

By labeling the left, right, ahead and behind states of a simple wave, we thus get the characterization

$$u_r - u_l = h_a - h_b \tag{2.8}$$

for all simple waves, where the wave is rarefying if $c(h_a) > c(h_b)$ (and thus $h_a > h_b$) and compressing if $c(h_b) > c(h_a)$; here $a = l$ and $b = r$ for a backward wave, and $a = r$ and $b = l$ for a forward wave. We shall say a simple wave is an *elementary wave* if h is monotone across the entire wave, so the wave is either rarefying or compressing. It is clear that a general simple wave can then be described as a set of adjacent elementary waves.

2.2. **Shocks.** It is well known that compressions cannot be sustained, and shocks will form in the solution. These are determined by the Rankine-Hugoniot and entropy conditions [7, 11]. The Rankine-Hugoniot equations for (1.7) are

$$\begin{aligned} \sigma\,[v(h)] &= -[u] \quad \text{and} \\ \sigma\,[u] &= [p(h)] \end{aligned} \tag{2.9}$$

where as usual, $[\cdot]$ denotes the jump and σ is the shock speed. Solving, we obtain

$$\begin{aligned} \sigma &= \mp S(h_0, h) \quad \text{and} \\ u - u_0 &= \mp K(h_0, h)\,\mathrm{sgn}(h - h_0), \end{aligned} \tag{2.10}$$

for the backward and forward shocks, respectively, where the absolute shock speed $S(h_1, h_2)$ is defined by

$$S(h_1, h_2) = \sqrt{\frac{p(h_2) - p(h_1)}{v(h_1) - v(h_2)}} > 0 \tag{2.11}$$

and the function $K(h_1, h_2)$ is defined by

$$K(h_1, h_2) = \sqrt{(p(h_2) - p(h_1))\,(v(h_1) - v(h_2))}. \tag{2.12}$$

As above, we view the shock curve as being parameterized by h. We use the Lax condition as our entropy condition, so that for backward shocks with left state $(h_0\ u_0)^t$, we want

$$-c(h_0) > -S(h_0, h) > -c(h), \tag{2.13}$$

which yields $h > h_0$ since $c(h)$ is increasing. Similarly, the forward shock curve is the branch of (2.10) with $h < h_0$.

As above, we combine our descriptions of forward and backward shocks by referring to the left, right, ahead and behind states, respectively. Thus for both families the shock curve is

$$u_r - u_l = -K(h_a, h_b), \tag{2.14}$$

the entropy condition holding provided the (absolute) sound speed is bigger behind the shock, so $h_b > h_a$.

2.3. **Centered Waves.** When solving the Riemann problem, we admit only centered waves, which are those emanating from a single discontinuity at the origin. These are shocks, which have no width, and rarefactions, all of whose characteristics meet at the origin. Using our descriptions (2.8) and (2.14) above, we combine these to describe the centered wave curves.

Defining the function $G : \mathbf{R}^2 \to \mathbf{R}$ by

$$G(h_1, h_2) = \begin{cases} h_1 - h_2, & \text{for} \quad h_1 \geq h_2, \\ -K(h_1, h_2), & \text{for} \quad h_1 \leq h_2, \end{cases} \tag{2.15}$$

we easily check that both forward and backward wave curves can be described by

$$u_r - u_l = G(h_a, h_b), \tag{2.16}$$

the wave being a rarefaction or shock respectively. Using this concise description of the waves allows us to understand the Riemann problem and wave interactions through the functions G and K.

2.4. **Wave strength.** In studying waves and their interactions, we want to measure the difference between the shock and rarefaction curves. It is convenient to work with a new function Θ which directly measures this difference. The function is defined by the identity

$$G(h_1, h_2) = h_1 - h_2 - 2\,\Theta(h_1, h_2), \tag{2.17}$$

so that $\Theta(h_1, h_2) = 0$ for $h_1 \geq h_2$, and

$$\Theta(h_1, h_2) = (K(h_1, h_2) + h_1 - h_2)/2 \quad \text{for} \quad h_1 \leq h_2. \tag{2.18}$$

It is clear that that Θ is supported on shocks, and is a measure of the nonlinearity of the function $K(h_1, h_2)$. We shall refer to Θ as the *shock error* function.

For a simple wave, we have

$$u_r - u_l = G(h_a, h_b) = h_a - h_b,$$

which provides a consistent measure of the (signed) strength of the wave. However, the second equality does not hold for a shock, and it is convenient to instead define the strength of any wave by

$$\Gamma(h_a, h_b) = h_a - h_b - \Theta(h_a, h_b). \tag{2.19}$$

Since Θ is supported on shocks, (2.19) provides a natural definition of strength for any elementary wave. With this definition, we follow the usual convention in that rarefactions have positive strength while shocks (and compressions) have negative strength. Also note that (2.16), (2.17) yield

$$2\,\Gamma(h_a, h_b) = (u_r - u_l) + (h_a - h_b), \tag{2.20}$$

that is the (absolute) wave strength is the average of the (absolute) change in the coordinates u and h across a wave. This is also the change in an appropriate Riemann invariant across the wave.

It is sometimes convenient to assign a single discrete wavespeed to simple waves, across which the actual wavespeed $c(h)$ varies continuously. In such cases, a natural choice is the Hugoniot speed $S(h_1, h_2)$ given in (2.11) for these simple waves. We shall call this the average speed of a simple wave.

2.5. **Scaling.** We now observe that because we have the monomial constitutive law (2.1), the wave curves are scalings of a single curve. We shall usually use the convention of denoting the scaled version of a function $G : \mathbf{R}^2 \to \mathbf{R}$ by $g : \mathbf{R} \to \mathbf{R}$, etc. To begin with, for $n \neq 0$, define

$$q_n(z) = \frac{1 - z^{-n}}{n}, \tag{2.21}$$

and from (2.2), we write

$$p(h_2) - p(h_1) = B_0\, h_2^{d+1}\, q_{d+1}(\frac{h_2}{h_1}) = -B_0\, h_1^{d+1}\, q_{d+1}(\frac{h_1}{h_2}), \tag{2.22}$$

and

$$v(h_1) - v(h_2) = \frac{h_1^{1-d}}{B_0}\, q_{d-1}(\frac{h_2}{h_1}) = -\frac{h_2^{1-d}}{B_0}\, q_{d-1}(\frac{h_1}{h_2}). \tag{2.23}$$

It follows that

$$K(h_1, h_2) = h_1\, k(\frac{h_2}{h_1}) = h_2\, k(\frac{h_1}{h_2}), \tag{2.24}$$

where k is the function defined by

$$k(z) = z^{\frac{d+1}{2}}\, \sqrt{q_{d+1}(z)\, q_{d-1}(z)}. \tag{2.25}$$

Similarly, we rewrite (2.15), (2.17) as

$$G(h_1,h_2) = h_1\ g(\frac{h_2}{h_1}) \quad\text{and}\quad \Theta(h_1,h_2) = h_1\ \theta(\frac{h_2}{h_1}), \tag{2.26}$$

where $g:\mathbf{R}\to\mathbf{R}$ and $\theta:\mathbf{R}\to\mathbf{R}$ are defined by

$$g(z) = 1 - z - 2\,\theta(z) = \begin{cases} 1-z, & \text{for}\quad z\le 1,\\ -k(z), & \text{for}\quad z\ge 1.\end{cases} \tag{2.27}$$

It is clear that the wave strength $\Gamma(h_1,h_2)$ scales similarly,

$$\Gamma(h_1,h_2) = h_1\ \gamma(\frac{h_2}{h_1}), \quad\text{where}$$
$$\gamma(z) = 1 - z - \theta(z) = (1 - z + g(z))/2. \tag{2.28}$$

We shall refer to γ as the scaled wave strength, and θ as the scaled shock error.

We now describe the centered wave curves (2.16). For definiteness, fix the ahead state $(h_a\ u_a)^t$, and draw the locus of states behind a forward wave. For the forward 2-waves, we have $u_r = u_a$, and we get

$$u_b = u_a + h_a\ g(\frac{h_b}{h_a}).$$

Clearly changing u_a simply translates the wave curve vertically, so we can take $u_a = 0$. We can now see that u_b is a similarity scaling of $g(z)$ by h_a. It is clear that the same geometry applies for the wave curves drawn with any fixed state b, l or r. In particular, the shock curves steepen nearer to the vacuum $h = 0$ while preserving the same shape. Graphs of the wave curves are shown in Figure 1. The first diagram is the locus of backward states that can be joined to a given ahead state, while the second is the locus of ahead states with a fixed behind state; as usual, subscripts indicate the direction of the wave. The third diagram shows the shock error $\Theta(h_a,h_b)$ for a given forward shock. Note the steeper curve in the third diagram: this is because h_a is smaller there than in the others, and reflects the scaling of the curves.

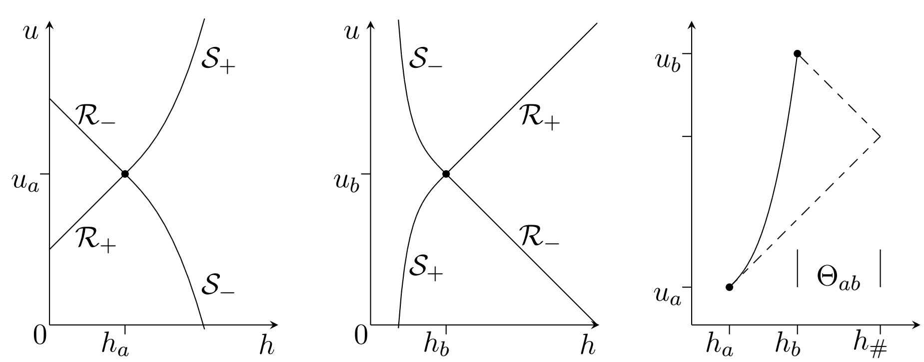

FIGURE 1. Wave curves

2.6. **Properties of Wave Curves.** We now recall the local property that wave curves are C^2, and describe some global properties of the wave curves which are used in the sequel. We begin by analyzing the scaled functions of a single variable.

Lemma 1. *The scaled functions $g(z)$ and $\gamma(z)$ are C^2, monotone decreasing and convex down. The scaled shock error $\theta(z)$ is supported on the interval $[1,\infty)$, monotone non-decreasing and convex up.*

Proof. It is immediate from (2.27) and (2.28) that for $z<1$,

$$g'(z)=\gamma'(z)=-1 \quad \text{and} \quad \theta'(z)=0,$$

so it suffices to consider $z\geq 1$, provided we check the appropriate limits as $z\to 1^+$. We shall work with the function $k(z)$ defined in (2.25); the Lemma will follow because each function differs from k by linear operations for $z\geq 1$.

For $z>1$, write

$$\kappa(z)\equiv \log k(z)=\frac{d+1}{2}\,\log z+\frac{1}{2}\,\log q_{d+1}(z)+\frac{1}{2}\,\log q_{d-1}(z)$$

so that $k(z)=e^{\kappa(z)}$,

$$\begin{aligned} k'(z)&=e^{\kappa(z)}\,\kappa'(z) \quad \text{and}\\ k''(z)&=e^{\kappa(z)}\,\left(\kappa'(z)^2+\kappa''(z)\right). \end{aligned} \tag{2.29}$$

Noting that $q_n'(z)=1/zz^n$ for each $n\neq 0$, we compute

$$\kappa'(z)=\frac{1}{2\,z}\,(d+1+\tau_{d+1}+\tau_{d-1})>0, \tag{2.30}$$

where we have set

$$\tau_n(z)=\frac{1}{z^n\,q_n(z)}=\frac{n}{z^n-1}.$$

This in turn satisfies

$$\tau_n'(z)=\frac{-1}{z}\,\left(n\,\tau_n(z)+\tau_n^2(z)\right),$$

for $n>0$ and $z>1$. Differentiating (2.30), we thus obtain

$$\begin{aligned} \kappa''(z)=&\frac{-1}{2\,z^2}\,\Big(d+1+\tau_{d+1}+\tau_{d-1}\\ &+(d+1)\,\tau_{d+1}+\tau_{d+1}^2+(d-1)\,\tau_{d-1}+\tau_{d-1}^2\Big). \end{aligned} \tag{2.31}$$

Now using (2.30) and (2.31) in (2.29) and manipulating, we get

$$\begin{aligned} \frac{4\,z^2}{k(z)}\,k''(z)&=4\,z^2\,\left(\kappa'(z)^2+\kappa''(z)\right)\\ &=d^2-(\tau_{d+1}-\tau_{d-1}+1)^2. \end{aligned} \tag{2.32}$$

To estimate this, note that we can write $1/\tau_n(z) = \int_1^z \bar{z}^{n-1}\,d\bar{z}$, and for $z > 1$, we have

$$\int_1^z \frac{\bar{z}^{d+1}}{\bar{z}} d\bar{z} > \int_1^z \frac{\bar{z}^{d-1}}{\bar{z}} d\bar{z} > \frac{1}{z^2} \int_1^z \frac{\bar{z}^{d+1}}{\bar{z}} d\bar{z},$$

which yield the inequalities

$$\tau_{d+1}(z) < \tau_{d-1}(z) < z^2\,\tau_{d+1}(z) < z^{d+1}\tau_{d+1}(z).$$

Also, since

$$\tau_n(z) + n = z^n\;\tau_n(z),$$

we have

$$d + \tau_{d+1} - \tau_{d-1} + 1 = z^{d+1}\;\tau_{d+1} - \tau_{d-1} > 0.$$

Moreover,

$$d - \tau_{d+1} + \tau_{d-1} - 1 = d - 1 + \tau_{d-1} - \tau_{d+1} > 0,$$

and substituting these into (2.32), we deduce that $k(z)$ is convex.

Finally, l'Hôpital's rule implies that $k(z)\,\tau_{d\pm1}(z) \to 1$ as $z \to 1$, and, setting $z = 1 + h$ and expanding, we see that

$$\begin{aligned}
\tau_n(z) - \tau_m(z) &= \frac{n}{(1+h)^n - 1} - \frac{m}{(1+h)^m - 1} \\
&= \frac{1/h}{1 + \frac{n-1}{2}h + (h^2)} - \frac{1/h}{1 + \frac{m-1}{2}h + (h^2)} \\
&\to \frac{m-n}{2} \quad \text{as} \quad z \to 1.
\end{aligned}$$

We thus conclude from (2.29), (2.30), (2.32) that

$$k'(1) = 1 \quad \text{and} \quad k''(1) = 0.$$

This yields $g'(1) = \gamma'(1) = -1$ and $\theta'(1) = 0$ and $g''(1) = \gamma''(1) = \theta''(1) = 0$, as desired. □

We now apply the scaling properties to deduce information on the global wave curves. We state our results in terms of the shock error Θ, although they can equally be stated in terms of G or wave strength Γ. These results are not new: indeed, equation (2.34) appears in [11], albeit in a different form than stated here.

Corollary 1. *The shock error function Θ is C^2, convex and monotone non-increasing in the first variable and non-decreasing in the second. We have*

$$\Theta_{;1}(h_1, h_2) + \Theta_{;2}(h_1, h_2) < 0, \tag{2.33}$$

and moreover, if $h_1 \le h_2 \le h_3$, then also

$$0 \le \Theta(h_1, h_2) + \Theta(h_2, h_3) \le \Theta(h_1, h_3). \tag{2.34}$$

Analogous statements hold for the functions G and Γ.

Proof. It is clear from the scaling laws (2.24), (2.26), and (2.28) that the functions are C^2 for $h_i > 0$. We thus calculate the partial derivatives of these functions. Although the functions G, Γ and Θ are defined piecewise, their continuity on the diagonal $h_1 = h_2$ follows by continuity of the corresponding scaled functions at $z = 1$. We again work with $K(h_1, h_2)$, assuming that $h_1 < h_2$. A routine calculation shows that

$$\begin{aligned} K_{;2}(h_1,h_2) &= k'(\frac{h_2}{h_1}) \quad \text{and} \\ K_{;1}(h_1,h_2) &= k(\frac{h_2}{h_1}) - \frac{h_2}{h_1}\, k'(\frac{h_2}{h_1}), \end{aligned} \tag{2.35}$$

and

$$\begin{aligned} K_{;22}(h_1,h_2) &= \frac{1}{h_1}\, k''(\frac{h_2}{h_1}), \\ K_{;12}(h_1,h_2) &= -\frac{1}{h_1}\,\frac{h_2}{h_1}\, k''(\frac{h_2}{h_1}), \quad \text{and} \\ K_{;11}(h_1,h_2) &= \frac{1}{h_1}\,(\frac{h_2}{h_1})^2\, k''(\frac{h_2}{h_1}), \end{aligned} \tag{2.36}$$

provided $h_1 > 0$. The Hessian of $K(h_1, h_2)$ is thus

$$HK = \frac{1}{h_1}\, k''(z) \begin{pmatrix} 1 & -z \\ -z & z^2 \end{pmatrix}, \tag{2.37}$$

where $z = h_2/h_1$, which is positive semi-definite, so that $K(h_1, h_2)$ is convex; (2.18) immediately also implies convexity of $\Theta(h_1, h_2)$.

Since $k'(1) = 1$, convexity of k yields

$$K_{;2}(h_1,h_2) > 1 \quad \text{and} \quad \Theta_{;2}(h_1,h_2) > 0$$

for $h_2 > h_1$, so that $\Theta(h_1, h_2)$ is increasing as a function of h_2. Moreover, using (2.35) and the Mean Value Theorem, we have

$$\begin{aligned} 2\,(\Theta_{;1}(h_1,h_2) + \Theta_{;2}(h_1,h_2)) &= K_{;1}(h_1,h_2) + K_{;2}(h_1,h_2) \\ &= (1-z)\, k'(z) + k(z) - k(1) \\ &= (1-z)\, (k'(z) - k'(\bar z)) < 0, \end{aligned}$$

for some $\bar z$ between 1 and $z \equiv h_2/h_1$. This proves (2.33) and in particular implies $\Theta(h_1, h_2)$ and $K(h_1, h_2)$ are decreasing in h_1.

Finally, suppose we are given $h_1 \le h_2 \le h_3$. Then

$$K(h_1,h_2) + K(h_2,h_3) - K(h_1,h_3) = \int_{h_1}^{h_2} \int_{h_2}^{h_3} K_{;12}(\bar k, \bar h)\, d\bar h\, d\bar k,$$

and the integrand is negative, by (2.36) and $\bar k \le \bar h$. Thus we conclude that if $h_1 \le h_2 \le h_3$, then

$$0 \le K(h_1,h_2) + K(h_2,h_3) \le K(h_1,h_3).$$

Analogous inequalities hold for G and Θ, namely

$$0 \ge G(h_1,h_2) + G(h_2,h_3) \ge G(h_1,h_3), \quad \text{and}$$
$$0 \le \Theta(h_1,h_2) + \Theta(h_2,h_3) \le \Theta(h_1,h_3),$$

since these differ from K by linear operations. The last of these is (2.34), and the proof is complete. □

For completeness, we note that in the isothermal case (2.4), several critical simplifications occur. First, the sound speed $c(h)$ and density $1/v = e^{h/\alpha}$ never vanish, so that there is no vacuum state. After simplifying, (2.12) and (2.11) become

$$K(h_1,h_2) = 2\,\alpha\,\sinh\Big|\frac{h_2-h_1}{2\,\alpha}\Big| \quad \text{and}$$
$$S(h_1,h_2) = \alpha\, e^{h_1/2\alpha}\, e^{h_2/2\alpha} = \sqrt{c(h_1)\,c(h_2)}.$$

Thus each of $G(h_a,h_b)$, $\Theta(h_a,h_b)$ and $\Gamma(h_a,h_b)$ depend only on the difference $h_b - h_a$, and all wave curves are simply translates, rather than scalings, of one fixed curve. This simplified wave curve structure is crucial for Nishida's argument [9].

We now have several ways to classify waves as expansive or compressive, which we summarize here:

Corollary 2. *For an elementary centered wave described by*

$$u_r - u_l = G(h_a,h_b) \quad \text{or} \quad u_r - u_l = h_a - h_b,$$

the wave is a rarefaction if any (and therefore all) of the following equivalent inequalities hold:

$$h_a > h_b; \quad p(h_a) > p(h_b); \quad c(h_a) > c(h_b); \quad v(h_a) < v(h_b);$$
$$G(h_a,h_b) > 0; \quad \Gamma(h_a,h_b) > 0; \quad u_r > u_l.$$

On the other hand, if any (and thus all) of the inequalities are reversed, then the wave is either a simple compression or a shock.

3. The Riemann problem

We now combine the above descriptions of forward and backward waves to solve the Riemann problem with arbitrary left and right states [11]. Thus we are given two states $(h_l\ u_l)^t$ and $(h_r\ u_r)^t$, and we must identify a middle state $(h_*\ u_*)^t$, which connects to $(h_l\ u_l)^t$ and $(h_r\ u_r)^t$ via backward and forward centered waves, respectively.

According to (2.16), if $(h_l\ u_l)^t$ is joined to $(h_*\ u_*)^t$ by a backward centered wave, then we have

$$u_* - u_l = G(h_l,h_*),$$

while if $(h_*\ u_*)^t$ is joined to $(h_r\ u_r)^t$ by a forward wave, then

$$u_r - u_* = G(h_r,h_*).$$

Eliminating u_*, we get the equation

$$u_r - u_l = G(h_l, h_*) + G(h_r, h_*) \equiv f(h_*), \tag{3.1}$$

and we wish to solve for h_*. From (2.17) and Corollary 1, the function f defined in (3.1) is differentiable and monotone decreasing,

$$f'(h_*) = G_{;2}(h_l, h_*) + G_{;2}(h_r, h_*) < 0.$$

Thus (3.1) can be uniquely solved provided $u_r - u_l$ is in the range of f, which is clearly determined by that of G.

Fix h_0, and first consider $h > h_0$, corresponding to a shock wave. Using (2.2), we easily see that

$$G(h_0, h) = -K(h_0, h) \to -\infty \quad \text{as} \quad h \to \infty,$$

so the range of G is unbounded from below. On the other hand, for a centered rarefaction, we have $0 < h < h_0$ and

$$G(h_0, h) = h_0 - h < h_0,$$

so G is bounded from above. Thus the range of f defined in (3.1) is the interval $(-\infty, h_l + h_r)$, and we get a unique solution to the Riemann problem provided that the one-sided condition

$$u_r - u_l < h_l + h_r \tag{3.2}$$

holds. Moreover, by the implicit function theorem, the intermediate state $(h_*\ u_*)^t$ has the same continuity of the function f, which is also that of G. We conclude that $(h_*\ u_*)^t$ is C^2 as a function of left and right states.

3.1. **The Vacuum.** The failure of the one-sided condition (3.2) does not mean that a solution cannot be found: rather, it heralds the appearance of a vacuum [11, 8]. The vacuum occurs when $h = 0$, or equivalently the pressure $p(h)$ and sound speed $c(h)$ vanish, or the specific volume $v = v(h)$ becomes infinite. In the characteristic plane (x, t), our self-similar solution is constant along characteristics,

$$\pm c(h(x,t)) = x/t, \quad \text{and} \quad u(x,t) = u_a \pm G(h_a, h(x,t)). \tag{3.3}$$

Since the vacuum corresponds to sound speed $c(h) = 0$, it must therefore lie on the positive t-axis, $x = 0$. For $t > 0$, $x \neq 0$, the solution is finite and given by (3.3).

We now suppose that (3.2) is violated, and construct a solution containing the vacuum as follows. First, note that the vacuum cannot be the state behind a shock: indeed, the entropy condition (2.13) yields

$$c(h_b) > S(h_a, h_b) > c(h_a), \quad \text{so that} \quad c(h_b) > 0$$

for forward and backward waves, see [8, 13]. Thus if the Riemann problem admits a vacuum, the forward and backward centered waves must both be rarefactions.

For $x < 0$, the gas rarefies along a backward wave, and in the wedge $-c(h_l)\,t \le x < 0$, we have

$$c(h(x,t)) = -x/t \quad \text{and} \quad u(x,t) - u_l = h_l - h(x,t), \tag{3.4}$$

and so we get the limits

$$h(x,t) \to 0 \quad \text{and} \quad u(x,t) \to u_l + h_l \quad \text{as} \quad x \to 0-\,. \tag{3.5}$$

Similarly, in the wedge $0 < x \le c(h_r)\,t$, we have

$$\begin{aligned} c(h(x,t)) = x/t \quad \text{and} \quad u_r - u(x,t) = h_r - h(x,t), \quad \text{with} \\ h(x,t) \to 0 \quad \text{and} \quad u(x,t) \to u_r - h_r \quad \text{as} \quad x \to 0+\,. \end{aligned} \tag{3.6}$$

We conclude that for any fixed $t > 0$, h decreases for $x < 0$ and increases for $x > 0$, and that u increases for all $x \ne 0$. On the t-axis $x = 0$, the velocity has a jump with left and right limits given by

$$u_- \equiv u(0-,t) = u_l + h_l \le u_r - h_r = u(0+,t) \equiv u_+,$$

respectively, and so $u(x,t)$ is monotone increasing and bounded as a function of x.

Although h is finite and bounded, the specific volume $v = v(h)$ is infinite, and we refer to this when studying the vacuum. To take into account the jump in u, we take v to be a Radon measure, while the other variables remain bounded. This measure is singular only at the vacuum, so its singular part is supported on the t-axis $x = 0$. Since in addition v must be locally integrable, this singular part must have the form

$$\nu = w(t)\;\delta(x).$$

Since the velocity u is defined (a.e.) above, we can find the weight $w(t)$ by solving the equation

$$v_t - u_x = 0 \tag{3.7}$$

in the sense of distributions. By construction, the equation is satisfied away from the t-axis $x = 0$. On this axis, (3.7) reduces to

$$\frac{dw}{dt} = u_+ - u_-, \tag{3.8}$$

so that the singular part of $v(x,t)$ is the self-similar measure

$$\nu = (u_+ - u_-)\,t\;\delta(x) = (u_+ - u_-)\;\delta(x/t). \tag{3.9}$$

Although the specific volume v is unbounded, it is locally integrable in space. Indeed, for t fixed,

$$\begin{aligned} \int_{-c(h_l)\,t}^{-\epsilon\,t} v(x,t)\;dx &= x\,v(x,t)\,\Big|_{-c(h_l)\,t}^{-\epsilon\,t} - \int_{v(h_l)}^{v(-\epsilon\,t,t)} x\;dv \\ &= -t\;c\,v\Big|_{v(h_l)}^{v(-\epsilon\,t,t)} + t\int_{v(h_l)}^{v(-\epsilon\,t,t)} c\;dv \\ &\le t\;(c(h_l)\;v(h_l) + h_l), \end{aligned}$$

for all ϵ, where we have integrated by parts and used (1.5) and (3.4). A similar estimate holds for the forward rarefaction, and we get

$$\int_{-c(h_l)\,t}^{c(h_r)\,t} v(x,t)\,dx = t\,(c(h_l)\,v(h_l) + h_l) \\ + (u_+ - u_-)\,t + t\,(c(h_r)\,v(h_r) + h_r). \tag{3.10}$$

We have proved the following classical theorem, which we state in terms of the specific volume v [11].

Theorem 1. *Given constant left and right states $(v_l\ u_l)^t$ and $(v_r\ u_r)^t$, respectively, there is a unique self-similar solution $(v(x,t), u(x,t))^t$ to the Riemann problem. If condition (3.2) holds,*

$$u_r - u_l < h(v_l) + h(v_r),$$

there is an intermediate state $(v_*\ u_*)^t$ which is a C^2 function of the data. If (3.2) fails, then for each fixed $t > 0$, the velocity

$$u(x,t) \in L^\infty \cap BV \cap L^1_{loc}$$

is a bounded monotone increasing function, while $v(x,t)$ is a Radon measure whose singular part is the Dirac measure (3.9). Moreover this solution is Lipschitz continuous in time as a distribution in L^1_{loc}.

3.2. **Bound on intermediate states.** Because we are studying pairwise interactions of waves, we wish to consider more possibilities than those offered by the Riemann problem alone. For example, consider the interaction of a rarefaction with a shock wave of the same family. In a standard difference approximation such as Glimm's scheme or the Front Tracking method, the interaction is regarded as taking place instantaneously at a single point, and the interaction is resolved by solving a new Riemann problem [4, 1]. Although this is a good approximation for shocks and weak rarefactions, it is not appropriate for strong elementary waves, which have finite width. Thus in resolving the interaction, we wish to allow some of the waves to have nonzero width, as appropriate.

This extension is easily handled by the observation that shocks are the only waves that have zero width, and thus any wave of finite width is necessarily simple. On the other hand, the states across the simple waves are described linearly by (2.8). Thus the states in these interactions can be easily resolved, and we do not resolve the actual wave profiles, which would require integration of the characteristics themselves.

We describe this situation by supplying an extra datum χ, a zero width or shock indicator, which is 1 if a wave has zero width, and 0 otherwise. We allow elementary waves, which can be rarefactive or compressive (but not both), and we wish to resolve the intermediate state, without resolving the wave profiles. Recall that the shock error appears only for shocks, that is when $\chi = 1$. According to (2.8), (2.14), (2.17), any wave can thus be described by

$$u_r - u_l = h_a - h_b - 2\,\chi_w\,\Theta(h_a, h_b), \tag{3.11}$$

where χ is the zero width indicator,

$$\chi_w = \begin{cases} 1, & w = 0 \\ 0, & w > 0, \end{cases}$$

and w is the spatial width of the wave. Clearly the term $\chi_w\,\Theta(h_a, h_b)$ appears only when there is a shock, which implies both conditions

$$w = 0 \quad \text{and} \quad h_a < h_b,$$

so that if either of these fail, the states across the wave are described linearly.

We describe the solution of the resultant "extended Riemann problem," and give an upper bound for the intermediate state. Provided (3.2) holds, we are looking for an intermediate state $(h_*\ u_*)^t$, which satisfies both

$$\begin{aligned} u_* - u_l &= h_l - h_* - 2\,\chi_{w_-}\,\Theta(h_l, h_*) \quad \text{and} \\ u_r - u_* &= h_r - h_* - 2\,\chi_{w_+}\,\Theta(h_r, h_*). \end{aligned} \tag{3.12}$$

Lemma 2. *Given extended Riemann data $(h_l\ u_l)^t$, $(h_r\ u_r)^t$ including widths $\{w_-, w_+\}$, to be resolved into the intermediate state $(h_*\ u_*)^t$, we set*

$$h_\# = (u_l - u_r + h_l + h_r)/2. \tag{3.13}$$

If $h_\# \le 0$, a vacuum is present in the solution, while otherwise, a unique solution h_* exists and is bounded,

$$0 < h_* \le h_\#.$$

Moreover, this inequality is strict if and only if at least one of the outgoing waves is a shock (and in particular $w_-\,w_+ = 0$).

Proof. Eliminating u_* from (3.12) and using (3.13) gives

$$h_* + \chi_{w_-}\,\Theta(h_l, h_*) + \chi_{w_+}\,\Theta(h_r, h_*) = h_\#. \tag{3.14}$$

Since $\Theta(h., h_*) \ge 0$, we have $h_* \le h_\#$ whenever there is no vacuum, i.e. whenever $h_\# > 0$. Moreover, (3.14) has the trivial solution $h_* = h_\#$ only if both Θ terms vanish, i.e. if there are no shocks. $\square$

For completeness, we now recall the well-known invariant region [11, 3]. According to (3.12), since $\Theta(h_1, h_2) \ge 0$ for all h_1, h_2, the solution of the (extended) Riemann problem satisfies both

$$\begin{aligned} u_* + h_* &\le u_l + h_l \le \max\{u_l + h_l, u_r + h_r\} \quad \text{and} \\ u_* - h_* &\ge u_r - h_r \ge \min\{u_l - h_l, u_r - h_r\}. \end{aligned}$$

Since the general solution consists of waves and their interactions, pieced together, and these respect the inequalities, given Cauchy data $(h_0(x)\ u_0(x))^t$, we obtain the bounds

$$\begin{aligned} u(x,t) + h(x,t) &\le \sup_x \{u_0(x) + h_0(x)\} \quad \text{and} \\ u(x,t) - h(x,t) &\ge \inf_x \{u_0(x) - h_0(x)\} \end{aligned} \tag{3.15}$$

for any reasonable approximation of the solution. Moreover, since these bounds are independent of the approximation parameter, we conclude that they hold in the limit, provided of course that we show that the limit exists. In particular, we get the *a priori* upper bound

$$2\,h(x,t) \le \sup_x \{u_0(x) + h_0(x)\} - \inf_x \{u_0(x) - h_0(x)\} \tag{3.16}$$

for the thermodynamic variable h. This is the easy case: we also need to find wave interaction estimates which are uniform as $h \to 0$.

4. Pairwise Glimm Interactions

We now analyze pairwise wave interactions. Recall that Glimm interactions resolve the states adjacent to the various waves while ignoring the actual characteristic patterns. As above, we shall distinguish between compressions and shock waves by use of extra data. Simple waves thus include both rarefactions and compressions, and we'll denote waves as $\mathcal{S}_\pm$, $\mathcal{C}_\pm$, $\mathcal{R}_\pm$, referring to forward and backward shocks, compressions and rarefactions, respectively. It is clear that interactions are symmetric for forward and backward waves, and we will isolate the forward waves when considering a single family; the corresponding statements for backward waves follow immediately.

Recall from (3.11) that a wave is described by

$$u_r - u_l = h_a - h_b - 2\,\chi\,\Theta(h_a, h_b),$$

and the (signed) strength of the wave is

$$\Gamma(h_a, h_b) = h_a - h_b - \chi\,\Theta(h_a, h_b), \tag{4.1}$$

where the zero width indicator $\chi = 1$ if the wave is a shock, and zero otherwise, and the subscripts refer to the right, left, ahead and behind states, respectively.

Recall that elementary waves are either rarefactive or compressive, but not both. Thus, the splitting of a simple wave, or joining of two or more such waves along characteristics can be considered a trivial interaction (or more precisely no interaction!) in which wave strengths add or subtract, and no wave is reflected.

We briefly describe which (pairwise) interactions can take place. First, a self-interaction occurs when a compression collapses. Next, two waves of the same family may merge, provided at least one of them is a shock. Finally, two waves of opposite families may cross. We treat the collapse of a compression as a merge of two (compressive) simple waves, so there are essentially only two cases. We do not consider composite interactions, such as forward and backward compressions focussing at the same point, because these can be regarded as superpositions of simpler interactions.

4.1. **Merge of forward waves.** We first consider the interaction of a pair of forward waves, given by

$$u_m - u_l = h_m - h_l - 2\,\chi_{ml}\,\Theta(h_m, h_l) \quad \text{and}$$
$$u_r - u_m = h_r - h_m - 2\,\chi_{rm}\,\Theta(h_r, h_m), \tag{4.2}$$

respectively, and depicted in Figures 2 and 3. In order for the waves to interact, either at least one of the waves is a shock, or both waves are compressions focussing at a single point; this is a self-interaction. Moreover, in order for the waves to meet, the wavespeed of the left wave must exceed that of the right:

$$S(h_m, h_l) > S(h_r, h_m),$$

the wavespeed being given by (2.11), and we have used the Hugoniot speed to approximate the average speed of a simple wave. Since S is symmetric and increases in each variable, we necessarily have the condition $h_l > h_r$ on the outside states. We characterize the incoming waves by comparing h_m to these states.

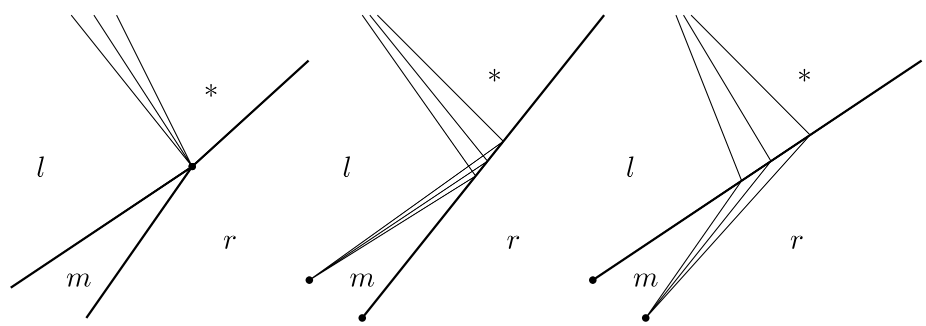

FIGURE 2. Merges of forward waves in (x,t)-space

In solving the extended Riemann problem for $(h_*\ u_*)^t$ to find the resulting waves, we recall that the outgoing forward (transmitted) wave cannot be a simple compression, so $\chi_{r*} = 1$, while the reflected backward wave has the spatial width of the interaction, which is the larger of the widths of the two incident waves. In particular, if one of the forward waves is a rarefaction, the interaction takes place over a finite time interval, and the reflected wave cannot be a shock. The outgoing waves are thus given by

$$u_* - u_l = h_l - h_* - 2\,\chi_{l*}\,\Theta(h_l, h_*) \quad \text{and}$$
$$u_r - u_* = h_r - h_* - 2\,\Theta(h_r, h_*), \tag{4.3}$$

where $\chi_{l*} = 0$ if one of the incident waves is a rarefaction.

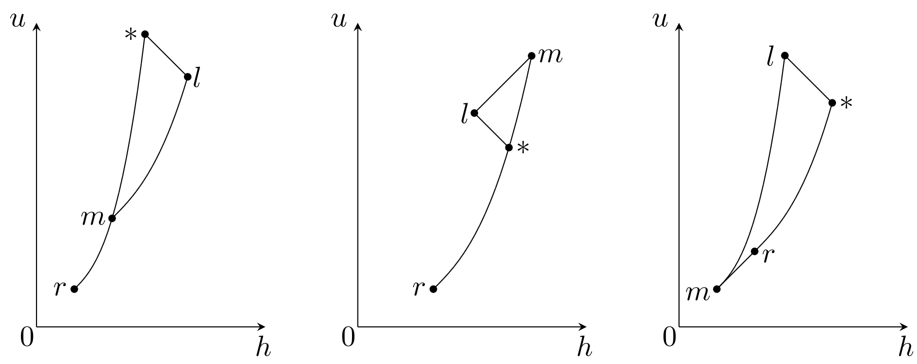

FIGURE 3. Merges of forward waves in (h, u)-space

We eliminate u in (4.2), (4.3) to get

$$\begin{aligned} h_* + \Theta(h_r, h_*) + \chi_{l*}\, \Theta(h_l, h_*) \\ = h_l + \chi_{ml}\, \Theta(h_m, h_l) + \chi_{rm}\, \Theta(h_r, h_m), \end{aligned} \tag{4.4}$$

which in turn implies

$$\begin{aligned} h_* - h_l + \chi_{l*}\, \Theta(h_l, h_*) + \Theta(h_r, h_*) - \Theta(h_r, h_l) \\ = \chi_{ml}\, \Theta(h_m, h_l) + \chi_{rm}\, \Theta(h_r, h_m) - \Theta(h_r, h_l). \end{aligned}$$

Now by Corollary 1, the LHS has the sign of $h_* - h_l$, so the type of the reflected wave is determined by the sign of the RHS. If $h_l > h_m > h_r$, so that both incident waves are compressive, then (2.34) implies that this RHS is negative, and so $h_l > h_*$ and the reflected backward wave is a rarefaction. On the other hand, if $h_m < h_r$ then the right wave is a rarefaction and the left is necessarily a shock, so $\chi_{ml} = 1$ and the RHS is

$$\Theta(h_m, h_l) - \Theta(h_r, h_l) > 0;$$

similarly, if $h_m > h_l$ the RHS is $\Theta(h_r, h_m) - \Theta(h_r, h_l) > 0$. In both of these latter cases, $h_* > h_l$ and the reflected wave is compressive, but since one of the incident waves is a rarefaction, it has positive width and $\chi_{l*} = 0$. Thus in all cases the term $\chi_{l*}\, \Theta(h_l, h_*)$ in (4.4) vanishes. In terms of wave strength, (4.1) and (4.4) yield

$$\begin{aligned} -\Gamma(h_r, h_*) &= h_* - h_r + \Theta(h_r, h_*) \\ &= h_l - h_r + \chi_{ml}\, \Theta(h_m, h_l) + \chi_{rm}\, \Theta(h_r, h_m) > 0, \end{aligned}$$

since $h_l > h_r$. It follows from Corollary 2 that $h_* > h_r$, and the transmitted wave is necessarily a shock. Moreover, it follows immediately from (4.1) that

$$\Gamma(h_r, h_*) = \Gamma(h_m, h_l) + \Gamma(h_r, h_m),$$

so that the forward wave strengths add *exactly*. We summarize the foregoing in the following theorem.

Theorem 2. *When two forward waves merge (or a compression collapses), a shock of that family results and a simple backward wave is reflected. Moreover, the wave strengths add linearly,*

$$\Gamma(h_r, h_*) = \Gamma(h_m, h_l) + \Gamma(h_r, h_m), \tag{4.5}$$

while the reflected wave has signed strength

$$\Gamma(h_l, h_*) = \Theta(h_r, h_*) - \chi_{ml}\, \Theta(h_m, h_l) - \chi_{rm}\, \Theta(h_r, h_m). \tag{4.6}$$

If both incident waves are compressive, the reflected wave is a rarefaction, while if one incident wave is a rarefaction, then the reflected wave is a compression. Analogous statements hold for the merge of backward waves.

Three merges of forward waves are illustrated in Figures 2 and 3, in characteristic and state space, respectively. The first diagram shows two shocks merging, while the others are a rarefaction catching up to a shock and a shock catching up to a rarefaction, respectively.

4.2. **Crossing waves.** We now consider crossing forward and backward waves. We refer to the states using the (directional) subscripts w, s, e and n, so that the incoming forward and backward waves are given by

$$\begin{aligned} u_s - u_w &= h_s - h_w - 2\, \chi_{sw}\, \Theta(h_s, h_w) \quad \text{and} \\ u_e - u_s &= h_s - h_e - 2\, \chi_{se}\, \Theta(h_s, h_e), \end{aligned}$$

respectively, as depicted in Figures 4 and 5. Since we have excluded composite interactions in which one wave may collapse, each wave has a continuous width through the interaction, and we can write the outgoing waves as

$$\begin{aligned} u_n - u_w &= h_w - h_n - 2\, \chi_{wn}\, \Theta(h_w, h_n) \quad \text{and} \\ u_e - u_n &= h_e - h_n - 2\, \chi_{en}\, \Theta(h_e, h_n), \end{aligned}$$

where

$$\chi_{en} = \chi_{sw} \quad \text{and} \quad \chi_{wn} = \chi_{se}.$$

Solving for h_n, we get

$$\begin{aligned} h_n &+ \chi_{en}\, \Theta(h_e, h_n) + \chi_{wn}\, \Theta(h_w, h_n) \\ &= h_e + h_w - h_s + \chi_{sw}\, \Theta(h_s, h_w) + \chi_{se}\, \Theta(h_s, h_e), \end{aligned} \tag{4.7}$$

which yields both

$$\begin{aligned} \Gamma(h_e, h_n) - \chi_{wn}\, \Theta(h_w, h_n) &= \Gamma(h_s, h_w) - \chi_{se}\, \Theta(h_s, h_e) \quad \text{and} \\ \Gamma(h_w, h_n) - \chi_{en}\, \Theta(h_e, h_n) &= \Gamma(h_s, h_e) - \chi_{sw}\, \Theta(h_s, h_w). \end{aligned} \tag{4.8}$$

Rewriting this as

$$\begin{aligned} \Gamma(h_e, h_n) &- \chi_{se}\, (\Theta(h_w, h_n) - \Theta(h_w, h_e)) \\ &= \Gamma(h_s, h_w) - \chi_{se}\, (\Theta(h_s, h_e) - \Theta(h_w, h_e)), \end{aligned}$$

and again using Corollary 1, the LHS has the sign of $h_n - h_e$, while the RHS has that of $h_w - h_s$. It follows that the outgoing forward wave is the same type as the incoming forward wave. Clearly the same conclusion holds for the backward waves.

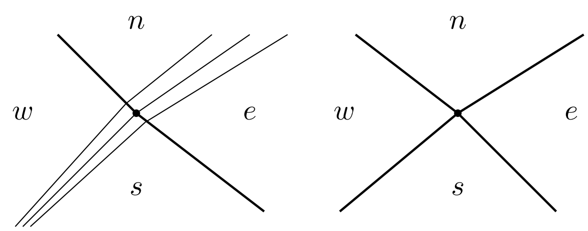

FIGURE 4. Waves crossing in (x, t)-space

Note that if one wave of the waves is simple, then $\chi = 0$ and (4.8) implies that the other wave has unchanged strength across the interaction. In particular if both waves are simple, the interaction is linear in terms of wave strengths, although the underlying characteristics change nonlinearly. On the other hand, if the forward wave crosses a shock, then its strength changes as

$$\Gamma(h_e, h_n) - \Gamma(h_s, h_w) = \Theta(h_w, h_n) - \Theta(h_s, h_e). \tag{4.9}$$

We shall show in Corollary 3 below that this has the same sign as $\Gamma(h_s, h_w)$, which in turn implies that a simple wave's strength *increases* after it crosses a shock.

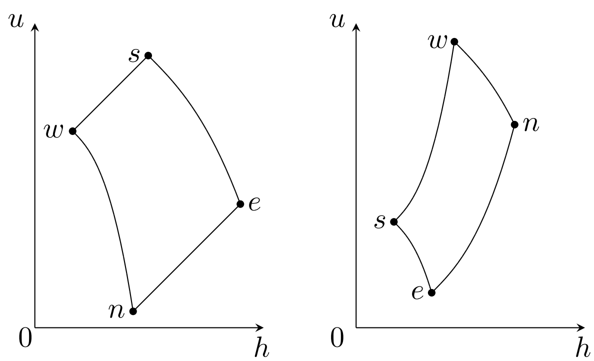

FIGURE 5. Waves crossing in (h, u)-space

Figures 4 and 5 illustrate two interactions in which waves cross: on the left a forward rarefaction crosses a backward shock, and on the right we see two shocks crossing. We have proved the following theorem:

Theorem 3. *If a wave of one family crosses a simple (rarefaction or compression) wave of the opposite family, its strength is unchanged during and after the interaction. If it crosses a shock of the other family, it emerges stronger, and the difference in its wave strength is given exactly by (4.9).*

In particular, no wave may change type by crossing a wave of the opposite family.

We have seen that a shock crossing an opposite rarefaction undergoes no change in strength. In particular, if the rarefaction extends up to the vacuum, $h = 0$, the shock will still have the same strength as it meets the vacuum. The shock is absorbed into the vacuum, while changing the edge velocity of the vacuum. The author addresses this situation in detail in the upcoming paper [15].

4.3. **Crossing Shocks.** We have seen that a shock preserves its strength when crossing a simple wave, but not when it crosses an opposite shock. We now consider that situation: a forward 2-shock with $(h_s\ u_s)^t$ ahead and $(h_w\ u_w)^t$ behind meets a backward 1-shock with state $(h_e\ u_e)^t$ behind, resulting in two outgoing shocks with the resultant behind state $(h_n\ u_n)^t$, as in the second part of Figure 5. Here we give an initial estimate of the outgoing wave strength. In Section 6, we will give a more precise estimate of the interaction.

We describe each of the waves using (2.14), and eliminate the velocities to get the equation

$$G(h_s, h_w) + G(h_s, h_e) = G(h_w, h_n) + G(h_e, h_n),$$

in which we solve for h_n. Since the incident waves are shocks, the remarks following (4.8) show that the outgoing waves are also shocks, and we have

$$h_n > h_w > h_s \quad \text{and} \quad h_n > h_e > h_s. \tag{4.10}$$

Thus for all four waves, we have $G(h_a, h_b) = -K(h_a, h_b)$, and we rewrite the shock crossing interaction as

$$K(h_s, h_w) + K(h_s, h_e) = K(h_w, h_n) + K(h_e, h_n). \tag{4.11}$$

Using the scaling law (2.24), this becomes

$$k(\frac{h_w}{h_s}) + k(\frac{h_e}{h_s}) = \frac{h_w}{h_s}\ k(\frac{h_n}{h_w}) + \frac{h_e}{h_s}\ k(\frac{h_n}{h_e}), \tag{4.12}$$

which gives $\frac{h_n}{h_s}$ as a function of two variables $\frac{h_w}{h_s}$ and $\frac{h_e}{h_s}$.

First, if $h_w = h_e$, then (4.12) becomes

$$k(\frac{h_w}{h_s}) = \frac{h_w}{h_s}\ k(\frac{h_n}{h_e}) > k(\frac{h_n}{h_e}),$$

so that $h_n/h_e < h_w/h_s$ since $k(z)$ is increasing for $z > 1$. On the other hand, suppose that

$$h_n = \frac{h_e\, h_w}{h_s},$$

so that $\frac{h_n}{h_e} = \frac{h_w}{h_s}$ and $\frac{h_n}{h_e} = \frac{h_e}{h_s}$. In this case, (4.12) becomes

$$(1 - \frac{h_e}{h_s})\ k(\frac{h_w}{h_s}) = (\frac{h_w}{h_s} - 1)\ k(\frac{h_e}{h_s}),$$

and since the coefficients have opposite sign, this can happen only if $h_w = h_s$ or $h_e = h_s$, so that one of the incident shocks vanishes. Thus by continuity, we conclude that $h_n < h_e\ h_w/h_s$ for all crossing interactions of nonzero shocks, and in particular

$$\frac{h_n}{h_w} < \frac{h_e}{h_s} \quad \text{and} \quad \frac{h_n}{h_e} < \frac{h_w}{h_s}. \tag{4.13}$$

Thus after the interaction, the scaled shock strengths decrease,

$$|\gamma(\frac{h_n}{h_w})| < |\gamma(\frac{h_e}{h_s})| \quad \text{and} \quad |\gamma(\frac{h_n}{h_e})| < |\gamma(\frac{h_w}{h_s})|, \tag{4.14}$$

while the usual shock strength increases, but we have the bounds

$$\begin{aligned} |\Gamma(h_s, h_e)| <&|\Gamma(h_w, h_n)| < \frac{h_w}{h_s}\ |\Gamma(h_s, h_e)| \quad \text{and} \\ |\Gamma(h_s, h_w)| <&|\Gamma(h_e, h_n)| < \frac{h_e}{h_s}\ |\Gamma(h_s, h_w)|. \end{aligned} \tag{4.15}$$

We shall refine these estimates in Section 6 below.

5. Bounds for Reflections

We have seen that the nonlinear effects of wave interactions can be described exactly, and that with the correct choice of wave strength, waves of the same family add linearly. We also have exact expressions for the reflected waves, and for the change in strength when a simple wave crosses a shock. Here we analyze the reflected waves in more detail, and obtain bounds and monotonicity conditions.

5.1. **Wave strength as variable.** The main difficulty we encounter in our treatment of reflected waves is finding bounds for them: these are differences of the type

$$\Theta(h_r, h_*) - \Theta(h_m, h_l) - \Theta(h_r, h_m),$$

say, from (4.6). It is clear from (2.18), (2.12) and (2.2) that for fixed behind state h_b, the shock error $\Theta(h_a, h_b) \to \infty$ as $h_a \to 0$, so it is not obvious that the reflection is uniformly bounded. In order to obtain bounds, we express the various quantities associated with a wave in terms of the wave strength, and use that to estimate reflected waves.

Recalling the definition of wave strength (2.19),

$$\Gamma(h_a, h_b) = h_a - h_b - \Theta(h_a, h_b),$$

Corollary 1 yields

$$\begin{aligned} \Gamma_{;1}(h_a, h_b) &= 1 - \Theta_{;1}(h_a, h_b) \geq 1 \quad \text{and} \\ \Gamma_{;2}(h_a, h_b) &= -1 - \Theta_{;2}(h_a, h_b) \leq -1 \end{aligned}$$

for all h_a, h_b. It follows that we can regard h_a as fixed and treat h_b as a function of the wave strength Γ, and similarly define h_a in terms of Γ

(and h_b). That is, we implicitly define the functions $h_b = \Phi(h, \mathcal{Z})$ and $h_a = \Psi(h, \mathcal{Z})$ by the identities

$$\mathcal{Z} = \Gamma(h, \Phi(h, \mathcal{Z})) = h - \Phi(h, \mathcal{Z}) - \Theta(h, \Phi(h, \mathcal{Z})) \tag{5.1}$$

and

$$\mathcal{Z} = \Gamma(\Psi(h, \mathcal{Z}), h) = \Psi(h, \mathcal{Z}) - h - \Theta(\Psi(h, \mathcal{Z}), h), \tag{5.2}$$

respectively.

Note that for fixed wave strength $\mathcal{Z}$, the functions Φ and Ψ are inverse functions with regard to the state variable:

$$h_b = \Phi(h_a, \mathcal{Z}) \quad \text{iff} \quad h_a = \Psi(h_b, \mathcal{Z}). \tag{5.3}$$

Similarly, if we fix one state, then the other state and wave strength are inverses:

$$\Gamma(h_a, h_b) = \mathcal{Z} \quad \text{iff} \quad \Phi(h_a, \mathcal{Z}) = h_b \quad \text{iff} \quad \Psi(h_b, \mathcal{Z}) = h_a.$$

We will refer to the functions Φ and Ψ as the behind and ahead state functions, respectively.

Having described the states across a wave in terms of the wave strength, we now describe the shock error $\Theta(h_a, h_b)$ in the same way: that is, we set

$$\Omega(h, \mathcal{Z}) = \Theta(h, \Phi(h, \mathcal{Z})), \tag{5.4}$$

where we have fixed the ahead state $h = h_a$. We can similarly write the shock error by fixing the behind state h_b, but we shall not do so explicitly.

Recalling that Θ is supported on shocks, we note that for $\mathcal{Z} \geq 0$, we have the linear relations

$$\Phi(h, \mathcal{Z}) = \mathcal{Z} - h \quad \text{and} \quad \Psi(h, \mathcal{Z}) = \mathcal{Z} + h,$$

and $\Omega(h, \mathcal{Z}) = 0$.

Finally, we observe that we can extend these functions up to the vacuum. Indeed, the wave adjacent to the vacuum is simple, and a shock crossing a simple wave has constant strength; thus we can take the limit as $h \to 0$ in the above functions. There is clearly no difficulty when $\mathcal{Z} \geq 0$. Now consider a shock of fixed strength $\mathcal{Z} < 0$ approaching the vacuum by passing through an adjacent (opposite) rarefaction: we let h be the state ahead of the shock as it passes through the rarefaction, so $\Phi(h, \mathcal{Z})$ is the corresponding state behind the shock. Since the shock is absorbed into the vacuum, the behind state has limit

$$\Phi(h, \mathcal{Z}) \to 0 \quad \text{as} \quad h \to 0,$$

and since $\Psi = \Phi^{-1}$ as functions of h, we have also

$$\Psi(h, \mathcal{Z}) \to 0 \quad \text{as} \quad h \to 0.$$

From (5.1), (5.4), Ω can also be written as

$$\Omega(h, \mathcal{Z}) = h - \Phi(h, \mathcal{Z}) - \mathcal{Z}, \tag{5.5}$$

so taking the limit as $h \to 0$ we get

$$\Omega(0, \mathcal{Z}) = -\mathcal{Z}, \tag{5.6}$$

where $\mathcal{Z} < 0$.

5.2. **Effect of scaling.** The behind state Φ and shock error Ω scale by the ahead state h as did our earlier functions: using (2.26) in (5.1) yields

$$\frac{\mathcal{Z}}{h} = \frac{\Gamma(h,\Phi)}{h} = 1 - \frac{\Phi}{h} - \theta(\frac{\Phi}{h}),$$

where $\Phi = \Phi(h,\mathcal{Z})$. Thus setting $\zeta = \mathcal{Z}/h$ and defining the *scaled behind state* $\phi(\gamma)$ implicitly by the relation

$$\zeta = \gamma(\phi(\zeta)) = 1 - \phi(\gamma) - \theta \circ \phi(\gamma), \tag{5.7}$$

we have the expected scaling relation

$$\Phi(h,\mathcal{Z}) = h\ \phi(\zeta) = h\ \phi(\mathcal{Z}/h). \tag{5.8}$$

Comparing (2.28) and (5.7), it follows that the scaled behind state ϕ is the inverse function of the scaled wave strength $\gamma(z)$, as expected. Referring to (5.4), the shock error also scales as

$$\Omega(h,\mathcal{Z}) = h\ \varpi(\zeta) = h\ \varpi(\mathcal{Z}/h), \quad \text{where} \quad \varpi = \theta \circ \phi \tag{5.9}$$

gives the scaled shock error as a function of scaled wave strength.

We can also derive a scaling law for the ahead state, as follows. Using the scaling law (5.8) in (5.3), we have $h_a = \Psi(h_b,\mathcal{Z})$ if and only if

$$\frac{h_b}{h_a} = \phi(\frac{\mathcal{Z}}{h_a}), \quad \text{or} \quad \frac{\mathcal{Z}}{h_a} = \gamma(\frac{h_b}{h_a}).$$

Now setting

$$\eta = \frac{\mathcal{Z}}{h_b}, \quad y = \psi(\eta) = \frac{h_a}{h_b} \quad \text{and} \quad z = \frac{1}{y},$$

this becomes

$$\frac{\eta}{\psi(\eta)} = \gamma(\frac{1}{\psi(\eta)}), \tag{5.10}$$

or alternatively $y = \psi(\eta) = 1/z(\eta)$, where $z = z(\eta)$ is implicitly given by

$$\eta\ z = \gamma(z). \tag{5.11}$$

Thus, if ψ is implicitly defined by (5.10), it follows that Ψ has the scaling

$$\Psi(h_b,\mathcal{Z}) = h_b\ \psi(\frac{\mathcal{Z}}{h_b}), \tag{5.12}$$

analogous to (5.8), (5.9). Alternatively, we can express Ψ as an homogeneous function directly for $\mathcal{Z} < 0$ by writing (2.18), (2.19), (2.24) as

$$\eta = (y - 1 - k(y))/2,$$

where $\eta = \mathcal{Z}/h_b$ and $y = \psi(\eta) = \Psi/h_b < 1$; also note $k(y) = y\ k(\frac{1}{y})$.

Figure 5.2 shows the various scaled functions defined here: the first picture shows wave strength $\gamma(z)$ and shock error $\theta(z)$ as functions of the (scaled) behind state; the second shows behind state $\phi(\gamma)$ and shock error $\varpi(\gamma)$ as functions of wave strength γ; and the third shows the ahead state $\psi(\eta)$ as a function of wave strength, now scaled by the behind state.

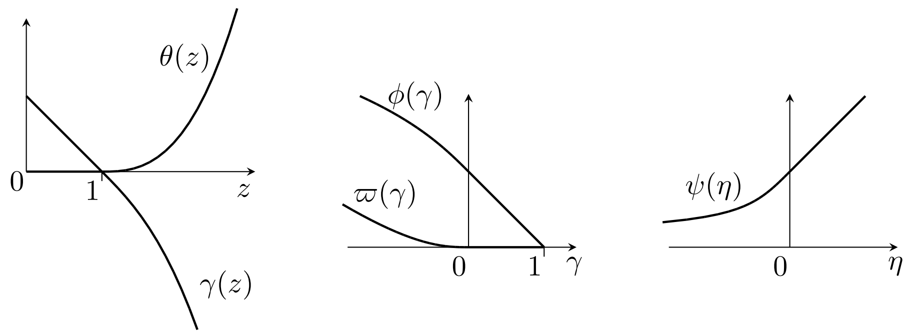

FIGURE 6. Scaled opposite states and shock error functions

5.3. **Derivatives and Convexity.** We now use the scaling rules to calculate the derivatives of the opposite state and shock error, expressed in terms of wave strength. Recall that the nonlinearities are supported only on shocks, so the derivatives are trivial for $\mathcal{Z} \geq 0$.

A routine calculation as in the proof of Corollary 1 gives formulae analogous to (2.35), (2.36): since $F(h, \mathcal{Z}) = h\, f(\zeta)$ with $\zeta = \mathcal{Z}/h$, we have

$$\begin{aligned} F_{;\mathcal{Z}}(h, \mathcal{Z}) &= f'(\zeta) \quad \text{and} \\ F_{;h}(h, \mathcal{Z}) &= f(\zeta) - \zeta\, f'(\zeta), \end{aligned} \tag{5.13}$$

and

$$\begin{aligned} F_{;\mathcal{Z}\mathcal{Z}}(h, \mathcal{Z}) &= \frac{1}{h}\, f''(\zeta), \\ F_{;h\mathcal{Z}}(h, \mathcal{Z}) &= -\frac{1}{h}\, \zeta\, f''(\zeta), \quad \text{and} \\ F_{;hh}(h, \mathcal{Z}) &= \frac{1}{h}\, \zeta^2\, f''(\zeta), \end{aligned} \tag{5.14}$$

where F is each of Φ, Ψ and Ω, respectively, with f the corresponding scaled function. We note here that $f''(\zeta) = 0$ for $\zeta \geq 0$, and all three second derivatives have the same sign for $\zeta < 0$. In particular, as with (2.37), F has the convexity of the scalar function f.

It thus suffices to calculate the derivatives of the scaled functions $\gamma(z)$, $\phi(\zeta)$, $\psi(\eta)$ and $\varpi(\zeta)$. We express these derivatives using Corollary 1.

Lemma 3. *The functions Φ, Ψ and Ω are convex, C^2 functions defined for $h \geq 0$ and for all $\mathcal{Z}$, and whose derivatives satisfy the bounds*

$$\begin{aligned} -1 \leq{} &\Phi_{;\Upsilon}(h, \mathcal{Z}) \leq 0, \\ 0 \leq{} &\Psi_{;\Upsilon}(h, \mathcal{Z}) \leq 1, \quad \textit{and} \\ -1 \leq{} &\Omega_{;\Upsilon}(h, \mathcal{Z}) \leq 0, \end{aligned} \tag{5.15}$$

and similarly

$$\begin{aligned} 1 \leq & \Phi_{;h}(h,\mathcal{Z}), \\ 0 \leq & \Psi_{;h}(h,\mathcal{Z}) \leq 1, \quad \textit{and} \\ & \Omega_{;h}(h,\mathcal{Z}) \leq 0. \end{aligned} \tag{5.16}$$

Moreover, for any $\mathcal{Z} \geq 0$*, we have* $\Omega(h,\mathcal{Z}) = 0$ *and*

$$\Phi(h,\mathcal{Z}) = h - \mathcal{Z} \quad \textit{and} \quad \Psi(h,\mathcal{Z}) = h + \mathcal{Z},$$

while if $\mathcal{Z} < 0$ *and* $h > 0$*, all inequalities are strict.*

Proof. It is convenient to treat the cases of rarefactions and shocks separately. For $0 \leq z = \frac{h_b}{h_a} \leq 1$, or equivalently $0 \leq \zeta \leq 1$, corresponding to a rarefaction, we have

$$\gamma(z) = 1 - z, \quad \phi(\zeta) = 1 - \zeta \quad \text{and} \quad \psi(\eta) = \eta + 1,$$

while also $\xi(z) = \varpi(\zeta) = 0$. Thus clearly

$$\gamma'(z) = \phi'(\zeta) = -1 = \psi'(\eta) \quad \text{and} \quad \theta'(z) = \varpi'(\zeta) = 0,$$

and all second derivatives vanish. This is expected as states vary linearly across simple waves.

In the case of shocks, we have the equivalent conditions $\zeta < 0$, $z > 1$ and $y = \frac{1}{z} < 1$. According to (2.27), (2.28), for $z > 1$ we have

$$\gamma(z) + \theta(z) = 1 - z \quad \text{and} \quad \gamma(z) - \theta(z) = g(z) = -k(z).$$

Solving, we have

$$\gamma(z) = \frac{1 - z - k(z)}{2} \quad \text{and} \quad \theta(z) = \frac{1 - z + k(z)}{2}, \tag{5.17}$$

and so we immediately have

$$\gamma'(z) = -\frac{1 + k'(z)}{2} \quad \text{and} \quad \theta'(z) = \frac{k'(z) - 1}{2},$$

and

$$\gamma''(z) = -\frac{k''(z)}{2} \quad \text{and} \quad \theta''(z) = \frac{k''(z)}{2}.$$

Next, since $\phi = \gamma^{-1}$, we have

$$\phi'(\zeta) = \frac{1}{\gamma'(z)} = \frac{-2}{1 + k'(z)},$$

where we have set $z = \phi(\zeta)$, and

$$\phi''(\zeta) = \frac{-\gamma''(z)}{\gamma'(z)^3} = \frac{-4\, k''(z)}{(1 + k'(z))^3}.$$

Similarly, since $\varpi = \theta \circ \phi$, we calculate directly that

$$\varpi'(\zeta) = \theta'(\phi(\zeta))\ \phi'(\zeta) = -\frac{k'(z)-1}{k'(z)+1}, \quad \text{and}$$
$$\varpi''(\zeta) = \frac{4\ k''(z)}{(1+k'(z))^3} = -\phi''(\zeta).$$

Finally, we use (5.11) to calculate the derivatives of ψ. We have

$$\psi'(\eta) = \frac{-z'(\eta)}{z^2} \quad \text{and} \quad z + \eta\ z'(\eta) = \gamma'(z)\ z'(\eta),$$

which yields

$$\psi'(\eta) = \frac{1}{\gamma(z) - z\ \gamma'(z)} = \frac{2}{1-k(z)+z\ k'(z)}.$$

Differentiating again and simplifying, we get

$$\psi''(\eta) = \frac{-z^3\ \gamma''(z)}{(\gamma(z)-z\ \gamma'(z))^3} = \frac{4\ z^3\ k''(z)}{(1-k(z)+z\ k'(z))^3},$$

where $z = 1/\psi(\eta) > 1$. The Lemma now follows by substituting these values into (5.13) and (5.14). □

Corollary 3. *A simple wave's strength increases after it crosses a shock of the opposite family.*

Proof. Without loss of generality, we suppose that the forward wave $\Gamma(h_s, h_w)$ is simple and the backward wave $\Gamma(h_s, h_e)$ is a shock, where we have used the labels from Section 4. According to the (4.8) and the discussion thereof, the outgoing backward wave is also a shock and its strength is unchanged,

$$\Gamma(h_w, h_n) = \Gamma(h_s, h_e) \equiv \mathcal{A} < 0.$$

The change of the simple wave is given by (4.9), namely

$$\begin{aligned}\Gamma(h_e, h_n) - \Gamma(h_s, h_w) &= \Theta(h_w, h_n) - \Theta(h_s, h_e)\\ &= \Omega(h_w, \mathcal{A}) - \Omega(h_s, \mathcal{A})\\ &= (h_w - h_s)\ \Omega_{;h}(k, \mathcal{A})\ ,\end{aligned}$$

where we have used (5.4) and the Mean Value Theorem. Similarly, we have

$$h_w - h_s = \Phi(h_s, \Gamma(h_s, h_w)) - \Phi(h_s, 0) = \Phi_{;\Upsilon}(h_s, \mathcal{B})\ \Gamma(h_s, h_w),$$

for some $\mathcal{B}$ between 0 and $\Gamma(h_s, h_w)$, so that

$$\Gamma(h_e, h_n) = \Gamma(h_s, h_w)\ \ (1 + \Omega_{;h}(k, \mathcal{A})\ \Phi_{;\Upsilon}(h_s, \mathcal{B}))$$

and the result follows by (5.15), (5.16). □

6. Crossing shocks

We now study the interaction of crossing shocks of arbitrary strength. The main issue here is that we need estimates which are uniform in the density as the vacuum is approached. We consider the configuration shown in the second part of Figure 5: a forward 2-shock with $(h_s\ u_s)^t$ ahead and $(h_w\ u_w)^t$ behind meets a backward 1-shock with state $(h_e\ u_e)^t$ behind, resulting in two outgoing shocks with the resultant behind state $(h_n\ u_n)^t$.

6.1. **Shock Interaction Estimate.**

It is convenient to work in scaled variables which are balanced to reflect the amount of symmetry in the interaction. That is, we define

$$\mu = \frac{\sqrt{h_w\, h_e}}{h_s}, \quad \rho = \sqrt{\frac{h_e}{h_w}} \quad \text{and} \quad \nu = \frac{h_n}{\sqrt{h_w\, h_e}}, \tag{6.1}$$

so that

$$\frac{h_w}{h_s} = \frac{\mu}{\rho}, \quad \frac{h_e}{h_s} = \mu\,\rho, \quad \frac{h_n}{h_w} = \nu\,\rho, \quad \text{and} \quad \frac{h_n}{h_e} = \frac{\nu}{\rho}. \tag{6.2}$$

Clearly μ is a measure of the average size of the incoming shocks, while ρ measures their relative sizes. According to (4.10), we have the constraints

$$\frac{1}{\mu} < \rho < \mu \quad \text{and} \quad \frac{1}{\nu} < \rho < \nu.$$

Moreover, our earlier estimate (4.13) translates to $\nu < \mu$, and so we have

$$\frac{1}{\mu} < \frac{1}{\nu} < \rho < \nu < \mu. \tag{6.3}$$

In these variables, equation (4.12) becomes

$$\frac{1}{\mu}\, k(\frac{\mu}{\rho}) + \frac{1}{\mu}\, k(\mu\,\rho) = \frac{1}{\rho}\, k(\nu\,\rho) + \rho\, k(\frac{\nu}{\rho}) \tag{6.4}$$

where we regard $\nu = \nu(\mu, \rho)$. It is clear that the interaction is symmetric, that is $\nu(\mu, 1/\rho) = \nu(\mu, \rho)$.

Recalling (2.25), (2.21), we define

$$r(z) = \sqrt{q_{d+1}(z)\, q_{d-1}(z)}, \tag{6.5}$$

so that $k(z) = z^{\frac{d+1}{2}}\, r(z)$. Our equation (6.4) can then be written

$$\mu^{\frac{d-1}{2}} \left[\rho^{-\frac{d+1}{2}}\, r(\frac{\mu}{\rho}) + \rho^{\frac{d+1}{2}}\, r(\mu\,\rho)\right] = \nu^{\frac{d+1}{2}} \left[\rho^{\frac{d-1}{2}}\, r(\nu\,\rho) + \rho^{-\frac{d-1}{2}}\, r(\frac{\nu}{\rho})\right]. \tag{6.6}$$

Before solving equation (6.6), we derive some properties of $r(z)$.

Lemma 4. *The function $r(z)$ is monotone increasing and concave, with limiting values*

$$r(1) = 0, \quad r'(1) = 1, \quad r''(1) = -(d+1) \quad \textit{and}$$
$$\lim_{z\to\infty} r(z) = \frac{1}{\sqrt{d^2-1}}\,.$$

Moreover, for all $z > 1$ we have the bounds

$$q_d(z) < r(z) < \frac{d}{\sqrt{d^2-1}}\, q_d(z)\,, \tag{6.7}$$

and the ratio $\frac{r(z)}{q_d(z)}$ is monotone increasing.

Proof. Recall from (2.21) that q_n is defined by

$$q_n(z) = \frac{1-z^{-n}}{n},$$

so we can write

$$n\, q_n(z) = 1 - z^{-n} \quad \text{and}$$
$$q_n(z) = \int_1^z \bar{z}^{-n-1}\, d\bar{z} = \int_0^{\log z} e^{-nx} dx. \tag{6.8}$$

We regard these as functions of both z and n. It is clear that for each n, $q_n(z)$ is monotone increasing and concave as a function of z. On the other hand, fixing z and regarding these as functions of n, we see that they are monotone increasing and decreasing, respectively.

The concavity and monotonicity of $r(z)$ is an immediate consequence of the following:

Claim 1. *Given positive functions $f_1(x)$ and $f_2(x)$, their geometric average $F(x) = \sqrt{f_1(x)\, f_2(x)}$ is concave whenever*

$$f_1(x)''\, f_2(x) + f_1(x)\, f_2''(x) < 0\,;$$

in particular, if f_1 and f_2 are concave, so is F. To prove this, we differentiate:

$$F(x)^2 = f_1(x)\, f_2(x)\,,$$
$$2\, F\, F' = f_1'\, f_2 + f_1\, f_2' \quad \text{and}$$
$$2\, F'^2 + 2F\, F'' = f_1''\, f_2 + 2\, f_1'\, f_2' + f_1\, f_2''\,,$$

which after rearranging yields

$$4\, F^3\, F'' = 2\, f_1\, f_2\, (f_1''\, f_2 + f_1\, f_2'') - (f_1'\, f_2 - f_1\, f_2')^2\,,$$

and the claim follows.

The limiting values of r are now calculated from those of $q_n(z)$: by l'Hôpital's rule, $q_n/r \to 1$ as $z \to 1^+$ for any n, and by above,

$$2\, r\, r'' \approx q_{d+1}''\, q_{d-1} + q_{d+1}\, q_{d-1}''\,,$$

so $r'' \to (q''_{d+1} + q''_{d-1})/2 = -(d+1)$ as $z \to 1$.

We now compare $r(z)$ to $q_d(z)$. To this end, write

$$q_n(z) = z^{-n/2}\,\frac{z^{n/2} - z^{-n/2}}{n},$$

and make the change of variables

$$w \equiv \log\sqrt{z}\,, \quad \text{so that} \quad z^{\frac{1}{2}} = e^{w}\,, \tag{6.9}$$

and

$$q_n(z) = e^{-n\,w}\,\frac{2\,\sinh n\,w}{n} \;\equiv\; \widehat{q_n}\,(w)\,. \tag{6.10}$$

In these variables, we calculate

$$\frac{r}{q_d} = \frac{\sqrt{b_{d+1}(w)\;b_{d-1}(w)}}{b_d(w)}\,, \tag{6.11}$$

where we have set

$$b_n(w) \equiv \frac{\sinh n\,w}{n}\,. \tag{6.12}$$

We use the shorthand

$$\begin{aligned} c_1 = \cosh w, &\quad s_1 = \sinh w \\ c_d = \cosh dw, &\quad s_d = \sinh dw\,, \end{aligned} \tag{6.13}$$

and recall the identities

$$\cosh^2 x - \sinh^2 x = 1 \quad \text{and}$$
$$\sinh(x_1 \pm x_2) = \sinh x_1 \cosh x_2 \pm \cosh x_1 \sinh x_2\,.$$

Using these in (6.11), (6.12), we get

$$\begin{aligned} \frac{r}{q_d} &= \frac{\sqrt{s_d^2\;c_1^2 - c_d^2\;s_1^2}}{s_d}\;\frac{d}{\sqrt{d^2-1}} \\ &= \frac{\sqrt{s_d^2\;(1+s_1^2) - (1+s_d^2)\;s_1^2}}{s_d}\;\frac{d}{\sqrt{d^2-1}} \\ &= \sqrt{1 - \left(\frac{s_1}{s_d}\right)^2}\;\frac{d}{\sqrt{d^2-1}}\,. \end{aligned} \tag{6.14}$$

It now follows by the chain rule that r/q_d is monotone increasing in z if and only if s_d/s_1 is increasing in w. We calculate

$$\begin{aligned} \left(\frac{s_d}{s_1}\right)' &= \frac{d\;c_d\;s_1 - s_d\;c_1}{s_1^2} \\ &= \frac{\frac{d-1}{2}(s_d\,c_1 + c_d\,s_1) - \frac{d+1}{2}(s_d\,c_1 - c_d\,s_1)}{s_1^2} \\ &= \frac{(d-1)\sinh(d+1)\,w - (d+1)\sinh(d-1)\,w}{2\;s_1^2} \ge 0\,, \end{aligned} \tag{6.15}$$

the last inequality following by convexity of $\sinh x$ for $x \geq 0$, and the proof is complete. $\square$

We now return to equation (6.6) which defines $\nu = \nu(\mu, \rho)$.

Theorem 4. *There are positive constants K_1 and K_2 such that the solution $\nu = \nu(\mu, \rho)$ satisfies the uniform bounds*

$$K_1\, \mu^{\frac{d-1}{d+1}}\, J(\rho)^{\frac{2}{d+1}} \leq \nu(\mu, \rho) \leq K_2\, \mu^{\frac{d-1}{d+1}}\, J(\rho)^{\frac{2}{d+1}} \tag{6.16}$$

valid for all μ and ρ with $1/\mu < \rho < \mu$. Here $J(\rho)$ is the function defined by

$$J(\rho) \equiv \frac{\rho^{\frac{d+1}{2}} + \rho^{-\frac{d+1}{2}}}{\rho^{\frac{d-1}{2}} + \rho^{-\frac{d-1}{2}}}. \tag{6.17}$$

Throughout this section, we use the convention that Q_j is some quantity depending on μ and ρ, and K_j is a positive constant depending only on the parameter d, and hence on the ideal gas constant γ.

Proof. We use the upper and lower bounds of (6.7) on either side of (6.6) to replace $r(\cdot)$ with $q_d(\cdot)$. This yields

$$\begin{aligned} &\mu^{\frac{d-1}{2}} \left[\rho^{-\frac{d+1}{2}}\, q_d(\frac{\mu}{\rho}) + \rho^{\frac{d+1}{2}}\, q_d(\mu\,\rho)\right] \\ &\qquad \leq \frac{d}{\sqrt{d^2-1}}\, \nu^{\frac{d+1}{2}} \left[\rho^{\frac{d-1}{2}}\, q_d(\nu\,\rho) + \rho^{-\frac{d-1}{2}}\, q_d(\frac{\nu}{\rho})\right], \end{aligned}$$

and similarly

$$\begin{aligned} &\frac{d}{\sqrt{d^2-1}}\, \mu^{\frac{d-1}{2}} \left[\rho^{-\frac{d+1}{2}}\, q_d(\frac{\mu}{\rho}) + \rho^{\frac{d+1}{2}}\, q_d(\mu\,\rho)\right] \\ &\qquad \geq \nu^{\frac{d+1}{2}} \left[\rho^{\frac{d-1}{2}}\, q_d(\nu\,\rho) + \rho^{-\frac{d-1}{2}}\, q_d(\frac{\nu}{\rho})\right]. \end{aligned}$$

Therefore, if we define

$$Q_1 \equiv \frac{\rho^{-\frac{d+1}{2}}\, q_d(\frac{\mu}{\rho}) + \rho^{\frac{d+1}{2}}\, q_d(\mu\,\rho)}{\rho^{\frac{d-1}{2}}\, q_d(\nu\,\rho) + \rho^{-\frac{d-1}{2}}\, q_d(\frac{\nu}{\rho})}, \tag{6.18}$$

where $\nu = \nu(\mu, \rho)$, then we have

$$\frac{\sqrt{d^2-1}}{d}\, \mu^{\frac{d-1}{2}}\, Q_1 \leq \nu^{\frac{d+1}{2}} \leq \frac{d}{\sqrt{d^2-1}}\, \mu^{\frac{d-1}{2}}\, Q_1\,. \tag{6.19}$$

Now using the definition (2.21) of $q_d(z)$ in (6.18) and simplifying, we get

$$\begin{aligned} Q_1 &= \frac{\rho^{-\frac{d+1}{2}} + \rho^{\frac{d+1}{2}} - \mu^{-d}\left(\rho^{\frac{d-1}{2}} + \rho^{-\frac{d-1}{2}}\right)}{\rho^{\frac{d-1}{2}} + \rho^{-\frac{d-1}{2}} - \nu^{-d}\left(\rho^{-\frac{d+1}{2}} + \rho^{\frac{d+1}{2}}\right)} \\ &= \frac{J(\rho) - \mu^{-d}}{1 - \nu^{-d}\, J(\rho)} \\ &= J(\rho)\, \frac{1 - \mu^{-d}/J(\rho)}{1 - \nu^{-d}\, J(\rho)} \\ &\equiv J(\rho)\, Q_2\,, \end{aligned} \tag{6.20}$$

where $J(\rho)$ is given by (6.17).

It remains to find upper and lower bounds for

$$Q_2 = \frac{1 - \mu^{-d}/J}{1 - \nu^{-d}\, J} \equiv \frac{1-y}{1-x}\,, \tag{6.21}$$

where $J = J(\rho)$ and we have set

$$y = \mu^{-d}/J \quad \text{and} \quad x = \nu^{-d}\, J\,.$$

Note that $J(1/\rho) = J(\rho)$, and it is not hard to see that for all $\rho > 0$, we have

$$1 \le J \le \max\{\rho, 1/\rho\}\,.$$

Also, by (6.3), we have $\nu \le \mu$, so that for all ρ,

$$\mu^{-d}/J \le \mu^{-d}\, J \le \nu^{-d}\, J, \quad \text{so} \quad y \le x \le 1\,,$$

which immediately gives the lower bound

$$Q_2 \ge 1\,.$$

To show that Q_2 is bounded above, we must show that if $x \to 1^-$ in (6.21), then also $y \to 1$. To this end, we may regard (6.4) as defining a C^2 function $\mu = \mu(\nu, \rho)$: this follows directly from the Implicit Function Theorem, if we note that

$$z\, k'(z) - k(z) > 0 \quad \text{for all} \quad z > 1\,,$$

by convexity of k. Now, if $x \to 1^-$, then

$$1 \ge \nu^{-d+1} \ge \nu^{-d}\, J = x \quad \text{so} \quad \nu \to 1^+,$$

while also

$$1/\nu \le \rho \le \nu \quad \text{so} \quad \rho \to 1\,.$$

Since also $\mu(1,1) = 1$, by continuity we have

$$y = \mu^{-d}/J \to 1 \quad \text{as} \quad x \to 1\,,$$

from which we conclude Q_2 is bounded. Indeed, we obtain $Q_2 \to 1$ as $x \to 1$ using l'Hôpital's rule, as follows. By continuity, we may take $\rho = 1$, so $J = 1$ and

$$Q_2 = \frac{1 - \mu^{-d}}{1 - \nu^{-d}} \to \frac{\partial \mu}{\partial \nu}(1,1) = 1\,,$$

the last equality being evaluated by implicitly differentiating (6.4).

Substituting the bounds for Q_2 into (6.19), (6.20) now yields (6.16), with constants

$$K_1 = \left(\frac{\sqrt{d^2-1}}{d}\right)^{\frac{2}{d+1}} = \left(1-\frac{1}{d^2}\right)^{\frac{1}{d+1}} \quad \text{and}$$

$$K_2 = (\max Q_2)^{\frac{2}{d+1}} \big/ K_1\,, \tag{6.22}$$

and the proof is complete. □

We remark that although our upper bound for Q_2 can in principle be arbitrarily large, it is in practice rather small. This can be seen by "bootstrapping" the lower bound, as follows: after some manipulation, the lower bound of (6.16) gives

$$x = \nu^{-d}\,J \le K_1^{-d}\,\left(\mu^{-d}/J\right)^{\frac{d-1}{d+1}} = K_1^{-d}\,y^{\frac{d-1}{d+1}}\,, \tag{6.23}$$

and so by bounding y away from 1, we get associated bounds on Q_2. We can regard this as a way of measuring the strength of an interaction: if y is far from 1, the interaction is nonlinear and this estimate can be used, but for y near 1 we need to use a linear estimate as in the proof, because since

$$1-\frac{1}{d^3} < K_1 < 1,$$

the estimate (6.23) is not strong enough to bound x away from 1. In computations, for $d=6$, corresponding to gamma law $\gamma = 1.4$, we have $Q_2 < 1.1$, and for $d = 2$, corresponding to $\gamma = 3$, we get $Q_2 < 1.35$.

6.2. **Properties of Scaled Wave Strength.** We wish to use the bound on $\nu = \nu(\mu,\rho)$ to estimate the outgoing wave strengths. To do so, we bound the scaled wave strength $\gamma(z)$ by a power of z, as we did for $k(z)$. Following (2.21) and (2.25), for $z > 1$ we write

$$-2\,\gamma(z) \equiv z^{\frac{d+1}{2}}\,t(z), \tag{6.24}$$

so that $t(z)$ can also be written

$$t(z) = -2\,\gamma(z)\,z^{-\frac{d+1}{2}} = -2\,\gamma(z)\,\frac{r(z)}{k(z)}\,.$$

Lemma 5. *The function $t(z)$ has a unique critical point z_*, is concave increasing on $[1,z_*]$ and monotone decreasing on $[z_*,\infty]$, and has limits*

$$t(1) = 0, \quad t'(1) = 2 \quad \textit{and} \quad t(\infty) = \frac{1}{\sqrt{d^2-1}}\,.$$

If we define the point $z_\#$ by

$$t(z_\#) \equiv \frac{1}{\sqrt{d^2-1}} \equiv t_\#\,, \tag{6.25}$$

then we have the bounds

$$t_\# \le t(z) \le t_* \equiv t(z_*) \quad \textit{whenever} \quad z \ge z_\#\,, \tag{6.26}$$

and

$$\frac{t_\#}{z_\# - 1}\,(z-1) \le t(z) \le 2\,(z-1) \quad \textit{for} \quad 1 \le z \le z_\#\,. \tag{6.27}$$

Moreover, $t(z)$ satisfies the bounds

$$\frac{d}{\sqrt{d^2-1}}\,q_d(z) < t(z) < \min\Big\{2, \frac{d^2}{\sqrt{d^2-1}}\Big\}\,q_d(z)\,, \tag{6.28}$$

for all $z \ge 1$.

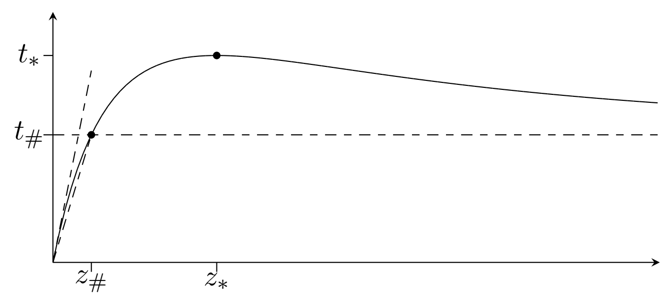

FIGURE 7. The function $t(z)$ for $d = 6$.

Proof. Using (5.17), (2.21), we write

$$\begin{aligned} t(z) &= -2\,\gamma(z)\,z^{-\frac{d+1}{2}} \\ &= \big(k(z) + z - 1\big)\;z^{-\frac{d+1}{2}} \\ &= r(z) + z^{-\frac{d-1}{2}} - z^{-\frac{d+1}{2}} \\ &= r(z) + z^{-\frac{d}{2}}\,\Big(z^{\frac12} - z^{-\frac12}\Big)\,. \end{aligned} \tag{6.29}$$

First, since $z^{-\frac{d-1}{2}}$ is the dominant nonlinear term for large z, we set

$$x \equiv z^{-\frac{d-1}{2}} \quad \text{so that} \quad z = x^{-\frac{2}{d-1}}\,, \tag{6.30}$$

with $0 < x \le 1$, and we write $t(z) = \widehat{t}\,(x)$, with

$$\widehat{t}\,(x) = \frac{1}{\sqrt{d^2-1}}\,\sqrt{(1-x^{2\,\gamma})(1-x^2)} + x - x^\gamma\,, \tag{6.31}$$

where we have used (6.30) and (6.5) in (6.29), and $\gamma \equiv \frac{d+1}{d-1}$ is the gas constant (2.3). Note that $\widehat{t}\,(x)$ extends continuously to $x = 0$ and we calculate

$$\widehat{t}\,(0) = \frac{1}{\sqrt{d^2-1}}\,, \quad \widehat{t}\,(1) = 0 \quad \text{and} \quad \widehat{t}\,'(0) = 1\,.$$

Since $r'(1) = 1$, we have $t'(1) = 2$, and the chain rule yields

$$\widehat{t}'(1) = \frac{-4}{d-1} < 0.$$

Next, since each of the functions $1 - x^{2\gamma}$, $1 - x^2$ and $x - x^\gamma$ is concave, by Claim 1, so is $\widehat{t}(x)$. It follows that $\widehat{t}$ has a unique maximum $x_* \in (0,1)$, and that $\widehat{t}'(x) > 0$ for $x < x_*$ and $\widehat{t}'(x) < 0$ for $x > x_*$. This implies that $t(z)$ has a unique critical point z_*, and that t is increasing for $z < z_*$ and decreasing for $z > z_*$. Moreover, we calculate

$$t''(z) = \widehat{t}''(x) \left(\frac{dx}{dz}\right)^2 + \widehat{t}'(x)\, \frac{d^2 z}{dx^2}\,,$$

and $t(z)$ is concave whenever this is negative; in particular, $t(z)$ is concave for $x > x_*$, corresponding to $z < z_*$.

Thus far, we know that $t(z)$ increases from 0 at $z = 1$ to its maximum of t_* at z_*, and then decreases to $\frac{1}{\sqrt{d^2-1}}$ as $z \to \infty$. Therefore, if we define $z_\#$ and $t_\#$ by

$$t(z_\#) \equiv \frac{1}{\sqrt{d^2-1}} \equiv t_\#\,,$$

which is (6.25), then we have

$$t_\# \le t(z) \le t_* \quad \text{whenever} \quad z \ge z_\#\,.$$

On the other hand, for $z < z_\#$, we obtain linear estimates for $t(z)$, as follows. Since $z_\# < z_*$, we know that $t(z)$ is concave on the interval $[1, z_\#]$. Thus its graph lies below the tangent at $z = 1$, and above the secant joining the points $(1,0)$ and $(z_\#, t_\#)$. That is, for any $1 \le z \le z_\#$, we have

$$\frac{t_\#}{z_\# - 1}\,(z-1) \le t(z) \le 2\,(z-1)\,.$$

In Figure 7, we show the graph of $t(z)$ for $d = 6$; in this case, it is easy to compute $(z_*, t_*) \approx (1.54, 0.27)$ and $(z_\#, t_\#) \approx (1.13, 0.17)$, with associated scaled wave strength $\gamma_\# \approx -0.13$.

Finally, to compare $t(z)$ to $q_d(z)$, we recall the change of variables (6.9), namely $w = \log\sqrt{z}$, and write (6.29) as

$$\frac{t(z)}{q_d(z)} = \frac{r}{q_d} + e^{-dw}\,\frac{2\sinh w}{q_d}\,,$$

which by (6.10), (6.11), (6.14) and (6.12) can be written as

$$\begin{aligned}\frac{t}{q_d} &= \frac{\sqrt{b_{d+1}(w)\, b_{d-1}(w)}}{b_d(w)} + \frac{b_1(w)}{b_d(w)}\\ &= \sqrt{1 - \left(\frac{s_1}{s_d}\right)^2}\,\frac{d}{\sqrt{d^2-1}} + \frac{d\, s_1}{s_d}\,,\end{aligned}$$

where we have again used the shorthand (6.13). Thus, if we set

$$y \equiv d\,\frac{s_1}{s_d} = \frac{b_1(w)}{b_d(w)}\,,$$

then

$$z \geq 1 \iff 0 \leq w \leq \infty \iff 1 \geq y \geq 0\,,$$

and

$$\frac{t}{q_d} = f(y) \equiv y + \sqrt{\frac{d^2 - y^2}{d^2 - 1}}\,, \tag{6.32}$$

and according to (6.9), (6.15), t/q_d decreases when $f(y)$ increases. Thus we calculate the extreme values of $f(y)$ on the interval $[0, 1]$. First note that

$$f(1) = 2 \quad \text{and} \quad f(0) = \frac{d}{\sqrt{d^2 - 1}}\,,$$

and we calculate

$$f'(y) = 1 - \frac{1}{\sqrt{d^2 - 1}}\,\frac{y}{\sqrt{d^2 - y^2}}\,,$$

so that

$$f'(y) = 0 \quad \text{if and only if} \quad y = \sqrt{d^2 - 1}\,,$$

and the maximum of f is

$$f(\sqrt{d^2 - 1}) = \frac{d^2}{\sqrt{d^2 - 1}}\,.$$

Since this is increasing in d, and we are restricting to $y \in [0, 1]$, we conclude that

$$\frac{d}{\sqrt{d^2 - 1}} \leq \frac{t}{q_d} \leq \min\big\{2, \frac{d^2}{\sqrt{d^2 - 1}}\big\}\,,$$

this minimum being 2 if $d \geq \sqrt{2}$, which in turn corresponds to gas constant $\gamma \leq 3 + 2\sqrt{2}$. □

In order to get estimates for ρ and μ, we must estimate ϕ, which according to (5.7) is the inverse function of the scaled wave strength γ. Thus, regarding $\gamma < 0$ as the variable, we write (6.24) as

$$2\,|\gamma| = \phi(\gamma)^{\frac{d+1}{2}}\,t(\phi)\,. \tag{6.33}$$

Recalling that $(z_\#, t_\#)$ is given by (6.25), it is convenient to set

$$\gamma_\# \equiv \phi^{-1}(z_\#)\,, \tag{6.34}$$

so that

$$\gamma_\# \leq \gamma \leq 0 \quad \text{iff} \quad 1 \leq \phi(\gamma) \leq z_\#\,,$$

and, in view of the following Lemma, we shall regard such a γ as a *weakly nonlinear* wave.

Lemma 6. *The (invertible) function $\phi(\gamma)$ satisfies the global lower bound*

$$\phi(\gamma) \geq K_3\, |\gamma|^{\frac{2}{d+1}} \quad \textit{for} \quad \gamma \leq 0\,, \tag{6.35}$$

while if $\gamma \leq \gamma_\# < 0$, then we have the upper bound

$$\phi(\gamma) \leq K_4\, |\gamma|^{\frac{2}{d+1}}\,. \tag{6.36}$$

On the other hand, if $\gamma_\# \leq \gamma \leq 0$, then

$$\widehat{K}_3\, |\gamma| \leq \phi(\gamma) - 1 \leq |\gamma|\,, \tag{6.37}$$

where $\widehat{K}_3 \equiv \frac{z_\# - 1}{|\gamma_\#|}$.

Proof. By Lemma 5, for all $\gamma < 0$ we have

$$2\,|\gamma| \leq \phi(\gamma)^{\frac{d+1}{2}}\, t_*\,,$$

which implies the global lower bound (6.35), with constant

$$K_3 = \left(\frac{2}{t_*}\right)^{\frac{2}{d+1}}\,.$$

Since $\phi(0) = 1$, an upper bound analogous to (6.35) cannot hold for all $\gamma < 0$, but it will hold for $|\gamma|$ suitably bounded away from 0. To see this, suppose that $|\gamma|$ is large enough that

$$\phi(\gamma) \geq z_\#\,, \quad \text{so that} \quad t(\phi(\gamma)) \geq t_\#\,,$$

which by (6.35) is guaranteed if

$$|\gamma| \geq \left(\frac{t_\#}{K_3}\right)^{\frac{d+1}{2}}\,.$$

Then (6.33) becomes

$$2\,|\gamma| = \phi(\gamma)^{\frac{d+1}{2}}\, t(\phi) \geq \phi(\gamma)^{\frac{d+1}{2}}\, t_\#\,,$$

and so we have

$$\phi(\gamma) \leq K_4\, |\gamma|^{\frac{2}{d+1}}\,,$$

which is (6.36), with

$$K_4 \equiv \left(\frac{2}{t_\#}\right)^{\frac{2}{d+1}}\,.$$

On the other hand, for

$$1 \leq \phi(\gamma) \leq z_\#\,,$$

which we think of as a weak or linear wave, then since $\gamma \approx 0$, a linear estimate is more appropriate. First we note that since ϕ is the inverse function of γ which is decreasing and concave, so also is ϕ, see Figure 5.2. Thus on the interval $[\gamma_\#, 0]$, the graph of ϕ lies between the tangent of ϕ at 0 and the secant determined by the interval. That is,

$$1 + \frac{z_\# - 1}{\gamma_\#}\,\gamma \leq \phi(\gamma) \leq 1 - \gamma \quad \text{for any} \quad \gamma_\# \leq \gamma \leq 0\,,$$

which is (6.37). $\square$

We note that the particular choice of $t_\#$ is not crucial in the Lemma, but was chosen as the largest lower bound of $t(z)$ for large z. In fact, the same conclusions hold for any $\gamma_\dagger$ satisfying $|\gamma_\dagger| \le |\gamma_\#|$, provided we define $z_\dagger$ and $t_\dagger$ by

$$z_\dagger = \phi(\gamma_\dagger) \quad \text{and} \quad t_\dagger = t(z_\dagger)\,. \tag{6.38}$$

Corollary 4. *For any $\gamma_\dagger$ with $0 > \gamma_\dagger \ge \gamma_\#$, the conclusions of Lemma 6 hold, with $\gamma_\#$ replaced by $\gamma_\dagger$ and modified constants*

$$K_4 = \left(\frac{2}{t_\dagger}\right)^{\frac{2}{d+1}} \quad \textit{and} \quad \widehat{K}_3 = \frac{z_\dagger - 1}{\gamma_\dagger}\,.$$

6.3. **Effect on Wave Strengths.** We now return to the shock interaction problem, and describe the interaction in terms of scaled and unscaled wave strengths. We first use Theorem 4 to estimate the outgoing scaled wave strengths $\gamma(h_n/h_e) = \gamma(\nu/\rho)$ and $\gamma(h_n/h_w) = \gamma(\nu\,\rho)$.

Lemma 7. *We have the estimates*

$$\begin{aligned} \gamma(\nu\,\rho) &= \gamma(\mu\,\rho)\,\frac{J(\rho)}{\mu}\,Q_3(\mu,\rho) \quad \textit{and} \\ \gamma(\nu/\rho) &= \gamma(\mu/\rho)\,\frac{J(\rho)}{\mu}\,Q_3(\mu,1/\rho)\,, \end{aligned} \tag{6.39}$$

where Q_3 is uniformly bounded,

$$0 < K_5 \le Q_3(\mu,x) \le K_6. \tag{6.40}$$

Proof. Referring to Theorem 4, we regard $\nu = \nu(\mu,\rho)$, and define

$$Q_4(\mu,\rho) \equiv \frac{\nu^{\frac{d+1}{2}}}{\mu^{\frac{d-1}{2}}\,J(\rho)}\,.$$

Then $Q_4(\mu,\rho) = Q_4(\mu,1/\rho)$, and, according to (6.16), Q_4 is uniformly bounded,

$$K_1^{\frac{d+1}{2}} \le Q_4(\mu,\rho) \le K_2^{\frac{d+1}{2}}\,. \tag{6.41}$$

Thus we have, for $x = \rho$ or $1/\rho$,

$$(\nu\,x)^{\frac{d+1}{2}} = (\mu\,x)^{\frac{d+1}{2}}\,\frac{J}{\mu}\,Q_4(\mu,x)\,,$$

and so, by (6.24),

$$\gamma(\nu\,x) = \gamma(\mu\,x)\,\frac{J}{\mu}\,Q_4(\mu,x)\,\frac{t(\nu\,x)}{t(\mu\,x)}\,.$$

Equations (6.39) follow directly if we define

$$Q_3(\mu,x) = Q_4(\mu,x)\,\frac{t(\mu\,x)}{t(\nu\,x)}\,, \tag{6.42}$$

with $x = 1/\rho$ and ρ, respectively. Using (6.41) and (6.28), it suffices to show that the ratio

$$\frac{q_d(\mu\, x)}{q_d(\nu\, x)}$$

is uniformly bounded away from 0 for $x = \rho$ or $1/\rho$. First, since q_d is increasing, $\mu \geq \nu$ again implies

$$\frac{q_d(\mu\, x)}{q_d(\nu\, x)} \geq 1\,.$$

On the other hand, as in the proof of Theorem 4, this ratio is bounded away from $q_d(\nu\, x) = 0$, that is $\nu\, x = 1$. But, as we have seen,

$$\mu \to 1 \quad \text{as} \quad \nu \to 1,$$

and again, by l'Hôpital's rule, we have

$$\lim_{\nu\to 1} \frac{q_d(\mu\, x)}{q_d(\nu\, x)} = \lim_{\nu\to 1} \frac{\partial\mu}{\partial\nu} = 1\,.$$

Thus again by continuity, the fraction $q_d(\mu\, x)/q_d(\nu\, x)$ is uniformly bounded, and (6.40) follows from (6.42) and (6.41), with

$$K_5 = K_1^{\frac{d+1}{2}} = \sqrt{1 - \frac{1}{d^2}}\,,$$

by (6.22), and the proof is complete. $\square$

Corollary 5. *The unscaled wave strengths satisfy the estimates*

$$\begin{aligned} \Gamma(h_e, h_n) &= \Gamma(h_s, h_w)\ J(\rho)\ \rho\ Q_3(\mu, 1/\rho) \quad \textit{and} \\ \Gamma(h_w, h_n) &= \Gamma(h_s, h_e)\ \frac{J(\rho)}{\rho}\ Q_3(\mu, \rho). \end{aligned} \tag{6.43}$$

Proof. Recalling the scaling law (2.28) and the relations (6.2), and applying (6.39), we have

$$\begin{aligned} \Gamma(h_e, h_n) &= h_e\ \gamma(\frac{h_n}{h_e}) = h_s\ \mu\,\rho\ \gamma(\frac{\nu}{\rho}) \\ &= h_s\ \gamma(\frac{\mu}{\rho})\ \rho\, J(\rho)\ Q_3(\mu, 1/\rho) \\ &= \Gamma(h_s, h_w)\ \rho\, J(\rho)\ Q_3(\mu, 1/\rho), \end{aligned}$$

and the second relation follows similarly. $\square$

Finally, in order to fully describe the shock interaction in terms of wave strengths, we must express the ratio ρ in those terms. Thus, suppose we are given incident wave strengths $\mathcal{A} = \Gamma(h_s, h_e)$ and $\mathcal{B} = \Gamma(h_s, h_w)$. From (6.1), and using (5.3), (5.8), we define the scaled wave strengths

$$\alpha \equiv \frac{\mathcal{A}}{h_s} \quad \text{and} \quad \beta \equiv \frac{\mathcal{B}}{h_s}\,, \tag{6.44}$$

so that

$$\rho = \sqrt{\frac{h_e}{h_w}} = \sqrt{\frac{\Phi(h_s,\mathcal{A})}{\Phi(h_s,\mathcal{B})}} = \sqrt{\frac{\phi(\mathcal{A}/h_s)}{\phi(\mathcal{B}/h_s)}} = \sqrt{\frac{\phi(\alpha)}{\phi(\beta)}}\,, \tag{6.45}$$

and similarly

$$\mu = \sqrt{\phi(\alpha)\;\phi(\beta)}\,. \tag{6.46}$$

We can now state our main theorem of shock interactions. We consider the interaction of two shocks with incident strengths $\mathcal{A} = \Gamma(h_s, h_e)$ and $\mathcal{B} = \Gamma(h_s, h_w)$ and corresponding scaled strengths

$$\alpha = \gamma(\frac{h_e}{h_s}) = \gamma(\mu\,\rho) \quad \text{and} \quad \beta = \gamma(\frac{h_w}{h_s}) = \gamma(\frac{\mu}{\rho}) \tag{6.47}$$

as in (6.2), (6.44). We denote the outgoing waves (and their strengths) by

$$\begin{aligned} \mathcal{A}' = \Gamma(h_w, h_n)\,, &\quad \mathcal{B}' = \Gamma(h_e, h_n)\,, \\ \alpha' = \gamma(\frac{h_n}{h_w}) = \gamma(\nu\;\rho) \quad \text{and} &\quad \beta' = \gamma(\frac{h_n}{h_e}) = \gamma(\frac{\nu}{\rho})\,, \end{aligned} \tag{6.48}$$

respectively.

Theorem 5. *The outgoing scaled wave strengths satisfy the estimates*

$$\begin{aligned} K_7\;|\alpha|\;\frac{1}{\eta} \le |\alpha'| \le K_6\;|\alpha|\;\frac{1}{\eta} \quad &\textit{and} \\ K_7\;|\beta|\;\frac{1}{\eta} \le |\beta'| \le K_6\;|\beta|\;\frac{1}{\eta}\,, & \end{aligned} \tag{6.49}$$

where

$$\eta = \min\{\phi(\alpha),\,\phi(\beta)\} > 1.$$

If $|\mathcal{A}| \ge |\mathcal{B}|$, then the unscaled wave strength $\mathcal{A}'$ satisfies

$$K_7\;|\mathcal{A}| \le |\mathcal{A}'| \le K_6\;|\mathcal{A}|\,, \tag{6.50}$$

while for $\mathcal{B}$ we have the three cases: if $|\mathcal{B}| \ge |\gamma_\#|\,h_s$,

$$K_8\;|\mathcal{A}|^{\frac{2}{d+1}}\;|\mathcal{B}|^{\frac{d-1}{d+1}} \le |\mathcal{B}'| \le K_9\;|\mathcal{A}|^{\frac{2}{d+1}}\;|\mathcal{B}|^{\frac{d-1}{d+1}}\,; \tag{6.51}$$

next, if $|\mathcal{B}| \le |\gamma_\#|\;h_s \le |\mathcal{A}|$,

$$\widehat{K}_8\;\frac{|\mathcal{A}|^{\frac{2}{d+1}}|\mathcal{B}|}{h_s^{\frac{2}{d+1}}} \le |\mathcal{B}'| \le \widehat{K}_9\;\frac{|\mathcal{A}|^{\frac{2}{d+1}}|\mathcal{B}|}{h_s^{\frac{2}{d+1}}} \le \overline{K}_9\;|\mathcal{A}|^{\frac{2}{d+1}}\;|\mathcal{B}|^{\frac{d-1}{d+1}}\,; \tag{6.52}$$

and finally, if $|\mathcal{B}| \le |\mathcal{A}| \le |\gamma_\#|\;h_s$, then

$$\widetilde{K}_8\;|\mathcal{B}| \le |\mathcal{B}'| \le \widetilde{K}_9\;|\mathcal{B}|\,. \tag{6.53}$$

On the other hand, if $|\mathcal{B}| \ge |\mathcal{A}|$, then by symmetry we get the same estimates with the positions of $\mathcal{A}$ and $\mathcal{B}$ reversed.

Note that our estimates are uniform in the middle state h_s, and in particular, by continuity, they hold up to the vacuum. Indeed, for fixed $\mathcal{A}$ and $\mathcal{B}$, as h_s decreases the scaled wave strength increases, and (6.51) applies. In (6.49), we can use Lemma 6 to bound $1/\eta$ from above and below in terms of $\min\{|\alpha|, |\beta|\}$.

Proof. According to Lemma 7 and Corollary 5, we need to find estimates for the quantities

$$\frac{J(\rho)}{\mu}\,, \quad J(\rho)\,\rho \quad \text{and} \quad \frac{J(\rho)}{\rho}\,,$$

in terms of the incident wave strengths $\mathcal{A}$ and $\mathcal{B}$ and middle state h_s. Moreover, we want these estimates to be uniform as $h_s \to 0$.

Recall that $J(\rho)$ is defined by (6.17), so that

$$\frac{J(\rho)}{\rho} = \frac{1+\rho^{-(d+1)}}{1+\rho^{-(d-1)}} \equiv j(\rho)\,, \tag{6.54}$$

so that $j(\rho) \le 1$ for $\rho \ge 1$; moreover, since $j(\rho) \to 1 = j(1)$ as $\rho \to \infty$, it follows that for some $\rho_* > 1$, we have

$$j(\rho) \ge j_* \equiv j(\rho_*) \quad \text{for all} \quad \rho \ge 1\,.$$

It follows that for $\rho \ge 1$,

$$j_* \le \frac{J(\rho)}{\rho} \le 1 \quad \text{and} \quad j_*\,\rho^2 \le J(\rho)\,\rho \le \rho^2\,, \tag{6.55}$$

while if $\rho \le 1$, so that $1/\rho \ge 1$,

$$\begin{aligned} \frac{j_*}{\rho^2} \le \frac{J(\rho)}{\rho} &= J\Big(\frac{1}{\rho}\Big)\,\frac{1}{\rho} \le \frac{1}{\rho^2} \quad \text{and} \\ j_* \le J(\rho)\,\rho &= \frac{J(1/\rho)}{1/\rho} \le 1\,. \end{aligned}$$

Similarly, we get the estimates

$$\begin{aligned} j_*\,\frac{\rho}{\mu} \le \frac{J(\rho)}{\mu} \le \frac{\rho}{\mu} \quad &\text{if} \quad \rho \ge 1\,, \quad \text{and} \\ \frac{j_*}{\mu\,\rho} \le \frac{J(\rho)}{\mu} \le \frac{1}{\mu\,\rho} \quad &\text{if} \quad \rho \le 1\,. \end{aligned} \tag{6.56}$$

Now by (6.46) and (6.45), we express μ and ρ in terms of the wave strengths as

$$\begin{aligned} \mu\,\rho = \phi(\alpha)\,, \qquad\qquad & \frac{\mu}{\rho} = \phi(\beta)\,, \\ \rho = \sqrt{\frac{\phi(\alpha)}{\phi(\beta)}} \quad \text{and} \quad & \mu = \sqrt{\phi(\alpha)\,\phi(\beta)}\,. \end{aligned} \tag{6.57}$$

Thus, if $|\alpha| \ge |\beta|$, then $\rho \ge 1$ and (6.56) becomes

$$\frac{j_*}{\phi(\beta)} \le \frac{J(\rho)}{\mu} \le \frac{1}{\phi(\beta)}\,,$$

and using this and (6.48) in Lemma 7 yields

$$K_6\,\frac{\alpha}{\phi(\beta)} \le \alpha' \le j_* K_5\,\frac{\alpha}{\phi(\beta)} < 0\,, \quad \text{and}$$

$$K_6\,\frac{\beta}{\phi(\beta)} \le \beta' \le j_* K_5\,\frac{\beta}{\phi(\beta)} < 0\,.$$

Similarly, if $|\beta| \ge |\alpha|$, then

$$K_6\,\frac{\alpha}{\phi(\alpha)} \le \alpha' \le j_* K_5\,\frac{\alpha}{\phi(\alpha)} < 0\,, \quad \text{and}$$

$$K_6\,\frac{\beta}{\phi(\alpha)} \le \beta' \le j_* K_5\,\frac{\beta}{\phi(\alpha)} < 0\,.$$

The inequalities (6.49) follow immediately with $K_7 \equiv j_*\,K_5$.

We now suppose $|\mathcal{A}| \ge |\mathcal{B}|$, so that $\rho \ge 1$ and (6.55), (6.57) gives

$$j_* \le \frac{J(\rho)}{\rho} \le 1 \quad \text{and} \quad j_*\,\frac{\phi(\alpha)}{\phi(\beta)} \le J(\rho)\,\rho \le \frac{\phi(\alpha)}{\phi(\beta)}\,,$$

and Corollary 5 yields

$$K_6\,\mathcal{A} \le \mathcal{A}' \le j_* K_5\,\mathcal{A} < 0\,, \quad \text{and}$$

$$K_6\,\mathcal{B}\,\frac{\phi(\alpha)}{\phi(\beta)} \le \mathcal{B}' \le j_* K_5\,\mathcal{B}\,\frac{\phi(\alpha)}{\phi(\beta)} < 0\,.$$

which immediately yields (6.50) and

$$j_* K_5\,|\mathcal{B}|\,\frac{\phi(\alpha)}{\phi(\beta)} \le |\mathcal{B}'| \le K_6\,|\mathcal{B}|\,\frac{\phi(\alpha)}{\phi(\beta)}\,. \tag{6.58}$$

It remains to estimate the ratio $\frac{\phi(\alpha)}{\phi(\beta)}$ from above and below in terms of the unscaled wave strengths $|\mathcal{A}|$ and $|\mathcal{B}|$. Since $\alpha = \mathcal{A}/h_s$ and $\beta = \mathcal{B}/h_s$, we refer to Lemma 6, and separately consider the three cases

$$|\gamma_{\#}|\,h_s \le |\mathcal{B}|\,, \quad |\mathcal{B}| \le |\gamma_{\#}|\,h_s \le |\mathcal{A}|\,, \quad \text{and} \quad |\mathcal{A}| \le |\gamma_{\#}|\,h_s\,.$$

In the first case, in which both $|\mathcal{A}|$ and $|\mathcal{B}|$ are large relative to h_s, we use (6.35), (6.36) to get

$$\frac{K_3}{K_4}\,\left|\frac{\alpha}{\beta}\right|^{\frac{2}{d+1}} \le \frac{\phi(\alpha)}{\phi(\beta)} \le \frac{K_4}{K_3}\,\left|\frac{\alpha}{\beta}\right|^{\frac{2}{d+1}}\,,$$

but we know $\frac{\alpha}{\beta} = \frac{\mathcal{A}}{\mathcal{B}}$, and using this in (6.58) yields (6.51), with constants

$$K_8 \equiv \frac{j_*\,K_5\,K_3}{K_4} \quad \text{and} \quad K_9 \equiv \frac{K_6\,K_4}{K_3}\,.$$

Next, if $|\beta| \le |\gamma_{\#}| \le |\alpha|$, we use (6.37) for β to get

$$\frac{K_3\,|\alpha|^{\frac{2}{d+1}}}{1+|\beta|} \le \frac{\phi(\alpha)}{\phi(\beta)} \le \frac{K_4\,|\alpha|^{\frac{2}{d+1}}}{1+\,\widehat{K}_3|\beta|}\,,$$

so that

$$\frac{K_3}{1+|\gamma_{\#}|}\,\frac{|\mathcal{A}|^{\frac{2}{d+1}}}{h_s^{\frac{2}{d+1}}} \le \frac{\phi(\alpha)}{\phi(\beta)} \le K_4\,\frac{|\mathcal{A}|^{\frac{2}{d+1}}}{h_s^{\frac{2}{d+1}}}\,.$$

Using this in (6.58) yields the first part of (6.52), with constants

$$\widehat{K}_8 \equiv \frac{j_*\,K_5\,K_3}{1+|\gamma_{\#}|} \quad \text{and} \quad \widehat{K}_9 \equiv K_6\,K_4\,.$$

Now note that

$$\frac{|\mathcal{A}|^{\frac{2}{d+1}}|\mathcal{B}|}{h_s^{\frac{2}{d+1}}} = |\mathcal{A}|^{\frac{2}{d+1}}\,|\mathcal{B}|^{\frac{d-1}{d+1}}\,\left(\frac{|\mathcal{B}|}{h_s}\right)^{\frac{2}{d+1}} \le |\mathcal{A}|^{\frac{2}{d+1}}\,|\mathcal{B}|^{\frac{d-1}{d+1}}\,\Big|\gamma_{\#}\Big|^{\frac{2}{d+1}}\,,$$

which completes (6.52) if we set

$$\overline{K}_9 \equiv \widehat{K}_9\,\Big|\gamma_{\#}\Big|^{\frac{2}{d+1}}\,.$$

In the third case, we use (6.37) for both α and β to get

$$\frac{1+\widehat{K}_3\,|\alpha|}{1+|\beta|} \le \frac{\phi(\alpha)}{\phi(\beta)} \le \frac{1+|\alpha|}{1+\widehat{K}_3\,|\beta|}\,,$$

and since $|\beta|\le|\alpha|\le|\gamma_{\#}|$, this implies

$$\frac{1+|\alpha|}{1+\widehat{K}_3\,|\beta|} \le 1+|\gamma_{\#}| \quad \text{and}$$

$$\frac{1+\widehat{K}_3\,|\alpha|}{1+|\beta|} \ge \frac{1+\widehat{K}_3\,|\alpha|}{1+|\alpha|} \ge \frac{1+\widehat{K}_3\,|\gamma_{\#}|}{1+|\gamma_{\#}|}\,.$$

Thus if we set

$$\widetilde{K}_8 \equiv j_*\,K_5\,\frac{1+\widehat{K}_3\,|\gamma_{\#}|}{1+|\gamma_{\#}|} \quad \text{and} \quad \widetilde{K}_9 \equiv K_6\,(1+|\gamma_{\#}|)\,,$$

then (6.53) follows, and the proof is complete. □

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Department of Mathematics & Statistics, University of Massachusetts