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LARGE TIME DECAY OF SOLUTIONS TO ISENTROPIC GAS DYNAMICS WITH SPHERICAL SYMMETRY

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Abstract. We consider the large time behavior of solutions to isentropic gas dynamics with spherical symmetry. In the present paper, we show the decay of the pressure in particular. To do this, we investigate approximate solutions constructed by a difference scheme.

Keywords: Isentropic gas dynamics; spherical symmetry; decay; the Lax-Friedrichs scheme.

1. Introduction

This paper is concerned with isentropic gas dynamics with spherical symmetry in an exterior domain:

$$\begin{cases} \rho_t + m_r = -\dfrac{2}{r}m, \quad r = |\vec{x}| \geq 1, \quad \vec{x} \in \mathbf{R}^3, \\ m_t + \left(\dfrac{m^2}{\rho} + p(\rho)\right)_r = -\dfrac{2}{r}\dfrac{m^2}{\rho}, \end{cases} \tag{1.1}$$

where ρ, m and p are density, momentum and the pressure of gas, respectively. For the non-vacuum state $\rho > 0$, $v := m/\rho$ is velocity. For the polytropic gas, $p(\rho) := \rho^\gamma/\gamma$, where γ is a constant satisfying $1 < \gamma \leq 5/3$.

Then we consider the initial-boundary value problem (1.1) with initial and boundary data:

$$(\rho, m)|_{t=0} = (\rho_0(x), m_0(x)), \quad m|_{r=1} = 0. \tag{1.2}$$

By using a vector $u := {}^t(\rho, m) = {}^t(\rho, \rho v)$, (1.1)–(1.2) can be written as the simple form:

$$\begin{cases} u_t + f(u)_r = g(r,u), \\ u|_{t=0} = u_0(r), \quad m|_{r=1} = 0. \end{cases} \tag{1.3}$$

Related results. First, we survey the one-dimensional Cauchy problem for isentropic gas dynamics. DiPerna [8] proved the global existence by the vanishing viscosity method and a compensated compactness argument. DiPerna applied the method of compensated compactness to systems first, for the special case where $\gamma = 1+2/n$ and n is an odd integer ≥ 5. Subsequently, Ding, Chen and Luo [6] and Chen [2] extended the analysis to any γ in $(1, 5/3]$ by the Lax-Friedrichs scheme. On the other hand, for large time behavior, Glimm and Lax [9] showed that if initial data are constant outside a finite interval and have locally bounded total variation and small oscillation, then the total variation of the Glimm solution decays to zero at the rate $t^{-1/2}$. Perthame [12] discussed a different type of decay estimate by using a kinetic formulation. In Tsuge [15], this decay estimate was proved for a weak solution as the limit of the approximate solutions in Chen [2].

Next, we refer to the spherically symmetric case. Makino and Takeno [11] proved the local existence of weak solutions by the fractional step Lax-Friedrichs scheme and the method of compensated compactness. In Chen [4], under the following condition of initial data

$$0 \leq \{\rho_0(r)\}^\theta/\theta \leq v_0(r) < +\infty, \tag{1.4}$$

the global existence was proved. In Tsuge [14] and [16], under a much weaker condition than (1.4), the existence theorem was obtained by the modified Godunov scheme. On the other hand, the large time behavior of spherically symmetric solutions has not received much attention compared with that of solutions to the one-dimensional Cauchy problem.

Now, in this paper, we assume the following:

(H) If initial data (ρ_0, v_0) satisfy

$$0 \leq \rho_0(r) \leq B, \quad |v_0(r)| \leq B, \tag{1.5}$$

for any $T > 0$, there exists a constant $B(T)$ such that approximate solutions (ρ^Δ, v^Δ) of (1.1)–(1.2) constructed by the fractional step Lax-Friedrichs scheme in Section 3 satisfy

$$0 \leq \rho^\Delta(r,t) \leq B(T), \quad |v^\Delta(r,t)| \leq B(T), \quad 0 \leq t \leq T. \tag{1.6}$$

Remark 1.1.

(1) If we assume (H), the approximate solutions (ρ^Δ, v^Δ) defined in Chen [4] convergence to a global solution to (1.1)–(1.2) by the methods of compensated compactness. However, (H) has not yet been proved. On the other hands, in view of Chen and Glimm [5], our assumption (H) seems to be natural.

(2) If we assume (1.4) instead of (1.5), from the argument in Chen [4], we can obtain the bound of fractional step Lax-Friedrichs approximate solutions. Therefore, we do not need the assumption (H) in this case.

Let us define the mechanical energy $\eta(u)$ and its flux $q(u)$ as follows:

$$\eta(u)=\frac{m^2}{\rho}+\frac{p(\rho)}{\theta},\quad q(u)=\left(\frac{m^2}{\rho}+\frac{\gamma p(\rho)}{\theta}\right)\frac{m}{\rho},\quad \left(\theta:=\frac{\gamma-1}{2}\right).$$

Then our main theorem is stated as follows:

Theorem 1.2. *Under the assumption (H), we suppose that, for $B\geq 0$ and $C_0\geq 0$, initial data satisfy* (1.5) *and*

$$\int_1^\infty r^4\rho_0(r)dr\leq C_0,\quad \int_1^\infty r^2\eta(u_0(r))dr\leq C_0. \tag{1.7}$$

Then, for a.e. $t>1$, an entropy weak solution to (1.1)–(1.2) *as the limit of the fractional step Lax-Friedrichs approximate solutions satisfies*

$$\int_1^\infty r^2p(\rho(r,t))dr\leq\frac{\theta}{t^{3(\gamma-1)}}\left(\int_1^\infty(r^4+2r)\rho_0(r)dr+\frac{5-3\gamma}{2}\int_1^\infty r^2\eta(u_0(r))dr\right). \tag{1.8}$$

Outline of this paper. In this paper, we show the decay of entropy weak solutions obtained by the fractional step Lax-Friedrichs scheme. Entropy weak solutions satisfy the energy inequality

$$\eta(u)_t+q(u)_r\leq -2q(u)/r \tag{1.9}$$

in the distributional sense.

Then fundamental calculations $(r^4+2r)t^\beta\times(1.1)_1-2(r^3-1)t^{\beta+1}\times(1.1)_2$ $+r^2t^{\beta+2}\times(1.9)$ and $(r^4+2r)\times(1.1)_1-2(r^3-1)t\times(1.1)_2+r^2t^2\times(1.9)$ yield

$$U_t+V_r\leq W,\quad X_t+Y_r\leq Z \tag{1.10}$$

respectively, where $\beta:=3\gamma-5$ and functionals U,V,W,X,Y and Z are defined as follows:

$$\begin{aligned}
U(r,t,u)=&(r^4+2r)t^\beta\rho-2(r^3-1)t^{\beta+1}m+r^2t^{\beta+2}\eta(u),\\
V(r,t,u)=&(r^4+2r)t^\beta m-2(r^3-1)t^{\beta+1}\left(\frac{m^2}{\rho}+p(\rho)\right)+r^2t^{\beta+2}q(u),\\
W(r,t,u)=&\beta(r^4+2r)t^{\beta-1}\rho-2\beta(r^3-1)t^\beta m-\left(2r^2+4/r\right)t^{\beta+1}\frac{m^2}{\rho}\\
&-6r^2t^{\beta+1}p(\rho)+(\beta+2)r^2t^{\beta+1}\eta(u),\\
X(r,t,u)=&t^{-\beta}U,\ Y(r,t,u)=t^{-\beta}V,\ Z(r,t,u)=(5-3\gamma)r^2tp(\rho)/\theta-\frac{4}{r}t\frac{m^2}{\rho}.
\end{aligned}$$

Since $W(r,t,u)\leq 0$, integrating $(1.10)_1$ over the region $[1,\infty)\times[1,t]$, we have

$$\int_1^\infty U(r,t)dr\leq\int_1^\infty U(r,1)dr. \tag{1.11}$$

On the other hand, we notice that $U(r,1) = X(r,1)$ and $p(\rho)/\theta \leq \eta(u)$. Then, integrating $(1.10)_2$ over the region $[1,\infty) \times [0,1]$, from the energy inequality, we have

$$\int_1^\infty U(r,1)dr \leq \int_1^\infty (r^4 + 2r)\rho_0(r)dr + \frac{(5-3\gamma)}{2}\int_1^\infty r^2\eta(u_0(r))dr. \quad (1.12)$$

Since $r^2 t^{3(\gamma-1)} p(\rho)/\theta \leq U(r,t,u)$, combining (1.11) and (1.12), we have (1.8) in our main theorem.

However the above argument is formal, because we implicitly use the following facts:

(a) Solutions are smooth.
(b) $r^4\rho(r,t) \to 0$ as $r \to \infty$.

Solutions of (1.1) starting out from smooth initial data eventually develop discontinuities propagations as shock waves. Consequently, we must understand solutions in the sense of distribution. Therefore we cannot use the fact (a). In addition, it is not easy to prove (b), even if we assume $(1.7)_1$.

In order to overcome these difficulties, we investigate not weak solutions directly but approximate solutions constructed by the fractional step Lax-Friedrichs scheme (see [4,11]). Indeed, the analysis of the approximate solutions is much easier than that of weak solutions. Therefore, for the approximate solutions, we first deduce the decay estimates corresponding to the above formal calculus (see Lemmas 4.1 and 4.2). Since the approximate solutions convergence to a weak solution of (1.1) by the method of compensated compactness, these estimates consequently yield (1.8).

In Section 2, we recollect the homogeneous system of isentropic gas dynamics and state the Rankine-Hugoniot condition, the entropy condition and the Riemann problem.

In Section 3, we construct approximate solutions by the fractional step Lax-Friedrichs scheme and derive some recurrence relations.

In Section 4, we study decay estimates of the approximate solutions. By applying the methods of compensated compactness to the approximate solutions, we conclude Theorem 1.2.

2. Preliminary

In this section, we first review some results of the Riemann solutions for the homogeneous system of gas dynamics:

$$u_t + f(u)_r = 0. \quad (2.1)$$

Discontinuous solutions arise for (2.1). The jump discontinuity in a weak solution to (2.1) must satisfy the following Rankine-Hugoniot condition:

$$\lambda(u - u_0) = f(u) - f(u_0), \quad (2.2)$$

where λ is the propagation speed of the discontinuity and $u_0 = (\rho_0, m_0)$ and $u = (\rho, m)$ are the corresponding left and right states respectively. Furthermore a jump discontinuity is called a *shock* if it satisfies the entropy condition:

$$\lambda(\eta(u) - \eta(u_0)) - (q(u) - q(u_0)) \geq 0. \tag{2.3}$$

Next, let us recall the Riemann problem for (2.1) with

$$u|_{t=0} = \begin{cases} u_-, & r < r_0, \\ u_+, & r > r_0, \end{cases} \tag{2.4}$$

and the Riemann initial-boundary value problem for (2.1) with

$$u|_{t=0} = u_+, \quad m|_{r=1} = 0, \quad r \geq 1,\ t \geq 0, \tag{2.5}$$

where $\rho_\pm \geq 0$ and $m_\pm$ are constants satisfying $|m_\pm| \leq C\rho_\pm$. These solutions consist of rarefaction waves and shocks (see, for example, [1] or [13]).

3. Lax-Friedrichs approximate solutions

In this section, we first introduce the fractional step Lax-Friedrichs scheme and construct the Lax-Friedrichs approximate solutions. We next deduce some recurrence relations.

3.1. *The fractional step Lax-Friedrichs scheme*

For any $T > 0$, we construct approximate solutions by the fractional step Lax-Friedrichs scheme in the region $r \geq 1,\ 0 \leq t \leq T$. We denote the approximate solutions by $u^\Delta(r,t) = (\rho^\Delta(r,t), m^\Delta(r,t))$. In addition, we choose the space mesh length Δr, time mesh length Δt satisfying $0 < \Delta t < 1$ and

$$\Lambda := \max_{i=1,2}\left(\sup_{0\leq\rho\leq B(T),\ |v|\leq B(T)} |\lambda_i(u)|\right) \leq \frac{\Delta r}{\Delta t} \leq 2\Lambda, \tag{3.1}$$

where λ_1 and λ_2 are characteristic speeds defined as $\lambda_1 := v - \rho^\theta,\ \lambda_2 := v + \rho^\theta$.

Now, to construct $u^\Delta(r,t)$, we introduce the following notation:

$$(j,n) \in \mathbf{Z}_{\geq 0} \times \mathbf{Z}_{\geq 0},\ J_n := \{j \in \mathbf{Z}_{\geq 0};\ n + j = \text{even}\},\ r_j := j\Delta r + 1,\ t_n := n\Delta t.$$

Step 1. First, we define

$$u^\Delta(r, -0) = u_0(r)\chi_{R_0}(r),$$

where

$$\chi_{R_0}(r) = \begin{cases} 1, & r \leq R_0, \\ 0, & r \geq R_0. \end{cases} \tag{3.2}$$

Step 2. Second, we assume that $u^\Delta(r,t)$ is defined for $t < t_n$. Then we define $u_j^n = (\rho_j^n, m_j^n)$ as follows:

(1) For $j \in J_n$ and $j \neq 0, 2$, we set

$$u_j^n := \frac{1}{2\Delta r}\int_{r_{j-1}}^{r_{j+1}} u^\Delta(r, t_n - 0)dr, \quad r_{j-1} \leq r \leq r_{j+1};$$

(2) If n is even, for $j = 0, 2$, we set

$$u_0^n = u_2^n := \frac{1}{3\Delta r}\int_{r_0}^{r_3} u^\Delta(r, t_n - 0)dr, \quad r_0 \leq r \leq r_3.$$

Step 3. In the strip $t_n \leq t < t_{n+1}$, we define $\overline{u}^\Delta(r, t)$ as follows:

(1) In the region $r_j \leq r < r_{j+2}$, for j with $j \geq 1$ and $j \in J_n$, the solution of the Riemann problem at $r = r_{j+1}$:

$$\begin{cases} u_t + f(u)_r = 0, \quad r_j \leq r < r_{j+2}, \\ u|_{t=t_n} = \begin{cases} u_j^n, & r < r_{j+1}, \\ u_{j+2}^n, & r > r_{j+1}; \end{cases} \end{cases}$$

(2) In the region $1 \leq r < r_1$, the solution of the Riemann initial boundary value problem at $r = 1$:

$$\begin{cases} u_t + f(u)_r = 0, \quad 1 \leq r < r_1, \\ m|_{r=1} = 0, \quad u|_{t=t_n} = \begin{cases} u_1^n, & \text{if } n \text{ is odd}, \\ u_2^n, & \text{if } n \text{ is even}. \end{cases} \end{cases} \tag{3.3}$$

For $j = 1, 2$, the solution to (3.3) is constructed as follows:

(a) If $\rho_j^n > 0$ and $v_j^n \leq 0$, there exists u_b^n with $v_b^n = 0$ from which u_j^n is connected by a 2-shock curve.
(b) If $v_j^n \geq 0$ and $z(u_j^n) \leq 0$, then there exists u_b^n with $v_b^n = 0$ from which u_j^n is connected by a 2-rarefaction curve.
(c) If $v_j^n \geq 0$ and $z(u_j^n) \geq 0$, then there exists u_* with $\rho_* = 0$ from which u_j^n is connected by a 2-rarefaction, and u_* and u_b^n with $\rho_b^n = v_b^n = 0$ are connected by the vacuum.
(d) If $v_j^n \leq 0$ and $\rho_j^n = 0$, then u_b^n with $\rho_b^n = v_b^n = 0$ is connected from u_j^n by the vacuum.
Here $z(u)$ is the Riemann invariant defined as $z(u) = m/\rho - \rho^\theta/\theta$.

(3) If n is even, in the region $r_1 \leq r < r_2$, we set $\overline{u}^\Delta(r, t) = u_0^n = u_2^n$.

Step 4. Finally, in the region where $1 \leq r$, $t_n \leq t < t_{n+1}$, by the fractional step method (see [7]), we define

$$u^\Delta(r, t) = \overline{u}^\Delta(r, t) + g(r, \overline{u}^\Delta(r, t))(t - t_n). \tag{3.4}$$

Remark 3.1. By (1.6), the above approximate solutions satisfy the Courant-Friedrichs-Lewy condition:

$$\Lambda := \max_{i=1,2}\left(\sup_{0\leq t\leq T, 1\leq r} |\lambda_i(u^\Delta(r, t))|\right) \leq \frac{\Delta r}{\Delta t} \leq 2\Lambda. \tag{3.5}$$

3.2. *Recurrence relations for Lax-Friedrichs approximate solutions*

In this subsection, we introduce recurrence relations derived from Lax-Friedrichs approximate solutions. These relations shall be used in the next section.

First we define the following notation:

(1) For $j \in J_n$ and $j \neq 0, 2$, we set

$$\overline{u}_j^n := \frac{1}{2\Delta r}\int_{r_{j-1}}^{r_{j+1}} \overline{u}^{\Delta}(r, t_n - 0)dr, \quad r_{j-1} \leq r \leq r_{j+1};$$

(2) If n is even, for $j = 0, 2$, we set

$$\overline{u}_0^n = \overline{u}_2^n := \frac{1}{3\Delta r}\int_{r_0}^{r_3} \overline{u}^{\Delta}(r, t_n - 0)dr, \quad r_0 \leq r \leq r_3.$$

Then we observe that the approximate solutions satisfy the following Lemma:

Lemma 3.2.

1. *For* $j \in J_{n+1}^* := \{j \in J_n;\ j \geq 3\}$, *we have*

$$\begin{aligned}
\overline{\rho}_j^{n+1} &= \frac{\rho_{j+1}^n + \rho_{j-1}^n}{2} - \frac{\Delta t}{2\Delta r}(m_{j+1}^n - m_{j-1}^n),\\
\overline{m}_j^{n+1} &= \frac{m_{j+1}^n + m_{j-1}^n}{2} - \frac{\Delta t}{2\Delta r}\left\{\frac{(m_{j+1}^n)^2}{\rho_{j+1}^n} + p(\rho_{j+1}^n) - \frac{(m_{j-1}^n)^2}{\rho_{j-1}^n} - p(\rho_{j-1}^n)\right\}\\
\textit{and}\quad \eta(\overline{u}_j^{n+1}) &\leq \frac{\eta(u_{j+1}^n) + \eta(u_{j-1}^n)}{2} - \frac{\Delta t}{2\Delta r}\left\{q(u_{j+1}^n) - q(u_{j-1}^n)\right\}.
\end{aligned}$$

2. *If n is even, we have*

$$\begin{aligned}
\overline{\rho}_1^{n+1} &= \frac{\rho_2^n + \rho_0^n}{2} - \frac{\Delta t}{2\Delta r}m_2^n,\\
\overline{m}_1^{n+1} &= \frac{m_2^n + m_0^n}{2} - \frac{\Delta t}{2\Delta r}\left\{\frac{(m_2^n)^2}{\rho_2^n} + p(\rho_2^n) - p(\rho_b^n)\right\}\\
\textit{and}\quad \eta(\overline{u}_1^{n+1}) &\leq \frac{\eta(u_2^n) + \eta(u_0^n)}{2} - \frac{\Delta t}{2\Delta r}q(u_2^n).
\end{aligned}$$

3. *If n is odd, we have*

$$\begin{aligned}
\overline{\rho}_2^{n+1} = \overline{\rho}_0^{n+1} &= \frac{\rho_3^n + 2\rho_1^n}{3} - \frac{\Delta t}{3\Delta r}m_3^n,\\
\overline{m}_2^{n+1} = \overline{m}_0^{n+1} &= \frac{m_3^n + 2m_1^n}{3} - \frac{\Delta t}{3\Delta r}\left\{\frac{(m_3^n)^2}{\rho_3^n} + p(\rho_3^n) - p(\rho_b^n)\right\}\\
\textit{and}\quad \eta(\overline{u}_2^{n+1}) = \eta(\overline{u}_0^{n+1}) &\leq \frac{\eta(u_3^n) + 2\eta(u_1^n)}{3} - \frac{\Delta t}{3\Delta r}q(u_3^n).
\end{aligned}$$

Here ρ_b^n is stated Step 3 in Subsection 3.1.

Proof. Applying the Green formula to the Riemann solution in each cell, from (2.2) and (2.3), we obtain Lemma 3.2. □

Remark 3.3. From the definition of our approximate solutions, we have the following:

(1) For $j \in J_n$ and $j \neq 0, 2$,

$$u_j^{n+1} = \overline{u}_j^{n+1} - \frac{\Delta t}{2\Delta r}\int_{r_{j-1}}^{r_{j+1}} g(s, \overline{u}^{\Delta}(s, t_{n+1}-0))ds.$$

(2) If n is odd, for $j = 0, 2$,

$$u_2^{n+1} = u_0^{n+1} = \overline{u}_0^{n+1} - \frac{\Delta t}{3\Delta r}\int_{r_0}^{r_3} g(s, \overline{u}^{\Delta}(s, t_{n+1}-0))ds.$$

4. Decay estimates of the fractional step Lax-Friedrichs approximate solutions

In this section, we derive decay estimates of the Lax-Friedrichs approximate solutions defined in the previous section.

First we define

$$\begin{aligned}
F(r,t,u) &= (r^4 + 2r)\rho - 2(r^3 - 1)t\, m + r^2 t^2 \eta(u),\\
G(r,t,u) &= (r^4 + 2r)m - 2(r^3 - 1)t\left(\frac{m^2}{\rho} + p(\rho)\right) + r^2 t^2 q(u),\\
H(r,t,u) &= (5 - 3\gamma) r^2 t\,\frac{p(\rho)}{\theta} - \frac{4}{r} t\,\frac{m^2}{\rho}
\end{aligned}$$

and $I_j = \{r \in \mathbf{R};\ 1 \leq r,\ r_{j-1} \leq r < r_{j+1}\}$ $(j \in \mathbf{Z}_{\geq 0})$. Then we notice that $I_0 = \{r \in \mathbf{R};\ 1 \leq r < r_1\}$. Moreover, for any $T > 1$, we set $N = [T/\Delta t]$, where $[x]$ is the greatest integer not greater than x. Finally, by the Landau symbols such as $O(\Delta r)$, we henceforth denote quantities whose moduli satisfy a uniform bound depending only on T, $B(T)$ in (1.6), C_0 in (1.7) and R_0 in (3.2).

4.1. *Estimate of* u_j^n

Our goal in this subsection is to prove the following lemma:

Lemma 4.1.
Let N_1 be the smallest integer such that $N_1 \Delta t \geq 1$. Then we have

$$\begin{aligned}
&\sum_{j \in J_N} \int_{I_j} (t_N)^{\beta} F(r, t_N, u_j^N) dr\\
&\leq \int_1^{\infty} (r^4 + 2r^2)\rho_0(r) dr + \frac{(5 - 3\gamma)N_1(N_1 - 1)}{2}(\Delta t)^2 \int_1^{\infty} r^2 \eta(u_0(r)) dr + O(\sqrt{\Delta r}),
\end{aligned}$$

where $\beta = 3\gamma - 5$.

Proof. We assume that there exist $M_n(\overline{u}^{\Delta})$ and $S_n(\overline{u}^{\Delta})$ such that

$$\sum_{k=0}^{N-1} M_k(\overline{u}^{\Delta}) = O(\sqrt{\Delta r}), \quad \sum_{k=0}^{N-1} S_k(\overline{u}^{\Delta}) = O(\sqrt{\Delta r}), \tag{4.1}$$

$$\sum_{j\in J_{n+1}}\int_{I_j} r^2\eta(u_j^{n+1})dr \le \sum_{j\in J_n}\int_{I_j} r^2\eta(u_j^n)dr + M_n(\overline{u}^\Delta) + O((\Delta r)^2) \tag{4.2}$$

and

$$\begin{aligned}\sum_{j\in J_{n+1}}\int_{I_j} F(r,t_{n+1},u_j^{n+1})dr \le \sum_{j\in J_n}\int_{I_j} F(r,t_n,u_j^n)dr + \sum_{j\in J_n}\int_{I_j} \Delta t H(r,t_n,u_j^n)dr \\ + S_n(\overline{u}^\Delta) + O((\Delta r)^2).\end{aligned} \tag{4.3}$$

The proof of (4.1)–(4.3) is postponed to Appendix A.

Now, for an integer n such that $t_n = n\Delta t \ge 1$, multiplying both sides of (4.3) by $(t_{n+1})^{3\gamma-5}$, we have

$$\begin{aligned}&\sum_{j\in J_{n+1}}\int_{I_j} (t_{n+1})^{3\gamma-5}F(r,t_{n+1},u_j^{n+1})dr \\ &\le \sum_{j\in J_n}\int_{I_j} (t_{n+1})^{3\gamma-5}\{((r^3-1)/r - r\,t_n v_j^n)^2 + (4r^3-1)/r^2\}\rho_j^n dr \\ &\quad + \sum_{j\in J_n}\int_{I_j} (t_{n+1})^{3\gamma-5}(t_n)^2 r^2 p(\rho_j^n)/\theta dr + \sum_{j\in J_n}\int_{I_j} (t_{n+1})^{3\gamma-5}\Delta t H(r,t_n,u_j^n)dr \\ &\quad + (t_{n+1})^{3\gamma-5}S_n(\overline{u}^\Delta) + O((\Delta r)^2) \\ &\le \sum_{j\in J_n}\int_{I_j} (t_n)^{3\gamma-5}\{((r^3-1)/r - r\,t_n v_j^n)^2 + (4r^3-1)/r^2\}\rho_j^n dr \\ &\quad + \sum_{j\in J_n}\int_{I_j} (t_n)^{3\gamma-5}(t_n)^2 r^2 p(\rho_j^n)/\theta dr + S_n(\overline{u}^\Delta) + O((\Delta r)^2) \\ &= \sum_{j\in J_n}\int_{I_j} (t_n)^{3\gamma-5}F(r,t_n,u_j^n)dr + S_n(\overline{u}^\Delta) + O((\Delta r)^2).\end{aligned}$$

Here, for the second term on the right–hand side of the first inequality (respectively, the first, third and fourth terms on the right–hand side of the first inequality), we have used $(t_{n+1})^{3\gamma-5} = (t_n)^{3\gamma-5} + (3\gamma-5)(t_n)^{3\gamma-6}\Delta t + (t_n)^{3\gamma-7}O((\Delta t)^2)$ (respectively, $(t_{n+1})^{3\gamma-5} \le (t_n)^{3\gamma-5}$, $(t_{n+1})^{3\gamma-5} \le 1$). On the other hand, from (4.2) and (4.3), we obtain

$$\begin{aligned}&\sum_{j\in J_{N_1}}\int_{I_j} F(r,t_{N_1},u_j^{N_1})dr \\ &\le \sum_{j\in J_0}\int_{I_j} F(r,0,u_j^0)dr + \sum_{n=0}^{N_1-1}\sum_{j\in J_n}\int_{I_j} \Delta t H(r,t_n,u_j^n)dr + \sum_{k=0}^{N_1-1} S_k(\overline{u}^\Delta) + O(\Delta r)\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{j\in J_0}\int_{I_j} F(r,0,u_j^0)dr + (5-3\gamma)\sum_{n=0}^{N_1-1}\sum_{j\in J_n}\int_{I_j} n(\Delta t)^2 r^2\eta(r,t_n,u_j^n)dr \\
&\quad + \sum_{k=0}^{N_1-1} S_k(\overline{u}^\Delta) + O(\Delta r) \\
&\leq \sum_{j\in J_0}\int_{I_j} F(r,0,u_j^0)dr + (5-3\gamma)\sum_{n=0}^{N_1-1} n(\Delta t)^2 \sum_{j\in J_0}\int_{I_j} r^2\eta(r,0,u_j^0)dr \\
&\quad + \sum_{k=0}^{N_1-1} (M_k+S_k)(\overline{u}^\Delta) + O(\Delta r) \\
&\leq \sum_{j\in J_0}\int_{I_j} F(r,0,u_j^0)dr + (5-3\gamma)\frac{N_1(N_1-1)}{2}(\Delta t)^2\sum_{j\in J_0}\int_{I_j} r^2\eta(r,0,u_j^0)dr \\
&\quad + \sum_{k=0}^{N_1-1} (M_k+S_k)(\overline{u}^\Delta) + O(\Delta r).
\end{aligned}
$$

From the above two inequalities, we deduce

$$
\begin{aligned}
&\sum_{j\in J_N}\int_{I_j} (t_N)^\beta F(r,t_N,u_j^N)dr \\
&\quad\leq \sum_{j\in J_{N_1}}\int_{I_j} (t_{N_1})^\beta F(r,t_{N_1},u_j^{N_1})dr + \sum_{k=N_1}^{N-1} S_k(\overline{u}^\Delta) + O(\Delta r) \\
&\quad\leq \sum_{j\in J_{N_1}}\int_{I_j} F(r,t_{N_1},u_j^{N_1})dr + \sum_{k=N_1}^{N-1} S_k(\overline{u}^\Delta) + O(\Delta r) \\
&\quad\leq \sum_{j\in J_0}\int_{I_j} F(r,0,u_j^0)dr + (5-3\gamma)\frac{N_1(N_1-1)}{2}(\Delta t)^2\sum_{j\in J_0}\int_{I_j} r^2\eta(r,0,u_j^0)dr \\
&\qquad + \sum_{k=0}^{N_1-1} M_k(\overline{u}^\Delta) + \sum_{k=0}^{N-1} S_k(\overline{u}^\Delta) + O(\Delta r).
\end{aligned}
$$

Moreover, we have

$$
\sum_{j\in J_0}\int_{I_j} F(r,0,u_j^0)dr = \sum_{j\in J_0}\int_{I_j} (r^4+2r)\rho_j^0 dr = \int_1^\infty (r^4+2r)\rho_0(r)dr + O(\Delta r)
$$

and (by using the Jensen inequality)

$$
\sum_{j\in J_0}\int_{I_j} r^2\eta(u_j^0)dr \leq \int_1^\infty r^2\eta(u_0(r))dr + O(\Delta r).
$$

Therefore, from (4.1), we obtain Lemma 4.1. □

4.2. *Estimates of $u^{\Delta}(r,t)$ in $t_N \le t < t_{N+1}$*

In this subsection, we prove the following lemma:

Lemma 4.2. *For T with $t_N \le T < t_{N+1}$, we have*

$$\int_1^{\infty} T^{\beta} F(r,T,u^{\Delta}(r,T))dr \le \sum_{j \in J_N} \int_{I_j} (t_N)^{\beta} F(r,t_N,u_j^N)dr + O(\Delta r),$$

where $\beta := 3\gamma - 5 \le 0$.

This lemma is deduced from the following lemma:

Lemma 4.3.

1. *For $j \in J_N^*$, we have*

$$\int_{I_j} T^{\beta} F(r,T,u^{\Delta}(r,t))dr \le \int_{r_j}^{r_{j+1}} (t_N)^{\beta} F(r,t_N,u_{j+1}^N)dr + \int_{r_{j-1}}^{r_j} (t_N)^{\beta} F(r,t_N,u_{j-1}^N)dr$$
$$- \left(\int_{t_N}^{T} t^{\beta} G(r_{j+1},t,u_{j+1}^N)dt - \int_{t_N}^{T} t^{\beta} G(r_{j-1},t,u_{j-1}^N)dt \right)$$
$$+O((\Delta r)^2). \tag{4.4}$$

2. *If N is even, we have*

$$\int_{I_1} T^{\beta} F(r,T,u^{\Delta}(r,t))dr \le \int_{I_1} (t_N)^{\beta} F(r,t_N,u_2^N)dr - \int_{t_N}^{T} t^{\beta} G(r_2,t,u_2^N)dt$$
$$+ O((\Delta r)^2).$$

3. *If N is odd, we have*

$$\int_{r_0}^{r_3} T^{\beta} F(r,T,u^{\Delta}(r,t))dr \le \int_{r_2}^{r_3} (t_N)^{\beta} F(r,t_N,u_3^N)dr + \int_{r_0}^{r_2} (t_N)^{\beta} F(r,t_N,u_1^N)dr$$
$$- \int_{t_N}^{T} t^{\beta} G(r_3,t,u_3^N)dt + O((\Delta r)^2).$$

Proof. Let us prove (4.4). We consider the region $r_{j-1} \le r \le r_{j+1},\ t_N \le t \le T$. Recall that $\overline{u}^{\Delta}(r,t)$ is a Riemann solution in this region. For simplicity, we consider the case where one shock arise. Let the propagation speed of the shock be λ. Then the ray $x - j\Delta r - 1 = \lambda(t - t_N)$ divides the region into two parts.

On the other hand, if $\overline{u}^{\Delta}(r,t)$ is a smooth solution to (2.1), from (3.4), we obtain

$$\{t^{\beta} F(r,t,u^{\Delta})\}_t + \{t^{\beta} G(r,t,u^{\Delta})\}_r = \beta t^{\beta-1} \overline{\rho}^{\Delta} \left\{ rt\overline{v}^{\Delta} - \tfrac{(r^3-1)}{r} \right\}^2 - \tfrac{4}{r} t^{\beta+1} \tfrac{(\overline{m}^{\Delta})^2}{\overline{\rho}^{\Delta}}$$
$$+\beta(4r - 1/r^2)t^{\beta-1}\overline{\rho}^{\Delta} + O(\Delta r)$$
$$\le O(\Delta r). \tag{4.5}$$

Then we notice that $\overline{u}^{\Delta}(r,t)$ is continuous and piecewise smooth in each region of the divided two parts and $u^{\Delta}(r,t) = \overline{u}^{\Delta}(r,t) + O(\Delta r)$. Applying the Green formula to (4.5), we conclude

$$\begin{aligned}
&\text{the left-hand side of (4.4)}\\
&\quad\leq \text{the right-hand side of (4.4)}\\
&\qquad+\int_{t_N}^{T}\big[t^{\beta}\{G(r(t),t,\overline{u}^{\Delta}(r(t)+0,t))-G(r(t),t,\overline{u}^{\Delta}(r(t)-0,t))\}\\
&\qquad-\lambda t^{\beta}\{F(r(t),t,\overline{u}^{\Delta}(r(t)+0,t))-F(x(t),t,\overline{u}^{\Delta}(r(t)-0,t))\}\big]\,dt+O((\Delta r)^2)\\
&\quad\leq O((\Delta r)^2)\quad\text{(from (2.2) and (2.3))},
\end{aligned}$$

where $r(t):=j\Delta r+1+\lambda(t-t_N)$.

In view of $G(r_0,t,u_b^N)=0$ (u_b^N is defined in Step 3 of Subsection 3.1.), we can deduce the other inequalities in a similar manner to (4.4). □

By virtue of the methods of compensated compactness for the approximate solutions (see [10] and [11]), there exists a subsequence u^{Δ_k} such that $(\Delta r)_k\to 0$ and u^{Δ_k} tends to a weak solution to (1.3) almost everywhere as $k\to\infty$. On the other hand, we observe that

$$\begin{aligned}
F(r,t,u)&=t^{\beta}\rho\left[\left\{rtv-(r^3-1)/r\right\}^2+(4r^3-1)/r^2\right]+r^2t^{3(\gamma-1)}p(\rho)/\theta\\
&\geq r^2t^{3(\gamma-1)}p(\rho)/\theta.
\end{aligned}$$

Then we apply Lemmas 4.1 and 4.2 to the above subsequence. Since we can obtain (1.8) for an arbitrary R_0 in (3.2), we conclude Theorem 1.2.

Appendix A. Proof of formulas (4.1)–(4.3)

To complete the proof of Lemma 4.1, it remains to prove (4.1)–(4.3).

We first consider ρ_j^n. By Lemma 3.2 and Remark 3.3, we have

$$\sum_{j\in J_{n+1}}\int_{I_j}(r^4+2r)\rho_j^{n+1}dr=A_n+B_n,$$

where

(1) if n is even,

$$\begin{aligned}
A_n&=\sum_{j\in J_{n+1}^*}\int_{I_j}(r^4+2r)\frac{\rho_{j+1}^n+\rho_{j-1}^n}{2}dr-\sum_{j\in J_{n+1}^*}\int_{I_j}(r^4+2r)\frac{\Delta t}{2\Delta r}(m_{j+1}^n-m_{j-1}^n)dr\\
&\quad+\int_{I_1}(r^4+2r)\rho_2^n dr-\int_{I_1}(r^4+2r)\frac{\Delta t}{2\Delta r}m_2^n dr,\\
B_n&=-\sum_{j\in J_{n+1}}\int_{I_j}(r^4+2r)\frac{\Delta t}{2\Delta r}\left\{\int_{I_j}\frac{2}{s}\,\overline{m}^{\Delta}(s,t_{n+1}-0)ds\right\}dr;
\end{aligned}$$

(2) if n is odd,

$$\begin{aligned}
A_n =& \sum_{j\in J^*_{n+1}} \int_{I_j} (r^4+2r)\frac{\rho^n_{j+1}+\rho^n_{j-1}}{2}dr - \sum_{j\in J^*_{n+1}} \int_{I_j} (r^4+2r)\frac{\Delta t}{2\Delta r}(m^n_{j+1}-m^n_{j-1})dr \\
&+ \int_{r_0}^{r_3} (r^4+2r)\frac{\rho^n_3+2\rho^n_1}{3}dr - \int_{r_0}^{r_3} (r^4+2r)\frac{\Delta t}{3\Delta r}m^n_3 dr, \\
B_n =& -\sum_{j\in J^*_{n+1}} \int_{I_j} (r^4+2r)\frac{\Delta t}{2\Delta r}\left\{\int_{I_j} \frac{2}{s}\,\overline{m}^\Delta(s,t_{n+1}-0)ds\right\}dr \\
&- \int_{r_0}^{r_3} (r^4+2r)\frac{\Delta t}{3\Delta r}\left\{\int_{r_0}^{r_3} \frac{2}{s}\,\overline{m}^\Delta(s,t_{n+1}-0)ds\right\}dr.
\end{aligned}$$

For simplicity, we consider only the case where n is even. The other case is similar to this one. We first compute A_n.

$$\begin{aligned}
A_n =& \sum_{j\in J_{n+1}} \left\{\int_{I_{j+2}} (r^4+2r)dr + \int_{I_j} (r^4+2r)dr\right\}\frac{\rho^n_{j+1}}{2} \\
&+ \sum_{j\in J_{n+1}} \left\{\int_{I_{j+2}} (r^4+2r)dr - \int_{I_j} (r^4+2r)dr\right\}\frac{\Delta t}{2\Delta r}m^n_{j+1} \\
&+ \int_{I_1} (r^4+2r)\frac{\rho^n_0}{2} \quad (\text{Recall that } \rho^n_2=\rho^n_0.) \\
=& \sum_{j\in J_n} \left\{\int_{I_{j+1}} (r^4+2r)dr + \int_{I_{j-1}} (r^4+2r)dr\right\}\frac{\rho^n_j}{2}dr \\
&+ \sum_{j\in J_n} \left\{\int_{I_{j+1}} (r^4+2r)dr - \int_{I_{j-1}} (r^4+2r)dr\right\}\frac{\Delta t}{2\Delta r}m^n_j - \int_{I_1} (r^4+2r)dr\frac{\Delta t}{2\Delta r}m^n_0 \\
=& \sum_{j\in J_n} \int_{I_j} (r^4+2r)\rho^n_j dr + \sum_{j\in J_n} \int_{I_j} (4r^3+2)\Delta t\, m^n_j dr + O((\Delta r)^2).
\end{aligned}$$

Next we compute B_n. From (3.2) and finite propagation (3.5), we can choose R_T large enough such that Supp $u^\Delta \subset [1,R_T]\times[0,T]$. Then we have

$$\begin{aligned}
B_n =& -\sum_{j\in J_n} \int_{I_j} (r^4+2r)\frac{\Delta t}{2\Delta r}\left\{\int_{I_j} \frac{2}{s}\,\overline{m}^\Delta(s,t_n+0)ds\right\}dr + K_n(u^\Delta) \\
=& -\sum_{j\in J_n} \int_{I_j} (2r^3+4)\Delta t\, m^n_j dr + K_n(u^\Delta) + O((\Delta r)^2),
\end{aligned}$$

where

$$\begin{aligned}
K_n(u^\Delta) &= \sum_{j\in J_n}\int_{I_j}(r^4+2r)\frac{\Delta t}{2\Delta r}\left\{\int_{I_j}\frac{2}{s}\,\overline{m}^\Delta(s,t_n+0)ds\right\}dr \\
&\quad -\sum_{j\in J_{n+1}}\int_{I_j}(r^4+2r)\frac{\Delta t}{2\Delta r}\left\{\int_{I_j}\frac{2}{s}\,\overline{m}^\Delta(s,t_{n+1}-0)ds\right\}dr \\
&= \int_1^{R_T}(2r^3+4)\Delta t\left\{\overline{m}^\Delta(r,t_n+0)-\overline{m}^\Delta(r,t_{n+1}-0)\right\}dr+O((\Delta r)^2).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\sum_{j\in J_{n+1}}\int_{I_j}(r^4+2r)\rho_j^{n+1}dr &= \sum_{j\in J_n}\int_{I_j}(r^4+2r)\rho_j^n dr+\sum_{j\in J_n}\int_{I_j}2(r^3-1)\Delta t\, m_j^n dr \\
&\quad + K_n(u^\Delta)+O((\Delta r)^2).
\end{aligned} \tag{A.1}$$

Similarly, we have the estimate of m_j^n:

$$\begin{aligned}
&\sum_{j\in J_{n+1}}\int_{I_j}2(r^3-1)t_{n+1}\, m_j^{n+1}dr \\
&\quad = \sum_{j\in J_n}\int_{I_j}2(r^3-1)t_{n+1}\, m_j^n dr+\sum_{j\in J_n}\int_{I_j}(2r^2+4/r)t_{n+1}\Delta t\frac{(m_j^n)^2}{\rho_j^n}dr \\
&\quad + \sum_{j\in J_n}\int_{I_j}6r^2t_{n+1}\Delta t\, p(\rho_j^n)dr+t_{n+1}L_n(u^\Delta)+O((\Delta r)^2)
\end{aligned} \tag{A.2}$$

and the estimate of $\eta(u_j^n)$:

$$\begin{aligned}
\sum_{j\in J_{n+1}}\int_{I_j}r^2(t_{n+1})^2\eta(u_j^{n+1})dr &\le \sum_{j\in J_n}\int_{I_j}r^2(t_{n+1})^2\eta(u_j^n)dr+(t_{n+1})^2M_n(u^\Delta) \\
&\quad + O((\Delta r)^2),
\end{aligned} \tag{A.3}$$

where

$$\begin{aligned}
L_n(u^\Delta) &= \int_1^{R_T}4\left(r^2-\frac{1}{r}\right)\Delta t\left\{\frac{\left(\overline{m}^\Delta\right)^2}{\overline{\rho}^\Delta}+p(\overline{\rho}^\Delta)\right\}(r,t_n+0)dr \\
&\quad -\int_1^{R_T}4\left(r^2-\frac{1}{r}\right)\Delta t\left\{\frac{\left(\overline{m}^\Delta\right)^2}{\overline{\rho}^\Delta}+p(\overline{\rho}^\Delta)\right\}(r,t_{n+1}-0)dr, \\
M_n(u^\Delta) &= \int_1^{R_T}2r\Delta t\left\{q(\overline{u}^\Delta(r,t_n+0))-q(\overline{u}^\Delta(r,t_{n+1}-0))\right\}dr.
\end{aligned}$$

From (A.1)–(A.3), we have

$$\sum_{j\in J_{n+1}} \int_{I_j} F(r, t_{n+1}, u_j^{n+1})dr$$
$$\leq \sum_{j\in J_n} \int_{I_j} F(r, t_n, u_j^n)dr + \sum_{j\in J_n} \int_{I_j} \Delta t\, H(r, t_n, u_j^n)dr + S_n(u^\Delta) + O((\Delta r)^2),$$

where $S_n(u^\Delta) := \left\{K_n + t_{n+1}L_n + (t_{n+1})^2 M_n\right\}(u^\Delta)$. Inequality (4.2) can be prove in a similar manner.

Finally we prove (4.1). To achieve this, we introduce the following proposition:

Proposition Appendix A.1.

$$\sum_{k=0}^{N-1} \int_1^{R_T} |\overline{u}^\Delta(r, t_k - 0) - \overline{u}^\Delta(r, t_k + 0)|^2 dr \leq C.$$

This can be found in [7, (4.18)] and [11, (9)].

From the above proposition and the Schwarz inequality, we have

$$\sum_{k=0}^{N-1} (t_{k+1})^2 M_k(u^\Delta)$$
$$= \Delta t \sum_{k=0}^{N-1} \int_1^{R_T} 2r\left\{(t_k)^2 q(\overline{u}^\Delta(r, t_k - 0)) - (t_{k+1})^2 q(\overline{u}^\Delta(r, t_k + 0))\right\} dr + O(\Delta r)$$
$$= \Delta t \sum_{k=0}^{N-1} (t_k)^2 \int_1^{R_T} 2r\left\{q(\overline{u}^\Delta(r, t_k - 0)) - q(\overline{u}^\Delta(r, t_k + 0))\right\} dr + O(\Delta r)$$

(Notice that $t_{k+1} - t_k = O(\Delta r)$.)

$$\leq C\Delta t \sum_{k=0}^{N-1} \int_1^{R_T} \left|\overline{u}^\Delta(r, t_k - 0) - \overline{u}^\Delta(r, t_k + 0)\right| dr + O(\Delta r)$$
$$= O(\sqrt{\Delta r}).$$

We can similarly compute the other terms, i.e., $K_n(u^\Delta)$ and $t_n L_n(u^\Delta)$.

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