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Convergent finite element methods for compressible barotropic Stokes systems

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Abstract. We propose finite element methods for compressible barotropic Stokes systems. We state convergence results for these methods and outline their proofs. The principal tools of the proofs are higher integrability estimates for the discrete density, equations for the discrete effective viscous flux, and renormalized formulations of the numerical method for the density equation.

1. Introduction

In this contribution we consider mixed type systems of the form

$$\partial_t \varrho + \operatorname{div}(\varrho \boldsymbol{u}) = 0, \quad \text{in } (0,T)\times\Omega, \tag{1.1}$$

$$-\mu\Delta\boldsymbol{u} - \lambda D \operatorname{div}\boldsymbol{u} + Dp(\varrho) = \boldsymbol{f}, \quad \text{in } (0,T)\times\Omega, \tag{1.2}$$

with initial data

$$\varrho|_{t=0} = \varrho_0, \quad \text{on } \Omega. \tag{1.3}$$

Here Ω is a simply connected, bounded, open, polygonal domain in $\mathbb{R}^N$ $(N = 2,3)$, with Lipschitz boundary $\partial\Omega$, and $T > 0$ is a final time. The unknowns are the density $\varrho = \varrho(t,\boldsymbol{x}) \geq 0$ and the velocity $\boldsymbol{u} = \boldsymbol{u}(t,\boldsymbol{x}) \in \mathbb{R}^N$, with $t \in (0,T)$ and $\boldsymbol{x} \in \Omega$. We denote by div and D the usual spatial divergence and gradient operators and by Δ the Laplace operator.

The pressure $p(\varrho)$ is governed by the equation of state $p(\varrho) = a\varrho^\gamma$, $a > 0$ (Boyle's law). Typical values of γ range from a maximum of $\frac{5}{3}$ for monoatomic gases, through $\frac{7}{5}$ for diatomic gases *including air*, to lower values close to 1 for polyatomic gases at high temperatures. We will assume that $\gamma \geq 1$. Furthermore, the viscosity coefficients μ, λ are assumed to be constant and to satisfy $\mu > 0, N\lambda + 2\mu \geq 0$.

At the boundary $\partial\Omega$, the system (1.1)–(1.2) is supplemented either with the homogenous Dirichlet condition

$$\boldsymbol{u} = 0, \quad \text{on } (0,T)\times\partial\Omega, \tag{1.4}$$

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or with the Navier–slip condition

$$\boldsymbol{u}\cdot\nu = 0, \quad \operatorname{curl}\boldsymbol{u}\times\nu = 0, \quad \text{on } (0,T)\times\partial\Omega. \tag{1.5}$$

System (1.1)–(1.2) can be motivated in several ways. Firstly, it can be used as a model equation for the barotropic compressible Navier–Stokes equations. This is a reasonable approximation for strongly viscous fluids for which convection can be neglected. Secondly, in [**7**, Section 5.2, Remark 5.8], Lions construct solutions to the barotropic compressible Navier–Stokes equations using solutions of the system (1.1)–(1.2). Finally, by setting $\gamma = 1$, $\boldsymbol{f} = 0$, and $\mu = 0$, the system (1.1)–(1.2) is exactly on the same form as the model derived in [**8**] for the dynamics of vortices in Ginzburg–Landau theories in superconductivity.

Among many others, the semi–stationary system (1.1)–(1.3) has been studied by Lions in [**7**, Section 8.2] where he proves the existence of weak solutions and some higher regularity results.

The plan of this contribution is to summarize some results [**4, 5, 6**] from an ongoing project to develop convergent numerical methods for multi-dimensional compressible viscous flow models. We construct numerical methods that comply with the mathematical framework developed for the compressible Navier–Stokes equations by Lions [**7**] and Feireisl [**3**]. Over the years, several numerical methods appropriate for compressible viscous gas flow have been proposed. Except for some one-dimensional situations (cf. Zhao and Hoff [**9, 10**]), it is not known, however, that these methods converge to a weak solution as the discretization parameters tend to zero. Convergence analysis for the compressible Navier–Stokes system is made difficult by the non–linearities in the convection and pressure terms and their interaction. As a first step towards establishing convergence of numerical methods for the full system, we consider simplified systems that contain some of the difficulties but not all. In that respect (1.1)–(1.2) provides an example.

The finite element methods presented here are designed to satisfy the properties needed to apply the weak convergence techniques used in the global existence theory for the compressible Navier–Stokes equations. Although the simplified system (1.1)–(1.2) contain additional structures rendering the solutions more regular than those of the full Navier–Stokes system, we strive to employ techniques that can potentially be extended to the full system. More specifically, our finite element methods are designed such that Hodge decompositions of the velocity, $\boldsymbol{u} = \operatorname{curl}\boldsymbol{\xi} + Dz$, can be achieved and described at the discrete level. This is important since then a discrete equation for the effective viscous flux, $(\lambda+\mu)\operatorname{div}\boldsymbol{u} - p(\varrho)$, can easily be extracted from the numerical scheme. It is the properties of this quantity that leads to strong convergence of the numerical density function; the major obstacle to proving convergence of a numerical method. Formally, multiplying the equation (1.2) with $\boldsymbol{u}$, integrating by parts, and using the continuity equation multiplied with $\frac{1}{\gamma-1}p'(\varrho)$ one obtains the energy relation

$$\frac{d}{dt}\int_\Omega \frac{p(\varrho)}{\gamma-1}\,dx + \int_\Omega \mu|D\boldsymbol{u}|^2 + \lambda|\operatorname{div}\boldsymbol{u}|^2\,dx = \int_\Omega \boldsymbol{f}\boldsymbol{u}\,dx.$$

A similar relation holds for our finite element methods, which reveals the rather weak a priori estimates that are available to us. Indeed, it is now clear that a major obstacle is to obtain enough compactness on the numerical density ϱ_h to conclude that $p(\varrho_h) \rightharpoonup p(\varrho)$; of course, this is equivalent to $\varrho_h \to \varrho$ almost everywhere.

The remaining part of this contribution is organized as follows: We collect some preliminary material, including the notion of weak solutions, in Section 2. In Section 3 we present a finite element method for the semi–stationary Stokes system in primitive variables. We state a convergence result for this method and comment on its proof. This method is fully developed and analyzed in [**5**]. In Section 4, we present and analyze an alternative finite element method [**4**] for the same system. This method is, however, restricted to the case of the Navier–slip boundary condition (1.5). Finally, we conclude this contribution by presenting a convergent finite element method for the Stokes approximation equations, which generalizes the system (1.1)–(1.2) by adding an additional time derivative term $\partial_t u$ to the equation for the velocity.

2. Preliminary material

Throughout the text we make frequent use of the divergence and curl operators and denote these by div and curl, respectively. In the 2D case we denote both the rotation operator taking scalars into vectors and the curl operator taking vectors into scalars by curl. We make use of the spaces

$$\begin{aligned}\boldsymbol{W}^{\mathrm{div},2}(\Omega) &= \left\{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega) : \operatorname{div}\boldsymbol{v} \in L^2(\Omega)\right\},\\ \boldsymbol{W}^{\mathrm{curl},2}(\Omega) &= \left\{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega) : \operatorname{curl}\boldsymbol{v} \in \boldsymbol{L}^2(\Omega)\right\},\end{aligned}$$

where ν denotes the unit outward pointing normal vector on $\partial\Omega$. If $\boldsymbol{v} \in \boldsymbol{W}^{\mathrm{div},2}(\Omega)$ satisfies $\boldsymbol{v}\cdot\nu|_{\partial\Omega} = 0$, we write $\boldsymbol{v} \in \boldsymbol{W}_0^{\mathrm{div},2}(\Omega)$. Similarly, $\boldsymbol{v} \in \boldsymbol{W}_0^{\mathrm{curl},2}(\Omega)$ means $\boldsymbol{v} \in \boldsymbol{W}^{\mathrm{div},2}(\Omega)$ and $\boldsymbol{v}\times\nu|_{\partial\Omega} = 0$. In two dimensions, $\boldsymbol{w}$ is a scalar function and the space $\boldsymbol{W}_0^{\mathrm{curl},2}(\Omega)$ is to be understood as $W_0^{1,2}(\Omega)$. To define weak solutions, we shall use the space

$$\mathcal{W} = \left\{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega) : \operatorname{div}\boldsymbol{v} \in L^2(\Omega), \operatorname{curl}\boldsymbol{v} \in \boldsymbol{L}^2(\Omega), \boldsymbol{v}\cdot\nu|_{\partial\Omega} = 0\right\},$$

which coincides with $\boldsymbol{W}_0^{\mathrm{div},2}(\Omega) \cap \boldsymbol{W}^{\mathrm{curl},2}(\Omega)$. The space $\mathcal{W}$ is equipped with the norm $\|\boldsymbol{v}\|_{\mathcal{W}}^2 = \|\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)}^2 + \|\operatorname{div}\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)}^2 + \|\operatorname{curl}\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)}^2$. It is known that $\|\cdot\|_{\mathcal{W}}$ is equivalent to the H^1 norm on the space $\left\{v \in H^1(\Omega) : \boldsymbol{v}\cdot\nu|_{\partial\Omega} = 0\right\}$.

Next we introduce the notion of weak solutions.

Definition 2.1 (Weak solutions). A pair $(\varrho, \boldsymbol{u})$ of functions constitutes a weak solution of the semi-stationary compressible Stokes system (1.1)–(1.2) with initial data (1.3) provided that:

(1) $(\varrho, \boldsymbol{u}) \in L^\infty(0,T;L^\gamma(\Omega)) \times L^2(0,T;\mathcal{W}(\Omega))$,

(2) $\partial_t\varrho + \operatorname{div}(\varrho\boldsymbol{u}) = 0$ in the weak sense, i.e, $\forall\phi \in C^\infty([0,T)\times\overline{\Omega})$,

$$\int_0^T\int_\Omega \varrho\left(\phi_t + \boldsymbol{u}D\phi\right)\,dxdt + \int_\Omega \varrho_0\phi|_{t=0}\,dx = 0; \tag{2.1}$$

(3) $-\mu\Delta\boldsymbol{u} - \lambda D\operatorname{div}\boldsymbol{u} + Dp(\varrho) = \boldsymbol{f}$ in the weak sense, i.e, $\forall\boldsymbol{\phi} \in \boldsymbol{C}^\infty([0,T)\times\overline{\Omega})$ for which $\boldsymbol{\phi}\cdot\nu = 0$ on $(0,T)\times\partial\Omega$,

$$\int_0^T\int_\Omega \mu\operatorname{curl}\boldsymbol{u}\operatorname{curl}\boldsymbol{\phi} + \left[(\mu+\lambda)\operatorname{div}\boldsymbol{u} - p(\varrho)\right]\operatorname{div}\boldsymbol{\phi}\,dxdt = \int_0^T\int_\Omega \boldsymbol{f}\boldsymbol{\phi}\,dxdt, \tag{2.2}$$

Whenever the Dirichlet boundary condition (1.4) is part of the problem, we require that $\boldsymbol{u}\times\nu = 0$ on $(0,T)\times\partial\Omega$ in (1) and moreover that (2.2) holds for test functions satisfying $\phi = 0$ on $(0,T)\times\partial\Omega$.

3. A non–conforming finite element method

Following [**5**], in this section we present a finite element method for the system (1.1)–(1.2) appropriate for both the Dirichlet boundary condition (1.4) and the Navier–slip boundary condition (1.5).

For discretization of the velocity we will use the *Crouzeix–Raviart* element space. Consequently, the finite element method is non–conforming in the sense that the velocity approximation space is not a subspace of the corresponding continuous space, $\boldsymbol{W}^{1,2}(\Omega)$. Moreover, we will use a non–standard finite element formulation. More precisely, the formulation implicitly use the identity

$$\int_\Omega D\boldsymbol{u}D\boldsymbol{v}\;dx = \int_\Omega \operatorname{curl}\boldsymbol{u}\operatorname{curl}\boldsymbol{v} + \operatorname{div}\boldsymbol{u}\operatorname{div}\boldsymbol{v}\;dx, \tag{3.1}$$

valid for all $\boldsymbol{u} \in \mathcal{W}(\Omega)$ satisfying any of the two boundary conditions (1.5) and (1.4). However, as the method is non–conforming, this identity does not hold discretely (as a sum over elements). Still, at the discrete level, the form on the right-hand side of (3.1) is used. In contrast to the standard situation where the form on the left–hand side of (3.1) is used, this discretization does not converge unless additional terms controlling the discontinuities of the velocity are added [**1**]:

$$\sum_{\Gamma\in\Gamma_h}\frac{1}{|\Gamma|}\int_\Gamma [\![\boldsymbol{u}\cdot\nu]\!]_\Gamma[\![\boldsymbol{v}\cdot\nu]\!]_\Gamma + [\![\boldsymbol{u}\times\nu]\!]_\Gamma[\![\boldsymbol{v}\times\nu]\!]_\Gamma\;dS(x),$$

where Γ_h is the set of faces and $[\![\cdot]\!]_\Gamma$ denotes the jump over the edge Γ.

The advantage with this formulation is that it enables Hodge decompositions of the numerical velocity field. By writing $\boldsymbol{u} = \operatorname{curl}\boldsymbol{\xi} + Dz$ the Laplace operator can be split into a curl part and a divergence part plus certain jump terms. This is very convenient in the convergence analysis of the method. A discrete equation for the effective viscous flux can then be easily obtained. The reader is encouraged to consult [**5**] for the details

Given a time step $\Delta t > 0$, we discretize the time interval $[0,T]$ in terms of the points $t^m = m\Delta t$, $m = 0,\dots,M$, where we assume that $M\Delta t = T$. Regarding the spatial discretization, we let $\{E_h\}_h$ be a shape regular family of tetrahedral meshes of Ω, where h is the maximal diameter. It will be a standing assumption that h and Δt are related such that $\Delta t = ch$, for some constant c. For each h, let Γ_h denote the set of faces in E_h.

We need to introduce some additional notation for discontinuous Galerkin schemes. Concerning the boundary ∂E of an element E, we write f_+ for the trace of the function f achieved from within the element E and f_- for the trace of f achieved from outside E. Concerning an edge Γ that is shared between two elements E_- and E_+, we will write f_+ for the trace of f achieved from within E_+ and f_- for the trace of f achieved from within E_-. Here E_- and E_+ are defined such that ν points from E_- to E_+, where ν is fixed (throughout) as one of the two possible normal components on each edge Γ throughout the discretization. We also write $[\![f]\!]_\Gamma = f_+ - f_-$ for the jump of f across the edge Γ, while forward time-differencing of f is denoted by $[\![f^m]\!] = f^{m+1} - f^m$ and $d_t^h[f^m] = \frac{[\![f^m]\!]}{\Delta t}$.

We will approximate the density in the space of piecewise constants on E_h and we denote this space by $Q_h(\Omega)$. For approximation of the velocity we will use the

Crouzeix–Raviart [**2**] element space

$$\boldsymbol{V}_h(\Omega) = \left\{ \boldsymbol{v}_h; \boldsymbol{v}_h|_E \in \mathcal{P}_1^N(E),\ \forall E \in E_h,\ \int_\Gamma [\![\boldsymbol{v}_h]\!] \, dS(x) = 0,\ \forall \Gamma \in \Gamma_h \right\}.$$

To incorporate boundary conditions, we let degrees of freedom of $\boldsymbol{V}_h(\Omega)$ vanish at the boundary. That is, for Navier boundary condition (1.5) we require

$$\int_\Gamma \boldsymbol{v}_h \cdot \nu \, dS(x) = 0, \quad \forall \Gamma \in \Gamma_h \cap \partial\Omega, \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h,$$

and for the Dirichlet boundary condition (1.4),

$$\int_\Gamma \boldsymbol{v}_h \, dS(x) = 0, \quad \forall \Gamma \in \Gamma_h \cap \partial\Omega, \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h.$$

To the space $\boldsymbol{V}_h(\Omega)$ we associate the semi–norm

$$\begin{aligned} |\boldsymbol{v}_h|^2_{\boldsymbol{V}_h(\Omega)} &= \| \operatorname{curl}_h \boldsymbol{v}_h \|^2_{L^2(\Omega)} + \| \operatorname{div}_h \boldsymbol{v}_h \|^2_{L^2(\Omega)} \\ &\quad + \frac{h^\epsilon}{|\Gamma|} \sum_{\Gamma \in \Gamma_h} \| [\![\boldsymbol{v}_h \cdot \nu]\!] \|^2_{L^2(\Gamma)} + \| [\![\boldsymbol{v}_h \times \nu]\!] \|^2_{L^2(\Gamma)}, \end{aligned}$$

and the corresponding norm

$$\|\boldsymbol{v}_h\|^2_{\boldsymbol{V}_h(\Omega)} = \|\boldsymbol{v}_h\|^2_{L^2(\Omega)} + |\boldsymbol{v}_h|^2_{\boldsymbol{V}_h(\Omega)}.$$

Here, curl_h and div_h denotes the curl and divergence operators, respectively, taken inside each element. The scaling parameter $\epsilon > 0$ is required to prove convergence of the finite element method. The size of ϵ will affect the accuracy of the method and it should therefore be fixed very small in practical computations [**5**].

Before stating the finite element method, we recall from [**5**] the following basic compactness result for approximations in $\boldsymbol{V}_h(\Omega)$.

LEMMA 3.1. *There exists a constant $C > 0$, depending only on the shape regularity of E_h and the size of Ω, such that for any $\xi \in \mathbb{R}^2$*

$$\|\boldsymbol{v}_h(\cdot) - \boldsymbol{v}_h(\cdot - \xi)\|_{\boldsymbol{L}^2(\Omega)} \le C|\xi|^{\frac{1}{2} - \frac{\epsilon}{4}} |\boldsymbol{v}_h|_{\boldsymbol{V}_h(\Omega)}, \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h(\Omega),$$

and $\|\boldsymbol{v}_h\|_{\boldsymbol{L}^2(\Omega)} \le C|\boldsymbol{v}_h|_{\boldsymbol{V}_h(\Omega)}$, $\forall \boldsymbol{v}_h \in \boldsymbol{V}_h(\Omega)$.

DEFINITION 3.2 (Finite element method). Let $\left\{\varrho_h^0(x)\right\}_{h>0}$ be a sequence (of piecewise constant functions) in $Q_h(\Omega)$ that satisfies $\varrho_h^0 > 0$ for each fixed $h > 0$ and $\varrho_h^0 \to \varrho^0$ a.e. in Ω and in $L^1(\Omega)$ as $h \to 0$. Set $\boldsymbol{f}_h := \Pi_h^Q \boldsymbol{f}$, where it is understood that $\Pi_h^Q \boldsymbol{f}$ projects $\boldsymbol{f}(t,x)$ onto constants both in time t and space x; for notational convenience we set $\boldsymbol{f}_h^m := \boldsymbol{f}_h(t^m, \cdot) \in Q_h(\Omega)$ for any $m = 0, \dots, M$.

Now, determine functions $(\varrho_h^m, \boldsymbol{u}_h^m) \in Q_h(\Omega) \times \boldsymbol{V}_h(\Omega)$, $m = 1, \dots, M$, such that for all $\phi_h \in Q_h(\Omega)$,

(3.2)

$$\int_\Omega d_t^h[\varrho_h^m]\phi_h \, dx - \Delta t \sum_{\Gamma \in \Gamma_h} \int_\Gamma \left(\varrho_-^m (\boldsymbol{u}_h^m \cdot \nu)^+ + \varrho_+^m (\boldsymbol{u}_h^m \cdot \nu)^- \right) [\![\phi_h]\!]_\Gamma \, dS(x) = 0.$$

and for all $\boldsymbol{v}_h \in \boldsymbol{V}_h(\Omega)$,

$$\begin{aligned} &\int_\Omega \mu \operatorname{curl}_h \boldsymbol{u}_h^m \operatorname{curl}_h \boldsymbol{v}_h + [(\mu + \lambda) \operatorname{div}_h \boldsymbol{u}_h^m - p(\varrho_h^m)] \operatorname{div}_h \boldsymbol{v}_h \, dx \\ &\quad + \sum_{\Gamma \in \Gamma_h} \frac{h^\epsilon}{|\Gamma|} \int_\Gamma [\![\boldsymbol{u}_h^m \cdot \nu]\!][\![\boldsymbol{v}_h \cdot \nu]\!] + [\![\boldsymbol{u}_h^m \times \nu]\!][\![\boldsymbol{v}_h \times \nu]\!] \, dS(x) = \int_\Omega \boldsymbol{f}_h^m \boldsymbol{v}_h \, dx, \end{aligned} \tag{3.3}$$

In (3.2), $(\boldsymbol{u}_h \cdot \nu)^+(x) = \max\left\{\frac{1}{|\Gamma|}\int_\Gamma \boldsymbol{u}_h \cdot \nu \; dS(x), 0\right\}$ and $(\boldsymbol{u}_h \cdot \nu)^-(x) = \min\left\{\frac{1}{|\Gamma|}\int_\Gamma \boldsymbol{u}_h \cdot \nu \; dS(x), 0\right\}$ for $x \in \Gamma$ and all $\Gamma \in \Gamma_h$.

The existence of a solution to the discrete equations (3.2)–(3.3) is proved in [**5**] by using a topological degree argument. In [**5**] it is also shown that the scheme preserves the total mass and that the density remains strictly positive provided that the initial density is strictly positive. Moreover, for any $m = 1, \dots, M$,

$$\begin{aligned} &\int_\Omega P(\varrho_h^m)\; dx + C\sum_{k=1}^m \Delta t \|\boldsymbol{u}_h^k\|_{\boldsymbol{V}_h(\Omega)}^2 \\ &\quad + \sum_{k=1}^m \int_\Omega P''(\varrho_{\dagger\dagger}^k)[\![\varrho_h^{k-1}]\!]^2 \; dx + \sum_{k=1}^m \sum_{\Gamma\in\Gamma_h} \Delta t \int_\Gamma P''(\varrho_\dagger^k)[\![\varrho_h^k]\!]_\Gamma^2 |\boldsymbol{u}_h^k \cdot \nu| \; dx \\ &\leq \int_\Omega P(\varrho_0)\; dx + \frac{1}{4C}\sum_{k=1}^m \Delta t \|\boldsymbol{f}_h^k\|_{\boldsymbol{L}^2(\Omega)}^2, \end{aligned}$$

where $P(\varrho) = \frac{p(\varrho)}{\gamma-1}$ if $\gamma > 1$ and $P(\varrho) = \varrho \log \varrho$ if $\gamma = 1$. Moreover, $\varrho_{\dagger\dagger}^k \in [\varrho_h^{k-1}, \varrho_h^k]$ and $\varrho_\dagger^k \in [\varrho_+^k, \varrho_-^k]$.

Next, for each fixed $h > 0$, we extend the numerical solution $\{(\varrho_h^m, \boldsymbol{u}_h^m)\}_{m=0}^M$ to the whole of $(0,T)\times\Omega$ by setting

$$(\varrho_h, \boldsymbol{u}_h)(t) = (\varrho_h^m, \boldsymbol{u}_h^m), \qquad t \in (t_{m-1}, t_m), \quad m = 1, \dots, M. \tag{3.4}$$

In addition, we set $\varrho_h(0) = \varrho_h^0$.

The main result of [**5**] is that the approximate solutions (3.4) converge to a weak solution of the semi–stationary Stokes system (1.1)–(1.2).

THEOREM 3.3. *Suppose* $\boldsymbol{f} \in \boldsymbol{L}^2((0,T)\times\Omega)$ *and* $\varrho_0 \in L^\gamma(\Omega)$*, if* $\gamma > 1$*, and* $\varrho_0 \log \varrho_0 \in L^1(\Omega)$*, if* $\gamma = 1$*. Let* $\{(\varrho_h, \boldsymbol{u}_h)\}_{h>0}$ *be a sequence of numerical solutions constructed according to* (3.4) *and Definition 3.2. Then, passing if necessary to a subsequence as* $h \to 0$*,* $\boldsymbol{u}_h \rightharpoonup \boldsymbol{u}$ *in* $L^2(0,T;\boldsymbol{L}^2(\Omega))$*,* $\varrho_h\boldsymbol{u}_h \rightharpoonup \varrho\boldsymbol{u}$ *in the sense of distributions on* $(0,T)\times\Omega$*, and* $\varrho_h \to \varrho$ *a.e. in* $(0,T)\times\Omega$*, where the limit pair* $(\varrho, \boldsymbol{u})$ *is a weak solution as stated in Definition 2.1.*

Comments on the proof of Theorem 3.3. In proving convergence to a weak solution of the continuity equation (1.1) the main step is to obtain convergence of the product $\varrho_h\boldsymbol{u}_h \rightharpoonup \varrho\boldsymbol{u}$ in the sense of distributions; this follows from an Aubin–Lions argument using the spatial compactness of the velocity established in Lemma 4.1 combined with the fact that $d_t^h[\varrho_h] \in_{\mathrm{b}} L^1(0,T;W^{-1,1}(\Omega))$.

To conclude convergence to a weak solution of the velocity equation (1.2), we need a higher integrability estimate for the numerical density. We achieve this by utilizing test functions $\boldsymbol{v}_h \in \boldsymbol{V}_h(\Omega)$ satisfying $\operatorname{div}\boldsymbol{v}_h = p(\varrho_h)$, thereby obtaining $p(\varrho_h) \in_{\mathrm{b}} L^2(0,T;L^2(\Omega))$. Next, we establish strong convergence of the density. This is obtained by first proving weak sequential continuity of the effective viscous flux. That is, first we establish that $\lim_{h\to 0}[(\lambda+\mu)\operatorname{div}\boldsymbol{u}_h - p(\varrho_h)]\,\varrho_h = \overline{(\mu+\lambda)\operatorname{div}\boldsymbol{u} - p(\varrho)\varrho}$, where the overbar denotes the weak limit. In this step, the div–curl structure of the scheme is utilized. In particular, we employ test functions $\boldsymbol{v}_h \in \boldsymbol{V}_h(\Omega)$ that satisfies $\operatorname{div}\boldsymbol{v}_h = \varrho_h$ and $\operatorname{curl}\boldsymbol{v}_h = 0$ on elements away from the boundary. Finally, using this and a renormalized version of the continuity scheme (3.2), we obtain strong convergence of the density.

4. A mixed finite element method

Following [**4**], we present an alternative finite element method appropriate for the Navier–slip boundary condition (1.5). The method is derived by introducing the vorticity $\boldsymbol{w} = \operatorname{curl} \boldsymbol{u}$ as an auxiliary variable and recasting (1.2) as

$$\mu \operatorname{curl} \boldsymbol{w} - (\lambda + \mu) D \operatorname{div} \boldsymbol{u} + Dp(\varrho) = \boldsymbol{f},$$

where also the identity $-\Delta = \operatorname{curl}\operatorname{curl} - D \operatorname{div}$ is used. This leads naturally to the following mixed formulation: Determine functions

$$(\boldsymbol{w}, \boldsymbol{u}) \in L^2(0,T; \boldsymbol{W}_0^{\mathrm{curl},2}(\Omega)) \times L^2(0,T; \boldsymbol{W}_0^{\mathrm{div},2}(\Omega))$$

such that

$$\begin{aligned} &\int_0^T \int_\Omega \mu \operatorname{curl} \boldsymbol{w} \boldsymbol{v} + [(\mu + \lambda) \operatorname{div} \boldsymbol{u} - p(\varrho)] \operatorname{div} \boldsymbol{v} \; dxdt = \int_0^T \int_\Omega \boldsymbol{f} \boldsymbol{v} \; dxdt, \\ &\int_0^T \int_\Omega \boldsymbol{w} \boldsymbol{\eta} - \operatorname{curl} \boldsymbol{\eta} \boldsymbol{u} \; dxdt = 0, \end{aligned} \tag{4.1}$$

for all $(\eta, \boldsymbol{v}) \in L^2(0,T; \boldsymbol{W}_0^{\mathrm{curl},2}(\Omega)) \times L^2(0,T; \boldsymbol{W}_0^{\mathrm{div},2}(\Omega))$. We make clear that if $(\varrho, \boldsymbol{w}, \boldsymbol{u})$ is a triple satisfying (2.1) and (4.1), then the pair $(\varrho, \boldsymbol{u})$ is also a weak solution according to Definition 2.1.

To obtain a stable numerical method, the mixed finite element formulation of (4.1) is posed with the velocity $\boldsymbol{v}_h$ in a div–conforming space $\boldsymbol{V}_h(\Omega) \subset \boldsymbol{W}_0^{\mathrm{div},2}(\Omega)$ and vorticity $\boldsymbol{w}_h$ in a curl–conforming space $\boldsymbol{W}_h(\Omega) \subset \boldsymbol{W}_0^{\mathrm{curl},2}(\Omega)$. There exists several such spaces, however here we will use the Nedelec spaces of first order and first kind. We choose these spaces for their simplicity and since the most natural choice of approximation space for the density is then the space of piecewise constants. We will continue to denote this space by $Q_h(\Omega)$.

This choice of finite element spaces is also very convenient since they can be related through the exact de Rham sequence

$$0 \xrightarrow{\subset} S_h \xrightarrow{\mathrm{grad}} \boldsymbol{W}_h \xrightarrow{\mathrm{curl}} \boldsymbol{V}_h \xrightarrow{\mathrm{div}} Q_h \longrightarrow 0.$$

Thus, we can use spaces orthogonal to the range of the previous operator, i.e.,

$$\boldsymbol{W}_h^{0,\perp} := \{\boldsymbol{w}_h \in \boldsymbol{W}_h; \operatorname{curl} \boldsymbol{w}_h = 0\}^\perp \cap \boldsymbol{W}_h, \quad \boldsymbol{V}_h^{0,\perp} := \{\boldsymbol{v}_h \in \boldsymbol{V}_h; \operatorname{div} \boldsymbol{v}_h = 0\}^\perp \cap \boldsymbol{V}_h,$$

to deduce the decompositions

$$\boldsymbol{W}_h = DS_h + \boldsymbol{W}_h^{0,\perp}, \quad \boldsymbol{V}_h = \operatorname{curl} \boldsymbol{W}_h + \boldsymbol{V}_h^{0,\perp},$$

together with the discrete Poincaré inequalities

$$\|\boldsymbol{v}_h\|_{\boldsymbol{L}^2(\Omega)} \leq C \|\operatorname{div} \boldsymbol{v}_h\|_{L^2(\Omega)}, \quad \|\boldsymbol{w}_h\|_{\boldsymbol{L}^2(\Omega)} \leq C \|\operatorname{curl} \boldsymbol{w}_h\|_{L^2(\Omega)},$$

Consequently, as with the previous method, the mixed finite element method also admits Hodge decompositions, which in turn implies that a discrete equation for the effective viscous flux can be derived.

We need the following compactness property of the space $\boldsymbol{V}_h^{0,\perp}$. The proof is given in [**4**, Appendix A].

Lemma 4.1. *Let $\{\boldsymbol{v}_h\}_{h>0}$ be a sequence in $\boldsymbol{V}_h^{0,\perp}$ such that $\|\operatorname{div} \boldsymbol{v}_h\|_{L^2(\Omega)} \leq C$, where the constant $C > 0$ is independent of h. Then, for any $\xi \in \mathbb{R}^N$,*

$$\|\boldsymbol{v}_h(x) - \boldsymbol{v}_h(x - \xi)\|_{\boldsymbol{L}^2(\Omega)} \leq C(|\xi|^{\frac{4-N}{2}} + |\xi|^2)^{\frac{1}{2}} \|\operatorname{div} \boldsymbol{v}_h\|_{L^2(\Omega)},$$

where the constant $C > 0$ is independent of both h and ξ.

Definition 4.2 (Mixed finite element method). Let ϱ_h^0 and $\boldsymbol{f}_h^m$ be as given in Definition 3.2. Determine functions

$$(\varrho_h^m, \boldsymbol{w}_h^m, \boldsymbol{u}_h^m) \in Q_h(\Omega) \times \boldsymbol{W}_h(\Omega) \times \boldsymbol{V}_h(\Omega), \quad m = 1, \dots, M,$$

such that for all $\phi_h \in Q_h(\Omega)$,

$$\int_\Omega d_t^h[\varrho_h^m]\phi_h \, dx - \Delta t \sum_{\Gamma \in \Gamma_h} \int_\Gamma \left(\varrho_-^m (\boldsymbol{u}_h^m \cdot \nu)^+ + \varrho_+^m (\boldsymbol{u}_h^m \cdot \nu)^- \right) [\phi_h]_\Gamma \, dS(x) = 0, \tag{4.2}$$

and for all $(\boldsymbol{\eta}_h, \boldsymbol{v}_h) \in \boldsymbol{W}_h(\Omega) \times \boldsymbol{V}_h(\Omega)$,

$$\begin{aligned} &\int_\Omega \mu \operatorname{curl} \boldsymbol{w}_h^m \boldsymbol{v}_h + [(\mu + \lambda) \operatorname{div} \boldsymbol{u}_h^m - p(\varrho_h^m)] \operatorname{div} \boldsymbol{v}_h \, dx = \int_\Omega \boldsymbol{f}_h^m \boldsymbol{v}_h \, dx, \\ &\int_\Omega \boldsymbol{w}_h^m \boldsymbol{\eta}_h - \boldsymbol{u}_h^m \operatorname{curl} \boldsymbol{\eta}_h \, dx = 0. \end{aligned} \tag{4.3}$$

In (4.2), $(\boldsymbol{u}_h \cdot \nu)^+ = \max\{\boldsymbol{u}_h \cdot \nu, 0\}$ and $(\boldsymbol{u}_h \cdot \nu)^- = \min\{\boldsymbol{u}_h \cdot \nu, 0\}$.

The existence of a solution to the discrete equations (4.2)–(4.3) is proved in [**4**]. Moreover, for any $m = 1, \dots, M$,

$$\begin{aligned} &\int_\Omega P(\varrho_h^m) \, dx + \sum_{k=1}^m \Delta t \|\boldsymbol{u}_h^k\|^2_{\boldsymbol{W}^{\mathrm{div},2}(\Omega)} + \sum_{k=1}^m \Delta t \|\boldsymbol{w}_h^k\|^2_{\boldsymbol{W}^{\mathrm{curl},2}(\Omega)} \\ &\quad + \sum_{k=1}^m \int_\Omega P''(\varrho_{\dagger\dagger}^k) [\![\varrho_h^{k-1}]\!]^2 \, dx + \sum_{k=1}^m \sum_{\Gamma \in \Gamma_h} \Delta t \int_\Gamma P''(\varrho_\dagger^k) [\![\varrho_h^k]\!]_\Gamma^2 |\boldsymbol{u}_h^k \cdot \nu| \, dx \\ &\leq \int_\Omega P(\varrho_0) \, dx + C \sum_{k=1}^m \Delta t \|\boldsymbol{f}_h^k\|^2_{\boldsymbol{L}^2(\Omega)}, \end{aligned}$$

where $\varrho_{\dagger\dagger}^k \in [\varrho_h^{k-1}, \varrho_h^k]$ and $\varrho_\dagger^k \in [\varrho_+^k, \varrho_-^k]$.

For each fixed $h > 0$, the numerical solution $\{(\varrho_h^m, \boldsymbol{w}_h^m, \boldsymbol{u}_h^m)\}_{m=0}^M$ is extended to the whole of $(0, T) \times \Omega$ by setting

$$(\varrho_h, \boldsymbol{w}_h, \boldsymbol{u}_h)(t) = (\varrho_h^m, \boldsymbol{w}_h^m, \boldsymbol{u}_h^m), \qquad t \in (t_{m-1}, t_m), \quad m = 1, \dots, M. \tag{4.4}$$

In addition, we set $\varrho_h(0) = \varrho_h^0$. The main result in [**4**] is that the sequence $\{\varrho_h, \boldsymbol{w}_h, \boldsymbol{u}_h\}_{h>0}$ converges to a weak solution in the sense of Definition 2.1.

Theorem 4.3. *Suppose $\boldsymbol{f} \in \boldsymbol{L}^2((0,T) \times \Omega)$, $\varrho_0 \in L^\gamma(\Omega)$ if $\gamma > 1$, and $\varrho_0 \log \varrho_0 \in L^1(\Omega)$ if $\gamma = 1$. Let $\{(\varrho_h, \boldsymbol{w}_h, \boldsymbol{u}_h)\}_{h>0}$ be a sequence of numerical solutions constructed according to* (4.4) *and Definition 4.2. Then, passing if necessary to a subsequence as $h \to 0$, $\boldsymbol{w}_h \rightharpoonup \boldsymbol{w}$ in $L^2(0, T; \boldsymbol{W}_0^{\mathrm{curl},2}(\Omega))$, $\boldsymbol{u}_h \rightharpoonup \boldsymbol{u}$ in $L^2(0, T; \boldsymbol{W}_0^{\mathrm{div},2}(\Omega))$, $\varrho_h \boldsymbol{u}_h \rightharpoonup \varrho \boldsymbol{u}$ in the sense of distributions on $(0, T) \times \Omega$, and $\varrho_h \to \varrho$ a.e. in $(0, T) \times \Omega$, where the limit triplet $(\varrho, \boldsymbol{w}, \boldsymbol{u})$ satisfies the mixed form* (4.1)*, and consequently $(\varrho, \boldsymbol{u})$ is a weak solution as stated in Definition 2.1.*

Comments on the proof of Theorem 4.3. The proof of convergence to a weak solution of the continuity equation (1.1) is similar to the corresponding step in the proof of Theorem 3.3. The difference is that the compactness of the velocity approximation now requires a different argument. In particular, Lemma 4.1 must be employed. The proof of convergence to a weak solution of the velocity equation (1.2) is also similar to the proof of Theorem 3.3. However, a difference is that the weak sequential continuity of the effective viscous flux can now be obtained by using test functions in $\boldsymbol{V}_h^{0,\perp}(\Omega)$ satisfying $\operatorname{div} \boldsymbol{v}_h = \varrho_h$. Strong convergence of the density is then obtained as in the proof of Theorem 3.3.

5. Extension to the Stokes approximation equations.

In this final section we present an extension of the previous finite element method to the following system

$$\partial_t \varrho + \operatorname{div}(\varrho \boldsymbol{u}) = 0, \quad \text{in } (0,T)\times\Omega \tag{5.1}$$

$$\overline{\varrho}\partial_t \boldsymbol{u} - \mu\Delta\boldsymbol{u} - \lambda D \operatorname{div}\boldsymbol{u} + Dp(\varrho) = 0, \quad \text{in } (0,T)\times\Omega, \tag{5.2}$$

where $\overline{\varrho} = \frac{1}{|\Omega|}\int_\Omega \varrho_0 \, dx$ denotes the average initial density. The equations (5.1)–(5.2) is known in the literature as the *Stokes approximation equations*. The system is almost identical to the compressible Stokes system (1.1)–(1.2), the difference being the inclusion of the time derivative term in (5.2).

The finite element method is similar to the mixed method of Section 4, and as such it is only applicable to the case of Navier–slip boundary conditions (1.5). Furthermore, for technical reasons, convergence is proved only in the case $\gamma > \frac{N}{2}$. The method is constructed and analyzed in [**6**].

Definition 5.1 (Numerical scheme). Let ϱ_h^0 be as given in Definition 3.2. Determine functions

$$(\varrho_h^m, \boldsymbol{w}_h^m, \boldsymbol{u}_h^m) \in Q_h(\Omega)\times \boldsymbol{W}_h(\Omega)\times\boldsymbol{V}_h(\Omega), \quad m = 1,\dots,M,$$

such that for all $\phi_h \in Q_h(\Omega)$,

$$\int_\Omega d_t^h[\varrho_h^m]\phi_h \, dx - \Delta t \sum_{\Gamma\in\Gamma_h}\int_\Gamma \left(\varrho_-^m(\boldsymbol{u}_h^m\cdot\nu)^+ + \varrho_+^m(\boldsymbol{u}_h^m\cdot\nu)^-\right)[\![\phi_h]\!]_\Gamma \, dS(x) = 0, \tag{5.3}$$

and for all $(\boldsymbol{\eta}_h, \boldsymbol{v}_h) \in \boldsymbol{W}_h(\Omega)\times\boldsymbol{V}_h(\Omega)$,

$$\begin{aligned}
&\int_\Omega d_t^h[\boldsymbol{u}_h^m]\boldsymbol{v}_h + \mu \operatorname{curl}\boldsymbol{w}_h^m\boldsymbol{v}_h + [(\mu+\lambda)\operatorname{div}\boldsymbol{u}_h^m - p(\varrho_h^m)]\operatorname{div}\boldsymbol{v}_h \, dx = 0,\\
&\int_\Omega \boldsymbol{w}_h^m\boldsymbol{\eta}_h - \boldsymbol{u}_h^m \operatorname{curl}\boldsymbol{\eta}_h \, dx = 0,
\end{aligned} \tag{5.4}$$

for $m = 1,\dots,M$.

Existence of a numerical solution and various properties of these solutions hold as in the previous section with only minor modifications. We extend $\{\varrho_h^k, \boldsymbol{w}_h^k, \boldsymbol{u}_h^k\}_{h>0}$ for $k = 1,\dots,M$ to functions $\{(\varrho_h, \boldsymbol{w}_h, \boldsymbol{u}_h)\}_{h>0}$ defined on all of $(0,T)\times\Omega$ as in (4.4). The main result in [**6**] is that the sequence $\{(\varrho_h, \boldsymbol{w}_h, \boldsymbol{u}_h)\}_{h>0}$ converges to a weak solution of the Stokes approximation equations (5.1)–(5.2). The notion of a weak solution is similar to that in Definition 2.1.

Theorem 5.2. *Suppose $\gamma > \frac{N}{2}$ and $\varrho_0 \in L^\gamma(\Omega)$. Let $\{(\varrho_h, \boldsymbol{w}_h, \boldsymbol{u}_h)\}_{h>0}$ be a sequence of numerical solutions constructed according to Definition 5.1. Then, passing if necessary to a subsequence as $h\to 0$, $\boldsymbol{w}_h \rightharpoonup \boldsymbol{w}$ in $L^2(0,T;\boldsymbol{W}_0^{\mathrm{curl},2}(\Omega))$, $\boldsymbol{u}_h \rightharpoonup \boldsymbol{u}$ in $L^2(0,T;\boldsymbol{W}_0^{\mathrm{div},2}(\Omega))$, $\varrho_h\boldsymbol{u}_h \rightharpoonup \varrho\boldsymbol{u}$ in the sense of distributions, and $\varrho_h \to \varrho$ a.e. in $(0,T)\times\Omega$, where the limit triplet $(\varrho, \boldsymbol{w}, \boldsymbol{u})$ is a weak solution to the Stokes approximation equations* (5.1)*–*(5.2).

Comments to the proof of Theorem 5.2. The proof of convergence is similar to the proof of Theorem 3.3. However, the proof of higher integrability on the density and the proof of weak sequential continuity of the effective viscous flux now requires additional arguments in order to handle the time derivative in (5.4). In particular, the continuity scheme (5.3) needs to be used to handle the term $\boldsymbol{u}\boldsymbol{v}_t$, where $\boldsymbol{v}_t \in \boldsymbol{V}^{0,\perp}(\Omega)$ satisfies $\operatorname{div} \boldsymbol{v}_t = \varrho_t$. For technical reasons, we must then require $\gamma > \frac{N}{2}$. Moreover, the higher integrability estimate for the density now gives $p(\varrho_h)\varrho_h \in_{\mathrm{b}} L^1(0,T;L^1(\Omega))$. In addition, in this case we need in fact strong convergence of the velocity, $\boldsymbol{u}_h \to \boldsymbol{u}$. This is obtained through an Aubin–Lions argument using the spatial compactness on the velocity together with weak control of $d_t^h[\boldsymbol{u}_h]$. Due to space limitations we refer the reader to [**6**] for details.

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