The Slow Erosion Limit in a Model of Granular Flow

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Abstract

We study a 2×2 system of balance laws that describes the evolution of a granular material (avalanche) flowing downhill. The original model was proposed by Hadeler and Kuttler [13]. The Cauchy problem for this system is studied by the authors in the recent papers [2, 16].

In this paper, we first consider an initial-boundary value problem. The boundary condition is given by the flow of the incoming material. For this problem we prove the global existence of BV solutions for a suitable class of data, with bounded by possibly large total variations.

We then study the "slow erosion (or deposition) limit". We show that, if the thickness of the moving layer remains small, then the profile of the standing layer depends only on the total mass of the avalanche flowing downhill, not on the time-law describing at which rate the material slides down. More precisely, in the limit as the thickness of the moving layer tends to zero, the slope of the mountain is provided by an entropy solution to a scalar integro-differential conservation law.

1 Introduction and main results

We consider a model for the flow of granular material, such as sand or gravel, which was proposed in [13]. The material is divided in two parts: a moving layer on top and a standing layer at the bottom. We denote by h the thickness of the moving layer, and by u the height of the standing layer. The normalized model in [13] takes the following form in space dimension ≤ 2,

$$\begin{cases} h_t &= \operatorname{div}(h\nabla u) - (1-|\nabla u|)h\,, \\ u_t &= \big(1-|\nabla u|\big)h\,. \end{cases}$$

Throughout the following, we study the case of one space dimension, and assume $u_x \geq 0$ (i.e., the slope of the mountain profile does not change sign). In terms of the variables $h \geq 0$, $p \doteq u_x > 0$ the system reads as

$$\begin{cases} h_t - (hp)_x &= (p-1)h\,, \\ p_t + \big((p-1)h\big)_x &= 0\,. \end{cases} \tag{1.1}$$

According to (1.1), the moving layer slides downhill, with speed equal to the slope of the standing layer. There is a critical slope, which in this normalized model is $p = 1$, with the following property. If $p > 1$ then grains initially at rest are hit by rolling grains of the moving layer and start moving as well. Hence the moving layer gets bigger. On the other hand, if $p < 1$, grains which are rolling can be deposited on the bed. Hence the moving layer becomes smaller.

A simple calculation shows that this system is strictly hyperbolic on the domain

$$\Omega = \{(h,p): \ h \geq 0\,,\ p > 0\}\,,$$

while at $(h,p) = (0,0)$ the two characteristic eigenvalues coincide. For a suitable class of initial data, in [16] one of the authors established the global existence of smooth solutions. In the recent work [2] we proved the global existence of large BV solutions to the Cauchy problem, for a class of initial data with bounded but possibly large total variation.

The purpose of the present paper is to study the "slow erosion (or deposition) limit", in the following sense: we wish to describe how the mountain profile evolves when the thickness of the moving layer approaches zero, but the total mass of sliding material remains positive and bounded. We will show that if the thickness of the moving layer remains small, then the profile of the standing layer depends only on the total mass of the avalanche flowing downhill, not on the time-law describing at which rate the material slides down.

The result stated in this paper is best formulated in connection with an initial-boundary value problem. Fix any point $\bar{x} \in \mathbb{R}$. By a translation of coordinates, it is not restrictive to assume $\bar{x} = 0$. On the domain $\mathbb{R}_- \doteq \{x < 0\}$, consider the initial-boundary value problem for (1.1), with initial data

$$h(0,x) = \bar{h}(x)\,, \qquad p(0,x) = \bar{p}(x)\,, \qquad x < 0 \tag{1.2}$$

and boundary conditions at $x = 0$

$$p(t,0)h(t,0) \ = \ F(t)\,. \tag{1.3}$$

Notice that here we are prescribing the incoming flux $F(t)$ of granular material, through the point $x = 0$. We shall assume that $\bar{h}$, $\bar{p} : \mathbb{R}_- \mapsto \mathbb{R}_+^* \doteq \{x \geq 0\}$ are non-negative functions with bounded variation, such that

$$\text{Tot.Var.}\{\bar{h}\} \leq M\,, \qquad \|\bar{h}\|_{\mathbf{L}^1(\mathbb{R}_-)} \leq M\,, \tag{1.4}$$

and

$$\text{Tot.Var.}\{\bar{p}\} \leq M\,, \qquad \|\bar{p} - 1\|_{\mathbf{L}^1(\mathbb{R}_-)} \leq M\,, \qquad \bar{p}(x) \geq p_0 > 0\,, \tag{1.5}$$

for some bounded positive constants M (possibly large) and p_0, and also that

$$F(t) \geq 0\,, \qquad \text{Tot.Var.}\{F\} \leq M\,, \qquad \int_0^\infty F(\tau)\,d\tau \leq M\,. \tag{1.6}$$

Our first result is on the global existence of large BV solutions for the initial boundary value problem, for sufficiently small $\|\bar{h}\|_{\mathbf{L}^\infty}$ and $\|F\|_{\mathbf{L}^\infty}$.

Theorem 1 ***(global existence for the initial-boundary value problem).*** *Given M, $p_0 > 0$, there exists $\delta > 0$ such that the assumptions (1.4), (1.5) and (1.6), together with*

$$\|\bar{h}\|_{\mathbf{L}^\infty} \leq \delta\,, \qquad \|F\|_{\mathbf{L}^\infty} \leq \delta \tag{1.7}$$

imply that the initial-boundary value problem (1.1), (1.2), (1.3) has a global weak solution $(h,p)(t,\cdot)$, with uniformly bounded total variation for all $t \geq 0$.

For the definition of weak solution to problem (1.1)–(1.3), see Def. 1 in Sect. 2. Note that, since $(h, p)(t, \cdot)$ has bounded variation, the limit as $x \to 0$ is well defined for all t. The boundary condition (1.3) is here intended to be satisfied for all $t > 0$, except at most countably many.

Theorem 1 is similar to the main result in [2], where the global existence of large BV solution was established for the Cauchy problem. This Theorem can therefore be proved in a similar fashion as in [2], taking into additional account of the boundary effect. Here one must take care of the reflecting waves and newly generated incoming waves at the boundary. Furthermore, since the second characteristic speed is not bounded away from 0 (i.e., the speed of the boundary), one must justify that the boundary condition is satisfied in the limit by the approximate solutions.

Note that the characteristic speeds λ_i satisfy $\lambda_1 < 0 \leq \lambda_2$. This indicates that the characteristics of the first family strictly enters the domain $x < 0$. Therefore, the scalar boundary condition (1.3) is sufficient for this problem. See also [12, 14, 1, 3, 10, 17] for existence and stability results in case of small BV data.

Our main interests in the present paper is to study the limiting behavior of the slope $p(\cdot)$, when the above $\mathbf{L}^\infty$ norms approach zero. This is of practical interest, because it describes how the mountain profile evolves, when the granular material pours down at a very slow rate. We thus consider a sequence of initial and boundary data, of the form

$$\begin{aligned} t = 0\,,\ x < 0: \qquad & h_\nu(0, x) \ = \ \bar{h}_\nu(x), \quad p_\nu(0, x) \ = \ \bar{p}(x) \\ x = 0: \qquad & p_\nu(t, 0) h_\nu(t, 0) \ = \ F_\nu(t)\,, \end{aligned} \tag{1.8}$$

with $\|\bar{h}_\nu\|_{\mathbf{L}^\infty} \to 0$ and $\|F_\nu\|_{\mathbf{L}^\infty} \to 0$ as $\nu \to \infty$, and assume that

$$0 < M' < \int_0^\infty F_\nu(\tau)\, d\tau \leq M \tag{1.9}$$

for some positive constant M'. Notice that for p we are always considering the same initial data. To obtain a well defined limit, let us re-parametrize time in terms of the total amount of mass flowing downstream across the point $x = 0$. This is achieved by setting

$$\mu_\nu(t) = \int_0^t F_\nu(\tau)\, d\tau\,. \tag{1.10}$$

The map $t \mapsto \mu_\nu(t)$ is continuous and non-decreasing, hence it has a generalized inverse:

$$t_\nu(\mu) = \min\ \{\tau \geq 0\,;\quad \mu_\nu(\tau) = \mu\}\,. \tag{1.11}$$

Because of (1.6), the above mapping is well defined for $\mu \in [0, M']$. Let now $h_\nu(t, x)$, $p_\nu(t, x)$ be the solution of the initial-boundary value problems (1.1), (1.8) given by Theorem 1 and define

$$\tilde{p}_\nu(\mu, x) \doteq p_\nu(t_\nu(\mu), x)\,. \tag{1.12}$$

In the above setting, our second main result reads

Theorem 2 ***(slow erosion/deposition limit).*** *Consider a family of initial and boundary data $\bar{h}_\nu$, $\bar{p}$, and F_ν, satisfying (1.4), (1.5), (1.6) and (1.9) with the same constants M, M', p_0 for every $\nu \in \mathbb{N}$. Assume that*

$$\|\bar{h}_\nu\|_{\mathbf{L}^\infty} \to 0\,, \qquad \|F_\nu\|_{\mathbf{L}^\infty} \to 0 \qquad as \quad \nu \to \infty\,. \tag{1.13}$$

For every ν sufficiently large, let $(h_\nu,\, p_\nu)$ be an entropy weak solutions of the initial-boundary value problem (1.1), (1.8) as from Theorem 1.

Then the re-parametrized solutions $\tilde{p}_\nu(\mu, x)$, defined as in (1.10)–(1.12), converge to a limit function $\tilde{p}(\mu, x)$ in the distance of $\mathbf{L}^\infty\Big([0, M']\,;\ \mathbf{L}^1(\mathbb{R}_-)\Big)$. The function $\tilde{p}$ provides a weak entropy solution to the scalar integro-differential conservation law

$$p_\mu + \left(\frac{p-1}{p}\cdot \exp\int_x^0 \frac{p(\mu,y)-1}{p(\mu,y)}\,dy\right)_x = 0\,, \tag{1.14}$$

with initial data

$$\tilde{p}(0,x) = \bar{p}(x) \qquad\qquad x<0\,. \tag{1.15}$$

Notice that for the initial-boundary value problem (1.14), (1.15) we are not specifying any boundary condition at $x = 0$. This is appropriate, because the characteristic speed is non-negative, and there is no incoming characteristic. Notice also that the initial data $\bar{h}_\nu$ play no role in the limit. Indeed, $\bar{h}_\nu \to 0$ uniformly.

Intuitively, the formula (1.14) can be explained as follows. Consider a small amount of mass $\Delta\mu$ pouring down initially from the point $x = 0$. Call $\Delta\mu(t)$ the size of this small avalanche and let $x(t)$ be its location, at time t. These satisfy the equations

$$\frac{d}{dt}\Delta\mu(x(t)) = (p-1)\,\Delta\mu\,, \qquad \frac{d}{dt}x(t) = -p\,.$$

We assume here that the slope p of the mountain varies very slowly in time. Calling $\Delta\mu(x)$ the size of the avalanche when it reaches the point $x < 0$, from the above equations we obtain

$$\frac{\partial}{\partial x}\Delta\mu(x) = \frac{dt}{dx}\cdot\frac{\partial}{\partial t}\Delta\mu(x(t)) = \frac{1}{-p(x)}\cdot(p(x)-1)\Delta\mu(x)\,.$$

Hence, solve this ODE for $\Delta\mu(x)$, we get

$$\Delta\mu(x) = \left(\exp\int_x^0 \frac{p(y)-1}{p(y)}\,dy\right)\,\Delta\mu(0)\,, \qquad x<0\,.$$

In turn, when this avalanche crosses the point x, it produces a change in the height of the mountain (i.e., in the height of the standing layer u) measured by

$$\Delta u(x) = \frac{(p-1)\Delta\mu(x)}{-p} = -\left(\frac{p(x)-1}{p(x)}\cdot\exp\int_x^0\frac{p(y)-1}{p(y)}\,dy\right)\,\Delta\mu(0)\,.$$

This formally gives a partial derivative

$$\frac{\partial u}{\partial\mu} = -\left(\frac{p(x)-1}{p(x)}\cdot\exp\int_x^0\frac{p(y)-1}{p(y)}\,dy\right)\,.$$

Differentiating the above equation w.r.t. x, and recalling $p = u_x$, we formally obtain (1.14).

Since the main technicality in [2] is similar to the proof of Theorem 1 in the current paper, we explain in some details of [2]. The system (1.1) is weakly linearly degenerate at the point $(h, p) = (0, 1)$. This means that the first characteristic field is genuinely nonlinear away from the line $p = 1$, but linearly degenerate along $p = 1$; similarly the second field is genuinely nonlinear away from the line $h = 0$, but linearly degenerate along $h = 0$.

Global existence of large BV solutions are established through global a priori bounds on the total variations of approximate solutions, where the wave interaction estimates are crucial. By measuring the wave strength in terms of the Riemann coordinates, sharper interaction estimates can be achieved. In more detail, at an interaction let σ_1 and σ_2 be the strengths of the incoming waves of family 1 and 2, respectively, and let σ_1' and σ_2' be the outgoing ones. One then has a cubic interaction estimate of the form

$$|\sigma_1' - \sigma_1| + |\sigma_2' - \sigma_2| = \mathcal{O}(1) \cdot |\sigma_1| \cdot |\sigma_2| \cdot (|\sigma_1| + |\sigma_2|) \,. \tag{1.16}$$

Assuming that the height of the moving layer h is sufficiently small, (1.16) can be much improved. Recall that the system is linearly degenerate along the straight line $h = 0$. In the region where $h > 0$ is very small, the second field of the system is "almost-Temple class". Rarefaction curve and shock curve through the same point are very close to each other. More precisely, let U_o be a point on the rarefaction curve of the second family through the point $U = (h, p)$. Then, there exists a point U^* on the 2-shock curve through U, which is very close to U_o, such that

$$|U^* - U_o| = \mathcal{O}(1) \cdot h^2.$$

This allows us to replace the estimate (1.16) with

$$|\sigma_1' - \sigma_1| + |\sigma_2' - \sigma_2| = \mathcal{O}(1) \cdot |\sigma_1| \cdot |\sigma_2| \cdot \|h\|_{\mathbf{L}^\infty} \,. \tag{1.17}$$

Besides (1.17), interaction estimates of waves from the same family are also improved as follows. If two 2-waves of strength σ_2 and $\tilde{\sigma}_2$ interact, then the strengths σ_1^+ and σ_2^+ of the outgoing waves satisfy

$$|\sigma_1^+| + \left|\sigma_2^+ - (\sigma_2 + \tilde{\sigma}_2)\right| = \mathcal{O}(1) \cdot h_l \cdot |\sigma_2 \, \tilde{\sigma}_2| \,. \tag{1.18}$$

If two 1-waves of size σ_1 and $\tilde{\sigma}_1$ interact, then the strengths σ_1^+ and σ_2^+ of the outgoing waves satisfy

$$\left|\sigma_1^+ - (\sigma_1 + \tilde{\sigma}_1)\right| + |\sigma_2^+| = \mathcal{O}(1) \cdot |p_l - 1| \left(|\sigma_1| + |\tilde{\sigma}_1|\right) \cdot |\sigma_1 \, \tilde{\sigma}_1| \,. \tag{1.19}$$

Here h_l and p_l denote the left state of interaction. Since $|\sigma_1| + |\tilde{\sigma}_1|$ is of order $\|h\|_{\mathbf{L}^\infty}$, we note that all three estimates (1.17)–(1.19) contain an additional factor $\|h\|_{\mathbf{L}^\infty}$, which is arbitrary small. Therefore we can assume that the total strength of all new waves produced by interactions is as small as we like. In essence, the change in the total variation is thus determined only by the source terms.

Furthermore, the source term involves the quadratic form $h(p-1)$. Here the quantities h and $p - 1$ have large, but bounded $\mathbf{L}^1$ norms. Moreover, they are transported with strictly different speeds. The total strength of the source term is thus expected to be $\mathcal{O}(1) \cdot \|h\|_{\mathbf{L}^1} \cdot \|p - 1\|_{\mathbf{L}^1}$. In addition, since h itself is a factor in the source term, one can obtain a uniform bound on the norm $\|h\|_{\mathbf{L}^\infty}$, valid for all times $t \geq 0$. By choosing weights which take into account the mass to be crossed in future, one can define suitable weighted functionals that are non-increasing in time, achieving the desired estimates, which lead to the global existence of large BV solutions.

The remainder of the paper is organized as follows. In Section 2 we prove the boundary estimates. These are used in the proof of Theorem 1, the global existence of large BV solutions for the initial boundary value problems, which is worked out in Section 3. In Section 4 we prove Theorem 2, showing that the slow erosion limit leads to a scalar integro-differential equation.

Besides the paper [13], we refer to [5], [11], [15] for other models of granular flow. Other examples of conservation laws with non-local flux can be found in [8] and references therein.

2 Boundary estimates

Writing the system of balance laws (1.1) in quasilinear form, the corresponding Jacobian matrix is

$$A(h,p) = \begin{pmatrix} -p & -h \\ p-1 & h \end{pmatrix}. \tag{2.1}$$

In the domain Ω where $h \geq 0$ and $p > 0$, the system is strictly hyperbolic with two real distinct eigenvalues $\lambda_1 < \lambda_2$, namely

$$\lambda_{1,2} = \frac{1}{2}\left[h - p \pm \sqrt{(p-h)^2 + 4h}\right].$$

The corresponding eigenvectors have the form

$$r_1(h,p) = \begin{pmatrix} 1 \\ \\ -\dfrac{\lambda_1+1}{\lambda_1} \end{pmatrix}, \qquad r_2(h,p) = \begin{pmatrix} -\dfrac{\lambda_2}{\lambda_2+1} \\ \\ 1 \end{pmatrix}, \tag{2.2}$$

and the directional derivatives are

$$r_1 \bullet \lambda_1 \;=\; -\frac{2(\lambda_1+1)}{\lambda_2-\lambda_1} \;\sim\; \frac{2(p-1)}{p}, \qquad\qquad r_2 \bullet \lambda_2 \;=\; -\frac{2\lambda_2}{\lambda_2-\lambda_1} \;\sim\; -2\frac{h}{p^2}$$

as $h \to 0$. We see that the first characteristic field is genuinely nonlinear away from the line $p = 1$ and the second field is genuinely nonlinear away from the line $h = 0$. So the system is weakly linearly degenerate at the point $(h,p) = (0,1)$.

We give now the definition of weak solution of the initial-boundary value problem (1.1)–(1.3), with initial data $(\bar{h}, \bar{p}) \in BV(\mathbb{R}_-)$ and boundary data $F \in BV(\mathbb{R}_+)$.

Definition 1 *We say that $(h,p) : [0,\infty) \times (-\infty, 0) \to \mathbb{R}^2$ is a weak solution to (1.1)–(1.3) if*

(i) the map $t \mapsto (h,p)(t,\cdot) \in \mathbf{L}^1_{loc}(\mathbb{R}_-)$ is continuous, and satisfies (1.2) at $t = 0$;

(ii) (h,p) satisfies (1.1) in the sense of distibutions on the domain $\{(t,x) :\ t > 0,\ x < 0\}$;

(iii) for all $t > 0$ except at most countably many, one has that

$$\lim_{x \to 0-} h(t,x)p(t,x) = F(t).$$

Approximate solutions to the initial-boundary value problem (1.1)–(1.3) are defined through an operator splitting method. Fix a time step $\Delta t \geq 0$ and consider the sequence of times $t_k = k\Delta t$. Let (h^Δ, p^Δ) be an approximate solution, constructed as follows. Boundary condition is approximated by piecewise constant inflow flux F on every interval (t_{k-1}, t_k). On each subinterval $[t_{k-1}\,,\ t_k[$ the functions (h^Δ, p^Δ) provide an approximate solution to the system of conservation laws

$$\left\{\begin{array}{rcl} h_t - (hp)_x &=& 0\,, \\ p_t + \big((p-1)h\big)_x &=& 0\,, \end{array}\right. \tag{2.3}$$

constructed by a wave-front tracking algorithm [6, 4, 7] for 2×2 systems. Moreover, in order to account for the source term, at each time t_k the functions are redefined in the following time step

$$\left\{\begin{array}{rcl} h^\Delta(t_k) &=& h^\Delta(t_k-) + \Delta t\ \big[p^\Delta(t_k-) - 1\big]\, h^\Delta(t_k-)\,, \\ p^\Delta(t_k) &=& p^\Delta(t_k-)\,. \end{array}\right. \tag{2.4}$$

As in [2], we introduce Riemann coordinates for this system and use them to measure the wave strengths. Given a point $(h,p) \in \mathbb{R}_+ \times \mathbb{R}_+$, let $(H, 0)$ be the point on the h-axis connected with (h,p) by a 2-characteristic curve, and let $(0, P)$ be the point on the p-axis connected to (h,p) by a 1-characteristic curve. Then the functions (H,P) form a coordinate systems of Riemann invariants of the point (h,p). The wave strengths are measured by the jumps between the corresponding Riemann coordinates of the left and right states (H for the 1^{st} family and P for the 2^{nd}).

In the remainder of this section we study how the strength of waves changes at the boundary $x = 0$ when waves are reflected or generated for the boundary value problem.

Note that the boundary condition (1.3) assigns a data to the flux of the first equation in (1.1), which satisfies

$$\nabla(hp) \cdot r_1(h,p) \neq 0$$

on Ω. This is an injectivity condition that is usually required for non-characteristic boundary value problems (see for instance [12]). In fact, it is used for having the unique solvability of the boundary Riemann problem and Lipschitz dependence on the boundary data, see the Lemma below.

Lemma 1 ***(Boundary estimates).***
(i) At a time τ where the flux F has a jump, a front of the first family is created. Its strength $|\sigma_h^+|$ satisfies

$$|\sigma_h^+| = \mathcal{O}(1) \cdot |F(\tau+) - F(\tau-)|\ . \tag{2.5}$$

(ii) At a time τ where a p-front of strength σ_p hits the boundary at $x = 0$, a new reflected front of the first family is created. Calling h_l the state to the left of the jump σ_p and σ_h^+ the size of the new jump, one has the estimate

$$|\sigma_h^+| = \mathcal{O}(1) \cdot h_l\, |\sigma_p|\,. \tag{2.6}$$

(iii) At each time t_k where the inductive step (2.4) is performed, a new h-wave σ_h^+ is created at $x = 0$. Calling (h,p) the state before the time step, one has the estimate

$$|\sigma_h^+| \ = \ \mathcal{O}(1) \cdot \Delta t \cdot h \cdot |p-1|\,. \tag{2.7}$$

Note that all the estimates in Lemma 1 contain either the factor $\|h\|_{\mathbf{L}^\infty}$ or $\|F\|_{\mathbf{L}^\infty}$, which are both bounded by δ. This means the reflecting waves and the newly generated waves at the boundary are both arbitrary small.

Proof. Denote by φ the flux function hp. Consider a h-rarefaction curve; we see that $\varphi = hp$ is monotone along this curve; indeed, recalling (2.1) and (2.2), one has

$$\nabla_{(h,p)}\varphi \cdot r_1 = -\lambda_1 \,,$$

which is positive for all $(h,p) \in \Omega$.

Consider now a h-shock curve $(h, p(h))$ through (h_o, p_o). By Rankine-Hugoniot conditions, we have

$$-(hp - h_o p_o) = s(h - h_o), \qquad (hp - h_o p_o) - (h - h_o) = s(p - p_o) \,.$$

By simple computation we obtain the shock curve of the 1^{st} family with left state (h_o, p_o), parametrized by h,

$$p(h) = p_o - \frac{s_1 + 1}{s_1}(h - h_0) \,, \qquad s_1 = \lambda_1(h, p_o) \,.$$

Computing the flux along this shock curve, we get

$$\varphi(h, p(h)) = hp(h) = h_o p_o - \lambda_1(h, p_o)(h - h_o) \,.$$

Then, the change of flux along this shock curve is

$$\frac{d}{dh}\varphi = -\partial_h \lambda_1(h, p_o) \cdot (h - h_o) - \lambda_1(h, p_o) \,.$$

We claim that $\frac{d}{dh}\varphi > 0$. Indeed, if $p_o > 1$, then $h < h_o$ and the quantity $\partial_h \lambda_1$ is positive, so we have

$$\frac{d}{dh}\varphi \geq -\lambda_1(h, p_o) > 0 \,. \tag{2.8}$$

Otherwise, if $p_o < 1$, then $h > h_o$ and $\partial_h \lambda_1(h, p_o) < 0$, so that (2.8) still holds. Finally, if $p_o = 1$ then $p = 1$ is the shock curve, and $\frac{d}{dh}\varphi = 1 > 0$ is obvious.

We conclude that φ is monotone along h-shock/rarefaction curve. Then, by continuity and the fact that $\partial_H h > 0$, one can use φ to parameterize the h-shock/rarefaction curve, and this change of parameterization (w.r.t. the parametrization by H) is locally bi-Lipschitz. Moreover, φ ranges from 0 to $+\infty$ along a h-shock/rarefaction curve.

(i) Since wave strength is measured in Riemann coordinate, (2.5) follows from the equivalence of two parameterizations, between H and φ.

(ii) The strength of the reflected h-wave is a function of h_l and σ_p, denote it as $\sigma_h^+ \doteq \Phi(h_l, \sigma_p)$. We claim this mapping is well-defined for $h_l \geq 0$. Indeed, if $h_l = 0$, we obviously have $\Phi(0, \sigma_p) = 0$. If $h_l > 0$, let (h_l, p_l) and (h_r, p_r) be the state on the left and right (respectively) of the incoming p-wave, and let (h_l, p_l) and (h_r^+, p_r^+) be those for the reflected h-wave. Since the flux is continuous along the boundary, we have $\varphi(h_r, p_r) = \varphi(h_r^+, p_r^+)$. Then along the h-shock/rarefaction curve issued at (h_l, p_l), there is a unique state (h_r^+, p_r^+) such that $\varphi(h_r^+, p_r^+) = \varphi(h_r, p_r)$. So the mapping $\Phi(h_l, \sigma_p)$ is well-defined. We also have $\Phi(h_l, 0) = 0$. By the continuity of the mapping Φ and the obvious identity $\Phi(0, \sigma_p) = 0$, we get (2.6).

(iii) Let (h, p), (h^+, p^+) be the state before and after the time step (2.4), respectively. From (2.4) we have $h^+ = h\,(1 + \Delta t(p - 1))$ and $p^+ = p$. The strength of the new h-wave is a function of h, p, h^+. Let's denote it again by $\sigma_h^+ \doteq \Phi(h, h^+; p)$. Again this mapping is well-defined and continuous. By the identity $\Phi(h, h; p) = 0$ we get $\sigma_h^+ = \mathcal{O}(1)(h^+ - h) = \mathcal{O}(1)\Delta t\, h\,(p - 1)$ and hence (2.7). $\square$

3 Global existence of large BV solutions for the initial boundary value problem; Proof of Theorem 1

In this section we study the global existence of solutions to the initial-boundary value problem for the system (1.1) on the half line $\{x < 0\}$, with initial data (1.2) at $t = 0$, and boundary data (1.3) at $x = 0$, proving Theorem 1.

We remark that as long as p remains strictly positive, the characteristic speeds satisfy $\lambda_1 < 0 \leq \lambda_2$, therefore the problem is locally well posed.

The main steps in the proof of Theorem 1 are very similar to those for Theorem 1 in [2]. We approximate the initial data and the boundary data with piecewise constant functions. The flux on the boundary is approximated by a piecewise constant function, constant on the time interval (t_{k-1}, t_k). On the time intervals (t_{k-1}, t_k) an approximate solution of the conservation laws (2.3) is constructed by front tracking, with constant flow at the boundary $x = 0$. At time $t = t_k$ the solution is updated by means of (2.4).

The following global a priori estimates will be derived for the approximate solutions:

- the norms $\|h(t,\cdot)\|_{\mathbf{L}^1}$ and $\|p(t,\cdot) - 1\|_{\mathbf{L}^1}$;
- the lower bound on p, i.e., the quantity $\inf_x p(t,x)$;
- the uniform bounds on h and p, i.e., the quantities $\|h(t,\cdot)\|_{\mathbf{L}^\infty}$ and $\|p(t,\cdot)\|_{\mathbf{L}^\infty}$;
- the total variations $\text{Tot.Var.}\{h(t,\cdot)\}$ and $\text{Tot.Var.}\{p(t,\cdot)\}$.

These will be established in Sections 3.1-3.4. At the end, we will put them together in Section 3.5 to prove Theorem 1. Since the proof is an extension of that in [2], we will make it rather brief, describing mainly the needed modifications.

3.1 The $\mathbf{L}^1$ bound on $p(t,\cdot) - 1$ and $h(t,\cdot)$.

This estimate follows the one in Section 4.1 in [2]. By the second conservation equation, the estimate

$$\|p(t,\cdot) - 1\|_{\mathbf{L}^1(\mathbb{R}_-)} \leq \|\bar{p} - 1\|_{\mathbf{L}^1(\mathbb{R}_-)} \qquad \text{for all } t \geq 0 \tag{3.1}$$

remains valid. To estimate the $\mathbf{L}^1$ norm of $h(t,\cdot)$, we define the weight W,

$$W(t,x) \doteq \exp\left\{\int_{-\infty}^{x} |p(t,y) - 1|\,dy\right\} = \exp\left\{\int_{-\infty}^{x} |q(t,y)|\,dy\right\}. \tag{3.2}$$

This weight accounts for the mass of $(p-1)$ to be encountered at the point x. Note that, using the second equation in (1.1), the Lipschitz function W satisfies the inequality

$$W_t + h\,W_x \leq 0\,, \tag{3.3}$$

and the following inequality holds in the sense of distributions

$$(Wh)_t - (Whp)_x \;\leq\; 0\,. \tag{3.4}$$

Now define the weighted functional

$$\widehat{\mathcal{I}}^h(t) \doteq \int_{-\infty}^{0} W(t,x)\,h(t,x)\,dx + \int_t^{\infty} W(\tau,0)\,F(\tau)\,d\tau\,.$$

From (3.4) we get that

$$\begin{aligned}\frac{d}{dt}\widehat{\mathcal{I}}^h(t) &= \frac{d}{dt}\left(\int_{-\infty}^0 W(t,x)\,h(t,x)\,dx\right) - W(t,0)F(t)\\ &\le (Wph)(t,0) - W(t,0)F(t) = 0\,,\end{aligned}$$

hence the above functional $\widehat{\mathcal{I}}^h$ is non-increasing in time. Therefore, using also (3.1), we have the following a priori estimate

$$\|h(t,\cdot)\|_{\mathbf{L}^1} \le \exp\{\|\bar p - 1\|_{\mathbf{L}^1}\} \cdot \left(\|\bar h\|_{\mathbf{L}^1} + \|F\|_{\mathbf{L}^1}\right)\,. \tag{3.5}$$

3.2 The lower bound on p and the $\mathbf{L}^\infty$ bounds on h and p.

These estimates can be derived in a similar way as in Section 4.2 and 4.3 in [2]. We first observe that, if all wave strengths are measured in terms of Riemann coordinates, then all the interaction estimates (1.17)–(1.19) and the boundary estimates (2.5)–(2.7) contain the additional factor $\|h\|_{\mathbf{L}^\infty}$. Therefore, if the norm $\|h\|_{\mathbf{L}^\infty}$ remains sufficiently small, we can assume that the total strength of all new waves produced by interactions is as small as we like. In essence, the change in the total variation and in the $\mathbf{L}^\infty$ norms of h, p is thus determined only by the source term in the first equation (1.1).

To achieve a lower bound on p, we define

$$P_{\inf}(t) \doteq \text{ess-}\inf_x P(t,x)\,.$$

For any smooth solutions of (1.1), ignore interaction effects, and consider a 2-characteristic, say $t \mapsto x_2(t)$, with $\dot x_2(t) = \lambda_2(t) \ge 0$. Since the 2-characteristics go out of the domain at $x = 0$, the boundary condition plays no role here. Then we have

$$\frac{d}{dt}P(t,x_2(t)) = \frac{\partial P}{\partial h}\cdot(p-1)h \ge 0\,.$$

Indeed, the geometry of the wave curves implies $\partial P/\partial h < 0$ when $p < 1$ and $\partial P/\partial h > 0$ when $p > 1$. This shows that the quantity $P_{\inf}$ is non-decreasing in time if the solution is smooth.

For the bound on the $\mathbf{L}^\infty$ norm and the total variation of the solution, we define the weight V^h

$$V^h(t,x) \doteq \exp\left(\kappa_1\cdot\int_{-\infty}^x |p(t,y)-1|\,dy\right) = W(t,x)^{\kappa_1}\,, \tag{3.6}$$

and the weight $\widehat V^p$

$$\widehat V^p(t,x) \doteq \exp\left(\kappa_2\cdot\int_x^0 W(t,y)h(t,y)\,dy + \kappa_2\cdot\int_t^\infty W(\tau,0)\,F(\tau)\,d\tau\right)\,. \tag{3.7}$$

From the analysis in Section 3.1, the quantities V^h, $\widehat V^p$ are a priori bounded. The estimate for $\|h(t,\cdot)\|_\infty$ follows in a similar way as in [2, Section 4.3], provided that the constant κ_1 is chosen large enough. For the bound on the quantity $\|p(t,\cdot)\|_\infty$, note that by (3.4) we have

$$\begin{aligned}&\frac{d}{dt}\left(\int_{x(t)}^0 W(t,y)h(t,y)\,dy + \int_t^\infty W(\tau,0)\,F(\tau)\,d\tau\right)\\ &\qquad \le \big(Whp(t,0) - Whp(t,x)\big) - Wh(t,x)\dot x - W(t,0)hp(t,0)\end{aligned}$$

and hence

$$\frac{d}{dt}\widehat{V}^p(t,x(t)) \quad \leq \quad -\kappa_2 \widehat{V}^p\, W\, h\, (p+\dot{x})\,. \tag{3.8}$$

Therefore, along a 2-characteristics $x_2(t)$, and for κ_2 large enough, one has

$$\frac{d}{dt}\big(\widehat{V}^p P\big)(t,x_2(t)) \quad \leq \quad -\kappa_2 \widehat{V}^p \cdot (p+\lambda_2)\cdot W\, h \cdot P + \widehat{V}^p \cdot \frac{\partial P}{\partial h}\,(p-1)h \;\leq\; 0\,.$$

This leads to a uniform, a priori bound on $\|p(t,\cdot)\|_\infty$.

3.3 Wave front interactions with no source term

This is the counter part for Section 4.4 in [2]. However, in the study of the effect of wave-front interactions for the initial boundary value problem, we also need to consider the case of reflecting waves from the boundary $x=0$. This is the case when a 2-wave of strength σ_p hits the boundary and travels out of the domain $x \leq 0$, and a 1-wave will be reflected into the domain $x \leq 0$. It is not restrictive to assume that no 2-waves would hit the boundary $x=0$ at exactly $t=t_k$ for any k (this can be achieved by an arbitrary small perturbation). Thanks to Lemma 1, the new 1-wave is small, whose strength is of order $\mathcal{O}(1)\cdot h_l|\sigma_p|$.

In order to handle the incoming wave at the boundary due to the jump in F, we define the total wave strength V as

$$V \doteq \sum_\alpha |\sigma_\alpha| + C_0 \;\text{Tot.Var.}\{F\,;\; [t,\infty[\,\}\,.$$

The interaction potential $\mathcal{Q}$ are defined in the same way as in Section 4.4 in [2], with

$$\mathcal{Q}(u) = \mathcal{Q}_{hh} + \mathcal{Q}_{pp} + \mathcal{Q}_{ph}\,.$$

Here the interaction potential of waves of the first family is defined as

$$\mathcal{Q}_{hh} = \sum_{i_\alpha = i_\beta = 1,\, x_\alpha < x_\beta} w_{\alpha,\beta}\, |\sigma_\alpha|\,|\sigma_\beta|\,.$$

Since this first characteristic field is not genuinely nonlinear along the line $\{p=1\}$, we insert here the factor $w_{\alpha,\beta}$ defined as follows. If σ_α and σ_β are two shocks, on the same side of the line $p=1$, then we set

$$w_{\alpha,\beta} = \delta_0 \cdot \min\;\Big\{|P_l^\alpha - 1|\,,\; |P_l^\beta - 1|\Big\}$$

where $P_l^\alpha = P(x_\alpha-)$, $P_l^\beta = P(x_\beta-)$ are the left limits of P (Riemann coordinate) at x_α, x_β respectively, and $\delta_0 > 0$ is a small constant to be determined. In all other cases, we set $w_{\alpha,\beta} = 0$.

The other parts of the interaction potential are defined as usual:

$$\mathcal{Q}_{pp} = \sum_{(\alpha,\beta)\in\mathcal{A}_2} |\sigma_\alpha|\,|\sigma_\beta| \tag{3.9}$$

is the interaction potential of waves of the second family. Here $\mathcal{A}_2$ denotes the set of couples of waves of the second family, with $x_\alpha < x_\beta$, at least one of which is a shock. Finally,

$$\mathcal{Q}_{ph} = \sum_{i_\alpha=2,\; i_\beta=1,\; x_\alpha<x_\beta} |\sigma_\alpha|\,|\sigma_\beta| \tag{3.10}$$

is the interaction potential among waves of different families. We then introduce the functional

$$\mathcal{S} \doteq V + c\mathcal{Q},$$

for some positive constant c.

At an interaction point, analysis in [2] Section 4.4 shows that $\mathcal{S}$ is decreasing if we choose $\|h\|_{\mathbf{L}^\infty}$ sufficiently small. We claim that the same is true at a reflection point and at where the boundary flux $F(t)$ has a jump.

Indeed, at a reflection point, we see that V decreases because the new generated 1-wave is of much smaller strength than the old 2-wave. Let ΔV be the change in V due to the reflection. Using (2.6), the change can be estimated by

$$\Delta V = -|\sigma_p| + \mathcal{O}(1) \cdot \|h\|_{\mathbf{L}^\infty} \cdot |\sigma_p| .$$

For the corresponding change in $\mathcal{Q}$, we see that $\mathcal{Q}_{hh}$ may increase by the amount $\mathcal{O}(1)\cdot\|h\|_{\mathbf{L}^\infty}|\sigma_p|$, and $\mathcal{Q}_{ph}$ may also increase by the same amount because the newly created 1-wave will be approaching all the 2-waves. The term $\mathcal{Q}_{pp}$ will decrease, because the 2-wave travels out of the domain. In summary, the change in $\mathcal{S}(t) = V(t) + c\mathcal{Q}(t)$ is estimated as

$$\Delta\mathcal{S} \;\leq\; -|\sigma_p| \;+\; \mathcal{O}(1)\cdot\|h\|_{\mathbf{L}^\infty}\cdot|\sigma_p| \;+\; c\,\mathcal{O}(1)\cdot\|h\|_{\mathbf{L}^\infty}|\sigma_p| \;\leq\; -\frac{|\sigma_p|}{2} \tag{3.11}$$

if $\|h\|_{\mathbf{L}^\infty}$ is assumed small enough.

For the case where the boundary flux F has a jump at time τ, by (2.5) in Lemma 1 the change in V can be estimated as

$$\Delta V = \mathcal{O}(1)|\Delta F| - C_0|\Delta F| \leq -\frac{C_0}{2}|\Delta F|$$

for C_0 large enough. For the corresponding change in $\mathcal{Q}$, we see that $\mathcal{Q}_{pp} = 0$ while $\mathcal{Q}_{hh} + \mathcal{Q}_{ph}$ may increase by the amount $\mathcal{O}(1)\cdot|\Delta F|$; however, this increase is balanced with the decrease of V, for C_0 large enough with respect to the constant c in the definition of $\mathcal{S}$.

We can conclude that the function $\mathcal{S}(t)$ is decreasing at every interaction and reflection point, and at the point where the boundary flux has a discontinuity.

3.4 New wave functionals for the system with the source term.

This is the counter part of Section 4.5 and 4.6 in [2]. To derive a-priori bounds on the total variation, we introduce the functional

$$Z(t,x) \doteq V^h(t,x)\cdot \widehat{V}^p(t,x) . \tag{3.12}$$

Recall that V^h and $\widehat{V}^p$ are defined in (3.6) and (3.7) respectively. Along a discontinuity $x(t)$, using (3.3) we get

$$\frac{d}{dt}V^h(t,x(t)) \;\leq\; -\kappa_1 V^h\cdot|p_l-1|\cdot(h_l-\dot{x}) .$$

Then by (3.8), we get

$$\frac{d}{dt}Z(t,x(t)) \;\leq\; -Z(t,x(t))\Big\{\kappa_1(h_l-\dot{x})|p_l-1| \;+\; \kappa_2(p_r+\dot{x})W(t,x(t))h_r\Big\} . \tag{3.13}$$

In particular, $Z(t,0)$ is decreasing in time:

$$\frac{d}{dt}Z(t,0) \quad\leq\quad -Z(t,0)\Big\{\kappa_1\, h(t,0)\,|p(t,0)-1| \;+\; \kappa_2\,(hp)(t,0)\,W(t,0)\Big\} \;\leq\; 0\,. \tag{3.14}$$

For the total wave strength and interaction potential, we introduce the functionals

$$\widehat{\mathcal{S}} \;=\; \widehat{V} + c\,\widehat{\mathcal{Q}}\,, \tag{3.15}$$

where

$$\widehat{V}(t) \;=\; \sum_\alpha Z(t,x_\alpha)\,|\sigma_\alpha| \;+\; C_0\,Z(t,0)\Big(1+\text{ Tot.Var.}\{F\,;\;[t,\infty[\,\}\Big)\,, \tag{3.16}$$

$$\begin{aligned}\widehat{\mathcal{Q}}(t) \;&=\; \widehat{\mathcal{Q}}_{hh}(t) \;+\; \widehat{\mathcal{Q}}_{pp}(t) \;+\; \widehat{\mathcal{Q}}_{ph}(t)\\ &\qquad + C_0\cdot\left(\sum_\alpha Z(t,x_\alpha)|\sigma_\alpha|\right)\cdot Z(t,0)\Big(1+\text{ Tot.Var.}\{F\,;\;[t,\infty[\,\}\Big)\,,\end{aligned} \tag{3.17}$$

and

$$\begin{aligned}\widehat{\mathcal{Q}}_{hh}(t) \;&=\; \sum_{i_\alpha=i_\beta=1,\,x_\alpha<x_\beta} w_{\alpha,\beta}Z(t,x_\alpha)|\sigma_\alpha|\,Z(t,x_\beta)|\sigma_\beta|\,,\\ \widehat{\mathcal{Q}}_{pp}(t) \;&=\; \sum_{(\alpha,\beta)\in\mathcal{A}_p} Z(t,x_\alpha)|\sigma_\alpha|\,Z(t,x_\beta)|\sigma_\beta|\,,\\ \widehat{\mathcal{Q}}_{ph}(t) \;&=\; \sum_{i_\alpha=2,i_\beta=1,\,x_\alpha<x_\beta} Z(t,x_\beta)|\sigma_\alpha|\,Z(t,x_\beta)|\sigma_\beta|\,.\end{aligned}$$

Here all summations range over wave fronts σ_α of the functions (h,p) at time t. Notice that now we have an explicit dependence on time, because of the boundary condition $F(\cdot)$.

We remark that, at times of interaction of two wave fronts or reflection at the boundary, the same arguments as in Section 3.3 would lead to the conclusion of a decreasing $\widehat{\mathcal{S}}$. Indeed, the presence of the weights $Z(x_\alpha)$ may only increase the sizes of the coefficients $\mathcal{O}(1)$ in the various estimates. This can be counter-balanced by choosing $\|h\|_{\mathbf{L}^\infty}$ sufficiently small. We omit the details.

Now we show that the newly defined functional $\widehat{\mathcal{S}}$ is non increasing from time t_{k-1} to t_k.

The change in $\widehat{V}$ from t_{k-1} to t_k. First we notice that the first term in (3.16) will decrease if we don't consider the boundary condition. The change in $\widehat{V}$ can be written

$$\Delta\widehat{V} \;=\; (\Delta\widehat{V})_1 \;+\; (\Delta\widehat{V})_2\,.$$

Here the first term $(\Delta\widehat{V})_1$ is the change in the first term in (3.16) without taking into account the boundary terms. This is estimated in [2]:

$$(\Delta\widehat{V})_1 \;\leq\; -c_1\Delta t\sum_{\alpha,\,i_\alpha=1} Z(x_\alpha(t_{k-1}))|\sigma_\alpha||p_l^\alpha-1| - c_2\Delta t\sum_{\beta,\,i_\beta=2} Z(x_\beta(t_{k-1}))|\sigma_\beta|h_r^\beta\,. \tag{3.18}$$

The second term $(\Delta\widehat{V})_2$ is caused by the boundary condition. At time t_k, a 1-wave is generated and it propagates into the domain $x\leq 0$. This new wave will increase the first term in (3.16).

However, the second term will decrease. Denote by $\bar{\sigma}_0$ the strength of the new wave. By (2.5) and (2.7), the strength $\bar{\sigma}_0$ can be bounded as

$$|\bar{\sigma}_0| \leq \mathcal{O}(1)\cdot|F(t_k)-F(t_{k-1})| + \mathcal{O}(1)\cdot\Delta t\cdot h(t_{k-1}).$$

By the lower bound on p and the definition of the flux $F = hp$, this is equivalent to

$$|\bar{\sigma}_0| \leq \mathcal{O}(1)\cdot\Big\{\,|F(t_k)-F(t_{k-1})| + \Delta t\cdot F(t_{k-1})\Big\}. \tag{3.19}$$

Now, define $Z_k \doteq Z(t_k,0)$ and $Z_{k-1} \doteq Z(t_{k-1},0)$, we have

$$(\Delta\widehat{V})_2 = Z_k|\bar{\sigma}_0| \;+\; C_0 I_v$$

where

$$I_v = (Z_k - Z_{k-1}) + (Z_k\,\text{Tot.Var.}\{F;[t_k,\infty[\,\} - Z_{k-1}\,\text{Tot.Var.}\{F;[t_{k-1},\infty[\,\})\;. \tag{3.20}$$

Let's estimate the first term in (3.20). Note that by (3.14) we have $Z_k \leq Z_{k-1}$. Furthermore, recalling that $W \geq 1$, $\kappa_2 > 1$, we have from (3.14)

$$\frac{d}{dt}Z(t,0) \leq -Z(t_{k-1})(hp)(t_{k-1},0) = -Z_{k-1}F(t_{k-1})$$

on the time interval $t \in (t_{k-1},t_k)$, since F is constant on the interval. This gives the following estimate

$$Z_k - Z_{k-1} \;\leq\; -\Delta t\cdot Z_{k-1}F(t_{k-1}) \leq 0\,. \tag{3.21}$$

For the last term in (3.20), we have

$$\begin{aligned}
Z_k\,&\text{Tot.Var.}\{F;[t_k,\infty[\,\} \;-\; Z_{k-1}\,\text{Tot.Var.}\{F;[t_{k-1},\infty[\,\} \\
&\leq\;\; Z_{k-1}\Big[\text{Tot.Var.}\{F;[t_k,\infty[\,\} \;-\; \text{Tot.Var.}\{F;[t_{k-1},\infty[\,\}\Big] \\
&\leq\;\; -Z_{k-1}\,|F(t_k)-F(t_{k-1})|\;.
\end{aligned}$$

Hence we obtain an estimate for I_v:

$$I_v \;\;\leq\;\; -Z_{k-1}\,\{\Delta t\,F(t_{k-1}) + |F(t_k)-F(t_{k-1})|\}\;. \tag{3.22}$$

Together with (3.19) and (3.21), we get an estimate for $(\Delta\widehat{V})_2$

$$(\Delta\widehat{V})_2 \;\;\leq\;\; Z_{k-1}\,\{\Delta t\,F(t_{k-1}) + |F(t_k)-F(t_{k-1})|\}\,\Big\{\mathcal{O}(1) \;-\; C_0\Big\}\,. \tag{3.23}$$

By choosing C_0 sufficiently large, we conclude that $(\Delta\widehat{V})_2 \leq 0$, therefore $\Delta\widehat{V} \leq (\Delta\widehat{V})_1$ which can be bounded as in (3.18).

The change in $\widehat{\mathcal{Q}}$ from t_{k-1} to t_k. Similarly, the change in $\widehat{\mathcal{Q}}$ consists of two parts:

$$\Delta\widehat{\mathcal{Q}} \;=\; (\Delta\widehat{\mathcal{Q}})_1 \;+\; (\Delta\widehat{\mathcal{Q}})_2\,,$$

where $(\Delta\widehat{\mathcal{Q}})_1$ is the part without considering the boundary effects, which is estimated in [2]:

$$(\Delta\widehat{\mathcal{Q}})_1 \quad \leq \quad \mathcal{O}(1)\Delta t \left\{ \sum_{i_\alpha=1} Z_\alpha |\sigma_\alpha| |p_l^\alpha - 1| + \sum_{i_\beta=2} Z_\beta |\sigma_\beta| h_l^\beta \right\} \cdot \left\{ \sum_\alpha Z_\alpha |\sigma_\alpha| \right\}, \qquad (3.24)$$

and $(\Delta\widehat{\mathcal{Q}})_2$ is caused by the boundary conditions. Let's consider the term $(\Delta\widehat{\mathcal{Q}})_2$. We see that the incoming 1-wave $\bar{\sigma}_0$ at the boundary at t_k will make $\widehat{\mathcal{Q}}_{hh}$ and $\widehat{\mathcal{Q}}_{ph}$ increase, however the last term in $\widehat{\mathcal{Q}}$ will decrease. This incoming wave does not affect the term $\widehat{\mathcal{Q}}_{pp}$. The total change can be estimated as

$$(\Delta\widehat{\mathcal{Q}})_2 \quad = \quad Z_k\, |\bar{\sigma}_0| \left\{ \sum_{i_\alpha=1} w_{0,\alpha}\, Z(t_k, x_\alpha)|\sigma_\alpha| + \sum_{i_\alpha=2} Z(t_k, x_\alpha)|\sigma_\alpha| \right\} + C_0 \left\{ \sum_\alpha Z(t_k, x_\alpha)|\sigma_\alpha| \right\} I_v$$

where I_v is given by (3.20). Now, using (3.21), (3.19) and (3.22), we get

$$\begin{aligned} (\Delta\widehat{\mathcal{Q}})_2 \quad &\leq \quad Z_{k-1}\, |\bar{\sigma}_0|\, \mathcal{O}(1) \left\{ \sum_\alpha Z(t_k, x_\alpha)|\sigma_\alpha| \right\} \;+\; C_0 \left\{ \sum_\alpha Z(t_k, x_\alpha)|\sigma_\alpha| \right\} I_v \\ &\leq \quad \left\{ \sum_\alpha Z(t_k, x_\alpha)|\sigma_\alpha| \right\} \cdot \left\{ \mathcal{O}(1)\, Z_{k-1}\, |\bar{\sigma}_0| + C_0 I_v \right\} \;\leq\; 0 \end{aligned}$$

for C_0 sufficiently large.

We conclude that $(\Delta\widehat{\mathcal{Q}})_2 \leq 0$ so $\Delta\widehat{\mathcal{Q}} \leq (\Delta\widehat{\mathcal{Q}})_1$ which is estimated by (3.24).

3.5 Putting things together to complete the proof.

The proof follows in a similar way as in [2] Section 5, where one goes through all the a priori estimates, and justify that the errors we neglected such as interactions, discretizations are vanishingly small. Therefore all the a priori estimates hold, and convergence follows. Here, however, care must be taken for the issues coming from the boundary condition, which we discuss below.

The extra error terms that we neglected in this proof is the incoming 1-wave from the boundary due to a jump in the flux F, and the reflecting 1-waves from the boundary.

For the lower bound of p, the incoming 1-wave caused by a jump in F has strength of order $\mathcal{O}(1) \cdot \Delta F$, thanks to the estimate (2.5). Then the change in P caused by this wave is of order $\mathcal{O}(1) \cdot (\Delta F)^2$. Summing over all time steps, the total change in P is of order

$$\mathcal{O}(1) \cdot \sum_j \Delta F(t_j)^2 \leq \mathcal{O}(1) \cdot \|F\|_{\mathbf{L}^\infty} \text{Tot.Var.}\{F\}\,.$$

Thanks to assumption (1.7), this change is of order $\mathcal{O}(1)\delta$, which is as small as we like. Furthermore, the discretization will introduce an error of order $\mathcal{O}(1) \cdot \Delta t \cdot \|h\|_{\mathbf{L}^\infty}$ in P.

The extra error term caused by the reflecting 1-waves from the boundary is of order $\mathcal{O}(1) \cdot \|h\|_{\mathbf{L}^\infty} |\Delta\widehat{S}|$, thanks to (2.6) and (3.11), and therefore the sum of these terms is vanishingly small.

Furthermore, one must verify that the boundary conditions are satisfied in the limit, in the sense of Def. 1. Here, due to the fact that the 2-characteristic speed is not bounded away from 0, large p-waves with speed close to 0 may accumulate at the boundary. Fortunately, the speed of a p-wave is proportional to h, which is a factor in the flux F, so the jump in F across such a p-wave is vanishingly small. By the assumption on the total variation on $F(t)$, we see that F is continuous in t except at countably many points. Consider a point $t = \tau$ where F is continuous, we consider two cases.

- First, if $F(\tau) > 0$, then due the lower bound on p we must have $h > 0$. Hence the characteristic speed of the second family is strictly bounded away from 0 in a time interval $I_\tau = [t_1, t_2]$ around τ. One can then use an argument similar to [1, Sect. 7] and conclude that the boundary condition is satisfied pointwise in $[t_1, t_2]$ except at most countably many times.

- Second, if $F(\tau) = 0$, then by continuity of F, for every ε there exists an interval $I_\tau = [t_1, t_2]$ around $t = \tau$ such that $F(t) \le \varepsilon$ for all $t \in I_\tau$. Then, by the lower bound on p ($p \ge p_0$), we have $h(t, 0) \le \frac{1}{p_0}\varepsilon$ for every $t \in I_\tau$. Now, the mass of h travels with speed $-p$, which is strictly negative and $-p \le -p_0$. We define the domain of dependence Ω_τ of I_τ for h

$$\Omega_\tau \doteq \{(t, x);\ t_1 \le t \le t_2,\ \ -p_0(t - t_1) \le x \le 0\} .$$

  Clearly, $(\tau, 0) \in \Omega_\tau$. The change of h in Ω_τ is proportional to the mass of $p - 1$ it crosses. Thanks to the bound on the $\mathbf{L}^1$ norm of $p - 1$, we have $h(t, x) \le \frac{c_0}{p_0}\varepsilon$ for all $(t, x) \in \Omega_\tau$ and some constant $c_0 > 0$. Then, by the uniform bound on p, we have

$$h(x, t)\, p(x, t) \le \|p\|_{\mathbf{L}^\infty} \frac{c_0}{p_0}\varepsilon, \quad \text{for } (x, t) \in \Omega_\tau . \tag{3.25}$$

  Since ε is arbitrary and the estimate (3.25) holds for every approximate solution (h, p), we get

$$\lim_{x \to 0-} h(\tau, x) p(\tau, x) = 0 = F(\tau) .$$

Therefore, we conclude that the boundary condition is satisfied by the approximate solution in the limit at almost every point, except at most countably many points. This completes the proof of Theorem 1.

4 The slow erosion limit; proof of Theorem 2

In this section we prove Theorem 2, showing that the rescaled solutions $\tilde{p}_\nu = \tilde{p}_\nu(\mu, x)$ converge to a solution $\tilde{p}$ of (1.14), (1.15). This goal will be achieved in several steps. Since the functions $q = p - 1$, $q_\nu \doteq p_\nu - 1$ are integrable, it is convenient to rewrite the equations (1.1), (1.14) in terms of these variables:

$$\begin{cases} (h_\nu)_t - ((q_\nu + 1)h_\nu)_x &= q_\nu h_\nu , \\ (q_\nu)_t + (h_\nu q_\nu)_x &= 0 , \end{cases} \tag{4.1}$$

$$q_\mu + \left(\frac{q}{q + 1} \cdot \exp \int_x^0 \frac{q(\mu, \xi)}{q(\mu, \xi) + 1}\, d\xi \right)_x = 0 . \tag{4.2}$$

The main steps in the proof are the followings.

1. Establish a Lipschitz-type dependence of q on the rescaled time variable for the solutions of (4.1). The estimate is uniform in ν.

2. Derive estimates on the flux function $\varphi = (q+1)h$ for the solutions of (4.1), uniformly in ν.

3. Show that the limit of the q-component of (4.1) is a weak solution of (4.2), using the estimates in the previous two steps. This is achieved by showing that it satisfies the weak formulation. One needs the strong convergence of the function $q_\nu/(q_\nu+1)$ and the weak convergence of the flux $(q_\nu+1)h_\nu$. The details are worked out in Sec. 4.3.

4. Check the entropy admissibility of the limit, achieved in Sec. 4.4.

We define an entropy weak solution of (4.2) as a map $q : [0, M'] \to \mathbf{L}^1(\mathbb{R}_-)$ which is Lipschitz continuous and satisfies

- $\text{ess-}\inf_x q(\mu, x) + 1 \geq p_0$ for some $p_0 > 0$, for all $\mu \in [0, M']$;
- $\sup_{\mu \in [0,M']} \|q(\mu, \cdot)\|_{\mathbf{L}^\infty} < \infty$;
- q is a weak entropy solution of

$$q_\mu + \left(k(\mu, x) \, \frac{q}{q+1} \right)_x = 0 \,, \tag{4.3}$$

where

$$k(\mu, x) \doteq \exp \left\{ \int_x^0 \frac{q(\mu, y)}{q(\mu, y) + 1} \, dy \right\} \,,$$

which is Lipschitz continuous on $[0, M'] \times \mathbb{R}_-$ thanks to the above assumptions.

Before we start the estimates on solutions of (4.1), we first notice that, given any $\varepsilon_0 > 0$, there exists $R > 0$ large enough so that

$$\int_{-\infty}^{-R} |\bar{p}(x) - 1| \, dx < \varepsilon_0 \,. \tag{4.4}$$

By (1.1), the flow speed for $|q| = |p - 1|$ is non-negative. Hence

$$\int_{-\infty}^{-R} |q_\nu(t, x)| \, dx \; = \; \int_{-\infty}^{-R} |p_\nu(t, x) - 1| \, dx \; < \; \varepsilon_0 \tag{4.5}$$

for all ν and all $t > 0$. To prove that

$$\lim_{\nu \to \infty} \left(\sup_{\tau \in [0, M']} \int_{-\infty}^0 |\tilde{q}_\nu(\tau, x) - \tilde{q}(\tau, x)| \; dx \right) = 0 \,,$$

it thus suffices to show that $\tilde{q}_\nu \to \tilde{q}$ in the space $\mathbf{L}^\infty\Big([0, M'] \,;\; \mathbf{L}^1([-R, 0])\Big)$, for any given $R > 0$.

4.1 Lipschitz dependence of p on rescaled time

In this step we establish a Lipschitz-type dependence of q on the rescaled time variable, see (4.10) below. Consider a solution (h, q) of the initial-boundary value problem

$$\begin{cases} h_t - ((q+1)h)_x &= qh\,, \\ q_t + \big(hq\big)_x &= 0\,, \end{cases} \tag{4.6}$$

with boundary condition

$$h(t,0)\,(q(t,0)+1) = F(t)\,.$$

Assume that $t' < t''$ and

$$\int_{t'}^{t''} F(t)\,dt = \delta > 0\,. \tag{4.7}$$

As a first step we derive an estimate on the difference $\|q(t'',\cdot) - q(t',\cdot)\|_{\mathbf{L}^1}$, showing that it is $\mathcal{O}(\delta)$ as $\delta \to 0$, $\|F\|_{\mathbf{L}^\infty} \to 0$ and $\|h\|_{\mathbf{L}^\infty} \to 0$. Define the function

$$g(x) \doteq \sup_{t\in[t',t'']} |q(t,x) - q(t',x)|\,. \tag{4.8}$$

We will establish the estimate

$$\int_{-R}^{0} g(x)\,dx \leq C_g \cdot \Big(\|h\|_{\mathbf{L}^\infty} + \delta\Big)\,, \tag{4.9}$$

where the constant C_g depends on R, the total variations of q, h and the $\mathbf{L}^1$ norms of h, F, but not on δ. We remark that (4.9) leads to

$$\|q(t'',\cdot) - q(t',\cdot)\|_{\mathbf{L}^1((-R,0))} \leq C_g \cdot \Big(\|h\|_{\mathbf{L}^\infty} + \int_{t'}^{t''} F(t)\,dt\Big)\,. \tag{4.10}$$

To prove (4.9), let (H, Q) be the Riemann coordinates corresponding to the solution (h, q), with $Q = P - 1$, as defined in Section 2. Observe that

$$\sup_{t,x} |Q(t,x) - q(t,x)| = \mathcal{O}(1) \cdot \|h\|_{\mathbf{L}^\infty}\,. \tag{4.11}$$

To prove (4.9) it suffices to establish the corresponding statement for the variable Q, namely

$$\int_{-R}^{0} G(x)\,dx = \mathcal{O}(1) \cdot \Big(\|h\|_{\mathbf{L}^\infty} + \delta\Big)\,, \tag{4.12}$$

where

$$G(x) \doteq \sup_{t\in[t',t'']} |Q(t,x) - Q(t',x)|\,.$$

Since the entropy weak solution is obtained as a limit of a sequence of front tracking approximations, it suffices to prove that the bound (4.12) is uniformly valid for all approximations. From now on we thus consider a piecewise constant approximation, constructed by the flux-splitting technique described by (2.3), (2.4) in Section 2, and use the Riemann variables (H, Q). We have the estimate

$$\int_{-R}^{0} G(x)\,dx \;\leq\; J_1 + J_2 + J_3\,, \tag{4.13}$$

where the J_i $(i = 1, 2, 3)$ are defined as follows. Let $t_\ell \in [t', t'']$ be the times at which the source term is inserted, as in (2.4). Moreover, for each $t \in [t', t'']$, let $x_\alpha(t)$, $i_\alpha \in \{1, 2\}$, be the locations of the wave-fronts at time t, belonging to the first or the second family. We then define

$$J_1 \doteq \int_{t'}^{t''} \sum_{i_\alpha=1} \Big|Q(t, x_\alpha+) - Q(t, x_\alpha-)\Big| \cdot |\dot{x}_\alpha(t)|\, dt\,,$$

$$J_2 \doteq \int_{t'}^{t''} \sum_{i_\alpha=2} \Big|Q(t, x_\alpha+) - Q(t, x_\alpha-)\Big| \cdot |\dot{x}_\alpha(t)|\, dt\,.$$

Here J_1 and J_2 account for the changes in Q due to the crossings of 1- and 2-waves, respectively. Finally, the contributions of the source terms at the times t_ℓ yield the term

$$J_3 \doteq \sum_{t_\ell \in [t', t'']} \int_{-R}^{0} |Q(t_\ell+, x) - Q(t_\ell-, x)|\, dx\,.$$

To estimate the first term J_1, we observe that along a rarefaction front of the first family, the jump in the second Riemann coordinate is zero. Along a shock front of the first family, the jump in the second Riemann coordinate is of second order w.r.t. the size of the shock. For any front of the first family, say located at x_α and with strength $|\sigma_\alpha|$, we thus have

$$\Big|Q(t, x_\alpha+) - Q(t, x_\alpha-)\Big| = \mathcal{O}(1) \cdot |\sigma_\alpha| \cdot \|h\|_{\mathbf{L}^\infty}\,.$$

This yields the estimate

$$\begin{aligned} J_1 &= \int_{-R}^{0} \left(\sum_{i_\alpha=1,\ x_\alpha(t)=x} \Big|Q(t, x_\alpha+) - Q(t, x_\alpha-)\Big| \cdot |\dot{x}_\alpha| \right) dx \\ &= \mathcal{O}(1) \cdot \|h\|_{\mathbf{L}^\infty} \cdot \int_{-R}^{0} \left(\sum_{i_\alpha=1,\ x_\alpha(t)=x} |\sigma_\alpha| \cdot |\dot{x}_\alpha| \right) dx \\ &= \mathcal{O}(1) \cdot R\, \|h\|_{\mathbf{L}^\infty}\,. \end{aligned} \tag{4.14}$$

Notice that in the above estimate we fix a point $x \in [-R, 0]$ and estimate the total strength of 1-fronts that cross the point x from right to left. This quantity is uniformly bounded, since it satisfies the same bounds as the total variation of h.

Observing that the shift in a p-front is proportional to the amount of h crossing the front from right to left, we have the bound for the second term J_2

$$\begin{aligned} J_2 &\leq \sup_t \left(\sum_{i_\alpha=2} \Big|Q(t, x_\alpha+) - Q(t, x_\alpha-)\Big| \right) \cdot \int_{t'}^{t''} \sum_{i_\alpha=2} |\dot{x}_\alpha(t)|\ dt \\ &= \mathcal{O}(1) \cdot \sup_t \Big(\text{Tot.Var.}\{Q(t, \cdot)\} \Big) \cdot \left(\int_{-R}^{0} h(t', x)\, dx + \int_{t'}^{t''} F(t)\, dt \right) \\ &= \mathcal{O}(1) \cdot \Big(R\|h\|_{\mathbf{L}^\infty} + \delta \Big)\,. \end{aligned} \tag{4.15}$$

Finally, to estimate the third term J_3, we note that we have

$$|Q(t_\ell+, x) - Q(t_\ell-, x)| = \mathcal{O}(1)\, |h(t_\ell+, x) - h(t_\ell-, x)| = \mathcal{O}(1)\Delta t |q(t_\ell-, x)| \cdot h(t_\ell-, x)\,.$$

By (2.5) we have

$$\begin{aligned} J_3 &= \mathcal{O}(1) \sum_{t_\ell \in [t',t'']} \int_{-R}^{0} \Delta t |q| h \, dx \\ &= \mathcal{O}(1) \cdot \int_{-R}^{0} h(t',x)\, dx \; + \; \mathcal{O}(1) \cdot \int_{t'}^{t''} F(t)\, dt \\ &= \mathcal{O}(1) \cdot \Big(R\, \|h\|_{\mathbf{L}^\infty} + \delta \Big). \end{aligned} \tag{4.16}$$

Together, the estimates (4.14)–(4.16) and (4.13) yield (4.12).

4.2 Estimates for the flux function

Next, we denote by φ the flux function of the equation for h,

$$\varphi = (q+1)h = ph\,.$$

For a given $\xi \leq 0$, consider the flux through the interval with endpoints (t', ξ) and $(t'',\, \xi)$, namely

$$\Phi(\xi) \doteq \int_{t'}^{t''} \varphi(t,\xi)\, dt = \int_{t'}^{t''} (q(t,\xi)+1)\, h(t,\xi)\, dt\,. \tag{4.17}$$

Observe that the previous quantity is well defined, non negative, continuous and bounded uniformly w.r.t. ξ. Before we give an estimate on Φ, we state a technical lemma, which is a variation of the Gronwall lemma.

Lemma 2 *Let $\xi \mapsto \Phi(\xi) \geq 0$ be a Lipschitz continuous function defined for $\xi \geq 0$. If*

$$\Phi(\xi_2) \leq \Phi(\xi_1) + \int_{\xi_1}^{\xi_2} [\beta(x)\, \Phi(x) + \epsilon]\; dx \qquad \text{for all} \;\; \xi_2 > \xi_1 > 0, \tag{4.18}$$

for some bounded, measurable function β and some constant $\epsilon > 0$, then

$$\Phi(\xi) \leq \left(\exp \int_0^\xi \beta(x)\, dx \right) \Phi(0) + M_1 \epsilon\,, \tag{4.19}$$

where $M_1 \geq 0$ is given as

$$M_1 = \frac{\exp(\xi \|\beta\|_{\mathbf{L}^\infty}) - 1}{\|\beta\|_{\mathbf{L}^\infty}}\,.$$

Similarly, if

$$\Phi(\xi_2) \geq \Phi(\xi_1) + \int_{\xi_1}^{\xi_2} [\alpha(x)\, \Phi(x) - \epsilon]\; dx \qquad \text{for all} \;\; \xi_2 > \xi_1 > 0, \tag{4.20}$$

for some bounded, measurable function α, then

$$\Phi(\xi) \geq \left(\exp \int_0^\xi \alpha(x)\, dx \right) \Phi(0) - M_2 \epsilon\,, \tag{4.21}$$

where $M_2 \geq 0$ is given as

$$M_2 = \frac{\exp(\xi \|\alpha\|_{\mathbf{L}^\infty}) - 1}{\|\alpha\|_{\mathbf{L}^\infty}}\,.$$

The proof of Lemma 2 is deferred to the Appendix.

Now we will derive the estimate on Φ. We integrate the first equation in (4.6) on the domain

$$\Omega_\xi = \{(t,x)\,; \quad x \in [\xi_1, \xi_2]\,, \quad t' < t < t''\}\,,$$

where $\xi_1 < \xi_2 \leq 0$, and we obtain

$$\Phi(\xi_2) \;=\; \Phi(\xi_1) \;-\; \int_{\Omega_\xi} \frac{q}{q+1}\,\varphi\; dt\, dx \;+\; \int_{\xi_1}^{\xi_2} \big(h(t'',x) - h(t',x)\big)\; dx\,. \tag{4.22}$$

The last term in (4.22) can be simply estimated as

$$\left| \int_{\xi_1}^{\xi_2} (h(t'',x) - h(t',x))\, dx \right| \leq \int_{\xi_1}^{\xi_2} 2\|h\|_{\mathbf{L}^\infty}\, dx\,. \tag{4.23}$$

In order to give an estimate on the second term in (4.22), let $\bar q : [\xi_1, \xi_2] \to \mathbb{R}$ be defined by

$$\bar q(x) = q(t', x)\,.$$

Since $\bar q$ is constant w.r.t. t, we have

$$\begin{aligned}
\int_{\Omega_\xi} \frac{q}{q+1}\,\varphi\; dt\, dx \;&=\; \int_{\Omega_\xi} \left(\frac{\bar q}{\bar q+1} + \left[\frac{q}{q+1} - \frac{\bar q}{\bar q+1} \right] \right) \varphi\; dt\, dx \\
&=\; \int_{\xi_1}^{\xi_2} \frac{\bar q(x)}{\bar q(x)+1} \cdot \Phi(x)\, dx \;+\; \int_{\Omega_\xi} \left[\frac{q}{q+1} - \frac{\bar q}{\bar q+1} \right] \varphi\; dt\, dx\,.
\end{aligned} \tag{4.24}$$

Recalling the definition (4.8) of $g(x)$, then by using the lower bound on $p = q+1$, the last term (4.24) can be estimated as

$$\left| \int_{\Omega_\xi} \left[\frac{q}{q+1} - \frac{\bar q}{\bar q+1} \right] \varphi\; dt\, dx \right| \leq C \int_{\xi_1}^{\xi_2} g(x)\Phi(x)\, dx\,. \tag{4.25}$$

Therefore, by the estimates (4.23) and (4.25), from (4.22) we get the following estimates on Φ:

$$\begin{aligned}
\Phi(\xi_2) \;&\leq\; \Phi(\xi_1) \;+\; \int_{\xi_1}^{\xi_2} \left(\left[-\frac{\bar q(x)}{\bar q(x)+1} + Cg(x) \right] \cdot \Phi(x) + 2\|h\|_{\mathbf{L}^\infty} \right) dx \\
\Phi(\xi_2) \;&\geq\; \Phi(\xi_1) \;+\; \int_{\xi_1}^{\xi_2} \left(\left[-\frac{\bar q(x)}{\bar q(x)+1} - Cg(x) \right] \cdot \Phi(x) - 2\|h\|_{\mathbf{L}^\infty} \right) dx\,.
\end{aligned}$$

In particular, observe that Φ is Lipschitz continuous. Applying our Lemma 2, with

$$\alpha(x) = -\frac{\bar q(x)}{\bar q(x)+1} - Cg(x), \qquad \beta(x) = -\frac{\bar q(x)}{\bar q(x)+1} + Cg(x), \qquad \epsilon = 2\|h\|_{\mathbf{L}^\infty}\,,$$

and reversing the roles of ξ and 0 in the integral, we get for any $\xi < 0$

$$\begin{aligned}
\Phi(0) \;&\geq\; \Phi(\xi) \cdot \exp \int_\xi^0 \alpha(x)\, dx - M_2 \epsilon\,, \\
\Phi(0) \;&\leq\; \Phi(\xi) \cdot \exp \int_\xi^0 \beta(x)\, dx + M_1 \epsilon\,,
\end{aligned}$$

and hence

$$\Phi(\xi) \quad \leq \quad (\Phi(0) + 2M_2\|h\|_{\mathbf{L}^\infty}) \cdot \exp \int_\xi^0 \left[\frac{\bar{q}(x)}{\bar{q}(x)+1} + Cg(x)\right] dx \tag{4.26}$$

$$\Phi(\xi) \quad \geq \quad (\Phi(0) - 2M_1\|h\|_{\mathbf{L}^\infty}) \cdot \exp \int_\xi^0 \left[\frac{\bar{q}(x)}{\bar{q}(x)+1} - Cg(x)\right] dx\,. \tag{4.27}$$

Introduce the map

$$k(t', x) \doteq \exp \int_x^0 \frac{q(t',\zeta)}{q(t',\zeta)+1}\, d\zeta\,,$$

and recall from (4.7) that $\Phi(0) = \delta$. If we assume that

$$\|h\|_{\mathbf{L}^\infty} = \mathcal{O}(1) \cdot \delta^2, \tag{4.28}$$

and use the estimate (4.9), then (4.26) becomes

$$\begin{aligned}\Phi(\xi) \quad &\leq \quad \delta \cdot k(t',\xi) \cdot \big(1 + 2M_2\mathcal{O}(1)\delta\big) \cdot \exp(CC_g(\|h\|_{\mathbf{L}^\infty} + \delta)) \\ &\leq \quad \delta \cdot k(t',\xi) \cdot \big(1 + \mathcal{O}(1)\delta\big) \\ &= \quad \delta k(t',\xi) \; + \; \mathcal{O}(1)\delta^2\,.\end{aligned} \tag{4.29}$$

Following a similar argument, (4.27) leads to

$$\Phi(\xi) \quad \geq \quad \delta k(t',\xi) \; - \; \mathcal{O}(1)\delta^2\,. \tag{4.30}$$

Putting together (4.29) and (4.30), we obtain the desired estimate

$$\big|\Phi(\xi) - \delta k(t',\xi)\big| \quad \leq \quad \mathcal{O}(1)\delta^2\,. \tag{4.31}$$

4.3 Convergence of (4.1) to a weak solution of (4.2)

Finally, we study the rescaled limit and show it provides a weak solution to the conservation law (4.2). For each ν sufficiently large, consider the incoming flux F_ν through the boundary, and recall the function $t_\nu(\mu)$ defined in (1.11), i.e.

$$t_\nu(\mu) \doteq \min\left\{t \geq 0\,; \quad \int_0^t F_\nu(s)\, ds = \mu\right\} \qquad\qquad \mu \in [0, M']\,,$$

and consider the rescaled functions

$$\tilde{q}_\nu(\mu, x) \doteq q_\nu(t_\nu(\mu), x)\,,$$

obtained by using μ as new time variable. These are well defined for $x \leq 0$, $\mu \in [0, M']$ and ν sufficiently large. Moreover, for any given $R > 0$ we have from (4.10) that

$$\|\tilde{q}_\nu(\mu_2,\cdot) - \tilde{q}(\mu_1,\cdot)\|_{\mathbf{L}^1((-R,0))} \quad \leq \quad C_g \cdot |\mu_2 - \mu_1| \; + \; \delta_\nu$$

with $\delta_\nu = C_g\|h_\nu\|_{\mathbf{L}^\infty} \to 0$. Hence the uniform bounds on the total variations of the functions q_ν imply that there exists a subsequence, still called $\tilde{q}_\nu$, converging to a BV function $\tilde{q} = \tilde{q}(\mu, x)$ in $\mathbf{L}^1([0, M'] \times [-R, 0])$, for any $R > 0$.

We claim that $\tilde q$ provides a weak solution to the conservation law (4.2). In detail, we claim that for any fixed $0 \leq \mu_1 < \mu_2 \leq M'$ and any test function $\psi \in \mathcal{C}^1_c(\mathbb{R}_-)$, we have

$$\int_{-R}^0 \psi(x)\,[\tilde q(\mu_2,x) - \tilde q(\mu_1,x)]\,dx = \int_{\mu_1}^{\mu_2}\int_{-R}^0 \psi_x(x)\left[\frac{\tilde q(\mu,x)}{\tilde q(\mu,x)+1}\cdot \tilde k(\mu,x)\right] dx\,d\mu\,, \tag{4.32}$$

where

$$\tilde k(\mu,x) \doteq \exp\int_x^0 \frac{\tilde q(\mu,\zeta)}{\tilde q(\mu,\zeta)+1}\,d\zeta\,. \tag{4.33}$$

Indeed, for every ν the second equation in (4.1) yields

$$\begin{aligned}&\int_{-R}^0 \psi(x)\,\Big[q_\nu(t_\nu(\mu_2),x) - q_\nu(t_\nu(\mu_1),x)\Big]\,dx \\ &\qquad = \int_{t_\nu(\mu_1)}^{t_\nu(\mu_2)}\int_{-R}^0 \psi_x(x)\frac{q_\nu(t,x)}{q_\nu(t,x)+1}\cdot\varphi_\nu(t,x)\,dx\,dt\,,\end{aligned} \tag{4.34}$$

where

$$\varphi_\nu(t,x) = (q_\nu(t,x)+1)\,h_\nu(t,x)$$

is the flux function.

Since the sequence $q_\nu(t_\nu(\mu),x)$ converges to $\tilde q(\mu,x)$, we have

$$\int_{-R}^0 \Big|q_\nu(t_\nu(\mu),\,x) - \tilde q(\mu,x)\Big|\,dx \to 0\,. \tag{4.35}$$

Hence the left hand side of (4.34) converges to the corresponding left hand side of (4.32).

We now define

$$\tilde k_\nu(\mu,x) \doteq \exp\int_x^0 \frac{q_\nu(t_\nu(\mu),\zeta)}{q_\nu(t_\nu(\mu),\zeta)+1}\,d\zeta = \exp\int_x^0 \frac{\tilde q_\nu(\mu,\zeta)}{\tilde q_\nu(\mu,\zeta)+1}\,d\zeta\,. \tag{4.36}$$

Observe that all functions $\tilde k_\nu$ are uniformly Lipschitz continuous. They converge to the function $\tilde k(\mu,x)$ defined in (4.33) uniformly on the rectangular domain $[0,M']\times[-R,0]$.

Next, we claim that for every x,μ',μ'', as $\nu\to\infty$ one has

$$\int_{t_\nu(\mu')}^{t_\nu(\mu'')} \varphi_\nu(t,x)\,dt = \int_{t_\nu(\mu')}^{t_\nu(\mu'')} (q_\nu(t,x)+1)\,h_\nu(t,x)\,dt \;\to\; \int_{\mu'}^{\mu''}\tilde k(\mu,x)\,d\mu\,. \tag{4.37}$$

Indeed, let $\delta > 0$ be given. Let $m \geq 1$ be the smallest integer such that $(\mu''-\mu')/m \leq \delta$, and let $\delta_0 \doteq (\mu''-\mu')/m$ so $\delta_0 \leq \delta$. Define the intermediate values

$$\mu_i = \mu' + i\cdot\frac{\mu''-\mu'}{m} \qquad\qquad i = 0,1,\ldots,m\,.$$

Recall now the analysis on the convergence of the flux, i.e., estimate (4.31) in Subsection 4.2. Let $\|h_\nu\|_{\mathbf{L}^\infty} \leq \mathcal{O}(1)\cdot\delta_0^2$; then, for every $x \leq 0$ and every i, estimate (4.31) gives

$$\left(\int_{t_\nu(\mu_{i-1})}^{t_\nu(\mu_i)} \varphi_\nu(t,x)\,dt\right) \;-\; \delta_0\,\tilde k_\nu(\mu_{i-1},x) = \mathcal{O}(1)\cdot\delta_0^2\,.$$

Furthermore, by the uniform Lipschitz continuity of $\tilde{k}_\nu$, we also have

$$\left|\delta_0 \tilde{k}_\nu(\mu_{i-1}, x) - \int_{\mu_{i-1}}^{\mu_i} \tilde{k}_\nu(\mu, x)\, d\mu\right| \le \int_{\mu_{i-1}}^{\mu_i} \left|\tilde{k}_\nu(\mu, x) - \tilde{k}_\nu(\mu_{i-1}, x)\right| \, d\mu \le \mathcal{O}(1)\delta_0^2 \,.$$

Combine the previous two inequalities, and use triangle inequality, we get

$$\begin{aligned}
&\left|\int_{t_\nu(\mu_{i-1})}^{t_\nu(\mu_i)} \varphi_\nu(t, x)\, dt - \int_{\mu_{i-1}}^{\mu_i} \tilde{k}_\nu(\mu, x)\, d\mu\right| \\
&\le \quad \left|\int_{t_\nu(\mu_{i-1})}^{t_\nu(\mu_i)} \varphi_\nu(t, x)\, dt - \delta_0 \tilde{k}_\nu(\mu_{i-1}, x)\right| + \left|\delta_0 \tilde{k}_\nu(\mu_{i-1}, x) - \int_{\mu_{i-1}}^{\mu_i} \tilde{k}_\nu(\mu, x)\, d\mu\right| \\
&\le \quad C\,\delta_0^2 \,,
\end{aligned}$$

for some constant C and all ν large enough. Summing over $i = 1, \ldots, m$, since $\delta > 0$ is arbitrary and $m = O(\delta^{-1})$, from the above inequality and the uniform convergence $\tilde{k}_\nu \to \tilde{k}$ we deduce (4.37). In turn, (4.37) yields the weak convergence

$$(\tilde{q}_\nu(\mu, x) + 1)\, \tilde{h}_\nu(\mu, x) \;\rightharpoonup\; \tilde{k}(\mu, x)$$

on the domain $[0, M'] \times [-R, 0]$. Together with the strong convergence

$$\frac{\tilde{q}_\nu(\mu, x)}{\tilde{q}_\nu(\mu, x) + 1} \quad \to \quad \frac{\tilde{q}(\mu, x)}{\tilde{q}(\mu, x) + 1} \,,$$

this implies that the right hand side of (4.34) converges to the right hand side of (4.32). Hence $\tilde{q}$ provides a weak solution of the conservation law (4.2).

4.4 Entropy admissibility

The fact that $\tilde{p}$ is also entropy-admissible follows from the fact that each shock in $\tilde{p}$ must be a limit of corresponding shocks in the rescaled solutions $\tilde{p}_\nu$. We claim that these shocks in $\tilde{p}_\nu$ are entropy-admissible, hence the same is true for the limit $\tilde{p}$.

To check the entropy admissibility for shocks in $\tilde{p}_\nu$, it suffices to check the directions of the jumps of the second family (p-wave) for the system (4.2). From the analysis of the state curves, we already know that the p-shocks jump upward. For the scalar integro-differential equation in (4.2), we see that the flux $f(q) = q/(q + 1)$ is strictly concave down for $q > -1$, since

$$f'(q) = \frac{1}{(q + 1)^2} \,, \qquad f''(q) = \frac{-2}{(q + 1)^3} < 0 \,.$$

Therefore admissible shocks must jump upward, which agrees with the direction of jumps in p-wave for (4.1). This completes the proof of Theorem 2. □

5 Concluding remarks

In this paper we show that the slow erosion limit of the granular flow model converges to a scalar integro-differential equation. However, we leave open the wellposedness of the scalar integro-differential equation (1.14). Due to the fact that the flux is a global function which involves an integral of the unknown variable, issues such as existence and uniqueness should need separate attention.

In essence, we are considering a scalar equation

$$p_\mu + (k(\mu, x) f(p))_x = 0 \tag{5.38}$$

in one space dimension. Here $k(\mu, x)$ is the global term

$$k(\mu, x) = \exp \int_x^0 \frac{p(\mu, y) - 1}{p(\mu, y)} dy \,,$$

and $f(p) = (p-1)/p$ is the local part of the flux. Due to the discontinuities in p, the global term $k(t, x)$ is only a Lipschitz function, thus recent results such as [9] can not be applied directly. Rewrite (5.38) into

$$p_\mu + k(\mu, x) f(p)_x = -k(\mu, x)_x f(p) \,. \tag{5.39}$$

Formally the increase of total variation of p is caused by the source term in (5.39). To obtain the BV bound for p, one needs a bound on the total variation of $k(\mu, x)_x$ for any given μ. From the particular formula of k, we have for some positive constant M

$$\text{Tot.Var.}\{k_x\} \leq M \text{Tot.Var.}\{p\} \,,$$

provided that p is uniformly bounded in $\mathbf{L}^\infty$ and $\mathbf{L}^1$, and $p > 0$ is bounded away from 0. Therefore, the evolution of the total variation of p in time μ follows

$$\frac{d}{d\mu} \text{Tot.Var}\{p\} \leq M \text{Tot.Var}\{p\} \,.$$

This means Tot.Var$\{p\}$ would increase exponentially in μ. Thus, for finite μ (which is our case, where μ is the total mass of avalanche, and is bounded by assumption), one has that Tot.Var$\{p\}$ is bounded. This provides a formal argument for BV bound on p. Therefore, wellposedness of BV solutions is expected. Details will be worked out in a forthcoming paper.

A Proof of Lemma 2

In this appendix we prove Lemma 2, stated in Subsection 4.2.

Let's first prove inequality (4.19). Let L be the Lipschitz constant for Φ. For a given integer N, define $\Delta\tau = \xi/N$, $\tau_i = i \cdot \Delta\tau$, and set $K_1 \doteq \|\beta\|_{\mathbf{L}^\infty} L \Delta\tau + \epsilon$. We now check by induction on $i = 0, 1, \ldots, N$, that

$$\Phi(\tau_i) \;\leq\; \left(\exp \int_0^{\tau_i} \beta(x)\, dx\right) \Phi(0) \;+\; K_1 \frac{\exp(\tau_i \|\beta\|_{\mathbf{L}^\infty}) - 1}{\|\beta\|_{\mathbf{L}^\infty}} \,. \tag{A.1}$$

Indeed, let (A.1) hold for $i = 1, \ldots, k-1$. Then (4.18) and the Lipschitz condition yield

$$\begin{aligned} \Phi(\tau_k) \;&\leq\; \Phi(\tau_{k-1}) + \int_{\tau_{k-1}}^{\tau_k} \beta(x) \Phi(\tau_{k-1})\, dx \;+\; \int_{\tau_{k-1}}^{\tau_k} \{\beta(x) \left[\Phi(x) - \Phi(\tau_{k-1})\right] + \epsilon\}\, dx \\ &\leq\; \Phi(\tau_{k-1}) \cdot \left(1 + \int_{\tau_{k-1}}^{\tau_k} \beta(x)\, dx\right) \;+\; \int_{\tau_{k-1}}^{\tau_k} \{|\beta(x)| \cdot L\Delta\tau + \epsilon\}\, dx \,. \end{aligned} \tag{A.2}$$

Using the relation $1 + x \le e^x$ (for all x) and the inductive assumption (A.1) with $i = k - 1$, and recalling that $\Phi \ge 0$, the first term in (A.2) is bounded by

$$\begin{aligned}\Phi(\tau_{k-1}) \cdot \left(1 + \int_{\tau_{k-1}}^{\tau_k} \beta(x)\,dx\right) &\le \Phi(\tau_{k-1}) \cdot \exp\left(\int_{\tau_{k-1}}^{\tau_k} \beta(x)\,dx\right) \\ &\le \left(\exp \int_0^{\tau_k} \beta(x)\,dx\right) \Phi(0) \; + \; K_1 \, \frac{\exp(\tau_k \|\beta\|_{\mathbf{L}^\infty}) - \exp(\Delta\tau \|\beta\|_{\mathbf{L}^\infty})}{\|\beta\|_{\mathbf{L}^\infty}} \, . \end{aligned} \tag{A.3}$$

The last term in (A.2) can be simply estimated by

$$\int_{\tau_{k-1}}^{\tau_k} \{|\beta(s)| \cdot L\Delta\tau + \epsilon\}\, ds \;\le\; K_1 \Delta\tau \;\le\; K_1 \frac{\exp(\Delta\tau \|\beta\|_{\mathbf{L}^\infty}) - 1}{\|\beta\|_{\mathbf{L}^\infty}} \, . \tag{A.4}$$

Putting together (A.3) and (A.4) into (A.2), we obtain (A.1) for $i = k$, completing the inductive step.

Now let $i = N$ in (A.1), one gets

$$\Phi(\xi) \;\le\; \exp\left(\int_0^\xi \beta(x)\,dx\right) \Phi(0) \; + \; \left\{\frac{\|\beta\|_{\mathbf{L}^\infty} L\, \xi}{N} + \epsilon\right\} \cdot \frac{\exp(\xi \|\beta\|_{\mathbf{L}^\infty}) - 1}{\|\beta\|_{\mathbf{L}^\infty}} \, .$$

Letting $N \to \infty$ we thus obtain (4.19).

The proof for (4.21) follows in a similar way, with some modifications. Set $K_2 \doteq \|\alpha\|_{\mathbf{L}^\infty} L\Delta\tau + \epsilon$. By induction on $i = 0, 1, \ldots, N$, one needs to check that

$$\Phi(\tau_i) \;\ge\; \left(\exp \int_0^{\tau_i} \alpha(x)\,dx\right) \Phi(0) \; - \; \exp(\Delta\tau \|\alpha\|_{\mathbf{L}^\infty}) \cdot K_2 \, \frac{\exp(\tau_i \|\alpha\|_{\mathbf{L}^\infty}) - 1}{\|\alpha\|_{\mathbf{L}^\infty}} \, . \tag{A.5}$$

Let (A.5) hold for $i = 1, \ldots, k - 1$. Here, (4.20) and the Lipschitz condition yield

$$\Phi(\tau_k) \;\;\ge\;\; \Phi(\tau_{k-1}) + \int_{\tau_{k-1}}^{\tau_k} \alpha(x)\Phi(\tau_k)\,dx \;-\; \int_{\tau_{k-1}}^{\tau_k} \{|\alpha(x)| \cdot L\Delta\tau + \epsilon\}\, dx \, ,$$

which gives

$$\left(1 - \int_{\tau_{k-1}}^{\tau_k} \alpha(x)\,dx\right) \Phi(\tau_k) \;\ge\; \Phi(\tau_{k-1}) \;-\; K_2 \Delta\tau \, .$$

Using the relation $1 - x \le e^{-x}$, we obtain

$$\exp\left(-\int_{\tau_{k-1}}^{\tau_k} \alpha(x)\,dx\right) \cdot \Phi(\tau_k) \;\ge\; \Phi(\tau_{k-1}) \;-\; K_2 \Delta\tau \, . \tag{A.6}$$

By the inductive assumption on $i = k - 1$, the r.h.s. in (A.6) satisfies

$$\begin{aligned} &\Phi(\tau_{k-1}) \;-\; K_2\Delta\tau \\ &\quad\ge\; \left(\exp \int_0^{\tau_{k-1}} \alpha(x)\,dx\right) \Phi(0) - K_2 \, \frac{\exp(\tau_k \|\alpha\|_{\mathbf{L}^\infty}) - \exp(\Delta\tau \|\alpha\|_{\mathbf{L}^\infty})}{\|\alpha\|_{\mathbf{L}^\infty}} \;-\; K_2\Delta\tau \\ &\quad\ge\; \left(\exp \int_0^{\tau_{k-1}} \alpha(x)\,dx\right) \Phi(0) - K_2 \, \frac{\exp(\tau_k \|\alpha\|_{\mathbf{L}^\infty}) - 1}{\|\alpha\|_{\mathbf{L}^\infty}} \, . \end{aligned}$$

Using this last inequality in (A.6), we obtain

$$\Phi(\tau_k) \;\geq\; \left(\exp\int_0^{\tau_k}\alpha(x)\,dx\right)\,\Phi(0) \;-\; \exp\left(\int_{\tau_{k-1}}^{\tau_k}\alpha(x)\,dx\right)\cdot K_2\,\frac{\exp(\tau_k\|\alpha\|_{\mathbf{L}^\infty})-1}{\|\alpha\|_{\mathbf{L}^\infty}}$$

that leads to (A.1) also for $i=k$, completing the inductive step.

Finally, setting $i=N$ in (A.5), one gets

$$\Phi(\xi) \;\geq\; \exp\left(\int_0^{\xi}\alpha(x)\,dx\right)\,\Phi(0) \;-\; \exp\left(\frac{\xi\|\alpha\|_{\mathbf{L}^\infty}}{N}\right)\cdot\left\{\frac{\|\alpha\|_{\mathbf{L}^\infty}L\,\xi}{N}+\epsilon\right\}\cdot\frac{\exp(\xi\|\alpha\|_{\mathbf{L}^\infty})-1}{\|\alpha\|_{\mathbf{L}^\infty}}\,,$$

and then letting $N\to\infty$ we complete the proof for (4.21).

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