NEW ENTROPY CONDITIONS FOR THE SCALAR CONSERVATION LAW WITH DISCONTINUOUS FLUX

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Abstract. We propose new Kruzhkov type entropy conditions for one dimensional scalar conservation law with a discontinuous flux in rather general form (no convexity or genuine linearity is needed). We prove existence and uniqueness of the entropy admissible weak solution to the corresponding Cauchy problem merely under the assumption that initial data belong to the BV-class. Such initial data enable us to prove that the sequence of solutions to a special vanishing viscosity approximation of the considered equation is, at the same time, the sequence of quasisolutions to a non-degenerate scalar conservation law. This provides existence of the solution admitting strong traces at the interface. The admissibility conditions are chosen so that a kind of crossing condition is satisfied which, together with existence of traces, provides uniqueness of the solution.

In the current contribution, we consider the following problem

$$\begin{cases} \partial_t u + \partial_x \left(H(x)f(u) + H(-x)g(u)\right) = 0, & (t,x) \in I\!R^+ \times I\!R \\ u|_{t=0} = u_0(x) \in BV(I\!R), & x \in I\!R \end{cases} \tag{1}$$

where u is the scalar unknown function; u_0 is an integrable initial function of bounded variation such that $a \leq u_0 \leq b$, $a, b \in I\!R$; H is the Heaviside function; and $f, g \in C^1_0(\mathbf{R})$ are such that $f(a) = f(b) = g(a) = g(b) = 0$.

Problems such as (1) are non-trivial generalization of scalar conservation law with smooth flux, and they describe different physical phenomena (flow in porous media, sedimentation processes, traffic flow, radar shape-from-shading problems, blood flow, gas flow in a variable duct...). Therefore, beginning with eighties (probably from [33]), problems of type (1) are under intensive investigations.

As usual in conservation laws, the Cauchy problem under consideration in general does not possess classical solution, and it can have several weak solutions. Since it is not possible to directly generalize standard theory of entropy admissible solutions [22], in order to choose a proper weak solution to (1) many admissibility conditions were proposed. We mention minimal jump condition [16], minimal variation condition and Γ condition [9, 10], entropy conditions [18, 1], vanishing capillary pressure limit [17], admissibility conditions via adapted entropies [5, 7] or via conditions at the interface [2, 11].

But, in every of the mentioned approaches, some structural hypothesis on the flux (such as convexity or genuine nonlinearity) or on the form of the solution (see

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[2]) were assumed. Such hypothesis were used to show existence of traces as well as existence or uniqueness of a weak solution to the considered problem.

Recently, in [25], we have proved existence and uniqueness in the multidimensional situation. Still, due to certain technical obstacles, admissible solutions selected in that paper are rather special.

Here, we propose admissibility conditions which involve much less restrictions than in previous works on the subject (excluding [25] where there are no restrictions), and we still can make many different stable semigroups depending on the physical situation under considerations. We only assume that $u_0 \in BV(I\!R)$ and

$$f \text{ and } g \text{ are not identically equal to zero on any subinterval } (\tilde{a}, \tilde{b}) \subset (a,b). \quad (2)$$

We remark that, in a view of recent preprint [4], the condition $u_0 \in BV(I\!R)$ can be omitted. We provide details in Remark 19.

Since one can find excellent overviews on the subject in many papers [4, 2, 6, 7, 11, 28] which are easily available via internet (e.g. www.math.ntnu.no/conservation), we shall restrict our attention on papers [18], [20], and [28] which are in the closest connection to our contribution.

In [18], degenerate parabolic equation with discontinuous flux is considered:

$$\begin{cases} \partial_t u + \partial_x \left(H(x) f(u) + H(-x) g(u) \right) = \partial_{xx} A(u), & (t,x) \in (0,T) \times I\!R \\ u|_{t=0} = u_0(x) \in BV(I\!R) \cap L^1(I\!R), & x \in I\!R, \end{cases}$$

where A is non-decreasing with $A(0) = 0$. Assuming that $A \equiv 0$ we obtain the problem of type (1). In order to obtain the uniqueness of a weak solution to the problem, the Kruzhkov type entropy admissibility condition [22] is used:

Definition 1. [18] Let u be a weak solution to problem (1).

We say that u is entropy admissible weak solution to (1) if the following entropy condition is satisfied for every fixed $\xi \in \mathbf{R}$:

$$\partial_t |u - \xi| + \partial_x \Big\{ \mathrm{sgn}(u - \xi) \Big[H(x)(f(u) - f(\xi)) + H(-x)(g(u) - g(\xi)) \Big] \Big\} \\ - |f(\xi) - g(\xi)| \delta(x) \leq 0 \ \text{ in } \ \mathcal{D}'(I\!R^+ \times I\!R).$$

Still, merely such entropy condition was insufficient to prove stability of the admissible weak solution to the considered problem. Two more things were necessary.

First, one needs the following technical assumption:

Crossing condition: For any states u, v the following crossing condition must hold:

$$f(u) - g(u) < 0 < f(v) - g(v) \Rightarrow u < v.$$

Geometrically, the crossing condition requires that either the graph of f and g do not cross, or the graph g lies above the graph of f to the left of the crossing point (see Figure 1). The functions f and g appearing in (1) do not necessarily satisfy the crossing conditions, but it is possible to transform them so that the crossing condition is satisfied (see Figure 2).

We remark that the crossing condition is bypassed in [18] by using so called adapted entropies (see [5]). Admissibility conditions that we introduce in Definition 9 can be considered as a generalization of the approach from [18] (see Remark 10). In a matter of fact, our approach shed (another) light on how adapted entropies enabled avoiding the crossing conditions.

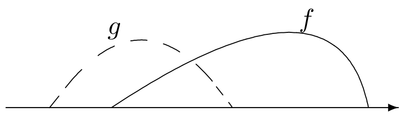

FIGURE 1. Functions f (normal line) and g (dashed line) satisfying the crossing condition.

Next, in [20] existence of strong traces at the interface $x = 0$ was necessary. We provide appropriate definition.

Definition 2. Let $W : I\!R \times I\!R^+ \to I\!R$ be a function that belongs to $L^\infty(I\!R \times I\!R^+)$. By the right and left traces of $W(\cdot, t)$ at the point $x = 0$ we understand functions $t \mapsto W(0\pm, t) \in L^\infty_{loc}(I\!R^+)$ that satisfy for a.e. $t \in I\!R^+$:

$$\text{esslim}_{x\uparrow 0} |W(x,t) - W(0+,t)| = 0, \quad \text{esslim}_{x\downarrow 0} |W(x,t) - W(0-,t)| = 0$$

Assuming the crossing condition and the existence of traces, we have the following theorem:

Theorem 3. [18] *Assume that weak solutions u and v to (1) with the initial conditions u_0 and v_0, respectively, satisfy entropy admissibility conditions from Definition 1 and admit left and right strong traces at the interface $x = 0$.*

Then for any $T, R > 0$ there exist constants $C, \bar{R} > 0$ such that:

$$\int_0^T \int_{-R}^R |v(t,x) - u(t,x)| dx dt \leq CT \int_{-\bar{R}}^{\bar{R}} |v_0(x) - u_0(x)| dx. \tag{3}$$

Remark 4. It is important to notice that Theorem 3 remains to hold if in (1), instead of $\partial_t u$, we put $\partial_t(\alpha(u)H(x) + \beta(u)H(-x))$, for some strictly increasing functions $\alpha, \beta : [a,b] \to [a,b]$. Indeed, since we did not put a function depending on $t \in I\!R^+$ under the derivative ∂_t, and since α and β are increasing functions (we can extract all the information on u knowing only $\beta(u)$ or $\alpha(u)$), we can safely use results from [20] on the equation $\partial_t(\alpha(u)H(x)+\beta(u)H(-x)) + \partial_x(f(u)H(x)+g(u)H(-x)) = 0$.

First, we shall explain how to force the crossing condition and existence of traces. We shall use the idea from [28]. In [28] the following problem was considered

$$\begin{aligned} \partial_t u + \partial_x f(\alpha(x,u)) =& 0, \\ u|_{t=0} =& u_0(x), \end{aligned} \tag{4}$$

where α is a function discontinuous in $x \in I\!R$ and strictly increasing with respect to u. Then we can write:

$$v = \alpha(x,u) \Rightarrow u = \beta(x,v).$$

Problem (4) becomes

$$\begin{aligned} \partial_t \beta(x,v) + \partial_x f(v) =& 0, \\ v|_{t=0} =& \alpha(x,u_0). \end{aligned} \tag{5}$$

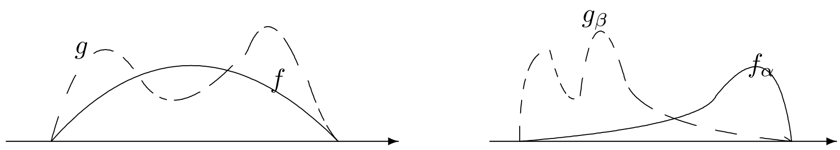

FIGURE 2. Functions f (normal line) and g (dashed line) on the left plot do not satisfy the crossing condition. On the other hand, for appropriate (highly convex) α and (highly concave) β, the functions $f_\alpha = f \circ \alpha$ and $g_\beta = g \circ \beta$ on the right plot satisfy the crossing conditions.

Thus, the discontinuity in x is removed out of the derivative in x, and we can apply standard vanishing viscosity approach:

$$\begin{aligned} \partial_t \beta(x, v_\varepsilon) + \partial_x f(v_\varepsilon) =& \varepsilon \partial_{xx} v_\varepsilon, \\ v|_{t=0} =& \alpha(x, u_0), \end{aligned} \tag{6}$$

to obtain the sequence (v_ε) strongly converging in $L^1_{loc}(I\!R \times I\!R^+)$ to a unique Kruzhkov admissible weak solution v of (5) which immediately gives uniqueness of appropriate weak solution to (4).

It is important to notice that the existence and uniqueness are actually obtained thanks to the appropriate choice of the viscosity term. Such choice enables the author to control the flux corresponding to (4).

Using this observation, we shall propose new admissibility conditions which will enable us to control the flux corresponding to (1) in an extent which will provide uniqueness in a rather general situation. Informally speaking, we shall consider the following vanishing viscosity regularization to (1):

$$\begin{cases} \partial_t u + \partial_x \left(H(x) f(u) + H(-x) g(u) \right) = \varepsilon \partial_{xx} (\tilde{\alpha}(u) H(x) + \tilde{\beta}(u) H(-x)), \\ u|_{t=0} = u_0(x), \end{cases} \tag{7}$$

where $\tilde{\alpha}, \tilde{\beta} : [a, b] \to [a', b']$ are smooth strictly increasing functions. In the sequel, without loosing on generality, we shall assume that $[a', b'] = [a, b]$.

Denote by α and β the inverse functions of the functions $\tilde{\alpha}$ and $\tilde{\beta}$, respectively. Introducing the change of the unknown function:

$$v = \tilde{\alpha}(u) H(x) + \tilde{\beta}(u) H(-x) \Rightarrow u = \alpha(v) H(x) + \beta(v) H(-x),$$

and denoting $f_\alpha = f \circ \alpha$ and $g_\beta = g \circ \beta$, we have from (7):

$$\begin{cases} \partial_t (\alpha(v) H(x) + \beta(v) H(-x)) + \partial_x \left(H(x) f_\alpha(v) + H(-x) g_\beta(v) \right) = \varepsilon \partial_{xx} v, \\ v|_{t=0} = \tilde{\alpha}(u_0) H(x) + \tilde{\beta}(u_0) H(-x). \end{cases} \tag{8}$$

So, instead of dealing with the flux $H(x) f(u) + H(-x) g(u)$, we deal with the new flux $H(x) f_\alpha(v) + H(-x) g_\beta(v)$. Clearly, by choosing appropriate functions α and β we can always make the new flux to satisfy "the crossing condition" (see Figure 2). From here, appealing on [18], we conclude that we need only existence of traces to obtain the uniqueness.

The question of existence of traces is rather serious in itself [23, 27, 34] and usually demands a kind of genuine nonlinearity condition:

Definition 5. Let $h : I\!R^2 \to I\!R$ and $f, g : I\!R \to I\!R$.

We say that the vector $(h(x,\lambda), H(x)f(\lambda) + H(-x)g(\lambda))$ is genuinely nonlinear if for almost every $x \in I\!R$ and every $(\xi_0, \xi_1) \in S^1$, $S^1 \subset I\!R^2$ is two dimensional sphere, the mapping

$$(a,b) \ni \lambda \mapsto \xi_0 h(x,\lambda) + \xi_1 \left(H(x)f(\lambda) + H(-x)g(\lambda)\right),$$

is different from a constant on any non-degenerate interval $(\alpha,\beta) \subset (a,b)$.

In order to formulate a necessary theorem about the existence of traces, we need the notion of the quasi-solution.

Definition 6. We say that the function $u \in L^\infty(I\!R^d)$ is a quasi-solution to the scalar conservation law

$$\mathrm{div}_x F(u) = 0, \quad x \in I\!R^d,$$

where $F = (F_1, \dots, F_d) \in C(I\!R^d; I\!R)$ if it satisfies for every $\xi \in I\!R$:

$$\mathrm{div}_x \mathrm{sgn}(u-\xi)(F(u) - F(\xi)) = \gamma_k \ \text{ in } \ \mathcal{D}'(I\!R^d),$$

where γ_k is a locally bounded Borel measure.

Next theorem can be found in [27]. We adapt it to our situation.

Theorem 7. [27] *Let $h, f \in C(I\!R)$.*

Suppose that the function u is a quasi-solution to

$$\partial_t h(u) + \partial_x f(u) = 0,$$

where the vector (h,f) is genuinely nonlinear.

Then, the function u admits right and left strong traces at $x = 0$.

Now, we can explain how to force the existence of traces for a solution to (1). At the same time, due to similarity in the approach, we shall make a plan how to deal with the existence question. We need the following theorem formulated so that it corresponds to our purposes. It uses the genuine nonlinearity condition similarly as Theorem 7.

Theorem 8. [26] *Assume that the vector $(h(x,u), H(x)f(u)+H(-x)g(u))$, $(x,u) \in I\!R \times I\!R$, is genuinely nonlinear in the sense of Definition 5.*

Then, the following statement holds:
Each family $(v_\varepsilon(t,x)) \in L^\infty(I\!R^+ \times I\!R)$, $a \le v_\varepsilon \le b$, $\varepsilon > 0$, such that for every $c \in I\!R$ the quantity

$$\begin{aligned}&\partial_t (H(v_\varepsilon - c)(h(x,v_\varepsilon) - h(x,c))) \\ &\quad + \partial_x \left(H(v_\varepsilon - c)((H(x)(f(v_\varepsilon) - f(c)) + H(-x)(g(v_\varepsilon) - g(c))))\right)\end{aligned} \tag{9}$$

is precompact in $W^{-1,2}_{\mathrm{loc}}(I\!R^+ \times I\!R)$, contains a subsequence convergent in $L^1_{\mathrm{loc}}(I\!R^+ \times I\!R)$.

In the case of a scalar conservation law with smooth flux, the proof of existence is based on the BV-estimates for a sequence of solutions to the corresponding Cauchy problem regularized with the vanishing viscosity. Such estimates are not available if the flux is discontinuous. Therefore, we need to apply more subtle arguments involving singular mapping [33], compensated compactness [20, 19, 31], difference schemes [2, 21, 18] or H-measures [14, 15, 26, 32].

In general, using e.g. the compensated compactness, it is possible to prove that the sequence (u_ε) of solutions to (7) weakly converges to a weak solution u of (1).

But, it is not possible to state that the weak solution satisfies wanted admissibility conditions. In order to be sure that u is admissible, in principle, we need to prove that the corresponding sequence (u_ε) strongly converges strongly in $L^1_{loc}(I\!R^+ \times I\!R)$ to u (still, not necessarily; see [29]). However, at least in the framework of the compensated compactness (or the H-measures which we are going to use), this can be proved only by assuming the genuine nonlinearity condition from Definition 5.

In order to overcome this obstacle we shall use an idea from [20] which is further developed in [3]. In [20, 3], existence of solution to a Cauchy problem of type (1) is proved. Roughly speaking, the key point of the proof is based on a lemma stating that if in (1) we assume $u_0 \in BV(I\!R)$, then, for the sequence (u_ε) of solutions to (7), it holds $\|\partial_t u_\varepsilon\|_{L^1(I\!R)} \leq const$ for every fixed $t, \varepsilon \in I\!R^+$. This actually means that for any function $h(x,\lambda)$, $x, \lambda \in I\!R$, which is Lipshitz continuous in λ, it holds $\|\partial_t h(x, u_\varepsilon)\|_{L^1(I\!R)} \leq const$ for every fixed $t, \varepsilon \in I\!R^+$.

Next, it is not difficult to prove that it holds for the sequence (u_ε) of solutions to (7)

$$\partial_t(H(u_\varepsilon - c)(u_\varepsilon - c)) + \partial_x \left(H(u_\varepsilon - c)\left(H(x)(f(u_\varepsilon) - f(c)) + H(-x)(g(u_\varepsilon) - g(c))\right)\right)$$

is precompact in $W^{-1,2}_{\rm loc}(I\!R^+ \times I\!R^d)$. However, since $(|\partial_t u_\varepsilon|)$ is the sequence bounded in the space of Radon measures, we also have:

$$\begin{aligned}&\partial_t(H(u_\varepsilon - c)\left(H(x)(h_R(u_\varepsilon) - h_R(c)) + H(-x)(h_L(u_\varepsilon) - h_L(c))\right)\\ &+ \partial_x \left(H(u_\varepsilon - c)\left(H(x)(f(u_\varepsilon) - f(c)) + H(-x)(g(u_\varepsilon) - g(c))\right)\right)\end{aligned}$$

is precompact in $W^{-1,2}_{\rm loc}(I\!R^+ \times I\!R^d)$ if $h_L, h_R \in {\rm Lip}(I\!R)$ (Lipschitz continuous functions). Furthermore, if we choose h_L and h_R so that the vector $(H(x)h_R(u) + H(-x)h_L(u), H(x)f(u) + H(-x)g(u))$ is genuinely nonlinear, we can apply Theorem 8 to conclude about strong L^1_{loc} precompactness of the family (u_ε). It is clear that a L^1_{loc} limit along a subsequence of the family (u_ε) will represent wanted admissible weak solution to (1). Furthermore, according to Theorem 7, we infer about the existence of traces on the interface $x = 0$ for the previously constructed weak solution which immediately gives the uniqueness.

1. New entropy admissibility conditions

In order to simplify the presentation, we shall assume that the mappings

$$\lambda \mapsto f(\lambda), \quad \lambda \mapsto g(\lambda), \tag{10}$$

are non-constant on any interval $(\alpha, \beta) \subset (a, b)$.

Together with (2), such condition provides that the vector $(H(x)f^2(\lambda) + H(-x)g^2(\lambda), H(x)f(\lambda) + H(-x)g(\lambda))$, $x, \lambda \in I\!R$, is genuinely nonlinear. Otherwise, we could take disjunct intervals $(\alpha_i^f, \beta_i^f) \subset (a, b)$, $i = 1, \dots, d_1$, where the function f is constant, and the disjunct intervals $(\alpha_i^g, \beta_i^g) \subset (a, b)$, $i = 1, \dots, d_2$, where the function g is constant. Then, we take the functions $\hat{f}, \hat{g} \in {\rm Lip}(I\!R)$:

$$\hat{f}(\lambda) = \begin{cases} 0, & \lambda \notin (\alpha_i^f, \beta_i^f), \quad i = 1, \dots, d_1, \\ (\lambda - \alpha_i^f)(\lambda - \beta_i^f), & \lambda \in (\alpha_i^f, \beta_i^f), \end{cases}$$

$$\hat{g}(\lambda) = \begin{cases} 0, & \lambda \notin (\alpha_i^g, \beta_i^g), \quad i = 1, \dots, d_2, \\ (\lambda - \alpha_i^g)(\lambda - \beta_i^g), & \lambda \in (\alpha_i^g, \beta_i^g). \end{cases}$$

Now, the vector $(H(x)(f^2+\hat{f})(u)+H(-x)(g^2+\hat{g})(u), H(x)f(u)+H(-x)g(u))$ is genuinely nonlinear and, if (10) is not satisfied, everywhere in the sequel we can choose it instead of the vector $(H(x)f^2(\lambda)+H(-x)g^2(\lambda), H(x)f(\lambda)+H(-x)g(\lambda))$.

Finally, we are ready to introduce the new entropy admissibility conditions.

Definition 9. Let u be a weak solution to problem (1). Let $\alpha, \beta : [a,b] \to [a,b]$ be smooth strictly increasing functions. Denote by $\tilde{\alpha}$ and $\tilde{\beta}$ the inverse functions to α and β, respectively.

We say that u is an (α,β)-admissible weak solution to (1) if the function $v = \tilde{\alpha}(u)H(x) + \tilde{\beta}(u)H(-x)$ satisfies the following entropy condition for every fixed $\xi \in \mathbf{R}$:

$$\begin{aligned}\partial_t\Big\{&\mathrm{sgn}(v-\xi)\Big[H(x)(\alpha(v)-\alpha(\xi))+H(-x)(\tilde{\beta}(v)-\tilde{\beta}(\xi))\Big]\Big\}\\&+\partial_x\Big\{\mathrm{sgn}(v-\xi)\Big[H(x)(f_\alpha(v)-f_\alpha(\xi))+H(-x)(g_\beta(v)-g_\beta(\xi))\Big]\Big\}\\&\qquad\qquad-|f_\alpha(\xi)-g_\beta(\xi)|\delta(x)\le 0,\end{aligned} \tag{11}$$

where, as before, $f_\alpha = f\circ\alpha$ and $g_\beta = g\circ\beta$.

Remark 10. Here, we will explain how the given definition of admissibility can be understood as a generalization of the admissibility conditions [7, Definition 3.1.]. Let us briefly recall the concept from [7]. First, we need the function c^{AB} (see [7, (11)]):

$$c^{AB}(x) = \begin{cases} A, & x\ge 0\\ B, & x<0\end{cases}.$$

In [7], the function c^{AB} is used to form the function $u\mapsto |u-c^{AB}(x)|$ which is an example of what is called an adapted entropy in [5]. Still, in [5], the existence of infinitely many adapted entropies was necessary to prove uniqueness (see also [28]) while in [7] only the entropy $u\mapsto |u-c^{AB}(x)|$ was enough for uniqueness (together with the classical Kruzhkov entropies out of the interface). The function c^{AB} is called a connection if it represents a weak solution to (1), i.e. if $f(A)=g(B)$. The following admissibility conditions were used in [7]:

Definition 11. [7, Definition 3.1.] (Entropy solution of type (A,B)). A measurable function $u : I\!R^+\times I\!R\to I\!R$, representing a weak solution to (1) is an entropy solution of type (A,B) if it satisfies the following conditions:

(D.1) $u\in L^\infty(I\!R^+\times I\!R)$; $u(t,x)\in[a,b]$ for a.e. $(t,x)\in I\!R^+\times I\!R$.

(D.2) For any test function $0\le\varphi\in\mathcal{D}([0,T)\times I\!R)$, $T>0$, which vanishes for $x\ge 0$, and any $\xi\in I\!R$, the following holds:

$$\int_0^T\int_{I\!R}\left(|u-\xi|\varphi_t+\mathrm{sgn}(u-\xi)(f(u)-f(\xi))\varphi_x\right)dxdt+\int_{I\!R}|u_0-\xi|\varphi(0,x)dx\ge 0,$$

and for any test function $0\le\varphi\in\mathcal{D}([0,T)\times I\!R)$, $T>0$, which vanishes for $x\le 0$

$$\int_0^T\int_{I\!R}\left(|u-\xi|\varphi_t+\mathrm{sgn}(u-\xi)(g(u)-g(\xi))\varphi_x\right)dxdt+\int_{I\!R}|u_0-\xi|\varphi(0,x)dx\ge 0,$$

(D.3) The following Kruzhkov-type entropy inequality holds for any test function $0 \leq \varphi \in \mathcal{D}([0,T) \times I\!\!R)$, $T > 0$,

$$\int_0^T \int_{I\!\!R} \Big(|u - c^{AB}(x)|\varphi_t$$
$$+\mathrm{sgn}(u - c^{AB}(x))(H(x)(f(u) - f(A)) + H(-x)(g(u) - g(B)))\varphi_x \Big) dxdt$$
$$+ \int_{I\!\!R} |u_0 - c^{AB}(x)|\varphi(0,x)dx \geq 0.$$

Now, assume that a function u is the entropy solution of type (A, B) in the sense of Definition 11. Take the functions α and β from Definition 9 so that for some constant c it holds $\alpha(c) = A$ and $\beta(c) = B$. Assume that the functions f_α and g_β satisfy the crossing conditions. Notice that, in the cases when uniqueness is proved in [7], we can always choose such α and β. Then, the function c^{AB} will represent a unique (α, β)-admissible weak solution to (1) in the sense of Definition 9. Taking another (α, β)-admissible weak solution to (1), say $u = \alpha(v)H(x) + \beta(v)H(-x)$, in the sense of Definition 9 and applying the procedure from [18], we reach to the following well known relation:

$$\begin{aligned} &\partial_t \mathrm{sgn}(v - c)\left(H(x)(\alpha(v) - \alpha(c)) + H(-x)(\beta(v) - \beta(c))\right) \\ &+ \partial_x \mathrm{sgn}(v - c)\left(H(x)(f_\alpha(v) - f_\alpha(c)) + H(-x)(g_\beta(v) - g_\beta(c))\right) \leq 0. \end{aligned} \tag{12}$$

Since α and β as well as their inverses $\tilde{\alpha}$ and $\tilde{\beta}$ are increasing functions. it holds

$$\mathrm{sgn}(v - c) = \mathrm{sgn}\left(\tilde{\alpha}(u) - \tilde{\alpha}(A)H(x) + \tilde{\beta}(u) - \tilde{\beta}(B)H(x)\right) = \mathrm{sgn}(u - c^{AB}).$$

From here, we see that (12) is actually condition (D.3) from Definition 11 meaning that the (α, β)-admissible weak solution u is, at the same time, an entropy solution of type (A, B) (conditions (D.1.) and (D.2.) from Definition 11 are easily checked).

The following theorem is the main theorem of the paper:

Theorem 12. *There exists a pair of function (α, β) from Definition 9 such that there exists a unique (α, β)-entropy admissible solution to (1).*

For such α and β any two (α, β)-entropy admissible solutions u and v to (1) satisfy (3).

In order to construct an (α, β)-entropy admissible solution to (1), we use non-standard vanishing viscosity approximation with regularized flux.

First, introduce the following change of the unknown function u:

$$u(t, x) = \tilde{\alpha}(v(t, x))H(x) + \tilde{\beta}(v(t, x))H(-x),$$

for the functions from Definition 9. Equation (1) becomes:

$$\partial_t \left(H(x)\alpha(v) + H(-x)\beta(v)\right) + \partial_x \left(H(x)f_\alpha(v) + H(-x)g_\beta(v)\right) = 0. \tag{13}$$

Then, take the following regularization of the Heaviside function H

$$H_\varepsilon(x) = \int_{-\infty}^{x/\varepsilon} \omega(z)dz,$$

where ω is a smooth even compactly supported function with total mass one.

Then, consider the following regularized problem:

$$\begin{aligned}&\partial_t\left(H_\varepsilon(x)\alpha(v_\varepsilon)+H_\varepsilon(-x)\beta(v_\varepsilon)\right)\\&+\partial_x\left(H_\varepsilon(x)f_\alpha(v_\varepsilon)+H_\varepsilon(-x)g_\beta(v_\varepsilon)\right)=\varepsilon\partial_{xx}v_\varepsilon.\end{aligned}\tag{14}$$

Obviously, for every fixed $\varepsilon>0$ quasilinear parabolic equation (14) augmented with the initial condition $v_\varepsilon\big|_{t=0}=(\tilde\alpha(u_0)H(x)+\tilde\beta(u_0)H(-x))\star\frac{1}{\varepsilon}\omega(\cdot/\varepsilon)\chi_\varepsilon(x)$, where χ_ε is a smooth function equal to one in the interval $(-1/\varepsilon,1/\varepsilon)$ and zero out of the interval $(-2/\varepsilon,2/\varepsilon)$, will have a unique smooth solution v_ε.

Since α and β are strictly increasing functions, slightly modifying the methodology from [7], we obtain the following three lemmas.

Lemma 13. [7, Lemma 4.1] *[L^∞-bound] There exists constant $c_0>0$ such that for all $t\in(0,T)$,*

$$\|v_\varepsilon(t,\cdot)\|_{L^\infty(I\!R)}\le c_0.$$

More precisely,

$$a\le v_\varepsilon\le b.$$

Lemma 14. [7, Lemma 4.2] *[Lipshitz regularity in time] There exists constant c_1, independent of ε, such that for all $t>0$,*

$$\iint_{I\!R}|\partial_t v_\varepsilon(\cdot,t)|\,dx\le c_1.$$

Lemma 15. [7, Lemma 4.3] *[Entropy dissipation bound] There exists a constant c_2 independent from ε such that*

$$\varepsilon\int_{I\!R}\left(\partial_x v_\varepsilon(t,x)\right)^2dx\le c_2,$$

for all $t>0$.

To proceed, we need Murat's lemma:

Lemma 16. [13] *Assume that the family (Q_ε) is bounded in $L^p(\Omega)$, $\Omega\subset I\!R^d$, $p>2$. Then,*

$$(\operatorname{div}Q_\varepsilon)_\varepsilon\in W^{-1,2}_{\rm c,loc}\quad\text{if}\quad\operatorname{div}Q_\varepsilon=p_\varepsilon+q_\varepsilon,$$

with $(q_\varepsilon)_\varepsilon\in W^{-1,2}_{\rm c,loc}(\Omega)$ and $(p_\varepsilon)_\varepsilon\in\mathcal{M}_{b,loc}(\Omega)$.

Now, we can prove a crucial lemma for obtaining the existence of the (α,β)-admissible weak solution to (1).

Lemma 17. *Denote for a fixed $\xi\in I\!R$:*

$$\begin{aligned}q(x,\lambda)&=H(\lambda-k)\Big(H(x)(f_\alpha(\lambda)-f_\alpha(\xi))+H(-x)(g_\beta(\lambda)-g_\beta(\xi))\Big),\\ \bar q(x,\lambda)&=H(\lambda-k)\Big(H(x)(f^2_\alpha(\lambda)-f^2_\alpha(\xi))+H(-x)(g^2_\beta(\lambda)-g^2_\beta(\xi))\Big),\\ q_{\alpha,\beta}(x,\lambda)&=H(\lambda-k)\Big(H(x)(\alpha(\lambda)-\alpha(\xi))+H(-x)(\beta(\lambda)-\beta(\xi))\Big).\end{aligned}\tag{15}$$

The family

$$\partial_t\bar q(x,v_\varepsilon)+\partial_x q(x,v_\varepsilon),\quad\varepsilon>0,\tag{16}$$

is precompact in $W^{-1,2}_{\rm loc}(I\!R^+\times I\!R)$.

Proof:
Denote $\eta'(\lambda)=H(\lambda-\xi)$. Define the entropy flux which corresponds to (14):

$$q^\varepsilon(x,\lambda)=H(\lambda-\xi)\Big(H_\varepsilon(x)(f_\alpha(\lambda)-f_\alpha(\xi))+H_\varepsilon(-x)(g_\beta(\lambda)-g_\beta(\xi))\Big),$$
$$q^\varepsilon_{\alpha,\beta}(x,\lambda)=H(\lambda-\xi)\Big(H_\varepsilon(x)(\alpha(\lambda)-\alpha(\xi))+H_\varepsilon(-x)(\beta(\lambda)-\beta(\xi))\Big).$$

Denote $\delta_\varepsilon(x)=H'_\varepsilon(x)$, $i=1,2$. After multiplying (14) by $\eta'(v_\varepsilon)$, we obtain in the sense of distributions:

$$\begin{aligned}&\partial_t q^\varepsilon_{\alpha,\beta}(x,v_\varepsilon)+\partial_x q^\varepsilon(x,v_\varepsilon)\\&=(\delta_\varepsilon(x)f_\alpha(\xi)-\delta_\varepsilon(x)g_\beta(\xi))+\varepsilon(\partial_x(v_{\varepsilon x}\eta'(v_\varepsilon))-(v_{\varepsilon x})^2\eta''(v_\varepsilon)\\&\leq \delta_\varepsilon(x)\,(f_\alpha(\xi)-g_\beta(\xi))+\varepsilon(\partial_x(v_{\varepsilon x}\eta'(v_\varepsilon)).\end{aligned}\tag{17}$$

From here, according to the Schwartz lemma for non-negative distributions, we conclude that there exists a positive Radon measure $\mu^\varepsilon_\xi(t,x)$ such that:

$$\begin{aligned}&\partial_t q^\varepsilon_{\alpha,\beta}(x,v_\varepsilon)+\partial_x q^\varepsilon(x,v_\varepsilon)\\&=\delta_\varepsilon\,(f_\alpha(\xi)-g_\alpha(\xi))+\varepsilon(\partial_x(v_{\varepsilon x}\eta'(v_\varepsilon))-\mu^\varepsilon_\xi(t,x).\end{aligned}\tag{18}$$

Rewrite expression (18) in the form:

$$\begin{aligned}&\partial_t\bar q(x,v_\varepsilon)+\partial_x q(x,v_\varepsilon)\\&=\partial_t\left(\bar q(x,v_\varepsilon)-q^\varepsilon_{\alpha,\beta}(x,v_\varepsilon)\right)+\partial_x\left(q^\varepsilon(x,v_\varepsilon)-q(x,v_\varepsilon)\right)\\&+\delta_\varepsilon\,(f_\alpha(\xi)-g_\beta(\xi))+\varepsilon(\partial_x(v_{\varepsilon x}\eta'(v_\varepsilon))-\mu^\varepsilon_\xi(t,x).\end{aligned}\tag{19}$$

Since, clearly, $q^\varepsilon(x,v_\varepsilon)-q(x,v_\varepsilon)\to 0$ as $\varepsilon\to 0$ pointwisely, we derive the statement of the lemma from the Lebesgue dominated convergence theorem, Lemmas 13-15, and Lemma 16. For details please consult [3, Theorem 2.6.]
□
From Lemma 17 and Theorem 8, it is easy to prove the following theorem:

Theorem 18. *For every functions (α,β) from Definition 9 there exists an (α,β)-entropy admissible weak solution to (1).*

Proof:
First, notice that the vector $(\bar q(x,\lambda),q(x,\lambda))$ from (15) is genuinely nonlinear. Indeed, for $x>0$ the vector reduces to $(f^2_\alpha(\lambda),f_\alpha(\lambda))$ and this is obviously genuinely nonlinear vector since, due to (2) and (10), for any $\xi_0,\xi_1\in I\!R$, it holds $\xi_0 f^2(\lambda)\neq\xi_1 f(\lambda)$ for a.e. $\lambda\in(a,b)$. Similarly, we conclude about the genuine nonlinearity for $x<0$.

Now, from Theorem 8 and Lemma 17, we conclude that the family (v_ε) of solutions to (14) is strongly precompact in $L^1_{loc}(I\!R^+\times I\!R)$. Denote by v the $L^1_{loc}(I\!R^+\times I\!R)$ limit along a subsequence of the family (v_ε). Clearly, $u=\alpha(v)H(x)+\beta(v)H(-x)$ will represent the (α,β)-admissible weak solution to (1).
□
Now, we can prove the main theorem of the paper.
Proof of Theorem 12:
First, choose functions α and β from Definition 9 such that the functions f_α and g_β satisfy the crossing condition (see Figure 2). Then, notice that from the construction (it is enough to let $\varepsilon\to 0$ in (19)) and Lemma 14, it follows that for an

(α,β)-admissible weak solution u to (1), the function $v=\alpha(u)H(x)+\beta(u)H(-x)$ is, at the same time, a quasi-solution to the problem:

$$\begin{aligned}&\partial_t\Big(H(x)f^2_\alpha(v)+H(-x)g^2_\beta(v)\Big)+\partial_x\Big(H(x)f_\alpha(v)+H(-x)g_\beta(v)\Big)=0,\\ &v|_{t=0}=\tilde\alpha(u_0)H(x)+\tilde\beta(u_0)H(-x).\end{aligned}$$

Since the vector $(H(x)f^2_\alpha(\lambda)+H(-x)g^2_\beta(\lambda), H(x)f_\alpha(\lambda)+H(-x)g_\beta(\lambda))$ is genuinely nonlinear, according to Theorem 7, the function v admits strong traces at the interface $x=0$.

Similarly, from the construction again and according to the choice of the function α and β, we see that v is an entropy admissible solution in the sense of Definition 1 to the Cauchy problem

$$\begin{aligned}&\partial_t\left(\tilde\alpha(v)H(x)+\tilde\beta(v)H(-x)\right)+\partial_x\left(H(x)f_\alpha(v)+H(-x)g_\alpha(v)\right)=0,\\ &v|_{t=0}=\alpha(u_0)H(x)+\beta(u_0)H(-x),\end{aligned}\tag{20}$$

where f_α and g_β satisfy the crossing condition.

According to Theorem 3, we conclude that v is a unique entropy admissible solution to (20) in the sense of Definition 1 implying that $u=\tilde\alpha(v)H(x)+\tilde\beta(v)H(-x)$ is a unique (α,β)-admissible weak solution to (1).

□

Remark 19. If the initial data $u_0\notin BV(I\!R)$ then, approximate the function u_0 by a sequence $(u_{0\delta})\in BV(I\!R)$ so that

$$u_0-u_{0\delta}\to 0 \ \text{ as } \ \delta\to 0$$

strongly in $L^1_{loc}(I\!R)$. Then, we find a unique (α,β)-admissible weak solution u_δ to (1) where $u_{t=0}=u_{0\delta}$ (given α and β for which we have uniqueness). According to Theorem 12, the family (u_δ) satisfy the following stability relation:

$$\int_0^T\int_{-R}^R|u_{\delta_1}-u_{\delta_2}|dxdt\leq T\int_{-\bar R}^{\bar R}|u_{0\delta_1}-u_{0\delta_2}|dx,$$

where R and T are arbitrary positive constants, and $\tilde R$ is a large constant depending on R, the functions f, g, α and β. Since the right-hand side of the latter expression is uniformly small with respect to δ_1 and δ_2, from the Cauchy criterion we conclude that there exists $u\in L^1_{loc}$ such that $u_\delta\to u$ strongly in $L^1_{loc}(I\!R^d)$. Clearly, the function u will represent an (α,β)-admissible weak solution to (1) without BV-assumption on u_0.

Furthermore, in [4], it is announced that uniqueness can be obtained only by assuming existence of strong traces of the flux $H(x)f(u)+H(-x)g(u)$.

On the other hand, such traces exist almost always (see [27, Theorem 1.2]) and we would have existence and uniqueness without any restrictions on the initial data.

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