Asymptotic behavior of global classical solutions to the mixed initial-boundary value problem for quasilinear hyperbolic systems with small BV data

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Abstract

In this paper, we investigate the asymptotic behavior of global classical solutions to the mixed initial-boundary value problem with small BV data for linearly degenerate quasilinear hyperbolic systems with general nonlinear boundary conditions in the half space $\{(t,x)|t \geq 0, x \geq 0\}$. Based on the existence result on the global classical solution, we prove that when t tends to the infinity, the solution approaches a combination of C^1 traveling wave solutions, provided that the C^1 norm of the initial and boundary data is bounded and the BV norm of the initial and boundary data is sufficiently small. Applications to quasilinear hyperbolic systems arising in physics and mechanics, particularly to the system describing the motion of the relativistic string in the Minkowski space-time R^{1+n}, are also given.

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1. Introduction and main result

Consider the following first order quasilinear hyperbolic system:

$$\frac{\partial u}{\partial t} + A(u)\frac{\partial u}{\partial x} = 0, \tag{1.1}$$

where $u = (u_1, \ldots, u_n)^T$ is the unknown vector function of (t, x) and $A(u)$ is an $n \times n$ matrix with suitably smooth elements $a_{ij}(u)$ $(i, j = 1, \ldots, n)$.

It is assumed that system (1.1) is strictly hyperbolic, i.e., for any given u on the domain under consideration, $A(u)$ has n real distinct eigenvalues

$$\lambda_1(u) < \lambda_2(u) < \ldots < \lambda_n(u). \tag{1.2}$$

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Let $l_i(u) = (l_{i1}(u), \dots, l_{in}(u))$ (resp. $r_i(u) = (r_{i1}(u), \dots, r_{in}(u))^T$) be a left (resp. right) eigenvector corresponding to $\lambda_i(u)(i = 1, \dots, n)$:

$$l_i(u)A(u) = \lambda_i(u)l_i(u) \qquad (\text{resp. } A(u)r_i(u) = \lambda_i(u)r_i(u)), \tag{1.3}$$

then we have

$$det|l_{ij}(u)| \neq 0 \qquad (\text{equivalently, } \det|r_{ij}(u)| \neq 0). \tag{1.4}$$

Without loss of generality, we may assume that on the domain under consideration

$$l_i(u)r_j(u) \equiv \delta_{ij} \quad (i, j = 1, \dots, n) \tag{1.5}$$

and

$$r_i^T(u)r_i(u) \equiv 1 \quad (i = 1, \dots, n), \tag{1.6}$$

where δ_{ij} stands for the Kronecker's symbol.

Clearly, all $\lambda_i(u)$, $l_{ij}(u)$ and $r_{ij}(u)(i, j = 1, \dots, n)$ have the same regularity as $a_{ij}(u)(i, j = 1, \dots, n)$.

We assume that on the domain under consideration, the eigenvalues satisfy the non-characteristic condition

$$\lambda_r(u) < 0 < \lambda_s(u) \qquad (r = 1, \dots, m;\ s = m + 1, \dots, n). \tag{1.7}$$

We also assume that on the domain under consideration, system (1.1) is linearly degenerate, i.e., each characteristic field is linearly degenerate in the sense of Lax:

$$\nabla\lambda_i(u)r_i(u) \equiv 0 \quad (i = 1, \dots, n). \tag{1.8}$$

Consider the mixed initial-boundary value problem with small BV data for system (1.1) in the half space

$$D = \{(t, x) \mid t \geq 0,\ x \geq 0\} \tag{1.9}$$

with the initial condition:

$$t = 0 : u = \varphi(x)\ (x \geq 0) \tag{1.10}$$

and the nonlinear boundary condition (cf. [10, 15-17])

$$x = 0 : v_s = G_s(\alpha(t), v_1, \dots, v_m) + h_s(t),\ \ s = m + 1, \dots, n \quad (t \geq 0), \tag{1.11}$$

where

$$v_i = l_i(u)u \quad (i = 1, \dots, n) \tag{1.12}$$

and

$$\alpha(t) = (\alpha_1(t), \dots, \alpha_k(t)).$$

Here, $G_s \in C^1(s = m + 1, \dots, n)$, $\varphi = (\varphi_1, \dots, \varphi_n)^T$, α and $h(\cdot) = (h_{m+1}(\cdot), \dots, h_n(\cdot)) \in C^1$ with bounded C^1 norm, such that

$$||\varphi(x)||_{C^1},\ ||\alpha(t)||_{C^1},\ ||h(t)||_{C^1} \leq M, \tag{1.13}$$

for some positive constant M (bounded but possibly large). Also, we assume that the conditions of C^1 compatibility are satisfied at the point $(0,0)$. Without loss of generality, we assume that

$$G_s(\alpha(t), 0, \dots, 0) \equiv 0 \quad (s = m+1, \dots, n). \tag{1.14}$$

Without loss of generality, we also assume that

$$\varphi(0) = 0. \tag{1.15}$$

In fact, by the following transformation

$$\widetilde{u} = u - \varphi(0), \tag{1.16}$$

we can always realize the above assumption.

Recently, Shao [14] proved the following global existence result on the classical solution:

Theorem A. Suppose that system (1.1) is strictly hyperbolic and linearly degenerate. Suppose furthermore that in a neighborhood of $u = 0$, $A(u) \in C^2$ and (1.7) holds. Suppose finally that $\varphi, \alpha, G_s, h_s (s = m+1, \dots, n)$ are all C^1 functions with respect to their arguments satisfying the conditions of C^1 compatibility at the point $(0,0)$. For any constant $M > 0$, there exists $\varepsilon > 0$ small enough such that, if (1.13)-(1.15) hold together with

$$\int_0^{+\infty} |\varphi'(x)| dx, \ \int_0^{+\infty} |\alpha'(t)| dt, \ \int_0^{+\infty} |h'(t)| dt \le \varepsilon, \tag{1.17}$$

then the mixed initial-boundary value problem (1.1) and (1.10)-(1.11) admits a unique global C^1 solution $u = u(t,x)$ in the half space $\{(t,x) | t \ge 0, x \ge 0\}$.

Our goal in this paper is to describe the asymptotic behavior of global classical solutions to the mixed initial-boundary value problem (1.1) and (1.10)-(1.11). Based on Theorem A, we shall prove the following theorem.

Theorem 1.1 (Asymptotic Behavior). Under the assumptions of Theorem A, for the mixed initial-boundary value problem (1.1) and (1.10)-(1.11), if

$$N \triangleq \max\{\int_0^{+\infty} |\varphi(x)| dx, \int_0^{+\infty} |h(t)| dt\} < +\infty, \tag{1.18}$$

then there exists a unique C^1vector-valued function $\phi(x) = (\phi_1(x), \dots, \phi_n(x))^T$ such that in the normalized coordinates

$$u(t,x) \to \sum_{i=1}^{n} \phi_i(x - \lambda_i(0)t) e_i \quad \text{as } t \to +\infty, \tag{1.19}$$

where

$$e_i = (0, \dots, 0, \overset{(i)}{1}, 0, \dots, 0)^T.$$

Moreover, $\phi_i(x) (i = 1, \dots, n)$ are global Lipschitz continuous, more precisely, there exists a positive constant κ_1 independent of ε, M, x_1 and x_2 such that

$$|\phi_i(x_1) - \phi_i(x_2)| \le \kappa_1 M |x_1 - x_2|, \qquad \forall\, x_1, x_2 \in \mathbf{R}. \tag{1.20}$$

Furthermore, if the derivatives of the initial and boundary data, i.e., $\varphi'(x)$, $\alpha'(t)$ and $h'(t)$, are global ρ-Hölder continuous, where $0 < \rho \le 1$, that is, there exists positive constants ς_1, ς_2 and ς_3 such that

$$|\varphi'(x_1) - \varphi'(x_2)| \le \varsigma_1 |x_1 - x_2|^{\rho}, \qquad \forall\, x_1, x_2 \in \mathbf{R}^+, \tag{1.21}$$

$$|\alpha'(t_1)-\alpha'(t_2)| \le \varsigma_2 |t_1-t_2|^\rho, \qquad \forall\, t_1,t_2 \in \mathbf{R}^+ \tag{1.22}$$

and

$$|h'(t_1)-h'(t_2)| \le \varsigma_3 |t_1-t_2|^\rho, \qquad \forall\, t_1,t_2 \in \mathbf{R}^+, \tag{1.23}$$

then $\phi'(x)$ is also global ρ-Hölder continuous and satisfies that

$$|\phi'(x_1)-\phi'(x_2)| \le \kappa_2 \varsigma (1+MN+\varepsilon)^\rho |x_1-x_2|^\rho + \kappa_2 M^2(1+\varepsilon)(1+MN+\varepsilon)|x_1-x_2|, \tag{1.24}$$

where κ_2 is a positive constant independent of $\varepsilon, M, N, \varsigma, x_1$ and x_2.

Remark 1.1. Suppose that system (1.1) is non-strictly hyperbolic but each characteristic has a constant multiplicity, say, on the domain under consideration,

$$\lambda_1(u) < \cdots < \lambda_m(u) < 0 < \lambda_m(u) < \ldots < \lambda_{p+1}(u) \equiv \cdots \equiv \lambda_n(u) \ \ (m \le p \le n). \tag{1.25}$$

Then, if there exist the normalized coordinates, the conclusion of Theorem 1.1 still holds (cf. [3-4, 9]).

The global existence of classical solution of the Cauchy problem for quasilinear hyperbolic systems has been established for linearly degenerate characteristics or weakly linearly degenerate characteristics with various smallness assumptions on the initial data by Bressan [1], Kong [6], Li et al [11-12], Zhou [18] and etc. On the other hand, for the asymptotic behavior of the classical solutions of the quasilinear hyperbolic systems, many results have also been obtained in the literature (for instance, see [3-5, 7, 13] and the references therein). In particular, Kong and Yang [7] firstly studied the asymptotic behavior of the classical solutions of the quasilinear hyperbolic systems with some decay initial data. However, it is well known that the BV space is a suitable framework for one-dimensional quasilinear hyperbolic systems (see Bressan [2]), the result in Bressan [1] suggests that one may achieve global smoothness even if the C^1 norm of the initial data is large. So the following question arises naturally: can we obtain the global existence and the asymptotic behavior of the classical solutions to the mixed initial-boundary value problem (1.1) and (1.10)-(1.11), provided that the BV norm of the initial and boundary data is suitably small? Here, it is important to mention that for the Cauchy problem case, this problem was solved by Bressan [1], Zhou [18], Dai and Kong [4]. However, due to the presence of a boundary, any waves with negative speed are expected to be reflected at the boundary, some additional difficulties appear. Therefore new proofs are required to overcome them. This makes our new analysis more complicated than that for the Cauchy problem case. The present paper can be viewed as a development of [1], [4] and [18]. The rest of the paper is organized as follows. For the sake of completeness, in Section 2 we recall John's formula on the decomposition of waves with some supplements. Section 3 is devoted to establishing some new estimates, these estimates will play an important role in the proof of main result. The main result, Theorem 1.1, is proved in Section 4. It is easy to see that Theorem 1.1 can be applied to all physical models discussed in Li and Wang [10] on the mixed initial boundary value problem for the system of the planar motion of an elastic string, provided that the BV norm of the initial and boundary data is suitably small, therefore we do not give the details in this paper. However, of particular interest is the system of the motion of the relativistic string in the Minkowski space-time R^{1+n}, as an application of Theorem 1.1, the asymptotic behavior of the classical solutions to the mixed initial-boundary value problem with small BV data for this system is presented in Section 5.

2. Preliminaries

Suppose that on the domain under consideration, system (1.1) is strictly hyperbolic and (1.5)-(1.6) hold.

Suppose that $A(u) \in C^2$. By Lemma 2.5 in [11], there exists an invertible C^3 transformation $u = u(\widetilde{u})$ $(u(0) = 0)$ such that in $\widetilde{u}$-space, for each $i = 1, \ldots, n$, the ith characteristic trajectory passing through $\widetilde{u} = 0$ coincides with the $\widetilde{u}_i$-axis at least for $|\widetilde{u}_i|$ small, namely,

$$\widetilde{r}_i(\widetilde{u}_i e_i) \equiv e_i, \quad \forall\ |\widetilde{u}_i| \text{ small } (i = 1, \ldots, n), \tag{2.1}$$

where

$$e_i = (0, \ldots, 0, \overset{(i)}{1}, 0, \ldots, 0)^T. \tag{2.2}$$

Such a transformation is called the normalized transformation and the corresponding unknown variables $\widetilde{u} = (\widetilde{u}_1, \ldots, \widetilde{u}_n)^T$ are called the normalized variables or normalized coordinates (see [12]).

Let

$$w_i = l_i(u) u_x \quad (i = 1, \ldots, n), \tag{2.3}$$

where

$$l_i(u) = (l_{i1}(u), \ldots, l_{in}(u))$$

denotes the ith left eigenvector.

By (1.5), it follows from (1.12) and (2.3) that

$$u = \sum_{k=1}^{n} v_k r_k(u) \tag{2.4}$$

and

$$u_x = \sum_{k=1}^{n} w_k r_k(u). \tag{2.5}$$

Let

$$\frac{d}{d_i t} = \frac{\partial}{\partial t} + \lambda_i(u) \frac{\partial}{\partial x} \tag{2.6}$$

be the directional derivative along the ith characteristic. We have (see [3-4, 12])

$$\frac{dv_i}{d_i t} = \sum_{j,k=1}^{n} \beta_{ijk}(u) v_j w_k \ \ (i = 1, \ldots, n), \tag{2.7}$$

where

$$\beta_{ijk}(u) = (\lambda_k(u) - \lambda_i(u)) l_i(u) \nabla r_j(u) r_k(u). \tag{2.8}$$

Hence, we have

$$\beta_{iji}(u) \equiv 0, \quad \forall\ j \tag{2.9}$$

and by (2.1), in the normalized coordinates we have

$$\beta_{ijj}(u_j e_j) \equiv 0, \quad \forall\ |u_j| \text{ small}, \ \forall\ j. \tag{2.10}$$

Noting (2.5), by (2.7) we have

$$\frac{\partial v_i}{\partial t} + \frac{\partial(\lambda_i(u) v_i)}{\partial x} = \sum_{j,k=1}^{n} B_{ijk}(u) v_j w_k \overset{def}{=} F_i(t, x), \tag{2.11}$$

or equivalently,

$$d[v_i(dx - \lambda_i(u)dt)] = \sum_{j,k=1}^{n} B_{ijk}(u)v_j w_k dt \wedge dx = F_i(t,x)dt \wedge dx, \tag{2.12}$$

where

$$B_{ijk}(u) = \beta_{ijk}(u) + \nabla\lambda_i(u)r_k(u)\delta_{ij}. \tag{2.13}$$

By (2.9), it is easy to see that

$$B_{iji}(u) \equiv 0, \qquad\qquad \forall\ i \neq j \tag{2.14}$$

and

$$B_{iii}(u) = \nabla\lambda_i(u)r_i(u), \qquad\qquad \forall\ i. \tag{2.15}$$

Moreover, by (2.10), in the normalized coordinates we have

$$B_{ijj}(u_j e_j) \equiv 0, \qquad \forall\ |u_j|\ \text{ small, } \forall\ i \neq j. \tag{2.16}$$

When the system is linearly degenerate, in the normalized coordinates, we have

$$B_{ijj}(u_j e_j) \equiv 0, \qquad \forall\ |u_j|\ \text{ small, } \forall\ j. \tag{2.17}$$

On the other hand, we have (see [3-4, 12])

$$\frac{dw_i}{d_i t} = \sum_{j,k=1}^{n} \gamma_{ijk}(u)w_j w_k \quad (i = 1, \ldots, n), \tag{2.18}$$

where

$$\gamma_{ijk}(u) = \frac{1}{2}\{(\lambda_j(u) - \lambda_k(u))l_i(u)\nabla r_k(u)r_j(u) - \nabla\lambda_i(u)r_j(u)\delta_{ik} + (j|k)\}, \tag{2.19}$$

in which $(j|k)$ denotes all the terms obtained by changing j and k in the previous terms. Hence, we have

$$\gamma_{ijj}(u) \equiv 0, \quad \forall\ j \neq i \tag{2.20}$$

and

$$\gamma_{iii}(u) \equiv -\nabla\lambda_i(u)r_i(u) \quad (i = 1, \ldots, n). \tag{2.21}$$

When the system is linearly degenerate, we have

$$\gamma_{ijj}(u) \equiv 0, \qquad \forall\ i, j. \tag{2.22}$$

Noting (2.5), by (2.18) we have

$$\frac{\partial w_i}{\partial t} + \frac{\partial(\lambda_i(u)w_i)}{\partial x} = \sum_{j,k=1}^{n} \Gamma_{ijk}(u)w_j w_k \stackrel{def}{=} G_i(t,x), \tag{2.23}$$

equivalently,

$$d[w_i(dx - \lambda_i(u)dt)] = \sum_{j,k=1}^{n} \Gamma_{ijk}(u)w_j w_k dt \wedge dx = G_i(t,x)dt \wedge dx, \tag{2.24}$$

where

$$\Gamma_{ijk}(u) = \frac{1}{2}(\lambda_j(u) - \lambda_k(u))l_i(u)[\nabla r_k(u)r_j(u) - \nabla r_j(u)r_k(u)]. \tag{2.25}$$

Hence, we have

$$\Gamma_{ijj}(u) \equiv 0, \qquad \forall\ i, j. \tag{2.26}$$

3. Uniform Estimates

In this section, we shall establish some new uniform estimates which play a key role in the proof of Theorem 1.1.

By Lemma 2.5 in [11], there exists a normalized transformation. Without loss of generality, we assume that $u = (u_1, \ldots, u_n)^T$ are already the normalized coordinates.

Noting (1.2) and (1.7), we have

$$\lambda_1(0) < \ldots < \lambda_m(0) < 0 < \lambda_{m+1}(0) < \ldots < \lambda_n(0). \tag{3.1}$$

Thus, there exist sufficiently small positive constants δ and δ_0 such that

$$\lambda_{i+1}(u) - \lambda_i(v) \geq \delta_0, \qquad \forall\ |u|,\ |v| \leq \delta\ \ (i = 1, \ldots, n-1), \tag{3.2}$$

$$|\lambda_i(u) - \lambda_i(v)| \leq \frac{\delta_0}{2}, \qquad \forall\ |u|, |v| \leq \delta\ \ (i = 1, \ldots, n) \tag{3.3}$$

and

$$|\lambda_i(0)| \geq \delta_0 \quad (i = 1, \ldots, n). \tag{3.4}$$

On the other hand, by Lemma 3.5 in [14], we know that on the domain of existence of the C^1 solution $u = u(t, x)$ to the mixed initial-boundary value problem (1.1) and (1.10)-(1.11), we have

$$|u(t, x)| \leq K_0 \varepsilon. \tag{3.5}$$

where $K_0 > 0$ is a constant independent of ε and M. Therefore, taking ε suitably small, we always have

$$|u(t, x)| \leq \delta. \tag{3.6}$$

For any fixed $T > 0$, let

$$U_\infty(T) = \sup_{0 \leq t \leq T} \sup_{x \in \mathbf{R}^+} |u(t, x)|, \tag{3.7}$$

$$V_\infty(T) = \sup_{0 \leq t \leq T} \sup_{x \in \mathbf{R}^+} |v(t, x)|, \tag{3.8}$$

$$W_\infty(T) = \sup_{0 \leq t \leq T} \sup_{x \in \mathbf{R}^+} |w(t, x)|, \tag{3.9}$$

$$U_1(T) = \sup_{0 \leq t \leq T} \int_0^{+\infty} |u(t, x)| dx, \tag{3.10}$$

$$V_1(T) = \sup_{0 \leq t \leq T} \int_0^{+\infty} |v(t, x)| dx, \tag{3.11}$$

$$W_1(T) = \sup_{0 \leq t \leq T} \int_0^{+\infty} |w(t, x)| dx, \tag{3.12}$$

$$\widetilde{U}_1(T) = \max_{i=1,\ldots,n} \max_{j \neq i} \sup_{C_j} \int_{C_j} |u_i| dt, \tag{3.13}$$

$$\widetilde{V}_1(T) = \max_{i=1,\ldots,n} \max_{j \neq i} \sup_{C_j} \int_{C_j} |v_i| dt, \tag{3.14}$$

$$\widetilde{W}_1(T) = \max_{i=1,\ldots,n} \max_{j \neq i} \sup_{C_j} \int_{C_j} |w_i| dt, \tag{3.15}$$

$$\overline{U}_1(T) = \max_{i=1,\ldots,n} \max_{j \neq i} \sup_{L_j} \int_{L_j} |u_i| dt, \tag{3.16}$$

$$\overline{V}_1(T) = \max_{i=1,\ldots,n} \max_{j \neq i} \sup_{L_j} \int_{L_j} |v_i| dt, \tag{3.17}$$

$$\overline{W}_1(T) = \max_{i=1,\ldots,n} \max_{j \neq i} \sup_{L_j} \int_{L_j} |w_i| dt, \tag{3.18}$$

where $|\cdot|$ stands for the Euclidean norm in $\mathbf{R^n}$, C_j stands for any given jth characteristic on the domain $[0,T] \times \mathbf{R}^+$, while L_j stands for any given ray with the slope $\lambda_j(0)$ on the domain $[0,T] \times \mathbf{R}^+$. Clearly, $V_\infty(T)$ is equivalent to $U_\infty(T)$.

Lemma 3.1. Under the assumptions of Theorem 1.1, on any given domain of existence $\{(t,x) | 0 \leq t \leq T, x \geq 0\}$ of the C^1 solution $u = u(t,x)$ to the mixed initial-boundary value problem (1.1) and (1.10)-(1.11), there exists a positive constant K_1 independent of ε, M, N and T such that

$$\int_0^{+\infty} |v_i(t,x)| dx \leq K_1 \Big\{ V_1(0) + \int_0^{+\infty} |h(t)| dt + \int_0^T \int_0^{+\infty} |F(t,x)| dx dt \Big\}, \forall\ t \leq T \quad (i = 1, \ldots, n), \tag{3.19}$$

provided that the right hand side of the inequality is bounded and $F = (F_1, F_2, \ldots, F_n)$.

Proof. To estimate $\int_0^{+\infty} |v_i(t,x)| dx$, we need only to estimate

$$\int_0^L |v_i(t,x)| dx \tag{3.20}$$

for any given $L > 0$ and then let $L \to +\infty$.

i) For $i = 1, \ldots, m$, for any fixed point (T, L), we draw the ith backward characteristic $x = x_i(t)$ $(0 \leq t \leq T)$ passing through this point:

$$\begin{cases} \frac{dx_i(t)}{dt} = \lambda_i(u(t, x_i(t))), \quad t \leq T, \\ x_i(T) = L. \end{cases} \tag{3.21}$$

From (2.11), we have

$$|v_i|_t + (\lambda_i(u)|v_i|)_x = \operatorname{sgn}(v_i) F_i(t,x). \tag{3.22}$$

Thus, noting (1.7), we get

$$\begin{aligned} \frac{d}{dt} \int_0^{x_i(t)} |v_i(t,x)| dx &= \int_0^{x_i(t)} \frac{\partial}{\partial t} |v_i(t,x)| dx + x_i'(t) |v_i(t, x_i(t))| \\ &= \int_0^{x_i(t)} \operatorname{sgn}(v_i) F_i(t,x) dx - \int_0^{x_i(t)} (\lambda_i(u)|v_i|)_x dx + x_i'(t) |v_i(t, x_i(t))| \\ &= \int_0^{x_i(t)} \operatorname{sgn}(v_i) F_i(t,x) dx - (\lambda_i(u(t, x_i(t))) - x_i'(t)) |v_i(t, x_i(t))| + \lambda_i(u(t,0)) |v_i(t,0)| \\ &= \int_0^{x_i(t)} \operatorname{sgn}(v_i) F_i(t,x) dx + \lambda_i(u(t,0)) |v_i(t,0)| \leq \int_0^{x_i(t)} |F_i(t,x)| dx. \end{aligned} \tag{3.23}$$

Then, it follows from (3.23) that

$$\int_0^{x_i(t)} |v_i(t,x)|dx \le \int_0^{x_i(0)} |v_i(0,x)|dx + \int_0^T \int_0^{x_i(t)} |F_i(t,x)|dxdt$$
$$\le \int_0^{+\infty} |v_i(0,x)|dx + \int_0^T \int_0^{+\infty} |F(t,x)|dxdt. \tag{3.24}$$

Thus

$$\int_0^L |v_i(t,x)|dx \le V_1(0) + \int_0^T \int_0^{+\infty} |F(t,x)|dxdt. \tag{3.25}$$

Letting $L \to +\infty$, we immediately get the assertion in (3.19).

(ii) For $i = m+1,\dots,n$, let $x = x_i(t,L)$ $(0 \le t \le T)$ be the ith forward characteristic passing through point $(0,L)$. Then, passing through point $(T,a)\,(a > x_i(T,L))$, we draw the ith backward characteristic $x = x_i(t)$ $(0 \le t \le T)$ which intersects the x-axis at a point $(0,x_{i0})$. By exploiting the same arguments as in (i), we can deduce that

$$\frac{d}{dt}\int_0^{x_i(t)} |v_i(t,x)|dx = \int_0^{x_i(t)} \operatorname{sgn}(v_i)F_i(t,x)dx + \lambda_i(u(t,0))|v_i(t,0)|$$
$$\le \int_0^{x_i(t)} |F_i(t,x)|dx + \lambda_i(u(t,0))|v_i(t,0)|. \tag{3.26}$$

Thus, noting (3.6), it follows from (3.26) that

$$\int_0^{x_i(t)} |v_i(t,x)|dx \le \int_0^{x_{i0}} |v_i(0,x)|dx + \int_0^T \lambda_i(u(t,0))|v_i(t,0)|dt + \int_0^T \int_0^{x_i(t)} |F_i(t,x)|dxdt$$
$$\le c_1\Big\{\int_0^{+\infty} |v_i(0,x)|dx + \int_0^T |v_i(t,0)|dt + \int_0^T \int_0^{+\infty} |F(t,x)|dxdt\Big\}, \tag{3.27}$$

where here and henceforth, $c_i (i = 1,2,\dots)$ will denote positive constants independent of ε, M, N and T. Noting (1.14), by (1.11), it is easy to see that

$$v_i(t,0) = \sum_{r=1}^m g_{ir}(t)v_r(t,0) + h_i(t), \tag{3.28}$$

where

$$g_{ir}(t) = \int_0^1 \frac{\partial G_i}{\partial v_r}(\alpha(t), \tau v_1(t,0),\dots,\tau v_m(t,0))d\tau. \tag{3.29}$$

Thus, noting (3.6), we have

$$\int_0^T |v_i(t,0)|dt = \sum_{r=1}^m \int_0^T |g_{ir}(t)v_r(t,0)|dt + \int_0^T |h_i(t)|dt$$
$$\le c_2\Big\{\sum_{r=1}^m \int_0^T |v_r(t,0)|dt + \int_0^{+\infty} |h(t)|dt\Big\}. \tag{3.30}$$

Then, passing through the point $A(T,0)$, we draw the rth characteristic $C_r (r \in \{1,\dots,m\})$ which intersects the x-axis at point $B(0,x_B)$. We rewrite (2.12) as

$$d(|v_r(t,x)|(dx - \lambda_r(u)dt)) = \operatorname{sgn}(v_r)F_r dxdt. \tag{3.31}$$

By (3.31), using Stokes' formula on the domain AOB, we have

$$\Big|\int_0^T |v_r(t,0)|(-\lambda_r(u)dt)\Big| \le \int_0^{x_B} |v_r(0,x)|dx + \int\int_{AOB} |F_r|dxdt$$
$$\le V_1(0) + \int_0^T \int_0^{+\infty} |F(t,x)|dxdt. \tag{3.32}$$

Noting (3.4) and (3.6), for sufficiently small $\delta > 0$, it is easy to see that

$$|\lambda_r(u)| \ge \frac{\delta_0}{2}. \tag{3.33}$$

Therefore, it follows from (3.32) that

$$\int_0^T |v_r(t,0)|dt \le c_3\Big\{V_1(0) + \int_0^T \int_0^{+\infty} |F(t,x)|dxdt\Big\}. \tag{3.34}$$

Combining (3.27) with (3.30) and (3.34), we obtain

$$\int_0^{x_i(t)} |v_i(t,x)|dx \le c_4\Big\{V_1(0) + \int_0^{+\infty} |h(t)|dt + \int_0^T \int_0^{+\infty} |F(t,x)|dxdt\Big\}. \tag{3.35}$$

Thus

$$\int_0^L |v_i(t,x)|dx \le c_4\Big\{V_1(0) + \int_0^{+\infty} |h(t)|dt + \int_0^T \int_0^{+\infty} |F(t,x)|dxdt\Big\}. \tag{3.36}$$

Letting $L \to +\infty$, we immediately get the assertion in (3.19). The proof of Lemma 3.1 is finished. □

Lemma 3.2. Under the assumptions of Lemma 3.1, on any given domain of existence $\{(t,x)|0 \le t \le T, x \ge 0\}$ of the C^1 solution $u = u(t,x)$ to the mixed initial-boundary value problem (1.1) and (1.10)-(1.11), there exists a positive constant K_2 independent of ε, M, N and T such that

$$\int_0^T \int_0^{+\infty} |v_i(t,x)||w_j(t,x)|dxdt \le K_2\Big(V_1(0) + \int_0^{+\infty} |h(t)|dt + \int_0^T \int_0^{+\infty} |F(t,x)|dxdt\Big)$$
$$\times\Big(W_1(0) + \int_0^{+\infty} (|\alpha'(t)| + |h'(t)|)dt + \int_0^T \int_0^{+\infty} |G(t,x)|dxdt\Big),$$
$$\forall\ i \ne j \quad (i,j = 1,\ldots,n), \tag{3.37}$$

provided that the right hand side of the inequality is bounded and $G = (G_1, G_2, \ldots, G_n)$.

Proof. To estimate

$$\int_0^T \int_0^{+\infty} |v_i(t,x)||w_j(t,x)|dxdt, \tag{3.38}$$

it is enough to estimate

$$\int_0^T \int_0^L |v_i(t,x)||w_j(t,x)|dxdt, \tag{3.39}$$

for any given $L > 0$ and then let $L \to +\infty$.

i) For $i, j \in \{1, \ldots, m\}$ and $i \ne j$, without loss of generality, we suppose that $i < j$, passing through point (T, L), we draw the ith backward characteristic $x = x_i(t)$ $(0 \le t \le T)$ which intersects the x-axis at a point $(0, x_{i0})$.

We introduce the "continuous Glimm's functional" (cf. [1, 18])

$$Q(t) = \int\int_{0<x<y<x_i(t)} |w_j(t,x)||v_i(t,y)|dxdy. \tag{3.40}$$

Then, it is easy to see that

$$
\begin{aligned}
\frac{dQ(t)}{dt} &= x_i'(t)|v_i(t,x_i(t))| \int_0^{x_i(t)} |w_j(t,x)|dx \\
&+ \int\int_{0<x<y<x_i(t)} \frac{\partial}{\partial t}(|w_j(t,x)|)|v_i(t,y)|dxdy \\
&+ \int\int_{0<x<y<x_i(t)} |w_j(t,x)|\frac{\partial}{\partial t}(|v_i(t,y)|)dxdy \\
&= x_i'(t)|v_i(t,x_i(t))| \int_0^{x_i(t)} |w_j(t,x)|dx \\
&- \int\int_{0<x<y<x_i(t)} \frac{\partial}{\partial x}(\lambda_j(u)|w_j(t,x)|)|v_i(t,y)|dxdy \\
&- \int\int_{0<x<y<x_i(t)} |w_j(t,x)|\frac{\partial}{\partial y}(\lambda_i(u)|v_i(t,y)|)dxdy \\
&+ \int\int_{0<x<y<x_i(t)} \mathrm{sgn}(w_j)G_j(t,x)|v_i(t,y)|dxdy \\
&+ \int\int_{0<x<y<x_i(t)} |w_j(t,x)|\ \mathrm{sgn}(v_i)F_i(t,y)dxdy \\
&= (x_i'(t) - \lambda_i(u(t,x_i(t))))|v_i(t,x_i(t))| \int_0^{x_i(t)} |w_j(t,x)|dx \\
&+\lambda_j(u(t,0))|w_j(t,0)| \int_0^{x_i(t)} |v_i(t,x)|dx \\
&- \int_0^{x_i(t)} (\lambda_j(u(t,x)) - \lambda_i(u(t,x)))|v_i(t,x)||w_j(t,x)|dx \\
&+ \int\int_{0<x<y<x_i(t)} \mathrm{sgn}(w_j)G_j(t,x)|v_i(t,y)|dxdy \\
&+ \int\int_{0<x<y<x_i(t)} |w_j(t,x)|\ \mathrm{sgn}(v_i)F_i(t,y)dxdy \\
&\le |\lambda_j(u(t,0))||w_j(t,0)| \int_0^{x_i(t)} |v_i(t,x)|dx - \delta_0 \int_0^{x_i(t)} |v_i(t,x)||w_j(t,x)|dx \\
&+ \int_0^{x_i(t)} |G_j(t,x)|dx \int_0^{x_i(t)} |v_i(t,x)|dx \\
&+ \int_0^{x_i(t)} |F_i(t,x)|dx \int_0^{x_i(t)} |w_j(t,x)|dx \\
&\le |\lambda_j(u(t,0))||w_j(t,0)| \int_0^{+\infty} |v_i(t,x)|dx - \delta_0 \int_0^{x_i(t)} |v_i(t,x)||w_j(t,x)|dx \\
&+ \int_0^{+\infty} |G_j(t,x)|dx \int_0^{+\infty} |v_i(t,x)|dx \\
&+ \int_0^{+\infty} |F_i(t,x)|dx \int_0^{+\infty} |w_j(t,x)|dx. \qquad (3.41)
\end{aligned}
$$

It then follows from Lemma 3.3 in Shao [14] and Lemma 3.1 that

$$\frac{dQ(t)}{dt} + \delta_0 \int_0^{x_i(t)} |v_i(t,x)||w_j(t,x)|dx$$

$$\le c_5\Big(|\lambda_j(u(t,0))||w_j(t,0)| + \int_0^{+\infty} |G(t,x)|dx\Big)\Big(V_1(0) + \int_0^{+\infty} |h(t)|dt + \int_0^T \int_0^{+\infty} |F(t,x)|dxdt\Big)$$
$$+c_6 \int_0^{+\infty} |F(t,x)|dx\Big(W_1(0) + \int_0^{+\infty} (|\alpha'(t)| + |h'(t)|)dt + \int_0^T \int_0^{+\infty} |G(t,x)|dxdt\Big). \tag{3.42}$$

Thus, noting (3.6), we have

$$\delta_0 \int_0^T \int_0^{x_i(t)} |v_i(t,x)||w_j(t,x)|dxdt$$

$$\le Q(0)+c_7\Big(\int_0^T |w_j(t,0)|dt+\int_0^T \int_0^{+\infty} |G(t,x)|dxdt\Big)\Big(V_1(0)+\int_0^{+\infty} |h(t)|dt+\int_0^T \int_0^{+\infty} |F(t,x)|dxdt\Big)$$
$$+c_6 \int_0^T \int_0^{+\infty} |F(t,x)|dxdt\Big(W_1(0) + \int_0^{+\infty} (|\alpha'(t)| + |h'(t)|)dt + \int_0^T \int_0^{+\infty} |G(t,x)|dxdt\Big). \tag{3.43}$$

Then, passing through $A(T,0)$, we draw the jth characteristic C_j which intersects the x-axis at a point $B(0,x_B)$. We rewrite (2.24) as

$$d(|w_j(t,x)|(dx - \lambda_j(u)dt)) = \operatorname{sgn}(w_j)G_j dxdt. \tag{3.44}$$

By (3.44), using Stokes' formula on the domain AOB, we have

$$\Big|\int_0^T |w_j(t,0)|(-\lambda_j(u)dt)\Big| \le \int_0^{x_B} |w_j(0,x)|dx + \int\int_{AOB} |G_j|dxdt$$
$$\le W_1(0) + \int_0^T \int_0^{+\infty} |G(t,x)|dxdt. \tag{3.45}$$

Thus, it follows from (3.33) that

$$\int_0^T |w_j(t,0)|dt \le c_8\{W_1(0) + \int_0^T \int_0^{+\infty} |G(t,x)|dxdt\}. \tag{3.46}$$

Then, noting

$$Q(0) \le \int_0^{+\infty} |v_i(0,x)|dx \int_0^{+\infty} |w_j(0,x)|dx, \tag{3.47}$$

it follows from (3.43) and (3.46) that

$$\delta_0 \int_0^T \int_0^{x_i(t)} |v_i(t,x)||w_j(t,x)|dxdt$$
$$\le c_9\Big(V_1(0) + \int_0^{+\infty} |h(t)|dt + \int_0^T \int_0^{+\infty} |F(t,x)|dxdt\Big)$$
$$\times\Big(W_1(0) + \int_0^{+\infty} (|\alpha'(t)| + |h'(t)|)dt + \int_0^T \int_0^{+\infty} |G(t,x)|dxdt\Big). \tag{3.48}$$

It thus follows

$$\int_0^T \int_0^{x_i(t)} |v_i(t,x)||w_j(t,x)|dxdt$$
$$\le c_{10}\Big(V_1(0) + \int_0^{+\infty} |h(t)|dt + \int_0^T \int_0^{+\infty} |F(t,x)|dxdt\Big)$$

$$\times\left(W_1(0)+\int_0^{+\infty}(|\alpha'(t)|+|h'(t)|)dt+\int_0^T\int_0^{+\infty}|G(t,x)|dxdt\right). \tag{3.49}$$

Hence

$$\begin{aligned}&\int_0^T\int_0^L |v_i(t,x)||w_j(t,x)|dxdt\\ &\le c_{10}\left(V_1(0)+\int_0^{+\infty}|h(t)|dt+\int_0^T\int_0^{+\infty}|F(t,x)|dxdt\right)\\ &\times\left(W_1(0)+\int_0^{+\infty}(|\alpha'(t)|+|h'(t)|)dt+\int_0^T\int_0^{+\infty}|G(t,x)|dxdt\right).\end{aligned} \tag{3.50}$$

and the desired conclusion follows by taking $L\to+\infty$.

ii) For $i\in\{m+1,\dots,n\}$ and $j\in\{1,\dots,m\}$, passing through point (T,L), we draw the jth backward characteristic $x=x_j(t)$ $(0\le t\le T)$ which intersects the x-axis at a point $(0,x_{j0})$.

We introduce the "continuous Glimm's functional" (cf. [1, 18])

$$Q(t)=\int\int_{0<y<x<x_j(t)}|w_j(t,x)||v_i(t,y)|dxdy. \tag{3.51}$$

Then, it is easy to see that

$$\begin{aligned}\frac{dQ(t)}{dt}&=x_j'(t)|w_j(t,x_j(t))|\int_0^{x_j(t)}|v_i(t,y)|dy\\ &+\int\int_{0<y<x<x_j(t)}\frac{\partial}{\partial t}(|w_j(t,x)|)|v_i(t,y)|dxdy\\ &+\int\int_{0<y<x<x_j(t)}|w_j(t,x)|\frac{\partial}{\partial t}(|v_i(t,y)|)dxdy\\ &=x_j'(t)|w_j(t,x_j(t))|\int_0^{x_j(t)}|v_i(t,y)|dy\\ &-\int\int_{0<y<x<x_j(t)}\frac{\partial}{\partial x}(\lambda_j(u)|w_j(t,x)|)|v_i(t,y)|dxdy\\ &-\int\int_{0<y<x<x_j(t)}|w_j(t,x)|\frac{\partial}{\partial y}(\lambda_i(u)|v_i(t,y)|)dxdy\\ &+\int\int_{0<y<x<x_j(t)}\mathrm{sgn}(w_j)G_j(t,x)|v_i(t,y)|dxdy\\ &+\int\int_{0<y<x<x_j(t)}|w_j(t,x)|\ \mathrm{sgn}(v_i)F_i(t,y)dxdy\\ &=(x_j'(t)-\lambda_j(u(t,x_j(t))))|w_j(t,x_j(t))|\int_0^{x_j(t)}|v_i(t,y)|dy\\ &+\lambda_i(u(t,0))|v_i(t,0)|\int_0^{x_j(t)}|w_j(t,x)|dx\\ &-\int_0^{x_j(t)}(\lambda_i(u(t,x))-\lambda_j(u(t,x)))|w_j(t,x)||v_i(t,x)|dx\\ &+\int\int_{0<y<x<x_j(t)}\mathrm{sgn}(w_j)G_j(t,x)|v_i(t,y)|dxdy\end{aligned}$$

$$
\begin{aligned}
&+\int\int_{0<y<x<x_j(t)} |w_j(t,x)|\ \mathrm{sgn}(v_i)F_i(t,y)dxdy \\
&\le \lambda_i(u(t,0))|v_i(t,0)|\int_0^{x_j(t)} |w_j(t,x)|dx - \delta_0\int_0^{x_j(t)} |w_j(t,x)||v_i(t,x)|dx \\
&\quad + \int_0^{x_j(t)} |G_j(t,x)|dx \int_0^{x_j(t)} |v_i(t,y)|dy \\
&\quad + \int_0^{x_j(t)} |F_i(t,y)|dy \int_0^{x_j(t)} |w_j(t,x)|dx \\
&\le \lambda_i(u(t,0))|v_i(t,0)|\int_0^{+\infty} |w_j(t,x)|dx - \delta_0\int_0^{x_j(t)} |w_j(t,x)||v_i(t,x)|dx \\
&\quad + \int_0^{+\infty} |G_j(t,x)|dx \int_0^{+\infty} |v_i(t,x)|dx \\
&\quad + \int_0^{+\infty} |F_i(t,x)|dx \int_0^{+\infty} |w_j(t,x)|dx. \qquad (3.52)
\end{aligned}
$$

It then follows from Lemma 3.3 in Shao [14] and Lemma 3.1 that

$$
\begin{aligned}
&\frac{dQ(t)}{dt} + \delta_0\int_0^{x_j(t)} |v_i(t,x)||w_j(t,x)|dx \\
&\le c_{11}\left(\lambda_i(u(t,0))|v_i(t,0)| + \int_0^{+\infty}|F(t,x)|dx\right)\left(W_1(0)+\int_0^{+\infty}(|\alpha'(t)|+|h'(t)|)dt + \int_0^T\int_0^{+\infty}|G(t,x)|dxdt\right) \\
&\quad + c_{12}\int_0^{+\infty}|G(t,x)|dx\left(V_1(0)+\int_0^{+\infty}|h(t)|dt+\int_0^T\int_0^{+\infty}|F(t,x)|dxdt\right). \qquad (3.53)
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\delta_0\int_0^T\int_0^{x_j(t)} |v_i(t,x)||w_j(t,x)|dxdt \\
&\le Q(0)+c_{11}\left(\int_0^T \lambda_i(u(t,0))|v_i(t,0)|dt + \int_0^T\int_0^{+\infty}|F(t,x)|dxdt\right)\left(W_1(0)+\int_0^{+\infty}(|\alpha'(t)|+|h'(t)|)dt + \int_0^T\int_0^{+\infty}|G(t,x)|dxdt\right) \\
&\quad + c_{12}\int_0^T\int_0^{+\infty}|G(t,x)|dxdt\left(V_1(0)+\int_0^{+\infty}|h(t)|dt+\int_0^T\int_0^{+\infty}|F(t,x)|dxdt\right). \qquad (3.54)
\end{aligned}
$$

By exploiting the same arguments as in Lemma 3.1, we can deduce that

$$
\begin{aligned}
\int_0^T \lambda_i(u(t,0))|v_i(t,0)|dt &\le c_{13}\int_0^T |v_i(t,0)|dt \le c_{14}\left\{\sum_{r=1}^m \int_0^T |v_r(t,0)|dt + \int_0^{+\infty}|h(t)|dt\right\} \\
&\le c_{15}\left(V_1(0)+\int_0^{+\infty}|h(t)|dt+\int_0^T\int_0^{+\infty}|F(t,x)|dxdt\right). \qquad (3.55)
\end{aligned}
$$

Then, noting

$$
Q(0) \le \int_0^{+\infty}|v_i(0,x)|dx\int_0^{+\infty}|w_j(0,x)|dx, \qquad (3.56)
$$

it follows from (3.54)-(3.55) that

$$
\delta_0\int_0^T\int_0^{x_j(t)} |v_i(t,x)||w_j(t,x)|dxdt
$$

$$
\begin{aligned}
&\le c_{16}\biggl(V_1(0)+\int_0^{+\infty}|h(t)|dt+\int_0^T\int_0^{+\infty}|F(t,x)|dxdt\biggr)\\
&\times\biggl(W_1(0)+\int_0^{+\infty}(|\alpha'(t)|+|h'(t)|)dt+\int_0^T\int_0^{+\infty}|G(t,x)|dxdt\biggr).
\end{aligned}\tag{3.57}
$$

It thus follows

$$
\begin{aligned}
&\int_0^T\int_0^{x_j(t)}|v_i(t,x)||w_j(t,x)|dxdt\\
&\le c_{17}\biggl(V_1(0)+\int_0^{+\infty}|h(t)|dt+\int_0^T\int_0^{+\infty}|F(t,x)|dxdt\biggr)\\
&\times\biggl(W_1(0)+\int_0^{+\infty}(|\alpha'(t)|+|h'(t)|)dt+\int_0^T\int_0^{+\infty}|G(t,x)|dxdt\biggr).
\end{aligned}\tag{3.58}
$$

Hence

$$
\begin{aligned}
&\int_0^T\int_0^{L}|v_i(t,x)||w_j(t,x)|dxdt\\
&\le c_{17}\biggl(V_1(0)+\int_0^{+\infty}|h(t)|dt+\int_0^T\int_0^{+\infty}|F(t,x)|dxdt\biggr)\\
&\times\biggl(W_1(0)+\int_0^{+\infty}(|\alpha'(t)|+|h'(t)|)dt+\int_0^T\int_0^{+\infty}|G(t,x)|dxdt\biggr).
\end{aligned}\tag{3.59}
$$

and the desired conclusion follows by taking $L\to+\infty$.

iii) For $i,j\in\{m+1,\dots,n\}$ and $i\neq j$, without loss of generality, we suppose that $i<j$. Let $x=x_i(t,L)$ $(0\le t\le T)$ be the ith forward characteristic passing through point $(0,L)$. Then, we draw the ith backward characteristic $x=x_i(t)$ $(0\le t\le T)$ passing through point $(T,a)\,(a>x_i(T,L))$.

We introduce the "continuous Glimm's functional" (cf. [1, 18])

$$
Q(t)=\int\int_{0<x<y<x_i(t)}|w_j(t,x)||v_i(t,y)|dxdy. \tag{3.60}
$$

Then, it is easy to see that

$$
\begin{aligned}
\frac{dQ(t)}{dt}&=x_i'(t)|v_i(t,x_i(t))|\int_0^{x_i(t)}|w_j(t,x)|dx\\
&+\int\int_{0<x<y<x_i(t)}\frac{\partial}{\partial t}(|w_j(t,x)|)|v_i(t,y)|dxdy\\
&+\int\int_{0<x<y<x_i(t)}|w_j(t,x)|\frac{\partial}{\partial t}(|v_i(t,y)|)dxdy\\
&=x_i'(t)|v_i(t,x_i(t))|\int_0^{x_i(t)}|w_j(t,x)|dx\\
&-\int\int_{0<x<y<x_i(t)}\frac{\partial}{\partial x}(\lambda_j(u)|w_j(t,x)|)|v_i(t,y)|dxdy\\
&-\int\int_{0<x<y<x_i(t)}|w_j(t,x)|\frac{\partial}{\partial y}(\lambda_i(u)|v_i(t,y)|)dxdy\\
&+\int\int_{0<x<y<x_i(t)}\operatorname{sgn}(w_j)G_j(t,x)|v_i(t,y)|dxdy\\
&+\int\int_{0<x<y<x_i(t)}|w_j(t,x)|\operatorname{sgn}(v_i)F_i(t,y)dxdy
\end{aligned}
$$

$$
\begin{aligned}
&= (x_i'(t)-\lambda_i(u(t,x_i(t))))|v_i(t,x_i(t))|\int_0^{x_i(t)}|w_j(t,x)|dx\\
&\quad +\lambda_j(u(t,0))|w_j(t,0)|\int_0^{x_i(t)}|v_i(t,x)|dx\\
&\quad -\int_0^{x_i(t)}(\lambda_j(u(t,x))-\lambda_i(u(t,x)))|v_i(t,x)||w_j(t,x)|dx\\
&\quad +\int\int_{0<x<y<x_i(t)}\mathrm{sgn}(w_j)G_j(t,x)|v_i(t,y)|dxdy\\
&\quad +\int\int_{0<x<y<x_i(t)}|w_j(t,x)|\,\mathrm{sgn}(v_i)F_i(t,y)dxdy\\
&\le \lambda_j(u(t,0))|w_j(t,0)|\int_0^{x_i(t)}|v_i(t,x)|dx-\delta_0\int_0^{x_i(t)}|v_i(t,x)||w_j(t,x)|dx\\
&\quad +\int_0^{x_i(t)}|G_j(t,x)|dx\int_0^{x_i(t)}|v_i(t,x)|dx\\
&\quad +\int_0^{x_i(t)}|F_i(t,x)|dx\int_0^{x_i(t)}|w_j(t,x)|dx\\
&\le \lambda_j(u(t,0))|w_j(t,0)|\int_0^{+\infty}|v_i(t,x)|dx-\delta_0\int_0^{x_i(t)}|v_i(t,x)||w_j(t,x)|dx\\
&\quad +\int_0^{+\infty}|G_j(t,x)|dx\int_0^{+\infty}|v_i(t,x)|dx\\
&\quad +\int_0^{+\infty}|F_i(t,x)|dx\int_0^{+\infty}|w_j(t,x)|dx. \qquad (3.61)
\end{aligned}
$$

It then follows from Lemma 3.3 in Shao [14] and Lemma 3.1 that

$$
\begin{aligned}
&\frac{dQ(t)}{dt}+\delta_0\int_0^{x_i(t)}|v_i(t,x)||w_j(t,x)|dx\\
&\le c_{18}\Big(\lambda_j(u(t,0))|w_j(t,0)|+\int_0^{+\infty}|G(t,x)|dx\Big)\Big(V_1(0)+\int_0^{+\infty}|h(t)|dt+\int_0^T\int_0^{+\infty}|F(t,x)|dxdt\Big)\\
&\quad +c_{19}\int_0^{+\infty}|F(t,x)|dx\Big(W_1(0)+\int_0^{+\infty}(|\alpha'(t)|+|h'(t)|)dt+\int_0^T\int_0^{+\infty}|G(t,x)|dxdt\Big). \qquad (3.62)
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\delta_0\int_0^T\int_0^{x_i(t)}|v_i(t,x)||w_j(t,x)|dxdt\\
&\le Q(0)+c_{18}\Big(\int_0^T\lambda_j(u(t,0))|w_j(t,0)|dt+\int_0^T\int_0^{+\infty}|G(t,x)|dxdt\Big)\Big(V_1(0)+\int_0^{+\infty}|h(t)|dt+\int_0^T\int_0^{+\infty}|F(t,x)|dxdt\Big)\\
&\quad +c_{19}\int_0^T\int_0^{+\infty}|F(t,x)|dxdt\Big(W_1(0)+\int_0^{+\infty}(|\alpha'(t)|+|h'(t)|)dt+\int_0^T\int_0^{+\infty}|G(t,x)|dxdt\Big). \qquad (3.63)
\end{aligned}
$$

Noting (3.6), by (3.25) in Shao [14], we have

$$
\int_0^T\lambda_j(u(t,0))|w_j(t,0)|dt\le c_{20}\int_0^T|w_j(t,0)|dt\le c_{20}\Big\{\sum_{r=1}^m\int_0^T|f_{jr}(t,u)w_r(t,0)|dt
$$

$$+\sum_{s=1}^{k}\int_0^T|\overline{f}_{js}(t,u)\alpha_s'(t)|dt+\sum_{l=m+1}^{n}\int_0^T|\widetilde{f}_{jl}(t,u)h_l'(t)|dt\Big\}$$

$$\leq c_{21}\Big\{\sum_{r=1}^{m}\int_0^T|w_r(t,0)|dt+\int_0^{+\infty}(|\alpha'(t)|+|h'(t)|)dt\Big\}. \tag{3.64}$$

Then, passing through $A(T,0)$, we draw the rth characteristic $C_r(r\in\{1,\dots,m\})$ which intersects the x-axis at a point $B(0,x_B)$. We rewrite (2.24) as

$$d(|w_i(t,x)|(dx-\lambda_i(u)dt))=\ \mathrm{sgn}(w_i)G_idxdt. \tag{3.65}$$

By (3.65), using Stokes' formula on the domain AOB, we have

$$\Big|\int_0^T|w_r(t,0)|(-\lambda_r(u)dt)\Big|\leq\int_0^{x_B}|w_r(0,x)|dx+\int\int_{AOB}|G_r|dxdt$$

$$\leq W_1(0)+\int_0^T\int_0^{+\infty}|G(t,x)|dxdt. \tag{3.66}$$

Thus, it follows from (3.33) that

$$\int_0^T|w_r(t,0)|dt\leq c_{22}\{W_1(0)+\int_0^T\int_0^{+\infty}|G(t,x)|dxdt\}. \tag{3.67}$$

Then, noting (3.64), we get

$$\int_0^T\lambda_j(u(t,0))|w_j(t,0)|dt\leq c_{23}\Big(W_1(0)+\int_0^{+\infty}(|\alpha'(t)|+|h'(t)|)dt+\int_0^T\int_0^{+\infty}|G(t,x)|dxdt\Big). \tag{3.68}$$

Thus, noting

$$Q(0)\leq\int_0^{+\infty}|v_i(0,x)|dx\int_0^{+\infty}|w_j(0,x)|dx, \tag{3.69}$$

it follows from (3.63) that

$$\delta_0\int_0^T\int_0^{x_i(t)}|v_i(t,x)||w_j(t,x)|dxdt$$

$$\leq c_{24}\Big(V_1(0)+\int_0^{+\infty}|h(t)|dt+\int_0^T\int_0^{+\infty}|F(t,x)|dxdt\Big)$$

$$\times\Big(W_1(0)+\int_0^{+\infty}(|\alpha'(t)|+|h'(t)|)dt+\int_0^T\int_0^{+\infty}|G(t,x)|dxdt\Big). \tag{3.70}$$

It thus follows

$$\int_0^T\int_0^{x_i(t)}|v_i(t,x)||w_j(t,x)|dxdt$$

$$\leq c_{25}\Big(V_1(0)+\int_0^{+\infty}|h(t)|dt+\int_0^T\int_0^{+\infty}|F(t,x)|dxdt\Big)$$

$$\times\Big(W_1(0)+\int_0^{+\infty}(|\alpha'(t)|+|h'(t)|)dt+\int_0^T\int_0^{+\infty}|G(t,x)|dxdt\Big). \tag{3.71}$$

Hence

$$\int_0^T\int_0^L|v_i(t,x)||w_j(t,x)|dxdt$$

$$\leq c_{25}\Big(V_1(0)+\int_0^{+\infty}|h(t)|dt+\int_0^T\int_0^{+\infty}|F(t,x)|dxdt\Big)$$

$$\times\left(W_1(0)+\int_0^{+\infty}(|\alpha'(t)|+|h'(t)|)dt+\int_0^T\int_0^{+\infty}|G(t,x)|dxdt\right). \tag{3.72}$$

and the desired conclusion follows by taking $L\to+\infty$.

iv) For $i\in\{1,\dots,m\}$ and $j\in\{m+1,\dots,n\}$, passing through the point (T,L), we draw the ith backward characteristic $x=x_i(t)$ $(0\le t\le T)$ which intersects the x-axis at a point $(0,x_{i0})$.

We introduce the "continuous Glimm's functional"

$$Q(t)=\int\int_{0<x<y<x_i(t)}|w_j(t,x)||v_i(t,y)|dxdy. \tag{3.73}$$

By exploiting the same arguments as in (iii), we can deduce that

$$\begin{aligned}
&\int_0^T\int_0^{x_i(t)}|v_i(t,x)||w_j(t,x)|dxdt\\
\le\ & c_{26}\left(V_1(0)+\int_0^{+\infty}|h(t)|dt+\int_0^T\int_0^{+\infty}|F(t,x)|dxdt\right)\\
\times&\left(W_1(0)+\int_0^{+\infty}(|\alpha'(t)|+|h'(t)|)dt+\int_0^T\int_0^{+\infty}|G(t,x)|dxdt\right).
\end{aligned} \tag{3.74}$$

Hence

$$\begin{aligned}
&\int_0^T\int_0^{L}|v_i(t,x)||w_j(t,x)|dxdt\\
\le\ & c_{26}\left(V_1(0)+\int_0^{+\infty}|h(t)|dt+\int_0^T\int_0^{+\infty}|F(t,x)|dxdt\right)\\
\times&\left(W_1(0)+\int_0^{+\infty}(|\alpha'(t)|+|h'(t)|)dt+\int_0^T\int_0^{+\infty}|G(t,x)|dxdt\right).
\end{aligned} \tag{3.75}$$

and the desired conclusion follows by taking $L\to+\infty$. The proof of Lemma 3.2 is finished. □

Lemma 3.3. Under the assumptions of Lemma 3.1, on any given domain of existence $\{(t,x)|0\le t\le T,x\ge0\}$ of the C^1 solution $u=u(t,x)$ to the mixed initial-boundary value problem (1.1) and (1.10)-(1.11), there exists a positive constant K_3 independent of ε, M, N and T such that

$$W_1(T),\widetilde{W}_1(T)\le K_3\varepsilon, \tag{3.76}$$

$$U_\infty(T),V_\infty(T)\le K_3\varepsilon \tag{3.77}$$

and

$$W_\infty(T)\le K_3M. \tag{3.78}$$

The proof can be found in Shao [14].

Lemma 3.4. Under the assumptions of Lemma 3.1, on any given domain of existence $\{(t,x)|0\le t\le T,x\ge0\}$ of the C^1 solution $u=u(t,x)$ to the mixed initial-boundary value problem (1.1) and (1.10)-(1.11), there exists a positive constant K_4 independent of ε, M, N and T such that

$$U_1(T),\widetilde{U}_1(T),\overline{U}_1(T),V_1(T),\widetilde{V}_1(T),\overline{V}_1(T)\le K_4N \tag{3.79}$$

and

$$\overline{W}_1(T)\le K_4\varepsilon. \tag{3.80}$$

Proof. We introduce

$$Q_W(T)=\sum_{j=1}^n\sum_{i\ne j}\int_0^T\int_0^{+\infty}|w_i(t,x)||w_j(t,x)|dxdt \tag{3.81}$$

and let

$$Q_V(T)=\sum_{j=1}^{n}\sum_{i\neq j}\int_0^T\int_0^{+\infty}|v_i(t,x)||w_j(t,x)|dxdt. \tag{3.82}$$

By (2.11) and (2.23), it follows from Lemma 3.2 that

$$\begin{aligned}Q_V(T)\le c_{27}&\bigg(V_1(0)+\int_0^{+\infty}|h(t)|dt+\int_0^T\int_0^{+\infty}|F(t,x)|dxdt\bigg)\\ &\cdot\bigg(W_1(0)+\int_0^{+\infty}(|\alpha'(t)|+|h'(t)|)dt+\int_0^T\int_0^{+\infty}|G(t,x)|dxdt\bigg).\end{aligned} \tag{3.83}$$

Noting (2.26), we have

$$\int_0^T\int_0^{+\infty}|G(t,x)|dxdt\le c_{28}Q_W(T). \tag{3.84}$$

Noting (2.17) and using Hadamard's formula, we obtain

$$\begin{aligned}F_i(t,x)&=\sum_{j,k=1}^{n}B_{ijk}(u)v_jw_k=\sum_{j=1}^{n}\sum_{k\neq j}B_{ijk}(u)v_jw_k+\sum_{j=1}^{n}B_{ijj}(u)v_jw_j\\ &=\sum_{j=1}^{n}\sum_{k\neq j}B_{ijk}(u)v_jw_k+\sum_{j=1}^{n}(B_{ijj}(u)-B_{ijj}(u_je_j))v_jw_j\\ &=\sum_{j=1}^{n}\sum_{k\neq j}B_{ijk}(u)v_jw_k+\sum_{j=1}^{n}\sum_{h\neq j}\bigg(\int_0^1\frac{\partial B_{ijj}(\tau u_1,\dots,\tau u_{j-1},u_j,\tau u_{j+1},\dots,\tau u_n)}{\partial u_h}d\tau\bigg)u_hv_jw_j.\end{aligned} \tag{3.85}$$

By (4.27)-(4.31) in Zhou [18], i.e.,

$$\sum_{h\neq j}|u_h|\le c_{29}\sum_{h\neq j}|v_h|,\ \ \forall\, j\in\{1,\dots,n\}, \tag{3.86}$$

we have

$$|F_i(t,x)|\le c_{30}\sum_{j=1}^{n}\sum_{k\neq j}|v_jw_k|,\quad \forall\, i\in\{1,\dots,n\}. \tag{3.87}$$

Thus, we get

$$\int_0^T\int_0^{+\infty}|F(t,x)|dxdt\le c_{31}Q_V(T). \tag{3.88}$$

Noting (1.18), (1.19), (3.84) and (3.88), using (3.69) in Shao [14], we obtain from (3.83) that

$$\begin{aligned}Q_V(T)&\le c_{32}\bigg(V_1(0)+\int_0^{+\infty}|h(t)|dt+Q_V(T)\bigg)\cdot\bigg(W_1(0)+\int_0^{+\infty}(|\alpha'(t)|+|h'(t)|)dt+Q_W(T)\bigg)\\ &\le c_{33}(N+Q_V(T))(\varepsilon+\varepsilon^2),\end{aligned} \tag{3.89}$$

Therefore

$$Q_V(T)\le c_{34}N\varepsilon. \tag{3.90}$$

We now estimate $V_1(T)$.

By Lemma 3.1, we have

$$\begin{aligned}\int_0^{+\infty}|v_i(t,x)|dx&\le K_1\bigg\{V_1(0)+\int_0^{+\infty}|h(t)|dt+\int_0^T\int_0^{+\infty}|F(t,x)|dxdt\bigg\}\\ &\le c_{35}\bigg\{V_1(0)+\int_0^{+\infty}|h(t)|dt+Q_V(T)\bigg\}\le c_{36}N.\end{aligned} \tag{3.91}$$

Therefore

$$V_1(T) \leq c_{37}N. \tag{3.92}$$

We next estimate $\overline{V}_1(T)$.

To estimate $\overline{V}_1(T)$, we need to estimate

$$\int_{L_j} |v_i(t,x)|dt,$$

where L_j stands for any given ray with the slope $\lambda_j(0)$ on the domain $[0,T] \times \mathbf{R}^+$. Without loss of generality, we assume that L_j intersects the x-axis with point $A(0,\alpha)$, and intersects the line $t = T$ with point B.

i) For $i = 1, \ldots, m$, passing through point B, we draw the ith backward characteristic C_i which intersects the x-axis at a point $C(0,\beta)$. For fixing the idea, suppose that $\alpha < \beta$.

We rewrite (2.12) as

$$d(|v_i(t,x)|(dx - \lambda_i(u)dt)) = \operatorname{sgn}(v_i)F_i dxdt. \tag{3.93}$$

By (3.93), using Stokes' formula on the domain ABC, we have

$$\begin{aligned}\left|\int_{L_j} |v_i(t,x)|(\lambda_j(0) - \lambda_i(u))dt\right| &\leq \int_\alpha^\beta |v_i(0,x)|dx + \int\int_{ABC} |F_i|dxdt \\ &\leq V_1(0) + c_{38}Q_V(T). \end{aligned}\tag{3.94}$$

In the definition of $\overline{V}_1(T)$, $j \neq i$, thus we have from (3.2) that

$$|\lambda_j(0) - \lambda_i(u)| \geq \delta_0. \tag{3.95}$$

Therefore, it follows that

$$\int_{L_j} |v_i(t,x)|dt \leq c_{39}\{V_1(0) + Q_V(T)\}. \tag{3.96}$$

(ii)For $i = m+1, \ldots, n$, we draw the ith backward characteristic C_i passing through point B. Here, there are only two possibilities:

(a)The ith backward characteristic C_i intersects the t-axis at a point $C(\beta,0)$. By (3.93), using Stokes' formula on the domain OABC, we have

$$\begin{aligned}\left|\int_{L_j} |v_i(t,x)|(\lambda_j(0) - \lambda_i(u))dt\right| &\leq \int_0^\alpha |v_i(0,x)|dx + \int_0^\beta |\lambda_i(u(t,0))||v_i(t,0)|dt + \int\int_{OABC} |F_i|dxdt \\ &\leq V_1(0) + c_{40}\left\{\int_0^\beta |v_i(t,0)|dt + Q_V(T)\right\}. \end{aligned}\tag{3.97}$$

Thus, it follows from (3.95) and (3.97) that

$$\int_{L_j} |v_i(t,x)|dt \leq c_{41}\left\{V_1(0) + \int_0^\beta |v_i(t,0)|dt + Q_V(T)\right\}. \tag{3.98}$$

Noting (3.6), by (3.28), we have

$$\int_0^\beta |v_i(t,0)|dt = \sum_{r=1}^m \int_0^\beta |g_{ir}(t)v_r(t,0)|dt + \int_0^\beta |h_i(t)|dt$$

$$\leq c_{42}\Big\{\sum_{r=1}^{m}\int_0^{\beta}|v_r(t,0)|dt+\int_0^{+\infty}|h(t)|dt\Big\}. \tag{3.99}$$

Then, passing through $C(\beta,0)$, we draw the rth characteristic $C_r(r\in\{1,\ldots,m\})$ which intersects the x-axis at point $D(0,x_D)$. By (3.31), using Stokes' formula on the domain COD, we have

$$\begin{aligned}\Big|\int_0^{\beta}|v_r(t,0)|(-\lambda_r(u)dt)\Big| &\leq \int_0^{x_D}|v_r(0,x)|dx+\int\int_{COD}|F_r|dxdt\\ &\leq V_1(0)+c_{43}Q_V(T).\end{aligned} \tag{3.100}$$

Thus, it follows from (3.33) that

$$\int_0^{\beta}|v_r(t,0)|dt\leq c_{44}\{V_1(0)+Q_V(T)\}. \tag{3.101}$$

Then, noting (3.98)-(3.99), we have

$$\int_{L_j}|v_i(t,x)|dt\leq c_{45}\Big\{V_1(0)+\int_0^{+\infty}|h(t)|dt+Q_V(T)\Big\}. \tag{3.102}$$

(b) The ith backward characteristic C_i intersects the x-axis at a point $(0,\beta)$. By exploiting the same arguments as in (i), we can deduce that

$$\int_{L_j}|v_i(t,x)|dt\leq c_{46}\{V_1(0)+Q_V(T)\}. \tag{3.103}$$

Combining (3.96) and (3.102), (3.103), we have

$$\overline{V}_1(T)\leq c_{47}\Big\{V_1(0)+\int_0^{+\infty}|h(t)|dt+Q_V(T)\Big\}\leq c_{48}N. \tag{3.104}$$

Similarly, replacing the ray L_j with the slope $\lambda_j(0)$ by the jth characteristic C_j, we get

$$\widetilde{V}_1(T)\leq c_{49}\Big\{V_1(0)+\int_0^{+\infty}|h(t)|dt+Q_V(T)\Big\}\leq c_{50}N. \tag{3.105}$$

We next estimate $\overline{U}_1(T)$ and $\widetilde{U}_1(T)$.

Noting (2.1), by Hadamard's formula we have

$$\begin{aligned}u_i &= \sum_{k=1}^{n}v_kr_k(u)e_i=v_i+\sum_{k=1}^{n}v_k\big(r_k(u)-r_k(u_ke_k)\big)e_i\\ &= v_i+\sum_{k=1}^{n}v_k\sum_{j\neq k}\Big(\int_0^1\frac{\partial r_k(\tau u_1,\ldots,\tau u_{k-1},u_k,\tau u_{k+1},\ldots,\tau u_n)}{\partial u_j}d\tau\Big)u_je_i\\ &= v_i+\sum_{k=1}^{n}\sum_{j\neq k}\rho_{ijk}(u)u_jv_k,\end{aligned} \tag{3.106}$$

where $\rho_{ijk}(u)$ are all C^1 functions of u, which are defined by

$$\rho_{ijk}(u)=\int_0^1\frac{\partial r_k(\tau u_1,\ldots,\tau u_{k-1},u_k,\tau u_{k+1},\ldots,\tau u_n)}{\partial u_j}e_id\tau,\quad \forall\, j\neq k. \tag{3.107}$$

Integrating (3.106) along the ray L_j with the slope $\lambda_j(0)$, we have

$$\int_{L_j}|u_i(t,x)|dt\leq \overline{V}_1(T)+c_{51}\{U_\infty(T)\overline{V}_1(T)+V_\infty(T)\overline{U}_1(T)\}. \tag{3.108}$$

Noting (3.77) and (3.104), we obtain from (3.108) that

$$\overline{U}_1(T) \le c_{52} N. \tag{3.109}$$

On the other hand, integrating (3.106) along the jth characteristic $C_j (j \neq i)$ gives

$$\int_{C_j} |u_i(t,x)| dt \le \widetilde{V}_1(T) + c_{53}\{U_\infty(T)\widetilde{V}_1(T) + V_\infty(T)\widetilde{U}_1(T)\}. \tag{3.110}$$

Noting (3.77) and (3.105), we get

$$\widetilde{U}_1(T) \le c_{54} N. \tag{3.111}$$

We finally estimate $\overline{W}_1(T)$.

To estimate $\overline{W}_1(T)$, we need to estimate

$$\int_{L_j} |w_i(t,x)| dt,$$

where L_j stands for any given ray with the slope $\lambda_j(0)$ on the domain $[0,T] \times \mathbf{R}^+$. Without loss of generality, we assume that L_j intersects the x-axis with point $A(0,\alpha)$, and intersects the line $t = T$ with point B.

i) For $i = 1, \ldots, m$, passing through point B, we draw the ith backward characteristic C_i which intersects the x-axis at a point $C(0,\beta)$. For fixing the idea, suppose that $\alpha < \beta$.

We rewrite (2.24) as

$$d(|w_i(t,x)|(dx - \lambda_i(u)dt)) = \operatorname{sgn}(w_i) G_i dx dt. \tag{3.112}$$

By (3.112), using Stokes' formula on the domain ABC, we have

$$\begin{aligned} \left| \int_{L_j} |w_i(t,x)|(\lambda_j(0) - \lambda_i(u)) dt \right| &\le \int_\alpha^\beta |w_i(0,x)| dx + \int\int_{ABC} |G_i| dx dt \\ &\le W_1(0) + c_{55} Q_W(T). \end{aligned} \tag{3.113}$$

In the definition of $\widetilde{W_1}(T)$, $j \neq i$, thus we have from (3.2) that

$$|\lambda_j(0) - \lambda_i(u)| \ge \delta_0. \tag{3.114}$$

Thus, it follows that

$$\int_{L_j} |w_i(t,x)| dt \le c_{56}\{W_1(0) + Q_W(T)\}. \tag{3.115}$$

(ii)For $i = m+1, \ldots, n$, we draw the ith backward characteristic C_i passing through point B. Here, there are only two possibilities:

(a)The ith backward characteristic C_i intersects the t-axis at a point $C(\beta, 0)$. By (3.112), using Stokes' formula on the domain OABC, we have

$$\begin{aligned} \left| \int_{L_j} |w_i(t,x)|(\lambda_j(0) - \lambda_i(u)) dt \right| &\le \int_0^\alpha |w_i(0,x)| dx + \int_0^\beta |\lambda_i(u(t,0))||w_i(t,0)| dt + \int\int_{OABC} |G_i| dx dt \\ &\le W_1(0) + c_{57}\left\{ \int_0^\beta |w_i(t,0)| dt + Q_W(T) \right\}. \end{aligned} \tag{3.116}$$

Thus, it follows from (3.114) and (3.116) that

$$\int_{L_j} |w_i(t,x)|dt \le c_{58}\Big\{W_1(0) + \int_0^\beta |w_i(t,0)|dt + Q_W(T)\Big\}. \tag{3.117}$$

Noting (3.6), by (3.25) in Shao [14], we have

$$\int_0^\beta |w_i(t,0)|dt = \sum_{r=1}^m \int_0^\beta |f_{ir}(t,u)w_r(t,0)|dt + \sum_{j=1}^k \int_0^\beta |\overline{f}_{ij}(t,u)\alpha_j'(t)|dt + \sum_{l=m+1}^n \int_0^\beta |\widetilde{f}_{il}(t,u)h_l'(t)|dt$$

$$\le c_{59}\Big\{\sum_{r=1}^m \int_0^\beta |w_r(t,0)|dt + \int_0^{+\infty} (|\alpha'(t)| + |h'(t)|)dt\Big\}. \tag{3.118}$$

Then, passing through $C(\beta,0)$, we draw the rth characteristic $C_r (r \in \{1,\dots,m\})$ which intersects the x-axis at point $D(0,x_D)$. By (3.112), using Stokes' formula on the domain COD, we have

$$\Big|\int_0^\beta |w_r(t,0)|(-\lambda_r(u)dt)\Big| \le \int_0^{x_D} |w_r(0,x)|dx + \int\int_{COD} |G_r|dxdt$$

$$\le W_1(0) + c_{60}Q_W(T). \tag{3.119}$$

Thus, it follows from (3.33) that

$$\int_0^\beta |w_r(t,0)|dt \le c_{61}\{W_1(0) + Q_W(T)\}. \tag{3.120}$$

Then, noting (3.117) and (3.118), we get

$$\int_{L_j} |w_i(t,x)|dt \le c_{62}\Big\{W_1(0) + \int_0^{+\infty} (|\alpha'(t)| + |h'(t)|)dt + Q_W(T)\Big\}. \tag{3.121}$$

(b) The ith backward characteristic C_i intersects the x-axis at a point $(0,\beta)$. By exploiting the same arguments as in (i), we can deduce that

$$\int_{L_j} |w_i(t,x)|dt \le c_{63}\{W_1(0) + Q_W(T)\}. \tag{3.122}$$

Combining (3.115) and (3.121), (3.122), we have

$$\overline{W}_1(T) \le c_{64}\Big\{W_1(0) + \int_0^{+\infty} (|\alpha'(t)| + |h'(t)|)dt + Q_W(T)\Big\} \le c_{65}\varepsilon. \tag{3.123}$$

On the other hand, using (3.91) and noting (2.4) and (3.6), we have

$$U_1(T) \le c_{66}N. \tag{3.124}$$

Taking K_3 suitably large and noting (3.92), (3.104)-(3.107) and (3.123)-(3.124), we obtain (3.79)-(3.80) immediately. Thus, the proof of Lemma 3.4 is finished. □

By Theorem 1.1 in Shao [14], combining Lemmas 3.3 and 3.4 gives

Lemma 3.5. Under the assumptions of Theorem 1.1, there exists a positive constant K_5 independent of ε, M and N such that

$$U_1(\infty), \widetilde{U}_1(\infty), \overline{U}_1(\infty), V_1(\infty), \widetilde{V}_1(\infty), \overline{V}_1(\infty) \le K_5 N, \tag{3.125}$$

$$W_1(\infty), \widetilde{W}_1(\infty), \overline{W}_1(\infty) \le K_5\varepsilon, \tag{3.126}$$

$$U_\infty(\infty), V_\infty(\infty) \le K_5\varepsilon, \tag{3.127}$$

$$W_\infty(\infty) \le K_5 M, \tag{3.128}$$

where

$$V_1(\infty) = \sup_{t\in\mathbf{R}^+}\int_0^{+\infty}|v(t,x)|dx, \tag{3.129}$$

etc.

Lemma 3.6. Under the assumptions of Theorem 1.1, there exists a positive constant K_6 independent of ε, M, N, t, α and β such that for arbitrary $\alpha,\beta\in\mathbf{R}^+$ and $(t,x_i(t,\alpha)),(t,x_i(t,\beta))\in\mathbf{R}^+\times\mathbf{R}^+$,

$$|u(t,x_i(t,\alpha)) - u(t,x_i(t,\beta))| \le K_6 M|\alpha-\beta|; \tag{3.130}$$

moreover, for any given C^1 function $g(u)$,

$$|g(u(t,x_i(t,\alpha))) - g(u(t,x_i(t,\beta)))| \le K_6 M|\alpha-\beta|, \tag{3.131}$$

where for arbitrary $\alpha\in\mathbf{R}^+$, $x=x_i(t,\alpha)$ stands for the ith characteristic passing through the point $(0,\alpha)$.

Proof. For fixing the idea we assume that $\alpha\le\beta$. Since the solution $u=u(t,x)$ is classical, i.e., $u\in C^1(\mathbf{R}^+\times\mathbf{R}^+)$, using Taylor's formula and noting (2.5) (3.6) and (3.128), we obtain

$$\begin{aligned}|u(t,x_i(t,\alpha)) - u(t,x_i(t,\beta))| &\le \sup_{x\in\mathbf{R}^+}\{|u_x(t,x)|\}\times\sup_{\xi\in\mathbf{R}^+}\left\{\Big|\frac{\partial x_i(t,\xi)}{\partial\xi}\Big|\right\}\times|\alpha-\beta|\\ &\le c_1W_\infty(t)\times\sup_{\xi\in\mathbf{R}^+}\left\{\Big|\frac{\partial x_i(t,\xi)}{\partial\xi}\Big|\right\}\times|\alpha-\beta|\\ &\le c_2M|\alpha-\beta|\times\sup_{\xi\in\mathbf{R}^+}\left\{\Big|\frac{\partial x_i(t,\xi)}{\partial\xi}\Big|\right\},\end{aligned} \tag{3.132}$$

where here and henceforth, $c_i(i=1,2,\dots)$ will denote positive constants independent of ε, M,N,t,α and β.

Noting (1.8) and (2.5), we have

$$\nabla\lambda_i(u)u_x = \nabla\lambda_i(u)\sum_{j=1}^n w_jr_j(u) = \sum_{j\ne i}[\nabla\lambda_i(u)r_j(u)]w_j. \tag{3.133}$$

Then, noting (3.126), we obtain

$$\int_0^t|(\nabla\lambda_i(u)u_x)(s,x_i(s,\xi))|ds \le c_3\widetilde{W}_1(t)\le c_4\varepsilon. \tag{3.134}$$

By (4.46) in Dai and Kong [4], i.e.,

$$\frac{\partial x_i(t,\xi)}{\partial\xi} = \mathbf{exp}\left\{\int_{\mathbf{0}}^{\mathbf{t}}(\nabla\lambda_{\mathbf{i}}(\mathbf{u})\mathbf{u_x})(\mathbf{s},\mathbf{x_i}(\mathbf{s},\xi))\mathbf{ds}\right\}. \tag{3.135}$$

we have

$$\sup_{(t,\xi)\in\mathbf{R}^+\times\mathbf{R}^+}\left\{\Big|\frac{\partial x_i(t,\xi)}{\partial\xi}\Big|\right\}\le e^{c_4\varepsilon}. \tag{3.136}$$

Substituting (3.136) into (3.132) yields (3.130) immediately. Finally, noting (3.130) we get (3.131) by Taylor's formula. The proof of Lemma 3.6 is finished. □

Similarly, we can prove the following lemma.

Lemma 3.7. Under the assumptions of Theorem 1.1, there exists a positive constant K_7 independent of ε, M, N, t, α and β such that for arbitrary $\alpha, \beta \in \mathbf{R}^-$ and for $i \in \{m+1, \ldots, n\}$,

$$|u(t, x_i(t, \alpha)) - u(t, x_i(t, \beta))| \le K_7 M |\alpha - \beta|; \tag{3.137}$$

moreover, for any given C^1 function $g(u)$,

$$|g(u(t, x_i(t, \alpha))) - g(u(t, x_i(t, \beta)))| \le K_7 M |\alpha - \beta|, \tag{3.138}$$

where for arbitrary $\alpha \in \mathbf{R}^-$, $x = x_i(t, \alpha) (i = m+1, \ldots, n)$ stands for the ith characteristic passing through the point $(\frac{\alpha}{-\lambda_i(0)}, 0)$.

For any fixed $T \ge 0$ and for arbitrary $\alpha, \beta \in \mathbf{R}^+$, we introduce

$$U_\alpha^\beta(T) = \max_{i=1,\ldots,n} \max_{j \ne i} \int_0^T |u_i(t, x_j(t, \alpha)) - u_i(t, x_j(t, \beta))| dt, \tag{3.139}$$

$$V_\alpha^\beta(T) = \max_{i=1,\ldots,n} \max_{j \ne i} \int_0^T |v_i(t, x_j(t, \alpha)) - v_i(t, x_j(t, \beta))| dt, \tag{3.140}$$

$$W_\alpha^\beta(T) = \max_{i=1,\ldots,n} \max_{j \ne i} \int_0^T |w_i(t, x_j(t, \alpha)) - w_i(t, x_j(t, \beta))| dt, \tag{3.141}$$

where for arbitrary $\alpha \in \mathbf{R}^+$, $x = x_j(t, \alpha)$ stands for any given jth characteristic passing through the point $(0, \alpha)$.

Lemma 3.8. Under the assumptions of Theorem 1.1, there exists a positive constant K_8 independent of ε, M, N, T, α and β such that

$$U_\alpha^\beta(T) \le K_8(MN + \varepsilon)|\alpha - \beta|, \tag{3.142}$$

$$V_\alpha^\beta(T) \le K_8(MN + \varepsilon)|\alpha - \beta| \tag{3.143}$$

and

$$W_\alpha^\beta(T) \le K_8 M(1 + \varepsilon)|\alpha - \beta|. \tag{3.144}$$

Proof. We first prove (3.143).

For arbitrary $\alpha, \beta \in \mathbf{R}^+$, let $C_j(\alpha)$ and $C_j(\beta)$ be the jth characteristics passing through the points $P_1 : (0, \alpha)$ and $P_2 : (0, \beta)$, respectively. For the sake of simplicity, we assume that $\alpha < \beta$. We denote by $P_4 : (T, x_j(T, \alpha))$ (respectively $P_3 : (T, x_j(T, \beta))$) the the intersection point of $C_j(\alpha)$ (respectively $C_j(\beta)$) with the straight line $t = T$.

We rewrite (2.12) as

$$d[\xi(t) v_i(dx - \lambda_i(u)dt)] = \xi(t) F_i(t, x) dx dt, \quad a.e., \tag{3.145}$$

where

$$\xi(t) = \operatorname{sgn}[(v_i(t, x_j(t, \alpha)) - v_i(t, x_j(t, \beta)))(\lambda_j(u)(t, x_j(t, \beta)) - \lambda_i(u)(t, x_j(t, \alpha)))].$$

By (3.145), using Green formula on the domain $P_1P_2P_3P_4$ bounded by the curves $C_j(\alpha)$, $C_j(\beta)$, the x-axis and the straight line $t = T$, we have (cf. [4])

$$\int\int_{P_1P_2P_3P_4} \xi(t) F_i(t, x) dt dx = \int_\alpha^\beta \xi(0) v_i(0, x) dx + \int_0^T \xi(t)[v_i(\lambda_j(u) - \lambda_i(u))](t, x_j(t, \beta)) dt$$

$$-\int_\alpha^\beta \xi(T)v_i(T,x_j(T,\gamma))\frac{\partial x_j(T,\gamma)}{\partial\gamma}d\gamma-\int_0^T\xi(t)[v_i(\lambda_j(u)-\lambda_i(u))](t,x_j(t,\alpha))dt,$$

i.e.,

$$\begin{aligned}&\int_0^T|(v_i(t,x_j(t,\alpha))-v_i(t,x_j(t,\beta)))(\lambda_j(u)(t,x_j(t,\beta))-\lambda_i(u)(t,x_j(t,\alpha)))|dt\\
&=\int_0^T\xi(t)v_i(t,x_j(t,\beta))[\lambda_i(u)(t,x_j(t,\alpha))-\lambda_i(u)(t,x_j(t,\beta))]dt\\
&-\int_0^T\xi(t)v_i(t,x_j(t,\alpha))[\lambda_j(u)(t,x_j(t,\alpha))-\lambda_j(u)(t,x_j(t,\beta))]dt\\
&+\int_\alpha^\beta[\xi(0)v_i(0,\gamma)-\xi(T)v_i(T,x_j(T,\gamma))\frac{\partial x_j(T,\gamma)}{\partial\gamma}]d\gamma-\int\int_{P_1P_2P_3P_4}\xi(t)F_i(t,x)dtdx.\end{aligned}\tag{3.146}$$

In the definition of $\widetilde{V}_\alpha^\beta(T)$, $j\neq i$, thus we have from (3.2) that

$$|\lambda_j(u)(t,x_j(t,\beta))-\lambda_i(u)(t,x_j(t,\alpha))|\geq\delta_0.\tag{3.147}$$

Therefore, noting (3.136), it follows from Lemma 3.5 and Lemma 3.6 that

$$\begin{aligned}&\int_0^T|v_i(t,x_j(t,\alpha))-v_i(t,x_j(t,\beta))|dt\\
&\leq\frac{1}{\delta_0}\left\{[c_5V_\infty(T)+2K_6M\widetilde{V}_1(T)]|\alpha-\beta|+\int\int_{P_1P_2P_3P_4}|F_i(t,x)|dtdx\right\}\\
&\leq c_6\left\{(MN+\varepsilon)|\alpha-\beta|+\int\int_{P_1P_2P_3P_4}|F_i(t,x)|dtdx\right\}.\end{aligned}\tag{3.148}$$

On the other hand, noting (3.87) and using (3.136) and (3.76)-(3.79), we have

$$\begin{aligned}&\int\int_{P_1P_2P_3P_4}|F_i(t,x)|dtdx\leq c_7\sum_{l=1}^n\sum_{k\neq l}\int\int_{P_1P_2P_3P_4}|v_lw_k|dtdx\\
&=c_7\sum_{l=1}^n\sum_{k\neq l}\int_0^T dt\int_{x_j(t,\alpha)}^{x_j(t,\beta)}|v_lw_k|dx\leq c_8\sum_{l=1}^n\sum_{k\neq l}\int_\alpha^\beta d\gamma\int_0^T|v_lw_k|(t,x_j(t,\gamma))dt\\
&\leq c_9\{V_\infty(T)\widetilde{W}_1(T)+W_\infty(T)\widetilde{V}_1(T)\}|\alpha-\beta|\leq c_{10}(MN+\varepsilon^2)|\alpha-\beta|.\end{aligned}\tag{3.149}$$

Substituting (3.149) into (3.148) gives

$$\int_0^T|v_i(t,x_j(t,\alpha))-v_i(t,x_j(t,\beta))|dt\leq c_{11}(MN+\varepsilon)|\alpha-\beta|,\quad\forall\, j\neq i.\tag{3.150}$$

This proves (3.143).

We next prove (3.144).

We rewrite (2.24) as

$$d[\xi(t)w_i(dx-\lambda_i(u)dt)]=\xi(t)G_i(t,x)dxdt,\quad a.e.,\tag{3.151}$$

where

$$\xi(t)=\ \mathrm{sgn}[(w_i(t,x_j(t,\alpha))-w_i(t,x_j(t,\beta)))(\lambda_j(u)(t,x_j(t,\beta))-\lambda_i(u)(t,x_j(t,\alpha)))].$$

By (3.151), using Green formula on the domain $P_1P_2P_3P_4$ bounded by the curves $C_j(\alpha)$, $C_j(\beta)$, the x-axis and the straight line $t = T$, we have (cf. [4])

$$\int_0^T |(w_i(t,x_j(t,\alpha)) - w_i(t,x_j(t,\beta)))(\lambda_j(u)(t,x_j(t,\beta)) - \lambda_i(u)(t,x_j(t,\alpha)))|dt$$

$$= \int_0^T \xi(t)w_i(t,x_j(t,\beta))[\lambda_i(u)(t,x_j(t,\alpha)) - \lambda_i(u)(t,x_j(t,\beta))]dt$$

$$- \int_0^T \xi(t)w_i(t,x_j(t,\alpha))[\lambda_j(u)(t,x_j(t,\alpha)) - \lambda_j(u)(t,x_j(t,\beta))]dt$$

$$+ \int_\alpha^\beta [\xi(0)w_i(0,\gamma) - \xi(T)w_i(T,x_j(T,\gamma))\frac{\partial x_j(T,\gamma)}{\partial\gamma}]d\gamma - \int\int_{P_1P_2P_3P_4} \xi(t)G_i(t,x)dtdx. \quad (3.152)$$

In the definition of $\widetilde{W}_\alpha^\beta(T)$, $j \neq i$, thus we have from (3.2) that

$$|\lambda_j(u)(t,x_j(t,\beta)) - \lambda_i(u)(t,x_j(t,\alpha))| \geq \delta_0. \quad (3.153)$$

Therefore, noting (3.136), it follows from Lemmas 3.5 and 3.6 that

$$\int_0^T |w_i(t,x_j(t,\alpha)) - w_i(t,x_j(t,\beta))|dt$$

$$\leq \frac{1}{\delta_0}\left\{[c_5W_\infty(T) + 2K_6M\widetilde{W}_1(T)]|\alpha-\beta| + \int\int_{P_1P_2P_3P_4} |G_i(t,x)|dtdx\right\}$$

$$\leq c_{12}\left\{M(1+\varepsilon)|\alpha-\beta| + \int\int_{P_1P_2P_3P_4} |G_i(t,x)|dtdx\right\}. \quad (3.154)$$

On the other hand, noting (2.26) and using (3.136) and Lemma 3.3, we have

$$\int\int_{P_1P_2P_3P_4} |G_i(t,x)|dtdx \leq c_{13}\sum_{l=1}^n\sum_{k\neq l}\int\int_{P_1P_2P_3P_4} |w_lw_k|dtdx$$

$$= c_{13}\sum_{l=1}^n\sum_{k\neq l}\int_0^T dt\int_{x_j(t,\alpha)}^{x_j(t,\beta)} |w_lw_k|dx \leq c_{14}\sum_{l=1}^n\sum_{k\neq l}\int_\alpha^\beta d\gamma\int_0^T |w_lw_k|(t,x_j(t,\gamma))dt$$

$$\leq c_{15}W_\infty(T)\widetilde{W}_1(T)|\alpha-\beta| \leq c_{16}M\varepsilon|\alpha-\beta|. \quad (3.155)$$

Substituting (3.155) into (3.154) gives

$$\int_0^T |w_i(t,x_j(t,\alpha)) - w_i(t,x_j(t,\beta))|dt \leq c_{17}M(1+\varepsilon)|\alpha-\beta|, \quad \forall\, j \neq i. \quad (3.156)$$

This proves (3.144).

We finally prove (3.142).

We rewrite (3.106) as

$$\xi(t)u_i = \xi(t)v_i + \xi(t)\sum_{k=1}^n\sum_{l\neq k}\rho_{ilk}(u)u_lv_k, \quad a.e., \quad (3.157)$$

where $\rho_{ilk}(u)$ is defined by (3.107) and

$$\xi(t) = \operatorname{sgn}[u_i(t,x_j(t,\alpha)) - u_i(t,x_j(t,\beta))].$$

Integrating (3.157) from 0 to T along the characteristics $C_j(\alpha)$: $x = x_j(t,\alpha)$ and $C_j(\beta)$: $x = x_j(t,\beta)$, respectively, and subtracting the last integral from the first integral gives (cf. [4])

$$\begin{aligned}\int_0^T |u_i(t,x_j(t,\alpha)) - u_i(t,x_j(t,\beta))|dt &= \int_0^T \xi(t)[v_i(t,x_j(t,\alpha)) - v_i(t,x_j(t,\beta))]dt\\
&+\sum_{k=1}^n\sum_{l\neq k}\int_0^T \xi(t)\{[\rho_{ilk}(u)u_lv_k](t,x_j(t,\alpha)) - [\rho_{ilk}(u)u_lv_k](t,x_j(t,\beta))\}dt\\
&= \int_0^T \xi(t)[v_i(t,x_j(t,\alpha)) - v_i(t,x_j(t,\beta))]dt\\
&+\sum_{k=1}^n\sum_{l\neq k}\int_0^T \xi(t)\{\rho_{ilk}(u)(t,x_j(t,\alpha)) - \rho_{ilk}(u)(t,x_j(t,\beta))\}[u_lv_k](t,x_j(t,\alpha))dt\\
&+\sum_{k=1}^n\sum_{l\neq k}\int_0^T \xi(t)\rho_{ilk}(u)(t,x_j(t,\beta))\{u_l(t,x_j(t,\alpha)) - u_l(t,x_j(t,\beta))\}v_k(t,x_j(t,\alpha))dt\\
&+\sum_{k=1}^n\sum_{l\neq k}\int_0^T \xi(t)[\rho_{ilk}(u)u_l](t,x_j(t,\beta))\{v_k(t,x_j(t,\alpha)) - v_k(t,x_j(t,\beta))\}dt. \qquad (3.158)\end{aligned}$$

Thus, noting (3.6) and using (1.12), (3.130) and (3.131), we obtain

$$\begin{aligned}\int_0^T |u_i(t,x_j(t,\alpha))-u_i(t,x_j(t,\beta))|dt \le V_\alpha^\beta(T)+c_{18}\{K_6M|\alpha-\beta|[U_\infty(T)\widetilde{V}_1(T)+V_\infty(T)\widetilde{U}_1(T)]+K_6M|\alpha-\beta|\widetilde{V}_1(T)\\
+U_\alpha^\beta(T)V_\infty(T) + V_\alpha^\beta(T)U_\infty(T) + K_6M|\alpha-\beta|\widetilde{U}_1(T)\}, \quad \forall\, j\neq i. \qquad (3.159)\end{aligned}$$

Then, using Lemma 3.5 and (3.143), we have

$$\int_0^T |u_i(t,x_j(t,\alpha)) - u_i(t,x_j(t,\beta))|dt \le c_{19}(MN+\varepsilon)|\alpha-\beta| + c_{20}K_5\varepsilon U_\alpha^\beta(T), \quad \forall\, j\neq i. \qquad (3.160)$$

It follows that

$$U_\alpha^\beta(T) \le c_{19}(MN+\varepsilon)|\alpha-\beta| + c_{20}K_5\varepsilon U_\alpha^\beta(T). \qquad (3.161)$$

This implies (3.142). The proof of Lemma 3.8 is finished. □

For arbitrary $\alpha,\beta\in\mathbf{R}^-$ and for any fixed $T \ge \max\limits_{j=m+1,\dots,n}\frac{\min\{\alpha,\beta\}}{-\lambda_j(0)}$, we introduce

$$\widetilde{U}_\alpha^\beta(T) = \max\left\{\max_{i=1,\dots,m}\max_{j=m+1,\dots,n}\int_{\frac{\min\{\alpha,\beta\}}{-\lambda_j(0)}}^T |u_i(t,x_j(t,\alpha)) - u_i(t,x_j(t,\beta))|dt,\right.$$
$$\left.\max_{\substack{i,j=m+1,\dots,n\\ j\neq i}}\int_{\frac{\min\{\alpha,\beta\}}{-\lambda_j(0)}}^T |u_i(t,x_j(t,\alpha)) - u_i(t,x_j(t,\beta))|dt\right\}, \qquad (3.162)$$

$$\widetilde{V}_\alpha^\beta(T) = \max\left\{\max_{i=1,\dots,m}\max_{j=m+1,\dots,n}\int_{\frac{\min\{\alpha,\beta\}}{-\lambda_j(0)}}^T |v_i(t,x_j(t,\alpha)) - v_i(t,x_j(t,\beta))|dt,\right.$$
$$\left.\max_{\substack{i,j=m+1,\dots,n\\ j\neq i}}\int_{\frac{\min\{\alpha,\beta\}}{-\lambda_j(0)}}^T |v_i(t,x_j(t,\alpha)) - v_i(t,x_j(t,\beta))|dt\right\}, \qquad (3.163)$$

$$\widetilde{W}_\alpha^\beta(T) = \max\left\{\max_{i=1,\dots,m}\max_{j=m+1,\dots,n}\int_{\frac{\min\{\alpha,\beta\}}{-\lambda_j(0)}}^T |w_i(t,x_j(t,\alpha)) - w_i(t,x_j(t,\beta))|dt,\right.$$

$$\max_{\substack{i,j=m+1,\dots,n\\ j\neq i}} \int_{\frac{\min\{\alpha,\beta\}}{-\lambda_j(0)}}^{T} |w_i(t,x_j(t,\alpha)) - w_i(t,x_j(t,\beta))|dt\Big\}, \tag{3.164}$$

where for arbitrary $\alpha \in \mathbf{R}^-$, $x = x_j(t,\alpha)(j = m+1,\dots,n)$ stands for any given jth characteristic passing through the point $(\frac{\alpha}{-\lambda_j(0)}, 0)$.

Lemma 3.9. Under the assumptions of Theorem 1.1, there exists a positive constant K_9 independent of ε, M, N, T, α and β such that

$$\widetilde{U}_\alpha^\beta(T) \le K_9(MN+\varepsilon)|\alpha-\beta|, \tag{3.165}$$

$$\widetilde{V}_\alpha^\beta(T) \le K_9(MN+\varepsilon)|\alpha-\beta| \tag{3.166}$$

and

$$\widetilde{W}_\alpha^\beta(T) \le K_9 M(1+\varepsilon)|\alpha-\beta|. \tag{3.167}$$

Proof. For arbitrary $\alpha,\beta \in \mathbf{R}^-$ and for $j \in \{m+1,\dots,n\}$, let $C_j(\alpha)$ and $C_j(\beta)$ be the jth characteristics passing through the points $(\frac{\alpha}{-\lambda_j(0)}, 0)$ and $(\frac{\beta}{-\lambda_j(0)}, 0)$, respectively. For the sake of simplicity, we assume that $\alpha < \beta$. We assume that $C_j(\alpha)$ (respectively $C_j(\beta)$) intersects the straight line $t = \frac{\alpha}{-\lambda_j(0)}$ with point $P_1 : (\frac{\alpha}{-\lambda_j(0)}, 0)$ (respectively $P_2 : (\frac{\alpha}{-\lambda_j(0)}, x_j(\frac{\alpha}{-\lambda_j(0)}, \beta))$), and intersects the straight line $t = T$ with point $P_4 : (T, x_j(T,\alpha))$ (respectively $P_3 : (T, x_j(T,\beta))$).

We first prove (3.166).

By (3.145), using Green formula on the domain $P_1P_2P_3P_4$ bounded by the curves $C_j(\alpha)$, $C_j(\beta)$, the straight lines $t = \frac{\alpha}{-\lambda_j(0)}$ and $t = T$, we have (cf. [4])

$$\begin{aligned}
\int\int_{P_1P_2P_3P_4} \xi(t)F_i(t,x)dtdx = &\int_\alpha^\beta \xi(\frac{\alpha}{-\lambda_j(0)})v_i(\frac{\alpha}{-\lambda_j(0)}, x_j(\frac{\alpha}{-\lambda_j(0)},\gamma))\frac{\partial x_j(\frac{\alpha}{-\lambda_j(0)},\gamma)}{\partial\gamma}d\gamma\\
&+\int_{\frac{\alpha}{-\lambda_j(0)}}^T \xi(t)[v_i(\lambda_j(u)-\lambda_i(u))](t,x_j(t,\beta))dt\\
&-\int_\alpha^\beta \xi(T)v_i(T,x_j(T,\gamma))\frac{\partial x_j(T,\gamma)}{\partial\gamma}d\gamma\\
&-\int_{\frac{\alpha}{-\lambda_j(0)}}^T \xi(t)[v_i(\lambda_j(u)-\lambda_i(u))](t,x_j(t,\alpha))dt,
\end{aligned}$$

i.e.,

$$\begin{aligned}
&\int_{\frac{\alpha}{-\lambda_j(0)}}^T |(v_i(t,x_j(t,\alpha)) - v_i(t,x_j(t,\beta)))(\lambda_j(u)(t,x_j(t,\beta)) - \lambda_i(u)(t,x_j(t,\alpha)))|dt\\
&= \int_{\frac{\alpha}{-\lambda_j(0)}}^T \xi(t)v_i(t,x_j(t,\beta))[\lambda_i(u)(t,x_j(t,\alpha)) - \lambda_i(u)(t,x_j(t,\beta))]dt\\
&- \int_{\frac{\alpha}{-\lambda_j(0)}}^T \xi(t)v_i(t,x_j(t,\alpha))[\lambda_j(u)(t,x_j(t,\alpha)) - \lambda_j(u)(t,x_j(t,\beta))]dt\\
&+ \int_\alpha^\beta \xi(\frac{\alpha}{-\lambda_j(0)})v_i(\frac{\alpha}{-\lambda_j(0)}, x_j(\frac{\alpha}{-\lambda_j(0)},\gamma))\frac{\partial x_j(\frac{\alpha}{-\lambda_j(0)},\gamma)}{\partial\gamma}d\gamma\\
&- \int_\alpha^\beta \xi(T)v_i(T,x_j(T,\gamma))\frac{\partial x_j(T,\gamma)}{\partial\gamma}d\gamma - \int\int_{P_1P_2P_3P_4} \xi(t)F_i(t,x)dtdx.
\end{aligned} \tag{3.168}$$

In the definition of $\widetilde{V}_\alpha^\beta(T)$, $j \neq i$, thus we have from (3.2) that

$$|\lambda_j(u)(t, x_j(t, \beta)) - \lambda_i(u)(t, x_j(t, \alpha))| \geq \delta_0. \tag{3.169}$$

Therefore, noting (3.136), it follows from Lemma 3.5 and Lemma 3.7 that

$$\int_{\frac{\alpha}{-\lambda_j(0)}}^T |v_i(t, x_j(t, \alpha)) - v_i(t, x_j(t, \beta))| dt$$

$$\leq \frac{1}{\delta_0}\left\{[c_{21} V_\infty(T) + 2K_6 M \widetilde{V}_1(T)]|\alpha - \beta| + \int\int_{P_1P_2P_3P_4} |F_i(t,x)| dt dx\right\}$$

$$\leq c_{22}\left\{(MN + \varepsilon)|\alpha - \beta| + \int\int_{P_1P_2P_3P_4} |F_i(t,x)| dt dx\right\}. \tag{3.170}$$

On the other hand, noting (3.87) and using (3.136) and (3.76)-(3.79), we have

$$\int\int_{P_1P_2P_3P_4} |F_i(t,x)| dt dx \leq c_{23} \sum_{l=1}^n \sum_{k \neq l} \int\int_{P_1P_2P_3P_4} |v_l w_k| dt dx$$

$$= c_{23} \sum_{l=1}^n \sum_{k\neq l} \int_{\frac{\alpha}{-\lambda_j(0)}}^T dt \int_{x_j(t,\alpha)}^{x_j(t,\beta)} |v_l w_k| dx \leq c_{24} \sum_{l=1}^n \sum_{k \neq l} \int_\alpha^\beta d\gamma \int_{\frac{\alpha}{-\lambda_j(0)}}^T |v_l w_k|(t, x_j(t,\gamma)) dt$$

$$\leq c_{25}\{V_\infty(T)\widetilde{W}_1(T) + W_\infty(T)\widetilde{V}_1(T)\}|\alpha - \beta| \leq c_{26}(MN + \varepsilon^2)|\alpha - \beta|. \tag{3.171}$$

Substituting (3.171) into (3.170) gives

$$\int_{\frac{\alpha}{-\lambda_j(0)}}^T |v_i(t, x_j(t,\alpha)) - v_i(t, x_j(t, \beta))| dt \leq c_{27}(MN + \varepsilon)|\alpha - \beta|,$$

$$\text{if } i \in \{1, \dots, m\}, j \in \{m+1, \dots, n\} \text{ or } i, j \in \{m+1, \dots, n\},\ j \neq i. \tag{3.172}$$

This proves (3.166).

We next prove (3.167).

By (3.151), using Green formula on the domain $P_1P_2P_3P_4$ bounded by the curves $C_j(\alpha)$, $C_j(\beta)$, the straight lines $t = \frac{\alpha}{-\lambda_j(0)}$ and $t = T$, we have (cf. [4])

$$\int_{\frac{\alpha}{-\lambda_j(0)}}^T |(w_i(t, x_j(t,\alpha)) - w_i(t, x_j(t,\beta)))(\lambda_j(u)(t, x_j(t,\beta)) - \lambda_i(u)(t, x_j(t,\alpha)))| dt$$

$$= \int_{\frac{\alpha}{-\lambda_j(0)}}^T \xi(t) w_i(t, x_j(t,\beta))[\lambda_i(u)(t, x_j(t,\alpha)) - \lambda_i(u)(t, x_j(t,\beta))] dt$$

$$- \int_{\frac{\alpha}{-\lambda_j(0)}}^T \xi(t) w_i(t, x_j(t,\alpha))[\lambda_j(u)(t, x_j(t,\alpha)) - \lambda_j(u)(t, x_j(t,\beta))] dt$$

$$+ \int_\alpha^\beta \xi(\frac{\alpha}{-\lambda_j(0)}) w_i(\frac{\alpha}{-\lambda_j(0)}, x_j(\frac{\alpha}{-\lambda_j(0)}, \gamma)) \frac{\partial x_j(\frac{\alpha}{-\lambda_j(0)}, \gamma)}{\partial \gamma} d\gamma$$

$$- \int_\alpha^\beta \xi(T) w_i(T, x_j(T,\gamma)) \frac{\partial x_j(T,\gamma)}{\partial\gamma} d\gamma - \int\int_{P_1P_2P_3P_4} \xi(t) G_i(t,x) dt dx. \tag{3.173}$$

In the definition of $\widetilde{W}_\alpha^\beta(T)$, $j \neq i$, thus we have from (3.2) that

$$|\lambda_j(u)(t, x_j(t,\beta)) - \lambda_i(u)(t, x_j(t,\alpha))| \geq \delta_0. \tag{3.174}$$

Thus, noting (3.136), it follows from Lemmas 3.5 and 3.7 that

$$\int_{\frac{\alpha}{-\lambda_j(0)}}^{T}|w_i(t,x_j(t,\alpha))-w_i(t,x_j(t,\beta))|dt$$

$$\le\frac{1}{\delta_0}\Big\{[c_{28}W_\infty(T)+2K_6M\widetilde{W}_1(T)]|\alpha-\beta|+\int\int_{P_1P_2P_3P_4}|G_i(t,x)|dtdx\Big\}$$

$$\le c_{29}\Big\{M(1+\varepsilon)|\alpha-\beta|+\int\int_{P_1P_2P_3P_4}|G_i(t,x)|dtdx\Big\}. \tag{3.175}$$

On the other hand, noting (2.26) and using (3.136) and Lemma 3.3, we have

$$\int\int_{P_1P_2P_3P_4}|G_i(t,x)|dtdx\le c_{13}\sum_{l=1}^{n}\sum_{k\ne l}\int\int_{P_1P_2P_3P_4}|w_lw_k|dtdx$$

$$=c_{13}\sum_{l=1}^{n}\sum_{k\ne l}\int_{\frac{\alpha}{-\lambda_j(0)}}^{T}dt\int_{x_j(t,\alpha)}^{x_j(t,\beta)}|w_lw_k|dx\le c_{30}\sum_{l=1}^{n}\sum_{k\ne l}\int_{\alpha}^{\beta}d\gamma\int_{\frac{\alpha}{-\lambda_j(0)}}^{T}|w_lw_k|(t,x_j(t,\gamma))dt$$

$$\le c_{31}W_\infty(T)\widetilde{W}_1(T)|\alpha-\beta|\le c_{32}M\varepsilon|\alpha-\beta|. \tag{3.176}$$

Substituting (3.176) into (3.175) gives

$$\int_{\frac{\alpha}{-\lambda_j(0)}}^{T}|w_i(t,x_j(t,\alpha))-w_i(t,x_j(t,\beta))|dt\le c_{33}M(1+\varepsilon)|\alpha-\beta|,$$

$$\text{if } i\in\{1,\dots,m\},j\in\{m+1,\dots,n\}\text{ or }i,j\in\{m+1,\dots,n\},\ j\ne i. \tag{3.177}$$

This proves (3.167).

We finally prove (3.165).

Integrating (3.157) from $\frac{\alpha}{-\lambda_j(0)}$ to T along the characteristics $C_j(\alpha)$: $x=x_j(t,\alpha)$ and $C_j(\beta)$: $x=x_j(t,\beta)$, respectively, and subtracting the last integral from the first integral gives (cf. [4])

$$\int_{\frac{\alpha}{-\lambda_j(0)}}^{T}|u_i(t,x_j(t,\alpha))-u_i(t,x_j(t,\beta))|dt=\int_{\frac{\alpha}{-\lambda_j(0)}}^{T}\xi(t)[v_i(t,x_j(t,\alpha))-v_i(t,x_j(t,\beta))]dt$$

$$+\sum_{k=1}^{n}\sum_{l\ne k}\int_{\frac{\alpha}{-\lambda_j(0)}}^{T}\xi(t)\{[\rho_{ilk}(u)u_lv_k](t,x_j(t,\alpha))-[\rho_{ilk}(u)u_lv_k](t,x_j(t,\beta))\}dt$$

$$=\int_{\frac{\alpha}{-\lambda_j(0)}}^{T}\xi(t)[v_i(t,x_j(t,\alpha))-v_i(t,x_j(t,\beta))]dt$$

$$+\sum_{k=1}^{n}\sum_{l\ne k}\int_{\frac{\alpha}{-\lambda_j(0)}}^{T}\xi(t)\{\rho_{ilk}(u)(t,x_j(t,\alpha))-\rho_{ilk}(u)(t,x_j(t,\beta))\}[u_lv_k](t,x_j(t,\alpha))dt$$

$$+\sum_{k=1}^{n}\sum_{l\ne k}\int_{\frac{\alpha}{-\lambda_j(0)}}^{T}\xi(t)\rho_{ilk}(u)(t,x_j(t,\beta))\{u_l(t,x_j(t,\alpha))-u_l(t,x_j(t,\beta))\}v_k(t,x_j(t,\alpha))dt$$

$$+\sum_{k=1}^{n}\sum_{l\ne k}\int_{\frac{\alpha}{-\lambda_j(0)}}^{T}\xi(t)[\rho_{ilk}(u)u_l](t,x_j(t,\beta))\{v_k(t,x_j(t,\alpha))-v_k(t,x_j(t,\beta))\}dt. \tag{3.178}$$

Thus, noting (3.6) and using (1.12), (3.137) and (3.138), we obtain

$$\int_{\frac{\alpha}{-\lambda_j(0)}}^{T}|u_i(t,x_j(t,\alpha))-u_i(t,x_j(t,\beta))|dt\le\widetilde{V}_\alpha^\beta(T)+c_{34}\{K_6M|\alpha-\beta|[U_\infty(T)\widetilde{V}_1(T)+V_\infty(T)\widetilde{U}_1(T)]$$

$$+K_6M|\alpha-\beta|\widetilde{V}_1(T)+\widetilde{U}_\alpha^\beta(T)V_\infty(T)+\widetilde{V}_\alpha^\beta(T)U_\infty(T)+K_6M|\alpha-\beta|\widetilde{U}_1(T)\},$$

$$\text{if } i\in\{1,\dots,m\}, j\in\{m+1,\dots,n\} \text{ or } i,j\in\{m+1,\dots,n\},\ j\neq i. \tag{3.179}$$

Then, using Lemma 3.5 and (3.166), we have

$$\int_{\frac{\alpha}{-\lambda_j(0)}}^T |u_i(t,x_j(t,\alpha))-u_i(t,x_j(t,\beta))|dt\le c_{35}(MN+\varepsilon)|\alpha-\beta|+c_{36}K_5\varepsilon\widetilde{U}_\alpha^\beta(T),$$

$$\text{if } i\in\{1,\dots,m\}, j\in\{m+1,\dots,n\} \text{ or } i,j\in\{m+1,\dots,n\},\ j\neq i. \tag{3.180}$$

It follows that

$$\widetilde{U}_\alpha^\beta(T)\le c_{35}(MN+\varepsilon)|\alpha-\beta|+c_{36}K_5\varepsilon\widetilde{U}_\alpha^\beta(T). \tag{3.181}$$

This implies (3.165). The proof of Lemma 3.9 is finished. □

For any fixed $T\ge 0$ and for arbitrary $\alpha,\beta\in\mathbf{R}^+$, we introduce

$$\overline{U}_\alpha^\beta(T)=\max_{i=1,\dots,n}\max_{j\neq i}\int_0^T|u_i(t,\alpha+\lambda_j(0)t)-u_i(t,\beta+\lambda_j(0)t)dt, \tag{3.182}$$

$$\overline{V}_\alpha^\beta(T)=\max_{i=1,\dots,n}\max_{j\neq i}\int_0^T|v_i(t,\alpha+\lambda_j(0)t)-v_i(t,\beta+\lambda_j(0)t)dt \tag{3.183}$$

and

$$\overline{W}_\alpha^\beta(T)=\max_{i=1,\dots,n}\max_{j\neq i}\int_0^T|w_i(t,\alpha+\lambda_j(0)t)-w_i(t,\beta+\lambda_j(0)t)dt. \tag{3.184}$$

Similarly, we can prove the following lemma.

Lemma 3.10. Under the assumptions of Theorem 1.1, there exists a positive constant K_{10} independent of ε, M, N, T, α and β such that

$$\overline{U}_\alpha^\beta(T)\le K_{10}(MN+\varepsilon)|\alpha-\beta|, \tag{3.185}$$

$$\overline{V}_\alpha^\beta(T)\le K_{10}(MN+\varepsilon)|\alpha-\beta| \tag{3.186}$$

and

$$\overline{W}_\alpha^\beta(T)\le K_{10}M(1+\varepsilon)|\alpha-\beta|. \tag{3.187}$$

Combining Lemmas 3.8, 3.9 and 3.10 gives

Lemma 3.11. Under the assumptions of Theorem 1.1, there exists a positive constant K_{11} independent of ε, M, N, α and β such that

$$U_\alpha^\beta(\infty),\widetilde{U}_\alpha^\beta(\infty),\overline{U}_\alpha^\beta(T),V_\alpha^\beta(\infty),\widetilde{V}_\alpha^\beta(\infty),\overline{V}_\alpha^\beta(T)\le K_{11}(MN+\varepsilon)|\alpha-\beta| \tag{3.188}$$

and

$$W_\alpha^\beta(\infty),\widetilde{W}_\alpha^\beta(\infty),\overline{W}_\alpha^\beta(T)\le K_{11}M(1+\varepsilon)|\alpha-\beta|. \tag{3.189}$$

4. Asymptotic behavior of the global classical solution–Proof of Theorem 1.1

This section is devoted to the study of asymptotic behavior of the global classical solution of the mixed initial-boundary value problem (1.1) and (1.10)-(1.11) and gives the proof of Theorem 1.1. Without loss of generality, we assume that $u=(u_1,\dots,u_n)^T$ are already the normalized coordinates.

Let

$$\frac{D}{D_it}=\frac{\partial}{\partial t}+\lambda_i(0)\frac{\partial}{\partial x}. \tag{4.1}$$

Noting (1.1) and (2.5), we have

$$\frac{Du}{D_i t} = \frac{\partial u}{\partial t} + \lambda_i(0)\frac{\partial u}{\partial x} = -A(u)\frac{\partial u}{\partial x} + \lambda_i(0)\frac{\partial u}{\partial x} = \sum_{j=1}^{n}(\lambda_i(0) - \lambda_j(u))w_j r_j(u). \tag{4.2}$$

Thus, noting (1.8), it follows that

$$\begin{aligned}\frac{Du_i}{D_i t} &= \frac{Du}{D_i t}e_i = \sum_{j\neq i}(\lambda_i(0) - \lambda_j(u))w_j r_j(u)e_i + (\lambda_i(0) - \lambda_i(u))w_i r_i(u)e_i \\ &= \sum_{j\neq i}(\lambda_i(0) - \lambda_j(u))w_j r_j(u)e_i + (\lambda_i(u_i e_i) - \lambda_i(u))w_i r_i(u)e_i. \end{aligned}\tag{4.3}$$

By Hadamard's formula, (4.3) can be rewritten as

$$\frac{Du_i}{D_i t} = \sum_{j\neq i} B_{ij}(u)w_j + \sum_{j\neq i}\Gamma_{ij}(u)u_j w_i, \tag{4.4}$$

where $B_{ij}(u)$ and $\Gamma_{ij}(u)$ are all C^1 functions of u, which are defined by

$$B_{ij}(u) = (\lambda_i(0) - \lambda_j(u))r_j(u)e_i, \quad \forall\, j\neq i \tag{4.5}$$

and

$$\Gamma_{ij}(u) = -r_i(u)e_i\int_0^1 \frac{\partial\lambda_i(\tau u_1,\dots,\tau u_{i-1},u_i,\tau u_{i+1},\dots,\tau u_n)}{\partial u_j}d\tau, \quad \forall\, j\neq i. \tag{4.6}$$

For any fixed $(t,x)\in \mathbf{R}^+\times\mathbf{R}^+$, define

$$\alpha = x - \lambda_i(0)t. \tag{4.7}$$

Thus, it follows from (4.4) that

$$\begin{cases} u_i(t,x) = u_i(t,\alpha+\lambda_i(0)t) = u_i(0,\alpha) + \sum\limits_{j\neq i}\int\limits_0^t \Big\{\Gamma_{ij}(u)u_j w_i + B_{ij}(u)w_j\Big\}(s,\alpha+\lambda_i(0)s)ds, \\ \qquad \text{if } \mathbf{i}\in\{\mathbf{1},\dots,\mathbf{m}\}; \\ u_i(t,x) = u_i(t,\alpha+\lambda_i(0)t) = u_i(0,\alpha) + \sum\limits_{j\neq i}\int\limits_0^t \Big\{\Gamma_{ij}(u)u_j w_i + B_{ij}(u)w_j\Big\}(s,\alpha+\lambda_i(0)s)ds, \\ \qquad \text{if } \alpha\in\mathbf{R}^+,\ \mathbf{i}\in\{\mathbf{m+1},\dots,\mathbf{n}\}; \\ u_i(t,x) = u_i(t,\alpha+\lambda_i(0)t) = u_i(\frac{\alpha}{-\lambda_i(0)},0) + \sum\limits_{j\neq i}\int\limits_{\frac{\alpha}{-\lambda_i(0)}}^t \Big\{\Gamma_{ij}(u)u_j w_i + B_{ij}(u)w_j\Big\}(s,\alpha+\lambda_i(0)s)ds, \\ \qquad \text{if } \alpha\in\mathbf{R}^-,\ \mathbf{i}\in\{\mathbf{m+1},\dots,\mathbf{n}\}. \end{cases} \tag{4.8}$$

Then, Lemma 3.5 implies that the integral in the right hand side of (4.8) converges absolutely when t tends to $+\infty$. Therefore, there exists a unique function $\phi_i(\alpha)$ such that

$$u_i(t,x)\to\phi_i(\alpha), \quad \text{as } t\to+\infty. \tag{4.9}$$

Moreover, using Lemma 3.5, we obtain that there exists a positive constant K_{12} independent of ε, M, N and α such that

$$|\phi_i(\alpha)| \le K_{12}(MN+\varepsilon). \tag{4.10}$$

Then from above we have proved the following lemma.

Lemma 4.1. For any $i \in \{1, \ldots, n\}$, the limit

$$\lim_{t \to +\infty} u_i(t, x) = \phi_i(\alpha) = \phi_i(x - \lambda_i(0)t) \tag{4.11}$$

exists and the limit function $\phi_i(\alpha)$ satisfies the estimate (4.10).

Lemma 4.2. Suppose that the limit

$$\lim_{t \to +\infty} w_i(t, \alpha + \lambda_i(0)t)$$

exists, then

$$\frac{d\phi_i(\alpha)}{d\alpha} = \lim_{t \to +\infty} w_i(t, \alpha + \lambda_i(0)t). \tag{4.12}$$

Proof. By the definition (4.11),

$$\begin{aligned}
\frac{d\phi_i(\alpha)}{d\alpha} &= \lim_{\triangle\alpha \to 0} \frac{\phi_i(\alpha + \triangle\alpha) - \phi_i(\alpha)}{\triangle\alpha} \\
&= \lim_{\triangle\alpha \to 0} \lim_{t \to +\infty} \frac{u_i(t, \alpha + \triangle\alpha + \lambda_i(0)t) - u_i(t, \alpha + \lambda_i(0)t)}{\triangle\alpha} \\
&= \lim_{t \to +\infty} \lim_{\triangle\alpha \to 0} \frac{u_i(t, \alpha + \triangle\alpha + \lambda_i(0)t) - u_i(t, \alpha + \lambda_i(0)t)}{\triangle\alpha} \\
&= \lim_{t \to +\infty} \frac{\partial u_i(t, \alpha + \lambda_i(0)t)}{\partial x} \\
&= \lim_{t \to +\infty} \sum_{j=1}^{n} w_j(t, \alpha + \lambda_i(0)t) r_j(u(t, \alpha + \lambda_i(0)t)) e_i \\
&= \lim_{t \to +\infty} \Big\{ \sum_{j=1}^{n} w_j (r_j(u) - r_j(u_j e_j)) e_i + w_i \Big\} (t, \alpha + \lambda_i(0)t) \\
&= \lim_{t \to +\infty} \Big\{ \sum_{j=1}^{n} \sum_{k \neq j} O_{ijk}(u) u_k w_j + w_i \Big\} (t, \alpha + \lambda_i(0)t),
\end{aligned} \tag{4.13}$$

where

$$O_{ijk}(u) = \int_0^1 \frac{\partial r_j(\tau u_1, \ldots, \tau u_{j-1}, u_j, \tau u_{j+1}, \ldots, \tau u_n)}{\partial u_k} e_i d\tau, \quad \forall\, k \neq j. \tag{4.14}$$

By Lemma 3.5, when $t \to +\infty$,

$$\sum_{j=1}^{n} \sum_{k \neq j} O_{ijk}(u) u_k w_j(t, \alpha + \lambda_i(0)t) \to 0, \tag{4.15}$$

uniformly for $\alpha \in \mathbf{R}$. Hence

$$\frac{d\phi_i(\alpha)}{d\alpha} = \lim_{t \to +\infty} w_i(t, \alpha + \lambda_i(0)t). \tag{4.16}$$

The proof of Lemma 4.2 is finished. $\square$

In what follows, we shall investigate the regularity of the limit function $\Phi_i(\alpha)$.

Case 1 : When $\alpha \in \mathbf{R}^+$, for any fixed $(t, \alpha + \lambda_i(0)t) \in \mathbf{R}^+ \times \mathbf{R}^+$.

Case I There exists a $\theta_i(t, \alpha) \in \mathbf{R}^+$ such that

$$\theta_i(t, \alpha) + \int_0^t \lambda_i(u(s, x_i(s, \theta_i(t, \alpha)))) ds = \alpha + \lambda_i(0)t, \tag{4.17}$$

namely,

$$\theta_i(t, \alpha) = \alpha + \int_0^t [\lambda_i(0) - \lambda_i(u(s, x_i(s, \theta_i(t, \alpha))))] ds, \tag{4.18}$$

where $x = x_i(s, \theta_i(t, \alpha))$ stands for the ith characteristic passing through the point $(0, \theta_i(t, \alpha))$, which is defined by

$$\begin{cases} \frac{dx_i(s,\theta_i(t,\alpha))}{ds} = \lambda_i(u(s, x_i(s, \theta_i(t, \alpha)))), \\ x_i(0, \theta_i(t, \alpha)) = \theta_i(t, \alpha). \end{cases} \tag{4.19}$$

Lemma 4.3. Under the assumptions of Theorem 1.1 and for any fixed $\alpha \in \mathbf{R}^+$, there exists a unique $\vartheta_i(\alpha)$ such that

$$\theta_i(t, \alpha) \to \vartheta_i(\alpha), \quad t \to +\infty \tag{4.20}$$

and

$$|\vartheta_i(\alpha) - \alpha| \le K_{13} N. \tag{4.21}$$

The $\vartheta_i(\alpha)$ defined above is global Lipschitz continuous with respect to α, i.e.,

$$|\vartheta_i(\alpha) - \vartheta_i(\beta)| \le [1 + K_{14}(MN + \varepsilon)]|\alpha - \beta|, \tag{4.22}$$

where K_{13} is a positive constant independent of ε, M, N and α, while K_{14} is another positive constant independent of ε, M, N, α and β.

Proof. Noting (1.8), we have from (4.18) that

$$\theta_i(t, \alpha) = \alpha + \int_0^t (\lambda_i(u_i e_i) - \lambda_i(u))(s, x_i(s, \theta_i(t, \alpha)))ds. \tag{4.23}$$

By Hadamard's formula, (4.23) can be rewritten as

$$\theta_i(t, \alpha) = \alpha + \sum_{j \ne i} \int_0^t (\Lambda_{ij}(u) u_j)(s, x_i(s, \theta_i(t, \alpha)))ds, \tag{4.24}$$

where

$$\Lambda_{ij}(u) = -\int_0^1 \frac{\partial \lambda_i(\tau u_1, \dots, \tau u_{i-1}, u_i, \tau u_{i+1}, \dots, \tau u_n)}{\partial u_j} d\tau, \quad \forall\, j \ne i. \tag{4.25}$$

Noting (3.125), we observe that the integral in the right hand side of (4.25) converges absolutely when t tends to $+\infty$. This implies that there exists a unique $\vartheta_i(\alpha)$ such that

$$\lim_{t \to +\infty} \theta_i(t, \alpha) = \vartheta_i(\alpha). \tag{4.26}$$

Thus,

$$\begin{aligned} \vartheta_i(\alpha) - \vartheta_i(\beta) &= \alpha - \beta + \sum_{j \ne i} \lim_{t \to +\infty} \int_0^t \{[\Lambda_{ij}(u) u_j](s, x_i(s, \theta_i(t, \alpha))) - [\Lambda_{ij}(u) u_j](s, x_i(s, \theta_i(t, \beta)))\} ds \\ &= \alpha - \beta + \sum_{j \ne i} \lim_{t \to +\infty} \left\{ \int_0^t [\Lambda_{ij}(u)(s, x_i(s, \theta_i(t, \alpha))) - \Lambda_{ij}(u)(s, x_i(s, \theta_i(t, \beta)))] u_j(s, x_i(s, \theta_i(t, \alpha)))ds \right. \\ &\quad \left. + \int_0^t \Lambda_{ij}(u)(s, x_i(s, \theta_i(t, \beta)))[u_j(s, x_i(s, \theta_i(t, \alpha))) - u_j(s, x_i(s, \theta_i(t, \beta)))]ds \right\}. \end{aligned} \tag{4.27}$$

Noting (4.18), we have

$$\frac{\partial \theta_i(t, \xi)}{\partial \xi} = \frac{1}{1 + \int_0^t (\nabla \lambda_i(u) u_x)(s, x_i(s, \theta_i(t, \xi))) \frac{\partial x_i(s, \theta_i(t, \xi))}{\partial x} ds}. \tag{4.28}$$

Then, it follows from (3.134) and (3.136) that

$$\sup_{(t,\xi)\in\mathbf{R}^+\times\mathbf{R}^+}\left\{\Big|\frac{\partial\theta_i(t,\xi)}{\partial\xi}\Big|\right\}\le\frac{1}{1-c_4\varepsilon e^{c_4\varepsilon}}. \tag{4.29}$$

Thus, using Lemmas 3.5, 3.6 and 3.11, we obtain from (4.27) that

$$\begin{aligned}|\vartheta_i(\alpha)-\vartheta_i(\beta)| &\le|\alpha-\beta|+c_1\{K_6M|\alpha-\beta|\widetilde{U}_1(\infty)+U_\alpha^\beta(\infty)\}\\ &\le|\alpha-\beta|+c_1\{K_6M|\alpha-\beta|\times K_5N+K_{11}(MN+\varepsilon)|\alpha-\beta|\}\\ &\le[1+K_{14}(MN+\varepsilon)]|\alpha-\beta|, \end{aligned}\tag{4.30}$$

where here and henceforth, as before, $c_i(i=1,2,\ldots)$ will denote some positive constants independent of ε, M, N, α and β.

Employing Lemma 3.5 again, we obtain from (4.24) that

$$|\theta_i(t,\alpha)-\alpha|\le c_2\widetilde{U}_1(t)\le c_2K_5N. \tag{4.31}$$

Letting $t\to+\infty$, we immediately get (4.21). The proof of Lemma 4.3 is finished. □

Case II There exists a $\theta_i(t,\alpha)\in\mathbf{R}^-$ such that

$$\int_{\frac{\theta_i(t,\alpha)}{-\lambda_i(0)}}^t\lambda_i(u(s,x_i(s,\theta_i(t,\alpha))))ds=\alpha+\lambda_i(0)t\quad(i=m+1,\ldots,n), \tag{4.32}$$

namely,

$$\theta_i(t,\alpha)=\alpha+\int_{\frac{\theta_i(t,\alpha)}{-\lambda_i(0)}}^t[\lambda_i(0)-\lambda_i(u(s,x_i(s,\theta_i(t,\alpha))))]ds, \tag{4.33}$$

where $x=x_i(s,\theta_i(t,\alpha))$ stands for the ith characteristic passing through the point $(\frac{\theta_i(t,\alpha)}{-\lambda_i(0)},0)$, which is defined by

$$\begin{cases}\frac{dx_i(s,\theta_i(t,\alpha))}{ds}=\lambda_i(u(s,x_i(s,\theta_i(t,\alpha)))),\\ x_i(\frac{\theta_i(t,\alpha)}{-\lambda_i(0)},0)=0.\end{cases} \tag{4.34}$$

Similar to Case I, we have the following lemma.

Lemma 4.4. Under the assumptions of Theorem 1.1, for any $i\in\{m+1,\ldots,n\}$ and for any fixed $\alpha\in\mathbf{R}^+$, there exists a unique $\vartheta_i(\alpha)$ such that

$$\theta_i(t,\alpha)\to\vartheta_i(\alpha),\qquad t\to+\infty \tag{4.35}$$

and

$$|\vartheta_i(\alpha)-\alpha|\le K_{15}N. \tag{4.36}$$

The $\vartheta_i(\alpha)$ defined above is global Lipschitz continuous with respect to α, i.e.,

$$|\vartheta_i(\alpha)-\vartheta_i(\beta)|\le[1+K_{16}(MN+\varepsilon)]|\alpha-\beta|, \tag{4.37}$$

where K_{15} is a positive constant independent of ε, M, N and α, while K_{16} is another positive constant independent of ε, M, N, α and β.

Proof. Noting (1.8) and using Hadamard's formula, we have from (4.33) that

$$\theta_i(t,\alpha)=\alpha+\int_{\frac{\theta_i(t,\alpha)}{-\lambda_i(0)}}^t(\lambda_i(u_ie_i)-\lambda_i(u))(s,x_i(s,\theta_i(t,\alpha)))ds$$

$$= \alpha + \sum_{j \neq i} \int_{\frac{\theta_i(t,\alpha)}{-\lambda_i(0)}}^{t} (\Lambda_{ij}(u)u_j)(s, x_i(s, \theta_i(t,\alpha)))ds, \tag{4.38}$$

where $\Lambda_{ij}(u)$ is defined by (4.25). Noting (3.125), we observe that the integral in the right hand side of (4.38) converges absolutely when t tends to $+\infty$. This implies that there exists a unique $\vartheta_i(\alpha)$ such that

$$\lim_{t \to +\infty} \theta_i(t,\alpha) = \vartheta_i(\alpha). \tag{4.39}$$

Therefore,

$$\begin{aligned} \vartheta_i(\alpha) - \vartheta_i(\beta) = \alpha - \beta + \lim_{t \to +\infty} \sum_{j \neq i} \Bigg(& \int_{\frac{\theta_i(t,\alpha)}{-\lambda_i(0)}}^{t} [\Lambda_{ij}(u)u_j](s, x_i(s, \theta_i(t,\alpha)))ds \\ & - \int_{\frac{\theta_i(t,\beta)}{-\lambda_i(0)}}^{t} [\Lambda_{ij}(u)u_j](s, x_i(s, \theta_i(t,\beta)))ds \Bigg). \end{aligned} \tag{4.40}$$

For the sake of simplicity, we assume that $\theta_i(t,\alpha) < \theta_i(t,\beta)$. Then we have

$$\begin{aligned} \vartheta_i(\alpha) - \vartheta_i(\beta) = \alpha - \beta + \lim_{t \to +\infty} \sum_{j \neq i} \Bigg\{ & \int_{\frac{\theta_i(t,\alpha)}{-\lambda_i(0)}}^{t} [\Lambda_{ij}(u)(s, x_i(s, \theta_i(t,\alpha))) \\ & - \Lambda_{ij}(u)(s, x_i(s, \theta_i(t,\beta)))] u_j(s, x_i(s, \theta_i(t,\alpha)))ds \\ + \int_{\frac{\theta_i(t,\alpha)}{-\lambda_i(0)}}^{t} & \Lambda_{ij}(u)(s, x_i(s, \theta_i(t,\beta)))[u_j(s, x_i(s, \theta_i(t,\alpha))) - u_j(s, x_i(s, \theta_i(t,\beta)))]ds \\ & - \int_{\frac{\theta_i(t,\beta)}{-\lambda_i(0)}}^{\frac{\theta_i(t,\alpha)}{-\lambda_i(0)}} [\Lambda_{ij}(u)u_j](s, x_i(s, \theta_i(t,\beta)))ds \Bigg\}. \end{aligned} \tag{4.41}$$

Thus, noting (4.29) and Lemmas 3.5, 3.7 and 3.11, using Taylor's formula and the integral mean value theorem, we obtain from (4.41) that

$$\begin{aligned} |\vartheta_i(\alpha) - \vartheta_i(\beta)| & \leq |\alpha - \beta| + c_3\{K_7 M |\alpha - \beta| \widetilde{U}_1(\infty) + \widetilde{U}_\alpha^\beta(\infty)\} + c_4 \varepsilon |\alpha - \beta| \\ & \leq |\alpha - \beta| + c_3\{K_7 M |\alpha - \beta| \times K_5 N + K_{11}(MN + \varepsilon)|\alpha - \beta|\} + c_4 \varepsilon |\alpha - \beta| \\ & \leq [1 + K_{16}(MN + \varepsilon)]|\alpha - \beta|. \end{aligned} \tag{4.42}$$

Employing Lemma 3.5 again, we obtain from (4.38) that

$$|\theta_i(t,\alpha) - \alpha| \leq c_5 \sum_{j \neq i} \int_0^t |u_j(s, x_i(s, \theta_i(t,\alpha)))|ds \leq c_6 \widetilde{U}_1(t) \leq c_6 K_5 N. \tag{4.43}$$

Letting $t \to +\infty$, we immediately get (4.36). The proof of Lemma 4.4 is finished. □

Lemma 4.5. For every $i \in \{1, \dots, n\}$, there exists a positive constant K_{17} independent of ε, M, N, α and β such that

$$|\phi_i(\alpha) - \phi_i(\beta)| \leq K_{17}(M + M^2 N + M\varepsilon)|\alpha - \beta|, \qquad \forall\, \alpha, \beta \in \mathbf{R}^+. \tag{4.44}$$

Proof. It follows from (4.17) and (4.32) that

$$u_i(t, \alpha + \lambda_i(0)t) = u_i(t, x_i(t, \theta_i(t,\alpha))), \tag{4.45}$$

where $x = x_i(s, \theta_i(t,\alpha))$ stands for the ith characteristic passing through either the point $(0, \theta_i(t,\alpha))$ or the point $(\frac{\theta_i(t,\alpha)}{-\lambda_i(0)}, 0)$. Then, noting (4.11) and using Lemma 4.3 and Lemma 4.4, we have

$$\begin{aligned}\phi_i(\alpha) - \phi_i(\beta) &= \lim_{t\to+\infty} u_i(t, \alpha + \lambda_i(0)t) - \lim_{t\to+\infty} u_i(t, \beta + \lambda_i(0)t)\\ &= \lim_{t\to+\infty} u_i(t, x_i(t, \theta_i(t,\alpha))) - \lim_{t\to+\infty} u_i(t, x_i(t, \theta_i(t,\beta)))\\ &= \lim_{t\to+\infty} \{u_i(t, x_i(t, \vartheta_i(\alpha))) - u_i(t, x_i(t, \vartheta_i(\beta)))\}. \qquad (4.46)\end{aligned}$$

Thus, using Taylor's formula and noting (2.5), (3.6), (3.128), (3.136) and (4.22), (4.37), we have

$$\begin{aligned}|\phi_i(\alpha) - \phi_i(\beta)| &\le \sup_{(t,x)\in\mathbf{R}^+\times\mathbf{R}^+} \Big\{\Big|\frac{\partial u_i(t,x)}{\partial x}\Big|\Big\} \sup_{(t,\xi)\in\mathbf{R}^+\times\mathbf{R}^+} \Big\{\Big|\frac{\partial x_i(t,\xi)}{\partial \xi}\Big|\Big\} |\vartheta_i(\alpha) - \vartheta_i(\beta)|\\ &\le c_7 W_\infty(\infty) \times e^{c_4\varepsilon} \times |\vartheta_i(\alpha) - \vartheta_i(\beta)|\\ &\le K_{17}(M + M^2N + M\varepsilon)|\alpha - \beta|, \qquad \forall\, \alpha, \beta \in \mathbf{R}^+. \qquad (4.47)\end{aligned}$$

The proof of Lemma 4.5 is finished. □

For arbitrary $\alpha, \beta \in \mathbf{R}^+$ and for any fixed $i \in \{1, \dots, n\}$, we introduce

$$W^i_{\alpha,\beta}(\infty) = \sup_{t\in\mathbf{R}^+} |w_i(t, x_i(t,\alpha)) - w_i(t, x_i(t,\beta))|, \qquad (4.48)$$

where for arbitrary $\alpha \in \mathbf{R}^+$, $x = x_i(t,\alpha)$ stands for the ith characteristic passing through the point $(0, \alpha)$.

Lemma 4.6. Under the assumptions of Theorem 1.1, for any $i \in \{1, \dots, n\}$ and for any fixed $\alpha \in \mathbf{R}^+$, the limit

$$\lim_{t\to+\infty} w_i(t, x_i(t,\alpha))$$

exists, denoted it by $\psi_i(\alpha)$, i.e.,

$$\lim_{t\to+\infty} w_i(t, x_i(t,\alpha)) = \psi_i(\alpha), \qquad \forall\, \alpha \in \mathbf{R}^+, \qquad (4.49)$$

where $x = x_i(t,\alpha)$ stands for the ith characteristic passing through the point $(0, \alpha)$. Moreover, $\psi_i(\alpha)$ is continuous with respect to $\alpha \in \mathbf{R}^+$ and satisfies that there exists a positive constant K_{18} independent of ε, M, N and α such that

$$|\psi_i(\alpha)| \le (1 + K_{18}\varepsilon)M, \qquad \forall\, \alpha \in \mathbf{R}^+. \qquad (4.50)$$

Also, there exists a positive constant K_{19} independent of ε, M, N, α and β such that

$$W^i_{\alpha,\beta}(\infty) \le (1 + K_{19}\varepsilon)|w_i(0,\alpha) - w_i(0,\beta)| + K_{19}M^2(1+\varepsilon)|\alpha - \beta|, \quad \forall\, \alpha, \beta \in \mathbf{R}^+. \qquad (4.51)$$

In particular, if (1.21) is satisfied, then there exists a positive constant K_{20} independent of ε, M, N, α and β such that

$$|\psi_i(\alpha) - \psi_i(\beta)| \le K_{20}\varsigma_1|\alpha - \beta|^\rho + K_{20}M^2(1+\varepsilon)|\alpha - \beta|, \quad \forall\, \alpha, \beta \in \mathbf{R}^+, \qquad (4.52)$$

where $0 < \rho \le 1$.

Proof. For any fixed $\alpha \in \mathbf{R}^+$ and for any $i \in \{1, \dots, n\}$, we have from (2.18) and (2.22) that

$$w_i(t, x_i(t,\alpha)) = w_i(0,\alpha) + \int_0^t \sum_{j=1}^n \sum_{k\ne j} \gamma_{ijk}(u) w_j w_k(s, x_i(s,\alpha))ds. \qquad (4.53)$$

Then, Lemma 3.5 implies that the integrals in the right hand side of (4.53) converge absolutely when t tends to $+\infty$. Thus, there exists a unique function $\psi_i(\alpha)$ such that

$$w_i(t, x_i(t,\alpha)) \to \psi_i(\alpha), \quad \text{as } t \to +\infty. \tag{4.54}$$

Moreover, we obtain from Lemma 3.5 and (4.53) that

$$|w_i(t, x_i(t,\alpha))| \le |w_i(0,\alpha)| + c_8 W_\infty(t)\widetilde{W}_1(t) \le (1 + K_{17}\varepsilon)M. \tag{4.55}$$

This implies (4.50).

By a direct computation, we have

$$\begin{aligned}
w_i(t, x_i(t,\alpha)) - w_i(t, x_i(t,\beta)) &= w_i(0,\alpha) - w_i(0,\beta) \\
+\sum_{j=1}^n \sum_{k\ne j} \int_0^t [\gamma_{ijk}(u) w_j w_k(s, x_i(s,\alpha)) &- \gamma_{ijk}(u) w_j w_k(s, x_i(s,\beta))]ds \\
&= w_i(0,\alpha) - w_i(0,\beta) \\
+\sum_{j=1}^n \sum_{k\ne j} \int_0^t \{\gamma_{ijk}(u)(s, x_i(s,\alpha)) &- \gamma_{ijk}(u)(s, x_i(s,\beta))\}[w_j w_k](s, x_i(s,\alpha))ds \\
+\sum_{j=1}^n \sum_{k\ne j} \int_0^t \gamma_{ijk}(u)(s, x_i(s,\beta))\{w_j(s, x_i(s,\alpha)) &- w_j(s, x_i(s,\beta))\} w_k(s, x_i(s,\alpha))ds \\
+\sum_{j=1}^n \sum_{k\ne j} \int_0^t [\gamma_{ijk}(u) w_j](s, x_i(s,\beta))\{w_k(s, x_i(s,\alpha)) &- w_k(s, x_i(s,\beta))\}ds.
\end{aligned} \tag{4.56}$$

Then, noting Lemmas 3.5, 3.6 and 3.8, we obtain

$$\begin{aligned}
|w_i(t, x_i(t,\alpha)) - w_i(t, x_i(t,\beta))| &\le |w_i(0,\alpha) - w_i(0,\beta)| + c_9\{K_6 M|\alpha-\beta| W_\infty(t)\widetilde{W}_1(t) \\
&+ W_\alpha^\beta(t) W_\infty(t) + \widetilde{W}_1(t) \sup_{t\in\mathbf{R}^+} |w_i(t, x_i(t,\alpha)) - w_i(t, x_i(t,\beta))|\} \\
&\le |w_i(0,\alpha) - w_i(0,\beta)| + c_9\{K_6 M|\alpha-\beta| \times K_5 M \times K_5\varepsilon \\
&+ K_8 M(1+\varepsilon)|\alpha-\beta| \times K_5 M + K_5\varepsilon \sup_{t\in\mathbf{R}^+} |w_i(t, x_i(t,\alpha)) - w_i(t, x_i(t,\beta))|\} \\
&\le |w_i(0,\alpha) - w_i(0,\beta)| + c_{10} M^2 (1+\varepsilon)|\alpha-\beta| \\
&+ c_9 K_5 \varepsilon \sup_{t\in\mathbf{R}^+} |w_i(t, x_i(t,\alpha)) - w_i(t, x_i(t,\beta))|.
\end{aligned} \tag{4.57}$$

Thus, (4.51) follows from (4.57) directly. Because $w_i(0,x)$ is continuous, it follows from (4.51) that $\psi_i(\alpha) \in C^0(\mathbf{R}^+)$.

If (1.21) holds, we see that $w_i(0,x)$ is globally ρ-Hölder continuous. (4.52) follows from (4.51) easily. The proof of Lemma 4.6 is finished. $\square$

For arbitrary $\alpha, \beta \in \mathbf{R}^-$ and for any fixed $i \in \{m+1, \ldots, n\}$, we introduce

$$W^i_{\alpha,\beta}(\infty) = \sup_{t\in\mathbf{R}^+} |w_i(t, x_i(t,\alpha)) - w_i(t, x_i(t,\beta))|, \tag{4.58}$$

where for arbitrary $\alpha \in \mathbf{R}^-$, $x = x_i(t,\alpha)(i = m+1, \ldots, n)$ stands for the ith characteristic passing through the point $(\frac{\alpha}{-\lambda_i(0)}, 0)$.

Similarly, we have the following lemma.

Lemma 4.7. Under the assumptions of Theorem 1.1, for any $i \in \{m+1, \dots, n\}$ and for any fixed $\alpha \in \mathbf{R}^-$, the limit

$$\lim_{t \to +\infty} w_i(t, x_i(t, \alpha))$$

exists, denoted it by $\psi_i(\alpha)$, i.e.,

$$\lim_{t \to +\infty} w_i(t, x_i(t, \alpha)) = \psi_i(\alpha), \qquad \forall\, \alpha \in \mathbf{R}^-, \tag{4.59}$$

where $x = x_i(t, \alpha)$ stands for the ith characteristic passing through the point $(\frac{\alpha}{-\lambda_i(0)}, 0)$. Moreover, $\psi_i(\alpha)$ is continuous with respect to $\alpha \in \mathbf{R}^-$ and satisfies that there exists a positive constant K_{21} independent of ε, M, N and α such that

$$|\psi_i(\alpha)| \le (1 + K_{21}\varepsilon)M, \qquad \forall\, \alpha \in \mathbf{R}^-. \tag{4.60}$$

Also, there exists a positive constant K_{22} independent of ε, M, N, α and β such that

$$W^i_{\alpha,\beta}(\infty) \le (1 + K_{22}\varepsilon)|w_i(\frac{\alpha}{-\lambda_i(0)}, 0) - w_i(\frac{\beta}{-\lambda_i(0)}, 0)| + K_{22}M^2(1+\varepsilon)|\alpha - \beta|, \quad \forall\, \alpha, \beta \in \mathbf{R}^-. \tag{4.61}$$

In particular, if (1.21)-(1.23) are satisfied, then there exists a positive constant K_{23} independent of ε, M, N, α and β such that

$$|\psi_i(\alpha) - \psi_i(\beta)| \le K_{23}\varsigma|\alpha - \beta|^\rho + K_{23}M^2(1+\varepsilon)|\alpha - \beta|, \ \ \forall\, \alpha, \beta \in \mathbf{R}^-, \tag{4.62}$$

where $0 < \rho \le 1$.

Proof. For any fixed $\alpha \in \mathbf{R}^-$ and for any $i \in \{m+1, \dots, n\}$, we have from (2.18) and (2.22) that

$$w_i(t, x_i(t, \alpha)) = w_i(\frac{\alpha}{-\lambda_i(0)}, 0) + \int_{\frac{\alpha}{-\lambda_i(0)}}^t \sum_{j=1}^n \sum_{k \ne j} \gamma_{ijk}(u) w_j w_k(s, x_i(s, \alpha)) ds. \tag{4.63}$$

Then, Lemma 3.5 indicates that the integrals in the right hand side of (4.63) converge absolutely when t tends to $+\infty$. Thus, the right hand side of (4.63) converges when t tends to $+\infty$. We denote the limit by $\psi_i(\alpha)$, i.e.,

$$\lim_{t \to +\infty} w_i(t, x_i(t, \alpha)) = \psi_i(\alpha).$$

It follows from Lemma 3.5 and (4.63) that

$$\begin{aligned} |w_i(t, x_i(t, \alpha))| &\le |w_i(\frac{\alpha}{-\lambda_i(0)}, 0)| + c_{11} \sum_{j=1}^n \sum_{k \ne j} \int_0^t |w_j w_k(s, x_i(s, \alpha))| ds \\ &\le |w_i(\frac{\alpha}{-\lambda_i(0)}, 0)| + c_{12} W_\infty(t) \widetilde{W}_1(t) \le (1 + K_{20}\varepsilon)M. \end{aligned} \tag{4.64}$$

This implies (4.60). Moreover,

$$\begin{aligned} w_i(t, x_i(t, \alpha)) - w_i(t, x_i(t, \beta)) = w_i(\frac{\alpha}{-\lambda_i(0)}, 0) + \int_{\frac{\alpha}{-\lambda_i(0)}}^t \sum_{j=1}^n \sum_{k \ne j} \gamma_{ijk}(u) w_j w_k(s, x_i(s, \alpha)) ds \\ - w_i(\frac{\beta}{-\lambda_i(0)}, 0) - \int_{\frac{\beta}{-\lambda_i(0)}}^t \sum_{j=1}^n \sum_{k \ne j} \gamma_{ijk}(u) w_j w_k(s, x_i(s, \beta)) ds. \end{aligned} \tag{4.65}$$

For the sake of simplicity, we assume that $\alpha < \beta$. Then we have

$$w_i(t, x_i(t, \alpha)) - w_i(t, x_i(t, \beta)) = w_i(\frac{\alpha}{-\lambda_i(0)}, 0) - w_i(\frac{\beta}{-\lambda_i(0)}, 0)$$

$$+\sum_{j=1}^{n}\sum_{k\neq j}\int_{\frac{\alpha}{-\lambda_i(0)}}^{t}\{\gamma_{ijk}(u)(s,x_i(s,\alpha))-\gamma_{ijk}(u)(s,x_i(s,\beta))\}[w_jw_k](s,x_i(s,\alpha))ds$$

$$+\sum_{j=1}^{n}\sum_{k\neq j}\int_{\frac{\alpha}{-\lambda_i(0)}}^{t}\gamma_{ijk}(u)(s,x_i(s,\beta))\{w_j(s,x_i(s,\alpha))-w_j(s,x_i(s,\beta))\}w_k(s,x_i(s,\alpha))ds$$

$$+\sum_{j=1}^{n}\sum_{k\neq j}\int_{\frac{\alpha}{-\lambda_i(0)}}^{t}[\gamma_{ijk}(u)w_j](s,x_i(s,\beta))\{w_k(s,x_i(s,\alpha))-w_k(s,x_i(s,\beta))\}ds$$

$$-\sum_{j=1}^{n}\sum_{k\neq j}\int_{\frac{\beta}{-\lambda_i(0)}}^{\frac{\alpha}{-\lambda_i(0)}}\gamma_{ijk}(u)w_jw_k(s,x_i(s,\beta))ds. \tag{4.66}$$

Thus, using the integral mean value theorem and Lemmas 3.5, 3.7 and 3.9, we obtain

$$\begin{aligned}
|w_i(t,x_i(t,\alpha))-w_i(t,x_i(t,\beta))| &\le |w_i(\frac{\alpha}{-\lambda_i(0)},0)-w_i(\frac{\beta}{-\lambda_i(0)},0)|+c_{13}\{K_7M|\alpha-\beta|W_\infty(t)\widetilde{W}_1(t)\\
&+\widetilde{W}_\alpha^\beta(t)W_\infty(t)+\widetilde{W}_1(t)\sup_{t\in\mathbf{R}^+}|w_i(t,x_i(t,\alpha))-w_i(t,x_i(t,\beta))|\}+c_{14}M^2|\alpha-\beta|\\
&\le |w_i(\frac{\alpha}{-\lambda_i(0)},0)-w_i(\frac{\beta}{-\lambda_i(0)},0)|+c_{13}\{K_7M|\alpha-\beta|\times K_5M\times K_5\varepsilon\\
&+K_9M(1+\varepsilon)|\alpha-\beta|\times K_5M+K_5\varepsilon\sup_{t\in\mathbf{R}^+}|w_i(t,x_i(t,\alpha))-w_i(t,x_i(t,\beta))|\}+c_{14}M^2|\alpha-\beta|\\
&\le |w_i(\frac{\alpha}{-\lambda_i(0)},0)-w_i(\frac{\beta}{-\lambda_i(0)},0)|+c_{15}M^2(1+\varepsilon)|\alpha-\beta|\\
&+c_{14}K_5\varepsilon\sup_{t\in\mathbf{R}^+}|w_i(t,x_i(t,\alpha))-w_i(t,x_i(t,\beta))|.
\end{aligned}\tag{4.67}$$

Then, (4.61) follows from (4.67) directly.

Noting (3.25) in Shao [14], we have

$$\begin{aligned}
w_i(\frac{\alpha}{-\lambda_i(0)},0)-w_i(\frac{\beta}{-\lambda_i(0)},0) &= \sum_{r=1}^{m}\left\{f_{ir}(\frac{\alpha}{-\lambda_i(0)},u(\frac{\alpha}{-\lambda_i(0)},0))-f_{ir}(\frac{\beta}{-\lambda_i(0)},u(\frac{\beta}{-\lambda_i(0)},0))\right\}w_r(\frac{\alpha}{-\lambda_i(0)},0)\\
&+\sum_{r=1}^{m}f_{ir}(\frac{\beta}{-\lambda_i(0)},u(\frac{\beta}{-\lambda_i(0)},0))\left\{w_r(\frac{\alpha}{-\lambda_i(0)},0)-w_r(\frac{\beta}{-\lambda_i(0)},0)\right\}\\
&+\sum_{j=1}^{k}\left\{\overline{f}_{ij}(\frac{\alpha}{-\lambda_i(0)},u(\frac{\alpha}{-\lambda_i(0)},0))-\overline{f}_{ij}(\frac{\beta}{-\lambda_i(0)},u(\frac{\beta}{-\lambda_i(0)},0))\right\}\alpha'_j(\frac{\alpha}{-\lambda_i(0)})\\
&+\sum_{j=1}^{k}\overline{f}_{ij}(\frac{\beta}{-\lambda_i(0)},u(\frac{\beta}{-\lambda_i(0)},0))\left\{\alpha'_j(\frac{\alpha}{-\lambda_i(0)})-\alpha'_j(\frac{\beta}{-\lambda_i(0)})\right\}\\
&+\sum_{l=m+1}^{n}\left\{\widetilde{f}_{il}(\frac{\alpha}{-\lambda_i(0)},u(\frac{\alpha}{-\lambda_i(0)},0))-\widetilde{f}_{il}(\frac{\beta}{-\lambda_i(0)},u(\frac{\beta}{-\lambda_i(0)},0))\right\}h'_l(\frac{\alpha}{-\lambda_i(0)})\\
&+\sum_{l=m+1}^{n}\widetilde{f}_{il}(\frac{\beta}{-\lambda_i(0)},u(\frac{\beta}{-\lambda_i(0)},0))\left\{h'_l(\frac{\alpha}{-\lambda_i(0)})-h'_l(\frac{\beta}{-\lambda_i(0)})\right\},
\end{aligned}\tag{4.68}$$

where $f_{ir},\overline{f}_{ij}$ and $\widetilde{f}_{il}$ are continuous functions of t and u.

Then, passing through the point $(\frac{\alpha}{-\lambda_i(0)}, 0)$, we draw the ray $L_r (r \in \{1, \dots, m\})$ with the slope $\lambda_r(0)$ which intersects the x-axis at point $(0, \frac{\lambda_r(0)}{\lambda_i(0)}\alpha)$. Integrating (2.18) along the ray L_r from 0 to $\frac{\alpha}{-\lambda_i(0)}$ and noting (2.22), we have

$$w_r(\frac{\alpha}{-\lambda_i(0)}, 0) = w_r(0, \frac{\lambda_r(0)}{\lambda_i(0)}\alpha) + \int_0^{\frac{\alpha}{-\lambda_i(0)}} \sum_{j=1}^{n}\sum_{k\neq j} \gamma_{rjk}(u) w_j w_k (s, \frac{\lambda_r(0)}{\lambda_i(0)}\alpha + \lambda_r(0)s)ds. \tag{4.69}$$

Similarly, we have

$$w_r(\frac{\beta}{-\lambda_i(0)}, 0) = w_r(0, \frac{\lambda_r(0)}{\lambda_i(0)}\beta) + \int_0^{\frac{\beta}{-\lambda_i(0)}} \sum_{j=1}^{n}\sum_{k\neq j} \gamma_{rjk}(u) w_j w_k (s, \frac{\lambda_r(0)}{\lambda_i(0)}\beta + \lambda_r(0)s)ds. \tag{4.70}$$

Therefore,

$$w_r(\frac{\alpha}{-\lambda_i(0)}, 0) - w_r(\frac{\beta}{-\lambda_i(0)}, 0) = w_r(0, \frac{\lambda_r(0)}{\lambda_i(0)}\alpha) - w_r(0, \frac{\lambda_r(0)}{\lambda_i(0)}\beta) + \int_0^{\frac{\alpha}{-\lambda_i(0)}} \sum_{j=1}^{n}\sum_{k\neq j} \gamma_{rjk}(u) w_j w_k (s, \frac{\lambda_r(0)}{\lambda_i(0)}\alpha + \lambda_r(0)s)ds$$

$$- \int_0^{\frac{\beta}{-\lambda_i(0)}} \sum_{j=1}^{n}\sum_{k\neq j} \gamma_{rjk}(u) w_j w_k (s, \frac{\lambda_r(0)}{\lambda_i(0)}\beta + \lambda_r(0)s)ds. \tag{4.71}$$

Noting Lemmas 3.5 and 3.11, using Taylor's formula and making use of the method of (4.67), we obtain

$$|w_r(\frac{\alpha}{-\lambda_i(0)}, 0) - w_r(\frac{\beta}{-\lambda_i(0)}, 0)| \leq |w_r(0, \frac{\lambda_r(0)}{\lambda_i(0)}\alpha) - w_r(0, \frac{\lambda_r(0)}{\lambda_i(0)}\beta)| + c_{16}\{M|\alpha - \beta| W_\infty(\infty)\overline{W}_1(\infty)$$

$$+ \overline{W}_\alpha^\beta(\infty) W_\infty(\infty)\} + c_{17} M^2 |\alpha - \beta|$$

$$\leq |w_r(0, \frac{\lambda_r(0)}{\lambda_i(0)}\alpha) - w_r(0, \frac{\lambda_r(0)}{\lambda_i(0)}\beta)| + c_{18} M^2 (1 + \varepsilon)|\alpha - \beta|. \tag{4.72}$$

Because $w_i(0, x)$ is continuous, noting (4.68), it follows from (4.61) and (4.72) that $\Psi_i(\alpha) \in C^0(\mathbf{R}^-)$.

If (1.21)-(1.23) hold, we see that $w_i(0, x)$ is globally ρ-Hölder continuous. (4.62) follows from (4.61), (4.68) and (4.72) easily. The proof of Lemma 4.7 is finished. □

Lemma 4.8. For every $i \in \{1, \dots, n\}$, the limit $\lim_{t\to+\infty} w_i(t, \alpha + \lambda_i(0)t)$ exists and

$$\lim_{t\to+\infty} w_i(t, \alpha + \lambda_i(0)t) = \psi_i(\vartheta_i(\alpha)) \in C^0(\mathbf{R}^+). \tag{4.73}$$

Moreover, if (1.21)-(1.23) are satisfied, then the following estimate holds

$$\left|\psi_i(\vartheta_i(\alpha)) - \psi_i(\vartheta_i(\beta))\right| \leq K_{24}\varsigma(1 + MN + \varepsilon)^\rho |\alpha - \beta|^\rho$$

$$+ K_{24} M^2 (1 + \varepsilon)(1 + MN + \varepsilon)|\alpha - \beta|, \qquad \forall\, \alpha, \beta \in \mathbf{R}^+, \tag{4.74}$$

where K_{24} is a positive constant independent of ε, M, N, ς, α and β.

Proof. It follows from (4.17) and (4.32) that

$$w_i(t, \alpha + \lambda_i(0)t) = w_i(t, x_i(t, \theta_i(t, \alpha))), \tag{4.75}$$

where $x = x_i(s, \theta_i(t, \alpha))$ stands for the ith characteristic passing through either the point $(0, \theta_i(t, \alpha))$ or the point $(\frac{\theta_i(t,\alpha)}{-\lambda_i(0)}, 0)$. Then, noting Lemmas 4.3 and 4.4, we have

$$\lim_{t\to+\infty} w_i(t, \alpha + \lambda_i(0)t) = \lim_{t\to+\infty} w_i(t, x_i(t, \theta_i(t, \alpha))) = \lim_{t\to+\infty} w_i(t, x_i(t, \vartheta_i(\alpha))), \tag{4.76}$$

and then by Lemmas 4.6 and 4.7,

$$\lim_{t\to+\infty} w_i(t,\alpha+\lambda_i(0)t) = \lim_{t\to+\infty} w_i(t, x_i(t,\vartheta_i(\alpha))) = \psi_i(\vartheta_i(\alpha)). \tag{4.77}$$

Since, by Lemmas 4.6, 4.7, 4.3 and 4.4, $\psi_i(\cdot)$ and $\vartheta_i(*)$ are continuous with respect to $\cdot$ and $*$ respectively, $\psi_i(\vartheta_i(\alpha))$ is a continuous function of $\alpha \in \mathbf{R}^+$. This proves (4.73).

Moreover, if (1.21)-(1.23) are satisfied, then using (4.22), (4.37), (4.52) and (4.62), we obtain (4.74) immediately. The proof of Lemma 4.8 is finished. □

Combining Lemmas 4.2 and 4.8 gives

Lemma 4.9. For every $i \in \{1,\dots,n\}$, it follows that

$$\frac{d\phi_i(\alpha)}{d\alpha} = \psi_i(\vartheta_i(\alpha)) \in C^0(\mathbf{R}^+). \tag{4.78}$$

Moreover, if (1.21)-(1.23) are satisfied, then the following estimate holds

$$\left|\frac{d\phi_i}{d\alpha}(\alpha) - \frac{d\phi_i}{d\alpha}(\beta)\right| \le K_{24}\varsigma(1+MN+\varepsilon)^\rho|\alpha-\beta|^\rho$$

$$+K_{24}M^2(1+\varepsilon)(1+MN+\varepsilon)|\alpha-\beta|, \qquad \forall\,\alpha,\beta\in\mathbf{R}^+, \tag{4.79}$$

where K_{24} is a positive constant independent of ε, M, N, ς, α and β.

Case 2 : When $\alpha \in \mathbf{R}^-$ and $\alpha = 0$, the similar results above can also be obtained.

Clearly, we get from **Case 1** and **Case 2** that

Lemma 4.10. For every $i \in \{1,\dots,n\}$, it follows that

$$\frac{d\phi_i(\alpha)}{d\alpha} = \psi_i(\vartheta_i(\alpha)) \in C^0(\mathbf{R}). \tag{4.80}$$

Moreover, if (1.21)-(1.23) are satisfied, then the following estimate holds

$$\left|\frac{d\phi_i}{d\alpha}(\alpha) - \frac{d\phi_i}{d\alpha}(\beta)\right| \le K_{25}\varsigma(1+MN+\varepsilon)^\rho|\alpha-\beta|^\rho$$

$$+K_{25}M^2(1+\varepsilon)(1+MN+\varepsilon)|\alpha-\beta|, \qquad \forall\,\alpha,\beta\in\mathbf{R}, \tag{4.81}$$

where K_{25} is a positive constant independent of ε, M, N, ς, α and β.

Proof of Theorem 1.1. The conclusion of Theorem 1.1 follows from Lemmas 4.1, 4.5 and 4.10 immediately. Thus, the proof of Theorem 1.1. is finished. □

5. An application of Theorem 1.1

In this section, we use the conclusion of Theorem 1.1 to consider the mixed initial boundary value problem for the system of the motion of the relativistic string in the Minkowski space-time R^{1+n}. Recall Kong et al's work [8] at first. We denote by $X = (t, x_1, \cdots, x_n)$ points in the $(1+n)$-dimensional Minkowski space R^{1+n}. Then the scalar product of two vectors X and $Y = (\widetilde{t}, y_1, \cdots, y_n)$ in R^{1+n} is defined by

$$X\cdot Y = \sum_{i=1}^n x_i y_i - t\widetilde{t}, \tag{5.1}$$

in particular,

$$X^2 = \sum_{i=1}^n x_i^2 - t^2. \tag{5.2}$$

The Lorentzian metric of R^{1+n} can be written as

$$ds^2 = \sum_{i=1}^{n} dx_i^2 - dt^2. \tag{5.3}$$

To describe the motion of the relativistic string in the $(1+n)$-dimensional Minkowski space R^{1+n}, we consider the local equation of an extremal timelike surface S in R^{1+n} taking the following parameter form in a suitable coordinate system (cf. [8]):

$$x_i = x_i(t, \theta) \quad (i = 1, \ldots, n). \tag{5.4}$$

Then, in the surface coordinates t and θ, the Lorentzian metric (5.3) is expressed as

$$ds^2 = (dt, d\theta) M (dt, d\theta)^T, \tag{5.5}$$

where,

$$M = \begin{pmatrix} |x_t|^2 - 1 & \langle x_t, x_\theta \rangle \\ \langle x_t, x_\theta \rangle & |x_\theta|^2 \end{pmatrix}, \tag{5.6}$$

in which $x = (x_1, \cdots, x_n)^T$ and

$$\langle x_t, x_\theta \rangle = \sum_{i=1}^{n} x_{i,t} x_{i,\theta}, \quad |x_t|^2 = \langle x_t, x_t \rangle \text{ and } |x_\theta|^2 = \langle x_\theta, x_\theta \rangle. \tag{5.7}$$

Since the surface S is C^2 and timelike, i.e.,

$$det\ M < 0, \tag{5.8}$$

equivalently,

$$\langle x_t, x_\theta \rangle^2 - (|x_t|^2 - 1)|x_\theta|^2 > 0, \tag{5.9}$$

it follows that the area element of the surface S is

$$dA = \sqrt{\langle x_t, x_\theta \rangle^2 - (|x_t|^2 - 1)|x_\theta|^2}\, dt d\theta. \tag{5.10}$$

The surface S is called to be extremal surface, if $x = x(t, \theta)$ is the critical point of the area functional

$$I = \int\int \sqrt{\langle x_t, x_\theta \rangle^2 - (|x_t|^2 - 1)|x_\theta|^2}\, dt d\theta. \tag{5.11}$$

The corresponding Euler-Lagrange equation is (cf. [8])

$$\Big(\frac{|x_\theta|^2 x_t - \langle x_t, x_\theta \rangle x_\theta}{\sqrt{\langle x_t, x_\theta \rangle^2 - (|x_t|^2 - 1)|x_\theta|^2}} \Big)_t - \Big(\frac{\langle x_t, x_\theta \rangle x_t - (|x_t|^2 - 1) x_\theta}{\sqrt{\langle x_t, x_\theta \rangle^2 - (|x_t|^2 - 1)|x_\theta|^2}} \Big)_\theta = 0. \tag{5.12}$$

Let

$$u = x_t, \quad v = x_\theta, \tag{5.13}$$

where $u = (u_1, \cdots, u_n)^T$ and $v = (v_1, \cdots, v_n)^T$, Then (5.12) can be equivalently rewritten as

$$\begin{cases} \Big(\frac{|v|^2 u - \langle u, v \rangle v}{\sqrt{\langle u, v \rangle^2 - (|u|^2 - 1)|v|^2}} \Big)_t - \Big(\frac{\langle u, v \rangle u - (|u|^2 - 1) v}{\sqrt{\langle u, v \rangle^2 - (|u|^2 - 1)|v|^2}} \Big)_\theta = 0, \\ v_t - u_\theta = 0. \end{cases} \tag{5.14}$$

We consider the mixed initial boundary value problem for system (5.14) with the initial condition

$$t = 0 : u = u_0(\theta), \quad v = \widetilde{v}_0 + v_0(\theta)(\theta \geq 0) \tag{5.15}$$

and the boundary condition

$$\theta = 0 : u = 0 \ \ (t \geq 0). \tag{5.16}$$

Here, $\widetilde{v}_0 = (\widetilde{v}_1^0, \cdots, \widetilde{v}_n^0)^T$ is a constant vector with $|\widetilde{v}_0| = \sqrt{(\widetilde{v}_1^0)^2 + \cdots + (\widetilde{v}_n^0)^2} > 0$, $(u_0(\theta)^T, v_0(\theta)^T) \in C^1$ with bounded C^1 norm, such that

$$||u_0(\theta)||_{C^0}, \ ||v_0(\theta)||_{C^0}, \ ||u_0'(\theta)||_{C^0}, \ ||v_0'(\theta)||_{C^0} \leq M, \tag{5.17}$$

for some positive constant M (bounded but possibly large). Also, we assume that the conditions of C^1 compatibility are satisfied at the point $(0, 0)$.

Let

$$U = \begin{pmatrix} u \\ v \end{pmatrix}. \tag{5.18}$$

Then, we can rewrite system (5.14) as

$$U_t + A(U)U_\theta = 0, \tag{5.19}$$

where

$$A(U) = \begin{bmatrix} -\frac{2\langle u,v\rangle}{|v|^2} I_{n\times n} & \frac{|u|^2-1}{|v|^2} I_{n\times n} \\ -I_{n\times n} & 0 \end{bmatrix}. \tag{5.20}$$

It is easy to see that in a neighborhood of $U_0 = \begin{pmatrix} 0 \\ \widetilde{v}_0 \end{pmatrix}$, (5.14) is a hyperbolic system with the following real eigenvalues:

$$\lambda_1(U) \equiv \cdots \equiv \lambda_n(U) = \lambda_- < 0 < \lambda_{n+1}(U) \equiv \cdots \equiv \lambda_{2n}(U) = \lambda_+, \tag{5.21}$$

where

$$\lambda_\pm = \frac{-\langle u, v\rangle \pm \sqrt{\langle u, v\rangle^2 - (|u|^2 - 1)|v|^2}}{|v|^2}. \tag{5.22}$$

The corresponding left and right eigenvectors are

$$l_i(U) = (e_i, \lambda_+ e_i) \ \ (i = 1, \ldots, n), \quad l_i(U) = (e_{i-n}, \lambda_- e_{i-n}) \ \ (i = n+1, \ldots, 2n) \tag{5.23}$$

and

$$r_i(U) = (-\lambda_- e_i, e_i)^T \ \ (i = 1, \ldots, n), \qquad r_i(U) = (-\lambda_+ e_{i-n}, e_{i-n})^T \ \ (i = n+1, \ldots, 2n) \tag{5.24}$$

respectively, where

$$e_i = (0, \ldots, 0, \overset{(i)}{1}, 0, \ldots, 0) \ \ (i = 1, \ldots, n). \tag{5.25}$$

When $n = 1$, (5.14) is a strictly hyperbolic system; while, when $n \geq 2$, (5.14) is a non-strictly hyperbolic system with characteristics with constant multiplicity. It is easy to see that all characteristic fields are linearly degenerate in the sense of Lax, i.e.,

$$\nabla\lambda_i(U) r_i(U) \equiv 0, \ \ (i = 1, \ldots, 2n), \tag{5.26}$$

see [8].

Let

$$V_i = l_i(U)(U - U_0) \quad (i = 1, \dots, 2n). \tag{5.27}$$

Then, the boundary condition (5.16) can be rewritten as

$$\theta = 0: \; V_{n+i} = -V_i, \;\; (i = 1, \dots, n). \tag{5.28}$$

Thus, we have the global classical solutions of the mixed initial-boundary value problem (5.14)-(5.16). More precisely, the following existence theorem was proved by Shao [14].

Theorem B. Suppose that u_0, v_0 are all C^1 functions with respect to their arguments, for which there is a constant $M > 0$ such that

$$||u_0(\theta)||_{C^0}, \; ||v_0(\theta)||_{C^0}, \; ||u_0'(\theta)||_{C^0}, \; ||v_0'(\theta)||_{C^0} \le M, \tag{5.29}$$

Suppose furthermore that the conditions of C^1 compatibility are satisfied at the point $(0, 0)$. Then there exists a small positive constant ε independent of M such that, if (5.29) holds together with

$$\int_0^{+\infty} |u_0'(\theta)| d\theta, \; \int_0^{+\infty} |v_0'(\theta)| d\theta \;\; \le \varepsilon, \tag{5.30}$$

then the mixed initial-boundary value problem (5.14)-(5.16) admits a unique global C^1 solution $U = U(t, \theta)$ in the half space $\{(t, \theta) | t \ge 0, \theta \ge 0\}$.

By Theorem A and Theorem 1.1 , we get the following theorem.

Theorem 5.1. Under the assumptions of Theorem B, for the mixed initial-boundary value problem (5.14)-(5.16), if

$$N \triangleq \max\{\int_0^{+\infty} |u_0(\theta)| d\theta, \int_0^{+\infty} |v_0(\theta)| d\theta\} < +\infty, \tag{5.31}$$

then there exists a unique C^1vector-valued function $\phi(\theta) = (\phi_1(\theta), \dots, \phi_{2n}(\theta))^T$ such that in the normalized coordinates

$$u_i(t, \theta) \to \phi_i(\theta + \frac{t}{|\widetilde{v}_0|}), \qquad t \to +\infty, \qquad i = 1, \dots, n, \tag{5.32}$$

$$v_i(t, \theta) \to \phi_{n+i}(\theta - \frac{t}{|\widetilde{v}_0|}), \qquad t \to +\infty, \qquad i = 1, \dots, n. \tag{5.33}$$

Moreover, $\phi_i(\theta) (i = 1, \dots, 2n)$ are global Lipschitz continuous, more precisely, there exists a positive constant κ_1 independent of ε, M, θ_1 and θ_2 such that

$$|\phi_i(\theta_1) - \phi_i(\theta_2)| \le \kappa_1 M |\theta_1 - \theta_2|, \qquad \forall \, \theta_1, \theta_2 \in \mathbf{R}. \tag{5.34}$$

Furthermore, if $u_0'(\theta)$ and $v_0'(\theta)$ are global ρ-Hölder continuous, where $0 < \rho \le 1$, that is, there exists a positive constant ς such that

$$|u_0'(\theta_1) - u_0'(\theta_2)| + |v_0'(\theta_1) - v_0'(\theta_2)| \le \varsigma |\theta_1 - \theta_2|^\rho, \qquad \forall \, \theta_1, \theta_2 \in \mathbf{R}^+, \tag{5.35}$$

then $\phi'(\theta)$ is also global ρ-Hölder continuous and satisfies that

$$|\phi'(\theta_1) - \phi'(\theta_2)| \le \kappa_2 \varsigma (1 + MN + \varepsilon)^\rho |\theta_1 - \theta_2|^\rho + \kappa_2 M^2 (1 + \varepsilon)(1 + MN + \varepsilon) |\theta_1 - \theta_2|, \tag{5.36}$$

where κ_2 is a positive constant independent of $\varepsilon, M, N, \varsigma, \theta_1$ and θ_2.

Remark 5.1. By Remark 1.5 in Dai and Kong [4], the normalized coordinates always exist for system (5.14).

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