GLOBAL L^∞ SOLUTIONS OF THE COMPRESSIBLE EULER EQUATIONS WITH SPHERICAL SYMMETRY

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Abstract. We study the compressible Euler equations with spherical symmetry surrounding a solid ball. For the spherically symmetric flow, the global existence of L^∞ entropy weak solutions has not yet been obtained except a special case. In this paper, we prove the existence of global solutions in the more general case. We construct approximate solutions by using a modified Godunov scheme. The main point is to obtain an L^∞ bound for the approximate solutions.

1. Introduction

Let us consider the Euler equations in an exterior domain $\{x \in \mathbf{R}^3; |\vec{x}| \geq 1\}$ with spherical initial data:

$$\begin{cases} \rho_t + \nabla \cdot \vec{m} = 0, \\ \vec{m}_t + \nabla \cdot \left(\dfrac{\vec{m} \otimes \vec{m}}{\rho} \right) + \nabla p = 0, \end{cases} \tag{1.1}$$

where ρ, $\vec{m}$ and p are density, momentum and the pressure of gas, respectively. For the non-vacuum state $\rho > 0$, $\vec{u} = \vec{m}/\rho$ is velocity. For the polytropic gas, $p(\rho) = \rho^\gamma/\gamma$, where $\gamma \in (1, 5/3]$ is the adiabatic exponent for usual gases.

Consider the initial-boundary value problem (1.1) with

$$(\rho, \vec{m})|_{t=0} = (\rho_0(\vec{x}), \vec{m}_0(\vec{x})) \quad \text{and} \quad \vec{m}|_{|\vec{x}|=1} = 0, \tag{1.2}$$

where $m_0(x)$ is a scalar function of $x = |\vec{x}| \geq 1$ and initial data have the following geometric structure

$$(\rho_0(\vec{x}), \vec{m}_0(\vec{x})) = \left(\rho_0(|\vec{x}|), m_0(|\vec{x}|) \frac{\vec{x}}{|\vec{x}|} \right). \tag{1.3}$$

We look for the solution of the form

$$(\rho(\vec{x}, t), \vec{m}(\vec{x}, t)) = \left(\rho(|\vec{x}|, t), m(|\vec{x}|, t) \frac{\vec{x}}{|\vec{x}|} \right). \tag{1.4}$$

We rewrite (1.1) as

$$\begin{cases} \rho_t + m_x = -\dfrac{2}{x} m, \\ m_t + \left(\dfrac{m^2}{\rho} + p(\rho) \right)_x = -\dfrac{2}{x} \dfrac{m^2}{\rho}, \quad p(\rho) = \rho^\gamma/\gamma, \end{cases} \tag{1.5}$$

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where $\rho(x,t)$ and $m(x,t)$ are scalar functions of $x=|\vec{x}|\geq 1$. These equations can be written as

$$\begin{cases} v_t+f(v)_x=g(x,v), \quad x\geq 1,\\ v|_{t=0}=v_0(x),\\ m_{x=1}=0, \end{cases} \tag{1.6}$$

where $v={}^t(\rho,m),\ f(v)={}^t(m,m^2/\rho+p(\rho)),\ g(x,v)={}^t\left(-\dfrac{2}{x}m,-\dfrac{2}{x}\dfrac{m^2}{\rho}\right)$. Moreover, we define $u:=m/\rho$. We then consider the initial-boundary value problem (1.6) with initial data $v_0\in L^\infty(x\geq 1)$.

Now let us recollect the known results for this problem. The local existence of a weak solution was proved in [MT]. In [CG], the global existence with arbitrary L^∞ data was discussed by introducing the shock capturing scheme. However, there are some defects in the proof (see [T, Section 8]). The author of the present paper cannot yet correct them. On the other hand, the global existence theorem with large L^∞ data satisfying the following condition:

$$0\leq\{\rho_0(x)\}^\theta/\theta\leq u_0(x)<+\infty \tag{1.7}$$

was obtained in [C1], where $\theta=(\gamma-1)/2$. However, unfortunately, this condition (1.7) is restrictive. For example, under the condition (1.7), we cannot consider the negative initial velocity. Moreover, when $t>0$, the density of the solution is always 0 at the boundary ($x=1$). Therefore, in this paper, we prove the global existence of a solution under the condition which is more general than (1.7).

When we prove the global existence, the main difficulty is to obtain L^∞ estimates of approximate solutions. More precisely, apparent source terms $g(x,v)$ in (1.6) are its cause. To do this, we devise the way of the construction of approximate solutions. The approximate solutions consist of steady state solutions of an auxiliary equation and yield our desired estimate. In Section 2, we state wave curves, the Riemann solutions and the theory of invariant regions. In Section 3, we construct approximate solutions. However, their construction is technical. Therefore we postpone the detail of its calculus to Appendix A and B. In Section 4, we derive their L^∞ estimates. In Section 5–6, we prove the compactness and convergence for approximate solutions. In Appendix C, we prove Lemma 6.1, which we shall use in Section 5.

First we define the Riemann invariants w,z, which play important roles in this paper, as

Definition 1.1.

$$w:=\frac{m}{\rho}+\frac{\rho^\theta}{\theta}=u+\frac{\rho^\theta}{\theta},\quad z:=\frac{m}{\rho}-\frac{\rho^\theta}{\theta}=u-\frac{\rho^\theta}{\theta}\quad (\theta:=(\gamma-1)/2).$$

Then our main theorem is as follows.

Theorem 1.1. *We assume that, for constants C_1 and C_2, initial density and momentum data $(\rho_0,m_0)\in L^\infty(x\geq 1)$ satisfy*

$$w(v_0(x))\leq C_1,\ z(v_0(x))\geq -C_2x^{-\frac{2(\gamma-1)}{\gamma+1}},\ 0\leq\rho_0(x). \tag{1.8}$$

Then, there exists C_3 depending only on C_1 and C_2 such that the initial-boundary value problem (1.6) *has a global entropy weak solution $(\rho(x,t),m(x,t))$ satisfying*

$$w(v(x,t))\leq C_3,\ z(v(x,t))\geq -C_2x^{-\frac{2(\gamma-1)}{\gamma+1}},\quad (x,t)\in\{x\geq 1\}\times\mathbf{R}_+.$$

Remark 1.1. [The condition of initial data]

(1) If $C_1 < -C_2$, (1.8) has no intersection near $x = 1$. Moreover, notice that (1.7) implies $z(v_0(x)) \geq 0$. When $C_2 \leq 0$, Theorem 1.1 is consequently contained in [C1]. Therefore, we henceforth assume that $C_1 \geq -C_2$ and $C_2 > 0$.

(2) $(1.8)_2$ is equivalent to $u_0(x) - (\rho_0(x))^\theta/\theta \geq -C_2 x^{-\frac{2(\gamma-1)}{\gamma+1}}$. Therefore we can give the negative initial velocity.

(3) If initial data have compact support, the condition (1.8) means that we can give arbitrary L^∞ data.

(4) The second term of (1.8) is restrictive. Because of this restriction, our main theorem cannot contain the trivial solutions, $\rho = const.$ and $u = 0$. If $\gamma = 1$ (i.e. the isothermal case), (1.8) means that we can give arbitrary L^∞ data. In fact, the global existence theorem in [MU2] can contain the trivial solutions. In this paper, we cannot let γ be 1. However, the closer γ is to 1, the weaker the restriction becomes.

The Riemann invariants have the following properties and relations:

Remark 1.2.

(i) $|w| \geq |z|,\ w \geq 0,$ when $u \geq 0$. $\quad |w| \leq |z|,\ z \leq 0,$ when $u \leq 0$.

(ii) $u = \dfrac{w+z}{2},\ \rho = \left(\dfrac{\theta(w-z)}{2}\right)^{1/\theta}.$

From the remarks above, the lower bound of z and the upper bound of w yield the bound of ρ and u.

2. Preliminary

In this section, we first review some results of the Riemann solutions for the homogeneous system of gas dynamics. Consider the homogeneous system

$$\begin{cases} \rho_t + m_x = 0, \\ m_t + \left(\dfrac{m^2}{\rho} + p(\rho)\right)_x = 0, \quad p(\rho) = \rho^\gamma/\gamma. \end{cases} \tag{2.1}$$

The characteristic speeds of the system are

$$\lambda_1 := u - \rho^\theta = \frac{m}{\rho} - c, \quad \lambda_2 := u + \rho^\theta = \frac{m}{\rho} + c,$$

where $c := \rho^\theta$ is the sound speed. For the characteristic speeds and the Riemann invariants, the following relations hold.

Remark 2.1.

$$z \leq \lambda_1, \quad \lambda_2 \leq w.$$

Moreover,

$$\lambda_1 \leq \theta z, \ \text{ when } u \leq 0. \quad \lambda_2 \geq \theta w, \ \text{ when } u \geq 0.$$

A pair of functions $(\eta, q) : \mathbf{R}^2 \to \mathbf{R}^2$ is called an entropy-entropy flux pair, if it satisfies an identity

$$\nabla q = \nabla \eta \nabla f. \tag{2.2}$$

Furthermore, if, for any fixed $m/\rho \in (-\infty,\infty)$, η vanishes on the vacuum $\rho = 0$, then η is called a *weak entropy.* For example, the mechanical energy-energy flux pair

$$\eta_* := \frac{1}{2}\frac{m^2}{\rho} + \frac{1}{\gamma(\gamma-1)}\rho^\gamma, \quad q_* := m\left(\frac{1}{2}\frac{m^2}{\rho^2} + \frac{\rho^{\gamma-1}}{\gamma-1}\right) \tag{2.3}$$

is a strictly convex weak entropy-entropy flux pair.

The jump discontinuity in a weak solutions to (2.1) must satisfy the following Rankine-Hugoniot conditions

$$\lambda(v - v_0) = f(v) - f(v_0),$$

where λ is the propagation speed of the discontinuity, $v_0 = (\rho_0, m_0)$ and $v = (\rho, m)$ are the corresponding left and right state respectively. This means that

$$\begin{cases} m - m_0 = \dfrac{m_0}{\rho_0}(\rho - \rho_0) \pm \sqrt{\dfrac{\rho}{\rho_0}\dfrac{p(\rho)-p(\rho_0)}{\rho-\rho_0}}(\rho-\rho_0), \\ \lambda = \dfrac{m-m_0}{\rho-\rho_0} = \dfrac{m_0}{\rho_0} \pm \sqrt{\dfrac{\rho}{\rho_0}\dfrac{p(\rho)-p(\rho_0)}{\rho-\rho_0}}. \end{cases}$$

A jump discontinuity is called a *shock* if it satisfies the entropy condition

$$\lambda(\eta(v) - \eta(v_0)) - (q(v) - q(v_0)) \geq 0$$

for any convex entropy pair (η, q).

There are two distinct types of rarefaction and shock curves in the isentropic gases. Given a left state (ρ_0, m_0) or (ρ_0, u_0), the possible states (ρ, m) or (ρ, u) that can be connected to (ρ_0, m_0) or (ρ_0, u_0) on the right by a rarefaction or a shock curve constitute a 1-rarefaction wave curve $R_1(v_0)$, a 2-rarefaction wave curve $R_2(v_0)$, a 1-shock curve $S_1(v_0)$ and a 2-shock curve $S_2(v_0)$:

$$R_1(v_0) : \begin{cases} m - m_0 = \dfrac{m_0}{\rho_0}(\rho-\rho_0) - \dfrac{\rho}{\theta}(\rho^\theta - \rho_0^\theta), \quad \rho < \rho_0, \\ \text{or} \\ u - u_0 = -\dfrac{1}{\theta}(\rho^\theta - \rho_0^\theta), \quad \rho < \rho_0, \end{cases}$$

$$R_2(v_0) : \begin{cases} m - m_0 = \dfrac{m_0}{\rho_0}(\rho-\rho_0) + \dfrac{\rho}{\theta}(\rho^\theta - \rho_0^\theta), \quad \rho > \rho_0, \\ \text{or} \\ u - u_0 = \dfrac{1}{\theta}(\rho^\theta - \rho_0^\theta), \quad \rho > \rho_0, \end{cases}$$

$$S_1(v_0) : \begin{cases} m - m_0 = \dfrac{m_0}{\rho_0}(\rho-\rho_0) - \sqrt{\dfrac{\rho}{\rho_0}\dfrac{p(\rho)-p(\rho_0)}{\rho-\rho_0}}(\rho-\rho_0), \\ \rho > \rho_0 > 0 \\ \text{or} \\ u - u_0 = -\sqrt{\dfrac{1}{\rho\rho_0}\dfrac{p(\rho)-p(\rho_0)}{\rho-\rho_0}}(\rho-\rho_0) \quad \rho > \rho_0 > 0, \end{cases}$$

$$S_2(v_0):\begin{cases} m-m_0=\dfrac{m_0}{\rho_0}(\rho-\rho_0)+\sqrt{\dfrac{\rho}{\rho_0}\dfrac{p(\rho)-p(\rho_0)}{\rho-\rho_0}}(\rho-\rho_0),\\ \rho<\rho_0\\ \text{or}\\ u-u_0=\sqrt{\dfrac{1}{\rho\rho_0}\dfrac{p(\rho)-p(\rho_0)}{\rho-\rho_0}}(\rho-\rho_0)\quad \rho<\rho_0,\end{cases}$$

respectively.

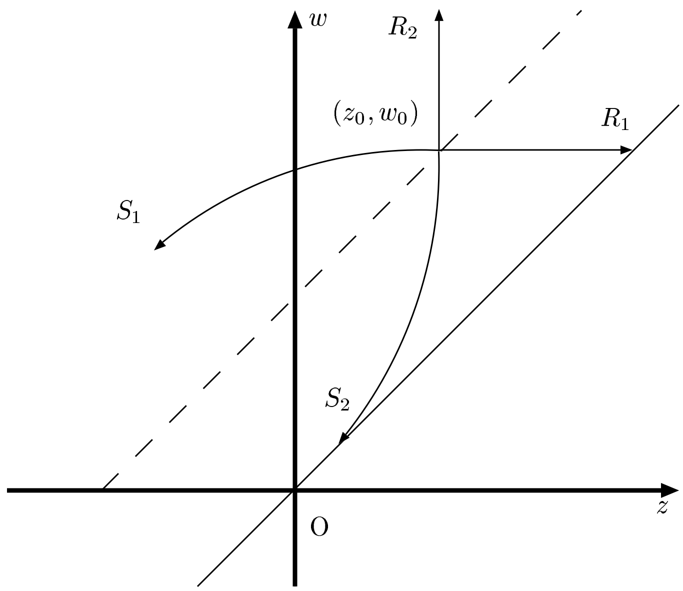

FIGURE 1. The rarefaction curves and the shock curves in (z, w)-plane.

Along the curve $R_1(v_0)$, we have $\left.\dfrac{dw}{dz}\right|_{R_1(v_0)}=0$, and along the curve $R_2(v_0)$, we have $\left.\dfrac{dz}{dw}\right|_{R_2(v_0)}=0$.

Along the curve $S_1(v_0)$, we have

$$\left.\frac{dw}{dz}\right|_{S_1(v_0)}=\frac{\gamma\rho^{\gamma+1}-(\gamma-1)\rho_0\rho^{\gamma}-\rho_0^{\gamma+1}-2\sqrt{(\rho^{\gamma}-\rho_0^{\gamma})(\rho-\rho_0)\gamma\rho_0\rho^{\gamma}}}{\gamma\rho^{\gamma+1}-(\gamma-1)\rho_0\rho^{\gamma}-\rho_0^{\gamma+1}+2\sqrt{(\rho^{\gamma}-\rho_0^{\gamma})(\rho-\rho_0)\gamma\rho_0\rho^{\gamma}}},$$

and along the curve $S_2(v_0)$, we have

$$\left.\frac{dw}{dz}\right|_{S_2(v_0)}=\frac{\gamma\rho^{\gamma+1}-(\gamma-1)\rho_0\rho^{\gamma}-\rho_0^{\gamma+1}+2\sqrt{(\rho^{\gamma}-\rho_0^{\gamma})(\rho-\rho_0)\gamma\rho_0\rho^{\gamma}}}{\gamma\rho^{\gamma+1}-(\gamma-1)\rho_0\rho^{\gamma}-\rho_0^{\gamma+1}-2\sqrt{(\rho^{\gamma}-\rho_0^{\gamma})(\rho-\rho_0)\gamma\rho_0\rho^{\gamma}}}.$$

Notice that

$$0\le\left.\frac{dw}{dz}\right|_{S_1(v_0)}\le 1,\quad \lim_{\rho\to\infty}\left.\frac{dw}{dz}\right|_{S_1(v_0)}=1$$

and

$$1 \leq \left.\frac{dw}{dz}\right|_{S_2(v_0)}, \quad \lim_{\rho\to 0}\left.\frac{dw}{dz}\right|_{S_2(v_0)} = 1.$$

Remark 2.2. Assume that there exists $C > 1$ such that

$$1/C \leq \rho/\rho_0 \leq C.$$

Then, considering w along $S_1(v_0)$, we have

$$\begin{aligned} w|_{S_1(v_0)} &= u_0 - \sqrt{\frac{1}{\rho\rho_0}\frac{p(\rho)-p(\rho_0)}{\rho-\rho_0}}(\rho-\rho_0) + \frac{\rho^\theta}{\theta} \\ &= u_0 + \frac{\rho_0^\theta}{\theta} + O(1)(\rho_0)^{\frac{\gamma-7}{2}}(\rho-\rho_0)^3, \end{aligned}$$

where $O(1)$ depends only on C.

Considering z along $S_2(v_0)$, we similarly have

$$\begin{aligned} z|_{S_2(v_0)} &= u_0 + \sqrt{\frac{1}{\rho\rho_0}\frac{p(\rho)-p(\rho_0)}{\rho-\rho_0}}(\rho-\rho_0) - \frac{\rho^\theta}{\theta} \\ &= u_0 - \frac{\rho_0^\theta}{\theta} + O(1)(\rho_0)^{\frac{\gamma-7}{2}}(\rho-\rho_0)^3, \end{aligned}$$

where $O(1)$ depends only on C. These representation show that S_1 (resp. S_2) and R_1 (resp. R_2) have a tangency of second order at the point (ρ_0, u_0).

2.1. **The Riemann solution.** Given a right state (ρ_0, m_0) or (ρ_0, u_0), the possible states (ρ, m) or (ρ, u) that can be connected to (ρ_0, m_0) or (ρ_0, u_0) on the left by a shock curve constitute 1-inverse shock curve $S_1^{-1}(v_0)$ and 2-inverse shock curve $S_2^{-1}(v_0)$:

$$S_1^{-1}(v_0): \begin{cases} m - m_0 = \dfrac{m_0}{\rho_0}(\rho-\rho_0) - \sqrt{\dfrac{\rho}{\rho_0}\dfrac{p(\rho)-p(\rho_0)}{\rho-\rho_0}}(\rho-\rho_0), \\ \underline{\rho < \rho_0} \\ \text{or} \\ u - u_0 = -\sqrt{\dfrac{1}{\rho\rho_0}\dfrac{p(\rho)-p(\rho_0)}{\rho-\rho_0}}(\rho-\rho_0) \quad \underline{\rho < \rho_0}, \end{cases}$$

and

$$S_2^{-1}(v_0): \begin{cases} m - m_0 = \dfrac{m_0}{\rho_0}(\rho-\rho_0) + \sqrt{\dfrac{\rho}{\rho_0}\dfrac{p(\rho)-p(\rho_0)}{\rho-\rho_0}}(\rho-\rho_0), \\ \underline{\rho > \rho_0 > 0} \\ \text{or} \\ u - u_0 = \sqrt{\dfrac{1}{\rho\rho_0}\dfrac{p(\rho)-p(\rho_0)}{\rho-\rho_0}}(\rho-\rho_0) \quad \underline{\rho > \rho_0 > 0}, \end{cases}$$

respectively.

Next we define a rarefaction shock. Given v_0, v on $S_i^{-1}(v_0)$ $(i = 1, 2)$, we call the piecewise constant solution to (2.1) which consists of the left and right states v_0, v a *rarefaction shock*. Here notice the following: although the inverse shock curve has the same form as the shock curve, the underline parts in $S_i^{-1}(v_0)$ is different from the corresponding parts in $S_i(v_0)$. Therefore the rarefaction shock does not satisfy the entropy condition.

The speed of the rarefaction shock above satisfies

The inverse Lax condition (see [B, Theorem 5.2])

$$\lambda_i(v_0) \leq \lambda_i \leq \lambda_i(v) \quad i = 1, 2. \tag{2.4}$$

We shall use a rarefaction shock in approximating a rarefaction wave. In addition, when we consider a rarefaction shock, we call the inverse shock curve connecting v_0 and v a *rarefaction shock curve* in particular.

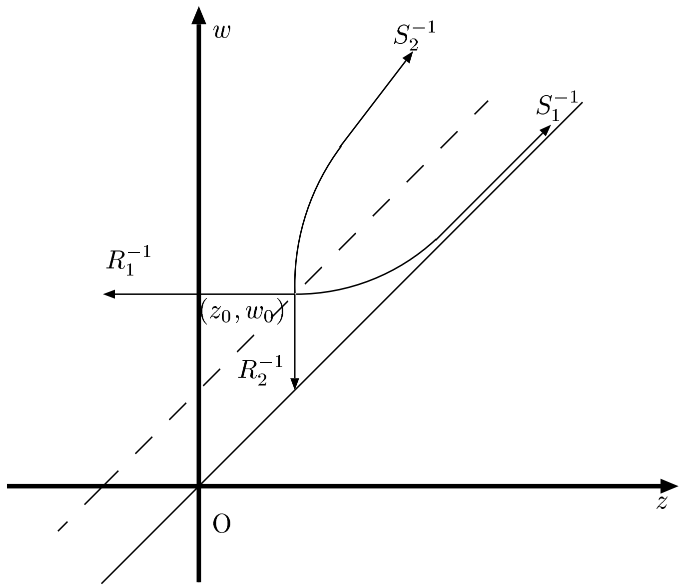

FIGURE 2. The inverse rarefaction curves and the inverse shock curves in (z, w)-plane.

From the properties of these curves in phase plane (z, w), we can construct a unique solution for the Riemann problem

$$v|_{t=0} = \begin{cases} v_-, & x < x_0, \\ v_+, & x > x_0, \end{cases} \tag{2.5}$$

and the Riemann initial boundary problem

$$v|_{t=0} = v_+, \quad m|_{x=1} = 0, \quad x \geq 1,\ t \geq 0, \tag{2.6}$$

where $x_0 \in (-\infty, \infty)$, $\rho_\pm \geq 0$ and $m_\pm$ are constants satisfying $|m_\pm| \leq C\rho_\pm$.

For the problem (2.1) and (2.5), we can consult [C3, Subsection 3.2]. For the problem (2.1) and (2.6), we draw a diagram by using the inverse wave curve of the second family and the vacuum as follows:

(1) If $\rho_+ > 0$ and $u_+ \leq 0$, there exists v_- with $u_- = 0$ from which v_+ is connected by a 2-shock curve.
(2) If $u_+ \geq 0$ and $z(v_+) \leq 0$, then there exists v_- with $u_- = 0$ from which v_+ is connected by a 2-rarefaction curve.

(3) If $u_+ \geq 0$ and $z(v_+) \geq 0$, then there exists v_* with $\rho_* = 0$ from which v_+ is connected by a 2-rarefaction, and v_* and v_- with $\rho_- = u_- = 0$ are connected by the vacuum.
(4) If $u_+ \leq 0$ and $\rho_+ = 0$, then v_- with $\rho_- = u_- = 0$ is connected from v_+ by the vacuum.

Then the following theorem holds [C3, Theorem 3.2].

Theorem 2.1. *There exists a unique piecewise smooth entropy solution* $(\rho(x,t), m(x,t))$ *containing the vacuum state* $(\rho = 0)$ *on the upper plane* $t > 0$ *for each problem of* (2.5) *and* (2.6) *satisfying*

(1) *For the Riemann problem* (2.5),

$$\begin{cases} w(\rho(x,t),m(x,t)) \leq \max(w(\rho_-,m_-),w(\rho_+,m_+)), \\ z(\rho(x,t),m(x,t)) \geq \min(z(\rho_-,m_-),z(\rho_+,m_+)), \\ w(\rho(x,t),m(x,t)) - z(\rho(x,t),m(x,t)) \geq 0. \end{cases}$$

(2) *For the Riemann initial boundary problem* (2.6),

$$\begin{cases} w(\rho(x,t),m(x,t)) \leq \max(w(\rho_+,m_+),-z(\rho_+,m_+)), \\ z(\rho(x,t),m(x,t)) \geq \min(z(\rho_+,m_+),0), \\ w(\rho(x,t),m(x,t)) - z(\rho(x,t),m(x,t)) \geq 0. \end{cases}$$

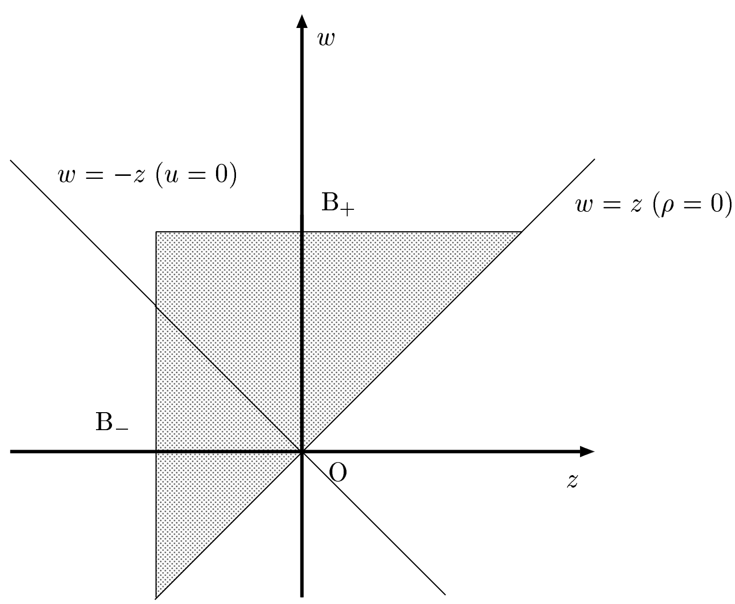

FIGURE 3. The invariant region in (w,z)-plane.

Such solutions also have the following properties:

Lemma 2.2. *For* $\mathrm{B}_+ \geq \mathrm{B}_-$, *the region* $\sum(\mathrm{B}_+,\mathrm{B}_-) = \{(\rho,\rho u) \in \mathbf{R}^2 : w = u + \rho^\theta/\theta,\ z = u - \rho^\theta/\theta,\ w \leq \mathrm{B}_+,\ z \geq \mathrm{B}_-,\ w - z \geq 0\}$ *is invariant with respect to both of the Riemann problem* (2.5) *and the average of the Riemann solutions in* x. *More precisely, if the Riemann data lie in* $\sum(\mathrm{B}_+,\mathrm{B}_-)$, *the corresponding*

Riemann solutions $(\rho(x,t), m(x,t)) = (\rho(x,t), \rho(x,t)u(x,t))$ *lie in* $\sum(\mathrm{B}_+, \mathrm{B}_-)$*, and their corresponding averages in* x *also in* $\sum(\mathrm{B}_+, \mathrm{B}_-)$*, namely*

$$\left(\frac{1}{b-a}\int_a^b \rho(x,t)dx, \frac{1}{b-a}\int_a^b m(x,t)dx\right) \in \sum(\mathrm{B}_+, \mathrm{B}_-).$$

Furthermore, for $\mathrm{B}_- \le 0 \le (\mathrm{B}_+ + \mathrm{B}_-)/2$*, the region* $\sum(\mathrm{B}_+, \mathrm{B}_-)$ *is invariant with respect to both of the Riemann initial-boundary problem* (2.6) *and the average of the corresponding Riemann solution in* x*.*

The proof of Lemma 2.2 can be found in [C3, Lemma 3.3].

Lemma 2.3. *Assume that the left state* v_- *and the right state* v_+ *satisfy the Rankine-Hugoniot conditions. Then, for an arbitrary weak entropy pair* (η, q)*, the following holds.*

$$\begin{aligned}&|\sigma(\eta(v_+) - \eta(v_-)) - (q(v_+) - q(v_-))| \\ &\qquad \le C\{\sigma(\eta_*(v_+) - \eta_*(v_-)) - (q_*(v_+) - q_*(v_-))\},\end{aligned}$$

where σ *is the corresponding propagation speed and the constant* C *depends only on* η *and* $\max(|\rho_\pm| + |m_\pm/\rho_\pm|)$*.*

The proof of Lemma 2.3 can be found in [C3, Lemma 3.5].

2.2. **Auxiliary inhomogeneous conservation laws.** Next, we consider the following inhomogeneous conservation laws

$$\begin{cases} \rho_t + m_x = -\dfrac{2}{x}m, \\ m_t + \left(\dfrac{m^2}{\rho} + p(\rho)\right)_x = -\dfrac{4\gamma}{(\gamma+1)x}\left(\dfrac{m^2}{\rho} + p(\rho)\right), \end{cases} \tag{2.7}$$

and the corresponding modified Riemann problem

$$v|_{t=0} = \begin{cases} v_- = (\tilde\rho_- x^{-\frac{4}{\gamma+1}}, \tilde m_- x^{-2}), & 1 < x < x_0, \\ v_+ = (\tilde\rho_+ x^{-\frac{4}{\gamma+1}}, \tilde m_+ x^{-2}), & x > x_0, \end{cases} \tag{2.8}$$

and the modified Riemann initial boundary problem of (2.7) with data

$$v|_{t=0} = (\tilde\rho_+ x^{-\frac{4}{\gamma+1}}, \tilde m_+ x^{-2}), \quad m|_{x=1} = 0, \quad x \ge 1,\ t \ge 0, \tag{2.9}$$

where $x_0 > 1$, $\tilde\rho_\pm \ge 0$ and $\tilde m_\pm$ are constants satisfying $|\tilde m_\pm| \le C\tilde\rho_\pm$.

It is easy to solve this problem. Because, setting $\rho = \tilde\rho x^{-\frac{4}{\gamma+1}}, m = \tilde m x^{-2}$ and $\xi = \frac{\gamma+1}{3\gamma-1}x^{\frac{3\gamma-1}{\gamma+1}}$, (2.7) becomes

$$\begin{cases} \tilde\rho_t + \tilde m_\xi = 0, \\ \tilde m_t + \left(\dfrac{\tilde m^2}{\tilde\rho} + p(\tilde\rho)\right)_\xi = 0. \end{cases} \tag{2.10}$$

These solutions satisfy the following lemma.

Lemma 2.4. *Let* $(\rho(x,t), m(x,t)) = (\tilde\rho(x,t)x^{-\frac{4}{\gamma+1}}, \tilde m(x,t)x^{-2})$ *be a solution of the Riemann problem* (2.8) *or the Riemann initial-boundary problem* (2.9)*. Then, for* $(\tilde\rho(x,t), \tilde m(x,t))$*, the statements of Lemma* 2.2 *holds.*

Remark 2.3.

When $\rho > 0$, $\rho = \tilde{\rho} x^{-\frac{4}{\gamma+1}}, m = \tilde{m} x^{-2}$ are equivalent to

$$w(\rho,m) = w(\tilde{\rho},\tilde{m}) x^{-\frac{2(\gamma-1)}{\gamma+1}}, z(\rho,m) = z(\tilde{\rho},\tilde{m}) x^{-\frac{2(\gamma-1)}{\gamma+1}}.$$

Remark 2.4. If $\tilde{v} = (\tilde{\rho},\tilde{m})$ is connected to $\tilde{v}_0 = (\tilde{\rho}_0,\tilde{m}_0)$ by a 1-shock (resp. a 2-shock), for arbitrary $x \geq 1$, $v = (\rho,m) = (\tilde{\rho} x^{-\frac{4}{\gamma+1}}, \tilde{m} x^{-2})$ is also connected to $v_0 = (\rho_0,m_0) = (\tilde{\rho}_0 x^{-\frac{4}{\gamma+1}}, \tilde{m}_0 x^{-2})$ by a 1-shock (resp. a 2-shock). These results from the fact that (2.7) and (2.10) have the same divergence form.

Finally, we consider the steady-state equations of (2.7):

$$\begin{cases} m_x = -\dfrac{2}{x} m, \\ \left(\dfrac{m^2}{\rho} + p(\rho)\right)_x = -\dfrac{4\gamma}{(\gamma+1)x}\left(\dfrac{m^2}{\rho} + p(\rho)\right) \end{cases} \tag{2.11}$$

subject to the condition

$$(\rho,m)|_{x=x_d} = (\rho_d,m_d) \ \text{ satisfying } \ |m_d| \leq C\rho_d. \tag{2.12}$$

Then we have a unique solution

$$\begin{aligned} &\rho(x) = \rho_d (x/x_d)^{-\frac{4}{\gamma+1}}, \ m = m_d (x/x_d)^{-2}, \\ &w(x) = w(\rho_d,m_d)(x/x_d)^{-\frac{2(\gamma-1)}{\gamma+1}} \text{ and } z(x) = z(\rho_d,m_d)(x/x_d)^{-\frac{2(\gamma-1)}{\gamma+1}}. \end{aligned} \tag{2.13}$$

The following lemma can be checked easily.

Lemma 2.5. *By choosing Δx small enough, the steady-state solution of* (2.11) *in* $[x_d - \Delta x/2, x_d + \Delta x/2]$, *with the condition $v|_{x=x_d} = v_d$, satisfies*

$$\frac{1}{\Delta x}\left|\int_{x_d - \frac{\Delta x}{2}}^{x_d + \frac{\Delta x}{2}} v(x) - v_d dx\right| = |v_d| O((\Delta x)^2), \quad |v(x) - v_d| = |v_d| O(\Delta x),$$

where bounds $O(\Delta x)$ and $O((\Delta x)^2)$ depend only on the bound of v_d.

In particular, if $\rho_d = o(1)$, the second equation implies

$$|v(x) - v_d| = o(\Delta x),$$

where $o(1)$ depend only on the bound of v_d and $o(1) \to 0$ as $\Delta x \to 0$.

3. The construction of approximate solutions

In this section, we construct approximate solutions. In the strip $0 \leq t \leq T$ for any fixed $T \in (0,\infty)$, we denote these approximate solutions by $v^{\Delta}(x,t) = (\rho^{\Delta}(x,t), m^{\Delta}(x,t))$. Let Δx and Δt be the space and time mesh length respectively.

Let us define the approximate solutions by using the modified Godunov scheme. Set

$$(j,n) \in \mathbf{Z}_+ \times \mathbf{Z}_{\geq 0}.$$

Moreover, for any fixed $0 < \varepsilon < 1$, we set $M_- = C_2$ and $M_+ \geq \max\{C_1, C_2\}$ such that

$$M_+^2 - M_-^2 - \frac{48}{\gamma+1}(M_+ + \varepsilon)M_- \geq 0. \tag{3.1}$$

Then, choose Δt small enough such that

$$\{(\gamma+11)M_+ + 12\varepsilon\}M_+\Delta t + 12(\gamma-1)M_+^2(M_+ + \varepsilon)(\Delta t)^2 \leq \varepsilon. \tag{3.2}$$

For M_+ and ε, we take Δx and Δt such that

$$\frac{\Delta x}{\Delta t} = 6(M_+ + \varepsilon). \tag{3.3}$$

First we define $v^{\Delta}(x,-0)$ by

$$v^{\Delta}(x,-0) := v_0(x).$$

Then we define $E_j^0(v)$ by

$$E_j^0(v) := \frac{1}{\Delta x}\int_{(j-1)\Delta x+1}^{j\Delta x+1} v^{\Delta}(x,-0)dx.$$

Next assume that $v^{\Delta}(x,t)$ is defined for $t < n\Delta t$. Then we define $E_j^n(v)$ by

$$E_j^n(v) := \frac{1}{\Delta x}\int_{(j-1)\Delta x+1}^{j\Delta x+1} v^{\Delta}(x,n\Delta t-0)dx.$$

Moreover, for $j \geq 1$, we define $v_j^n := (\rho_j^n, m_j^n)$ as follows.
We choose δ such that $1 < \delta < 1/(2\theta)$. If

$$E_j^n(\rho) := \frac{1}{\Delta x}\int_{(j-1)\Delta x+1}^{j\Delta x+1} \rho^{\Delta}(x,n\Delta t-0)dx < (\Delta x)^{\delta},$$

we define v_j^n by $v_j^n = 0$; otherwise, setting

$$\begin{aligned} w_j^n &:= \min\left\{w(E_j^n(v)),\ M_+ + \varepsilon\right\} \\ &\qquad\qquad\text{and} \\ z_j^n &:= \max\left\{z(E_j^n(v)),\ -M_-\{(j-1/2)\Delta x+1\}^{-\frac{2(\gamma-1)}{\gamma+1}}\right\}, \end{aligned} \tag{3.4}$$

we define v_j^n by

$$v_j^n := (\rho_j^n, m_j^n) := \left(\left\{\frac{\theta(w_j^n - z_j^n)}{2}\right\}^{1/\theta}, \left\{\frac{\theta(w_j^n - z_j^n)}{2}\right\}^{1/\theta}\frac{w_j^n + z_j^n}{2}\right).$$

Remark 3.1. The definition above implies

$$z(v_j^n) \geq -M_-\{(j-1/2)\Delta x+1\}^{-\frac{2(\gamma-1)}{\gamma+1}}, \quad w(v_j^n) \leq M_+ + \varepsilon. \tag{3.5}$$

3.1. **The construction of approximate solutions in the interior cell.** By using v_j^n defined above, we construct the approximate solutions with $v^{\Delta}((j-1/2)\Delta x + 1, n\Delta t + 0) = v_j^n$ in *the interior cell* $n\Delta t \leq t < (n+1)\Delta t$, $(j-1/2)\Delta x + 1 \leq x < (j+1/2)\Delta x + 1$ $(j = 1, 2, \ldots)$.

We first solve a Riemann problem with initial data (v_j^n, v_{j+1}^n). Call constants $v_{\mathrm{L}}(= v_j^n), v_{\mathrm{M}}, v_{\mathrm{R}}(= v_{j+1}^n)$ the left, middle and right states respectively. Then the following four cases occur.

- **Case 1** A 1-rarefaction wave and a 2-shock arise.
- **Case 2** A 1-shock and a 2-rarefaction wave arise.
- **Case 3** A 1-rarefaction wave and a 2-rarefaction arise.
- **Case 4** A 1-shock and a 2-shock arise.

We then construct approximate solutions $v^{\Delta}(x,t)$ by perturbing the Riemann solutions above. We separately consider the case where $v_{\rm M}$ is away from the vacuum and near the vacuum.

(A) The case where $v_{\rm M}$ is away from the vacuum

Let α be a constant satisfying $1/2<\alpha<1$. Then we can choose a positive value β small enough such that $\beta<\alpha$, $1/2+\beta/2<\alpha<1-2\beta$, $\beta<2/(\gamma+5)$ and $(9-3\gamma)\beta/2<\alpha$.

We first consider the case where $\rho_{\rm M}>(\Delta x)^{\beta}$, which means $v_{\rm M}$ is away from the vacuum. In this step, we consider Case 1 in particular. The construction of Case 2–4 are similar to that of Case 1.

Consider the case where a 1-rarefaction wave and a 2-shock arise as a Riemann solution with initial data (v^n_j, v^n_{j+1}). Assume that $v_{\rm L}, v_{\rm M}$ and $v_{\rm M}, v_{\rm R}$ are connected by a 1-rarefaction and a 2-shock curve respectively.

Step 1.

In order to approximate a 1-rarefaction wave by a piecewise constant *rarefaction fan*, we introduce the integer

$$p:=\max\left\{[\![(z_{\rm M}-z_{\rm L})/(\Delta x)^{\alpha}]\!]+1,2\right\},$$

where $z_{\rm L}=z(v_{\rm L}), z_{\rm M}=z(v_{\rm M})$ and $[\![x]\!]$ is the greatest integer not greater than x. Notice that

$$p=O((\Delta x)^{-\alpha}). \tag{3.6}$$

Define

$$z^*_1:=z_{\rm L},\ z^*_p:=z_{\rm M},\ w^*_i:=w_{\rm L}\ (i=1,\ldots,p),$$

and

$$z^*_i:=z_{\rm L}+(i-1)(\Delta x)^{\alpha}\ (i=1,\ldots,p-1).$$

We next introduce the rays $x=j\Delta x+1+\lambda_1(z^*_i,z^*_{i+1},w_{\rm L})(t-n\Delta t)$ separating finite constant states (z^*_i,w^*_i) $(i=1,\ldots,p)$, where

$$\lambda_1(z^*_i,z^*_{i+1},w_{\rm L}):=u(z^*_i,w_{\rm L})-S(\rho(z^*_{i+1},w_{\rm L}),\rho(z^*_i,w_{\rm L})),$$

$$\rho^*_i:=\rho(z^*_i,w_{\rm L}):=\left(\frac{\theta(w_{\rm L}-z^*_i)}{2}\right)^{1/\theta},\quad u^*_i:=u(z^*_i,w_{\rm L}):=\frac{w_{\rm L}+z^*_i}{2}$$

and

$$S(\rho,\rho_0):=\begin{cases}\sqrt{\dfrac{\rho(p(\rho)-p(\rho_0))}{\rho_0(\rho-\rho_0)}}, & \text{if } \rho\neq\rho_0,\\ \sqrt{p'(\rho_0)}, & \text{if } \rho=\rho_0.\end{cases} \tag{3.7}$$

We call this approximated 1-rarefaction wave a *1-rarefaction fan*.

Step 2.

In this step, we replace the constant states above with the following quasi-steady state solutions:

Definition 3.1. We define a *quasi-steady state solution* with data v_d at x_d as follows.

$$\mathbf{V}_{\mathrm{qs}}(x, x_d, v_d) := \begin{cases} \text{the solution } v(x) \text{ of (2.11)–(2.12),} \\ \ \text{when } \rho_d > (\Delta x)^\beta \text{ and } w(v_d) \le M_+; \\ v_d \text{ otherwise.} \end{cases}$$

Let $\bar{v}_{\mathrm{L}}(x)$ and $\bar{v}_{\mathrm{R}}(x)$ be $\mathbf{V}_{\mathrm{qs}}(x, (j-1/2)\Delta x+1, v_{\mathrm{L}})$ and $\mathbf{V}_{\mathrm{qs}}(x, (j+1/2)\Delta x+1, v_{\mathrm{R}})$ respectively. Set

$$\bar{v}_1(x) := \bar{v}_{\mathrm{L}}(x) \text{ and } x_1 := (j-1/2)\Delta x + 1. \tag{3.8}$$

First we determine a propagation speed σ_2 and $v_2 = (\rho_2, m_2)$ such that 1) the speed σ_2, the left and right states $\bar{v}_1(x_2), v_2$ satisfy the Rankine-Hugoniot conditions and 2) $z_2 := z(v_2) = z_2^*$, where $x_2 := j\Delta x + 1 + \sigma_2 \Delta t/2$. Then we fill up by $\bar{v}_1(x)$ the sector where $n\Delta t \le t < (n+1)\Delta t, (j-1/2)\Delta x + 1 \le x < j\Delta x + 1 + \sigma_2(t - n\Delta t)$ and set $\bar{v}_2(x) = \mathbf{V}_{\mathrm{qs}}(x, x_2, v_2)$.

Assume that v_k, $\bar{v}_k(x)$ and a propagation speed σ_k with $\sigma_{k-1} < \sigma_k$ are defined. Then we determine σ_{k+1} and $v_{k+1} = (\rho_{k+1}, m_{k+1})$ such that 1) the speed σ_{k+1}, the left and right states $\bar{v}_k(x_{k+1}), v_{k+1}$ satisfy the Rankine-Hugoniot conditions, 2) $z_{k+1} := z(v_{k+1}) = z_{k+1}^*$ and 3) $\sigma_k < \sigma_{k+1}$, where $x_{k+1} := j\Delta x + 1 + \sigma_{k+1}\Delta t/2$. Then we fill up by $\bar{v}_k(x)$ the sector where $n\Delta t \le t < (n+1)\Delta t, j\Delta x + 1 + \sigma_k(t - \Delta t) \le x < j\Delta x + 1 + \sigma_{k+1}(t - n\Delta t)$ and set $\bar{v}_{k+1}(x) = \mathbf{V}_{\mathrm{qs}}(x, x_{k+1}, v_{k+1})$. By induction we define v_i, $\bar{v}_i(x)$ and σ_i $(i = 1, \ldots, p-1)$. Finally we determine a propagation speed σ_p and $v_p = (\rho_p, m_p)$ such that 1) the speed σ_p, the left and right states $\bar{v}_{p-1}(x_p), v_p$ satisfy the Rankine Hugoniot conditions and 2) $z_p := z(v_p) = z_p^*$, where $x_p := j\Delta x + 1 + \sigma_p \Delta t/2$. We then fill up by $\bar{v}_{p-1}(x)$ and v_p the sector where $n\Delta t \le t < (n+1)\Delta t, j\Delta x + 1 + \sigma_{p-1}(t - n\Delta t) \le x < j\Delta x + 1 + \sigma_p(t - n\Delta t)$ and the line $n\Delta t \le t < (n+1)\Delta t, x = j\Delta x + 1 + \sigma_p(t - n\Delta t)$ respectively.

Given v_{L} and z_{M} with $z_{\mathrm{L}} \le z_{\mathrm{M}}$, we denote this piecewise quasi-steady state 1-rarefaction wave by $R_1^\Delta(z_{\mathrm{M}})(v_{\mathrm{L}})$. Notice that from the construction of $R_1^\Delta(z_{\mathrm{M}})(v_{\mathrm{L}})$ connects v_{L} and v_p with $z_p = z_{\mathrm{M}}$.

Now we fix $\bar{v}_{\mathrm{R}}(x)$ and $\bar{v}_{p-1}(x)$. Let σ_s be the propagation speed of the 2-shock connecting v_{M} and v_{R}. Choosing $\sigma_p^\diamond$ near to σ_p, $\sigma_s^\diamond$ near to σ_s and $v_{\mathrm{M}}^\diamond$ near to v_{M}, we fill up by a quasi-steady state solution $\bar{v}_{\mathrm{M}}^\diamond(x) = \mathbf{V}_{\mathrm{qs}}(x, j\Delta x + 1, v_{\mathrm{M}}^\diamond)$ the gap between $x = j\Delta x + 1 + \sigma_p^\diamond(t - n\Delta t)$ and $x = j\Delta x + 1 + \sigma_s^\diamond(t - n\Delta t)$, such that 1) $\sigma_{p-1} < \sigma_p^\diamond < \sigma_s^\diamond$, 2) the speed $\sigma_p^\diamond$, the left and right states $\bar{v}_{p-1}(x_p^\diamond), \bar{v}_{\mathrm{M}}^\diamond(x_p^\diamond)$ satisfy the Rankine-Hugoniot conditions and 3) so do the speed $\sigma_s^\diamond$, the left and right states $\bar{v}_{\mathrm{M}}^\diamond(x_s^\diamond), \bar{v}_{\mathrm{R}}(x_s^\diamond)$, where $x_p^\diamond := j\Delta x + 1 + \sigma_p^\diamond \Delta/2$ and $x_s^\diamond := j\Delta x + 1 + \sigma_s^\diamond \Delta/2$.

We denote this approximate Riemann solution which consists of quasi- steady state solutions by $\bar{v}^\Delta(x,t)$. In Appendix A, we will justify the construction above.

Remark 3.2. $\bar{v}^\Delta(x,t)$ satisfies the Rankine-Hugoniot conditions at the middle time of the cell, $t_{\mathrm{M}} := (n+1/2)\Delta t$.

Step 3.
Finally we define the desired $v^\Delta(x,t)$ in the cell $n\Delta t \le t < (n+1)\Delta t$, $(j-1/2)\Delta x + 1 \le x < (j+1/2)\Delta x + 1$ $(j = 1, 2, \ldots)$ by using $\bar{v}^\Delta(x,t)$ and the fractional step procedure. As mentioned above, notice that $\bar{v}^\Delta(x,t)$ consists of constants and steady state solutions of (2.11).

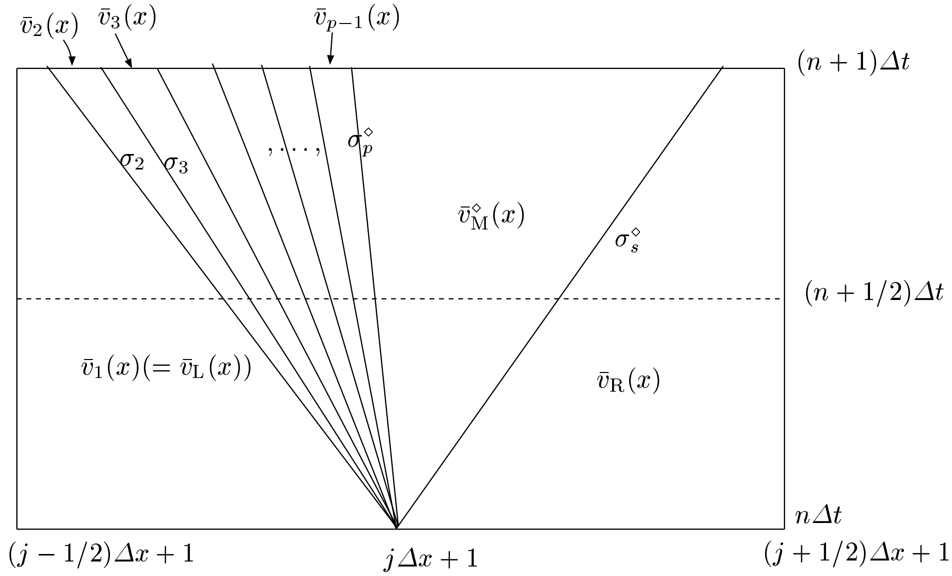

FIGURE 4. The approximate solution in the case where a 1-rarefaction and a 2-shock arise in the cell.

In the region where $\bar{v}^\Delta(x,t)$ is the steady state solution of (2.11), we define $v^\Delta(x,t)$ by

$$v^\Delta(x,t) := \bar{v}^\Delta(x,t) + h(x,\bar{v}^\Delta(x,t))(t-n\Delta t), \tag{3.9}$$

where $h(x,\bar{v}^\Delta(x,t)) := {}^t\left(0, \dfrac{2(\gamma-1)}{(\gamma+1)x}\dfrac{\{\bar{m}^\Delta(x,t)\}^2}{\bar{\rho}^\Delta(x,t)} + \dfrac{4\gamma}{(\gamma+1)x}p(\bar{\rho}^\Delta(x,t))\right)$.
We call (3.9) **Type I**.

Next, in the region where $\bar{v}^\Delta(x,t)$ is a constant, if $\bar{\rho}^\Delta(x,t) > (\Delta x)^\beta/2$ and $w(\bar{v}^\Delta(x,t)) > M_+$, we define $v^\Delta(x,t)$ by

$$v^\Delta(x,t) := \bar{v}^\Delta(x,t) + g(x,\bar{v}^\Delta(x,t))(t-n\Delta t). \tag{3.10}$$

We call (3.10) **Type II**.

Finally, in the region where $\bar{v}^\Delta(x,t)$ is a constant, if $\bar{\rho}^\Delta(x,t) \le (\Delta x)^\beta/2$, we define $v^\Delta(x,t)$ by

$$v^\Delta(x,t) := \bar{v}^\Delta(x,t). \tag{3.11}$$

We call (3.11) **Type III**.

We complete the construction of approximate solutions $v^\Delta(x,t)$ in the case (A).

(B) The case where the v_{M} is near the vacuum
In this step, we consider the case where $v_{\mathrm{M}} \le (\Delta x)^\beta$, which means that v_{M} is near the vacuum. In this case, we cannot construct approximate solutions in a similar fashion to the case (A). Therefore, we must define $v^\Delta(x,t)$ in the different way.

Case 1 A 1-rarefaction wave and a 2-shock arise.

Case 1.1 $\rho_{\mathrm{L}} > (\Delta x)^\beta$
Let $v_{\mathrm{L}}^{(1)}$ be a state satisfying $w(v_{\mathrm{L}}^{(1)}) = w(v_{\mathrm{L}})$ and $\rho_{\mathrm{L}}^{(1)} = (\Delta x)^\beta$.

(i) $z(v_{\rm M})-z(v_{\rm L}^{(1)})\leq(\Delta x)^{\alpha}$
Notice that $w(v_{\rm M})=w(v_{\rm L})=w(v_{\rm L}^{(1)})$. Then there exists $C>0$ such that $\rho_{\rm L}^{(1)}-\rho_{\rm M}\leq C(\Delta x)^{\alpha}$. Since $\alpha>\beta$, we then have $\rho_{\rm M}\geq 3(\Delta x)^{\beta}/4$. This case is reduced to the case (A).

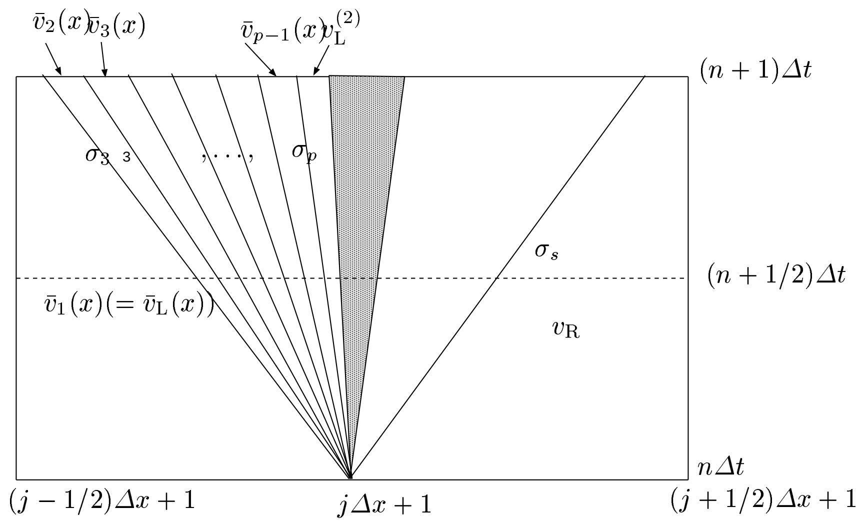

FIGURE 5. **Case 1.1 (ii)**: The approximate solution $\bar{v}^{\Delta}$ in the cell.

(ii) $z(v_{\rm M})-z(v_{\rm L}^{(1)})>(\Delta x)^{\alpha}$
Set

$$L_j:=-M_{-}\{(j+1/2)\Delta x+1\}^{-\frac{2(\gamma-1)}{\gamma+1}}.$$

Let $v_{\rm L}^{(2)}$ be a state connected to $v_{\rm L}$ on the right by $R_1^{\Delta}(\max\{z_{\rm L}^{(1)},L_j\})(v_{\rm L})$. Connecting the left and right states $v_{\rm L}^{(2)}$, $v_{\rm R}$ by rarefaction and shock curves, we define $\bar{v}^{\Delta}(x,t)$. Then, in the region where $\bar{v}^{\Delta}(x,t)$ is $R_1^{\Delta}(\max\{z_{\rm L}^{(1)},L_j\})(v_{\rm L})$, the definition of $v^{\Delta}(x,t)$ is similar to (A). In the region where $\bar{v}^{\Delta}(x,t)$ is the Riemann solution for $(v_{\rm L}^{(2)},v_{\rm R})$, we define $v^{\Delta}(x,t)$ by $v^{\Delta}(x,t):=\bar{v}^{\Delta}(x,t)$. Thus, for a Riemann solution near the vacuum, we define an approximate solution as this Riemann solution itself. We call this type approximate solution **Type III** like (3.11).

Case 1.2 $\rho_{\rm L}\leq(\Delta x)^{\beta}$

(i) $z(v_{\rm L})\geq L_j$
In this case, we define $v^{\Delta}(x,t)$ as a Riemann solution $(v_{\rm L},v_{\rm R})$.

(ii) $z(v_{\rm L})<L_j$
In this case, recalling $z(v_{\rm L})=z(v_j^n)\geq -M_{-}\{(j-1/2)\Delta x+1\}^{-\frac{2(\gamma-1)}{\gamma+1}}$, we can choose $x^{(3)}$ such that $(j-1/2)\Delta x+1\leq x^{(3)}\leq(j+1/2)\Delta x+1$ and

$$z(v_{\rm L})\left(\frac{x^{(3)}}{x_{\rm L}}\right)^{-\frac{2(\gamma-1)}{\gamma+1}}=L_j,$$

where $x_{\rm L} := (j-1/2)\varDelta x + 1$. Set

$$\rho_{\rm L}^{(3)} := \rho_{\rm L}\left(\frac{x^{(3)}}{x_{\rm L}}\right)^{-\frac{4}{\gamma+1}},\ m_{\rm L}^{(3)} := m_{\rm L}\left(\frac{x^{(3)}}{x_{\rm L}}\right)^{-2}. \tag{3.12}$$

Then it follows from (2.11) that

$$z(v_{\rm L}^{(3)}) = L_j. \tag{3.13}$$

Let $\lambda_1(v_{\rm L}^{(3)})$ be the 1-characteristic speed of $v_{\rm L}^{(3)}$. In the region where $n\varDelta t \le t < (n+1)\varDelta t$ and $(j-1/2)\varDelta x+1 \le x \le j\varDelta x+1+\lambda_1(v_{\rm L}^{(3)})(t-n\varDelta t)$, we define $\bar{v}^{\varDelta}(x,t)$ as a solution of (2.11) such that $\bar{v}^{\varDelta}((j-1/2)\varDelta x+1) = v_{\rm L}$. We next solve a Riemann problem $(v_{\rm L}^{(3)}, v_{\rm R})$. Since $v_{\rm R} \ge L_j$, a 1-shock does not arise. In the region where $n\varDelta t \le t < (n+1)\varDelta t$ and $j\varDelta x+1+\lambda_1(v_{\rm L}^{(3)})(t-n\varDelta t) \le x \le (j+1/2)\varDelta x+1$, we define $\bar{v}^{\varDelta}(x,t)$ as this Riemann solution. In the region where $\bar{v}^{\varDelta}(x,t)$ is the Riemann solution, we define $v^{\varDelta}(x,t)$ by $v^{\varDelta}(x,t) := \bar{v}^{\varDelta}(x,t)$; otherwise, the definition of $v^{\varDelta}(x,t)$ is similar to Type I.

Case 2 A 1-shock and a 2-rarefaction wave arise.

Case 2.1 $\rho_{\rm R} > (\varDelta x)^{\beta}$
Let $v_{\rm R}^{(1)}$ be a state satisfying $z(v_{\rm R}^{(1)}) = z(v_{\rm R})$ and $\rho_{\rm R}^{(1)} = (\varDelta x)^{\beta}$.
(i) $w(v_{\rm R}^{(1)}) - w(v_{\rm M}) \le (\varDelta x)^{\alpha}$
Notice that $z(v_{\rm R}^{(1)}) = z(v_{\rm R}) = z(v_{\rm M})$. Then there exists $C>0$ such that $\rho_{\rm R}^{(1)} - \rho_{\rm M} \le C(\varDelta x)^{\alpha}$. Since $\alpha > \beta$, we then have $\rho_{\rm M} \ge 3(\varDelta x)^{\beta}/4$. This case is reduced to the case (A).
(ii) $w(v_{\rm R}^{(1)}) - w(v_{\rm M}) > (\varDelta x)^{\alpha}$
Given $v_{\rm R}$ and $w(v_{\rm R}^{(1)})$ with $w(v_{\rm R}^{(1)}) \le w_{\rm R}$, we first construct a piecewise quasi-steady state 2-rarefaction wave $R_2^{\varDelta}(w(v_{\rm R}^{(1)}))(v_{\rm R})$ in a similar fashion to (A). Let $v_{\rm R}^{(2)}$ be a state connected to $v_{\rm R}$ on the left by $R_2^{\varDelta}(w(v_{\rm R}^{(1)}))(v_{\rm R})$. In this case, connecting the left and right states $v_{\rm L}, v_{\rm R}^{(2)}$ by rarefaction and shock curves, we define $\bar{v}^{\varDelta}(x,t)$. Then, in the region where $\bar{v}^{\varDelta}(x,t)$ is $R_2^{\varDelta}(z(v_{\rm R}^{(1)}))(v_{\rm R})$, the definition of $v^{\varDelta}(x,t)$ is similar to (A). In the region where $\bar{v}^{\varDelta}(x,t)$ is the Riemann solution for $(v_{\rm L}, v_{\rm R}^{(2)})$, we define $v^{\varDelta}(x,t)$ by $v^{\varDelta}(x,t) := \bar{v}^{\varDelta}(x,t)$.

Case 2.2 $\rho_{\rm R} \le (\varDelta x)^{\beta}$
In this case, we define $v^{\varDelta}(x,t)$ as a Riemann solution $(v_{\rm L}, v_{\rm R})$.

Case 3 A 1-rarefaction wave and a 2-rarefaction wave arise.
Throughout this case, if the vacuum arises, set $v_{\rm M}$ such that $\rho_{\rm M} = 0$ and $z(v_{\rm M}) = z(v_{\rm R})$; otherwise, set $v_{\rm M}$ such that $w(v_{\rm M}) = w(v_{\rm L})$ and $z(v_{\rm M}) = z(v_{\rm R})$. Let $v_{\rm R}^{(1)}$ be the state satisfying $z(v_{\rm R}^{(1)}) = z(v_{\rm R})$ and $\rho_{\rm R}^{(1)} = (\varDelta x)^{\beta}$.

Case 3.1 $w(v_{\rm R}^{(1)}) - w(v_{\rm M}) \le (\varDelta x)^{\alpha}$ and $\rho_{\rm R} > (\varDelta x)^{\beta}$
This case is reduced to (A).

Case 3.2 $w(v_{\rm R}^{(1)}) - w(v_{\rm M}) > (\varDelta x)^{\alpha}$ and $\rho_{\rm R} > (\varDelta x)^{\beta}$
Let $v_{\rm R}^{(2)}$ be the state connected to $v_{\rm R}$ on the left by $R_2^{\varDelta}(w(v_{\rm R}^{(1)}))(v_{\rm R})$.

(i) $\rho_{\rm L}>(\Delta x)^{\beta}$

Let $v_{\rm L}^{(1)}$ be the state satisfying $w(v_{\rm L}^{(1)})=w(v_{\rm L})$ and $\rho_{\rm L}^{(1)}=(\Delta x)^{\beta}$. Let $v_{\rm L}^{(2)}$ be the state connected to $v_{\rm L}$ on the right by $R_1^{\Delta}(\max\{z(v_{\rm L}^{(1)}),L_j\})(v_{\rm L})$. Connecting the left and right states $v_{\rm L}^{(2)},v_{\rm R}^{(2)}$ by rarefaction and shock curves, we define $\bar{v}^{\Delta}(x,t)$. In the region where $\bar{v}^{\Delta}(x,t)$ is the Riemann solution for $(v_{\rm L}^{(2)},v_{\rm R}^{(2)})$, we define $v^{\Delta}(x,t)$ by $v^{\Delta}(x,t):=\bar{v}^{\Delta}(x,t)$; otherwise, the definition of $v^{\Delta}(x,t)$ is similar to (A).

(ii) $\rho_{\rm L}<(\Delta x)^{\beta}$

(ii)–(a) $z(v_{\rm L})\geq\min\left\{z(v_{\rm R}^{(2)}),L_j\right\}$

Connecting the left and right states $v_{\rm L},v_{\rm R}^{(2)}$ by rarefaction and shock curves, we define $\bar{v}^{\Delta}(x,t)$. In the region where $\bar{v}^{\Delta}(x,t)$ is the Riemann solution for $(v_{\rm L},v_{\rm R}^{(2)})$, we define $v^{\Delta}(x,t)$ by $v^{\Delta}(x,t):=\bar{v}^{\Delta}(x,t)$; otherwise, the definition of $v^{\Delta}(x,t)$ is similar to (A).

(ii)–(b) $z(v_{\rm L})<\min\left\{z(v_{\rm R}^{(2)}),L_j\right\}$

We choose $x^{(3)}$ and $v_{\rm L}^{(3)}$ such that $(j-1/2)\Delta x+1\leq x^{(3)}\leq(j+1/2)\Delta x+1$ and

$$z(v_{\rm L}^{(3)})=z(v_{\rm L})\left(\frac{x^{(3)}}{x_{\rm L}}\right)^{-\frac{2(\gamma-1)}{\gamma+1}}=\min\left\{z(v_{\rm R}^{(2)}),L_j\right\},$$

$$\rho_{\rm L}^{(3)}=\rho_{\rm L}\left(\frac{x^{(3)}}{x_{\rm L}}\right)^{-\frac{4}{\gamma+1}},\quad m_{\rm L}^{(3)}=m_{\rm L}\left(\frac{x^{(3)}}{x_{\rm L}}\right)^{-2},$$

where $x_{\rm L}=(j-1/2)\Delta x+1$. Let $\lambda_1(v_{\rm L}^{(3)})$ be the 1-characteristic speed of $v_{\rm L}^{(3)}$. In the region where $n\Delta t\leq t<(n+1)\Delta t$ and $(j-1/2)\Delta x+1\leq x\leq j\Delta x+1+\lambda_1(v_{\rm L}^{(3)})(t-n\Delta t)$, we define $\bar{v}^{\Delta}(x,t)$ as a solution of (2.11) such that $\bar{v}^{\Delta}((j-1/2)\Delta x+1)=v_{\rm L}$. In the region where $n\Delta t\leq t<(n+1)\Delta t$ and $j\Delta x+1+\lambda_1(v_{\rm L}^{(3)})(t-n\Delta t)\leq x\leq(j+1/2)\Delta x+1$, we define $\bar{v}^{\Delta}(x,t)$ by connecting the left and right states $v_{\rm L}^{(3)},v_{\rm R}^{(2)}$ by rarefaction and shock curves. In the region where $\bar{v}^{\Delta}(x,t)$ is the Riemann solution for $(v_{\rm L}^{(3)},v_{\rm R}^{(2)})$, we define $v^{\Delta}(x,t)$ by $v^{\Delta}(x,t):=\bar{v}^{\Delta}(x,t)$. In the region where $\bar{v}^{\Delta}(x,t)$ is the steady state solution of (2.11), the definition of $v^{\Delta}(x,t)$ is the same as Type I. In the region where $\bar{v}^{\Delta}(x,t)$ is $R_2^{\Delta}(w(v_{\rm R}^{(1)}))(v_{\rm R})$, the definition of $v^{\Delta}(x,t)$ is the same as (A).

Case 3.3 $\rho_{\rm R}<(\Delta x)^{\beta}$

This case is similar to Case 1.1 and Case 1.2.

Case 4 A 1-shock and a 2-shock arise.

In this case, we define $v^{\Delta}(x,t)$ as a Riemann solution $(v_{\rm L},v_{\rm R})$.

We complete the construction of approximate solutions $v^{\Delta}(x,t)$ in the case (B). We postpone the validity of the construction above to Appendix B.

Through the case (B), in the region where $\bar{v}^{\Delta}(x,t)$ is a Riemann solution, we can construct $\bar{v}^{\Delta}(x,t)$ such that $\bar{\rho}^{\Delta}(x,t)<3(\Delta x)^{\beta}/2$. Then we notice the following.

Remark 3.3. When $\bar{\rho}^{\Delta}(x,t)=o(1)$ and $\bar{v}^{\Delta}(x,t)$ is a Riemann solution, for any function $\phi\in C_0^1(\{x\geq1\}\times\mathbf{R}_+)$, we find

$$\int_1^{\infty}\int_0^{\infty}\bar{v}^{\Delta}\phi_t+f(\bar{v}^{\Delta})\phi_x+g(x,\bar{v}^{\Delta})\phi\,dxdt=\int_1^{\infty}\int_0^{\infty}g(x,\bar{v}^{\Delta})\phi\,dxdt=o(1),$$

where $o(1)$ depends only on the uniform bound of $\bar{v}^{\Delta}$ and $o(1) \to 0$, as $\Delta x \to 0$.

3.2. **The construction of approximate solutions near the boundary.** In this section, we construct approximate solutions near the boundary $1 \leq x < 1 + \Delta x/2$. In the region where $1 \leq x < 1 + \Delta x/2$ and $n\Delta t \leq t < (n+1)\Delta t$, $\bar{v}^{\Delta}(x,t)$ is defined as follows.

When $u_1^n \geq 0$, we define $\bar{v}^{\Delta}(x,t)$ as the solution of the Riemann initial-boundary problem at $x = 1$:

$$\begin{cases} (2.1), \quad 1 \leq x < 1 + \Delta x/2, \quad n\Delta t \leq t < (n+1)\Delta t, \\ v|_{t=n\Delta t} = v_1^n, \quad m_{x=1} = 0; \end{cases}$$

When $u_1^n < 0$ and $\rho_1^n \geq (\Delta x)^{\beta}$, we first solve the modified Riemann initial-boundary problem at $x = 1$:

$$\begin{cases} (2.7), \quad 1 \leq x < 1 + \Delta x/2, \quad n\Delta t \leq t < (n+1)\Delta t, \\ v|_{t=n\Delta t} = \left(\rho_1^n \left(\frac{x}{1+\Delta x/2}\right)^{-\frac{4}{\gamma+1}}, m_1^n \left(\frac{x}{1+\Delta x/2}\right)^{-2}\right), \quad m_{x=1} = 0. \end{cases}$$

Here, notice that this solution has the form $(\tilde{\rho}(x,t)x^{-\frac{4}{\gamma+1}}, \tilde{m}(x,t)x^{-2})$ and a 2-shock arise in this case.

In order to make $\tilde{v}(x,t) = (\tilde{\rho}(x,t), \tilde{m}(x,t))$ self-similar, we set $\tilde{v}^{\dagger}(x,t)$ such that

$$\tilde{v}^{\dagger}(x,(n+1/2)\Delta t) = \tilde{v}(x,(n+1/2)\Delta t)$$

and extend the above to self-similar function

$$\tilde{v}^{\dagger}(x,t) = \psi\left((x-1)/(t-n\Delta t)\right),$$

where a certain function $\psi : \mathbf{R} \mapsto \mathbf{R}^2$.

In this case, we define $\bar{v}^{\Delta}(x,t) := (\tilde{\rho}^{\dagger}(x,t)x^{-\frac{4}{\gamma+1}}, \tilde{m}^{\dagger}(x,t)x^{-2})$ in the region where $1 \leq x < 1 + \Delta x/2$ and $n\Delta t \leq t < (n+1)\Delta t$.

When $u_1^n < 0$ and $\rho_1^n < (\Delta x)^{\beta}$, we define $\bar{v}^{\Delta}$ as a solution of the Riemann initial-boundary problem at $x = 1$:

$$\begin{cases} (2.1), \quad 1 \leq x < 1 + \Delta x/2, \quad n\Delta t \leq t < (n+1)\Delta t, \\ v|_{t=n\Delta t} = v_1^n, \quad m_{x=1} = 0. \end{cases}$$

We observe that $\bar{v}^{\Delta}$ satisfies $\bar{\rho}^{\Delta}(x,t) \leq C(\Delta x)^{(2-\gamma)\beta/2}$.

Then we define $v^{\Delta}(x,t)$ in the strip $n\Delta t \leq t < (n+1)\Delta t$ as follows.

If $u_1^n \geq 0$, we define $v^{\Delta}(x,t)$ by

$$v^{\Delta}(x,t) := \bar{v}^{\Delta}(x,t) + g(x, \bar{v}^{\Delta}(x,t))(t - n\Delta t).$$

If $u_1^n < 0$ and $\rho_1^n \geq (\Delta x)^{\beta}$, we define $v^{\Delta}(x,t)$ by

$$v^{\Delta}(x,t) := \bar{v}^{\Delta}(x,t) + h(x, \bar{v}^{\Delta}(x,t))(t - n\Delta t).$$

If $u_1^n < 0$ and $\rho_1^n < (\Delta x)^{\beta}$, we define $v^{\Delta}(x,t)$ by

$$v^{\Delta}(x,t) := \bar{v}^{\Delta}(x,t).$$

4. The L^∞ estimate of the approximate solutions

We estimate Riemann invariants of $v^\Delta(x,t)$ to use the invariant region theory. When we derive the estimate, the most difficulty is to treat inhomogeneous terms of (1.5). Let us return (1.5) to investigate the effect of source terms.

First we consider the case where the velocity is negative. Needless to say, the momentum is also negative. Then the first equation of (1.5) shows that the source term increases the density as time passes. Similarly, the second equation shows that the source term decreases the momentum as time passes. Therefore source terms increase the modulus of the density and the momentum as time passes. It is difficult to control this increase by source terms. On the other hand, when the velocity is nonnegative, the modulus of the density and the momentum decreases as time passes. It is easy to treat this case. In fact, if initial data satisfy the condition (1.7), the velocity of the corresponding solution is always nonnegative. Then we can obtain L^∞ estimates easily. Therefore, we must overcome difficulties of the case where the velocity is negative.

Recalling Remark 1.2, we shall derive the lower bound of $z(v^\Delta)$ and the upper bound of $w(v^\Delta)$. When the velocity is negative, the modulus of $z(v^\Delta)$ is larger than that of $w(v^\Delta)$ and $z(v^\Delta)$ is negative. So the really difficult point is the lower bound of $z(v^\Delta)$.

Our aim in this section is to deduce from (3.5) the following lemma:

Lemma 4.1.

$$\begin{aligned} &w(v^\Delta(x,(n+1)\Delta t-0)) \leq M_+ + \varepsilon + o(\Delta x),\\ &z(v^\Delta(x,(n+1)\Delta t-0)) \geq -M_- x^{-\frac{2(\gamma-1)}{\gamma+1}} - o(\Delta x), \end{aligned} \tag{4.1}$$

where $o(\Delta x)$ depends only on M_- and M_+.

Remark 4.1. Observing (3.4) and (4.1), we cut off the errors with the order $o(\Delta x)$. However these errors are so small that we can obtain H^{-1}-compactness of our approximate solutions (see (6.1)).

Throughout this paper, by Landau's symbols such as $O(\Delta x)$, $O((\Delta x)^2)$ and $o(\Delta x)$, we denote quantities whose moduli satisfy a uniform bound depending only on M_- and M_+ unless we specify them. In addition, for simplicity, we denote $w(\bar{v}_i(x))$ and $z(\bar{v}_i(x))$ by $\bar{w}_i(x)$ and $\bar{z}_i(x)$.

Now, in the previous section, separating some cases, we have constructed $v^\Delta(x,t)$. First let us observe the case (A), where $v_{\rm M}$ is away from the vacuum, in terms of L^∞ estimates:

- In **Case 1**, main difficulty is to obtain $(4.1)_1$ along R_1^Δ.
- In **Case 2**, main difficulty is to obtain $(4.1)_2$ along R_2^Δ.
- In **Case 3**, (4.1) follows that of Case 1 and Case 2.
- In **Case 4**, (4.1) is easier than that of the other cases.

On the other hand, we consider (B), where $v_{\rm M}$ is near the vacuum. Recall that $v^\Delta(x,t)$ consists of R_1^Δ, R_2^Δ and a Riemann solution. Since the estimates of R_1^Δ and R_2^Δ are similar to those of (A), we must derive (4.1) for the Riemann solution.

- In **Case 1**, we can deduce from Lemma 2.2 (4.1) easily.
- In **Case 2**, $(4.1)_2$ of Case 2.1 (ii) requires special care.
- In **Case 3**, (4.1) follows that of Case 1 and Case 2.
- In **Case 4**, we can deduce from Lemma 2.2 (4.1) easily.

Thus we treat (A) Case 1 and (B) Case 2.1 (ii) in particular. In (A) Case 1, we derive $(4.1)_1$ along R_1^Δ and estimate the other quasi-steady state solutions. In (B) Case 2 (ii), we derive $(4.1)_2$ along R_2^Δ and $(4.1)_2$ for a Riemann solution. We can estimate the other cases in a similar fashion to these two cases.

4.1. **Estimates of $\bar{v}^\Delta(x,t)$ in (A) Case 1.** In this step, we estimate $v^\Delta(x,t)$ in the case (A) of Subsection 3.1. In this case, each component of $\bar{v}^\Delta(x,t)$ has the following properties, which will be proved in Appendix A:

- $\sigma_i < \sigma_{i+1}$ $(i=1,\ldots,p-2), \sigma_{p-1} < \sigma_p^\diamond < \sigma_s^\diamond$. $\quad(4.2)$
- $\rho_i > (\Delta x)^\beta/2$ $(i=1,\ldots,p-1)$. $\quad(4.3)$
- Given data $z_i := z(v_i)$ and $w_i := w(v_i)$ at $x = x_i$, $\bar{v}_i(x) = \mathbf{V}_{\mathrm{qs}}(x,x_i,v_i)$ $(i=1,\ldots,p-1)$ i.e.

$$(\bar{z}_i(x),\bar{w}_i(x)) = \begin{cases} \left(z_i(x/x_i)^{-\frac{2(\gamma-1)}{\gamma+1}}, w_i(x/x_i)^{-\frac{2(\gamma-1)}{\gamma+1}}\right) \\ \quad \text{when } w_i \leq M_+ \text{ and } \rho_i > (\Delta x)^\beta/2, \\ (z_i,w_i) \quad \text{otherwise.} \end{cases} \tag{4.4}$$

- $\bar{w}_{i+1}(x_{i+1}) = w_{i+1} = \bar{w}_i(x_{i+1}) + O((\Delta x)^{3\alpha-(\gamma-1)\beta})$ $\quad(i=1,\ldots,p-2). \quad(4.5)$
- $|v_{\mathrm{M}}^\diamond - v_{\mathrm{M}}| = O((\Delta x)^{1-\frac{\gamma+1}{2}\beta})$. $\quad(4.6)$
- $\bar{v}_{\mathrm{M}}^\diamond(x) = \mathbf{V}_{\mathrm{qs}}(x, j\Delta x + 1, v_{\mathrm{M}}^\diamond)$.
- $\bar{v}_i(x_{i+1})$ and $\bar{v}_{i+1}(x_{i+1})$ are connected by a 1-rarefaction shock curve $(i=1,\ldots,p-2)$.
- $\bar{w}_{\mathrm{M}}^\diamond(x_p^\diamond) = \bar{w}_{p-1}(x_p^\diamond) + O((\Delta x)^{3\alpha+(\gamma-7)\beta/2})$. $\quad(4.7)$
- $\bar{v}_{p-1}(x_p^\diamond)$ and $\bar{v}_{\mathrm{M}}^\diamond(x_p^\diamond)$ are connected by a 1-shock or a 1-rarefaction shock curve.
- $\bar{v}_{\mathrm{M}}^\diamond(x_s^\diamond)$ and $\bar{v}_{\mathrm{R}}(x_s^\diamond)$ are connected by a 2-shock or a 2-rarefaction shock curve.

Now we derive (4.1) in the interior cell $n\Delta t \leq t < (n+1)\Delta t$, $(j-1/2)\Delta x + 1 \leq x \leq (j+1/2)\Delta x + 1$. To do this, we first consider components of $\bar{v}^\Delta(x,t)$.

Estimate of $\bar{w}_i(x)$ $(i=1,\ldots,p-1)$.
For $i=1,\ldots,p-1$, from (4.4), we have

$$\bar{w}_i(x) \leq \max\{w_i, 0\} \quad \text{for } x \geq x_i.$$

Therefore, recalling that $w_1 = w(v_j^n) \leq M_+ + \varepsilon$, from (3.6) and (4.5), we deduce that

$$\begin{aligned} \bar{w}_i(x) &\leq M_+ + \varepsilon + O((\Delta x)^{2\alpha-(\gamma-1)\beta}) \\ &\leq M_+ + \varepsilon + o(\Delta x) \quad \text{for } x \geq x_i. \end{aligned} \tag{4.8}$$

If $\bar{v}_i(x)$ is a constant, in the cell, it follows that

$$\bar{w}_i(x) \leq M_+ + \varepsilon + o(\Delta x). \tag{4.9}$$

If $\bar{v}_i(x)$ is a steady state solution of (2.11), from the definition of a quasi-steady state solution, we observe that $w_i \leq M_+$. Then, in the cell, (2.13) yields

$$\bar{w}_i(x) \leq \left\{ \frac{(j+1/2)\Delta x + 1}{(j-1/2)\Delta x + 1} \right\}^{\frac{2(\gamma-1)}{\gamma+1}} M_+. \tag{4.10}$$

Estimate of $\bar{w}_{\mathrm{M}}^{\diamond}(x)$.
If $w_{\mathrm{M}}^{\diamond} \leq M_+$ and $\rho_{\mathrm{M}}^{\diamond} > (\Delta x)^{\beta}/2$, this estimate is similar to (4.10); otherwise, combining the fact that $(9-3\gamma)\beta/2 < \alpha$, (4.7) and (4.8), we thus have

$$\bar{w}_{\mathrm{M}}^{\diamond}(x) = \bar{w}_{\mathrm{M}}^{\diamond}(x_p^{\diamond}) = \bar{w}_{p-1}(x_p^{\diamond}) + o(\Delta x) \leq M_+ + \varepsilon + o(\Delta x). \tag{4.11}$$

Estimate of $\bar{w}_{\mathrm{R}}(x)$.
If $w_{\mathrm{R}} \leq M_+$ and $\rho_{\mathrm{R}} > (\Delta x)^{\beta}/2$, this estimate is similar to (4.10); otherwise, recalling $w_{\mathrm{R}} = w(v_{j+1}^n) \leq M_+ + \varepsilon$, it follows that

$$\bar{w}_{\mathrm{R}}(x) = w_{\mathrm{R}} \leq M_+ + \varepsilon. \tag{4.12}$$

Estimate of $\bar{z}_i(x)$ $(i = 1, \dots, p-1)$.
Recall that $z_1 = z_{\mathrm{L}} = z_j^n \geq -M_-\{(j-1/2)\Delta x + 1\}^{-\frac{2(\gamma-1)}{\gamma+1}}$. Thus, when $i=1$, if $w_1 \leq M_+$ and $\rho_1 > (\Delta x)^{\beta}/2$, (4.4) with $i=1$ implies

$$\bar{z}_1(x) = z_1(x/x_1)^{-\frac{2(\gamma-1)}{\gamma+1}} \geq -M_- x^{-\frac{2(\gamma-1)}{\gamma+1}}; \tag{4.13}$$

otherwise,

$$\bar{z}_1(x) = z_1 \geq -M_-\{(j-1/2)\Delta x + 1\}^{-\frac{2(\gamma-1)}{\gamma+1}}. \tag{4.14}$$

Next we estimate $\bar{z}_i(x)$ $(i = 2, \dots, p-1)$. The construction of $\bar{v}_i(x)$ implies

$$\bar{z}_i(x) = \begin{cases} z_{\mathrm{L}} + (i-1)(\Delta x)^{\alpha}, & \text{when } w_i > M_+, \\ \{z_{\mathrm{L}} + (i-1)(\Delta x)^{\alpha}\}(x/x_i)^{-\frac{2(\gamma-1)}{\gamma+1}}, & \text{when } w_i \leq M_+. \end{cases}$$

Since $\alpha < 1$ and $z(v_{\mathrm{L}}) = z_j^n \geq -M_-\{(j-1/2)\Delta x + 1\}^{-\frac{2(\gamma-1)}{\gamma+1}}$,
on $I_j = [(j-1/2)\Delta x + 1,\ (j+1/2)\Delta x + 1]$, we have

$$\begin{aligned} \bar{z}_i(x) &\geq \left[-M_-\{(j-1/2)\Delta x + 1\}^{-\frac{2(\gamma-1)}{\gamma+1}} + (\Delta x)^{\alpha} \right] \\ &\quad \times \max\left\{ 1, \sup_{x \in I_j} (x/x_i)^{-\frac{2(\gamma-1)}{\gamma+1}} \right\} \\ &\geq \left[-M_-\{(j+1/2)\Delta x + 1\}^{-\frac{2(\gamma-1)}{\gamma+1}} - O(\Delta x) + (\Delta x)^{\alpha} \right] \\ &\quad \times \{1 + O(\Delta x)\} \\ &\geq -M_-\{(j+1/2)\Delta x + 1\}^{-\frac{2(\gamma-1)}{\gamma+1}}. \end{aligned} \tag{4.15}$$

Estimate of $\bar{z}_{\mathrm{R}}(x)$.
Set $x_{\mathrm{R}} = (j+1/2)\Delta x + 1$. Recall that $z_{\mathrm{R}} = z_{j+1}^n \geq -M_-\{(j+1/2)\Delta x + 1\}^{-\frac{2(\gamma-1)}{\gamma+1}}$. Thus, if $w_{\mathrm{R}} \leq M_+$ and $\rho_{\mathrm{R}} > (\Delta x)^{\beta}/2$, we have

$$\bar{z}_{\mathrm{R}}(x) = z_{\mathrm{R}}(x/x_{\mathrm{R}})^{-\frac{2(\gamma-1)}{\gamma+1}} \geq -M_- x^{-\frac{2(\gamma-1)}{\gamma+1}}; \tag{4.16}$$

otherwise,

$$\bar{z}_{\mathrm{R}}(x) = z_{\mathrm{R}} \geq -M_{-}\{(j+1/2)\Delta x + 1\}^{-\frac{2(\gamma-1)}{\gamma+1}}. \tag{4.17}$$

Estimate of $\bar{z}^{\diamond}_{\mathrm{M}}(x)$.
If $\bar{v}^{\diamond}_{\mathrm{M}}(x^{\diamond}_{s})$ and $\bar{v}_{\mathrm{R}}(x^{\diamond}_{s})$ are connected by a 2-shock curve, from (4.16) and (4.17), we have

$$\bar{z}^{\diamond}_{\mathrm{M}}(x^{\diamond}_{s}) \geq \bar{z}_{\mathrm{R}}(x^{\diamond}_{s}) \geq -M_{-}(x^{\diamond}_{s})^{-\frac{2(\gamma-1)}{\gamma+1}}. \tag{4.18}$$

On the other hand, we consider the case where $\bar{v}^{\diamond}_{\mathrm{M}}(x^{\diamond}_{s})$ and $\bar{v}_{\mathrm{R}}(x^{\diamond}_{s})$ are connected by a 2-rarefaction shock curve. First, recall that v_{M} and v_{R} are connected by not a 2-rarefaction shock curve but a 2-shock curve. Since $|\bar{v}^{\diamond}_{\mathrm{M}}(x^{\diamond}_{s}) - v^{\diamond}_{\mathrm{M}}| = O(\Delta x)$ and $|\bar{v}_{\mathrm{R}}(x^{\diamond}_{s}) - v_{\mathrm{R}}| = O(\Delta x)$, we then deduce from (4.6) that

$$|\bar{v}^{\diamond}_{\mathrm{M}}(x^{\diamond}_{s}) - \bar{v}_{\mathrm{R}}(x^{\diamond}_{s})| = O((\Delta x)^{1-(\gamma+1)\beta/2}). \tag{4.19}$$

Therefore, from Remark 2.2 and the fact that $\beta < 2/(\gamma+5)$, we conclude that

$$\bar{z}^{\diamond}_{\mathrm{M}}(x^{\diamond}_{s}) = \bar{z}_{\mathrm{R}}(x^{\diamond}_{s}) - O((\Delta x)^{3(1-\frac{\gamma+1}{2}\beta)+\frac{\gamma-7}{2}\beta}) \geq -M_{-}(x^{\diamond}_{s})^{-\frac{2(\gamma-1)}{\gamma+1}} - o(\Delta x).$$

If $w^{\diamond}_{\mathrm{M}} \leq M_{+}$ and $\rho^{\diamond}_{\mathrm{M}} > (\Delta x)^{\beta}$, we obtain

$$\bar{z}^{\diamond}_{\mathrm{M}}(x) = \bar{z}^{\diamond}_{\mathrm{M}}(x^{\diamond}_{s})(x/x^{\diamond}_{s})^{-\frac{2(\gamma-1)}{\gamma+1}} \geq -M_{-}x^{-\frac{2(\gamma-1)}{\gamma+1}} - o(\Delta x); \tag{4.20}$$

otherwise,

$$\begin{aligned}\bar{z}^{\diamond}_{\mathrm{M}}(x) = \bar{z}^{\diamond}_{\mathrm{M}}(\bar{x}^{\diamond}_{s}) &\geq -M_{-}(x^{\diamond}_{s})^{-\frac{2(\gamma-1)}{\gamma+1}} - o(\Delta x) \\ &\geq -M_{-}\{(j-1/2)\Delta x + 1\}^{-\frac{2(\gamma-1)}{\gamma+1}} - o(\Delta x).\end{aligned} \tag{4.21}$$

Estimate of $v^{\Delta}(x,t)$.
We derive (4.1) in the interior cell where $[(j-1/2)\Delta x + 1, (j+1/2)\Delta x + 1] \times [n\Delta t, (n+1)\Delta t)$.

We first estimate Type I. In this case, the corresponding $\bar{v}^{\Delta}(x,t)$ is the steady state solution of (2.11). Therefore $\bar{v}^{\Delta}(x,t)$ satisfies (4.10), (4.13), (4.15), (4.16) and (4.20).

From (4.13), (4.15), (4.16) and (4.20), we observe that

$$z(\bar{v}^{\Delta}(x,t)) \geq -M_{-}x^{-\frac{2(\gamma-1)}{\gamma+1}} - o(\Delta x).$$

Therefore, from Remark 1.2, (3.1) and (4.10), we deduce

$$\begin{aligned}|z(\bar{v}^{\Delta}(x,t))| &\leq \left\{\frac{(j+1/2)\Delta x + 1}{(j-1/2)\Delta x + 1}\right\}^{\frac{2(\gamma-1)}{\gamma+1}} M_{+}, \\ |w(\bar{v}^{\Delta}(x,t))| &\leq \left\{\frac{(j+1/2)\Delta x + 1}{(j-1/2)\Delta x + 1}\right\}^{\frac{2(\gamma-1)}{\gamma+1}} M_{+}, \\ &\qquad x \in [(j-1/2)\Delta x + 1, (j+1/2)\Delta x + 1].\end{aligned} \tag{4.22}$$

Estimate 1

$$
\begin{aligned}
w(v^\Delta(x,t)) &= w(\bar{v}^\Delta(x,t)) + \left\{ \frac{2(\gamma-1)}{(\gamma+1)x}\{\bar{u}^\Delta(x,t)\}^2 + \frac{4}{(\gamma+1)x}(\bar{\rho}^\Delta(x,t))^{\gamma-1} \right\} \\
&\quad \times (t - n\Delta t) \\
&\le \left\{ \frac{(j+1/2)\Delta x + 1}{(j-1/2)\Delta x + 1} \right\}^{\frac{2(\gamma-1)}{\gamma+1}} M_+ + \left\{ \frac{(j+1/2)\Delta x + 1}{(j-1/2)\Delta x + 1} \right\}^{\frac{4(\gamma-1)}{\gamma+1}} \\
&\quad \times \left\{ \frac{2(\gamma-1)}{(\gamma+1)} M_+^2 + \frac{(\gamma-1)^2}{(\gamma+1)} M_+^2 \right\} \Delta t \\
&\quad \text{(from Remark 1.2 and (4.22))} \\
&\le (1 + 2\Delta x)\{M_+ + (\gamma-1)M_+^2 \Delta t\} \text{ (from the fact that } \gamma \le 5/3) \\
&= \{1 + 12(M_+ + \varepsilon)\Delta t\}\{M_+ + (\gamma-1)M_+^2\Delta t\} \quad \text{(from (3.3))} \\
&\le M_+ + \{(\gamma+11)M_+ + 12\varepsilon\} M_+ \Delta t \\
&\quad + 12(\gamma-1)M_+^2 (M_+ + \varepsilon)(\Delta t)^2 \\
&\le M_+ + \varepsilon \quad \text{(from (3.2))}.
\end{aligned}
$$

Estimate 2

$$
\begin{aligned}
z(v^\Delta(x,t)) &= z(\bar{v}^\Delta(x,t)) + \left\{ \frac{2(\gamma-1)}{(\gamma+1)x}\{\bar{u}^\Delta(x,t)\}^2 + \frac{4}{(\gamma+1)x}(\bar{\rho}^\Delta(x,t))^{\gamma-1} \right\} \\
&\quad \times (t - n\Delta t) \\
&\ge z(\bar{v}^\Delta(x,t)) \ge -M_- x^{-\frac{2(\gamma-1)}{\gamma+1}} - o(\Delta x).
\end{aligned}
$$

We next estimate Type II. Notice that, from the definition of Type II, $\bar{v}^\Delta(x,t)$ is a constant in this case. Therefore the corresponding $\bar{v}^\Delta(x,t)$ satisfies (4.9), (4.11), (4.12), (4.14), (4.15), (4.17) and (4.21). Moreover, since $M_+ \ge M_-$, $z(\bar{v}^\Delta(x,t)) \ge -M_-$ and $w(\bar{v}^\Delta(x,t)) > M_+$, we have $\bar{u}^\Delta(x,t) \ge 0$.

Estimate 3

$$
\begin{aligned}
w(v^\Delta(x,t)) &= w(\bar{v}^\Delta(x,t)) - \frac{2}{x}(\bar{\rho}^\Delta(x,t))^\theta \bar{u}^\Delta(x,t)(t - n\Delta t) + O((\Delta x)^2) \\
&\le M_+ + \varepsilon + o(\Delta x).
\end{aligned}
$$

Estimate 4

We next estimate $z(v^\Delta(x,(n+1)\Delta t - 0))$. When $z(\bar{v}^\Delta(x,(n+1)\Delta t - 0)) \ge 0$,

$$
\begin{aligned}
&z(v^\Delta(x,(n+1)\Delta t - 0)) \ge z(\bar{v}^\Delta(x,(n+1)\Delta t - 0)) \\
&\quad + \frac{2}{x}(\bar{\rho}^\Delta(x,(n+1)\Delta t - 0))^\theta \bar{u}^\Delta(x,(n+1)\Delta t - 0)\Delta t - O((\Delta x)^2) \\
&\ge -o(\Delta x).
\end{aligned}
$$

When $z(\bar{v}^\Delta(x,(n+1)\Delta t - 0)) < 0$, notice that $-M_-\{(j-1/2)\Delta x + 1\}^{-\frac{2(\gamma-1)}{\gamma+1}} \le z(\bar{v}^\Delta(x,t)) < 0$. Moreover, since $\bar{v}^\Delta(x,t)$ is a constant in this case, from the definition of Type II, we observe that $w(\bar{v}^{\Delta}(x,t)) > M_{+}$. We then have

$$
\begin{aligned}
&z(v^{\Delta}(x,(n+1)\Delta t-0)) \geq z(\bar{v}^{\Delta}(x,(n+1)\Delta t-0)) \\
&\quad + \frac{2}{x}(\bar{\rho}^{\Delta}(x,(n+1)\Delta t-0))^{\theta}\bar{u}^{\Delta}(x,(n+1)\Delta t-0)\Delta t - O((\Delta x)^2) \\
&\geq -M_{-}\{(j-1/2)\Delta x+1\}^{-\frac{2(\gamma-1)}{\gamma+1}} \\
&\quad + \frac{2}{x}(\bar{\rho}^{\Delta}(x,(n+1)\Delta t-0))^{\theta}\bar{u}^{\Delta}(x,(n+1)\Delta t-0)\Delta t - O((\Delta x)^2) \\
&\geq -M_{-}x^{-\frac{2(\gamma-1)}{\gamma+1}} + M_{-}\frac{2(\gamma-1)}{\gamma+1}x^{-\frac{2(\gamma-1)}{\gamma+1}-1}\{(j-1/2)\Delta x+1-x\} \\
&\quad + \frac{2}{x}(\bar{\rho}^{\Delta}(x,(n+1)\Delta t-0))^{\theta}\bar{u}^{\Delta}(x,(n+1)\Delta t-0)\Delta t - O((\Delta x)^2) \\
&\geq -M_{-}x^{-\frac{2(\gamma-1)}{\gamma+1}} - \frac{2(\gamma-1)}{\gamma+1}\frac{1}{x}M_{-}\Delta x \\
&\quad + \frac{2}{x}(\bar{\rho}^{\Delta}(x,(n+1)\Delta t-0))^{\theta}\bar{u}^{\Delta}(x,(n+1)\Delta t-0)\Delta t - O((\Delta x)^2) \\
&\geq -M_{-}x^{-\frac{2(\gamma-1)}{\gamma+1}} - \frac{2(\gamma-1)}{\gamma+1}\frac{1}{x}M_{-}\Delta x \\
&\quad + \frac{\gamma-1}{4x}[\{\bar{w}^{\Delta}(x,(n+1)\Delta t-0)\}^2 - \{\bar{z}^{\Delta}(x,(n+1)\Delta t-0)\}^2]\Delta t \\
&\quad - O((\Delta x)^2) \\
&\geq -M_{-}x^{-\frac{2(\gamma-1)}{\gamma+1}} - \frac{2(\gamma-1)}{\gamma+1}\frac{1}{x}M_{-}\Delta x + \frac{\gamma-1}{4x}(M_{+}^2 - M_{-}^2)\Delta t - O((\Delta x)^2) \\
&\geq -M_{-}x^{-\frac{2(\gamma-1)}{\gamma+1}} - \frac{12(\gamma-1)}{\gamma+1}\frac{1}{x}M_{-}(M_{+}+\varepsilon)\Delta t \\
&\quad + \frac{\gamma-1}{4x}(M_{+}^2 - M_{-}^2)\Delta t - O((\Delta x)^2) \quad \text{(from (3.3))} \\
&\geq -M_{-}x^{-\frac{2(\gamma-1)}{\gamma+1}} + \frac{\gamma-1}{4x}\Delta t\left(M_{+}^2 - M_{-}^2 - \frac{48}{\gamma+1}(M_{+}+\varepsilon)M_{-}\right) - O((\Delta x)^2) \\
&\geq -M_{-}x^{-\frac{2(\gamma-1)}{\gamma+1}} - o(\Delta x) \quad \text{(from (3.1))}.
\end{aligned}
$$

Finally, we consider Type III. Since $\rho_i > (\Delta x)^{\beta}/2$ $(i = 1, \ldots, p-1)$ and $\rho_{\mathrm{M}}^{\diamond} > (\Delta x)^{\beta}/2$, it suffices to consider only $\bar{v}_{\mathrm{R}}(x)$. Then, in view of the definition of Type III, (4.12) and (4.17) yields (4.1).

Let us review Estimate 1–4. If the data of a quasi-steady state solution are in the area (I) (see Figure 6), we use Type I. If the data of a quasi-steady state solution are in the area (II), we use Type II. As long as we use Type I, we can obtain the lower bound of z of our approximate solutions (see Estimate II). Therefore, our approximate solutions cannot go over the line $z = -M_{-}x^{-\frac{2(\gamma-1)}{\gamma+1}}$. The rest is the estimate of w of our approximate solutions. However, as $w(v^{\Delta})$ becomes large, our approximate solutions go into the area where the velocity is positive ($w \geq z, w \geq -z$ in Figure 6). Here recall [C1]. If the velocity is nonnegative, we can prove the L^{∞} estimates easily. Therefore, our estimates are reduced to [C1] essentially (see Estimate 3–4). In fact, we use the same approximate solutions, Type II, as those of [C1] in this case. The main point of this paper is to use Type

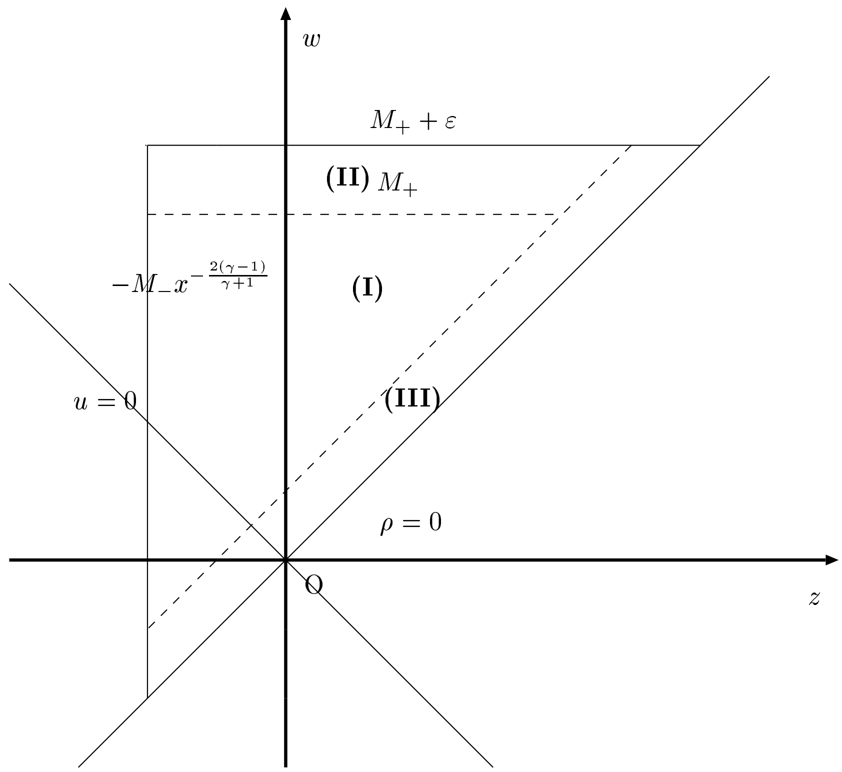

FIGURE 6. **Type I** and **Type II** in the (z, w)-plane.

I. Under $(1.8)_2$, Type I yields the most difficult estimate $(4.1)_2$, when the velocity is negative.

4.2. **Estimates of $v^{\Delta}(x, t)$ in (B) Case 2.1 (ii).** Defining

$$q := \max\left\{[\![(w_{\mathrm{R}} - w_{\mathrm{R}}^{(1)})/(\Delta x)^{\alpha}]\!] + 1, 2\right\},$$

we set

$$w_1^* = w_{\mathrm{R}},\ w_q^* = w_{\mathrm{R}}^{(1)},\ w_i^* = w_{\mathrm{R}} - (i-1)(\Delta x)^{\alpha} \quad (i = 1, \ldots, q-1).$$

Then notice that

$$q = O((\Delta x)^{-\alpha}). \tag{4.23}$$

Set

$$v_1 = v_{\mathrm{R}} \text{ and } x_1 = (j + 1/2)\Delta x + 1.$$

Then, there exist σ_i with $\sigma_{i+1} < \sigma_i$, $x_i := j\Delta x + 1 + \sigma_i \Delta t/2$ $(i = 2, \ldots, q)$ and v_i $(i = 1, \ldots, q)$ such that $\bar{v}^{\Delta}(x,t)$ consists of the following:

- $\bar{v}_1(x) := \mathbf{V}_{\mathrm{qs}}(x, (j+1/2)\Delta x + 1, v_1)$ in the sector $n\Delta t \leq t < (n+1)\Delta t$, $j\Delta x + 1 + \sigma_2(t - \Delta t) \leq x < (j+1/2)\Delta x + 1$.
- $\bar{v}_i(x) := \mathbf{V}_{\mathrm{qs}}(x, x_i, v_i)$ $(i = 2, \ldots, q-1)$ in the sector $n\Delta t \leq t < (n+1)\Delta t$, $j\Delta x + 1 + \sigma_{i+1}(t - \Delta t) \leq x < j\Delta x + 1 + \sigma_i(t - \Delta t)$.
- A Riemann solution for $(v_{\mathrm{L}}, v_q (= v_{\mathrm{R}}^{(2)}))$ in the sector $n\Delta t \leq t < (n+1)\Delta t$, $(j-1/2)\Delta x + 1 \leq x < j\Delta x + 1 + \sigma_q(t - \Delta t)$.

From the construction of (B) Case 2.1 (ii), notice that

$$v_q = v_{\mathrm{R}}^{(2)}.$$

Moreover each component of $\bar{v}^{\Delta}(x,t)$ has the following properties:

- $\rho_i > (\Delta x)^{\beta}/2$ $(i = 1, \ldots, q)$. (4.24)
- Given data z_i and w_i at $x = x_i$, $\bar{v}_i(x)$ $(i = 1, \ldots, q-1)$ is constructed as a quasi-steady state solution i.e.

$$(\bar{z}_i(x), \bar{w}_i(x)) = \begin{cases} \left(z_i (x/x_i)^{-\frac{2(\gamma-1)}{\gamma+1}}, w_i (x/x_i)^{-\frac{2(\gamma-1)}{\gamma+1}} \right) \\ \quad \text{when } w_i \leq M_{+} \text{ and } \rho_i > (\Delta x)^{\beta}/2, \\ (z_i, w_i) \quad \text{otherwise.} \end{cases} \tag{4.25}$$

- $\bar{z}_{i+1}(x_{i+1}) = z_{i+1} = \bar{z}_i(x_{i+1}) + O((\Delta x)^{3\alpha - (\gamma-1)\beta})$ $(i = 1, \ldots, q-2)$. (4.26)
- $\bar{v}_i(x_{i+1})$ and $\bar{v}_{i+1}(x_{i+1})$ are connected by a 2-rarefaction shock curve $(i = 1, \ldots, q-2)$.
- $z_q = \bar{z}_{q-1}(x_q) + O((\Delta x)^{3\alpha - (\gamma-1)\beta})$. (4.27)
- $\bar{v}_{q-1}(x_q)$ and v_q are connected by a 2-rarefaction shock curve.
- $|v_q - v_{\mathrm{R}}^{(1)}| = O(\Delta x)$. (4.28)

The proof of these properties is similar to that of R_1^{Δ} in Appendix A.

Estimate of $\bar{z}_i(x)$ $(i = 1, \ldots, q-1)$ and z_q.
For $i = 1, \ldots, q-1$, from (4.25), we have

$$\bar{z}_i(x) \geq \min\{ z_i (x/x_i)^{-\frac{2(\gamma-1)}{\gamma+1}}, 0 \} \quad \text{for } x \leq x_i.$$

Therefore, recalling that $z_1 = z_{\mathrm{R}} = z(v_{j+1}^n) \geq -M_{-}\{(j+1/2)\Delta x + 1\}^{-\frac{2(\gamma-1)}{\gamma+1}}$, from (4.23) and (4.26), we deduce that

$$\begin{aligned} \bar{z}_i(x) &\geq -M_{-} x^{-\frac{2(\gamma-1)}{\gamma+1}} - O((\Delta x)^{2\alpha - (\gamma-1)\beta}) \\ &\geq -M_{-} x^{-\frac{2(\gamma-1)}{\gamma+1}} - o(\Delta x) \quad \text{for } x \leq x_i. \end{aligned} \tag{4.29}$$

If $\bar{v}_i(x)$ is a steady state solution of (2.11), in the cell, it follows that

$$\bar{z}_i(x) \geq -M_{-} x^{-\frac{2(\gamma-1)}{\gamma+1}} - o(\Delta x). \tag{4.30}$$

If $\bar{v}_i(x)$ is a constant, in the cell, we have

$$\bar{z}_i(x) \geq -M_-\{(j-1/2)\Delta x+1\}^{-\frac{2(\gamma-1)}{\gamma+1}}. \tag{4.31}$$

Finally, from (4.27) and (4.29), we have

$$z_q \geq -M_-(x_q)^{-\frac{2(\gamma-1)}{\gamma+1}} - o(\Delta x). \tag{4.32}$$

Estimate of $\bar{w}_i(x)$ $(i=1,\ldots,q-1)$ and w_q.

Recall that $w_1 = w_{\mathrm{R}} = w_{j+1}^n \leq M_+ + \varepsilon$. Thus, when $i=1$, from (4.25) with $i=1$, if $\bar{v}_1(x)$ is a constant, it follows that

$$\bar{w}_1(x) \leq M_+ + \varepsilon. \tag{4.33}$$

If $\bar{v}_1(x)$ is a steady state solution of (2.11), from the definition of a quasi-steady state solution, we observe that $w_1 \leq M_+$. Then, in the cell, (2.13) yields

$$\bar{w}_1(x) \leq \left\{\frac{(j+1/2)\Delta x+1}{(j-1/2)\Delta x+1}\right\}^{\frac{2(\gamma-1)}{\gamma+1}} M_+. \tag{4.34}$$

Next we estimate $\bar{w}_i(x)$ $(i=2,\ldots,q-1)$. The construction of $\bar{v}_i(x)$ implies

$$\bar{w}_i(x) = \begin{cases} w(v_{\mathrm{R}}) - (i-1)(\Delta x)^\alpha, & \text{when } w_i > M_+, \\ \{w(v_{\mathrm{R}}) - (i-1)(\Delta x)^\alpha\}(x/x_i)^{-\frac{2(\gamma-1)}{\gamma+1}}, & \text{when } w_i \leq M_+. \end{cases}$$

Since $\alpha < 1$ and $w(v_{\mathrm{R}}) = w(v_{j+1}^n) \leq M_+ + \varepsilon$, on $I_j = [(j-1/2)\Delta x+1,$ $(j+1/2)\Delta x+1]$, we have

$$\begin{aligned} \bar{w}_i(x) &\leq [M_+ + \varepsilon - (\Delta x)^\alpha] \max\left\{1, \sup_{x\in I_j}(x/x_i)^{-\frac{2(\gamma-1)}{\gamma+1}}\right\} \\ &\leq [M_+ + \varepsilon - (\Delta x)^\alpha]\{1 + O(\Delta x)\} \\ &\leq M_+ + \varepsilon. \end{aligned} \tag{4.35}$$

Finally, from the definition of $v_{\mathrm{R}}^{(2)}$, we have

$$w_q = w(v_{\mathrm{R}}^{(2)}) = w(v_{\mathrm{R}}^{(1)}) \leq w(v_{\mathrm{R}}) \leq M_+ + \varepsilon. \tag{4.36}$$

Estimates of a Riemann solution for $(v_{\mathrm{L}}, v_{\mathrm{R}}^{(2)})$.

We denote by $v^R(x,t)$ a Riemann solution for $(v_{\mathrm{L}}, v_{\mathrm{R}}^{(2)})$. From our construction, in the sector $n\Delta t \leq t < (n+1)\Delta t$ and $(j-1/2)\Delta x + 1 \leq x < j\Delta x + 1 + \sigma_q(t-n\Delta t)$, we observe that $v^\Delta(x,t) = v^R(x,t)$.

Recalling that we consider Case 2, notice that $z(v_{\mathrm{L}}) \geq z(v_{\mathrm{R}})$. The definition of v_j^n $(=v_{\mathrm{L}})$ and v_{j+1}^n $(=v_{\mathrm{R}})$ thus implies

$$\begin{aligned} &-M_-\{(j+1/2)\Delta x+1\}^{-\frac{2(\gamma-1)}{\gamma+1}} \leq z(v_{\mathrm{R}}) \leq z(v_{\mathrm{L}}), \\ &w(v_{\mathrm{L}}) \leq M_+ + \varepsilon \text{ and } w(v_{\mathrm{R}}) \leq M_+ + \varepsilon. \end{aligned} \tag{4.37}$$

From Lemma 2.2 and (4.36), $v^R(x,t)$ satisfies

$$z(v^R(x,t)) \geq \min\{z_q, z(v_{\mathrm{L}})\}, \quad w(v^R(x,t)) \leq M_+ + \varepsilon. \tag{4.38}$$

We first consider the case where $z_q < 0$. In this case, since $\rho_{\mathrm{R}}^{(1)} = (\Delta x)^\beta$, from (4.28), we obtain $\rho_q = O((\Delta x)^\beta)$. Thus we have

$$\lambda_2(v_q) = z_q + (1+1/\theta)(\rho_q)^\theta < O((\Delta x)^{\theta\beta}).$$

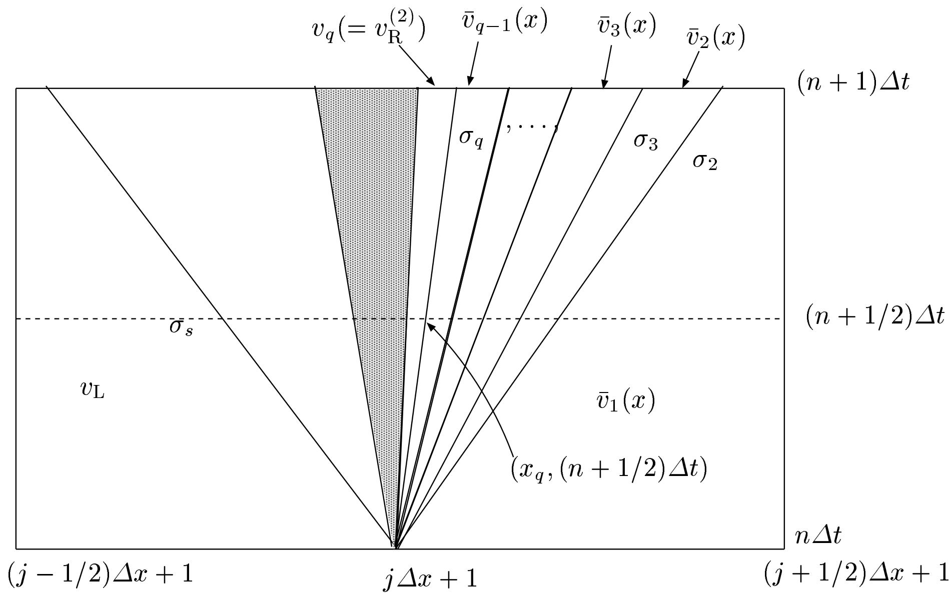

FIGURE 7. **Case 2.1** (ii): The approximate solution $\bar{v}^{\Delta}$ in the cell.

It follows from the inverse Lax condition that

$$\lambda_2(v_q) < \sigma_q < \lambda_2(\bar{v}_{q-1}(x_q)).$$

From our construction, we observe that $(\bar{z}_{q-1}(x_q), \bar{w}_{q-1}(x_q)) = (z_{q-1}, w_{q-1}) + O(\Delta x)$ and $w_q = w_{q-1} + O((\Delta x)^{\alpha})$. From (4.27) and the two equations above, we then have $|\lambda_2(\bar{v}_{q-1}(x_q)) - \lambda_2(v_q)| = O((\Delta x)^{\alpha})$. We thus conclude that

$$\sigma_q < O((\Delta x)^{\theta\beta}). \tag{4.39}$$

From (4.37)–(4.39), observing $M_- > 0$, we deduce that, for $(j-1/2)\Delta x + 1 \leq x < j\Delta x + 1 + \sigma_q \Delta t$,

$$\begin{aligned}
z(v^R(x,(n+1)\Delta t - 0)) &\geq -M_-(x_q)^{-\frac{2(\gamma-1)}{\gamma+1}} - o(\Delta x) \\
&= -M_-(x_q + \sigma_q\Delta t/2)^{-\frac{2(\gamma-1)}{\gamma+1}} \\
&\quad + M_-(x_q + \sigma_q\Delta t/2)^{-\frac{2(\gamma-1)}{\gamma+1}} - M_-(x_q)^{-\frac{2(\gamma-1)}{\gamma+1}} \\
&\quad - o(\Delta x) \\
&= -M_-(x_q + \sigma_q\Delta t/2)^{-\frac{2(\gamma-1)}{\gamma+1}} \\
&\quad - \frac{2(\gamma-1)}{\gamma+1} M_-(x_q + \tau\sigma_q\Delta t/2)^{-\frac{2(\gamma-1)}{\gamma+1}-1}\sigma_q\Delta t/2 \\
&\quad - o(\Delta x) \\
&\geq -M_-(j\Delta x + 1 + \sigma_q\Delta t)^{-\frac{2(\gamma-1)}{\gamma+1}} - o(\Delta x) \\
&\geq -M_- x^{-\frac{2(\gamma-1)}{\gamma+1}} - o(\Delta x),
\end{aligned} \tag{4.40}$$

where $0 < \tau < 1$.

On the other hand, if $z_q \geq 0$, we have

$$z(v^R(x,(n+1)\Delta t - 0)) \geq \min\{0, z(v_{\mathrm{L}})\}. \tag{4.41}$$

In view of (4.38), (4.40) and (4.41), we have (4.1) for $v^\Delta(x,t)$ $(= v^R(x,t))$ in the sector $n\Delta t \leq t < (n+1)\Delta t$ and $(j-1/2)\Delta x + 1 \leq x < j\Delta x + 1 + \sigma_q(t - n\Delta t)$.

From the estimate above, we can obtain (4.1) in a similar fashion to the case (A).

4.3. **Estimates near the boundary.** Finally we derive (4.1) for $v^\Delta(x,t)$ defined Subsection 3.2.

Estimates of $\bar{v}^\Delta(x,t)$.

First we estimate $\bar{v}^\Delta(x,t)$.

If $u_1^n \geq 0$, from Lemma 2.2, $w(\bar{v}^\Delta(x,t)) \leq M_+ + \varepsilon$ and $z(\bar{v}^\Delta(x,t)) \geq -M_-(1 + \Delta x/2)^{-\frac{2(\gamma-1)}{\gamma+1}}$.

On the other hand, if $u_1^n < 0$, since $M_- \leq M_+$, we have

$$w(v_1^n) = -z(v_1^n) + 2u_1^n < M_- \left(\Delta x/2 + 1\right)^{-\frac{2(\gamma-1)}{\gamma+1}} \leq M_+ \left(\Delta x/2 + 1\right)^{-\frac{2(\gamma-1)}{\gamma+1}}. \tag{4.42}$$

Then, if $\rho_1^n \geq (\Delta x)^\beta$ and $u_1^n < 0$, we deduce from Lemma 2.4 and (4.42) that $w(\bar{v}^\Delta(x,t)) \leq M_+ x^{-\frac{2(\gamma-1)}{\gamma+1}} \leq M_+$ and $z(\bar{v}^\Delta(x,t)) \geq -M_- x^{-\frac{2(\gamma-1)}{\gamma+1}}$; if $\rho_1^n < (\Delta x)^\beta$ and $u_1^n < 0$, Lemma 2.2 and (4.42) yields $w(\bar{v}^\Delta(x,t)) \leq M_+ \left(\Delta x/2 + 1\right)^{-\frac{2(\gamma-1)}{\gamma+1}} \leq M_+$ and $z(\bar{v}^\Delta(x,t)) \geq z(v_1^n) \geq -M_- \left(\Delta x/2 + 1\right)^{-\frac{2(\gamma-1)}{\gamma+1}}$.

Estimates of $v^\Delta(x,t)$.

Let us deduce from the estimates above (4.1). For the case where $u_1^n \geq 0$, we observe that $\bar{u}^\Delta(x,t) \geq 0$. Then, in the strip $n\Delta t \leq t < (n+1)\Delta t$, we have

$$\begin{aligned}
z(v^\Delta(x,t)) &= \bar{u}^\Delta(x,t) - \{\bar{\rho}^\Delta(x,t)\}^\theta/\theta \left\{1 - \frac{2}{x}\bar{u}^\Delta(x,t)(t - \Delta t)\right\}^\theta \\
&\geq z(\bar{v}^\Delta(x,t)) \\
&\geq -M_- \left(1 + \Delta x/2\right)^{-\frac{2(\gamma-1)}{\gamma+1}}, \\
w(v^\Delta(x,t)) &= \bar{u}^\Delta(x,t) + \{\bar{\rho}^\Delta(x,t)\}^\theta/\theta \left\{1 - \frac{2}{x}\bar{u}^\Delta(x,t)(t - \Delta t)\right\}^\theta \\
&\leq w(\bar{v}^\Delta(x,t)) \\
&\leq M_+ + \varepsilon.
\end{aligned}$$

The other cases are similar to the case (A).

Remark 4.2. Since $M_+ \geq M_-$, from Remark 2.1, Lemma 4.1 and (3.3), the following Courant-Friedrichs-Lewy condition:

$$4\Lambda := 4 \max_{i=1,2} \left(\sup_{1 \leq x,\, 0 \leq t \leq T} |\lambda_i(v^\Delta)| \right) \leq 4(M_+ + \varepsilon + o(\Delta x)) \leq 6(M_+ + \varepsilon) = \frac{\Delta x}{\Delta t}$$

holds, by choosing Δx small enough.

5. Local entropy estimates for the approximate solutions

Our aim in this section is to obtain local entropy estimates for the approximate solutions.

Lemma 5.1. *Consider a shock or a rarefaction shock curve with the left and right states $v_0=(\rho_0,m_0), v=(\rho,m)$. Then, along the wave curve,*

$$\begin{aligned}&|\sigma(\rho)(\eta(v(\rho))-\eta(v_0))-(q(v(\rho))-q(v_0))|\\&\quad\le C(\min(\rho,\rho_0))^{-1}|\rho-\rho_0|\sup_{s\in[\rho_0,\rho]}|\dot{\sigma}(s)(v(s)-v_0)^2|,\end{aligned}$$

for any C^2 weak entropy-entropy flux pair (η,q), where C is a constant depending only on the uniform bound of v and v_0.

Proof. Along a shock or a rarefaction shock curve, we have

$$\begin{aligned}m(\rho)&=\frac{m_0}{\rho_0}\rho\pm\sqrt{\frac{\rho}{\rho_0}\frac{p(\rho)-p(\rho_0)}{\rho-\rho_0}}(\rho-\rho_0),\\ \sigma(\rho)&=\frac{m(\rho)-m_0}{\rho-\rho_0}=\frac{m_0}{\rho_0}\pm\sqrt{\frac{\rho}{\rho_0}\frac{p(\rho)-p(\rho_0)}{\rho-\rho_0}}.\end{aligned}$$

Set

$$Q(\rho)=\sigma(\rho)(\eta(v(\rho))-\eta(v_0))-(q(v(\rho))-q(v_0)).$$

We first obtain

$$\dot{Q}(\rho)=\dot{\sigma}(\rho)(\eta(v(\rho))-\eta(v_0))+\sigma(\rho)\dot{\eta}(v(\rho))-\dot{q}(v(\rho)).$$

On the other hand, we observe that

$$\begin{cases}\dot{\sigma}(\rho)(v(\rho)-v_0)+\sigma(\rho)\dot{v}(\rho)=\dot{f}(v(\rho)), \text{ (the Rankine-Hugoniot conditions)}\\ \dot{q}(v(\rho))=\nabla q\cdot\dot{v}(\rho)=\nabla\eta\dot{f}(v(\rho)).\end{cases}$$

We then deduce that

$$\begin{aligned}\dot{Q}(\rho)&=\dot{\sigma}(\rho)\{\eta(v(\rho))-\eta(v_0)-\nabla\eta(v(\rho))(v(\rho)-v_0)\}\\&=-\dot{\sigma}(\rho)\int_0^1\tau\frac{d^2}{d\tau^2}\eta(v_0+\tau(v(\rho)-v_0))d\tau.\end{aligned}$$

Therefore, using the property of the Rankine-Hugoniot locus, we obtain

$$\begin{aligned}|Q(\rho)|&=\left|\int_{\rho_0}^{\rho}\dot{Q}(s)ds\right|\\&=\left|\int_{\rho_0}^{\rho}(-\dot{\sigma}(s))\{\eta(v(s))-\eta_0(v_0)-\nabla\eta(v(s))(v(s)-v_0)\}ds\right|\\&=\left|\int_{\rho_0}^{\rho}\dot{\sigma}(s)ds\int_0^1\tau\cdot{}^t(v(s)-v_0)\nabla^2\eta(v_0+\tau(v(s)-v_0))(v(s)-v_0)d\tau\right|\\&\le C\left|\int_{\rho_0}^{\rho}\dot{\sigma}(s)ds\int_0^1\tau\cdot{}^t(v(s)-v_0)\nabla^2\eta_*(v_0+\tau(v(s)-v_0))(v(s)-v_0)d\tau\right|\\&\le C\sup_{s\in[\rho_0,\rho]}|\dot{\sigma}(s)||\rho-\rho_0|(\min(\rho,\rho_0))^{-1}\sup_{s\in[\rho_0,\rho]}|(v(s)-v_0)^2|,\end{aligned}\tag{5.1}$$

where $0<\tau<1$. ∎

Now let us estimate entropies of $\bar{v}^{\Delta}(x,t)$. Recall that $\bar{v}^{\Delta}(x,t)$ is piecewise quasi-steady state, with discontinuities occurring finitely many straight lines in the $x-t$ plane. We first define the following:

Definition 5.1. In Case 1.2 (ii) and Case 3.2 (ii)–(b), $\bar{v}^{\Delta}(x,t)$ has discontinuity along $n\Delta t \le t < (n+1)\Delta t, x = j\Delta x + 1 + \lambda_1(v_{\mathrm{L}}^{(3)})(t-n\Delta t)$, which is called an *artificial discontinuity.*

Then Jump can be of three type:

$\mathcal{R}$: discontinuity connected by a rarefaction shock curve
$\mathcal{A}$: artificial discontinuity
$\mathcal{S}$: discontinuity connected by a shock curve

We prove lemmas for these jump. To do this, we introduce the following symbol: for a certain function $\psi(x,t)$ with the discontinuity occurring a straight line $(x(t),t)$, we define

$$[\psi(x(t),t)] := \psi(x(t)+0,t) - \psi(x(t)-0,t).$$

First we consider $\mathcal{R}$.

Lemma 5.2. *Consider each rarefaction front $x = x_{\mathcal{R}}(t)$ of $\mathcal{R}$, with the left and right states $\bar{v}^{\Delta}(x_{\mathcal{R}}(t)-0,t), \bar{v}^{\Delta}(x_{\mathcal{R}}(t)+0,t)$. We set $\sigma_{\mathcal{R}} = dx_{\mathcal{R}}(t)/dt$. Then there exists $C_1 > 0$ independent of Δx such that $\bar{\rho}^{\Delta}(x,t) \ge C_1(\Delta x)^{\beta}$ along each rarefaction front and*

$$\begin{aligned}&\left|\int_{n\Delta t}^{(n+1)\Delta t}\left\{\sigma_{\mathcal{R}}(t)\left[\eta(\bar{v}^{\Delta}(x_{\mathcal{R}}(t),t))\right] - \left[q(\bar{v}^{\Delta}(x_{\mathcal{R}}(t),t))\right]\right\}dt\right| \\ &\le C_2\left\{(\Delta x)^{1+3\alpha} + (\Delta x)^{3-\beta}\right\} \le o((\Delta x)^{2+\alpha}),\end{aligned} \tag{5.2}$$

for any C^2 weak entropy-entropy flux pair (η,q), where C_2 is a constant depending only on the uniform bound of $\bar{v}^{\Delta}(x,t)$.

Proof. We prove this lemma for $\mathcal{R}$ in R_1^{Δ}. The other cases can be treated similarly. Now the first part follows from (4.3).

Next, from the construction of R_1^{Δ}, it follows that $|z_i^* - z_{i+1}^*| \le C(\Delta x)^{\alpha}$. From (4.5), we then have

$$\left|\left[z(\bar{v}^{\Delta}(x_{\mathcal{R}}(t_{\mathrm{M}}),t_{\mathrm{M}}))\right]\right| = O((\Delta x)^{\alpha}), \quad \left|\left[w(\bar{v}^{\Delta}(x_{\mathcal{R}}(t_{\mathrm{M}}),t_{\mathrm{M}}))\right]\right| = O((\Delta x)^{\alpha}),$$

where $t_{\mathrm{M}} := (n+1/2)\Delta t$. Therefore, from (4.3), Remark 1.2 and the fact that $\alpha > \beta$, we deduce that

$$\begin{aligned}&1/2 < \left|\bar{\rho}^{\Delta}(x_{\mathcal{R}}(t_{\mathrm{M}})+0,t_{\mathrm{M}})/\bar{\rho}^{\Delta}(x_{\mathcal{R}}(t_{\mathrm{M}})-0,t_{\mathrm{M}})\right| < 3/2, \\ &\left|\left[\bar{v}^{\Delta}(x_{\mathcal{R}}(t_{\mathrm{M}}),t_{\mathrm{M}})\right]\right| = \left\{\bar{\rho}^{\Delta}(x_{\mathcal{R}}(t_{\mathrm{M}})+0,t_{\mathrm{M}})\right\}^{(3-\gamma)/2} O((\Delta x)^{\alpha}).\end{aligned}$$

On the other hand, recall that

$$|\nabla^2\eta| \le C/\rho, \tag{5.3}$$

where η is a C^2 weak entropy. Therefore, by taking Taylor's expansion for t at t_{M}, Lemma 5.1 and Remark 3.2 yields (5.2). □

Second, let us consider $\mathcal{A}$. In Case 1.2 (ii) and Case 3.2 (ii)–(b), we observe that

$$\rho_{\mathrm{L}} = O((\Delta x)^{\beta}),\ x^{(3)} - x_{\mathrm{L}} = O(\Delta x).$$

From the construction of $\bar{v}^{\Delta}(x,t)$ in this case and (2.11), on $n\Delta t \le t < (n+1)\Delta t$, we thus obtain

$$|\bar{v}^{\Delta}(x_{\mathcal{A}}(t)-0,t)-v_{\mathrm{L}}| = O((\Delta x)^{1+\beta}), \quad |v_{\mathrm{L}}^{(3)}-v_{\mathrm{L}}| = O((\Delta x)^{1+\beta}), \tag{5.4}$$

where $x_{\mathcal{A}}(t) = j\Delta x + 1 + \lambda_1(v_{\mathrm{L}}^{(3)})(t-n\Delta t)$. In view of $v_{\mathrm{L}}^{(3)} = \bar{v}^{\Delta}(x_{\mathcal{A}}(t)+0,t)$, we then have the following lemma for $\mathcal{A}$. We consider the amplitude of the discontinuity $x_{\mathcal{A}}(t)$. Since $|\nabla\eta| \le C$ and $|\nabla q| \le C$, we deduce from (5.4) the following lemma.

Lemma 5.3.

$$\begin{aligned}\left|\int_{n\Delta t}^{(n+1)\Delta t}\left\{\lambda_1(v_{\mathrm{L}}^{(3)})\left[\eta(\bar{v}^{\Delta}(x_{\mathcal{A}}(t),t)\right]-\left[q(\bar{v}^{\Delta}(x_{\mathcal{A}}(t),t))\right]\right\}dt\right| &= O((\Delta x)^{2+\beta})\\ &= o((\Delta x)^2),\end{aligned}$$

where (η,q) *is a* C^2 *weak entropy-entropy flux pair.*

Finally, we consider $\mathcal{S}$.

Lemma 5.4. *Consider a shock front* $x = x_{\mathcal{S}}(t)$ *of* $\mathcal{S}$*, with the left and right states* $\bar{v}^{\Delta}(x_{\mathcal{S}}(t)-0,t), \bar{v}^{\Delta}(x_{\mathcal{S}}(t)+0,t)$*. We set* $\sigma_{\mathcal{S}} = dx_{\mathcal{S}}(t)/dt$*. Then there is a constant* C *depending only on the uniform bound of* $\bar{v}^{\Delta}(x,t)$ *such that, along a shock front,*

$$\begin{aligned}&\int_{n\Delta t}^{(n+1)\Delta t}\left\{\sigma_{\mathcal{S}}(t)\left[\eta(\bar{v}^{\Delta}(x_{\mathcal{S}}(t),t))\right]-\left[q(\bar{v}^{\Delta}(x_{\mathcal{S}}(t)+0,t))\right]\right\}dt\\ &\ge -C(\Delta x)^{3-\beta} \ge -o((\Delta x)^2),\end{aligned} \tag{5.5}$$

and

$$\begin{aligned}&\left|\int_{n\Delta t}^{(n+1)\Delta t}\sigma_{\mathcal{S}}(t)\left[\eta(\bar{v}^{\Delta}(x_{\mathcal{S}}(t),t))\right]-\left[q(\bar{v}^{\Delta}(x_{\mathcal{S}}(t),t))\right]dt\right|\\ &\le C\int_{n\Delta t}^{(n+1)\Delta t}\left\{\sigma_{\mathcal{S}}(t)\left[\eta_*(\bar{v}^{\Delta}(x_{\mathcal{S}}(t),t))\right]-\left[q_*(\bar{v}^{\Delta}(x_{\mathcal{S}}(t)+0,t))\right]\right\}dt\\ &\quad + C((\Delta x)^{3-\beta}),\end{aligned} \tag{5.6}$$

for any C^2 *weak entropy-entropy pair* (η,q) *satisfying* (1.9) *and the mechanical energy-energy flux* (η_*,q_*) *defined by* (1.10).

Proof. From the definition of a quasi-steady state solution, in the region where $\bar{v}^{\Delta}(x,t)$ is a steady state solution of (2.11), we can assume that $\bar{\rho}^{\Delta}(x,t) > (\Delta x)^{\beta}/4$ by choosing Δx small enough. If $\bar{v}^{\Delta}(x,t)$ is the steady-state solution, by taking Taylor's expansion for t at t_{M}, from Lemma 2.3 and (5.3), we complete the proof of this lemma. □

6. H^{-1} COMPACTNESS ESTIMATES

We prove the H^{-1} compactness for the approximate solutions $(\rho^{\Delta}, m^{\Delta})$ of the initial-boundary problem (1.4) and (1.7).

First let us prove that $v^{\Delta}(x,t)$ satisfies

Stability

$$v_j^{n+1} = E_j^{n+1}(v) + o(\Delta x), \tag{6.1}$$

where $o(\Delta x)$ depends only on the uniform bound of $v^{\Delta}(x,t)$.

To do this, we introduce the following lemma:

Lemma 6.1. *We assume that* $v(x,t) = (\rho(x,t), m(x,t))$ *satisfies*

$$z(v(x,t)) \geq -M_- x^{-\frac{2(\gamma-1)}{\gamma+1}} - o(\Delta x), \quad w(v(x,t)) \leq M_+ + \varepsilon + o(\Delta x),$$

where $o(\Delta x)$ *depends only on* M_+ *and* M_-.

Set $E_j^t(v) = \dfrac{1}{\Delta x}\displaystyle\int_{(j-1)\Delta x+1}^{j\Delta x+1} v(x,t)dx, \quad j \in \mathbf{Z}_+$. *For* $1 < \delta < 1/(2\theta)$, *if*

$$E_j^t(\rho) \geq (\Delta x)^\delta,$$

choosing Δx *small enough, it follows that*

$$\begin{aligned} z(E_j^t(v)) &\geq -M_-\{(j-1/2)\Delta x + 1\}^{-\frac{2(\gamma-1)}{\gamma+1}} - o(\Delta x), \\ w(E_j^t(v)) &\leq M_+ + \varepsilon + o(\Delta x). \end{aligned}$$

We postpone the proof of this lemma to Appendix C.

Since initial data $v^\Delta(x,-0) = v_0(x)$ satisfy the assumption of Lemma 6.1, we deduce from Lemma 6.1 with $t = -0$ that

$$v_j^0 = E_j^0 + o(\Delta x).$$

Next, we applying Lemma 6.1 with $t = (n+1)\Delta t - 0$. In view of $\delta > 1$ and the definition of v_j^n, if $v^\Delta(x,t)$ satisfies (4.1), (6.1) holds. Therefore, by induction, (6.1) holds for any n.

We next introduce a basic lemma of functional analysis (see [C3]).

Lemma 6.2. *Let* $\Omega \subset \mathbf{R}^n$ *be a bounded and open set. Then*

$$\begin{aligned} &(\text{compact set of } W^{-1,q}(\Omega)) \cap (\text{bounded set of } W^{-1,r}(\Omega)) \\ &\qquad\qquad \subset (\text{compact set of } W^{-1,2}_{\text{loc}}(\Omega)), \end{aligned}$$

where q *and* r *are constants,* $1 < q \leq 2 < r < \infty$.

With Lemma 6.2, we have

Theorem 6.3. *We assume that* (ρ^Δ, m^Δ) *are the approximate solutions of the initial-boundary value problem* (1.4) *and* (1.7). *Then the measure sequence*

$$\eta(v^\Delta)_t + q(v^\Delta)_x$$

lies in a compact subset of $H^{-1}_{\text{loc}}(\Omega)$ *for any weak entropy pairs* (η, q), *where* $\Omega \subset \Pi_T$ *is any bounded and open set with a* C^1 *boundary, where* $\Pi_T = \{x \in \mathbf{R} : x \geq 1\} \times [0,T]$ *with any fixed* $T > 0$.

Proof. We first define $\bar{v}^\Delta(x,-0)$ by $\bar{v}^\Delta(x,-0) := v^\Delta(x,-0) = v_0(x)$.

Step 1. For any function $\phi \in C_0^1(\Pi_T)$, the entropy dissipation measures can be written in the form

$$\begin{aligned} &\int\int_{0 \leq t \leq T = m\Delta t} (\eta(v)\phi_t + q(v)\phi_x)dxdt \\ &= A(\phi) + L(\phi) + M(\phi) + N(\phi) + \sum(\phi), \end{aligned} \tag{6.2}$$

where

$$A(\phi)=\int\int_{\Pi_T}((\eta(v^\Delta)-\eta(\bar v^\Delta))\phi_t+(q(v^\Delta)-q(\bar v^\Delta))\phi_x)dxdt, \tag{6.3}$$

$$M(\phi)=\int_{-\infty}^{\infty}\phi(x,T)\eta(\bar v^\Delta(x,T))dx-\int_{-\infty}^{\infty}\phi(x,0)\eta(\bar v^\Delta(x,0))dx, \tag{6.4}$$

$$N(\phi)=\int\int_{\Pi_T}\nabla\eta(\bar v^\Delta)\sum_{\bar v^\Delta \text{ is Type I}}\left\{g(\bar v^\Delta)-h(\bar v^\Delta)\right\}\phi(x,t)dxdt, \tag{6.5}$$

$$L(\phi)=\sum_{j,n}\int_{(j-1)\Delta x+1}^{j\Delta x+1}(\eta(\bar v^n_-)-\eta(\bar v^n_+))\phi(x,n\Delta t)dx:=L_1(\phi)+L_2(\phi), \tag{6.6}$$

$$L_1(\phi)=\sum_{j,n}\phi^n_j\int_{(j-1)\Delta x+1}^{j\Delta x+1}(\eta(\bar v^n_-)-\eta(\bar v^n_+))dx, \tag{6.7}$$

$$L_2(\phi)=\sum_{j,n}\int_{(j-1)\Delta x+1}^{j\Delta x+1}(\eta(\bar v^n_-)-\eta(\bar v^n_+))(\phi-\phi^n_j)dx, \tag{6.8}$$

$$\sum(\phi)=\int_0^T\sum(\sigma[\eta]-[q])\phi(x(t),t)dt, \tag{6.9}$$

$\bar v^n_\pm=\bar v^\Delta(x,n\Delta t\pm 0)$, $\phi^n_j=\phi(j\Delta x+1,n\Delta t)$, the summention in $\sum(\phi)$ is taken over all discontinuities in $\bar v^\Delta$ at a fixed time t, σ is the propagating speed of the discontinuities.

Let $\mathcal{D}=(x(t),t)$ denote a discontinuity in $\bar v^\Delta(x,t)$, $[\eta]$ and $[q]$ denote the jump of $\eta(\bar v^\Delta(x,t))$ and $q(\bar v^\Delta(x,t))$ across $\mathcal{D}$ from left to right, respectively,

$$[\eta]=\eta(\bar v^\Delta(x(t)+0,t))-\eta(\bar v^\Delta(x(t)-0,t)),$$
$$[q]=q(\bar v^\Delta(x(t)+0,t))-q(\bar v^\Delta(x(t)-0,t)).$$

Step 2. Since the propagation speeds of the approximate solutions $v^\Delta(x,t)$ are finite, we can assume that

$$(\bar\rho^\Delta,\bar m^\Delta)|_{x\geq K+\Lambda T}=(0,0)$$

for sufficiently large $K>0$, without loss of generality. This implies

$$\int_{-\infty}^{\infty}\eta_*(\rho^\Delta(x,0),m^\Delta(x,0))dx<\infty.$$

We substitute $(\eta,q)=(\eta_*,q_*)$ and $\phi=1$ in the equality (6.2). For sufficiently large $K>0$, since $(\rho^\Delta,m^\Delta)|_{x\geq K+\Lambda T}=(0,0)$, we deduce that

$$\sum_n\int_1^\infty(\eta_*(\bar v^n_-)-\eta_*(\bar v^n_+))dx+\int_0^T\sum(\sigma[\eta_*]-[q_*])dt\leq C. \tag{6.10}$$

From Lemma 2.5, we obtain

$$\sum_{j,n}\int_{(j-1)\Delta x+1}^{j\Delta x+1}\int_0^1(1-\tau)\cdot{}^t(\bar v_+^n-v_j^n)\nabla^2\eta_*(v_j^n+\tau(\bar v_+^n-v_j^n))(\bar v_+^n-v_j^n)d\tau dx$$
$$\le C\sum_{j,n}\Delta x\frac{(|\rho_j^n|^2+|m_j^n|^2)(\Delta x)^2}{\rho_j^n(1-O(\Delta x))}. \tag{6.11}$$

Using Lemma 2.5 and (6.11), we conclude that

$$\begin{aligned}
&\sum_n\int_1^\infty(\eta_*(\bar v_-^n)-\eta_*(\bar v_+^n))dx\\
&=\sum_{j,n}\int_{(j-1)\Delta x+1}^{j\Delta x+1}(\eta_*(\bar v_-^n)-\eta_*(v_j^n))dx-\sum_{j,n}\int_{(j-1)\Delta x+1}^{j\Delta x+1}(\eta_*(\bar v_+^n)-\eta_*(v_j^n))dx\\
&=\sum_{j,n}\int_{(j-1)\Delta x+1}^{j\Delta x+1}\nabla\eta_*(v_j^n)(\bar v_-^n-v_j^n)dx-\sum_{j,n}\int_{(j-1)\Delta x+1}^{j\Delta x+1}\nabla\eta_*(v_j^n)(\bar v_+^n-v_j^n)dx\\
&\quad+\sum_{j,n}\int_{(j-1)\Delta x+1}^{j\Delta x+1}\int_0^1(1-\tau)\cdot{}^t(\bar v_-^n-v_j^n)\nabla^2\eta_*(v_j^n+\tau(\bar v_-^n-v_j^n))\\
&\quad\times(\bar v_-^n-v_j^n)d\tau dx\\
&\quad-\sum_{j,n}\int_{(j-1)\Delta x+1}^{j\Delta x+1}\int_0^1(1-\tau)\cdot{}^t(\bar v_+^n-v_j^n)\nabla^2\eta_*(v_j^n+\tau(\bar v_+^n-v_j^n))\\
&\quad\times(\bar v_+^n-v_j^n)d\tau dx+o(1)\\
&=\sum_{j,n}\int_{(j-1)\Delta x+1}^{j\Delta x+1}\int_0^1(1-\tau)\cdot{}^t(\bar v_-^n-v_j^n)\nabla^2\eta_*(v_j^n+\tau(\bar v_-^n-v_j^n))\\
&\quad\times(\bar v_-^n-v_j^n)d\tau dx+o(1)\quad\text{(from Stability)},
\end{aligned}\tag{6.12}$$

where $\bar v_\pm^n=\bar v^\Delta(x,n\Delta t\pm0)$ and notice that

$$\begin{aligned}
\sum_{j,n}\int_{(j-1)\Delta x+1}^{j\Delta x+1}\nabla\eta_*(v_j^n)(\bar v_-^n-v_j^n)dx&=\sum_{j,n}\nabla\eta_*(v_j^n)\int_{(j-1)\Delta x+1}^{j\Delta x+1}(\bar v_-^n-v_j^n)dx\\
&=\sum_{j,n}\Delta x\nabla\eta_*(v_j^n)(E_j^n(v)-v_j^n).
\end{aligned}$$

Using Lemmas 5.2 and 5.3, for $\mathcal{R}$ and $\mathcal{A}$, we have

$$\left|\int_0^T\sum_{\mathcal{R},\mathcal{A}}(\sigma[\eta_*]-[q_*])dt\right|\le o(1). \tag{6.13}$$

Here notice that the order of the number of $\mathcal{R}$ is $(\Delta x)^{-\alpha}$ in each cell. Similarly, using Lemma 5.4, for $\mathcal{S}$,

$$\int_0^T\sum_{\mathcal{S}}(\sigma[\eta_*]-[q_*])dt\ge -o(1). \tag{6.14}$$

Therefore, we have

$$\int_0^T\sum(\sigma[\eta_*]-[q_*])dt\ge -o(1) \tag{6.15}$$

for the convex entropy η_*. We conclude from (6.10) and (6.12)– (6.15) that

$$\int_0^T \sum(\sigma[\eta_*]-[q_*])dt \le C \tag{6.16}$$

and

$$\sum_{j,n}\int_{(j-1)\Delta x+1}^{j\Delta x+1}\int_0^1 (1-\tau)\cdot {}^t(\bar{v}_-^n - v_j^n)\nabla^2\eta_*(v_j^n+\tau(\bar{v}_-^n - v_j^n))(\bar{v}_-^n - v_j^n)d\tau dx \le C. \tag{6.17}$$

In particular, since $\nabla^2\eta_*(r,r) \ge c_0(r,r)$ for a constant $c_0>0$, we have

$$\sum_{\substack{j,n\\ |j\Delta x+1|\le K+\Lambda T}} \int_{(j-1)\Delta x+1}^{j\Delta x+1} |\bar{v}_-^n - v_j^n|^2 dx \le C(K). \tag{6.18}$$

Step 3. For Ω in Theorem 6.3 and any weak entropy pair (η,q), we deduce from (6.16)– (6.17), and Lemma 5.4 that

$$|M(\phi)| \le C\|\phi\|_{C(\Omega)}, \quad |N(\phi)| \le C\|\phi\|_{C(\Omega)},$$
$$\left|\sum(\phi)\right| \le C\|\phi\|_{C(\Omega)}\int_0^T (\sum(\sigma[\eta_*]-[q_*])dt \le C\|\phi\|_{C(\Omega)},$$

$$\begin{aligned}
|L_1(\phi)| &\le \left|\sum_{j,n}\phi_j^n\int_{(j-1)\Delta x+1}^{j\Delta x+1}(\eta(\bar{v}_-^n)-\eta(v_j^n))dx\right| \\
&\quad + \left|\sum_{j,n}\phi_j^n\int_{(j-1)\Delta x+1}^{j\Delta x+1}(\eta(\bar{v}_+^n)-\eta(v_j^n))dx\right| \\
&\le \|\phi\|_{C(\Omega)}\left\{\begin{array}{l}
\displaystyle\sum_{j,n}\int_{(j-1)\Delta x+1}^{j\Delta x+1}\int_0^1 (1-\tau)|{}^t(\bar{v}_-^n - v_j^n)\nabla^2\eta \\
\times(v_j^n+\tau(\bar{v}_-^n - v_j^n))(\bar{v}_-^n - v_j^n)|d\tau dx \\
\displaystyle +\sum_{j,n}\int_{(j-1)\Delta x+1}^{j\Delta x+1}\int_0^1 (1-\tau)|{}^t(\bar{v}_+^n - v_j^n)\nabla^2\eta \\
\times(v_j^n+\tau(\bar{v}_+^n - v_j^n))(\bar{v}_+^n - v_j^n)|d\tau dx + o(1)
\end{array}\right\} \\
&\le C\|\phi\|_{C(\Omega)}\left\{\sum_{j,n}\int_{(j-1)\Delta x+1}^{j\Delta x+1}\int_0^1 (1-\tau)|{}^t(\bar{v}_-^n - v_j^n)\right. \\
&\qquad\qquad \left.\times\nabla^2\eta_*(v_j^n+\tau(\bar{v}_-^n - v_j^n))(\bar{v}_-^n - v_j^n)|d\tau dx + o(1)\right\} \\
&\le C\|\phi\|_{C(\Omega)},
\end{aligned}$$

where the constant C depends only on the support of ϕ. Hence

$$\left|\left(M+N+L_1+\sum\right)(\phi)\right| \le C\|\phi\|_{C(\Omega)}.$$

From the arguments of [DC1] and [DC2] or [E2, Theorem 6], we thereby have

$$M + N + L_1 + \sum \quad \text{is relatively compact in } W^{-1,q_1}(\Omega), \tag{6.19}$$

where $1 < q_1 < 2$.

Furthermore, for any $\phi \in C_0^\alpha(\Omega),\ 1/2 < \alpha < 1$, we have

$$\begin{aligned}
|L_2(\phi)| &\le \sum_{j,n} \int_{(j-1)\Delta x+1}^{j\Delta x+1} |\phi(x, n\Delta t) - \phi_j^n|(|\eta(\bar{v}_-^n) - \eta(v_j^n)| \\
&\quad + |\eta(\bar{v}_+^n) - \eta(v_j^n)|)dx \\
&\le (\Delta x)^\alpha \|\phi\|_{C^\alpha(\Omega)} \\
&\quad \times \left\{ \sum_n \left(\sum_j \int_{(j-1)\Delta x+1}^{j\Delta x+1} |\eta(\bar{v}_-^n) - \eta(v_j^n)|^2 dx \right)^{\frac{1}{2}} + O(1) \right\} \\
&\le C(\Delta x)^{\alpha - \frac{1}{2}} \|\phi\|_{C^\alpha(\Omega)} \\
&\quad \times \left\{ \left(\sum_{j,n} \int_{(j-1)\Delta x+1}^{j\Delta x+1} |\bar{v}_-^n - v_j^n|^2 dx \right)^{\frac{1}{2}} + O(\sqrt{\Delta x}) \right\} \\
&\le C(\Delta x)^{\alpha - \frac{1}{2}} \|\phi\|_{C^\alpha(\Omega)}.
\end{aligned}$$

Using the Sobolev theorem $W^{1,p}(\Omega) \to C^\alpha(\bar{\Omega})$ $(0 < \alpha < 1 - 2/p)$, it follows that

$$|L_2(\phi)| \le C(\Delta x)^{\alpha - \frac{1}{2}} \|\phi\|_{W^{1,p}(\Omega)}, \quad p > \frac{2}{1-\alpha}.$$

Since $C_0^\infty(\Omega)$ is dense in $W_0^{1,p}(\Omega)$ and $\left(W_0^{1,p}(\Omega)\right)^* = W^{-1,q_2}(\Omega)$ $(q_2 = p/(p-1))$, we have

$$\|L_2\|_{W^{-1,q_2}(\Omega)} \le C(\Delta x)^{\alpha - \frac{1}{2}} \to 0, \quad \text{as } \Delta x \to 0 \tag{6.20}$$

for $1 < q_2 < 2/(1+\alpha)$.

Finally, for $A(\phi)$, we observe that

$$\begin{aligned}
|A(\phi)| &\le \int\int_{\Pi_T} (|\nabla\eta|_\infty + |\nabla q|_\infty)(|\phi_t| + |\phi_x|)|v^\Delta - \bar{v}^\Delta| dxdt \le C\Delta x \|\phi\|_{H^1(\Omega)} \\
&\le C\Delta x \|\phi\|_{W^{1,q_1^*}(\Omega)},
\end{aligned}$$

where $q_1^* = q_1/(q_1 - 1)$. The inequality above yields

$$\|A\|_{W^{-1,q_1}(\Omega)} \le C\Delta x \to 0, \quad \text{as } \Delta x \to 0. \tag{6.21}$$

It follows from (6.19)–(6.21) that

$$A + M + N + L + \sum \text{ is relatively compact in } W^{-1,q_0}(\Omega), \tag{6.22}$$

where $1 < q_0 := \min(q_1, q_2) < 2/(1+\alpha)$. Since $0 \le \rho \le C$ and $|m/\rho| \le C$, we obtain

$$A + M + N + L + \sum \text{ is bounded in } W^{-1,r}(\Omega)\ (r > 2). \tag{6.23}$$

We derive from (6.22)-(6.23) and Lemma 6.2

$$A + M + N + L + \sum \text{ is relatively compact in } H_{\text{loc}}^{-1}(\Omega), \tag{6.24}$$

which means that

$$\eta(v^{\Delta})_t + q(v^{\Delta})_x \text{ is relatively compact in } H^{-1}_{\mathrm{loc}}(\Omega).$$

This completes the proof of Theorem 6.3. □

7. Convergence and consistency

In Section 4 and Section 6, it is proved that the approximate solutions $(\rho^{\Delta}, m^{\Delta})$ of the initial-boundary problem (1.6) satisfy the following conditions:

(1) There is a constant $C > 0$ such that

$$0 \leq \rho^{\Delta}(x,t) \leq C, \quad \left|\frac{m^{\Delta}(x,t)}{\rho^{\Delta}(x,t)}\right| \leq C. \tag{7.1}$$

(2) The measure

$$\eta(v^{\Delta})_t + q(v^{\Delta})_x \text{ is compact in } H^{-1}_{\mathrm{loc}}(\Omega) \tag{7.2}$$

for all weak entropy pair (η, q), where $\Omega \subset \Pi_T$ is any bounded and open set with a C^1 boundary.

The compensated compactness framework (see [C2] and [C3]) ensures the strong compactness of the approximate solutions $v^{\Delta}(x,t)$ in $L^1_{\mathrm{loc}}(\Pi_T)$ for $1 < \gamma \leq 5/3$.

We first introduce a lemma of our approximate Riemann solutions.

Lemma 7.1. *Let $K \subset \mathbf{R}$ be any bounded set. Replace the steady state solution of* (2.11) *in our approximate Riemann solutions $\bar{v}^{\Delta}(x,t)$ with the corresponding data* (e.g. $v_i,\ i = 1, 2, \ldots, p-1,\ v_{\mathrm{R}},\ v^{\diamond}_{\mathrm{M}}$ in Subsection 3.1 Case (A)). *Then this modified approximate Riemann solutions $V^{\Delta}(x,t)$ is self-similar and satisfies the following*:

$$\sum_n \int_{(n-1)\Delta t}^{n\Delta t} \int_K |V^{\Delta}(x,t) - V^{\Delta}(x, n\Delta t - 0)|^2 dx dt = O(\Delta x),$$

where $O(\Delta x)$ depends on K.

The proof of Lemma 7.1 can be found in [MT, Proposition 3].

Using Lemma 7.1, we have

Theorem 7.2. *Assume that $(\rho^{\Delta}(x,t), m^{\Delta}(x,t))$ are the approximate solutions of the initial-boundary value problem* (1.6) *satisfying the conditions* (7.1)–(7.2). *Then there is a convergent subsequence in the approximate solutions such that*

$$(\rho^{\Delta_n}(x,t), m^{\Delta_n}(x,t)) \to (\rho(x,t), m(x,t)), \quad \text{a.e.} \tag{7.3}$$

The pair of functions $(\rho(x,t), m(x,t))$ is a global entropy solution of the initial-boundary value problem (1.6) *and satisfies*

$$0 \leq \rho(x,t) \leq C, \quad \left|\frac{m(x,t)}{\rho(x,t)}\right| \leq C \tag{7.4}$$

in the region Π_T.

Proof. Notice that for any convex weak entropy pair (η, q) and any nonnegative test function $\phi \in C_0^1(\Pi_T)$,

$$\begin{aligned}
&\int\int_{0\le t\le T=m\Delta t} (\eta(v^\Delta)\phi_t + q(v^\Delta)\phi_x - \nabla\eta(v^\Delta)g(v^\Delta)\phi)dxdt \\
&+\int_{-\infty}^{\infty} \eta(v_0(x))\phi(x,0)dx \\
&= I(\phi) + J(\phi) + \int_0^T \sum(\sigma[\eta]-[q])\phi(x(t),t) + E(\phi),
\end{aligned} \tag{7.5}$$

where

$$\begin{aligned}
I(\phi) = &\int\int_{\Pi_T} \phi_t(\eta(v^\Delta)-\eta(\bar v^\Delta)) + \phi_x(q(v^\Delta)-q(\bar v^\Delta)) - \phi(\nabla\eta(v^\Delta)g(v^\Delta) \\
&-\nabla\eta(\bar v^\Delta)g(\bar v^\Delta))dxdt,
\end{aligned} \tag{7.6}$$

$$\begin{aligned}
J(\phi) &= \sum_{j,n}\int_{(j-1)\Delta x+1}^{j\Delta x+1} (\eta(\bar v^n_-)-\eta(\bar v^n_+))\phi(x,n\Delta t)dx \\
&\quad+\int\int_{\Pi_T} \nabla\eta(\bar v^\Delta)\tilde g(\bar v^\Delta)\phi(x,t)dxdt \\
&= J_1(\phi)+J_2(\phi),
\end{aligned} \tag{7.7}$$

$$J_1(\phi) = \sum_{j,n}\phi_j^n\int_{(j-1)\Delta x+1}^{j\Delta x+1}(\eta(\bar v^n_-)-\eta(\bar v^n_+))dx + \int\int_{\Pi_T}\nabla\eta(\bar v^\Delta)\tilde g(\bar v^\Delta)\phi(x,t)dxdt, \tag{7.8}$$

$$J_2(\phi) = \sum_{j,n}\int_{(j-1)\Delta x+1}^{j\Delta x+1}(\eta(\bar v^n_-)-\eta(\bar v^n_+))(\phi-\phi_j^n)dx, \tag{7.9}$$

where

$$\tilde g(\bar v^\Delta) = \begin{cases} h(\bar v^\Delta) & \text{when} \bar v^\Delta \text{ is Type I}, \\ g(\bar v^\Delta) & \text{when} \bar v^\Delta \text{ is Type II}, \\ 0 & \text{when} \bar v^\Delta \text{ is Type III}, \end{cases}$$

$v^n_\pm = v^\Delta(x, n\Delta t \pm 0)$, $\bar v^n_\pm = \bar v^\Delta(x, n\Delta t\pm 0)$, $\phi_j^n = \phi(j\Delta x+1, n\Delta t)$, the summation of the third term in (7.5) is taken over all discontinuities in $\bar v^\Delta$ at a fixed t, σ is the propagating speed of the discontinuity and $E(\phi)$ is the error near the vacuum in the construction of approximate solutions of Type III, namely

$$E(\phi) = \int\int_{\Pi_T}\sum_{\bar v^\Delta \text{ is Type III}} \nabla\eta(\bar v^\Delta)g(\bar v^\Delta)\phi(x,t)dxdt.$$

In view of Remark 3.2, we have

$$|E(\phi)| \le o(1)\|\phi\|_{H^1(\Omega)}.$$

Since $v^\Delta - \bar v^\Delta = O(\Delta x)$, $I \to 0$ as $\Delta x \to 0$ by Lebesgue's dominated convergence theorem.

Taking Taylor's expansion of $(\sigma[\eta]-[q])\phi(x(t),t)$ for t at $(n+1/2)\Delta t$, we obtain, for $\mathcal{R}$,

$$\left|\int_{n\Delta t}^{(n+1)\Delta t}(\sigma[\eta]-[q])\,\phi(x(t),t)dt\right|\leq o((\Delta x)^{\alpha+2})$$

and, for $\mathcal{S}$,

$$\int_{n\Delta t}^{(n+1)\Delta t}(\sigma[\eta]-[q])\,\phi(x(t),t)dt\geq -o((\Delta x)^{2})$$

in a similar fashion to Lemmas 5.2 and 5.4. Then we have

$$\int_{0}^{T}\sum(\sigma[\eta]-[q])\phi(x(t),t)dt\geq -o(1)\|\phi\|_{C(\Omega)}.$$

On the other hand, notice that for $(j-1)\Delta x+1\leq x\leq j\Delta x+1$

$$v_j^n=\frac{1}{\Delta x}\int_{(j-1)\Delta x+1}^{j\Delta x+1}\bar{v}_-^n dx+\frac{\Delta t}{\Delta x}\int_{(j-1)\Delta x+1}^{j\Delta x+1}\tilde{g}(\bar{v}_-^n)dx.$$

Then since η is convex, from a similar argument of (6.12),

$$\begin{aligned}
J_1(\phi)&=\sum_{j,n}\phi_j^n\int_{(j-1)\Delta x+1}^{j\Delta x+1}(\eta(\bar{v}_-^n)-\eta(v_j^n))dx\\
&\quad+\sum_{j,n}\phi_j^n\int_{(j-1)\Delta x+1}^{j\Delta x+1}(\eta(v_j^n)-\eta(\bar{v}_+^n))dx\\
&\quad+\int\int_{\Pi_T}\nabla\eta(\bar{v}^{\Delta})\tilde{g}(\bar{v}^{\Delta})\phi(x,t)dxdt\\
&\geq\sum_{j,n}\phi_j^n\int_{(j-1)\Delta x+1}^{j\Delta x+1}\nabla\eta(v_j^n)(\bar{v}_-^n-v_j^n)dx\\
&\quad+\int\int_{\Pi_T}\nabla\eta(\bar{v}^{\Delta})\tilde{g}(\bar{v}^{\Delta})\phi(x,t)dxdt+o(1)\\
&=-\Delta t\sum_{j,n}\phi_j^n\int_{(j-1)\Delta x+1}^{j\Delta x+1}\nabla\eta(v_j^n)\tilde{g}(\bar{v}_-^n)dx\\
&\quad+\int\int_{\Pi_T}\nabla\eta(\bar{v}^{\Delta})\tilde{g}(\bar{v}^{\Delta})\phi(x,t)dxdt+o(1)\\
&=J_{11}+J_{12}+J_{13}+o(1),
\end{aligned}$$

where

$$\begin{aligned}
J_{11}&=\sum_{j,n}\phi_j^n\int_{(n-1)\Delta t}^{n\Delta t}\int_{(j-1)\Delta x+1}^{j\Delta x+1}(\nabla\eta(\bar{v}^{\Delta})\tilde{g}(\bar{v}^{\Delta})-\nabla\eta(v_j^n)\tilde{g}(v_j^n))dxdt,\\
J_{12}&=\sum_{j,n}\phi_j^n\int_{(n-1)\Delta t}^{n\Delta t}\int_{(j-1)\Delta x+1}^{j\Delta x+1}\nabla\eta(v_j^n)(\tilde{g}(v_j^n)-\tilde{g}(\bar{v}_-^n))dxdt,\\
J_{13}&=\sum_{j,n}\int_{(n-1)\Delta t}^{n\Delta t}\int_{(j-1)\Delta x+1}^{j\Delta x+1}(\nabla\eta(\bar{v}^{\Delta})\tilde{g}(\bar{v}^{\Delta})-\nabla\eta(v_j^n)\tilde{g}(\bar{v}_-^{\Delta}))\\
&\quad\times(\phi(x,t)-\phi_j^n)dxdt.
\end{aligned}$$

From Lemmas 2.5, the order of the difference between $\bar{v}^{\Delta}$ and the corresponding piecewise constant approximate Riemann solutions $V^{\Delta}(x,t)$ (see Lemma 7.1), which consists of data of quasi-steady state solutions, is Δx. Therefore, noting $|\nabla^2\eta| \le C/\rho$, $|\tilde{g}| \le C\rho, |\nabla_v \tilde{g}| \le C, |\nabla\eta| \le C$, from (6.18) and Lemma 7.1,

$$|J_{11}| \le C\sum_{j,n}\phi_j^n \int_{(n-1)\Delta t}^{n\Delta t}\int_{(j-1)\Delta x+1}^{j\Delta x+1} |\bar{v}^{\Delta} - v_j^n| dxdt = O(\sqrt{\Delta x}).$$

Since $|\nabla_v \tilde{g}| \le C, |\nabla\eta| \le C$, from (6.18), we can obtain

$$|J_{12}| \le C\sum_{j,n}\phi_j^n \int_{(n-1)\Delta t}^{n\Delta t}\int_{(j-1)\Delta x+1}^{j\Delta x+1} |\bar{v}_-^n - v_j^n| dxdt = O(\sqrt{\Delta x})$$

and

$$|J_{13}| = O(\Delta x).$$

Therefore $J_1 \to 0$ as $\Delta x \to 0$.

Furthermore, for any $\phi \in C_0^1(\Omega)$, we have

$$\begin{aligned}
|J_2(\phi)| &\le \sum_{j,n}\int_{(j-1)\Delta x+1}^{j\Delta x+1} |\phi(x, n\Delta t) - \phi_j^n|(|\eta(\bar{v}_-^n) - \eta(v_j^n)| \\
&\quad + |\eta(\bar{v}_+^n) - \eta(v_j^n)|)dx \\
&\le \Delta x\|\phi\|_{C^1(\Omega)}\left\{\sum_n\left(\sum_j\int_{(j-1)\Delta x+1}^{j\Delta x+1} |\eta(\bar{v}_-^n) - \eta(v_j^n)|^2 dx\right)^{\frac{1}{2}} + O(1)\right\} \\
&\le C\sqrt{\Delta x}\|\phi\|_{C^1(\Omega)}\left\{\left(\sum_{j,n}\int_{(j-1)\Delta x+1}^{j\Delta x+1} |\bar{v}_-^n - v_j^n|^2 dx\right)^{\frac{1}{2}} + O(\sqrt{\Delta x})\right\} \\
&\le C\sqrt{\Delta x}\|\phi\|_{C^1(\Omega)} \to 0, \quad \text{as } \Delta x \to 0.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\int\int_{0\le t\le T=m\Delta t} (\eta(v^{\Delta})\phi_t + q(v^{\Delta})\phi_x - \nabla\eta(v^{\Delta})g(v^{\Delta})\phi)dxdt \\
&\qquad + \int_{-\infty}^{\infty}\eta(\bar{v}^{\Delta}(x))\phi(0,x)dx \\
&\quad \ge -o(1)(\|\phi\|_{C^1(\Omega)} + \|\phi\|_{H^1(\Omega)}) \to 0, \quad \Delta x \to 0.
\end{aligned} \tag{7.10}$$

Taking the limit $\Delta x \to 0$ on both sides of (7.10) and using Lebesgue's dominated convergence theorem, we verify that the limit function $v = (\rho, m)$ satisfies

$$\eta(v)_t + q(v)_x + \nabla\eta(v)g(v) \le 0, \tag{7.11}$$

in the sense of distributions. Choosing $\eta(v) = \pm\rho, \pm m$, we immediately find that $v(x,t)$ is a weak solution. Using the standard procedure (cf. [S]), we conclude that the limit function $v(x,t)$ satisfies the entropy condition (1.12) along any shock wave. This completes the proof of Theorem 7.2. □

Appendix A. The validity of the construction of approximate solutions and their estimates in the case (A)

We prove that the construction of approximate solutions and their estimates in the case (A) are valid. Recall that, in this case, a 1-rarefaction and a 2-shock arise as Riemann solutions with initial data (v_j^n, v_{j+1}^n) and the middle state satisfies $\rho_{\mathrm{M}} \geq (\Delta x)^{\beta}$.

We shall replace the constants of a 1-rarefaction fan with quasi-steady state solutions. More precisely, we determine speeds $\sigma_i,\ i = 2, \ldots, p$ and quasi-steady state solutions $\bar{v}_i(x),\ i = 2, \ldots, p$ between $l_i : x = j\Delta x + 1 + \sigma_i(t - n\Delta t)$ and $l_{i+1} : x = j\Delta x + 1 + \sigma_{i+1}(t - n\Delta t)$.

First, let us determine the speed σ_2 and the quasi-steady state solution $\bar{v}_2(x)$. Let $F_2(\sigma_2)$ be $\sigma_2 - \bar{u}_1 + S(\rho_2, \bar{\rho}_1)$ at $x = x_2 := j\Delta x + 1 + \sigma_2 \Delta t/2$, where v_2 is the possible state that can be connected to $\bar{v}_1(x_2)$ on the right by a 1-rarefaction shock curve such that $z_2 := z(v_2) = z_2^*$ and $\bar{v}_1(x)$ is defined in (3.8). In this step, we determine σ_2 and v_2 in Subsection 3.1 (A) Step 2. To do this, we find σ_2 such that $F_2(\sigma_2) = 0$, by using the following theorem:

Theorem A.1. *Let $U \in \mathbf{R}^n$ be an open set. Let $\vec{x} \mapsto f(\vec{x})$ be a C^1 map from U into $\mathbf{R}^n$. Assume that there exist positive numbers ε, Δ_0, M such that the following hold : 1) $\varepsilon \leq \Delta_0/(2M)$; 2) $|f(\vec{x}_0)| \leq \varepsilon$; 3) $Df(\vec{x}_0)$ is invertible; 4) $\|Df(\vec{x}) - Df(\vec{x}_0)\| \leq 1/(2M)$ for $|\vec{x} - \vec{x}_0| \leq \Delta_0$; 5) $\left\|\{Df(\vec{x}_0)\}^{-1}\right\| \leq M$, where $\|A\| = \sup_{|\vec{x}|=1} |A\vec{x}|$ for $n \times n$ matrix A.*

Then there exists a unique solution $\vec{x}$ of

$$f(\vec{x}) = 0$$

in $|\vec{x} - \vec{x}_0| \leq \Delta_0$ such that

$$|\vec{x} - \vec{x}_0| \leq 2M\varepsilon.$$

First, from Remark 2.2, along a 1-shock or a 1-rarefaction shock curve, we have the following lemma:

Lemma A.2. *If there exist $C_1 > 0$ and $C_2 > 1$ independent of Δx such that $\min\{\rho, \rho_0\} \geq C_1(\Delta x)^{\beta}$ and $1/C_2 \leq \rho/\rho_0 \leq C_2$,*

$$w - w_0 = O((\Delta x)^{-(\gamma-1)\beta})(z - z_0)^3$$

holds, where (w, z) is connected to (w_0, z_0) on the right by a 1-rarefaction shock or a 1-shock curve and $O(\Delta x)$ depends only on C_1 and C_2.

Let $v_{0,2}$ be the possible state that can be connected to $\bar{v}_1(x_2^*)$ on the right by a 1-rarefaction shock curve such that $z_{0,2} = z_2^*$, where $x_2^* = j\Delta x + 1 + \lambda_1(z_1^*, z_2^*, w_{\mathrm{L}})\Delta t/2$. We deduce from Lemma A.2 that

$$w_{0,2} := w(v_{0,2}) = \bar{w}_1(x_2^*) + O((\Delta x)^{3\alpha-(\gamma-1)\beta}).$$

Since $2\alpha - (\gamma - 1)\beta > 1$, we have

$$F_2(\lambda_1(z_1^*, z_2^*, w_{\mathrm{L}})) = O(\Delta x),\ F_2'(\lambda_1(z_1^*, z_2^*, w_{\mathrm{L}})) > 1/2.$$

Applying Theorem A.1 with $\vec{x} = \sigma_2$ and $\vec{x}_0 = \lambda_1(z_1^*, z_2^*, w_{\mathrm{L}})$, we have a solution σ_2 such that

$$\sigma_2 = \bar{u}_1 - S(\rho_2, \bar{\rho}_1) \text{ at } x = x_2 \tag{A.1}$$

and

$$|\sigma_2 - \lambda_1(z_1^*, z_2^*, w_{\rm L})| = O(\Delta x). \tag{A.2}$$

By using this solution σ_2, we can determine v_2 such that 1) $z(v_2) = z_2^*$ and 2) the speed σ_2, the left and right states $\bar{v}_1(x_2), v_2$ satisfy the Rankine-Hugoniot conditions, where $x_2 := j\Delta x + 1 + \sigma_2 \Delta t/2$. Moreover, v_2 satisfies

$$\rho_2 \geq (\Delta x)^\beta/2, \quad |\rho_2/\bar{\rho}_1(x_2) - 1| = o(1). \tag{A.3}$$

Then we define a quasi-steady state solution $\bar{v}_2$ by

$$\bar{v}_2(x) := \mathbf{V}_{\rm qs}(x, x_2, v_2).$$

Since v_2 and $\bar{v}_1(x_2)$ are connected by a 1-rarefaction shock curve, from Lemma A.2, we obtain

$$\bar{w}_2(x_2) = w_2 = \bar{w}_1(x_2) + O((\Delta x)^{3\alpha - (\gamma-1)\beta}). \tag{A.4}$$

Now, we assume that we can determine the speed σ_k, v_k and $\bar{v}_k(x)$ $(k = 2, \ldots, i)$ satisfying the following (♠):

- $z_k := z(v_k) = z_k^*, w_k := w(v_k).$ (A.5)
- $|\sigma_k - \lambda_1(w_{\rm L}, z_{k-1}^*, z_k^*)| = O(\Delta x).$ (A.6)
- $x_k = j\Delta x + 1 + \sigma_k \Delta t/2.$
- v_k and $\bar{v}_{k-1}(x_k)$ are connected by a 1-rarefaction shock curve.
- $\bar{v}_k(x) = \mathbf{V}_{\rm qs}(x, x_k, v_k).$ (A.7)
- $\bar{w}_k(x_k) := w(\bar{v}_k(x_k)) = w_k = \bar{w}_{k-1}(x_k) + O((\Delta x)^{3\alpha-(\gamma-1)\beta}).$ (A.8)
- $\rho_k \geq (\Delta x)^\beta/2, \quad |\rho_k/\bar{\rho}_{k-1}(x_k) - 1| = o(1).$ (A.9)

In view of (A.8), since $p = O((\Delta x)^{-\alpha})$ and $2\alpha - (\gamma-1)\beta > 1$, we have

$$\begin{aligned}
|w_i - w_{\rm L}| &= |\bar{w}_i(x_i) - w_{\rm L}| = |\bar{w}_i(x_i) - \bar{w}_1(x_1)| \\
&\leq \sum_{k=1}^{i-1} |\bar{w}_k(x_{k+1}) - \bar{w}_k(x_k)| + O((\Delta x)^{2\alpha-(\gamma-1)\beta}) \\
&= O(\Delta x).
\end{aligned} \tag{A.10}$$

It follows from (A.10) that

$$\rho_i = \rho(z_i^*, w_{\rm L}) + (\rho(z_i^*, w_{\rm L}))^{\frac{3-\gamma}{2}} O(\Delta x) + O((\Delta x)^2) \tag{A.11}$$

and

$$u_i - u(z_i^*, w_{\rm L}) = \frac{w_i + z_i^*}{2} - \frac{w_{\rm L} + z_i^*}{2} = O(\Delta x). \tag{A.12}$$

Moreover, recalling that $\alpha < 1$, it follows from (A.6) that $\sigma_{k-1} < \sigma_k$. Then, by induction and the argument for $i = 2$, we can determine the speed σ_{i+1}, v_{i+1} and $\bar{v}_{i+1}(x)$ satisfying (♠) and complete $R_1^\Delta(z_{\rm M})(v_{\rm L})$. Notice that (A.6) and (A.10)–(A.12) imply the order of the difference between the 1-rarefaction fan and $R_1^\Delta(z_{\rm M})(v_{\rm L})$ is Δx.

Now we fix $\bar{v}_{\rm R}(x)$ and $\bar{v}_{p-1}(x)$. Choosing $\sigma_p^\diamond$ near to σ_p, $\sigma_s^\diamond$ near to σ_s and $v_{\rm M}^\diamond$ near to $v_{\rm M}$, we fill up the gap between $x = j\Delta x + 1 + \sigma_p^\diamond(t - n\Delta t)$ and $x = j\Delta x + 1 +$

$\sigma_s^\diamond(t-n\Delta t)$, by a quasi-steady state solution $\bar{v}_{\rm M}^\diamond(x)$ satisfying $\bar{v}_{\rm M}^\diamond(j\Delta x+1)=v_{\rm M}^\diamond$. First let $\sigma_p^\diamond=\sigma_p^\diamond(v_{\rm M}^\diamond)$ and $\sigma_s^\diamond=\sigma_s^\diamond(v_{\rm M}^\diamond)$ be solutions of the equations

$$\sigma_p^\diamond=\bar{u}_{p-1}-S(\bar{\rho}_{\rm M}^\diamond,\bar{\rho}_{p-1})\text{ at }x=x_p^\diamond:=j\Delta x+1+\sigma_p^\diamond\Delta t/2$$
$$\text{and}$$
$$\sigma_s^\diamond=\bar{u}_{\rm R}+S(\bar{\rho}_{\rm M}^\diamond,\bar{\rho}_{\rm R})\text{ at }x=x_s^\diamond:=j\Delta x+1+\sigma_s^\diamond\Delta t/2$$

respectively.

Let $F_s(\sigma_s^\diamond,\rho_{\rm M}^\diamond,u_{\rm M}^\diamond)$ be

$$\bar{u}_{\rm R}+S(\bar{\rho}_{\rm M}^\diamond,\bar{\rho}_{\rm R})\text{ at }x=x_s^\diamond:=j\Delta x+1+\sigma_s^\diamond\Delta t/2.$$

In the coming argument, we shall need the lower bound of $\rho_{\rm R}$ in the case where $\rho_{\rm R}$ is much smaller than $\rho_{\rm M}$. Therefore, when $\rho_{\rm R}\leq\rho_{\rm M}/2$, we first derive the estimate. It follows from Lax condition that

$$\lambda_2(v_{\rm M},v_{\rm R})=u_{\rm R}+\sqrt{\frac{\rho_{\rm M}}{\rho_{\rm R}}\frac{p(\rho_{\rm M})-p(\rho_{\rm R})}{\rho_{\rm M}-\rho_{\rm R}}}\leq u_{\rm M}+\rho_{\rm M}^\theta=\lambda_2(v_{\rm M}).$$

We then have

$$\begin{aligned}C\geq\left(u_{\rm M}-u_{\rm R}+\rho_{\rm M}^\theta\right)&=\sqrt{\frac{\rho_{\rm M}}{\rho_{\rm R}}\frac{p(\rho_{\rm M})-p(\rho_{\rm R})}{\rho_{\rm M}-\rho_{\rm R}}}\\ &\geq\sqrt{\frac{\rho_{\rm M}}{\rho_{\rm R}}\frac{(\rho_{\rm M})^\gamma/\gamma\,\{1-(\rho_{\rm R}/\rho_{\rm M})^\gamma\}}{\rho_{\rm M}}}\geq\sqrt{\frac{\{1-(1/2)^\gamma\}(\rho_{\rm M})^\gamma/\gamma}{\rho_{\rm R}}},\end{aligned}\tag{A.13}$$

where C depends only on M_- and M_+. We consequently obtain $\rho_{\rm R}\geq(\rho_{\rm M})^\gamma/C$.

We thus have

$$\begin{aligned}\left.\frac{\partial F_s(\sigma_s^\diamond,\rho_{\rm M}^\diamond,u_{\rm M}^\diamond)}{\partial\sigma_s^\diamond}\right|_{\substack{\sigma_s^\diamond=\sigma_s,\rho_{\rm M}^\diamond=\rho_{\rm M},\\ u_{\rm M}^\diamond=u_{\rm M}}}&=\frac{\partial\bar{u}_{\rm R}(x)}{\partial x}\frac{\Delta t}{2}+\frac{\partial S(\bar{\rho}_{\rm M}^\diamond,\bar{\rho}_{\rm R})}{\partial\bar{\rho}_{\rm R}}\frac{\partial\bar{\rho}_{\rm R}(x)}{\partial x}\frac{\Delta t}{2}\\ &\quad+\left.\frac{\partial S(\bar{\rho}_{\rm M}^\diamond,\bar{\rho}_{\rm R})}{\partial\bar{\rho}_{\rm M}^\diamond}\frac{\partial\bar{\rho}_{\rm M}^\diamond(x)}{\partial x}\frac{\Delta t}{2}\right|_{\substack{\sigma_s^\diamond=\sigma_s,\rho_{\rm M}^\diamond=\rho_{\rm M},\\ u_{\rm M}^\diamond=u_{\rm M}}}\\ &=O(\Delta x).\end{aligned}$$

Similarly, we have

$$\begin{aligned}\left.\frac{\partial F_s(\sigma_s^\diamond,\rho_{\rm M}^\diamond,u_{\rm M}^\diamond)}{\partial\rho_{\rm M}^\diamond}\right|_{\substack{\sigma_s^\diamond=\sigma_s,\rho_{\rm M}^\diamond=\rho_{\rm M},\\ u_{\rm M}^\diamond=u_{\rm M}}}&=\left.\frac{\partial S(\bar{\rho}_{\rm M}^\diamond,\bar{\rho}_{\rm R})}{\partial\bar{\rho}_{\rm M}^\diamond}\frac{\partial\bar{\rho}_{\rm M}^\diamond(x)}{\partial\rho_{\rm M}^\diamond}\right|_{\substack{\sigma_s^\diamond=\sigma_s,\rho_{\rm M}^\diamond=\rho_{\rm M},\\ u_{\rm M}^\diamond=u_{\rm M}}}\\ &=O((\rho_{\rm M})^{-1})\end{aligned}$$

and

$$\left.\frac{\partial F_s(\sigma_s^\diamond,\rho_{\rm M}^\diamond,u_{\rm M}^\diamond)}{\partial u_{\rm M}^\diamond}\right|_{\substack{\sigma_s^\diamond=\sigma_s,\rho_{\rm M}^\diamond=\rho_{\rm M},\\ u_{\rm M}^\diamond=u_{\rm M}}}=O(1).$$

Here we introduce the following theorem:

Theorem A.3. *Let $U\in\mathbf{R}^n,V\in\mathbf{R}$ be open sets. Let $(\vec{x},y)\mapsto f(\vec{x},y)$ be a C^1 map from $U\times V$ into $\mathbf{R}$. Let us consider the equation*

$$y=f(x_1,\ldots,x_n,y)\tag{A.14}$$

near $(\vec{x}, y) = (\vec{x}_0, y_0)$. *Assume that there exist positive numbers* $\varepsilon, \Delta_1, \Delta_2, M$ *such that the following hold*: 1) $\varepsilon \le \Delta_1/3$ *and* $\Delta_2 \le \Delta_1/(3M)$; 2) $|y_0 - f(\vec{x}_0, y_0)| \le \varepsilon$; 3) *If* $|y - y_0| \le \Delta_1$ *and* $|\vec{x} - \vec{x}_0| \le \Delta_1$, *then* $|\partial_y f(\vec{x}, y)| \le L \le 1/3$ *and* $|\partial_{\vec{x}} f| \le M$.

Then there exists a unique solution $y = y(\vec{x})$ *of* (A.14) *for* $|y - y_0| \le \Delta_1$ *and* $|\vec{x} - \vec{x}_0| \le \Delta_2$ *such that*

$$|y(\vec{x}) - f(\vec{x}_0, y_0)| \le \frac{1}{1-L}(M|\vec{x} - \vec{x}_0| + |y_0 - f(\vec{x}_0, y_0)|). \tag{A.15}$$

Moreover y *is* C^1.

Since

$$F_s(\sigma_s^\diamond, \rho_{\mathrm{M}}^\diamond, u_{\mathrm{M}}^\diamond)\Big|_{\substack{\sigma_s^\diamond=\sigma_s, \rho_{\mathrm{M}}^\diamond=\rho_{\mathrm{M}}, \\ u_{\mathrm{M}}^\diamond = u_{\mathrm{M}}}} - \sigma_s = O(\Delta x),$$

applying Theorem A.3 with $\vec{x} = (\rho_{\mathrm{M}}^\diamond, u_{\mathrm{M}}^\diamond)$, $y = \bar{\sigma}_s$, $\vec{x}_0 = (\rho_{\mathrm{M}}, u_{\mathrm{M}})$ and $y_0 = \sigma_s$, we have

$$\sigma_s^\diamond = \sigma_s + O(\Delta x) + O((\rho_{\mathrm{M}})^{-1}|\rho_{\mathrm{M}}^\diamond - \rho_{\mathrm{M}}| + |u_{\mathrm{M}}^\diamond - u_{\mathrm{M}}|),$$

where

$$\sigma_s = u_{\mathrm{R}} + S(\rho_{\mathrm{M}}, \rho_{\mathrm{R}}).$$

Let $F(\sigma_p^\diamond, \rho_{\mathrm{M}}^\diamond, u_{\mathrm{M}}^\diamond)$ be

$$\bar{u}_{p-1} - S(\bar{\rho}_{\mathrm{M}}^\diamond, \bar{\rho}_{p-1}) \text{ at } x = x_p^\diamond := j\Delta x + 1 + \sigma_p^\diamond \Delta t/2.$$

Set

$$\sigma_{\mathrm{Rw}}^* = u_{p-1} - S(\rho_{\mathrm{M}}, \rho_{p-1}).$$

We then observe that

$$F(\sigma_p^\diamond, \rho_{\mathrm{M}}^\diamond, u_{\mathrm{M}}^\diamond)\Big|_{\substack{\sigma_p^\diamond=\sigma_{\mathrm{Rw}}^*, \rho_{\mathrm{M}}^\diamond=\rho_{\mathrm{M}}, \\ u_{\mathrm{M}}^\diamond = u_{\mathrm{M}}}} - \sigma_{\mathrm{Rw}}^* = O(\Delta x).$$

Moreover we have

$$\frac{\partial F(\sigma_p^\diamond, \rho_{\mathrm{M}}^\diamond, u_{\mathrm{M}}^\diamond)}{\partial \sigma_p^\diamond}\Big|_{\substack{\sigma_p^\diamond=\sigma_{\mathrm{Rw}}^*, \rho_{\mathrm{M}}^\diamond=\rho_{\mathrm{M}}, \\ u_{\mathrm{M}}^\diamond = u_{\mathrm{M}}}} = O(\Delta x).$$

Similarly, we have

$$\frac{\partial F(\sigma_p^\diamond, \rho_{\mathrm{M}}^\diamond, u_{\mathrm{M}}^\diamond)}{\partial \rho_{\mathrm{M}}^\diamond}\Big|_{\substack{\sigma_p^\diamond=\sigma_{\mathrm{Rw}}^*, \rho_{\mathrm{M}}^\diamond=\rho_{\mathrm{M}}, \\ u_{\mathrm{M}}^\diamond = u_{\mathrm{M}}}} = O((\rho_{\mathrm{M}})^{\frac{\gamma-3}{2}})$$

and

$$\frac{\partial F(\sigma_p^\diamond, \rho_{\mathrm{M}}^\diamond, u_{\mathrm{M}}^\diamond)}{\partial u_{\mathrm{M}}^\diamond}\Big|_{\substack{\sigma_p^\diamond=\sigma_{\mathrm{Rw}}^*, \rho_{\mathrm{M}}^\diamond=\rho_{\mathrm{M}}, \\ u_{\mathrm{M}}^\diamond = u_{\mathrm{M}}}} = O(1).$$

Applying Theorem A.3 with $\vec{x} = (\rho_{\mathrm{M}}^\diamond, u_{\mathrm{M}}^\diamond)$, $y = \sigma_p^\diamond$, $\vec{x}_0 = (\rho_{\mathrm{M}}, u_{\mathrm{M}})$ and $y_0 = \sigma_{\mathrm{Rw}}^*$, we have

$$\sigma_p^\diamond = \sigma_{\mathrm{Rw}}^* + O(\Delta x) + O((\rho_{\mathrm{M}})^{\frac{\gamma-3}{2}}|\rho_{\mathrm{M}}^\diamond - \rho_{\mathrm{M}}| + |u_{\mathrm{M}}^\diamond - u_{\mathrm{M}}|). \tag{A.16}$$

Now we have obtained two functions of $\rho_{\mathrm{M}}^\diamond$ and $u_{\mathrm{M}}^\diamond$, $\sigma_p^\diamond = \sigma_p^\diamond(\rho_{\mathrm{M}}^\diamond, u_{\mathrm{M}}^\diamond)$ and $\sigma_s^\diamond = \sigma_s^\diamond(\rho_{\mathrm{M}}^\diamond, u_{\mathrm{M}}^\diamond)$. Next let us determine $\rho_{\mathrm{M}}^\diamond$ and $u_{\mathrm{M}}^\diamond$.

We first obtain

$$\begin{aligned}
\left.\frac{\partial\sigma_p^\diamond}{\partial\rho_{\mathrm{M}}^\diamond}\right|_{\rho_{\mathrm{M}}^\diamond=\rho_{\mathrm{M}},u_{\mathrm{M}}^\diamond=u_{\mathrm{M}}} &= -\frac{\partial}{\partial\bar{\rho}_{\mathrm{M}}^\diamond}S(\bar{\rho}_{\mathrm{M}}^\diamond,\bar{\rho}_{p-1})(1+O(\Delta x))\Big|_{\rho_{\mathrm{M}}^\diamond=\rho_{\mathrm{M}},u_{\mathrm{M}}^\diamond=u_{\mathrm{M}}},\\
\left.\frac{\partial\sigma_s^\diamond}{\partial\rho_{\mathrm{M}}^\diamond}\right|_{\rho_{\mathrm{M}}^\diamond=\rho_{\mathrm{M}},u_{\mathrm{M}}^\diamond=u_{\mathrm{M}}} &= \frac{\partial}{\partial\bar{\rho}_{\mathrm{M}}^\diamond}S(\bar{\rho}_{\mathrm{M}}^\diamond,\bar{\rho}_{\mathrm{R}})(1+O(\Delta x))\Big|_{\rho_{\mathrm{M}}^\diamond=\rho_{\mathrm{M}},u_{\mathrm{M}}^\diamond=u_{\mathrm{M}}},\\
\left.\frac{\partial\sigma_p^\diamond}{\partial u_{\mathrm{M}}^\diamond}\right|_{\rho_{\mathrm{M}}^\diamond=\rho_{\mathrm{M}},u_{\mathrm{M}}^\diamond=u_{\mathrm{M}}} &= O(\Delta x), \quad \left.\frac{\partial\sigma_s^\diamond}{\partial u_{\mathrm{M}}^\diamond}\right|_{\rho_{\mathrm{M}}^\diamond=\rho_{\mathrm{M}},u_{\mathrm{M}}^\diamond=u_{\mathrm{M}}} = O(\Delta x).
\end{aligned}$$

On the other hand, the Rankine-Hugoniot conditions on $t=(n+1/2)\Delta t$ are reduced to $\Lambda_1=\Lambda_2=0$, where

$$\begin{aligned}
\Lambda_1 &:= \bar{m}_{\mathrm{M}}^\diamond-\bar{m}_{p-1}-\sigma_p^\diamond(\bar{\rho}_{\mathrm{M}}^\diamond-\bar{\rho}_{p-1}) \text{ at } x=x_p^\diamond,\\
\Lambda_2 &:= \bar{m}_{\mathrm{R}}-\bar{m}_{\mathrm{M}}^\diamond-\sigma_s^\diamond(\bar{\rho}_{\mathrm{R}}-\bar{\rho}_{\mathrm{M}}^\diamond) \text{ at } x=x_s^\diamond.
\end{aligned}$$

Then we can find

$$\Lambda_1\big|_{\rho_{\mathrm{M}}^\diamond=\rho_{\mathrm{M}},u_{\mathrm{M}}^\diamond=u_{\mathrm{M}}}=O(\Delta x),\ \Lambda_2\big|_{\rho_{\mathrm{M}}^\diamond=\rho_{\mathrm{M}},u_{\mathrm{M}}^\diamond=u_{\mathrm{M}}}=O(\Delta x).$$

Next we estimate the derivatives of Λ_1 and Λ_2. We first obtain

$$\begin{aligned}
\left.\frac{\partial\Lambda_1}{\partial\rho_{\mathrm{M}}^\diamond}\right|_{\substack{\rho_{\mathrm{M}}^\diamond=\rho_{\mathrm{M}},\\ u_{\mathrm{M}}^\diamond=u_{\mathrm{M}}}} &= (\bar{\rho}_{\mathrm{M}}^\diamond-\bar{\rho}_{p-1})\frac{\partial}{\partial\bar{\rho}_{\mathrm{M}}^\diamond}S(\bar{\rho}_{\mathrm{M}}^\diamond,\bar{\rho}_{p-1})-\sigma_p^\diamond+\bar{u}_{\mathrm{M}}^\diamond\Big|_{\substack{\rho_{\mathrm{M}}^\diamond=\rho_{\mathrm{M}},\\ u_{\mathrm{M}}^\diamond=u_{\mathrm{M}}}}+O(\Delta x),\\
\left.\frac{\partial\Lambda_2}{\partial\rho_{\mathrm{M}}^\diamond}\right|_{\substack{\rho_{\mathrm{M}}^\diamond=\rho_{\mathrm{M}},\\ u_{\mathrm{M}}^\diamond=u_{\mathrm{M}}}} &= (\bar{\rho}_{\mathrm{M}}^\diamond-\bar{\rho}_{\mathrm{R}})\frac{\partial}{\partial\bar{\rho}_{\mathrm{M}}^\diamond}S(\bar{\rho}_{\mathrm{M}}^\diamond,\bar{\rho}_{\mathrm{R}})+\sigma_s^\diamond-\bar{u}_{\mathrm{M}}^\diamond\Big|_{\substack{\rho_{\mathrm{M}}^\diamond=\rho_{\mathrm{M}},\\ u_{\mathrm{M}}^\diamond=u_{\mathrm{M}}}}+O(\Delta x),\\
\left.\frac{\partial\Lambda_1}{\partial u_{\mathrm{M}}^\diamond}\right|_{\substack{\rho_{\mathrm{M}}^\diamond=\rho_{\mathrm{M}},\\ u_{\mathrm{M}}^\diamond=u_{\mathrm{M}}}} &= \bar{\rho}_{\mathrm{M}}^\diamond\Big|_{\substack{\rho_{\mathrm{M}}^\diamond=\rho_{\mathrm{M}},\\ u_{\mathrm{M}}^\diamond=u_{\mathrm{M}}}}+O(\Delta x),\ \left.\frac{\partial\Lambda_2}{\partial u_{\mathrm{M}}^\diamond}\right|_{\substack{\rho_{\mathrm{M}}^\diamond=\rho_{\mathrm{M}},\\ u_{\mathrm{M}}^\diamond=u_{\mathrm{M}}}}=-\bar{\rho}_{\mathrm{M}}^\diamond\Big|_{\substack{\rho_{\mathrm{M}}^\diamond=\rho_{\mathrm{M}},\\ u_{\mathrm{M}}^\diamond=u_{\mathrm{M}}}}+O(\Delta x).
\end{aligned}$$

These yield

$$\det\left.\begin{bmatrix}\dfrac{\partial\Lambda_1}{\partial\rho_{\mathrm{M}}^\diamond} & \dfrac{\partial\Lambda_1}{\partial u_{\mathrm{M}}^\diamond}\\[2ex] \dfrac{\partial\Lambda_2}{\partial\rho_{\mathrm{M}}^\diamond} & \dfrac{\partial\Lambda_2}{\partial u_{\mathrm{M}}^\diamond}\end{bmatrix}\right|_{\substack{\rho_{\mathrm{M}}^\diamond=\rho_{\mathrm{M}},\\ u_{\mathrm{M}}^\diamond=u_{\mathrm{M}}}} = -\bar{\rho}_{\mathrm{M}}^\diamond\left\{\Delta+O((\Delta x)^\alpha)\right\}\big|_{\rho_{\mathrm{M}}^\diamond=\rho_{\mathrm{M}},u_{\mathrm{M}}^\diamond=u_{\mathrm{M}}},$$

where

$$\Delta=-(\bar{\rho}_{\mathrm{R}}-\bar{\rho}_{\mathrm{M}}^\diamond)\frac{\partial}{\partial\bar{\rho}_{\mathrm{M}}^\diamond}S(\bar{\rho}_{\mathrm{M}}^\diamond,\bar{\rho}_{\mathrm{R}})+S(\bar{\rho}_{\mathrm{R}},\bar{\rho}_{\mathrm{M}}^\diamond)+S(\bar{\rho}_{\mathrm{M}}^\diamond,\bar{\rho}_{\mathrm{M}}^\diamond)\Big|_{\rho_{\mathrm{M}}^\diamond=\rho_{\mathrm{M}},u_{\mathrm{M}}^\diamond=u_{\mathrm{M}}}.$$

Since

$$\begin{aligned}
\Delta &= (\bar{\rho}_{\mathrm{M}}^\diamond-\bar{\rho}_{\mathrm{R}})\frac{\partial}{\partial\bar{\rho}_{\mathrm{M}}^\diamond}S(\bar{\rho}_{\mathrm{M}}^\diamond,\bar{\rho}_{\mathrm{R}})+S(\bar{\rho}_{\mathrm{R}},\bar{\rho}_{\mathrm{M}}^\diamond)+S(\bar{\rho}_{\mathrm{M}}^\diamond,\bar{\rho}_{\mathrm{M}}^\diamond)\Big|_{\rho_{\mathrm{M}}^\diamond=\rho_{\mathrm{M}},u_{\mathrm{M}}^\diamond=u_{\mathrm{M}}}\\
&= \frac{1}{2}S(\bar{\rho}_{\mathrm{R}},\bar{\rho}_{\mathrm{M}}^\diamond)+\frac{1}{2}\frac{1}{S(\bar{\rho}_{\mathrm{R}},\bar{\rho}_{\mathrm{M}}^\diamond)}p'(\bar{\rho}_{\mathrm{M}}^\diamond)+S(\bar{\rho}_{\mathrm{M}}^\diamond,\bar{\rho}_{\mathrm{M}}^\diamond)\Big|_{\rho_{\mathrm{M}}^\diamond=\rho_{\mathrm{M}},u_{\mathrm{M}}^\diamond=u_{\mathrm{M}}}\\
&\geq C(\Delta x)^{\frac{\gamma-1}{2}\beta},
\end{aligned}$$

we thus conclude that

$$\left\| \begin{pmatrix} \dfrac{\partial \Lambda_1}{\partial \rho_{\mathrm{M}}^{\diamond}} & \dfrac{\partial \Lambda_1}{\partial u_{\mathrm{M}}^{\diamond}} \\[2ex] \dfrac{\partial \Lambda_2}{\partial \rho_{\mathrm{M}}^{\diamond}} & \dfrac{\partial \Lambda_2}{\partial u_{\mathrm{M}}^{\diamond}} \end{pmatrix}^{-1} \right\|_{\substack{\rho_{\mathrm{M}}^{\diamond}=\rho_{\mathrm{M}},\\ u_{\mathrm{M}}^{\diamond}=u_{\mathrm{M}}}} = O((\Delta x)^{-\frac{\gamma+1}{2}\beta}).$$

Applying Theorem A.1, we have a solution $v_{\mathrm{M}}^{\diamond}$ satisfying

$$|\rho_{\mathrm{M}}^{\diamond} - \rho_{\mathrm{M}}| + |u_{\mathrm{M}}^{\diamond} - u_{\mathrm{M}}| = O((\Delta x)^{1-\frac{\gamma+1}{2}\beta}). \tag{A.17}$$

Finally we derive (4.7) and $\sigma_{p-1} < \sigma_p^{\diamond} < \sigma_s^{\diamond}$. Since $\rho_{\mathrm{M}} \geq (\Delta x)^{\beta}$ and $|\rho_{p-1} - \rho_{\mathrm{M}}| = O((\Delta x)^{\alpha})$, from (A.17), we have

$$\rho_{\mathrm{M}}^{\diamond} = \rho_{p-1}(1+o(1)) = \rho_{\mathrm{M}}(1+o(1)), \quad \rho_{\mathrm{M}}^{\diamond} > (\Delta x)^{\beta}/2.$$

This shows $|\rho_{\mathrm{M}}^{\diamond}/\rho_{p-1} - 1| = o(1)$. Observing that $\bar{\rho}_{p-1}(x_p^{\diamond}) = \rho_{p-1} + O(\Delta x)$ and $\bar{\rho}_{\mathrm{M}}^{\diamond}(x_p^{\diamond}) = \rho_{\mathrm{M}}^{\diamond} + O(\Delta x)$, (4.7) follows from Remark 2.2 and (A.17). $\sigma_{p-1} < \sigma_p^{\diamond}$ follows from the fact that $\alpha < 1 - 2\beta$, (A.16) and (A.17). Since $\sigma_p^{\diamond}$ and $\sigma_s^{\diamond}$ are speeds of the different families, recalling $\rho_{\mathrm{M}} \geq (\Delta x)^{\beta}$, we can obtain $\sigma_p^{\diamond} < \sigma_s^{\diamond}$.

Appendix B. The validity of the construction of approximate solutions and their estimates in the case (B)

We consider $\bar{v}^{\Delta}(x,t)$ in the case (B). In this case, $\bar{v}^{\Delta}(x,t)$ consists of a Riemann solution and a piecewise quasi-steady state rarefaction wave such as $R_1^{\Delta}(z_{\mathrm{L}}^{(1)})(v_{\mathrm{L}})$.

In this section, we estimate $\bar{v}^{\Delta}(x,t)$, in particular, in the region where $\bar{v}^{\Delta}(x,t)$ are Riemann solutions. Here notice that, if $\bar{v}^{\Delta}(x,t)$ is a Riemann solution, from our construction, $\bar{v}^{\Delta}(x,t) = v^{\Delta}(x,t)$. On the other hand, we cannot connect a piecewise quasi-steady state rarefaction wave and a shock wave of the same family. Therefore we must check the following:

- in the region where $v^{\Delta}(x,t)$ is a Riemann solution, $v^{\Delta}(x,t)$ satisfies (4.1) and $\rho^{\Delta}(x,t) < 3(\Delta x)^{\beta}/2$.
- when we connect a piecewise quasi-steady state rarefaction R_1^{Δ} (resp. R_2^{Δ}) and a Riemann solution, a 1-shock (resp. a 2-shock) does not arise as the Riemann solution.

Case 1.1 $\rho_l > (\Delta x)^{\beta}$

In this case, we first observe that

$$\rho_{\mathrm{R}} < (\Delta x)^{\beta}, \quad z(v_{\mathrm{R}}) \geq L_j. \tag{B.1}$$

(ii) $z(v_{\mathrm{M}}) - z(v_{\mathrm{L}}^{(1)}) > (\Delta x)^{\alpha}$

We separately consider two cases.

(ii)–(a) $z(v_{\mathrm{L}}^{(2)}) \geq L_j$

In this case, since $\max\{z_{\mathrm{L}}^{(1)}, L_j\} = z_{\mathrm{L}}^{(1)}$, the right state $v_{\mathrm{L}}^{(2)}$ is connected to the left state v_{L} by $R_1^{\Delta}(z_{\mathrm{L}}^{(1)})(v_{\mathrm{L}})$. On the other hand, from the argument of the case (A), we deduce that

$$|w(v_{\mathrm{L}}^{(2)}) - w(v_{\mathrm{L}}^{(1)})| = O(\Delta x), \;\; |\rho_{\mathrm{L}}^{(2)} - \rho_{\mathrm{L}}^{(1)}| = (\rho_{\mathrm{L}}^{(1)})^{1-\theta} O(\Delta x) \tag{B.2}$$

(from (A.10) and (A.11) with $i = p$)

and

$$w(v_{\mathrm{L}}^{(2)}) \leq M_+ + \varepsilon + o(\Delta x). \tag{B.3}$$

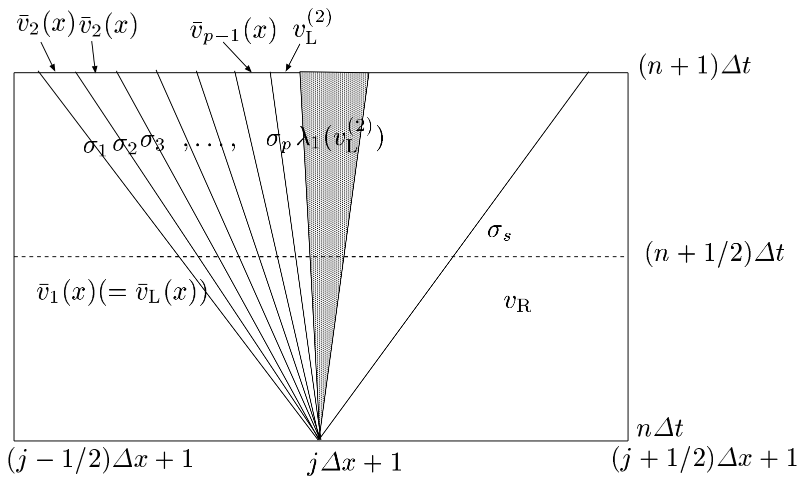

FIGURE 8. **Case 1.1 (ii)–(a)**: The approximate solution $\bar{v}^{\Delta}$ in the cell.

Since $\alpha < 1$, from $(\mathrm{B}.2)_1$ and the assumption of Case 1.1 (ii), a 1-shock does not arise for a Riemann problem $(v_{\mathrm{L}}^{(2)}, v_{\mathrm{R}})$. We then solve the Riemann problem $(v_{\mathrm{L}}^{(2)}, v_{\mathrm{R}})$.

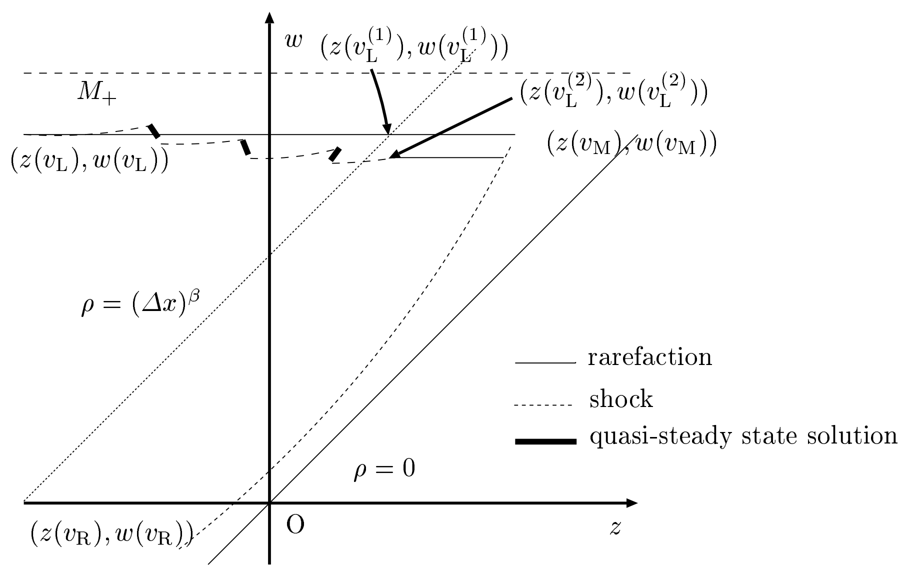

FIGURE 9. **Case 1.1 (ii)–(a)**: The approximate solution $\bar{v}^{\Delta}$ in the case where a 1-rarefaction and a 2-shock arise in (z, w)-plane.

Let σ_p be the propagation speed of the first 1-rarefaction shock on the right in $R_1^\Delta(z_{\rm L}^{(1)})(v_{\rm L})$ (see Figure 8). Then, notice that from (2.4) $\sigma_p \le \lambda_1(v_{\rm L}^{(2)})$. Therefore, we can connect this Riemann solution and $R_1^\Delta(z_{\rm L}^{(1)})(v_{\rm L})$.

From Lemma 2.2, (B.1) and (B.2)$_2$, recalling that $\rho_{\rm L}^{(1)} = (\Delta x)^\beta$, this Riemann solution $v^R(x,t)$ satisfies $\rho^R(x,t) < 3(\Delta x)^\beta/2$,

$$z(v^R(x,t)) \ge L_j, \quad w(v^R(x,t)) \le M_+ + \varepsilon + o(\Delta x). \tag{B.4}$$

Therefore, from the construction of $\bar{v}^\Delta(x,t)$ and (B.4), we have (4.1).

(ii)–(b) $z(v_{\rm L}^{(1)}) < L_j$

In this case, since $\max\{z_{\rm L}^{(1)}, L_j\} = L_j$, the right state $v_{\rm L}^{(2)}$ is connected to the left state $v_{\rm L}$ by $R_1^\Delta(L_j)(v_{\rm L})$ and $z(v_{\rm L}^{(2)}) = L_j$. Now, the definition of $v_{\rm L} = v_j^n$ implies $z(v_{\rm L}) \ge -M_-\{(j-1/2)\Delta x + 1\}^{-\frac{2(\gamma-1)}{\gamma+1}}$. On the other hand, from the definition of $v_{\rm L}^{(1)}$ and the assumption of Case 1.1, we find $z(v_{\rm L}^{(1)}) \ge z(v_{\rm L})$. Observing that $z(v_{\rm L}^{(2)}) = L_j > z(v_{\rm L}^{(1)})$, we have $|z(v_{\rm L}^{(2)}) - z(v_{\rm L}^{(1)})| = O(\Delta x)$. Therefore, since $\rho_{\rm L}^{(1)} = (\Delta x)^\beta$, from (A.10) with $i = p$, we thus have $(\Delta x)^\beta/2 < \rho_{\rm L}^{(2)} < 3(\Delta x)^\beta/2$.

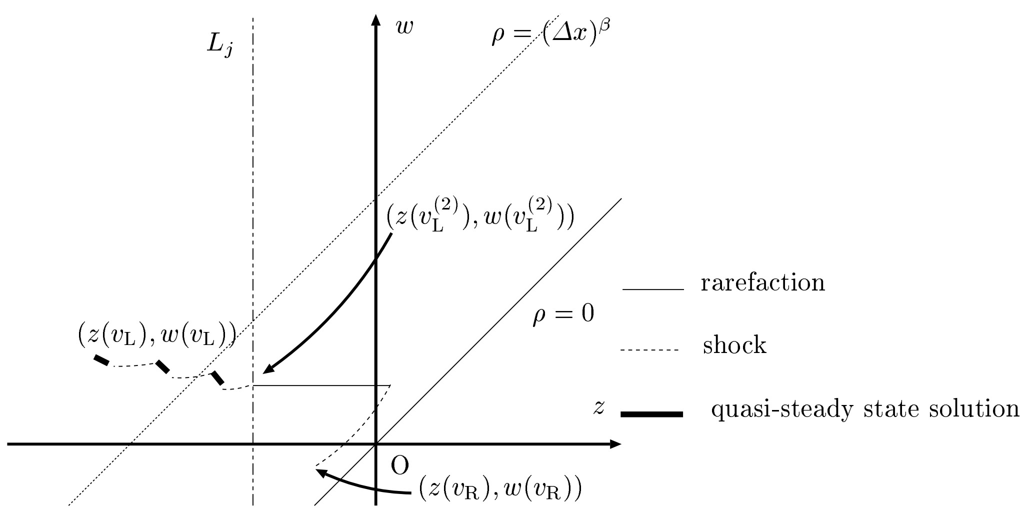

FIGURE 10. **Case 1.1 (ii)–(b)**: The approximate solution $\bar{v}^\Delta$ in (z,w)-plane.

We then solve a Riemann problem $(v_{\rm L}^{(2)}, v_{\rm R})$. Since $z(v_{\rm R}) \ge L_j$, a 1-shock does not arise.

We next estimate this Riemann solution. We deduce from the argument of the case (A) and the definition of $v_{\rm L}^{(2)}$ that $w(v_{\rm L}^{(2)}) \le M_+ + \varepsilon + o(\Delta x)$. From Lemma 2.2, the Riemann solution $v^R(x,t)$ satisfies $\rho^R(x,t) < 3(\Delta x)^\beta/2$,

$$z(v^R(x,t)) \ge L_j, \quad w(v^R(x,t)) \le M_+ + \varepsilon + o(\Delta x). \tag{B.5}$$

Therefore, from the construction of $\bar{v}^{\Delta}(x,t)$ and (B.5), we have (4.1).

Case 1.2 $\rho_{\mathrm{L}} < (\Delta x)^{\beta}$

(i) $z(v_{\mathrm{L}}) \geq L_j$
In this case, from (B.1), we have (4.1).

(ii) $z(v_{\mathrm{L}}) < L_j$
We solve a Riemann problem $(v_{\mathrm{L}}^{(3)}, v_{\mathrm{R}})$. Since $z(v_{\mathrm{R}}) \geq L_j$, a 1-shock does not arise. On the other hand, from the definition of $v_{\mathrm{L}}^{(3)}$, we find $w(v_{\mathrm{L}}^{(3)}) \leq M_+ + \varepsilon$. From (B.1), we then have (4.1).

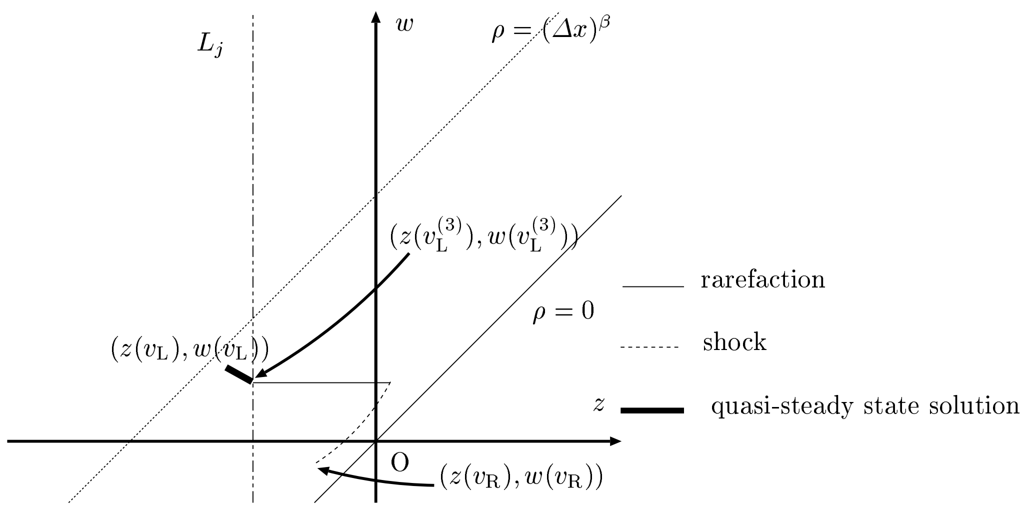

FIGURE 11. **Case 1.2 (ii)**: The approximate solution $\bar{v}^{\Delta}$ in (z, w)-plane.

Case 2.1
In this case, notice that $\rho_{\mathrm{L}} < (\Delta x)^{\beta}$ and $z(v_{\mathrm{L}}) \geq z(v_{\mathrm{R}})$. The definition of v_j^n $(= v_{\mathrm{L}})$ and v_{j+1}^n $(= v_{\mathrm{R}})$ thus implies

$$L_j \leq z(v_{\mathrm{R}}) \leq z(v_{\mathrm{L}}),\ w(v_{\mathrm{L}}) \leq M_+ + \varepsilon \text{ and } w(v_{\mathrm{R}}) \leq M_+ + \varepsilon. \tag{B.6}$$

(ii) $w(v_{\mathrm{R}}^{(1)}) - w(v_{\mathrm{M}}) > (\Delta x)^{\alpha}$ and $\rho_{\mathrm{R}} \geq (\Delta x)^{\beta}$

Then, notice that $|z(v_{\mathrm{R}}^{(2)}) - z(v_{\mathrm{R}}^{(1)})| = O(\Delta x)$. Since $\alpha < 1$, from the assumption of Case 2.1 (ii), a 2-shock does not arise for a Riemann problem $(v_{\mathrm{L}}, v_{\mathrm{R}}^{(2)})$. The estimate of this Riemann solution can be found in Section 4.

(iii) $\rho_{\mathrm{R}} < (\Delta x)^{\beta}$
In this case, from Lemma 2.2 and (B.6), we conclude that $L_j \leq z(v^{\Delta}(x,t))$ and $w(v^{\Delta}(x,t)) \leq M_+ + \varepsilon$.

Case 3.2 $w(v_{\mathrm{R}}^{(1)}) - w(v_{\mathrm{M}}) > (\Delta x)^{\alpha}$ and $\rho_{\mathrm{R}} \geq (\Delta x)^{\beta}$

(i) $\rho_{\mathrm{L}} \geq (\Delta x)^{\beta}$
We separately consider two cases.

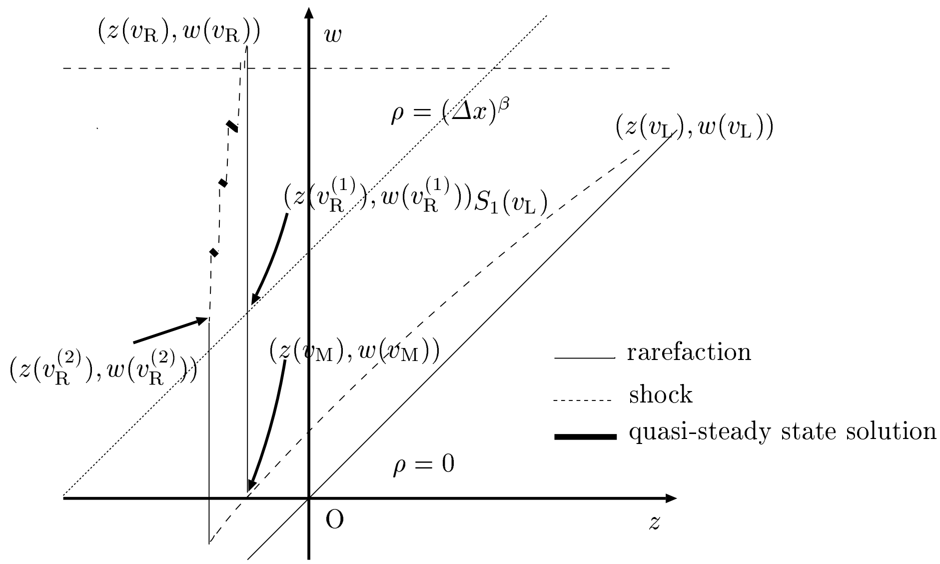

FIGURE 12. **Case 2.1** (ii): The approximate solution $\bar{v}^\Delta$ in the case where a 1-shock and a 2-rarefaction wave arise in (z, w)-plane.

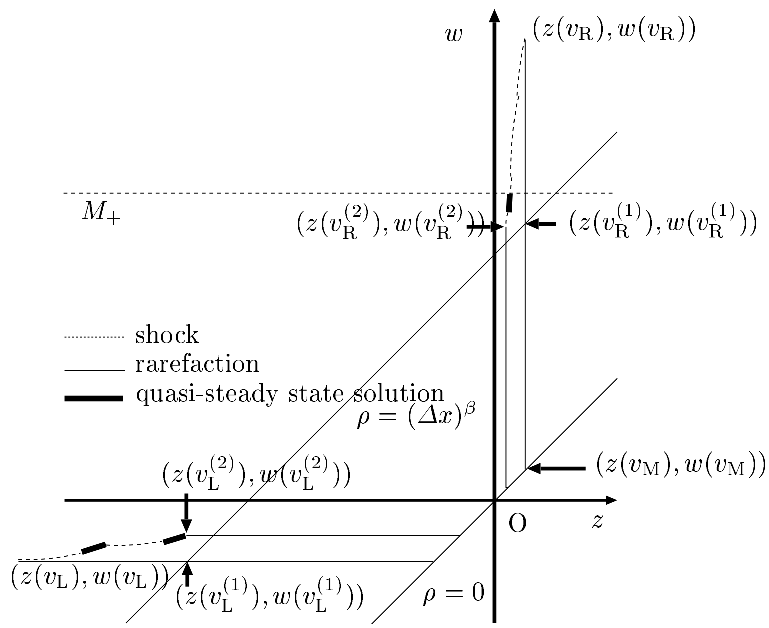

FIGURE 13. **Case 3.2 (i)–(a)**: The approximate solution $\bar{v}^\Delta$ in (z, w)-plane.

(i)–(a) $z(v_{\rm L}^{(1)})\geq L_j$
Since $|\rho_{\rm L}^{(1)}-\rho_{\rm L}^{(2)}|=(\rho_{\rm L}^{(1)})^{1-\theta}O(\varDelta x)$ (recall (A.11) with $i=p$), $|w_{\rm L}^{(1)}-w_{\rm L}^{(2)}|=O(\varDelta x)$, $|z_{\rm R}^{(1)}-z_{\rm R}^{(2)}|=O(\varDelta x)$, $|\rho_{\rm R}^{(1)}-\rho_{\rm R}^{(2)}|=(\rho_{\rm R}^{(1)})^{1-\theta}O(\varDelta x)$, $\rho_{\rm L}^{(1)}=\rho_{\rm R}^{(1)}$ and $\alpha<1$, we have

$$\begin{aligned}w(v_{\rm R}^{(2)})-w(v_{\rm L}^{(2)})&\geq w(v_{\rm R}^{(1)})-w(v_{\rm L}^{(1)})-O(\varDelta x)\\&\geq w(v_{\rm M})-w(v_{\rm L})+(\varDelta x)^{\alpha}-O(\varDelta x)\\&\quad\text{(from the assumption of Case 3.2)}\\&\geq(\varDelta x)^{\alpha}-O(\varDelta x)>(\varDelta x)^{\alpha}/2.\end{aligned}\tag{B.7}$$

$$\begin{aligned}z(w_{\rm R}^{(2)})-z(v_{\rm L}^{(2)})&=w(v_{\rm R}^{(2)})-w(v_{\rm L}^{(2)})-2(\rho_{\rm R}^{(2)})^{\theta}/\theta+2(\rho_{\rm L}^{(2)})^{\theta}/\theta\\&\geq w(v_{\rm R}^{(1)})-w(v_{\rm L}^{(1)})-O(\varDelta x)\\&\geq w(v_{\rm M})-w(v_{\rm L})+(\varDelta x)^{\alpha}-O(\varDelta x)\\&\geq(\varDelta x)^{\alpha}-O(\varDelta x)>(\varDelta x)^{\alpha}/2.\end{aligned}\tag{B.8}$$

Here, since

$$|1-\rho_{\rm L}^{(2)}/\rho_{\rm L}^{(1)}|=(\rho_{\rm L}^{(1)})^{-\theta}O(\varDelta x)=o(1)\quad\text{(from the fact that }\rho_{\rm L}^{(1)}=(\varDelta x)^{\beta}),$$

notice that

$$\begin{aligned}(\rho_{\rm L}^{(2)})^{\theta}/\theta&=(\rho_{\rm L}^{(1)})^{\theta}/\theta+\left\{\tau\rho_{\rm L}^{(1)}+(1-\tau)\rho_{\rm L}^{(2)}\right\}^{\theta-1}(\rho_{\rm L}^{(2)}-\rho_{\rm L}^{(1)})\\&=(\rho_{\rm L}^{(1)})^{\theta}/\theta+\left\{\tau\rho_{\rm L}^{(1)}+(1-\tau)\rho_{\rm L}^{(2)}\right\}^{\theta-1}(\rho_{\rm L}^{(1)})^{1-\theta}O(\varDelta x)\\&=(\rho_{\rm L}^{(1)})^{\theta}/\theta+\left\{\tau+(1-\tau)\rho_{\rm L}^{(2)}/\rho_{\rm L}^{(1)}\right\}^{\theta-1}O(\varDelta x)\\&=(\rho_{\rm L}^{(1)})^{\theta}/\theta+O(\varDelta x),\end{aligned}\tag{B.9}$$

where $0<\tau<1$. Similarly notice that

$$(\rho_{\rm R}^{(2)})^{\theta}/\theta=(\rho_{\rm R}^{(1)})^{\theta}/\theta+O(\varDelta x).\tag{B.10}$$

We solve a Riemann problem $(v_{\rm L}^{(2)},v_{\rm R}^{(2)})$. From (B.7) and (B.8), a 1-rarefaction wave and a 2-rarefaction wave arise. Then estimate of this Riemann solution can be checked by using (B.8)– (B.10).

(i)–(b) $z(v_{\rm L}^{(1)})<L_j$
In this case, since $\max\{z_{\rm L}^{(1)},L_j\}=L_j$, the right state $v_{\rm L}^{(2)}$ is connected to the left state $v_{\rm L}$ by $R_1^{\varDelta}(L_j)(v_{\rm L})$ and $z(v_{\rm L}^{(2)})=L_j$. We then obtain $(\varDelta x)^{\beta}/2<\rho_{\rm L}^{(2)}<3(\varDelta x)^{\beta}/2$, $|z_{\rm L}^{(2)}-z_{\rm L}^{(1)}|=O(\varDelta x)$ and $|w_{\rm L}^{(2)}-w_{\rm L}^{(1)}|=O(\varDelta x)$ in a similar fashion to Case 1.1 (ii)–(b). From (B.7), we have

$$\begin{aligned}w(v_{\rm R}^{(2)})-w(v_{\rm L}^{(2)})&\geq w(v_{\rm R}^{(1)})-w(v_{\rm L}^{(1)})-O(\varDelta x)>0,\\z(v_{\rm R}^{(2)})-z(v_{\rm L}^{(2)})&\geq z(v_{\rm R}^{(1)})-z(v_{\rm L}^{(1)})-O(\varDelta x)\\&=w(v_{\rm R}^{(1)})-w(v_{\rm L}^{(1)})-O(\varDelta x)>0.\end{aligned}$$

Therefore, a 1-rarefaction and a 2-rarefaction arise. The estimate of this Riemann solution can be obtained easily.

(ii) $\rho_{\mathrm{L}} < (\Delta x)^{\beta}$

In this case, from the assumption of Case 3.2, we have

$$\begin{aligned} w(v_{\mathrm{R}}^{(2)}) - w(v_{\mathrm{L}}) &\geq w(v_{\mathrm{R}}^{(2)}) - w(v_{\mathrm{M}}) \geq w(v_{\mathrm{R}}^{(1)}) - w(v_{\mathrm{M}}) - O(\Delta x) \\ &\geq (\Delta x)^{\alpha} - O(\Delta x) > (\Delta x)^{\alpha}/2. \end{aligned} \tag{B.11}$$

(ii)–(a) $z(v_{\mathrm{L}}) \geq \min\left\{z(v_{\mathrm{R}}^{(2)}), L_j\right\}$

We solve a Riemann problem $(v_{\mathrm{L}}, v_{\mathrm{R}}^{(2)})$. From (B.11), a 2-shock does not arise. The estimate of this Riemann solution is similar to Case 2.1.

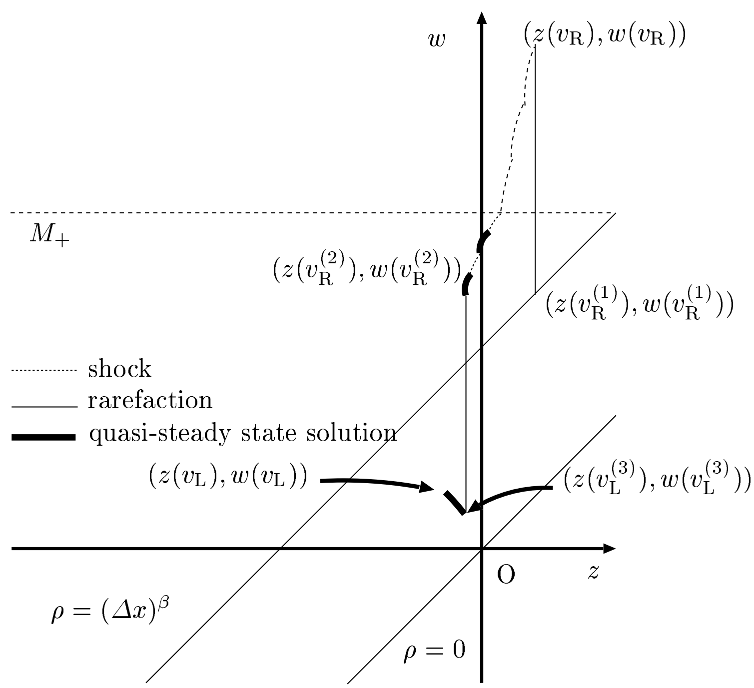

FIGURE 14. **Case 3.2 (ii)–(b)**: The approximate solution $\bar{v}^{\Delta}$ in (z, w)-plane.

(ii)–(b) $z(v_{\mathrm{L}}) < \min\left\{z(v_{\mathrm{R}}^{(2)}), L_j\right\}$

We solve a Riemann problem $(v_{\mathrm{L}}^{(3)}, v_{\mathrm{R}}^{(2)})$. Then, since $|w(v_{\mathrm{L}}^{(3)}) - w(v_{\mathrm{L}})| = O(\Delta x)$, from (B.11), a 2-shock does not arise. The estimate of this Riemann solution is similar to Case 2.1.

Appendix C. Proof of Lemma 6.1

Proof. Set

$$\rho(x,t) := \tilde{\rho}(x,t)x^{-\frac{4}{\gamma+1}},\ m(x,t) := \tilde{m}(x,t)x^{-2},\ x_{\mathrm{M}} := (j-1/2)\Delta x + 1,$$

$$E_j^t(\rho) := \frac{1}{\Delta x}\int_{(j-1)\Delta x+1}^{j\Delta x+1} \tilde{\rho}(x,t)x^{-\frac{4}{\gamma+1}}dx,\quad E_j^t(m) := \frac{1}{\Delta x}\int_{(j-1)\Delta x+1}^{j\Delta x+1} \tilde{m}(x,t)x^{-2}dx.$$

We then deduce from Lemma 2.2 that

$$w(E_j^t(\rho), E_j^t(m)) \le M_+ + \varepsilon + o(\Delta x). \tag{C.1}$$

Therefore our goal in this lemma is to prove

$$z(E_j^t(\rho), E_j^t(m)) \ge -M_- x_{\mathrm{M}}^{-\frac{2(\gamma-1)}{\gamma+1}} - o(\Delta x),$$

where

$$\begin{aligned}
&z(E_j^t(\rho), E_j^t(m)) \\
&= E_j^t(m)/E_j^t(\rho) - \{E_j^t(\rho)\}^\theta/\theta \\
&= \frac{\dfrac{1}{\Delta x}\displaystyle\int_{(j-1)\Delta x+1}^{j\Delta x+1} \tilde{m}(x,t)x^{-2}dx - \left(\dfrac{1}{\Delta x}\displaystyle\int_{(j-1)\Delta x+1}^{j\Delta x+1} \tilde{\rho}(x,t)x^{-\frac{4}{\gamma+1}}dx\right)^{\theta+1}/\theta}{\dfrac{1}{\Delta x}\displaystyle\int_{(j-1)\Delta x+1}^{j\Delta x+1} \tilde{\rho}(x,t)x^{-\frac{4}{\gamma+1}}dx}
\end{aligned} \tag{C.2}$$

and Landau's symbol $o(\Delta x)$ depends only on M_+ and M_-.

Step 1.

We find

$$\begin{aligned}
E_j^t(\rho) &= \frac{1}{\Delta x}\int_{(j-1)\Delta x+1}^{j\Delta x+1} \tilde{\rho}(x,t)x^{-2}x^{\frac{2(\gamma-1)}{\gamma+1}}dx \\
&= x_{\mathrm{M}}^{\frac{2(\gamma-1)}{\gamma+1}} \frac{1}{\Delta x}\int_{(j-1)\Delta x+1}^{j\Delta x+1} \tilde{\rho}(x,t)x^{-2}dx \\
&\quad + \frac{2(\gamma-1)}{\gamma+1} x_{\mathrm{M}}^{\frac{2(\gamma-1)}{\gamma+1}-1} \frac{1}{\Delta x}\int_{(j-1)\Delta x+1}^{j\Delta x+1} (x-x_{\mathrm{M}})\tilde{\rho}(x,t)x^{-2}dx \\
&\quad + \frac{(\gamma-1)}{\gamma+1}\left(\frac{2(\gamma-1)}{\gamma+1}-1\right) \frac{1}{\Delta x}\int_{(j-1)\Delta x+1}^{j\Delta x+1} (x-x_{\mathrm{M}})^2\tilde{\rho}(x,t)x^{-2} \\
&\quad \times \{x_{\mathrm{M}} + \tau_1(x)(x-x_{\mathrm{M}})\}^{\frac{2(\gamma-1)}{\gamma+1}-2}dx \\
&= x_{\mathrm{M}}^{\frac{2(\gamma-1)}{\gamma+1}} \frac{1}{\Delta x}\int_{(j-1)\Delta x+1}^{j\Delta x+1} \tilde{\rho}(x,t)x^{-2}dx \\
&\quad + \frac{2(\gamma-1)}{\gamma+1} x_{\mathrm{M}}^{-\frac{4}{\gamma+1}-1} \frac{1}{\Delta x}\int_{(j-1)\Delta x+1}^{j\Delta x+1} (x-x_{\mathrm{M}})\tilde{\rho}(x,t)dx \\
&\quad + O((\Delta x)^2)\frac{1}{\Delta x}\int_{(j-1)\Delta x+1}^{j\Delta x+1} \tilde{\rho}(x,t)x^{-\frac{4}{\gamma+1}}dx \\
&\qquad \text{(taking Taylor's expansion of } x^{-2} \text{ at } x_{\mathrm{M}}),
\end{aligned}$$

where $0 < \tau_1(x) < 1$.

Substituting the equation above for (C.2), we obtain

$$
\begin{aligned}
&z(E_j^t(\rho), E_j^t(m)) \\
&= \frac{\frac{1}{\Delta x}\int_{(j-1)\Delta x+1}^{j\Delta x+1} \tilde{m}(x,t)x^{-2}dx - \left(\frac{1}{\Delta x}\int_{(j-1)\Delta x+1}^{j\Delta x+1} \tilde{\rho}(x,t)x^{-\frac{4}{\gamma+1}}dx\right)^{\theta+1}/\theta}{x_{\mathrm{M}}^{\frac{2(\gamma-1)}{\gamma+1}}\frac{1}{\Delta x}\int_{(j-1)\Delta x+1}^{j\Delta x+1} \tilde{\rho}(x,t)x^{-2}dx} \\
&\quad - \frac{\frac{1}{\Delta x}\int_{(j-1)\Delta x+1}^{j\Delta x+1} \tilde{m}(x,t)x^{-2}dx - \left(\frac{1}{\Delta x}\int_{(j-1)\Delta x+1}^{j\Delta x+1} \tilde{\rho}(x,t)x^{-\frac{4}{\gamma+1}}dx\right)^{\theta+1}/\theta}{\left(x_{\mathrm{M}}^{\frac{2(\gamma-1)}{\gamma+1}}\frac{1}{\Delta x}\int_{(j-1)\Delta x+1}^{j\Delta x+1} \tilde{\rho}(x,t)x^{-2}dx\right)^2} \\
&\quad \times \frac{2(\gamma-1)}{\gamma+1}x_{\mathrm{M}}^{-\frac{4}{\gamma+1}-1}\frac{1}{\Delta x}\int_{(j-1)\Delta x+1}^{j\Delta x+1}(x-x_{\mathrm{M}})\tilde{\rho}(x,t)dx + O((\Delta x)^2).
\end{aligned} \tag{C.3}
$$

Set

$$
\begin{aligned}
\mu := &\frac{2(\gamma-1)}{(\theta+1)(\gamma+1)}\frac{1}{\left(\frac{1}{\Delta x}\int_{(j-1)\Delta x+1}^{j\Delta x+1} \tilde{\rho}(x,t)x^{-\frac{4}{\gamma+1}}dx\right)^{\theta}} \\
&\times \frac{\frac{1}{\Delta x}\int_{(j-1)\Delta x+1}^{j\Delta x+1} \tilde{m}(x,t)x^{-2}dx - \left(\frac{1}{\Delta x}\int_{(j-1)\Delta x+1}^{j\Delta x+1} \tilde{\rho}(x,t)x^{-\frac{4}{\gamma+1}}dx\right)^{\theta+1}/\theta}{x_{\mathrm{M}}^{\frac{2(\gamma-1)}{\gamma+1}}\frac{1}{\Delta x}\int_{(j-1)\Delta x+1}^{j\Delta x+1} \tilde{\rho}(x,t)x^{-2}dx}.
\end{aligned} \tag{C.4}
$$

Then assume that the following holds.

$$
\begin{aligned}
(E_j^t(\rho))^{\theta+1} \leq &\frac{1}{\Delta x}\int_{(j-1)\Delta x+1}^{j\Delta x+1}(\tilde{\rho}(x,t))^{\theta+1}x^{-2}dx \\
&- \mu(\theta+1)\left(\frac{1}{\Delta x}\int_{(j-1)\Delta x+1}^{j\Delta x+1} \tilde{\rho}(x,t)x^{-\frac{4}{\gamma+1}}dx\right)^{\theta} \\
&\times x_{\mathrm{M}}^{-\frac{4}{\gamma+1}-1}\frac{1}{\Delta x}\int_{(j-1)\Delta x+1}^{j\Delta x+1}(x-x_{\mathrm{M}})\tilde{\rho}(x,t)dx \\
&+ o(\Delta x)\frac{1}{\Delta x}\int_{(j-1)\Delta x+1}^{j\Delta x+1} \tilde{\rho}(x,t)x^{-\frac{4}{\gamma+1}}dx.
\end{aligned} \tag{C.5}
$$

This estimate shall be proved in step 2–4. Then, substituting (C.5) for (C.3), we conclude that

$$
\begin{aligned}
z(E_j^t(\rho), E_j^t(m)) &\geq \frac{\dfrac{1}{\Delta x}\displaystyle\int_{(j-1)\Delta x+1}^{j\Delta x+1}\tilde{\rho}(x,t)x^{-2}[\tilde{u}(x,t)-\{\tilde{\rho}(x,t)\}^{\theta}/\theta]dx}{x_{\mathrm{M}}^{\frac{2(\gamma-1)}{\gamma+1}}\dfrac{1}{\Delta x}\displaystyle\int_{(j-1)\Delta x+1}^{j\Delta x+1}\tilde{\rho}(x,t)x^{-2}dx}-o(\Delta x)\\
&\geq -M_{-}x_{\mathrm{M}}^{-\frac{2(\gamma-1)}{\gamma+1}}-o(\Delta x).
\end{aligned}
$$

Therefore we must prove (C.5). Separating three steps, we derive this estimate.

Step 2.

First, let us consider μ in (C.4). We prove that

$$
|\mu| \leq C(\Delta x)^{-\theta\delta},
$$

where C depends only on M_{+} and M_{-}.

It suffices to derive the bound of

$$
B := \frac{\dfrac{1}{\Delta x}\displaystyle\int_{(j-1)\Delta x+1}^{j\Delta x+1}\tilde{m}(x,t)x^{-2}dx-\left(\dfrac{1}{\Delta x}\displaystyle\int_{(j-1)\Delta x+1}^{j\Delta x+1}\tilde{\rho}(x,t)x^{-\frac{4}{\gamma+1}}dx\right)^{\theta+1}/\theta}{x_{\mathrm{M}}^{\frac{2(\gamma-1)}{\gamma+1}}\dfrac{1}{\Delta x}\displaystyle\int_{(j-1)\Delta x+1}^{j\Delta x+1}\tilde{\rho}(x,t)x^{-2}dx}.
$$

The upper bound:

$$
\begin{aligned}
B &\leq \frac{\dfrac{1}{\Delta x}\displaystyle\int_{(j-1)\Delta x+1}^{j\Delta x+1}\tilde{m}(x,t)x^{-2}dx}{x_{\mathrm{M}}^{\frac{2(\gamma-1)}{\gamma+1}}(j\Delta x+1)^{-\frac{2(\gamma-1)}{\gamma+1}}\dfrac{1}{\Delta x}\displaystyle\int_{(j-1)\Delta x+1}^{j\Delta x+1}\tilde{\rho}(x,t)x^{-\frac{4}{\gamma+1}}dx}\\
&\leq x_{\mathrm{M}}^{-\frac{2(\gamma-1)}{\gamma+1}}(j\Delta x+1)^{\frac{2(\gamma-1)}{\gamma+1}}E_j^t(m)/E_j^t(\rho)\\
&\leq x_{\mathrm{M}}^{-\frac{2(\gamma-1)}{\gamma+1}}(j\Delta x+1)^{\frac{2(\gamma-1)}{\gamma+1}}w(E_j^t(\rho),E_j^t(m))\\
&\leq x_{\mathrm{M}}^{-\frac{2(\gamma-1)}{\gamma+1}}(j\Delta x+1)^{\frac{2(\gamma-1)}{\gamma+1}}M_{+}\quad\text{(from Lemma 2.2)}.
\end{aligned}
$$

Here choosing Δx small enough such that $\Delta x \leq 2$, we have

$$
\begin{aligned}
x_{\mathrm{M}}^{-\frac{2(\gamma-1)}{\gamma+1}}(j\Delta x+1)^{\frac{2(\gamma-1)}{\gamma+1}} &= \{(j-1/2)\Delta x+1\}^{-\frac{2(\gamma-1)}{\gamma+1}}(j\Delta x+1)^{\frac{2(\gamma-1)}{\gamma+1}}\\
&= \left(1+\frac{\frac{1}{2}\Delta x}{(j-1/2)\Delta x+1}\right)^{\frac{2(\gamma-1)}{\gamma+1}} \leq 2^{\frac{2(\gamma-1)}{\gamma+1}}. \qquad \text{(C.6)}
\end{aligned}
$$

The upper bound follows from the inequalities above.

The lower bound:
using Jensen's inequality, we have

$$
\begin{aligned}
B &\geq \frac{\dfrac{1}{\Delta x}\displaystyle\int_{(j-1)\Delta x+1}^{j\Delta x+1} \tilde{m}(x,t)x^{-2}dx - \dfrac{1}{\Delta x}\displaystyle\int_{(j-1)\Delta x+1}^{j\Delta x+1} (\tilde{\rho}(x,t))^{\theta+1}/\theta x^{-2}dx}{x_{\mathrm{M}}^{\frac{2(\gamma-1)}{\gamma+1}}\dfrac{1}{\Delta x}\displaystyle\int_{(j-1)\Delta x+1}^{j\Delta x+1} \tilde{\rho}(x,t)x^{-2}dx} \\
&\geq \frac{\dfrac{1}{\Delta x}\displaystyle\int_{(j-1)\Delta x+1}^{j\Delta x+1} \tilde{\rho}(x,t)x^{-2}x^{\frac{2(\gamma-1)}{\gamma+1}}x_{\mathrm{M}}^{-\frac{2(\gamma-1)}{\gamma+1}}[\tilde{u}(x,t)-\{\tilde{\rho}(x,t)\}^{\theta}/\theta]x^{-\frac{2(\gamma-1)}{\gamma+1}}dx}{\dfrac{1}{\Delta x}\displaystyle\int_{(j-1)\Delta x+1}^{j\Delta x+1} \tilde{\rho}(x,t)x^{-2}dx} \\
&\geq -M_{-}x_{\mathrm{M}}^{-\frac{2(\gamma-1)}{\gamma+1}} - o(\Delta x) \quad \text{(from the assumption of this lemma and (C.6)).}
\end{aligned}
$$

Step 3.
In this step, we derive

$$
E_j^t(\rho) \leq C_1,\ |E_j^t(m)/E_j^t(\rho)| \leq C_2, \tag{C.7}
$$

where C_1 and C_2 depend only on M_+ and M_-.
When $E_j^t(m) \leq 0$, from Jensen's inequality, we have

$$
\begin{aligned}
&z(E_j^t(\rho),E_j^t(m)) = E_j^t(m)/E_j^t(\rho) - \{E_j^t(\rho)\}^{\theta}/\theta \\
&= \frac{\dfrac{1}{\Delta x}\displaystyle\int_{(j-1)\Delta x+1}^{j\Delta x+1} \tilde{m}(x,t)x^{-2}dx - \left(\dfrac{1}{\Delta x}\displaystyle\int_{(j-1)\Delta x+1}^{j\Delta x+1} \tilde{\rho}(x,t)x^{-\frac{4}{\gamma+1}}dx\right)^{\theta+1}/\theta}{\dfrac{1}{\Delta x}\displaystyle\int_{(j-1)\Delta x+1}^{j\Delta x+1} \tilde{\rho}(x,t)x^{-\frac{4}{\gamma+1}}dx} \\
&\geq \frac{\dfrac{1}{\Delta x}\displaystyle\int_{(j-1)\Delta x+1}^{j\Delta x+1} \tilde{\rho}(x,t)x^{-\frac{4}{\gamma+1}}[\tilde{u}(x,t)-\{\tilde{\rho}(x,t)\}^{\theta}/\theta]x^{-\frac{2(\gamma-1)}{\gamma+1}}dx}{\dfrac{1}{\Delta x}\displaystyle\int_{(j-1)\Delta x+1}^{j\Delta x+1} \tilde{\rho}(x,t)x^{-\frac{4}{\gamma+1}}dx} \\
&\geq -M_{-} - o(\Delta x) \quad \text{(from the argument of Step 2).}
\end{aligned} \tag{C.8}
$$

From (C.1), (C.8) and Remark 1.2, we have the bound of $w(E_j^t(\rho),E_j^t(m))$ and $z(E_j^t(\rho),E_j^t(m))$. Therefore, we conclude (C.7).
Step 4.
We next consider the first equation of (C.3):

$$
\left(\frac{1}{\Delta x}\int_{(j-1)\Delta x+1}^{j\Delta x+1} \tilde{\rho}(x,t)x^{-\frac{4}{\gamma+1}}dx\right)^{\theta+1}.
$$

We first find

$$
\begin{aligned}
E_j^t(\rho)&=\frac{1}{\Delta x}\int_{(j-1)\Delta x+1}^{j\Delta x+1}\tilde{\rho}(x,t)x^{\mu-\frac{4}{\gamma+1}}x^{-\mu}dx\\
&=x_{\mathrm{M}}^{-\mu}\frac{1}{\Delta x}\int_{(j-1)\Delta x+1}^{j\Delta x+1}\tilde{\rho}(x,t)x^{\mu-\frac{4}{\gamma+1}}dx\\
&\quad-\mu x_{\mathrm{M}}^{-\mu-1}\frac{1}{\Delta x}\int_{(j-1)\Delta x+1}^{j\Delta x+1}(x-x_{\mathrm{M}})\tilde{\rho}(x,t)x^{\mu-\frac{4}{\gamma+1}}dx\\
&\quad+\frac{\mu(\mu+1)}{2}\frac{1}{\Delta x}\int_{(j-1)\Delta x+1}^{j\Delta x+1}(x-x_{\mathrm{M}})^2\tilde{\rho}(x,t)x^{\mu-\frac{4}{\gamma+1}}\\
&\quad\times\{x_{\mathrm{M}}+\tau_2(x)(x-x_{\mathrm{M}})\}^{-\mu-2}dx\\
&:=I_0-I_1+I_2,
\end{aligned}
$$

where $0<\tau_2(x)<1$.

We next estimate I_1 and I_2 as follows:

$$
\begin{aligned}
|I_2|&\leq C|\mu(\mu+1)|(\Delta x)^2\frac{1}{\Delta x}\int_{(j-1)\Delta x+1}^{j\Delta x+1}\tilde{\rho}(x,t)x^{-\frac{4}{\gamma+1}}\left(\frac{j\Delta x+1}{(j-1)\Delta x+1}\right)^{|\mu|}dx\\
&\leq C|\mu(\mu+1)|(\Delta x)^2\frac{1}{\Delta x}\int_{(j-1)\Delta x+1}^{j\Delta x+1}\tilde{\rho}(x,t)x^{-\frac{4}{\gamma+1}}\left(1+\frac{\Delta x}{(j-1)\Delta x+1}\right)^{|\mu|}dx\\
&=o(\Delta x)\frac{1}{\Delta x}\int_{(j-1)\Delta x+1}^{j\Delta x+1}\tilde{\rho}(x,t)x^{-\frac{4}{\gamma+1}}dx\quad\text{(from Step 2)},\\
I_1&=\mu x_{\mathrm{M}}^{-\frac{4}{\gamma+1}-1}\frac{1}{\Delta x}\int_{(j-1)\Delta x+1}^{j\Delta x+1}(x-x_{\mathrm{M}})\tilde{\rho}(x,t)dx\\
&\quad+\left(\mu-\frac{4}{\gamma+1}\right)\mu x_{\mathrm{M}}^{-\mu-1}\frac{1}{\Delta x}\int_{(j-1)\Delta x+1}^{j\Delta x+1}(x-x_{\mathrm{M}})^2\tilde{\rho}(x,t)\\
&\quad\times\{x_{\mathrm{M}}+\tau_3(x)(x-x_{\mathrm{M}})\}^{\mu-\frac{4}{\gamma+1}-1}dx\\
&\quad\text{(taking Taylor's expansion of } x^{\mu-\frac{4}{\gamma+1}}\text{ at } x_{\mathrm{M}})\\
&:=J_1+J_2,\\
|J_2|&\leq C\left|\left(\mu-\frac{4}{\gamma+1}\right)\mu\right|(\Delta x)^2\\
&\quad\times\frac{1}{\Delta x}\int_{(j-1)\Delta x+1}^{j\Delta x+1}\tilde{\rho}(x,t)\left(\frac{j\Delta x+1}{(j-1)\Delta x+1}\right)^{|\mu|}x^{-\frac{4}{\gamma+1}}dx\\
&=o(\Delta x)\frac{1}{\Delta x}\int_{(j-1)\Delta x+1}^{j\Delta x+1}\tilde{\rho}(x,t)x^{-\frac{4}{\gamma+1}}dx.
\end{aligned}
$$

Therefore, we have

$$\begin{aligned}E_j^t(\rho) &= x_{\mathrm{M}}^{-\mu}\frac{1}{\Delta x}\int_{(j-1)\Delta x+1}^{j\Delta x+1}\tilde{\rho}(x,t)x^{\mu-\frac{4}{\gamma+1}}dx \\ &\quad - \mu x_{\mathrm{M}}^{-\frac{4}{\gamma+1}-1}\frac{1}{\Delta x}\int_{(j-1)\Delta x+1}^{j\Delta x+1}(x-x_{\mathrm{M}})\tilde{\rho}(x,t)dx \\ &\quad + o(\Delta x)\frac{1}{\Delta x}\int_{(j-1)\Delta x+1}^{j\Delta x+1}\tilde{\rho}(x,t)x^{-\frac{4}{\gamma+1}}dx. \end{aligned} \tag{C.9}$$

On the other hand, we find

$$x_{\mathrm{M}}^{-\frac{4}{\gamma+1}}\frac{1}{\Delta x}\int_{(j-1)\Delta x+1}^{j\Delta x+1}(x-x_{\mathrm{M}})\tilde{\rho}(x,t)dx = O(\Delta x)\frac{1}{\Delta x}\int_{(j-1)\Delta x+1}^{j\Delta x+1}\tilde{\rho}(x,t)x^{-\frac{4}{\gamma+1}}dx. \tag{C.10}$$

From (C.9) and (C.10), we deduce that

$$\begin{aligned}(E_j^t(\rho))^{\theta+1} &= \Bigg(x_{\mathrm{M}}^{-\mu}\frac{1}{\Delta x}\int_{(j-1)\Delta x+1}^{j\Delta x+1}\tilde{\rho}(x,t)x^{\mu-\frac{4}{\gamma+1}}dx \\ &\quad -\mu x_{\mathrm{M}}^{-\frac{4}{\gamma+1}-1}\frac{1}{\Delta x}\int_{(j-1)\Delta x+1}^{j\Delta x+1}(x-x_{\mathrm{M}})\tilde{\rho}(x,t)dx\Bigg)^{\theta+1} \\ &\quad + o(\Delta x)\frac{1}{\Delta x}\int_{(j-1)\Delta x+1}^{j\Delta x+1}\tilde{\rho}(x,t)x^{-\frac{4}{\gamma+1}}dx \\ &= \left(x_{\mathrm{M}}^{-\mu}\frac{1}{\Delta x}\int_{(j-1)\Delta x+1}^{j\Delta x+1}\tilde{\rho}(x,t)x^{\mu-\frac{4}{\gamma+1}}dx\right)^{\theta+1} \\ &\quad + (\theta+1)\Bigg(x_{\mathrm{M}}^{-\mu}\frac{1}{\Delta x}\int_{(j-1)\Delta x+1}^{j\Delta x+1}\tilde{\rho}(x,t)x^{\mu-\frac{4}{\gamma+1}}dx \\ &\quad -\tau_4\mu x_{\mathrm{M}}^{-\frac{4}{\gamma+1}-1}\frac{1}{\Delta x}\int_{(j-1)\Delta x+1}^{j\Delta x+1}(x-x_{\mathrm{M}})\tilde{\rho}(x,t)dx\Bigg)^{\theta} \\ &\quad \times -\mu x_{\mathrm{M}}^{-\frac{4}{\gamma+1}-1}\frac{1}{\Delta x}\int_{(j-1)\Delta x+1}^{j\Delta x+1}(x-x_{\mathrm{M}})\tilde{\rho}(x,t)dx \\ &\quad + o(\Delta x)\frac{1}{\Delta x}\int_{(j-1)\Delta x+1}^{j\Delta x+1}\tilde{\rho}(x,t)x^{-\frac{4}{\gamma+1}}dx. \end{aligned} \tag{C.11}$$

Since

$$
\begin{aligned}
&x_{\mathrm{M}}^{-\mu}\frac{1}{\Delta x}\int_{(j-1)\Delta x+1}^{j\Delta x+1}\tilde{\rho}(x,t)x^{\mu-\frac{4}{\gamma+1}}dx\\
&=\frac{1}{\Delta x}\int_{(j-1)\Delta x+1}^{j\Delta x+1}\tilde{\rho}(x,t)x^{-\frac{4}{\gamma+1}}dx\\
&\quad+\mu x_{\mathrm{M}}^{-\mu}\frac{1}{\Delta x}\int_{(j-1)\Delta x+1}^{j\Delta x+1}(x-x_{\mathrm{M}})\tilde{\rho}(x,t)\{x_{\mathrm{M}}+\tau_5(x)(x-x_{\mathrm{M}})\}^{\mu-1}x^{\mu-\frac{4}{\gamma+1}}dx\\
&=\frac{1}{\Delta x}\int_{(j-1)\Delta x+1}^{j\Delta x+1}\tilde{\rho}(x,t)x^{-\frac{4}{\gamma+1}}dx+o(\sqrt{\Delta x})\frac{1}{\Delta x}\int_{(j-1)\Delta x+1}^{j\Delta x+1}\tilde{\rho}(x,t)x^{-\frac{4}{\gamma+1}}dx,
\end{aligned}
$$

substituting this equation for the second equation in (C.11) and using (C.10), we obtain

$$
\begin{aligned}
(E_j^t(\rho))^{\theta+1}=&\left(x_{\mathrm{M}}^{-\mu}\frac{1}{\Delta x}\int_{(j-1)\Delta x+1}^{j\Delta x+1}\tilde{\rho}(x,t)x^{\mu-\frac{4}{\gamma+1}}dx\right)^{\theta+1}\\
&-\mu(\theta+1)\left(\frac{1}{\Delta x}\int_{(j-1)\Delta x+1}^{j\Delta x+1}\tilde{\rho}(x,t)x^{-\frac{4}{\gamma+1}}dx\right)^{\theta}\\
&\times x_{\mathrm{M}}^{-\frac{4}{\gamma+1}-1}\frac{1}{\Delta x}\int_{(j-1)\Delta x+1}^{j\Delta x+1}(x-x_{\mathrm{M}})\tilde{\rho}(x,t)dx\\
&+o(\Delta x)\frac{1}{\Delta x}\int_{(j-1)\Delta x+1}^{j\Delta x+1}\tilde{\rho}(x,t)x^{-\frac{4}{\gamma+1}}dx.
\end{aligned}
\tag{C.12}
$$

Considering (C.12), from Jensen's inequality, we have

$$
\begin{aligned}
&\left(x_{\mathrm{M}}^{-\mu}\frac{1}{\Delta x}\int_{(j-1)\Delta x+1}^{j\Delta x+1}\tilde{\rho}(x,t)x^{\mu-\frac{4}{\gamma+1}}dx\right)^{\theta+1}\\
&=\left(\frac{x_{\mathrm{M}}^{-\mu}\dfrac{1}{\Delta x}\displaystyle\int_{(j-1)\Delta x+1}^{j\Delta x+1}\tilde{\rho}(x,t)x^{\mu-\frac{\theta+1}{\theta}\mu-\frac{4}{\gamma+1}}x^{\frac{\theta+1}{\theta}\mu}dx}{\dfrac{1}{\Delta x}\displaystyle\int_{(j-1)\Delta x+1}^{j\Delta x+1}x^{\frac{\theta+1}{\theta}\mu}dx}\right)^{\theta+1}\\
&\quad\times\left(\frac{1}{\Delta x}\int_{(j-1)\Delta x+1}^{j\Delta x+1}x^{\frac{\theta+1}{\theta}\mu}dx\right)^{\theta+1}\\
&\leq\frac{1}{\Delta x}\int_{(j-1)\Delta x+1}^{j\Delta x+1}(\tilde{\rho}(x,t))^{\theta+1}x^{-2}dx\left(x_{\mathrm{M}}^{-\frac{\theta+1}{\theta}\mu}\frac{1}{\Delta x}\int_{(j-1)\Delta x+1}^{j\Delta x+1}x^{\frac{\theta+1}{\theta}\mu}dx\right)^{\theta}.
\end{aligned}
\tag{C.13}
$$

Since

$$
\begin{aligned}
& x_{\mathrm{M}}^{-\frac{\theta+1}{\theta}\mu} \frac{1}{\Delta x} \int_{(j-1)\Delta x+1}^{j\Delta x+1} x^{\frac{\theta+1}{\theta}\mu} dx \\
&= x_{\mathrm{M}}^{-\frac{\theta+1}{\theta}\mu} \frac{1}{\Delta x} \int_{(j-1)\Delta x+1}^{j\Delta x+1} x_{\mathrm{M}}^{\frac{\theta+1}{\theta}\mu} dx \\
&\quad + x_{\mathrm{M}}^{-\frac{\theta+1}{\theta}\mu} \frac{\theta+1}{\theta}\mu \frac{1}{\Delta x} \int_{(j-1)\Delta x+1}^{j\Delta x+1} x_{\mathrm{M}}^{\frac{\theta+1}{\theta}\mu-1}(x-x_{\mathrm{M}}) dx \\
&\quad + x_{\mathrm{M}}^{-\frac{\theta+1}{\theta}\mu} \frac{\theta+1}{\theta}\mu \left(\frac{\theta+1}{\theta}\mu - 1\right) \\
&\quad \times \frac{1}{\Delta x} \int_{(j-1)\Delta x+1}^{j\Delta x+1} \{x_{\mathrm{M}} + \tau_6(x)(x-x_{\mathrm{M}})\}^{\frac{\theta+1}{\theta}\mu-2} (x-x_{\mathrm{M}})^2 dx \\
&= 1 + o(\Delta x),
\end{aligned}
$$

(C.13) yields

$$
\begin{aligned}
& \left(x_{\mathrm{M}}^{-\mu} \frac{1}{\Delta x} \int_{(j-1)\Delta x+1}^{j\Delta x+1} \tilde{\rho}(x,t) x^{\mu - \frac{4}{\gamma+1}} dx \right)^{\theta+1} \\
&\qquad \le \frac{1}{\Delta x} \int_{(j-1)\Delta x+1}^{j\Delta x+1} (\tilde{\rho}(x,t))^{\theta+1} x^{-2} dx (1 + o(\Delta x)). \qquad \text{(C.14)}
\end{aligned}
$$

From (C.12) and (C.14), we obtain (C.5) and complete the proof of lemma 6.1. □

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