Regularity of entropy solutions to nonconvex scalar conservation laws with monotone initial data

Wang Jinghua *

Abstract

We prove that for a given strictly increasing initial datum in C^k,the solution of the initial value problem is piecewise C^k smooth except for flux functions of nonconvex conservation laws in a certain subset of C^{k+1} of first category,defined in the range of the initial datum.

Keywords:piecewise smooth solutions;noncovex conservation laws;a set of first category.

Mathematics Subject Classification 2000: 35L65,35B65.

1 Introduction

Consider the initial value problem for the nonconvex scalar hyperbolic conservation law,

$$\begin{cases} u_t + f(u)_x = 0 & \text{in } \mathbb{R} \times (0, \infty) \\ u = \phi & \text{on } \mathbb{R} \times \{t = 0\}. \end{cases} \tag{1.1}$$

We assume that the initial function ϕ is C^k smooth,bounded and satisfies

$$\phi'(x) > 0, \qquad \forall x \in \mathbb{R}, \tag{1.2}$$

and the flux function f is C^{k+1} smooth,defined in the range of the initial datum,$3 \le k \le \infty$. In general,the initial value problem (1.1) does not admit a global smooth solution even if the initial datum is smooth, but for arbitrary bounded measurable initial datum a unique global entropy solution does exist.

The piecewise smoothness of entropy solution of convex conservation laws has been studied by many authors, e.g. Chen-Zhang [2], Dafermos [3, 4], Lax [9], Li & Wang [10, 11], Oleinik [13], Schaeffer [14], Tadmor & Tassa [15] and Tang & Wang & Zhao [20].For the structure of nonconvex conservation laws: Dafermos [5] studied the regularity and large time behavior of a conservation law with one inflection point by a direct approach, without making appeal to particular construction scheme;Kruzhkov and Petroyan [8] studied large time behavior of conservation laws using the explicit solution given by Hopf [7]. T.-P.Liu [12] studied admissible solutions of n by n systems of strictly hyperbolic conservation laws and proved that the random choice method

*Institute of Systems Sciences, Academy of Mathematics and System Sciences, The Chinese Academy of Sciences, Beijing 100080, P. R. China. Email: jwang@amss.ac.cn.

approximates discontinuities sharply and yields admissible,in a rough sense,piecewise continuous solutions and the study of the Riemann problem is sufficient in understanding the local and large time behavior of the solution.The results in [12] are new even for scalar conservation law for which one need only to assume the initial data to be of bounded variation and the second derivative of the flux function $f(\cdot)$ has isolated zeros.

The main results of this work will be obtained by the maximization process of $I(x,t,\cdot)$ introduced by Hopf [7] as follows:

$$I(x,t,u) = -\Phi^*(u) + xu - tf(u),\ (x,t) \in \mathbb{R} \times (0,\infty) \tag{1.3}$$

where $\Phi^*(u) = \sup\limits_{y\in\mathbb{R}} (yu - \Phi(y))$, $\Phi(y) = \int_0^y \phi(x)dx$. It follows from Bardi and Evans [1] and Kruzhkov and Petroyan [8] that $U(x,t) = \sup\limits_{u\in(\phi_-,\ \phi_+)} I(x,t,u)$ is the viscosity solution of the initial value problem of Hamilton-Jacobi equation,

$$\begin{cases} U_t + f(U_x) = 0 & \text{in } \mathbb{R} \times (0,\infty) \\ U = \Phi & \text{on } \mathbb{R} \times \{t = 0\}, \end{cases}$$

where $\phi_\pm = \lim\limits_{x\to\pm\infty} \phi(x)$ and then $u(x,t) = U_x(x,t)$,defined almost everywhere in $\mathbb{R} \times (0,\infty)$, is the entropy solution of (1.1).

The mapping $u = \phi(y)$ is one to one and onto from $(-\infty,+\infty)$ to (ϕ_-,ϕ_+) because of (1.2).Then it follows from the fact Hamilton-Jacobi equation is of hyperbolic type with finite domain of dependence that

$$U(x,t) = \sup_{u\in(\phi_-,\ \phi_+)} I(x,t,u) = \sup_{y\in\mathbb{R}} I(x,t,\phi(y)) = \max_{y\in\mathbb{R}} I(x,t,\phi(y)) = \max_{u\in(\phi_-,\ \phi_+)} I(x,t,u). \tag{1.4}$$

The maximizing value of $I(x,t,\cdot)$,u,must be a critical point of $I(x,t,\cdot)$,the solution of the equation

$$I_u(x,t,u) = -\phi^{-1}(u) + x - tf'(u) = -y + x - tf'(\phi(y)) = 0. \tag{1.5}$$

In fact in this paper we give an independent proof that $u(x,t)$,the maximizing function of $I(x,t,\cdot)$, is the solution of (1.1). We prove that for a given bounded C^k initial datum satisfying (1.2) then $u(x,t)$,the solution of (1.1),is given by the maximizing function of $I(x,t,\cdot)$ except for flux functions of nonconvex conservation laws in a subset of C^{k+1} of first category ,defined in the range of the initial datum and the solution is piecewise smooth.

Definition 1.1 The solution $u(x,t)$ is said to be C^k piecewise smooth if every bounded subset of $H = \mathbb{R} \times (0,\infty)$ intersects at most a finite number of shocks, every shock is piecewise C^{k+1} smooth curve, and $u(x,t)$ is C^k smooth on the complement of the shock set.

Definition 1.2 Let u_0 be a maximizing value for $I(x_0,t_0,\cdot)$, u_0 is called nondegenerate (degenerate resp.) if $\frac{\partial}{\partial y} I_u(x_0,t_0,\phi(y)) \mid_{y=y_0} \neq 0 (= 0\ resp.)$,where $u_0 = \phi(y_0)$.Let $u(x,t)$ be a maximizing function for $I(x,t,\cdot), (x,t) \in \theta$, where θ is an open set,$u(x,t)$ is called nondegenerate if for each $(x',t') \in \theta$, $u(x',t')$ is a nondegenerate maximizing value for $I(x',t',\cdot)$

We define the following subsets of H in which the maximization process can be used to study the local structure of the solution of (1.1).

Let $\mathscr{M}_{(x,t)}$ be the set consisting of all the maximizing values for $I(x,t,\cdot)$ and $\mathscr{N}_{(x,t)}$ be the set consisting of all the degenerate maximizing values of $I(x,t,\cdot)$,and

$$\begin{cases} \Gamma_1 = \{(x,t)|\ \exists \text{ two nondegenerate values of } \mathscr{M}_{(x,t)}\}, \\ \bar{\Gamma}_1 = \{(x,t)|\ \exists \text{ two connected components of } \mathscr{M}_{(x,t)}\}, \\ \Gamma_0^{(f)} = \{(x,t)|\ \exists \text{ unique connected component of } \mathscr{N}_{(x,t)}\}, \\ \Gamma_0^{(c)} = \{(x,t)|\ \exists\ n \text{ connected components of } \mathscr{M}_{(x,t)}, \text{ where } n \geq 3\}, \\ U = \{(x,t)|\ \exists \text{ unique nondegenerate maximizing value for } I(x,t,\cdot)\}. \end{cases} \tag{1.6}$$

For study of piecewise smoothness of the solutions we introduce a subset of $C^{k+1}(\phi_-,\phi_+)$ for a given initial datum ϕ as follows:

$$\begin{aligned} \Omega \;=\; & \{f \in C^{k+1}(\phi_-,\phi_+) \mid \exists\, y \in \mathbb{R} \text{ such that } (f'(\phi(y)))' < 0, (f'(\phi(y)))'' = 0, \\ & \cdots, (f'(\phi(y)))^{(k)} = 0, (k \geq 3)\ \ \& \\ & \exists\, \xi \in (y, y+\delta) \text{ such that } (f'(\phi(y)))'' < 0 \text{ or/and} \\ & \exists\, \eta \in (y-\delta, y) \text{ such that } (f'(\phi(y)))'' > 0,\ \forall\ \delta > 0\}. \end{aligned} \tag{1.7}$$

This paper is arranged as follows.We study the local structure of the solutions in Section 2.We prove that any point $(x_0,t_0) \in U$ has a neighborhood such that the solution of (1.1) is C^k smooth in the neighborhood;any point $(x_0,t_0) \in \Gamma_1$ has a neighborhood θ such that there is a C^{k+1} smooth shock $x = \gamma(t)$ passing through (x_0,t_0) in θ and $\theta \setminus \gamma(t) \subset U$.Then we show that for $f \in C^{k+1}(\phi_-,\phi_+) \setminus \Omega$ and K is a compact set $\subset H$ there are finitely many components of $\mathscr{N}_K = \{\mathscr{N}_{(x,t)} \mid \ \ (x,t) \in K\}$ and there are finitely many points in $\Gamma_0^{(c)} \cap K$. Under the assumption $f \in C^{k+1}(\phi_-,\phi_+) \setminus \Omega$ we show that any point $(x_0,t_0) \in \bar{\Gamma}_1 \setminus \Gamma_1$ has a neighborhood θ such that there is a shock $x = \gamma(t)$ passing though (x_0,t_0),and both $\gamma(t), t < t_0$ and $\gamma(t), t > t_0$ are C^{k+1} smooth in θ and $\theta \setminus \gamma(t) \subset U$; any point $(x_0,t_0) \in \Gamma_0^{(f)}$ has a neighborhood θ such that a unique C^{k+1} smooth shock $\gamma^+(t)$ emanating at (x_0,t_0) and $\theta \setminus (\gamma^+(t) \cup (x_0,t_0)) \subset U$;and any point $(x_0,t_0) \in \Gamma_0^{(c)}$ has a neighborhood θ such that there are n C^{k+1} smooth shocks $(x = \gamma_i(t), i = 1,\cdots,n)$, $n-1$ terminating and one emanating at (x_0,t_0) and $\theta \setminus (\bigcup_{i=1}^{n} \gamma_i(t) \cup (x_0,t_0)) \subset U$. In section 3 we will prove that Ω is a subset of first category of $C^{k+1}(\phi_-,\phi_+)$ and $H = U \cup \bar{\Gamma}_1 \cup \Gamma_0^{(f)} \cup \Gamma_0^{(c)}$ and for a given bounded C^k initial datum satisfying (1.2) then the solution of (1.1) is piecewise smooth when $f \in C^{k+1}(\phi_-,\phi_+) \setminus \Omega$.As a by-product we see that there are no contact discontinuities and no centers of centered rarefaction waves for $t > 0$ in the solution of (1.1).

2 Local solution structure

In this section we study the local structure of the solutions of (1.1),mentioned in last section.We record here the following relations that will be needed later:

$$\frac{\partial}{\partial y} I_u(x,t,\phi(y)) = -1 - t(f'(\phi(y)))'. \tag{2.1}$$

$$\frac{\partial^m}{\partial y^m} I_u(x,t,\phi(y)) = -t(f'(\phi(y)))^{(m)}, \tag{2.2}$$

where $1 < m \leq k$.

2.1 Some useful lemmas

First we give two lemmas which hold for all the flux functions $f \in C^{k+1}(\phi_-, \phi_+)$ for a given initial datum ϕ.

Lemma 2.1 *Let U and Γ_1 be set defined in (1.6) and let $u(x,t)$ be the maximizing function of $I(x,t,\cdot)$ defined by (1.3). Then we have*
- *U is an open subset of H, and $u(x,t)$ is C^k smooth solution of (1.1) on U.*
- *Any point $(x_0,t_0) \in \Gamma_1$ has a neighborhood θ of (x_0,t_0) such that $\Gamma_1 \cap \theta$ is a C^{k+1} smooth shock $x = \gamma(t)$ passing through (x_0,t_0). $u(x,t)$ is C^ksmooth on both components of $\theta \backslash \Gamma_1$ and is solution of (1.1).*

Proof:The most parts of proof of the lemma is similar to relevant parts in lemma 1.1 and lemma 1.2 in [14] and $u(x,t)$ is solution of (1.1) on U can be deduced from that $u(x,t)$ is C^k smooth on U and (1.5).

We only prove that $x = \gamma(t)$ is a shock here.For $(x_0,t_0) \in \Gamma_1$, suppose that $u_1 < u_2$ are two nondegenerate maximizing values for $I(x_0,t_0,\cdot)$. Then by the implicit function theorem there exists some neighborhood θ of (x_0,t_0) such that for $(x,t) \in \theta$,the maximum of $I(x,t,\cdot)$ is assumed at either $u_1(x,t)$ or $u_2(x,t)$ or both,and $u_i(x,t)$ is C^k smooth and near u_i , $i = 1,2$. Hence every point of θ belongs to U or Γ_1. Here $\Gamma_1 \cap \theta$ is a curve of discontinuity $x = \gamma(t)$ defined by the following equation

$$I(x,t,u_2(x,t)) - I(x,t,u_1(x,t)) = 0. \tag{2.3}$$

It follows from (2.3) that the jump relation

$$\gamma'(t) = \sigma(u_1(x,t),u_2(x,t)) := [f(u_1(x,t)) - f(u_2(x,t))]/[u_1(x,t) - u_2(x,t)] \tag{2.4}$$

is satisfied along $x = \gamma(t)$.Suppose that $(\bar{x},\bar{t})$ is a point in $x = \gamma(t)$.It follows from Lemma 4 in [8] that for any $u \in (\bar{u}_1,\bar{u}_2)$,where $\bar{u}_1 = u_1(\gamma(\bar{t})-,\bar{t}) < \bar{u}_2 = u_2(\gamma(\bar{t})+,\bar{t})^{(*)}$,

$$\sigma(\bar{u}_1,u) \geq \sigma(\bar{u}_1,\bar{u}_2) + \frac{1}{\bar{t}}\left(\frac{\Phi^*(\bar{u}_1) - \Phi^*(\bar{u}_2)}{\bar{u}_1 - \bar{u}_2} - \frac{\Phi^*(u) - \Phi^*(\bar{u}_1)}{u - \bar{u}_1}\right). \tag{2.5}$$

Let u tends to $\bar{u}_1$ in (2.5) we have

$$f'(\bar{u}_1) \geq \sigma(\bar{u}_1,\bar{u}_2) + \frac{1}{\bar{t}}\left(\frac{\Phi^*(\bar{u}_1) - \Phi^*(\bar{u}_2)}{\bar{u}_1 - \bar{u}_2} - \Phi^{*'}(\bar{u}_1)\right). \tag{2.6}$$

It follows from (1.2) that $\Phi''(y) > 0, y \in \mathbb{R}$.Thus $\Phi^*(u)$ as a Legendre transform of Φ is strictly convex.Therefore the last term in (2.6) is positive,which implies

$$f'(\bar{u}_1) > \sigma(\bar{u}_1,\bar{u}_2) \tag{2.7}$$

Similarly

$$\sigma(\bar{u}_1,\bar{u}_2) > f'(\bar{u}_2). \tag{2.8}$$

So $x = \gamma(t)$ is a shock follows from (2.7) and (2.8). $\square$

$^{(*)}$The inequality $u_1(\gamma(\bar{t})-, \bar{t}) < u_2(\gamma(\bar{t})+, \bar{t})$ can be deduced from the lemma 1 in [8].

It is interesting to note from lemma 2.1 that there are no contact discontinuities in the solutions of (1.1) if the initial datum is strictly monotone and smooth.

The formula (1.5) implies that $I_u(x, t, u) = 0$ is equivalent to $x = y + tf'(u)$. Therefore if u is the critical value of $I(x, t, \cdot)$ then $y = \phi^{-1}(u) = x - tf'(u)$ is the emanating point at $t = 0$ of the characteristic passing through (x, t) with the slope of $f'(u)$.On the other hand for $(x, t) \in C$,

$$C := \{(x, t) \mid x = y + tf'(\phi(y)), t > 0\}, \tag{2.9}$$

then $u = \phi(y)$ is a critical value of $I(x, t, \cdot)$,which is a candidate of maximizing values for$I(x, t, \cdot)$. Naturally, it may be asked if $\phi(y)$ is a maximizing value for $I(x, t, \cdot)$ for $(x, t) \in C$. The following lemma gives an answer.

Lemma 2.2 *Assume $\phi(y)$ is bounded, C^k smooth and satisfies (1.2) and let*

$$C = \{(x, t) : x = y + tf'(\phi(y)), t > 0\}.$$

Then one of the following statements must hold:

- *case (1) $\phi(y)$ is the unique maximizing value for $I(x, t, \cdot)$,$\forall (x, t) \in C$;or*
- *case (2) there exists a point $(x_1, t_1) \in C$ such that $\phi(y)$ is either the unique degenerate maximizing value for $I(x_1, t_1, \cdot)$ or one of at least two maximizing values for $I(x_1, t_1, \cdot)$.Then $\phi(y)$ is the unique nondegenerate maximizing value for $I(x, t, \cdot)$ for $(x, t) \in C^- = C \cap \{(x, t) : t_1 > t > 0\}$ while $\phi(y)$ is no longer the maximizing value for $I(x, t, \cdot)$ for $(x, t) \in C^+ = C \cap \{(x, t) : t > t_1\}$.*

Proof: First we claim that $\phi(y)$ will be no longer a maximizing value for $I(x, t, \cdot)$ for $(x, t) \in C^+$ if there exist at least two maximizing values for $I(x_1, t_1, \cdot)$ and $\phi(y)$ is one of them. Let $\phi(\tilde{y})$,nearby $\phi(y)$,be another maximizing value for $I(x_1, t_1, \cdot)$.Thus

$$I(x_1, t_1, \phi(y))) = I(x_1, t_1, \phi(\tilde{y})) \tag{2.10}$$

It follows from (2.7) and (2.8) that

$$f'(\phi(y))(\phi(y) - \phi(\tilde{y})) < f(\phi(y)) - f(\phi(\tilde{y})),$$

therefore for $t > t_1$

$$f'(\phi(y))(\phi(y) - \phi(\tilde{y}))(t - t_1) < (f(\phi(y)) - f(\phi(\tilde{y})))(t - t_1). \tag{2.11}$$

and then

$$(\phi(y) - \phi(\tilde{y})))(x - x_1) < (f(\phi(y)) - f'(\phi(\tilde{y})))(t - t_1), \tag{2.12}$$

where $(x - x_1) = f'(\phi(y))(t - t_1)$ is used. Adding both sides of (2.10) and (2.12) and noting the definition of $I(x_1, t_1, \cdot)$ in (1.3) gives $I(x, t, \phi(y)) < I(x, t, \phi(\tilde{y}))$.Thus $\phi(y)$ is no longer a maximizing value of $I(x, t, \cdot)$ for $(x, t) \in C^+$.

If $\phi(y)$ is a degenerate maximizing value for $I(x_1, t_1, \cdot)$,namely

$$\frac{\partial}{\partial y} I_u(x_1, t_1, \phi(y)) = -1 - t_1 (f'(\phi(y)))' = 0.$$

Consequently, $I_u(x, t, \phi(y)) = 0$ and $I_{uu}(x, t, \phi(y)) = \frac{\partial}{\partial y} I_u(x_1, t_1, \phi(y))\phi'(y)^{-1} > 0$ for $(x, t) \in C^+$, which implies that $\phi(y)$ is a local minimizing value for $I(x, t, \cdot)$ for $(x, t) \in C^+$. Obviously, $\phi(y)$ is not a maximizing value for $I(x, t, \cdot)$ for $(x, t) \in C^+$. Then by the facts that for $(x, t) \in C$ (t is sufficiently small), $I(x, t, \cdot)$ has a unique critical point $\phi(y)$ and $I_{uu}(x, t, \phi(y)) < 0$,the proof is complete. $\square$

The statement same as the lemma 2.2 was obtained for convex conservation laws by Li & Wang [10].

Remark 2.1 *We define $C^- := \{(x, t) \mid x = y + tf'(\phi(y)), t > 0\}$ in case (1) and $C^- := C \cap \{(x, t) \mid t_1 > t > 0\}$ in case (2) in lemma2.2 as a regular characteristic.Then the facts that any two regular characteristics can not cross follows from lemma 2.2 and any point in a regular characteristic has a neighborhood belonging to U follows from lemma 2.1.*

Now we will give several lemmas related to the restriction on f, the flux function of the equation in (1.1) given by (1.7).We set

$$\begin{aligned} L: &= \{y \in \mathcal{L} \mid \\ &\quad \exists\, \xi \in (y, y+\delta) \text{ such that } (f'(\phi(\xi)))'' < 0 \text{ or/and} \\ &\quad \exists\, \eta \in (y-\delta, y) \text{ such that } (f'(\phi(\eta)))'' > 0,\ \forall\, \delta > 0\}, \end{aligned} \tag{2.13}$$

where

$$\begin{aligned} \mathcal{L}: &= \{y \in \mathbb{R} \mid (f'(\phi(y)))' < 0, (f'(\phi(y)))'' = 0, \\ &\quad \cdots, (f'(\phi(y)))^{(k)} = 0, (k \geq 3)\}. \end{aligned} \tag{2.14}$$

It is easy to know that$f \in C^{k+1}(\phi_-, \phi_+) \setminus \Omega$ if and only if $L \neq \emptyset$,where Ω is given by (1.7).

Lemma 2.3 *Suppose that $L = \emptyset$ and $[y_1, y_2]$ is a connected component of $\mathcal{L}$. Then there exists $\delta > 0$ such that*

$$(f'(\phi(y)))' < 0,\ y \in [y_1 - \delta, y_1]\ \&\ (f'(\phi(y)))' \textit{strictly decreases on } (y_1 - \delta, y_1) \tag{2.15}$$

$$(f'(\phi(y)))' < 0,\ y \in [y_2, y_2 + \delta]\ \&\ (f'(\phi(y)))' \textit{strictly increases on } (y_2, y_2 + \delta). \tag{2.16}$$

Proof: $L = \emptyset$ implies that there exists $\delta > 0$ such that

$$(f'(\phi(y)))' < 0, (f'(\phi(y)))'' \leq 0, y \in [y_1 - \delta, y_1], \tag{2.17}$$

$$(f'(\phi(y)))' < 0, (f'(\phi(y)))'' \geq 0, y \in [y_2, y_2 + \delta]. \tag{2.18}$$

We claim that $(f'(\phi(y)))''$ does not vanish on any interval in $[y_2, y_2 + \delta]$. Otherwise there is a interval,say,$[a, b] \subseteq [y_2, y_2 + \delta]$ such that $(f'(\phi(y)))'' = 0, y \in [a, b]$,which implies that$(f'(\phi(y)))''' =$

$0, \cdots, (f'(\phi(y)))^{(k)} = 0, (k \geq 3), y \in [a, b]$. Therefore $[a, b]$ is a subset of a connected component of $\mathcal{L}$. It follows from $[y_1, y_2]$ is a connected component of $\mathcal{L}$ that $y_2 < a$ .Hence we can assume that a is the left endpoint of the connected component of $\mathcal{L}$. Thus (2.17) implies there exists $\delta_1 > 0$ with $\delta_1 \leq a - y_2$ such that $(f'(\phi(y)))'' \leq 0, y \in [a - \delta_1, a]$,which together with (2.18) imply that $(f'(\phi(y)))''$ vanishes on $[a - \delta_1, b]$. It contradicts to the assumption a is the left endpoint of the connected component of $\mathcal{L}$.
Similarly we have $(f'(\phi(y)))''$ does not vanish on any interval in $[y_1 - \delta, y_1]$. $\square$

Lemma 2.4 *Suppose $L = \emptyset$ and K is a compact set $\subset H$, then there are finitely many connected components of $\mathscr{N}_K = \bigcup_{(x,t) \in K} \mathscr{N}_{(x,t)}$.*

Proof:Each connected component of $\mathscr{N}_K$ is either a closed interval or a point since $\mathscr{N}_{(x,t)} \bigcap \mathscr{N}_{(x',t')} = \emptyset, (x, t) \neq (x', t')$. Let $[u_i^-, u_i^+]$ with $u_i^- \leq u_i^+$ be a connected component of $\mathscr{N}_K$. It follows from definition 1.2 and (2.1) and (2.2) and lemma 2.2 that one of the following two cases must hold.
The case (1): $[u_i^-, u_i^+]$ is connected component of $\mathscr{N}_{(x,t)}$ and $[y_i^-, y_i^+] \subset \mathcal{L}$ where $y_i^\pm = x - tf'(u_i^\pm) = \phi^{-1}(u_i^\pm)$. Thus $[y_i^-, y_i^+]$ is a connected component of $\mathcal{L}$. Then there is δ_i such that (2.15) and (2.16) hold by lemma 2.3 and $(f'(\phi(y)))' < -1/2T, y \in [y_i - \delta_i, y_i] \cup [y_i^-, y_i^+ \delta_i]$. Here and later on $T = \max\{t \,|(x, t) \in K\}$.
The case (2): $u_i^- = u_i^+ = u_i \in \mathscr{N}_{(x,t)}$ and $y_i \notin \mathcal{L}$ where $y_i = x - tf'(u_i) = \phi^{-1}(u_i)$. Then

$$(f'(\phi(y_i)))' = -1/t, (f'(\phi(y_i)))^{(l)} = 0, (f'(\phi(y_i)))^{(m)} > 0 \text{ for } 1 < l < m \leq k, m \text{ is odd.} \quad (2.19)$$

By Talor expansion it follows from (2.19) that there exists $\delta_i > 0$ such that

$$(f'(\phi(y)))' < -\frac{1}{2T}, y \in [y_i - \delta_i, y_i] \;\&\; (f'(\phi(y)))'' < 0,\ y \in (y_i - \delta_i, y_i),$$

$$(f'(\phi(y)))' < -\frac{1}{2T}, y \in [y_i, y_i + \delta_i] \;\&\; (f'(\phi(y)))'' > 0,\ y \in (y_i, y_i + \delta_i);$$

therefore

$$(f'(\phi(y)))' < -\frac{1}{2T},\ y \in [y_i - \delta_i, y_i] \;\&\; (f'(\phi(y)))' \text{strictly decreases on } (y_i - \delta_i, y_i) \quad (2.20)$$

$$(f'(\phi(y)))' < -\frac{1}{2T},\ y \in [y_i, y_i + \delta_i] \;\&\; (f'(\phi(y)))' \text{strictly increases on } (y_i, y_i + \delta_i). \quad (2.21)$$

Next we claim there are finitely many connected components of $\mathscr{N}_K$. If not,without loss of generality, assume that there exists a sequence $\{[y_i^-, y_i^+]\}$,which is strictly increasing and bounded because $y_i^\pm = x - tf'(u_i^\pm)$ and K is a compact set. Then it must be convergent to a point y_0 from the left.The point y_0,as cluster point of y_i^+ and y_i^-,is a point of a connected component of $\mathcal{L}$,here the assumption K is a compact set is used to make $(f'(\phi(y_0)))' < 0 \leqq -1/2T$.According to (2.15) for case (1) there exists $\delta > 0$ such that

$$(f'(\phi(y)))' \text{strictly decreases on } (y_0 - \delta, y_0) \quad (2.22')$$

if y_0 is the connected component or the left endpoint of the connected component as an interval. If y_0 is not the left endpoint of the connected component as an interval,say $[a, b]$,then we have

$$(f'(\phi(y)))'' = 0 \text{ on } [a, y_0] \quad (2.23').$$

On other the hand $y_i^+ \in [y_0-\delta, y_0]$ ($y_i^+ \in [a, y_0]$ resp.) for i is big enough, there exists $\delta_i > 0$ such that

$$(f'(\phi(y)))'\text{strictly increases on } (y_i^+, y_i^+ + \delta_i) \tag{2.24$'$}$$

due to (2.16) for case (1) and (2.21) for case (2).We obtain contradiction from (2.24′) and (2.22′) or from (2.24′) and (2.23′) respectively. □

As a consequence of this lemma we see that under that under the assumptions of lemma 2.4, there are finitely many points in $(\Gamma_0^{(f)} \cup (\bar{\Gamma}_1 \setminus \Gamma_1)) \cap K$.

Lemma 2.5 *Suppose that* $L = \emptyset$. *Then for each point* $(x,t) \in H$, *there are finitely many connected components of* $\mathscr{M}_{(x,t)}$.

Proof:It follows from lemma 2.4 that there are finitely many connected components of $\mathscr{N}_{(x,t)}$. Therefore we only have to prove that there are finitely many nondegenerate maximizing values belonging to $\mathscr{M}_{(x,t)} \setminus \mathscr{N}_{(x,t)}$. If not,assume that there are infinitely many nondegenerate maximizing values ,say,$\{u_i, i = 1, 2, \cdots, \}$ with $u_i < u_{i+1}$ belonging to $\mathscr{M}_{(x,t)} \setminus \mathscr{N}_{(x,t)}$. It follows from (2.1) and definition1.2 that

$$\frac{\partial}{\partial y} I_u(x, t, \phi(y_j)) = -1 - t(f'(\phi(y_j)))' < 0, j = i, i+1 \tag{2.22}$$

where $y_j = x - tf'(u_j) = \phi^{-1}(u_j)$. Then there exists $\bar{u}_i \in (u_i, u_{i+1})$ such that $I(x, t, \cdot)$ assumes minimum at $\bar{u}_i$.Thus we have

$$I_{uu}(x, t, \phi(\bar{y}_i)) = \phi'(\bar{y}_i)^{-1} \frac{\partial}{\partial y} I_u(x, t, \phi(\bar{y}_i)) = (-1 - t(f'(\phi(\bar{y}_i)))')\phi'(\bar{y}_i)^{-1} \geq 0 \tag{2.23}$$

where $\bar{y}_i = x - tf'(\bar{u}_i) = \phi^{-1}(\bar{u}_i) \in (y_i, y_{i+1})$. It follows from (2.22) and (2.23) that there exists $[y_i', y_i''] \subset (y_i, y_{i+1})$ with $y_i' \leq y_i''$ such that $-1 - t(f'(\phi(y)))'$ assumes nonnegative maximum on $[y_i', y_i'']$ and

$$\begin{gathered}(f'(\phi(y)))' \leq -1/t, (f'(\phi(y)))'' = 0 \text{ on } [y_i', y_i''] \\ \exists\, \xi \in (y_i' - \delta, y_i') \text{ such that } (f'(\phi(\xi)))'' < 0\, \& \\ \exists\, \eta \in (y_i'', y_i'' + \delta) \text{ such that } (f'(\phi(\eta)))'' > 0, \forall \delta > 0.\end{gathered} \tag{2.24}$$

The sequence $\{y_i', y_i''\}$ is strictly increasing and bounded, then it must be convergent to a point y_0 from the left. y_0,as cluster point of y_i' and y_i'',is a point of a connected component of $\mathcal{L}$.According to (2.15) there exists $\delta > 0$ such that

$$(f'(\phi(y)))'\text{strictly decreases on } (y_0 - \delta, y_0) \tag{2.25}$$

if y_0 is the connected component of $\mathcal{L}$ or the left endpoint of the connected component of $\mathcal{L}$ as an interval.On other the hand $y_i'' \in (y_0 - \delta, y_0)$ for i is big enough.then we obtain contradiction from (2.24) and (2.25). If y_0 is not the left endpoint of the connected component $\mathcal{L}$ as an interval,say $[a, b]$,then we have $(f'(\phi(y)))'' = 0$ on $[a, y_0]$,which contradicts to (2.24) for i is big enough such that $y_i'' \in [a, y_0]$. □

The result by Li and Wang [11] the same as lemma 2.5 was obtained for convex conservation laws under the hypothesis that ϕ is locally finite to f.

Corollary 2.1 *Suppose $L = \emptyset$ and K is a compact set $\subset H$,then there are finitely many points in $\Gamma_0^{(c)} \cap K$.*

Proof:Assume there are infinitely many points in $\Gamma_0^{(c)} \cap K$. By lemma 2.4 there are finitely many points (x'_i, t'_i) in $\Gamma_0^{(c)} \cap K, i = 1, 2, \cdots, n$ such that there are degenerate maximizing values of $\mathscr{M}_{(x'_i,t'_i)}, i = 1, 2, \cdots, n$.Therefore there are infinitely many points $\{(x_i, t_i) \in \Gamma_0^{(c)} \cap K, i = 1, 2, \cdots\}$ such that there are no degenerate maximizing values of $\mathscr{M}_{(x_i,t_i)}, i = 1, 2, \cdots$. Thus by the definition of $\Gamma_0^{(c)}$ given by (1.7) there are at least three nondegenerate maximizing values of each $\mathscr{M}_{(x_i,t_i)}, i = 1, 2, \cdots$. Then it follows from remark 2.1 and the proof in lemma 2.5 for (2.24) that for any n points belonging $\Gamma_0^{(c)} \cap K$ there are at least $n+1$ intervals or/and points $[y'_i, y''_i]$ satisfying

$$(f'(\phi(y)))' \leq -1/T, (f'(\phi(y)))'' = 0 \text{ on } [y'_i, y''_i]$$
$$\exists\, \xi \in (y'_i - \delta, y'_i) \text{ such that } (f'(\phi(\xi)))'' < 0 \ \ \&$$
$$\exists\, \eta \in (y''_i, y''_i + \delta) \text{ such that } (f'(\phi(\eta)))'' > 0, \forall \delta > 0.$$

with $y'_i \leq y''_i, i = 1, 2, \ldots, m$ where $n+1 \leq m$ and $[y'_i, y''_i] \cap [y'_j, y''_j] = \emptyset, i \neq j$. Note that the compactness of the set K plays the role in proof of this corollary as it does in lemma 2.4.Then similar to the part of proof after (2.24) in lemma 2.5 we get a contradiction to the assumption of infinity of points in $\Gamma_0^{(c)} \cap K$. $\square$

Lemma 2.6 *Suppose $L = \emptyset$. If $[u_1^-, u_1^+]$ and $[u_2^-, u_2^+]$ are two neighboring connected components of $\mathscr{M}_{(x_0,t_0)}$,where $u_1^- \leq u_1^+ < u_2^- \leq u_2^+$, then there exists a neighborhood θ of (x_0, t_0) such that $\theta \cap G \cap \Gamma_1$ is a C^{k+1} smooth shock $x = \gamma^-(t)$ terminating at (x_0, t_0) and $(\theta \cap G) \setminus \Gamma_1 \subset U$. Here G is an open triangle region bounded by $t = 0$, the characteristics $x = \phi^{-1}(u_1^+) + tf'(u_1^+)$ and $x = \phi^{-1}(u_2^-) + tf'(u_2^-)$.*

Proof:By the remark 2.1 we have that for any maximizing value u for $I(x, t, \cdot)$,$(x, t) \in G$,then $\phi^{-1}(u) \in J$ where $J = (\phi^{-1}(u_1^+), \phi^{-1}(u_2^-))$. We will prove that for a given $\delta > 0$ there is a neighborhood θ of (x_0, t_0) such that for any maximizing value u for $I(x, t, \cdot)$,$(x, t) \in \theta \cap G$,then $\phi^{-1}(u)$ belongs to J_1 or/and J_2.Here $J_1 = (\phi^{-1}(u_1^+), \phi^{-1}(u_1^+) + \delta)$ and $J_2 = (\phi^{-1}(u_2^-) - \delta, \phi^{-1}(u_2^-))$. If not, then there exists a sequence $(x_n, t_n) \in \theta \cap G$ converging to (x_0, t_0) and a sequence u_n belonging to $[\phi^{-1}(u_1^+) + \delta, \phi^{-1}(u_2^-) - \delta]$ such that

$$I(x_n, t_n, u_n) = \max_{u \in J} I(x_n, t_n, u) \quad (n = 1, 2, \ldots).$$

Since the set $[\phi^{-1}(u_1^+) + \delta, \phi^{-1}(u_2^-) - \delta]$ is compact and we can select a subsequence of u_n,written again as u_n,converges to u_0 belonging to the set $[\phi^{-1}(u_1^+) + \delta, \phi^{-1}(u_2^-) - \delta]$ .Then

$$I(x_0, t_0, u_0) = \lim_{n \to \infty} I(x_n, t_n, u_n) = I(x_0, t_0, u_0),$$

since $I(x, t, u)$ is a continuous function of x, t and u;and $m(x, t) = \max_{u \in J} I(x, t, u)$ is continuous of x and t. This contradicts to that $[u_1^-, u_1^+]$ and $[u_2^-, u_2^+]$ are two neighboring connected components of $\mathscr{M}_{(x_0,t_0)}$.

We claim that if δ is small there is a neighborhood θ of (x_0, t_0) such that $I(x, t, \cdot)$ has a unique nondegenerate (local) maximizing function $u_1(x, t) \in (u_1^+, \phi(\phi^{-1}(u_1^+) + \delta))$,$(x, t) \in \theta \cap G$. There

are two cases to be considered.
Case(1) $u_1^+ \in \mathscr{N}_{(x_0,t_0)}$.Thus $-1-t_0(f'(\phi(y)))'=0, y=\phi^{-1}(u_1^+)$ by (2.1). Then according to (2.16) and (2.21) there is $\delta>0$ such that

$$(f'(\phi(y)))'<0 \text{ and } -1-t_0(f'(\phi(y)))'<0\,, y\in J_1 \tag{2.26}$$

Therefore direct computation deduces that any two characteristics emanating from J_1 can not intersect at the time less than or equal to t_0.
Case(2) $u_1^+ \in \mathscr{M}_{(x_0,t_0)} \setminus \mathscr{N}_{(x_0,t_0)}$. Thus $-1-t_0(f'(\phi(y)))'<0, y=\phi^{-1}(u_1^+)$ by (2.1).Just choosing $\delta>0$ small such that

$$-1-t_0(f'(\phi(y)))'<0, y\in J_1 \tag{2.27}$$

then any two characteristics emanating from J_1 can not intersect at the time less than or equal to t_0. Thus the characteristic $x=\phi^{-1}(u_1^+)+\delta+tf'(\phi(\phi^{-1}(u_1^+)+\delta)), t>0$ intersects the characteristic $x=\phi^{-1}(u_2^-)+tf'(u_2^-), t>0$ at the time $t_1<t_0$; and for $\bar{t}\in(t_1,t_0)$,the mapping $x=y+\bar{t}f'(\phi(y))$ is bijective and continuous from $(\phi^{-1}(u_1^+),\bar{y})$ to $(\bar{x_1},\bar{x_2})$ where $\bar{x_1}=\phi^{-1}(u_1^+)+\bar{t}f'((u_1^+)$, $\bar{x_2}=\phi^{-1}(u_2^-))+\bar{t}f'(u_2^-)$ and $\bar{x_2}=\bar{y}+\bar{t}f'(\phi(\bar{y}))$. In summary for each $(x,t)\in\theta\cap G\cap\{t>t_1\}$ there is a unique characteristic $x=y+tf'(\phi(y))$ from J_1 passing though it, namely there is a unique $y\in J_1$ such that $I_u(x,t,u)=0$ and $I_{uu}(x,t,u)<0, u=\phi(y)$. Here the fact $-1-t(f'(\phi(y)))'<0, t<t_0, y\in J_1$,which follows from (2.26) and (2.27), is used.Thus by the implicit function theorem there exists a unique nondegenerate (local) maximizing function $u_1(x,t)$ for $I(x,t,\cdot)$ in $\theta\cap G\cap\{t>t_1\}$ and $u_1(x,t)\in(u_1^+,\phi(\phi^{-1}(u_1^+)+\delta))$ is C^k smooth.

By the same way,we have that if δ is small there is a neighborhood θ of (x_0,t_0) such that the characteristic $x=\phi^{-1}(u_2^-)-\delta+tf'(\phi(\phi^{-1}(u_2^-)-\delta)), t>0$ intersects the characteristic $x=\phi^{-1}(u_1^+)+tf'(u_1^+), t>0$ at the time $t_2<t_0$.There exists a unique nondegenerate (local) maximizing function $u_2(x,t)$ for $I(x,t,\cdot)$ in $\theta\cap G\cap\{t>t_2\}$ and $u_2(x,t)\in(\phi(\phi^{-1}(u_2^-)-\delta),u_2^-)$ is C^k smooth.

We choose θ such that $\tilde{t}=\inf\limits_{(x,t)\in\theta\cap G} t\geq\max(t_1,t_2)$ Thus there are two nondegenerate (local) maximizing functions $u_1(x,t)$ and $u_2(x,t)$ in $\theta\cap G$.We consider the function

$$F(x,t,u_1(x,t),u_2(x,t))=I(x,t,u_1(x,t))-I(x,t,u_2(x,t))$$

in $\theta\cap G$.Then we have

$$F(x,t,u_1(x,t),u_2(x,t))>0, x=\phi^{-1}(u_1^+)+tf'(u_1^+))+0, t\in(\tilde{t},t_0),$$

$$F(x,t,u_1(x,t),u_2(x,t))<0, x=\phi^{-1}(u_2^-)+tf'(u_2^-))-0, t\in(\tilde{t},t_0),$$

and

$$\frac{\partial}{\partial x}F(x,t,u_1(x,t),u_2(x,t))=u_1(x,t)-u_2(x,t)<0, x\in(\phi^{-1}(u_1^+)+tf'(u_1^+),\phi^{-1}(u_2^-)+tf'(u_2^-)), t\in(\tilde{t},t_0).$$

Thus for each $t'\in(\tilde{t},t_0)$,there is a unique x' such that

$$F(x',t',u_1(x',t'),u_2(x',t'))=0$$

,which means $(x',t')\in\Gamma_1$ by the fact $-1-t'(f'(\phi(y)))'<0, y\in J_1$ and J_2,deduced from (2.26) and (2.27).Then the lemma follows from lemma 2.1. □

2.2 Local piecewise smooth solutions

Theorem 2.1 *Suppose $L = \emptyset$.Any point $(x_0, t_0) \in \Gamma_0^{(f)}$ has a neighborhood θ such that $\theta' \cap \Gamma_1$ is a C^{k+1} smooth shock $x = \gamma^+(t)$ emanating at (x_0, t_0) and $\theta' \setminus \Gamma_1 \subset U$ where $\theta' = \theta \setminus (x_0, t_0)$.*

Proof:The assumption $(x_0, t_0) \in \Gamma_0^{(f)}$ is equivalent to that $\mathscr{N}_{(x_0,t_0)}$ has a unique connected component,say $[u^-, u^+]$. Similar to lemma 2.6 by the facts functions $I(x, t, u)$ and $m(x, t) = \max_{u \in (\phi_-, \phi_+)} I(x, t, u)$ are continuous,we can show that for a given $\delta > 0$ then there is a neighborhood θ of (x_0, t_0) such that for any maximizing value u for $I(x, t, \cdot)$,$(x, t) \in \theta \setminus G$ then $\phi^{-1}(u)$ belongs to J_1 or/and J_2.Here $J_1 := (\phi^{-1}(u^-) - \delta, \phi^{-1}(u^-))$, $J_2 := (\phi^{-1}(u^+), \phi^{-1}(u^+) + \delta)$ and G is a closed triangle region bounded by $t = 0$,the characteristics $x = \phi^{-1}(u^+) + tf'(u^+)$ and $x = \phi^{-1}(u^-) + tf'(u^-)$.

Consider a family of characteristics from J_1:$\{x = \xi + tf(\phi(\xi)), t > 0, \xi \in J_1\}$. According to (2.15) and (2.20) there is $\delta > 0$ such that

$$(f'(\phi(y)))' < 0 \text{ , } y \in J_1 \ \& \ (f'(\phi(y)))' \text{strictly decreases on } J_1. \tag{2.28}$$

Then for each characteristic $x = \xi + tf(\phi(\xi)), t > 0, \xi \in J_1$, we define the degenerate point $(x(\xi), t(\xi))$ of it such that $I_{uu}(x(\xi), t(\xi), \phi(\xi)) = 0$. Thus the locus of the degenerate points of the characteristics emanating from J_1 can be written in the following form

$$\begin{cases} x(\xi) = \xi - \dfrac{f'(\phi(\xi))}{(f'(\phi(\xi)))'} \\ t(\xi) = -\dfrac{1}{(f'(\phi(\xi)))'} \end{cases} \qquad \xi \in J_1. \tag{2.29}$$

It follows from (2.28) that the locus of the degenerate points defined by (2.29) is smooth curve:$x = x_r(t), t > t_0$ from (x_0, t_0) with corresponding slope $f'(\phi(\xi)), \xi \in J_1$ and $x_r'(t_0 + 0) = f'(u^-)$ and $x = x_r(t)$ is strictly convex. Similarly the locus of the degenerate points of the characteristics emanating from J_2 is a smooth curve:$x = x_l(t), t > t_0$ from (x_0, t_0) with corresponding slope $f'(\phi(\xi)), \xi \in J_2$ and $x_l'(t_0 + 0) = f'(u^+)$ and $x = x_l(t)$ is strictly concave. The assumption $[u^-, u^+]$ is a unique connected component of $\mathscr{N}_{(x_0,t_0)}$ implies $f'(u^-) \geq f'(u^+)$, which together with convexity of $x = x_r(t)$ and concavity of $x = x_l(t)$ deduce that $x = x_r(t) - x_l(t) > 0, t > t_0$ in θ.

We claim that

$$\begin{aligned} &\text{there exist a unique smooth nondegenerate (local) maximizing function } u_1(x,t) \in \bar{J}_1 \\ &\text{for } I(x,t,\cdot), (x,t) \in \theta' \bigcap (\{x < x_r(t), t > t_0\} \bigcup \{x < \phi^{-1}(u^-) + tf'(u^-), t \leq t_0\}), \text{where} \\ &\bar{J}_1 := (\phi(\phi^{-1}(u^-) - \delta), u^-). \end{aligned} \tag{2.30}$$

It follows from (2.28) that each characteristic $x = \bar{\xi} + tf'(\phi(\bar{\xi})), \bar{\xi} \in J_1$ can only intersect a characteristic from a point ξ in J_1 with $\xi < \bar{\xi}$ after the time $\bar{t}$,at which $x_r(\bar{t}) = \bar{\xi} + \bar{t}f'(\phi(\bar{\xi}))$.Thus any two characteristics emanating from ξ_1, ξ_2 in J_1 with $\xi_1 < \xi_2 < \bar{\xi}$ can only intersect after the time $\bar{t}$. Then the mapping $x = \xi + \bar{t}f'(\phi(\xi))$ from $(\phi^{-1}(u^-) - \delta, \bar{\xi})$ to $(\bar{x}, x_r(\bar{t}))$ is bijective and continuous ,where $\bar{x} = \phi^{-1}(u^-) - \delta + \bar{t}f'(\phi(\phi^{-1}(u^-) - \delta))$ and $I_u(x, \bar{t}, \phi(\xi)) = 0, x = \xi +$

$\bar{t}f'(\phi(\xi))$, $\xi\in[\phi^{-1}(u^-)-\delta,\bar{\xi})$. By the way similar to prove (2.26) in lemma 2.6 in light of (2.1), (2.28) and $I_{uu}(x_r(\bar{t}),\bar{t},\phi(\bar{\xi}))=0$ we have

$$I_{uu}(\xi+\bar{t}f'(\phi(\xi)),\bar{t},\phi(\xi))<0,\qquad \xi\in(\phi^{-1}(u^-)-\delta,\bar{\xi}),$$

thus

$$I_{uu}(\xi+tf'(\phi(\xi)),t,\phi(\xi))<0,\qquad \xi\in(\phi^{-1}(u^-)-\delta,\bar{\xi}),t\in(0,\bar{t}).$$

Then by the implicit function theorem we see that (2.30) holds. Similarly we have

$$\begin{aligned}&\text{there exist a unique smooth nondegenerate (local) maximizing function } u_2(x,t)\in\bar{J}_2\\&\text{for } I(x,t,\cdot),(x,t)\in\theta'\bigcap(\{x>x_l(t),t>t_0\}\bigcup\{x>\phi^{-1}(u^+)+tf'(u^+),t\le t_0\}),\text{where}\\&\bar{J}_2:=(u^+,\phi(\phi^{-1}(u^+)+\delta)).\end{aligned}\tag{2.31}$$

The fact $(\theta'\cap G)\subset U$ follows from lemma 2.2.That $u_1(x,t)$ is the unique nondegenerate maximizing function for $I(x,t,\cdot)$ in $\theta'\cap\ \{(x,t)|x\le x_l(t),t>t_0$ and $x<\phi^{-1}(u^-)+tf'(u^-),t\le t_0\}$ is deduced from (2.30),(2.31) and the characteristic from J_2 is tangent to the concave curve $x=x_l(t),t>t_0$ from right.Similarly $u_2(x,t)$ is the unique nondegenerate maximizing function for$I(x,t,\cdot)$ in $\theta'\cap\ \{(x,t)|x\ge x_r(t),t>t_0$ and $x>\phi^{-1}(u^+)+tf'(u^+),t\le t_0\}$.The remaining part in θ' to be considered is $\theta'\cap\{(x,t)\,|\,x_l(t)<x<x_r(t),t>t_0\}$.Set the function

$$F(x,t,u_1(x,t),u_2(x,t))=I(x,t,u_1(x,t))-I(x,t,u_2(x,t))$$

in $\theta'\cap\{(x,t)\,|\,x_l(t)<x<x_r(t),t>t_0\}$.Then It follows from (2.30) and (2.31) that

$$F(x,t,u_1(x,t),u_2(x,t))>0,x=x_l(t)+0,t>t_0$$

$$F(x,t,u_1(x,t),u_2(x,t))<0,x=x_r(t)-0,t>t_0$$

and

$$\frac{\partial}{\partial x}F(x,t,u_1(x,t),u_2(x,t))=u_1(x,t)-u_2(x,t)<0,x\in(x_l(t),x_r(t)),t>t_0.$$

Therefore for each $t'>t_0$,there is a unique $x'\in(x_l(t'),x_r(t'))$ such that

$$F(x',t',u_1(x',t'),u_2(x',t'))=0$$

,which implies $(x',t')\in\Gamma_1$ by (2.30) and (2.31).Then the lemma follows from lemma 2.1. $\square$

Theorem 2.2 *Assume $L=\emptyset$.Any point $(x_0,t_0)\in\Gamma_0^{(c)}$ has a neighborhood θ such that $\Gamma_1\cap\theta$ consists of,say n,shocks,one emanating at (x_0,t_0) and the other $n-1$ terminating at (x_0,t_0).Moreover the $(\theta'\setminus\Gamma_1)\subset U$, where $\theta'=\theta\setminus\{(x_0,t_0)\}$.*

Proof:In virtue of lemma 2.5 we have that there are finitely many, say n,connected components $[u_1^-,u_1^+],[u_2^-,u_2^+],\cdots,[u_n^-,u_n^+]$ of $\mathscr{M}_{(x_0,t_0)}$,where $(u_1^-\le u_1^+<u_2^-\le u_2^+\cdots<u_n^-\le u_n^+)$ and $n\ge 3$.Set

$$\begin{aligned}l_i^-&:x=\phi^{-1}(u_i^-)+tf'(u_i^-),0<t<t_0,\\l_i^+&:x=\phi^{-1}(u_i^+)+tf'(u_i^+),0<t<t_0,\end{aligned}$$

where $i=1,\cdots,n$. In light of lemma 2.6, there exists a neighborhood θ of (x_0,t_0) such that there exists $n-1$ shocks terminating at (x_0,t_0) denoted as $x=\gamma_i^-(t), i=1,2,\cdots,n-1$ in $\theta\cap G_{i+1}^i$.Here G_{i+1}^i is an open triangle region bounded by the line $t=0$,the characteristics l_i^+ and l_{i+1}^- and $(\theta\cap G_{i+1}^i)\setminus\Gamma_1\subset U$.

By the similar way used in Theorem 2.1, there exists a shock emanating at (x_0,t_0) denoted as $x=\gamma^+(t)$ in $\theta\setminus G_n^1$ and $(\theta\setminus G_n^1)\setminus\Gamma_1\subset U$.Here G_n^1 is a closed triangle region formed by the line $t=0$ and the characteristics l_1^- and l_n^+. $\square$

Since the argument in theorem 2.2 works for the case $(x_0,t_0)\in\bar{\Gamma}_1\setminus\Gamma_1$. We just show how the shock $x=\gamma(t)$ behaves at t_0.Assume $[u_1^-,u_1^+],[u_2^-,u_2^+]$ are two connected components of $\mathscr{M}_{(x_0,t_0)}$ with $u_1^-\le u_1^+<u_2^-\le u_2^+$. The regularity of $x=\gamma(t)$ at t_0 can be given by the following two cases.
Case(1) $u_i^-=u_i^+=u_i, i=1,2$ and u_1 or/and u_2 belongs to $\mathscr{N}_{(x_0,t_0)}$:Then $x=\gamma(t)$ is only C^1 at $t=t_0$ since $u_1(x,t)$ and $u_2(x,t)$ are continuous,but $u_{1x}(x,t_0)\to\infty$ as $x\to x_0-0$ if u_1 belongs to $\mathscr{N}_{(x_0,t_0)}$ or/and $u_{2x}(x,t_0)\to\infty$ as $x\to x_0+0$ if u_2 belongs to $\mathscr{N}_{(x_0,t_0)}$.Here $u_1(x,t)$ $(u_2(x,t))$ is the maximizing function $u(x,t)$ for $I(x,t,\cdot)$ restricted in the left (right) part of $x=\gamma(t)$ in a neighborhood of (x_0,t_0).
Case(2) $u_1^-<u_1^+$ or/and $u_2^-<u_2^+$:In general $\gamma'(t_0-0)=\sigma(u_1^+,u_2^-)\neq\sigma(u_1^-,u_2^+)=\gamma'(t_0+0)$.So $x=\gamma(t)$ is only continuous at $t=t_0$.Thus we have

Corollary 2.2 *Suppose $L=\emptyset$.Any point $(x_0,t_0)\in\bar{\Gamma}_1\setminus\Gamma_1$ has a neighborhood θ such that $\bar{\Gamma}_1\cap\theta$ is a shock $x=\gamma(t)$ passing through (x_0,t_0) and $x=\gamma(t)$ is C^{k+1} smooth at each point except for (x_0,t_0) and $\theta\setminus\bar{\Gamma}_1\subset U$.*

3 Piecewise Smoothness

In this section we will show that for a given C^k smooth,bounded function $\phi(\cdot)$ satisfying (1.2),the solution of (1.1) is piecewise smooth except for flux functions $f(\cdot)$ in a subset of first category in $C^{k+1}(\phi_-,\ \phi_+)$. We will show that Ω given by (1.7) is a subset of first category in $C^{k+1}(\phi_-,\ \phi_+)$. To this end it suffices to show that

$$\Omega_1=\{f\in C^{k+1}(\phi_-,\phi_+)\mid\exists\, y\in\mathbb{R}\text{ such that }\phi'(y)>0,(f'(\phi(y)))^{(m)}=0,m=2,3\}$$

is a subset of first category in $C^{k+1}(\phi_-,\ \phi_+)$ in lemma 3.1 since $\Omega\subset\Omega_1$. The proof of lemma 3.1 is a slight modification of theorem 5.2 in [3]. Then generically,solutions of (1.1) are piecewise smooth is given by theorem 3.1.

Lemma 3.1 *For a given C^k smooth,bounded function $\phi(\cdot)$ satisfying (1.2), then the set of functions $f(\cdot)$ in $C^{k+1}(\phi_-,\phi_+)$,which satisfy*

$$\phi'(\bar{y})>0, \tag{3.1}$$

$$(f'(\phi(y)))^{(m)}=0,\ m=2,3,\ y=\bar{y}, \tag{3.2}$$

for some $\bar{y}\in\mathbb{R}$,is of a first category in $C^{k+1}(\phi_-,\phi_+)$. Here $3\le k\le\infty$.

Proof:To this end it suffices to prove that for any fixed interval $[a, b]$ and any fixed number $\delta > 0$ the set of functions $f(\cdot)$,for which

$$\phi'(\bar{y}) \geq \delta \tag{3.3}$$

$$(f'(\phi(y)))^{(m)} = 0,\ m = 2, 3,\ y = \bar{y}, \tag{3.4}$$

for some $\bar{y} \in [a, b]$,is closed and nowhere dense in $C^{k+1}(\phi_-, \phi_+)$.

This set is indeed closed because if $\{f_n(\cdot)\}$ is any sequence of functions in $C^{k+1}(\phi_-, \phi_+)$ for which $\phi'(y) \geq \delta, (f_n'(\phi(y)))^{(m)} = 0,\ m = 2, 3, y = y_n$,for some $y_n \in [a, b]$, and $\{f_n(\cdot)\}$ tends to $f(\cdot)$ in $C^{k+1}(\phi_-, \phi_+)$,then (3.3),(3.4) will hold for $f(\cdot)$ with $\bar{y}$,any cluster point of $\{y_n\}$.

In order to show that the above set is nowhere dense,we fix $f(\cdot)$ in $C^{k+1}(\phi_-, \phi_+)$ and proceed to show that there are functions in $C^{k+1}(\phi_-, \phi_+)$ arbitrarily near $f(\cdot)$, for which (3.3),(3.4) do not jointly hold for any $\bar{y} \in [a, b]$.

The set of y with $\phi'(y) > \delta/2$ is an open covering of the compact set $\{y \in [a, b] | \phi'(y) \geq \delta\}$.Consequently,there is a finite subcovering. We can thus find numbers $a \leq a_1 < b_1 \cdots < a_n < b_n \leq b$ with following properties: $\phi'(y) \geq \delta/2$ for $y \in [a_i, b_i]$,$i = 1, \cdots, n$; $\phi'(a_i) = \delta/2, i = 1, \cdots, n$,unless $i = 1$ and $a_1 = a$; $\phi'(b_i) = \delta/2, i = 1, \cdots, n$,unless $i = n$ and $b_n = b$;then the set $\{y \in [a, b] | \phi'(y) \geq \delta\}$ is contained in $\bigcup_{i=1}^{n} [a_i, b_i]$.

With each interval $[a_i, b_i]$ we associate $\epsilon_i < (a_i - b_{i-1})/2, i = 2 \cdots, n$ and $\epsilon_1 > 0, \epsilon_{n+1} > 0$; $\phi'(y) \geq \delta/4$ if $y \in [a_i - \epsilon_i, b_i + \epsilon_{i+1}], 1, \cdots, n$. We construct C^∞ smooth function $\psi_i(y)$ with the following properties: $\psi_i(y)$ is near zero in C^∞;the support of $\psi_i(y)$ is contained in the interval $[a_i - \epsilon_i, b_i + \epsilon_{i+1}]$;all critical points (if any) of the function $(f'(\phi(y)))' + \psi_i(y)$ on the interval $[a_i, b_i]$ are nondegenerate.(*)

We construct now the function $g(\cdot)$ in $C^{k+1}(\phi_-, \phi_+)$ as follows:We define $g(z)$ is near $f(z)$ in $C^{k+1}(\phi_-, \phi_+)$ for $z \notin \bigcup_{i=1}^{n} [\phi(a_i - \epsilon_i), \phi(b_i + \epsilon_{i+1})]$. On $[\phi(a_i - \epsilon_i), \phi(b_i + \epsilon_{i+1})], i = 1 \cdots, n$, $g(\cdot)$ is defined as the solution of the boundary value problem

$$\begin{cases} \frac{d}{dy} g'(\phi(y)) = (f'(\phi(y)))' + \psi_i(y) \\ g(\phi(a_i - \epsilon_i)) = f(\phi(a_i - \epsilon_i)) \\ g(\phi(b_i + \epsilon_{i+1})) = f(\phi(b_i + \epsilon_{i+1})) \end{cases} \tag{3.5}$$

By the transformation $z = \phi(y)$ the boundary value problem (3.5) can be rewritten as following boundary value problem

$$\begin{cases} \frac{d^2}{dz^2} g(z) = f''(z) + (\phi'(\phi^{-1}(z)))^{-1} \psi_i(\phi^{-1}(z)) \\ g(\phi(a_i - \epsilon_i)) = f(\phi(a_i - \epsilon_i)) \\ g(\phi(b_i + \epsilon_{i+1})) = f(\phi(b_i + \epsilon_{i+1})) \end{cases} \tag{3.6}$$

We note that this construction of $g(\cdot)$ is possible since $\phi'(y) \geq \delta/4$ for $y \in [a_i - \epsilon_i, b_i + \epsilon_{i+1}], 1, \cdots, n$, we will have,for $\psi_i(\cdot)$ sufficiently near zero,a well-posed problem in (3.6) and the resulting solution $g(\cdot)$ will be near $f(\cdot)$ on $[\phi(a_i - \epsilon_i), \phi(b_i + \epsilon_{i+1})]$.In fact $g(\cdot)$ can be given explicitly by integrating the both side of the equation two times with the two boundary conditions in (3.6). We recall that the set of $y \in [a, b]$ with $\phi'(y)) \geq \delta$ is contained in $\bigcup_{i=1}^{n} [a_i, b_i]$. On the other hand, by virtue of the equation in (3.5) and the construction of $\psi_i(\cdot)$, all critical points (if any) of $\frac{d}{dy}g'(\phi(y))$ on $[a_i, b_i]$ are nondegenerate. We have thus constructed functions arbitrarily near $f(\cdot)$ for which (3.3), (3.4) do not jointly hold for any $y \in [a, b]$.Therefore the set of $f(\cdot)$ in C^{k+1} for which (3.3),(3.4) hold for some $\bar{y} \in [a, b]$ is nowhere dense in C^{k+1}. $\square$

$^{(*)}$The existence of such a function $\psi_i(\cdot)$ has been shown by Dafermos in [3]

Theorem 3.1 *For a given bounded,C^ksmooth initial datum ϕ satisfying (1.2),solution of the initial value problem (1.1) is piecewise smooth if the flux function $f \in C^{k+1}(\phi_-, \phi_+) \setminus \Omega$.*

Proof:We recall that $f \in C^{k+1}(\phi_-, \phi_+) \setminus \Omega \iff L = \emptyset$.Thus by virtue of lemma 2.5 we have that $H = U \cup \bar{\Gamma}_1 \cup \Gamma_0^{(f)} \cup \Gamma_0^{(c)}$. By lemma 2.1 $\Gamma = \bar{\Gamma}_1 \cup \Gamma_0^{(f)} \cup \Gamma_0^{(c)}$ is a closed subset of H and Γ is covered by neighborhoods of the type described in lemma 2.1,corollary 2.2,theorem 2.1 and theorem 2.2.For any compact set $K \subset H$, by choosing a finite subcovering of $K \cap \Gamma$ we see that $K \cap \Gamma$ consists of the union of a finite number of shock and each shock is piecewise C^{k+1} . Therefore the maximizing function $u(x, t)$ is piecewise C^k smooth.Morever for small t,$u(x, t)$ is smooth solution and assumes the correct initial datum as $t \to 0$. $\square$

In fact we can deduce that for any compact set $K \subset H$ there are finitely many formation points of shocks in K,i.e.there are finite points in $\Gamma_0^{(f)} \cap K$,by lemma 2.4 and theorem 2.1;there are finitely many points in K at which a shock or some shocks fail to be C^{k+1} smooth by lemma 2.4,corollary 2.1, corollary 2.2 and theorem 2.2.Thus piecewise smoothness of the solution of (1.1) can also be reached.In other words lemma 2.4 and corollary 2.1 tell that there are finite points in $K \setminus (U \cup \Gamma_1)$.By virtue of the lemma 2.1,there is θ, a neighborhood of $(x_0, t_0) \in U$,the solution of (1.1) is C^k smooth in θ, or there is θ,a neighborhood of $(x_0, t_0) \in \Gamma_1$,there is a C^{k+1} shock $\gamma(t)$ passing though (x_0, t_0) and the solution of (1.1) is C^k smooth in $\theta \setminus \gamma(t)$. The structure of solution of (1.1) in θ ,a neighborhood of $(x_0, t_0) \in K \cap (\Gamma_0^{(f)} \cup (\bar{\Gamma} \setminus \Gamma_1) \cup \Gamma_0^{(c)}) = K \setminus (U \cup \Gamma_1)$ can be given by theorem 2.1 or corollary 2.2 or theorem 2.2 respectively.

4 Concluding remarks

The results in this paper still hold if assumption (1.2) on the initial datum $\phi(\cdot)$ can be relaxed to that $\phi'(\cdot)$ is nonnegative and does not vanish identically on any interval.To this end just note that there is one to one correspondence between a maximizing value for $I(x, t, \cdot)$ and a maximizing value for $I(x, t, \phi(\cdot))$,and y with $\phi'(y) = 0$ is a critical point for $I(x, t, \phi(\cdot))$ but is not a candidate for maximizing value for $I(x, t, \phi(\cdot))$ if $I_u(x, t, u) = -\phi^{-1}(u) + x - tf'(u) = -y + x - tf'(\phi(y)) \neq 0$.

We can use the conclusion of lemma 2.3 to construct Ω,a subset of $C^{k+1}(\phi_-,\ \phi_+)$ of first category,such that the solution of (1.1) is piecewise smooth if the flux function $f \in C^{k+1}(\phi_-, \phi_+) \setminus \Omega$ as follows:Let $[y_1, y_2]$ be any connected component of $\mathcal{L}$ defined in (2.14),then let

$$\begin{aligned}\Omega \quad = \quad & \{f \in C^{k+1}(\phi_-, \phi_+) \mid \forall\, \delta > 0, \exists\, \delta' > 0, \delta' < \delta \\ & \text{such that}\,(f'(\phi(\xi)))'\text{does not strictly increases on } (y_2, y_2 + \delta') \\ & \text{or/and } \ (f'(\phi(\eta)))'\text{does not strictly decreases on } (y_1 - \delta', y_1)\ .\}\end{aligned}$$

In this work we proved that if the flux functions of nonconvex scalar conservation laws do not belong to a very small subset of $C^{k+1}(\phi_-, \phi_+)$ then the solutions of the initial value problems

(1.1) are *piecewise C^k smooth* if a given initial datum $\phi(\cdot)$ is strictly monotone and C^k smooth. It is important to understand the conditions under which the solution of the initial value problems (1.1) is *piecewise smooth* since most practical cases deal with the *piecewise smooth* solutions. For this reason, there are many studies on approximation methods for conservation laws whose solutions are *piecewise smooth*. For example, for systems of conservation laws, Goodman and Xin [6] proved that the viscosity methods approximating *piecewise smooth* solutions with finitely many *noninteracting* shocks have a local first-order rate of convergence away from the shocks; for convex conservation laws, the global rate of convergence for the viscosity methods can be obtained [18] and the point-wise rate of convergence for the viscosity methods has been obtained [16, 17]; for nonconvex conservation laws,the global rate of convergence for the viscosity methods can be obtained under the *assumption* that the solutions are *piecewise smooth* [19].

Acknowledgment. The author was supported by National Natural Foundation of China under grant NO.10671116 and NO.10871133.

References

[1] M.Bardi and L.C.Evans,On Hopf's formulas for solutions of Hamilton-Jacobi equations,Nonlinear Anal.Theory.Methods&Applications,**8**(1984),1373-1381.

[2] S. Chen and Z. Zhang, Formation of shock for first order quasilinear equations, *Acta Sci. Nat. Univ. Fudan*, **8** (1963), 13-22 (in Chinese).

[3] C.M. Dafermos, Characteristics in hyperbolic conservation laws, Nonlinear Analysis and Mechanics, Heriot-Watt Symposium, Vol 1, 1-58, ed. R.J. Knops. London, Pitman, 1977.

[4] C.M. Dafermos, Generalized characteristics and the structure of solutions of hyperbolic conservation laws.Indiana *U.Math.J.* 26(1977),1097-1119.

[5] C.M. Dafermos, Regularity and large time behavior of solutions of a conservation law without convexity.*Proc.Royal.Soc.Edinburgh* 99K(1985),201-239.

[6] J. Goodman and Z. Xin, Viscous limits for piecewise smooth solutions to systems of conservation laws, *Arch. Rational Mech. Anal.* 121 (1992), pp. 235-265.

[7] E. Hopf, Generalized solutions of nonlinear equations of first order, *J.Math.Mech.* 14(1965), 951-973.

[8] S.N.Kruzhkov and N.S.Petroyan,Aymptotic behaviorof solutionsof Cuachy problemfor first order nonlinear equations,Uspekhi Mat.Mauk, **42**(1987),3-40.(in Russian)

[9] P.D. Lax, Hyperbolic systems of conservation laws II, *Comm. Pure. Appl. Math.* **10** (1957), 537-566.

[10] B. Li and J. Wang, The global qualitative study of solutions to a conservation law(I), *Sci. Special Math. Issue* (1979), 12-24 (in Chinese).

[11] B. Li and J. Wang, The global qualitative study of solutions to a conservation law (II), *Sci. Special Math. Issue* (1979), 25-38 (in Chinese).

[12] T.-P.Liu,Admissible solutions of hyperbolic conservation laws, *Mem.Amer.Math.Soc.***240** (1981)

[13] O. K. Oleinik, Discontinuous solutions of nonlinear differential equations, "Amer. Math. Soc. Transl.," Vol 26,pp.95-172, Amer. Math. Soc., Providence, 1963

[14] D.G. Schaeffer, K regularity theorem for conservation laws, *Adv. Math.* **11** (1973), 368-386

[15] E. Tadmor and T. Tassa, On the piecewise smoothness of entropy solutions to scalar conservation laws, *Commun. in PDEs* 18 (1993), pp. 1631-1652.

[16] E. Tadmor and T. Tang, Pointwise convergence rate for scalar conservation laws with piecewise smooth solutions, *SIAM J. Numer. Anal.* **36** (1999) 1739-1758.

[17] E. Tadmor and T. Tang, Pointwise error estimates for relaxation approximations to conservation laws, SIAM J. Math. Anal. **32** (2001), 870-886.

[18] T. Tang and Z.-H. Teng, Viscosity methods for piecewise smooth solutions to scalar conservation laws, *Math. Comp.* bf 66(1997), pp. 495-526.

[19] T. Tang , Z.-H. Teng and Z.-P. Xin, Fractional rate of convergence for viscous approximation to nonconvex conservation laws, *SIAM J. Math. Anal.* **35** (2003), pp. 98-122.

[20] T.Tang,J.Wang and Y.Zhao,On the piecewise smoothness of entropy solutions to scalar conservation laws for larger class of initial data,JHDE, **4**(2007),pp369-389.