ZERO DIFFUSION-DISPERSION-SMOOTHING LIMITS FOR SCALAR CONSERVATION LAW WITH DISCONTINUOUS FLUX FUNCTION

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Abstract. We consider multi-dimensional conservation laws with discontinuous flux, which are regularized with vanishing diffusion and dispersion terms and with smoothing of the flux discontinuities. We use the approach of H-measures [17] to investigate the zero diffusion-dispersion-smoothing limit.

Content

1. Introduction

We consider the convergence of smooth solutions $u = u^{\varepsilon}(t,x)$ with $(t,x) \in \mathbf{R}^+ \times \mathbf{R}^d$ of the nonlinear partial differential equation

$$\partial_t u + \operatorname{div}_x f_{\varrho}(t,x,u) = \varepsilon \operatorname{div}_x b(\nabla u) + \delta \sum_{j=1}^{d} \partial^3_{x_j x_j x_j} u \tag{1}$$

as $\varepsilon \to 0$ and $\delta = \delta(\varepsilon)$, $\varrho = \varrho(\varepsilon) \to 0$. Here

$$\sup_{u \in \mathbf{R}} ||f_{\varrho}(t,x,u) - f(t,x,u)||_{L^p_{\rm loc}(\mathbf{R}^+ \times \mathbf{R}^d)} \to 0, \quad \varrho \to 0, \quad p > 2,$$

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for the Caratheodory flux vector $f \in C(\mathbf{R}; BV(\mathbf{R}^+_t \times \mathbf{R}^d_x))$. The aim is to show convergence to a weak solution of the corresponding hyperbolic conservation law:

$$\partial_t u + \operatorname{div}_x f(t,x,u) = 0, \quad u = u(t,x),\ x \in \mathbf{R}^d,\ t \geq 0. \tag{2}$$

We refer to this problem as the zero diffusion-dispersion-smoothing limit.

In the case when the flux f is at least Lipschitz continuous, it is well known that the Cauchy problem corresponding to (2) has unique admissible entropy solution in the sense of Kruzhkov [11] (or measure valued solution in the sense of DiPerna [3]). The situation is more complicated when the flux is discontinuous and it has been the subject of intensive investigations in recent years (see, e.g., [9] and references therein). The one-dimensional case of the problem is widely investigated using several approaches (numerical techniques [9, 1], compensated compactness [23, 10], kinetic approach [15, 2]). In the multidimensional case there are only a few results concerning existence of a weak solution. In [8] existence is obtained by a two-dimensional variant of compensated compactness, while in [24] the approach of H-measures [17] is used for the case of arbitrary space dimensions. Still, many open questions remain such as the uniqueness and stability of solutions.

A problem that has not yet been studied in the context of conservation laws with discontinuous flux, and which is the topic of the present paper, is that of zero diffusion-dispersion limits. When the flux is independent of the spatial and temporal positions, the study of zero diffusion-dispersion limits was initiated in [21] and further addressed in numerous works by LeFloch et al. (e.g., [12, 14, 13]). The compensated compactness method is the basic tool used in the one dimensional situation for the so-called limiting case in which the diffusion and dispersion parameters are in an appropriate balance, while for the case in which diffusion dominates dispersion, the notion of measure valued solutions [3, 22] is used. More recently, in [7] the limiting case has also been analyzed using the kinetic approach and velocity averaging [19].

The remaining part of this paper is organized as follows: In Section 2 we collect some basic a priori estimates for smooth solutions of (1). In Section 3 we look into the diffusion-dispersion-smoothing limit for multidimensional conservation laws with a flux vector which is discontinuous with respect to spatial variable. In doing so we rely on the a priori estimates from the previous section in combination with Panov's H-measures approach [17]. Finally, in Section 3 we restrict ourselves to the one dimensional case for which we obtain slightly stronger results using the compensated compactness method.

2. A priori inequalities

Assume that the flux f in equation (1) is smooth in all variables. Consider a sequence $(u_{\varepsilon,\delta})_{\varepsilon,\delta}$ of solutions of:

$$\partial_t u + \operatorname{div}_x f(t,x,u) = \varepsilon \operatorname{div}_x b(\nabla u) + \delta \sum_{j=1}^{d} \partial^3_{x_j x_j x_j} u, \tag{3}$$

$$u(x,0) = u_0(x), \quad x \in \mathbf{R}^d. \tag{4}$$

We assume that $(u_{\varepsilon,\delta})_{\varepsilon,\delta}$ has enough regularity so that all formal computations below are correct. So, following Schonbek [21], we assume that for every $\varepsilon, \delta > 0$ we have $u_{\varepsilon,\delta} \in L^\infty([0,T]; H^4(\mathbf{R^d}))$.

Later on, we will assume that the initial data u_0 depends on ε. In this section, we shall determine a priori inequalities for the solutions of problem (3), (4).

To simplify the notation we will write u_ε instead of $u_{\varepsilon,\delta}$.

We shall need the following assumptions on the diffusion term $b(\lambda) = (b_1(\lambda), \dots, b_n(\lambda))$.

(H1) For some positive constants C_1, C_2 we have:

$$C_1|\lambda|^2 \leq \lambda \cdot b(\lambda) \leq C_2|\lambda|^2 \text{ for all } \lambda \in \mathbf{R}^d.$$

(H2) The gradient matrix $Db(\lambda)$ is a positive definite matrix, uniformly in $\lambda \in \mathbf{R}^d$, i.e., for every $\lambda, \varrho \in \mathbf{R}^d$, there exists a positive constant C_3 such that we have:

$$\varrho^T Db(\lambda)\varrho \geq C_3|\varrho|^2.$$

We use the following notation:

$$|D^2u|^2 = \sum_{i,k=1}^d |\partial^2_{x_i x_k} u|^2.$$

In the sequel, for a vector valued function $g = (g_1, \dots, g_d)$ defined on $\mathbf{R}^+ \times \mathbf{R}^d \times \mathbf{R}$, we denote:

$$|g|^2 = \sum_{i=1}^d |g_i|^2.$$

The partial derivative ∂_{x_i} in the point (t, x, u), where u possibly depends on (t, x), is defined by the formula:

$$\partial_{x_i} g(t, x, u(t, x)) = (D_{x_i} g(t, x, \lambda))|_{\lambda = u(t,x)}.$$

In particular, the total derivative D_{x_i} and the partial derivative ∂_{x_i} are connected by the identity

$$D_{x_i} g(t, x, u) = \partial_{x_i} g(t, x, u) + \partial_u g(t, x, u)\partial_{x_i} u.$$

Finally we use

$$\operatorname{div}_x g(t, x, u) = \sum_{i=1}^d D_{x_i} g_i(t, x, u), \quad g = (g_1, \dots, g_d),$$

$$\Delta_x q(t, x, u) = \sum_{i=1}^d D^2_{x_i x_i} q(t, x, u), \quad q \in C^2(\mathbf{R}^+ \times \mathbf{R}^d \times \mathbf{R}).$$

With the previous conventions, we introduce the following assumption on the flux vector f:

(H3) The growth of the velocity variable u and the spatial derivative of the flux f is such that for some $C, \alpha > 0$ we have

$$\sum_{i=1}^d |\partial_u f_i(t, x, u)| \leq C,$$

$$\sum_{i,j=1}^d |\partial_{x_i} f_j(t, x, u)| \leq \frac{\mu(t, x)}{1 + |u|^{1+\alpha}},$$

where $\mu \in \mathcal{M}(\mathbf{R}^+ \times \mathbf{R}^d)$ is a bounded measure (and, accordingly, the above inequality is understood in the sense of measures).

Now, we can prove the following theorem:

Theorem 1. *Suppose that the flux function $f = f(t, x, u)$ satisfies (H3) and that it is Lipschitz continuous on $\mathbf{R}^+ \times \mathbf{R}^d \times \mathbf{R}$. Assume also that initial data u_0 belongs to $L^2(\mathbf{R}^d)$. Under conditions (H1)–(H2) the sequence of solutions $(u_\varepsilon)_{\varepsilon>0}$ of (3)–(4) for every $t \in [0, T]$ satisfies the following inequalities:*

$$
\begin{aligned}
&\int_{\mathbf{R}^d} |u_\varepsilon(t,x)|^2 dx + \varepsilon \int_0^t \int_{\mathbf{R}^d} |\nabla u_\varepsilon(t',x)|^2 dx dt' \\
&\quad \leq C_4 \left(\int_{\mathbf{R}^d} |u_0(x)|^2 dx - \int_0^t \int_{\mathbf{R}^d} \int_0^{u_\varepsilon(t',x)} \operatorname{div}_x f(t',x,v) dv dx dt' \right),
\end{aligned} \tag{5}
$$

and

$$
\begin{aligned}
&\varepsilon^2 \int_{\mathbf{R}^d} |\nabla u_\varepsilon(t,x)|^2 dx + \varepsilon^3 \int_0^t \int_{\mathbf{R}^d} |D^2 u_\varepsilon(t',x)|^2 dx dt' \\
&\quad \leq C_5 \Big(\varepsilon^2 \int_{\mathbf{R}^d} |\nabla u_0(x)|^2 dx + \varepsilon \int_0^t \int_{\mathbf{R}^d} \sum_{k=1}^d |\partial_{x_k} f(t',x,u_\varepsilon(t',x))|^2 dx dt' \\
&\qquad\qquad + ||\partial_u f||^2_{L^\infty(\mathbf{R}^+\times\mathbf{R}^d\times\mathbf{R})} \Big),
\end{aligned} \tag{6}
$$

for some constants C_4 and C_5.

Proof: We follow the procedure from [7]. Given a smooth function $\eta = \eta(u)$, $u \in \mathbf{R}$, we define

$$
q_i(t,x,u) = \int_0^u \eta'(v) \partial_v f_i(t,x,v) dv, \quad i = 1, \dots, d.
$$

If we multiply (3) by $\eta'(u)$, it becomes:

$$
\begin{aligned}
&\partial_t \eta(u_\varepsilon) + \sum_{i=1}^d \partial_{x_i} q_i(t,x,u_\varepsilon) \\
&\qquad - \sum_{i=1}^d \int_0^{u_\varepsilon} \partial^2_{x_i v} f_i(t,x,v) \eta'(v) dv + \sum_{i=1}^d \eta'(u_\varepsilon) \partial_{x_i} f_i(t,x,u_\varepsilon) \\
&\quad = \varepsilon \sum_{i=1}^d \partial_{x_i} (\eta'(u_\varepsilon) b_i(\nabla u_\varepsilon)) - \varepsilon \eta''(u_\varepsilon) \sum_{i=1}^d b_i(\nabla u_\varepsilon) \partial_{x_i} u_\varepsilon \\
&\qquad + \delta \sum_{i=1}^d \partial_{x_i} (\eta'(u_\varepsilon) \partial^2_{x_i x_i} u_\varepsilon) - \frac{\delta}{2} \eta''(u_\varepsilon) \sum_{i=1}^d \partial_{x_i} (\partial_{x_i} u_\varepsilon)^2.
\end{aligned} \tag{7}
$$

Choosing here $\eta(u) = \frac{u^2}{2}$ and integrating over $[0,t) \times \mathbf{R}^d$ we get:

$$
\begin{aligned}
&\int_{\mathbf{R}^d} |u_\varepsilon(t,x)|^2 dx + \varepsilon \int_0^t \int_{\mathbf{R}^d} \nabla u_\varepsilon(t',x) \cdot b(\nabla u_\varepsilon(t',x)) dx dt' \\
&= \int_{\mathbf{R}^d} |u_0(x)|^2 dx + \sum_{j=1}^d \int_0^t \int_{\mathbf{R}^d} \int_0^{u_\varepsilon(t',x)} v D^2_{x_j v} f_j(t',x,v) dv dx dt' \\
&\quad - \sum_{i=1}^d \int_0^t \int_{\mathbf{R}^d} u_\varepsilon(t',x) \partial_{x_i} f_i(t',x,u_\varepsilon(t',x)) dx dt'
\end{aligned} \tag{8}
$$

$$= \int_{\mathbf{R}^d} |u_0(x)|^2 dx - \sum_{i=1}^d \int_0^t \int_{\mathbf{R}^d} \int_0^{u_\varepsilon(t',x)} \partial_{x_i} f_i(t',x,v) dv dx dt',$$

where the second equality sign is justified by the following partial integration

$$\begin{aligned}
&\int_0^t \int_{\mathbf{R}^d} \int_0^{u_\varepsilon} v D^2_{x_j v} f_j(t',x,v) dv dx dt' \\
&= \int_0^t \int_{\mathbf{R}^d} u_\varepsilon \partial_{x_i} f_i(t',x,u_\varepsilon) dx dt' - \int_0^t \int_{\mathbf{R}^d} \int_0^{u_\varepsilon} \partial_{x_i} f_i(t',x,v) dv dx dt'.
\end{aligned}$$

Now inequality (5) follows from (8), using (H1).

As for inequality (6), we start by using (8), viz.

$$\begin{aligned}
&\int_{\mathbf{R}^d} |u_\varepsilon(t,x)|^2 dx + \varepsilon \int_0^t \int_{\mathbf{R}^d} \nabla u_\varepsilon(t',x) \cdot b(\nabla u_\varepsilon(t',x)) dx dt' \\
&= \int_{\mathbf{R}^d} |u_0(x)|^2 dx - \sum_{i=1}^d \int_0^t \int_{\mathbf{R}^d} \int_0^{u_\varepsilon(t',x)} \partial_{x_i} f_i(t',x,v) dv dx dt', \\
&\leq \int_{\mathbf{R}^d} |u_0(x)|^2 dx + \sum_{i=1}^d \int_0^t \int_{\mathbf{R}^d} \int_0^{u_\varepsilon(t',x)} |\partial_{x_i} f_i(t',x,v)| dv dx dt' \\
&\leq \int_{\mathbf{R}^d} |u_0(x)|^2 dx + \int_0^t \int_{\mathbf{R}^d} \int_{\mathbf{R}} \frac{\mu(t',x)}{1+|v|^{1+\alpha}} dv dx dt' \\
&= \int_{\mathbf{R}^d} |u_0(x)|^2 dx + C \int_0^t \int_{\mathbf{R}^d} \mu(t',x) dx dt',
\end{aligned}$$

where $C = \int_{\mathbf{R}} \frac{dv}{1+|v|^{1+\alpha}}$.

From here, using (H3), we conclude in particular that

$$\varepsilon \int_0^t \int_{\mathbf{R}^d} |\nabla u_\varepsilon(t',x)|^2 dx dt' \leq C_{11} \tag{9}$$

for some constant C_{11} independent of ε.

Next we differentiate (3) with respect to x_k and multiply the expression by $\partial_{x_k} u$. Integrating over $\mathbf{R}^d$, using partial integration and then summing over $k = 1, \ldots, d$ we get:

$$\begin{aligned}
&\frac{1}{2} \int_{\mathbf{R}^d} \partial_t |\nabla u_\varepsilon|^2 dx - \sum_{k=1}^d \int_{\mathbf{R}^d} (\nabla \partial_{x_k} u_\varepsilon) \cdot (\partial_{x_k} f(t,x,u_\varepsilon) + \partial_u f \partial_{x_k} u_\varepsilon)\, dx \\
&\qquad = -\varepsilon \sum_{k=1}^d \int_{\mathbf{R}^d} (\nabla \partial_{x_k} u_\varepsilon)^T Db(\nabla u_\varepsilon)(\nabla \partial_{x_k} u_\varepsilon) dx.
\end{aligned}$$

Integrating this over $[0,t]$ and using the Cauchy–Schwarz inequality and condition (H2) we find:

$$\begin{aligned}
&\frac{1}{2} \int_{\mathbf{R}^d} |\nabla u_\varepsilon(t,\ \cdot\ )|^2 dx + \varepsilon C_3 \sum_{k=1}^d \int_0^t \int_{\mathbf{R}^d} |\nabla \partial_{x_k} u_\varepsilon|^2 dx dt' \\
&\leq \frac{1}{2} \int_{\mathbf{R}^d} |\nabla u_0|^2 dx
\end{aligned}$$

$$+\sum_{k=1}^{d}||\nabla(\partial_{x_k}u_\varepsilon)||_{L^2(\mathbf{R}^+\times\mathbf{R}^d)}||\partial_{x_k}f(\,\cdot\,,\,\cdot\,,u_\varepsilon)+\partial_u f\partial_{x_k}u_\varepsilon||_{L^2(\mathbf{R}^+\times\mathbf{R}^d)}.$$

Then, using Young's inequality (the constant C_3 is the same as above):

$$ab\le\frac{C_3\varepsilon}{2}a^2+\frac{C_6}{\varepsilon}b^2,\quad a,b\in\mathbf{R},$$

where C_3, C_6 are independent on ε, we can write

$$\begin{aligned}
&\frac12\int_{\mathbf{R}^d}|\nabla u_\varepsilon(t,\,\cdot\,)|^2dx+\varepsilon C_3\sum_{k=1}^{d}\int_0^t\int_{\mathbf{R}^d}|\nabla\partial_{x_k}u_\varepsilon|^2dxdt'\\
&\quad\le\frac12\int_{\mathbf{R}^d}|\nabla u_0|^2dx+C_3\frac{\varepsilon}{2}\sum_{k=1}^{d}\int_0^t\int_{\mathbf{R}^d}|\nabla\partial_{x_k}u_\varepsilon|^2dxdt'\\
&\qquad+\frac{C_6}{\varepsilon}\int_0^t\int_{\mathbf{R}^d}\sum_{k=1}^{d}\Big|\partial_{x_k}f(t',x,u_\varepsilon)+\partial_u f\partial_{x_k}u_\varepsilon\Big|^2dxdt'.
\end{aligned}$$

Multiplying this by ε^2, using $(a+b)^2\le 2a^2+2b^2$, and applying (9), we conclude:

$$\begin{aligned}
&\frac{\varepsilon^2}{2}\int_{\mathbf{R}^d}|\nabla u_\varepsilon(t,\,\cdot\,)|^2dx+C_3\frac{\varepsilon^3}{2}\int_{\mathbf{R}^d}\int_0^t|D^2u_\varepsilon|^2dxdt'\\
&\quad\le\frac{\varepsilon^2}{2}\int_{\mathbf{R}^d}|\nabla u_0|^2dxdt'\\
&\qquad+2\varepsilon C_6\int_0^t\int_{\mathbf{R}^d}\sum_{k=1}^{d}|\partial_{x_k}f(t',x,u_\varepsilon(t',x))|^2dxdt'\\
&\qquad+C_6C_{11}||\partial_u f||^2_{L^\infty(\mathbf{R}^+\times\mathbf{R}^d\times\mathbf{R})}.
\end{aligned}$$

This inequality is actually inequality (6) when we take $C_5=\frac{2\max\{1,2C_6,C_6C_{11}\}}{\min\{1,C_3\}}$.

❑

3. The multidimensional case

Consider the following initial-value problem: Find $u=u(t,x)$ such that

$$\partial_t u+\operatorname{div}_x f(t,x,u)=0, \tag{10}$$

$$u(x,0)=u_0(x),\quad x\in\mathbf{R}^d, \tag{11}$$

where $u_0\in L^2(\mathbf{R}^d)$ is given initial data.

For the flux $f=(f_1,\dots,f_d)$ we need the following assumption, denoted **(H4)**:

(H4a) For the flux $f=f(t,x,u)$, $(t,x,u)\in\mathbf{R}^+\times\mathbf{R}^d\times\mathbf{R}$, we assume that $f\in C(\mathbf{R};BV(\mathbf{R}^+\times\mathbf{R}^d))$ and that for every $l\in\mathbf{R}^+$ we have $\max_{u\in[-l,l]}f(t,x,u)\in L^p_{\rm loc}(\mathbf{R}^+\times\mathbf{R}^d)$, $p>2$.

(H4b) There exists a sequence $f_\varrho=(f_{1\varrho},\dots,f_{d\varrho})$, $\varrho\in(0,1)$, such that $f_\varrho=f_\varrho(t,x,u)\in C^1(\mathbf{R}^+\times\mathbf{R}^d\times\mathbf{R})$, satisfying for some $p>2$ and every $l\in\mathbf{R}^+$:

$$\lim_{\varrho\to0}\max_{z\in[-l,l]}||f_\varrho(\,\cdot\,,\,\cdot\,,z)-f(\,\cdot\,,\,\cdot\,,z)||_{L^p(\mathbf{R}^+\times\mathbf{R}^d)}=0, \tag{12a}$$

$$\sum_{i=1}^{d}|\partial_{x_i}f_{i\varrho}(t,x,u)|\le\frac{\mu_1(t,x)}{1+|u|^{1+\alpha}}, \tag{12b}$$

$$\varrho^3 \sum_{i=1}^d |\partial_{x_i} f_{i\varrho}(t,x,u)|^2 \le \mu_2(t,x), \tag{12c}$$

$$\sum_{i=1}^d |\partial_u f_{i\varrho}(t,x,u)| \le \frac{C}{\beta(\varrho)}, \tag{12d}$$

$$\sum_{i=1}^d |\partial^2_{x_i u} f_{i\varrho}(t,x,u)| \le \frac{\mu_3(t,x)}{1+|u|^{1+\alpha}}, \tag{12e}$$

where $\mu_i \in \mathcal{M}(\mathbf{R}^+ \times \mathbf{R}^d)$, $i = 1,2,3$, are bounded measures.

In the case when we have only vanishing diffusion it is usually possible to obtain uniform L^∞ bound for the corresponding sequence of solutions under relatively mild assumptions on the flux and initial data (see, e.g., [8, 17]). In the case when we have both vanishing diffusion and vanishing dispersion, we must assume more on the flux in order to obtain even much weaker bounds (see Theorem 3). We remark that demand on controlling the flux at infinity is rather usual in the case of conservation laws with vanishing diffusion and dispersion (see, e.g., [7, 14, 13]).

Remark 2. For an arbitrary compactly supported, nonnegative $\varphi_1 \in C_0^\infty(\mathbf{R}^+ \times \mathbf{R}^d)$ and $\varphi_2 \in C_0^\infty(\mathbf{R})$ with total mass one, denote by

$$\varphi_\varrho(z,u) = \frac{1}{\varrho^{d+1}} \varphi_1(\frac{z}{\varrho}) \frac{1}{\beta(\varrho)} \varphi_2(\frac{u}{\beta(\varrho)}),$$

$z \in \mathbf{R}^+ \times \mathbf{R}^d$ and $u \in \mathbf{R}$, where β is a positive function tending to zero as $\varrho \to 0$. In the case when the flux $f \in C(\mathbf{R}; BV(\mathbf{R}^+ \times \mathbf{R}^d)) \cap BV(\mathbf{R} \times \mathbf{R}^+ \times \mathbf{R}^d))$ is locally bounded, straightforward computation shows that the sequence $f_\varrho = f \star \varphi_\varrho = (f_{1\varrho}, \dots, f_{d\varrho})$ satisfies **(H4b)** with $\beta(\varrho) = \varrho$.

We also need to assume that the flux f is genuinely nonlinear, i.e., for every $(t,x) \in \mathbf{R}^+ \times \mathbf{R}^d$ and every $\xi \in \mathbf{R}^d \setminus \{0\}$, the mapping

$$\mathbf{R} \ni \lambda \mapsto \sum_{i=1}^d f_i(t,x,\lambda) \frac{\xi_i}{|\xi|} \tag{13}$$

is nonconstant on every non-degenerate interval of the real line.

We will analyze the vanishing diffusion-dispersion-smoothing limit of the problem

$$\partial_t u + \operatorname{div}_x f_\varrho(t,x,u) = \varepsilon \operatorname{div}_x b(\nabla u) + \delta \sum_{j=1}^d \partial^3_{x_j x_j x_j} u, \tag{14}$$

$$u(x,0) = u_{0,\varepsilon}(x), \quad x \in \mathbf{R}^d, \tag{15}$$

where the flux f_ϱ satisfies the conditions (H4b). We denote the solution of (14), (15) by $u_\varepsilon = u_\varepsilon(t,x)$. We assume that

$$\|u_{0,\varepsilon} - u_0\|_{L^2(\mathbf{R}^d)} \to 0 \text{ and } ||u_{0,\varepsilon}||_{L^2(\mathbf{R}^d)} + \varepsilon ||u_{0,\varepsilon}||_{H^1(\mathbf{R}^d)} \le C. \tag{16}$$

We also assume that $\varrho = \varrho(\varepsilon) \to 0$ and $\delta = \delta(\varepsilon) \to 0$ as $\varepsilon \to 0$. We want to prove that under certain conditions, a sequence of solutions $(u_\varepsilon)_{\varepsilon>0}$ of (14), (15) converges to a weak solution of problem (10), (11) as $\varepsilon \to 0$. To do this in the multidimensional case we use the approach of H-measures, introduced in [24] and further developed in [16, 17]. In the one dimensional case we use the compensated compactness method, following [21].

In order to accomplish the plan we need the following a priori estimates:

Theorem 3 (A priori inequalities)**.** *Suppose that the flux $f(t,x,u)$ satisfies (H4). Also assume that initial function u_0 satisfies (16). Under these conditions the sequence of smooth solutions $(u_\varepsilon)_{\varepsilon>0}$ of (14), (15) satisfies the following inequalities for every $t\in[0,T]$:*

$$\int_{\mathbf{R}^d}|u_\varepsilon(t,x)|^2dx+\varepsilon\int_0^t\int_{\mathbf{R}^d}|\nabla u_\varepsilon(x,s)|^2dxds\leq C_3\left(\int_{\mathbf{R}^d}|u_{0,\varepsilon}(x)|^2dx+C_{10}\right), \tag{17}$$

and

$$\begin{aligned}\varepsilon^2\int_{\mathbf{R}^d}|\nabla u_\varepsilon(t,x)|^2dx+\varepsilon^3\int_0^t\int_{\mathbf{R}^d}|D^2u_\varepsilon(t',x)|^2dxdt'\\ \leq C_4\left(\varepsilon^2\int_{\mathbf{R}^d}|\nabla u_{0,\varepsilon}(x)|^2dx+\frac{\varepsilon}{\varrho}C_{11}+\frac{C_{12}}{\beta(\varrho)^2}\right),\end{aligned} \tag{18}$$

for some constants C_{10},C_{11},C_{12} (the constants C_3,C_4 are from Theorem 1).

Proof: For every fixed ϱ, the function $f_\varrho=(f_{1\varrho},\dots,f_{d\varrho})$ is smooth, and, due to (H4), we see that f_ϱ satisfies (H3). This means that we can apply Theorem 1.

Replacing the flux f by f_ϱ from (14) and u_0 by $u_{0,\varepsilon}$ from (15) in (5) and (6), we get:

$$\begin{aligned}&\int_{\mathbf{R}^d}|u_\varepsilon(t,x)|^2dx+\varepsilon\int_0^t\int_{\mathbf{R}^d}|\nabla u_\varepsilon(x,s)|^2dxds \\ &\quad\leq C_3\Big(\int_{\mathbf{R}^d}|u_{0,\varepsilon}(x)|^2dx-\int_0^t\int_{\mathbf{R}^d}\int_0^{u_\varepsilon(t',x)}\operatorname{div}_x f_\varrho(t',x,v)dvdxdt'\Big),\end{aligned} \tag{19}$$

and

$$\begin{aligned}&\varepsilon^2\int_{\mathbf{R}^d}|\nabla u_\varepsilon(t,x)|^2dx+\varepsilon^3\int_0^t\int_{\mathbf{R}^d}|D^2u_\varepsilon(t',x)|^2dxdt' \\ &\quad\leq C_4\Big(\varepsilon^2\int_{\mathbf{R}^d}|\nabla u_{0,\varepsilon}(x)|^2dx+||\partial_uf_\varrho||^2_{L^\infty(\mathbf{R}^+\times\mathbf{R}^d\times\mathbf{R})} \\ &\qquad+\varepsilon\int_0^t\int_{\mathbf{R}^d}\sum_{k=1}^d\sum_{i=1}^d[\partial_{x_k}f_{i\varrho}(t',x,u_\varepsilon(t',x))]^2dxdt'\Big).\end{aligned} \tag{20}$$

To proceed, we use assumption (H4). We have:

$$\begin{aligned}&\int_0^t\int_{\mathbf{R}^d}\int_0^{u_\varepsilon(t',x)}\operatorname{div} f_{i\varrho}(t',x,v)dvdxdt' \\ &\qquad\leq\int_0^t\int_{\mathbf{R}^d}\int_{\mathbf{R}}\sum_{i=1}^d|\partial_{x_i}f_{i\varrho}(t',x,v)|dvdxdt \\ &\qquad\leq\int_0^t\int_{\mathbf{R}^d}\int_{\mathbf{R}}\frac{\mu_1(t,x)}{1+|v|^{1+\alpha}}dvdxdt\leq C_{10},\end{aligned} \tag{21}$$

which together with (19) immediately gives (17).

Similarly, combining (H4) and (20), and arguing as in (21), we get (18). ❑

In this section we shall inspect the convergence of a sequence $(u_\varepsilon)_{\varepsilon>0}$ of solutions to (14), (15) in the case when

$$b(\lambda_1,\dots,\lambda_d)=(\lambda_1,\dots,\lambda_d)$$

for the function b appearing in the right-hand side of (14). This is not an essential restriction, but we will use it in order to simplify the presentation.

Thus we use the following theorem which can be proved using the H-measures approach (see, e.g., [17, Corollary 2 and Remark 3]). We let θ denote the Heaviside function.

Theorem 4. [17] *Assume that the vector $f(t,x,u)$ is genuinely nonlinear in the sense of (13). Then each sequence $(v_\varepsilon(t,x))_{\varepsilon>0}\subset L^\infty(\mathbf{R}^+\times\mathbf{R}^d)$ such that for every $c\in\mathbf{R}$ the distribution*

$$\partial_t(\theta(v_\varepsilon-c)(v_\varepsilon-c))+\operatorname{div}_x\left(\theta(v_\varepsilon-c)(f(t,x,v_\varepsilon)-f(t,x,c))\right) \tag{22}$$

is precompact in H^{-1}_{loc}, contains a subsequence convergent in $L^1_{\mathrm{loc}}(\mathbf{R}^+\times\mathbf{R}^d)$.

We can now prove the following theorem.

Theorem 5. *Assume that the flux vector f is genuinely nonlinear in the sense of (13) and that it satisfies (H4). Furthermore, assume that*

$$\varrho^3=\varepsilon,\qquad \delta=\varepsilon^2\rho^2(\varepsilon)\ \text{with}\ \frac{\rho(\varepsilon)}{(\beta(\varepsilon^3))^2}\to 0\ \text{as}\ \varepsilon\to 0, \tag{23}$$

and that $u_{0,\varepsilon}$ satisfies (16). Then there exists a subsequence of solutions $(u_\varepsilon)_{\varepsilon>0}$ of (14)–(15) that converges to a weak solution of problem (10)–(11).

Proof: We shall use Theorem 4. Since it is well known that the sequence $(u_\varepsilon)_{\varepsilon>0}$ of solutions of problem (14)–(15) is not uniformly bounded, we cannot directly apply the conditions of Theorem 4.

Take an arbitrary C^2 function $S=S(u)$, $u\in\mathbf{R}$, and multiply the regularized equation (14) by $S'(u_\varepsilon)$. As usual, put

$$q(t,x,u)=\int_0^u S'(v)\partial_u f_\varrho\,dv,\qquad q=(q_1,\dots,q_d).$$

We easily find that

$$\begin{aligned}\partial_t S(u_\varepsilon)+\operatorname{div}_x q(t,x,u_\varepsilon)-\operatorname{div}_x q(t,x,v)|_{v=u_\varepsilon}+S'(u_\varepsilon)\operatorname{div}_x f_\varrho(t,x,v)|_{v=u_\varepsilon}\\ =\varepsilon\operatorname{div}_x\big(S'(u_\varepsilon)\nabla u_\varepsilon\big)-\varepsilon S''(u_\varepsilon)|\nabla u_\varepsilon|^2\\ +\delta\sum_{j=1}^d D_{x_j}(S'(u_\varepsilon)\partial^2_{x_jx_j}u_\varepsilon)-\delta\sum_{j=1}^d S''(u_\varepsilon)\partial_{x_j}u_\varepsilon\partial^2_{x_jx_j}u_\varepsilon.\end{aligned} \tag{24}$$

We will apply this formula repeatedly with different choices for $S(u)$.

In order to apply Theorem 4, we will consider a truncated sequence $(T_l(u_\varepsilon))_{\varepsilon>0}$, where the truncation function T_l is defined for every fixed $l\in\mathbf{N}$ as:

$$T_l(u)=\begin{cases}-l, & u\le -l,\\ u, & -l\le u\le l,\\ l, & u\ge l.\end{cases} \tag{25}$$

We shall prove that the sequence $(T_l(u_\varepsilon))_{\varepsilon>0}$ is precompact for every fixed l. Denote by u_l a subsequential limit (in L^1_{loc}) of the sequence $(T_l(u_\varepsilon))_{\varepsilon>0}$, which gives raise to a new sequence $(u_l)_{l>1}$ that we prove converges to a weak solution of (10)–(11).

To carry out this plan we must replace T_l by a C^2 regularization $T_{l,\sigma}\colon \mathbf{R}\to\mathbf{R}$. We define $T_{l,\sigma}\colon \mathbf{R}\to\mathbf{R}$ by $T_{l,\sigma}(0)=0$ and

$$T'_{l,\sigma}(u)=\begin{cases}1, & |u|<l-\sigma,\\ \frac{l-|u|}{\sigma}, & l-\sigma<|u|<l,\\ 0, & |u|>l.\end{cases} \tag{26}$$

Note that as $\sigma\to 0$ we have $T_{l,\sigma}(u)\to T_l(u)$ in $L^p_{\rm loc}$ for every $p<\infty$, where T_l is defined by (25).

Next we want to estimate $\|T''_{l,\sigma}(u_\varepsilon)\nabla u_\varepsilon\|_{L^2(\mathbf{R}^+\times\mathbf{R}^d)}$. To accomplish this, we insert the functions $T^{\pm}_{l,\sigma}$ for S in (24) where $T^{\pm}_{l,\sigma}$ are defined by $T^{\pm}_{l,\sigma}(0)=0$ and

$$(T^{+}_{l,\sigma})'(u)=\begin{cases}1, & u<l,\\ \frac{l+\sigma-u}{\sigma}, & l<u<l+\sigma,\\ 0, & u>l+\sigma,\end{cases}$$

$$(T^{-}_{l,\sigma})'(u)=\begin{cases}1, & u>-l,\\ \frac{l+\sigma+u}{\sigma}, & -l-\sigma<u<-l,\\ 0, & u<-l-\sigma.\end{cases}$$

Notice that

$$\begin{aligned}&(T^{\pm}_{l,\sigma}(u))'\le 1, \quad |T^{\pm}_{l,\sigma}(u)|\le |u|,\\ &T^{+}_{l,\sigma}(u)=T^{-}_{l,\sigma}(u) \ \text{ for } \ -l\le u\le l.\end{aligned} \tag{27}$$

By inserting $S(u)=-T^{+}_{l,\sigma}(u)$, $q=q_+(t,x,u)=-\int_0^u (T^{+}_{l,\sigma})'(v)\partial_u f_\varrho\,dv$ in (24) and integrating over $\Pi_t=[0,t]\times\mathbf{R}^d$ we get:

$$\begin{aligned}&-\int_{\mathbf{R}^d} T^{+}_{l,\sigma}(u_\varepsilon)dx+\int_{\mathbf{R}^d} T^{+}_{l,\sigma}(u_0)dx+\frac{\varepsilon}{\sigma}\iint_{\Pi_t\cap\{l<u_\varepsilon<l+\sigma\}}|\nabla u_\varepsilon|^2dxdt\\ &\quad=\iint_{\Pi_t}\operatorname{div}_x q_+(t,x,v)|_{v=u_\varepsilon}dxdt+\iint_{\Pi_t}(T^{+}_{l,\sigma})'(u_\varepsilon)\operatorname{div}_x f_\varrho(t,x,v)|_{v=u_\varepsilon}dxdt\\ &\qquad-\frac{\delta}{\sigma}\iint_{\Pi_t\cap\{l<u_\varepsilon<l+\sigma\}}\sum_{j=1}^d\partial_{x_j}u_\varepsilon\partial^2_{x_jx_j}u_\varepsilon dxdt.\end{aligned} \tag{28}$$

Similarly, for $S(u)=T^{-}_{l,\sigma}(u)$, $q=q_-(t,x,u)=\int_0^u (T^{-}_{l,\sigma})'(v)\partial_u f_\varrho\,dv$ we have from (24):

$$\begin{aligned}&\int_{\mathbf{R}^d} T^{-}_{l,\sigma}(u_\varepsilon)dx-\int_{\mathbf{R}^d} T^{-}_{l,\sigma}(u_0)dx+\frac{\varepsilon}{\sigma}\iint_{\Pi_t\cap\{-l-\sigma<u_\varepsilon<-l\}}|\nabla u_\varepsilon|^2dxdt\\ &\quad=\iint_{\Pi_t}\operatorname{div}_x q_-(t,x,v)|_{v=u_\varepsilon}dxdt-\iint_{\Pi_t}(T^{-}_{l,\sigma})'(u_\varepsilon)\operatorname{div}_x f_\varrho(t,x,v)|_{v=u_\varepsilon}dxdt\\ &\qquad+\frac{\delta}{\sigma}\iint_{\Pi_t\cap\{-l-\sigma<u_\varepsilon<-l\}}\sum_{j=1}^d\partial_{x_j}u_\varepsilon\partial^2_{x_jx_j}u_\varepsilon dxdt.\end{aligned} \tag{29}$$

Adding (28) and (29) we get:

$$
\begin{aligned}
&\frac{\varepsilon}{\sigma}\iint_{\Pi_t\cap\{l<|u_\varepsilon|<l+\sigma\}}|\nabla u_\varepsilon|^2dxdt\\
&=-\int_{\mathbf{R}^d}\Big(T^-_{l,\sigma}(u_\varepsilon)-T^+_{l,\sigma}(u_\varepsilon)\Big)\,dx+\int_{\mathbf{R}^d}\Big(T^-_{l,\sigma}(u_0)-T^+_{l,\sigma}(u_0)\Big)\,dx\\
&\quad+\iint_{\Pi_t}\operatorname{div}_x q_-(t,x,v)|_{v=u_\varepsilon}dxdt-\iint_{\Pi_t}\operatorname{div}_x q_+(t,x,v)|_{v=u_\varepsilon}dxdt\\
&\quad-\iint_{\Pi_t}(T^-_{l,\sigma})'(u_\varepsilon)\operatorname{div}_x f_\varrho(t,x,v)|_{v=u_\varepsilon}dxdt+\iint_{\Pi_t}(T^+_{l,\sigma})'(u_\varepsilon)\operatorname{div}_x f_\varrho(t,x,v)|_{v=u_\varepsilon}dxdt\\
&\quad-\frac{\delta}{\sigma}\iint_{\Pi_t\cap\{-l-\sigma<u_\varepsilon<-l\}}\sum_{j=1}^d\partial_{x_j}u_\varepsilon\partial^2_{x_jx_j}u_\varepsilon dxdt\\
&\quad-\frac{\delta}{\sigma}\iint_{\Pi_t\cap\{l<u_\varepsilon<l+\sigma\}}\sum_{j=1}^d\partial_{x_j}u_\varepsilon\partial^2_{x_jx_j}u_\varepsilon dxdt.
\end{aligned}
$$

From (27) and definition of q_- and q_+ it follows:

$$
\begin{aligned}
\frac{\varepsilon}{\sigma}\iint_{\Pi_t\cap\{l<|u_\varepsilon|<l+\sigma\}}|\nabla u_\varepsilon|^2dxdt\leq&\int_{|u_\varepsilon|>l}2|u_\varepsilon|dx+\int_{|u_0|>l}2|u_0|dx\\
&+2\iint_{\Pi_t}\int_{\mathbf{R}}\sum_{i=1}^d|D^2_{x_iv}f_{i\varrho}(t,x,v)|dvdxdt\\
&+2\iint_{\Pi_t}\sum_{i=1}^d|\partial_{x_i}f_{i\varrho}(t,x,u_\varepsilon)|dxdt\\
&+2\frac{\delta}{\sigma}\iint_{\Pi_t\cap\{l-\sigma<|u_\varepsilon|<l\}}\sum_{j=1}^d|\partial_{x_j}u_\varepsilon\partial^2_{x_jx_j}u_\varepsilon|dxdt.
\end{aligned}
\tag{30}
$$

Without loss of generality, we can assume that $l>1$. Having this in mind, we get from (H4) and (30):

$$
\begin{aligned}
&\frac{\varepsilon}{\sigma}\iint_{\Pi_t\cap\{l<|u_\varepsilon|<l+\sigma\}}|\nabla u_\varepsilon|^2dxdt\\
&\quad\leq\int_{|u_\varepsilon|>l}2|u_\varepsilon|^2dx+\int_{|u_0|>l}2|u_0|^2dx\\
&\qquad+2\iint_{\Pi_t}\int_{\mathbf{R}}\sum_{i=1}^d\frac{\mu_3(t,x)}{1+|v|^{1+\alpha}}dvdxdt\\
&\qquad+2\iint_{\Pi_t}\sum_{i=1}^d|\partial_{x_i}f_{i\varrho}(t,x,u_\varepsilon)|dxdt\\
&\qquad+2\frac{\delta}{\sigma}\iint_{\Pi_t\cap\{l<|u_\varepsilon|<l+\sigma\}}\sum_{j=1}^d|\partial_{x_j}u_\varepsilon\partial^2_{x_jx_j}u_\varepsilon|dxdt\\
&\quad\leq\quad\int_{\mathbf{R}^d}2\left(|u_\varepsilon(x,t)|^2+|u_0(x,t)|^2\right)dx
\end{aligned}
\tag{31}
$$

$$
\begin{aligned}
&+ K_1 + K_2 + 2\frac{\delta}{\sigma\varepsilon^2}\sum_{i=1}^d \|\varepsilon\partial_{x_i}u_\varepsilon\|_{L^2(\mathbf{R}^+\times\mathbf{R}^d)}\|\varepsilon^2\partial_{x_ix_i}u_\varepsilon\|_{L^2(\mathbf{R}^+\times\mathbf{R}^d)}\\
&\leq K_5 + \left(\frac{\delta^2}{\sigma^2\varepsilon^2(\beta(\varrho))^2} + \frac{\delta^2}{\sigma^2\varepsilon^4}\right)^{1/2} K_3K_4,
\end{aligned}
$$

where K_i, $i=1,\dots,5$, are constants such that (cf. (17) and (18)):

$$
\begin{aligned}
2\iint_{\Pi_t}\int_{\mathbf{R}}\sum_{i=1}^d \frac{\mu_3(t,x)}{1+|v|^{1+\alpha}}dvdxdt &\leq K_1,\\
2\iint_{\Pi_t}\sum_{i=1}^d |\partial_{x_i}f_{i\varrho}(t,x,u_\varepsilon)|dxdt &\leq K_2,\\
\sum_{i=1}^d \|\varepsilon\partial_{x_i}u_\varepsilon\|_{L^2(\mathbf{R}^+\times\mathbf{R}^d)} &\leq K_3,\\
\sum_{i=1}^d \|\varepsilon^2\partial_{x_ix_i}u_\varepsilon\|_{L^2(\mathbf{R}^+\times\mathbf{R}^d)} &\leq \left(\frac{1}{(\beta(\varrho))^2} + \frac{\varepsilon}{\varrho}\right)^{1/2} K_4,\\
\int_{\mathbf{R}^d} 2\left(|u_\varepsilon(x,t)|^2 + |u_0(x,t)|^2\right)dx + K_1 + K_2 &\leq K_5,\\
2\frac{\delta}{\sigma\varepsilon^2}\|\varepsilon\nabla u_\varepsilon\|_{L^2(\mathbf{R}^+\times\mathbf{R}^d)}\sum_{i=1}^d \|\varepsilon^2\partial_{x_ix_i}u_\varepsilon\|_{L^2(\mathbf{R}^+\times\mathbf{R}^d)} &\leq \left(\frac{\delta^2}{\sigma^2\varepsilon^4\beta^2(\varepsilon)} + \frac{\delta^2}{\sigma^2\varepsilon^4}\right)^{1/2} K_3K_4,
\end{aligned}
$$

where we in the last formula used the assumption $\varepsilon = \varrho$ from (23). These estimates follow from (H4) and the a priori estimates (17), (18). Thus, in view of (31),

$$
\frac{\varepsilon}{\sigma}\iint_{\Pi_t\cap\{l<|u_\varepsilon|<l+\sigma\}} |\nabla u_\varepsilon|^2 dxdt \leq K_5 + \left(\frac{\delta^2}{\sigma^2\varepsilon^2\beta^2(\varepsilon)} + \frac{\delta^2}{\sigma^2\varepsilon^4}\right)^{1/2} K_3K_4, \tag{32}
$$

which is the sought for estimate for $\|T''_{l,\sigma}(u_\varepsilon)\nabla u_\varepsilon\|_{L^2(\mathbf{R}^+\times\mathbf{R}^d)}$.

Next, take a function $U_\rho(z)$ satisfying $U_\rho(0)=0$ and

$$
U'_\rho(z) = \begin{cases} 0, & z<0,\\ \frac{z}{\rho}, & 0<z<\rho,\\ 1, & z>\rho. \end{cases}
$$

Clearly, U_ρ is convex, and we have $U'_\rho(z)\to\theta(z)$ in $L^p_{\mathrm{loc}}(\mathbf{R})$ as $\rho\to 0$, for any $p<\infty$; as before, θ denotes the Heaviside function.

Insert $S(u_\varepsilon) = U_\rho(T_{l,\sigma}(u_\varepsilon)-c)$ in (24). We get:

$$
\begin{aligned}
&\partial_t U_\rho(T_{l,\sigma}(u_\varepsilon)-c) + \operatorname{div}_x \int^{u_\varepsilon} U'_\rho(T_{l,\sigma}(v)-c)T'_{l,\sigma}(v)\partial_v f_\varrho(t,x,v)dv\\
&\quad = \int^{u_\varepsilon} U'_\rho(T_{l,\sigma}(v)-c)T'_{l,\sigma}(v)\operatorname{div}_x \partial_v f_\varrho(t,x,v)dv\\
&\qquad - U'_\rho(T_{l,\sigma}(u_\varepsilon)-c)T'_{l,\sigma}(u_\varepsilon)\operatorname{div}_x f_\varrho(t,x,v)|_{v=u_\varepsilon}\\
&\qquad + \varepsilon\Delta_x U_\rho(T_{l,\sigma}(u_\varepsilon)-c) - \varepsilon D^2_{uu}[U_\rho(T_{l,\sigma}(u_\varepsilon)-c)]|\nabla u_\varepsilon|^2\\
&\qquad + \delta\sum_{i=1}^d D_{x_i}\left(D_u[U_\rho(T_{l,\sigma}(u_\varepsilon)-c)]\partial^2_{x_ix_i}u_\varepsilon\right)
\end{aligned}
$$

$$- \delta \sum_{i=1}^{d} D_{uu}^2 \left[U_\rho(T_{l,\sigma}(u_\varepsilon) - c)\right] \partial_{x_i} u_\varepsilon \partial_{x_i x_i}^2 u_\varepsilon.$$

We rewrite the previous expression in the following manner:

$$\begin{aligned}
\partial_t\Big(\theta(T_l(u_\varepsilon) - c)(T_l(u_\varepsilon) - c)\Big) &+ \operatorname{div}_x \Big(\theta(T_l(u_\varepsilon) - c)(f(t,x,T_l(u_\varepsilon)) - f(t,x,c))\Big) \\
&= \partial_t\Big(\theta(T_l(u_\varepsilon) - c)(T_l(u_\varepsilon) - c) - U_\rho(T_{l,\sigma}(u_\varepsilon) - c)\Big) \\
&\quad + \operatorname{div}_x \Big(\theta(T_l(u_\varepsilon) - c)(f(t,x,T_l(u_\varepsilon)) - f(t,x,c)) \\
&\qquad\qquad - \int^{u_\varepsilon} U_\rho'(T_{l,\sigma}(v) - c) T_{l,\sigma}'(v) \partial_v f_\varrho(t,x,v) dv\Big) \\
&\quad + \int^{u_\varepsilon} U_\rho'(T_{l,\sigma}(v) - c) T_{l,\sigma}'(v) \operatorname{div}_x \partial_v f_\varrho(t,x,v) dv \\
&\quad - U_\rho'(T_{l,\sigma}(u_\varepsilon) - c) T_{l,\sigma}'(u_\varepsilon) \operatorname{div}_x f_\varrho(t,x,v)|_{v=u_\varepsilon} \\
&\quad + \varepsilon \Delta_x U_\rho(T_{l,\sigma}(u_\varepsilon) - c) - \varepsilon U_\rho''(T_{l,\sigma}(u_\varepsilon) - c)(T_{l,\sigma}'(u_\varepsilon))^2 |\nabla u_\varepsilon|^2 \\
&\quad - \varepsilon U_\rho'(T_{l,\sigma}(u_\varepsilon) - c) T_{l,\sigma}''(u_\varepsilon) |\nabla u_\varepsilon|^2 \\
&\quad + \delta \sum_{i=1}^{d} D_{x_i} \left(D_u[U_\rho(T_{l,\sigma}(u_\varepsilon) - c)] \partial_{x_i x_i}^2 u_\varepsilon\right) \\
&\quad - \delta \sum_{i=1}^{d} D_{uu}^2 \left[U_\rho(T_{l,\sigma}(u_\varepsilon) - c)\right] \partial_{x_i} u_\varepsilon \partial_{x_i x_i}^2 u_\varepsilon,
\end{aligned}$$

or

$$\begin{aligned}
\partial_t\Big(\theta(T_l(u_\varepsilon) - c)(T_l(u_\varepsilon) - c)\Big)& \\
&+ \operatorname{div}_x \Big(\theta(T_l(u_\varepsilon) - c)(f(t,x,T_l(u_\varepsilon)) - f(t,x,c))\Big) \\
= \Gamma_{1,\varepsilon} + \Gamma_{2,\varepsilon} &+ \Gamma_{3,\varepsilon} + \Gamma_{4,\varepsilon} + \Gamma_{5,\varepsilon} + \Gamma_{6,\varepsilon} + \Gamma_{7,\varepsilon}
\end{aligned} \tag{33}$$

where

$$\begin{aligned}
\Gamma_{1,\varepsilon} &= \partial_t\Big(\theta(T_l(u_\varepsilon) - c)(T_l(u_\varepsilon) - c) - U_\rho(T_{l,\sigma}(u_\varepsilon) - c)\Big), \\
\Gamma_{2,\varepsilon} &= \operatorname{div}_x \Big(\theta(T_l(u_\varepsilon) - c)(f(t,x,T_l(u_\varepsilon)) - f(t,x,c)) \\
&\qquad\qquad - \int^{u_\varepsilon} U_\rho'(T_{l,\sigma}(v) - c) T_{l,\sigma}'(v) \partial_v f_\varrho(t,x,v) dv\Big), \\
\Gamma_{3,\varepsilon} &= \int^{u_\varepsilon} U_\rho'(T_{l,\sigma}(v) - c) T_{l,\sigma}'(v) \operatorname{div}_x \partial_v f_\varrho(t,x,v) dv \\
&\quad - U_\rho'(T_{l,\sigma}(u_\varepsilon) - c) T_{l,\sigma}'(u_\varepsilon) \operatorname{div}_x f_\varrho(t,x,v)|_{v=u_\varepsilon}, \\
\Gamma_{4,\varepsilon} &= \varepsilon \Delta_x U_\rho(T_{l,\sigma}(u_\varepsilon) - c) \\
&\quad + \delta \sum_{i=1}^{d} D_{x_i} \left(D_u[U_\rho(T_{l,\sigma}(u_\varepsilon) - c)] \partial_{x_i x_i}^2 u_\varepsilon\right), \\
\Gamma_{5,\varepsilon} &= -\varepsilon U_\rho'(T_{l,\sigma}(u_\varepsilon) - c) T_{l,\sigma}''(u_\varepsilon) |\nabla u_\varepsilon|^2,
\end{aligned}$$

$$\Gamma_{6,\varepsilon} = -\delta \sum_{i=1}^{d} D_{uu}^2 \left[U_\rho(T_{l,\sigma}(u_\varepsilon) - c) \right] \partial_{x_i} u_\varepsilon \partial_{x_i x_i}^2 u_\varepsilon,$$

$$\Gamma_{7,\varepsilon} = -\varepsilon U_\rho''(T_{l,\sigma}(u_\varepsilon) - c)(T_{l,\sigma}'(u_\varepsilon))^2 |\nabla u_\varepsilon|^2.$$

To continue, we assume that σ depends on ε in the following way:

$$\sigma = \beta^2(\varepsilon^3). \tag{34}$$

From here, we shall prove that the sequence $(T_l(u_\varepsilon))_{\varepsilon>0}$ satisfies the conditions of Theorem 4. Accordingly, we need to prove that the left-hand side of (33) is precompact $H^{-1}_{\rm loc}(\mathbf{R}^+ \times \mathbf{R}^d)$.

To accomplish this, we use Murat's lemma ([5, Ch. 1, Cor. 1]). More precisely, we have to prove:

(i) When the left-hand side of (33) is written in the form $\operatorname{div} Q_\varepsilon$, we have $Q_\varepsilon \in L^p_{\rm loc}(\mathbf{R}^+ \times \mathbf{R}^d)$ for $p > 2$, and

(ii) The right-hand side of (33) is of the form $\mathcal{M}_{\rm loc,B} + H^{-1}_{\rm loc,c}$, where $\mathcal{M}_{\rm loc,B}$ denotes set of families which are locally bounded in the space of measures and $H^{-1}_{\rm loc,c}$ is set of families precompact in $H^{-1}_{\rm loc}$.

First, since $T_l(u_\varepsilon)$ is uniformly bounded by l, we see that (i) is satisfied.

To prove (ii), we consider each term on the right-hand side of (33). First we prove that:

$$\Gamma_{1,\varepsilon} = \partial_t \left(\theta(T_l(u_\varepsilon) - c)(T_l(u_\varepsilon) - c) - U_\rho(T_{l,\sigma}(u_\varepsilon) - c) \right) \in H^{-1}_{\rm loc,c}.$$

Trivially we have:

$$\begin{aligned}
&\theta(T_l(u_\varepsilon) - c)(T_l(u_\varepsilon) - c) - U_\rho(T_{l,\sigma}(u_\varepsilon) - c) \\
&\qquad = \theta(T_l(u_\varepsilon) - c)(T_l(u_\varepsilon) - c) - \theta(T_{l,\sigma}(u_\varepsilon) - c)(T_{l,\sigma}(u_\varepsilon) - c) \\
&\qquad\quad + \theta(T_{l,\sigma}(u_\varepsilon) - c)(T_{l,\sigma}(u_\varepsilon) - c) - U_\rho(T_{l,\sigma}(u_\varepsilon) - c).
\end{aligned}$$

Since the function $\theta(z-c)(z-c)$ is Lipschitz continuous in z with a Lipschitz constant one, and, according to definition of U_ρ, it holds $|U_\rho(z) - \theta(z)z| \leq \frac{1}{2}\rho$, we have from the last expression:

$$\begin{aligned}
&|\theta(T_l(u_\varepsilon) - c)(T_l(u_\varepsilon) - c) - U_\rho(T_{l,\sigma}(u_\varepsilon) - c)| \\
&\qquad\qquad \leq |T_l(u_\varepsilon) - T_{l,\sigma}(u_\varepsilon)| + \mathcal{O}(\rho) \leq \mathcal{O}(\sigma) + \mathcal{O}(\rho).
\end{aligned}$$

From this and assumptions (23) and (34) on $\sigma = \sigma(\varepsilon)$ and $\rho = \rho(\varepsilon)$ it follows that as $\varepsilon \to 0$

$$\theta(T_l(u_\varepsilon) - c)(T_l(u_\varepsilon) - c) - U_\rho(T_{l,\sigma}(u_\varepsilon) - c) \to 0$$

in $L^p_{\rm loc}$ for all $p < \infty$. Thus, (since we can take $p = 2$ as well) we see that $\Gamma_{1,\varepsilon} \in H^{-1}_{\rm loc,c}$.

Next we shall prove that

$$\begin{aligned}
\Gamma_{2,\varepsilon} = \operatorname{div}_x \Big(&\theta(T_l(u_\varepsilon) - c)(f(t,x,T_l(u_\varepsilon)) - f(t,x,c)) \\
&- \int^{u_\varepsilon} U_\rho'(T_{l,\sigma}(v) - c) T_{l,\sigma}'(v) \partial_v f_\varrho(t,x,v)\, dv \Big) \in H^{-1}_{\rm loc,c} + \mathcal{M}_{\rm loc,B}.
\end{aligned}$$

Indeed,

$$\theta(T_l(u_\varepsilon) - c)(f(t,x,T_l(u_\varepsilon)) - f(t,x,c))$$

$$
\begin{aligned}
&\qquad\qquad - \int^{u_\varepsilon} U_\rho'(T_{l,\sigma}(v)-c)T_{l,\sigma}'(v)\partial_v f_\varrho(t,x,v)dv \\
&= \theta(T_l(u_\varepsilon)-c)(f(t,x,T_l(u_\varepsilon))-f(t,x,c)) \\
&\qquad\qquad - \theta(T_{l,\sigma}(u_\varepsilon)-c)(f_\varrho(t,x,T_{l,\sigma}(u_\varepsilon))-f_\varrho(t,x,c)) \\
&\quad + \theta(T_{l,\sigma}(u_\varepsilon)-c)(f(t,x,T_{l,\sigma}(u_\varepsilon))-f(t,x,c)) \\
&\qquad\qquad - \int^{u_\varepsilon} U_\rho'(T_{l,\sigma}(v)-c)T_l'(v)\partial_v f_\varrho(t,x,v)dv \\
&\quad - \int^{u_\varepsilon} U_\rho'(T_{l,\sigma}(v)-c)(T_{l,\sigma}'(v)-T_l'(v))\partial_v f_\varrho(t,x,v)dv.
\end{aligned}
$$

Since $T_l'(u) \in \{0,1\}$,

$$
\begin{aligned}
&\int^{u_\varepsilon} U_\rho'(T_{l,\sigma}(v)-c)T_l'(v)\partial_v f_\varrho(t,x,v)dv \\
&\qquad = \int^{u_\varepsilon} U_\rho'(T_{l,\sigma}(v)-c)T_l'(v)\partial_v f_\varrho(t,x,T_l(v))dv,
\end{aligned}
$$

from which we conclude

$$
\begin{aligned}
&\theta(T_l(u_\varepsilon)-c)(f(t,x,T_l(u_\varepsilon))-f(t,x,c)) \\
&\qquad\qquad - \int^{u_\varepsilon} U_\rho'(T_{l,\sigma}(v)-c)T_{l,\sigma}'(v)\partial_v f_\varrho(t,x,v)dv \\
&= \theta(T_l(u_\varepsilon)-c)(f(t,x,T_l(u_\varepsilon))-f(t,x,c)) \\
&\qquad\qquad - \theta(T_{l,\sigma}(u_\varepsilon)-c)(f(t,x,T_{l,\sigma}(u_\varepsilon))-f(t,x,c)) \\
&\quad + \theta(T_{l,\sigma}(u_\varepsilon)-c)(f(t,x,T_{l,\sigma}(u_\varepsilon))-f(t,x,c)) \\
&\qquad\qquad - \int^{u_\varepsilon} U_\rho'(T_{l,\sigma}(v)-c)T_l'(v)\partial_v f_\varrho(t,x,T_l(v))dv \\
&\quad - \int^{u_\varepsilon} U_\rho'(T_{l,\sigma}(v)-c)(T_{l,\sigma}'(v)-T_l'(v))\partial_v f_\varrho(t,x,v)dv \\
&= \theta(T_l(u_\varepsilon)-c)(f(t,x,T_l(u_\varepsilon))-f(t,x,c)) \\
&\qquad\qquad - \theta(T_{l,\sigma}(u_\varepsilon)-c)(f(t,x,T_{l,\sigma}(u_\varepsilon))-f(t,x,c)) \\
&\quad + \theta(T_{l,\sigma}(u_\varepsilon)-c)(f(t,x,T_{l,\sigma}(u_\varepsilon))-f(t,x,c)) \\
&\qquad\qquad - \int^{u_\varepsilon} \theta(T_{l,\sigma}(v)-c)D_v[f_\varrho(t,x,T_l(v))]dv \\
&\quad - \int^{u_\varepsilon} U_\rho'(T_{l,\sigma}(v)-c)(T_{l,\sigma}'(v)-T_l'(v))\partial_v f_\varrho(t,x,v)dv \\
&\quad - \int^{u_\varepsilon} (U_\rho'(T_{l,\sigma}(v)-c)-\theta(T_{l,\sigma}(v)-c))T_l'(v)\partial_v f_\varrho(t,x,T_l(v))dv \\
&= \Gamma_{2,\varepsilon}^1 + \Gamma_{2,\varepsilon}^2 + \Gamma_{2,\varepsilon}^3,
\end{aligned} \tag{35}
$$

with

$$
\begin{aligned}
\Gamma_{2,\varepsilon}^1 &= \theta(T_l(u_\varepsilon)-c)(f(t,x,T_l(u_\varepsilon))-f(t,x,c)) \\
&\qquad\qquad - \theta(T_{l,\sigma}(u_\varepsilon)-c)(f(t,x,T_{l,\sigma}(u_\varepsilon))-f(t,x,c)),
\end{aligned}
$$

$$
\Gamma_{2,\varepsilon}^2 = \theta(T_{l,\sigma}(u_\varepsilon)-c)(f(t,x,T_{l,\sigma}(u_\varepsilon))-f(t,x,c))
$$

$$- \int^{u_\varepsilon} \theta(T_{l,\sigma}(v) - c) D_v[f_\varrho(t,x,T_l(v))]dv,$$

$$\begin{aligned}
\Gamma^3_{2,\varepsilon} = &- \int^{u_\varepsilon} U'_\rho(T_{l,\sigma}(v) - c)(T'_{l,\sigma}(v) - T'_l(v))\partial_v f_\varrho(t,x,v)dv \\
&- \int^{u_\varepsilon} (U'_\rho(T_{l,\sigma}(v) - c) - \theta(T_{l,\sigma}(v) - c))T'_l(v)\partial_v f_\varrho(t,x,T_l(v))dv.
\end{aligned}$$

Consider now each term on the right-hand side of (35). Since T_l is continuous function and $T_l(u) \in [-l, l]$, the function $f(t,x,T_l(u))$ is uniformly continuous in $u \in \mathbf{R}$. Therefore, we have pointwise on $\mathbf{R}^+ \times \mathbf{R}^d$:

$$\begin{aligned}
|\Gamma^1_{2,\varepsilon}| = |&\theta(T_l(u_\varepsilon) - c)(f(t,x,T_l(u_\varepsilon)) - f(t,x,c)) \\
&- \theta(T_{l,\sigma}(u_\varepsilon) - c)(f(t,x,T_{l,\sigma}(u_\varepsilon)) - f(t,x,c))| = o_\sigma(1)
\end{aligned}$$

where $o_\sigma(1) \to 0$ as $\sigma \to \infty$. Since $\max_{u\in[-l,l]} f(t,x,u) \in L^p_{\rm loc}(\mathbf{R}^+ \times \mathbf{R}^d)$, $p > 2$, Lebesgue's dominated convergence theorem yields $|\Gamma^1_{2,\varepsilon}| = o_{\sigma,L^p_{\rm loc}}(1)$ where $\int_{\mathbf{R}^+\times\mathbf{R}^d} |o_{\sigma,L^p}(1)|^p dxdt \to 0$ as $\sigma \to 0$. Thus, we conclude

$$\operatorname{div}_x \Gamma^1_{2,\varepsilon} \in H^{-1}_{\rm loc}(\mathbf{R}^+ \times \mathbf{R}^d). \tag{36}$$

We pass to $\Gamma^2_{2,\varepsilon}$. We have to make a case distinction depending on the relative size of c and l. Consider first the case when $|c| < l$, in which case we have $T_l(c) = c$, thus:

$$\begin{aligned}
|\Gamma^2_{2,\varepsilon}| &= |\theta(T_{l,\sigma}(u_\varepsilon) - c)(f(t,x,T_{l,\sigma}(u_\varepsilon)) - f(t,x,c)) \\
&\qquad - \int^{u_\varepsilon} \theta(T_{l,\sigma}(v) - c) D_v[f_\varrho(t,x,T_l(v))]dv| \\
&= |\theta(T_{l,\sigma}(u_\varepsilon) - c)(f(t,x,T_{l,\sigma}(u_\varepsilon)) - f(t,x,c)) \\
&\qquad - \theta(T_{l,\sigma}(u_\varepsilon) - c) \int_c^{u_\varepsilon} D_v[f_\varrho(t,x,T_l(v))]dv| \\
&= |\theta(T_{l,\sigma}(u_\varepsilon) - c)(f(t,x,T_{l,\sigma}(u_\varepsilon)) - f(t,x,c)) \\
&\qquad - \theta(T_{l,\sigma}(u_\varepsilon) - c)(f_\varrho(t,x,T_l(u_\varepsilon)) - f_\varrho(t,x,c))| \\
&\le |\theta(T_{l,\sigma}(u_\varepsilon) - c)(f(t,x,T_{l,\sigma}(u_\varepsilon)) - f_\varrho(t,x,T_l(u_\varepsilon)))| \\
&\quad + |\theta(T_{l,\sigma}(u_\varepsilon) - c)(f(t,x,c) - f_\varrho(t,x,c))| \\
&\le |\theta(T_{l,\sigma}(u_\varepsilon) - c)(f(t,x,T_{l,\sigma}(u_\varepsilon)) - f_\varrho(t,x,T_{l,\sigma}(u_\varepsilon)))| \\
&\quad + |\theta(T_{l,\sigma}(u_\varepsilon) - c)(f_\varrho(t,x,T_{l,\sigma}(u_\varepsilon)) - f_\varrho(t,x,T_l(u_\varepsilon)))| \\
&\quad + |\theta(T_{l,\sigma}(u_\varepsilon) - c)(f(t,x,c) - f_\varrho(t,x,c))| \\
&= o_{\varrho,L^p_{\rm loc}}(1) + o_{\sigma,L^p_{\rm loc}}(1) + o_{\varrho,L^p_{\rm loc}}(1),
\end{aligned} \tag{37}$$

where $o_\sigma(1)$ comes from (36) and term $|o_{\varrho,L^p_{\rm loc}}(1)|$ appears due to (H4b), (12a).

For $c > l$ we have $\theta(T_{l,\sigma}(u_\varepsilon) - c) \equiv 0$ and for $c < -l$ we have $\theta(T_{l,\sigma}(u_\varepsilon) - c) \equiv 1$. Thus, the problematic case is when $c < -l$. In this case, instead of (37) we have:

$$\begin{aligned}
\Gamma^2_{2,\varepsilon} &= \theta(T_{l,\sigma}(u_\varepsilon) - c)(f(t,x,T_{l,\sigma}(u_\varepsilon)) - f(t,x,c)) \\
&\qquad - \int^{u_\varepsilon} \theta(T_{l,\sigma}(v) - c) D_v[f_\varrho(t,x,T_l(v))]dv \\
&= f(t,x,T_{l,\sigma}(u_\varepsilon)) - f_\varrho(t,x,T_l(u_\varepsilon)) + f(t,x,l) - f(t,x,c)
\end{aligned}$$

implying

$$\operatorname{div}_x \Gamma^2_{2,\varepsilon} \in H^{-1}_{\mathrm{loc,c}} + \mathcal{M}_{\mathrm{loc,B}}, \tag{38}$$

since $f(t,x,T_{l,\sigma}(u_\varepsilon)) - f_\varrho(t,x,T_l(u_\varepsilon)) \to 0$ in $L^p_{\mathrm{loc}}(\mathbf{R}^+ \times \mathbf{R}^d)$ for p defined in (H4), and $f(t,x,l) - f(t,x,c) \in BV(\mathbf{R}^+ \times \mathbf{R}^d)$.

It remained to estimate $\Gamma^3_{2,\varepsilon}$. Noticing that $|U'_\rho|, |T_{l,\sigma}| \leq 1$ we get

$$\begin{aligned} &\Big|\int^{u_\varepsilon} U'_\rho(T_{l,\sigma}(v)-c)(T'_{l,\sigma}(v) - T'_l(v))\partial_v f_\varrho(t,x,v)dv\Big| \\ &\qquad\qquad \leq C\int_{\mathbf{R}} |T'_{l,\sigma}(v) - T'_l(v)|dv = \mathcal{O}\big(\frac{\sigma}{\beta(\varrho)}\big) \overset{(23),(34)}{=} \mathcal{O}(\beta(\varepsilon^3)), \end{aligned} \tag{39}$$

where C is the constant from (12d).

Similarly, from (12d) and since $|T'_l(v)| \leq 1$, we have:

$$\begin{aligned} &\Big|\int^{u_\varepsilon} (U'_\rho(T_{l,\sigma}(v)-c) - \theta(T_{l,\sigma}(v)-c))T'_l(v)\partial_v f_\varrho(t,x,T_l(v))dv\Big| \\ &\leq C\int^{u_\varepsilon} |U'_\rho(T_{l,\sigma}(v)-c) - \theta(T_{l,\sigma}(v)-c)|dv = \mathcal{O}\big(\frac{\rho}{\beta(\varrho)}\big) \overset{(23)}{=} \mathcal{O}(\beta(\varepsilon^3)), \end{aligned} \tag{40}$$

from which we conclude that $\Gamma^3_{2,\varepsilon}$ tends to zero in L^2_{loc}. From assumptions (23) and (34), as well as estimates (36)–(40), it follows that the expression from (35) tends to zero in L^2_{loc} from which it follows that $\Gamma_{2,\varepsilon} \in H^{-1}_{\mathrm{loc,c}}$.

The next term is

$$\begin{aligned} \Gamma_{3,\varepsilon} = &\int^{u_\varepsilon} U'_\rho(T_{l,\sigma}(v)-c)T'_{l,\sigma}(v)\operatorname{div}_x \partial_v f_\varrho(t,x,v)dv \\ &- U'_\rho(T_{l,\sigma}(u_\varepsilon)-c)T'_{l,\sigma}(u_\varepsilon)\operatorname{div}_x f_\varrho(t,x,v)|_{v=u_\varepsilon}. \end{aligned}$$

According to (H4) it is clear that $\Gamma_{3,\varepsilon} \in \mathcal{M}_{\mathrm{loc,B}}$. Indeed, since $|U'_\rho|, |T'_{l,\sigma}| \leq 1$ we have from (12b) and (12e):

$$|\Gamma_{3,\varepsilon}| \leq \int_{\mathbf{R}} \frac{\mu_3(t,x)}{1+|v|^{1+\alpha}}dv + \mu_1(t,x)$$

implying the claim.

Next, we claim that:

$$\Gamma_{4,\varepsilon} = \sum_{i=1}^d D_{x_i}\left(\varepsilon D_{x_i} U_\rho(T_{l,\sigma}(u_\varepsilon)-c) + \delta D_u[U_\rho(T_{l,\sigma}(u_\varepsilon)-c)]\partial^2_{x_i x_i} u_\varepsilon\right) \in H^{-1}_{\mathrm{loc,c}}.$$

Due to a priori estimates (17) and (18) and, again, the fact that $|T'_{l,\sigma}|, |U'_\rho| \leq 1$, we see that for every $i = 1,\dots,d$

$$\varepsilon D_{x_i} U_\rho(T_{l,\sigma}(u_\varepsilon)-c) + \delta D_u[U_\rho(T_{l,\sigma}(u_\varepsilon)-c)]\partial^2_{x_i x_i} u_\varepsilon \to 0$$

in $L^2(\mathbf{R}^+ \times \mathbf{R}^d)$. Therefore, $\Gamma_{4,\varepsilon} \in H^{-1}_{\mathrm{loc,c}}$.

Further, we claim that

$$\Gamma_{5,\varepsilon} = \varepsilon U'_\rho(T_{l,\sigma}(u_\varepsilon)-c)T''_{l,\sigma}(u_\varepsilon)|\nabla u_\varepsilon|^2 \in \mathcal{M}_{\mathrm{loc,B}}.$$

Since $|U'_\rho| \leq 1$ and $|T''_{l,\sigma}| \leq \frac{1}{\sigma}$ we have from (32)

$$\varepsilon \int_{\mathbf{R}^+ \times \mathbf{R}^d} |U'_\rho(T_{l,\sigma}(u_\varepsilon)-c)T''_{l,\sigma}(u_\varepsilon)|\,|\nabla u_\varepsilon|^2 dxdt$$

$$\leq \frac{\varepsilon}{\sigma}\int_{l<|u_\varepsilon|<l+\sigma}|\nabla u_\varepsilon|^2 dxdt \leq K_5 + \left(\frac{\delta^2}{\sigma^2\varepsilon^4}+\frac{\delta^2}{\sigma^2(\beta(\varrho))^2\varepsilon^4}\right)^{1/2} K_3K_4 \leq K_6,$$

for some constant K_6, according to assumptions (23) and (34) on $\delta=\delta(\varepsilon)$, $\sigma=\sigma(\varepsilon)$, $\varrho=\varrho(\varepsilon)$ and $\beta(\varrho)=\beta(\varepsilon^3)$. Thus, we see that $\Gamma_{5,\varepsilon}\in\mathcal{M}_{\mathrm{loc,B}}$.

Next, we need to show

$$\Gamma_{6,\varepsilon} = \delta\sum_{i=1}^d D^2_{uu}\left[U_\rho(T_{l,\sigma}(u_\varepsilon)-c)\right]\partial_{x_i}u_\varepsilon\partial^2_{x_ix_i}u_\varepsilon \in \mathcal{M}_{\mathrm{loc,B}}.$$

In view of a priori estimates (17), (18) and assumptions (23), (34), it holds

$$\iint_{\mathbf{R}^+\times\mathbf{R}^d}\delta\Big|\sum_{i=1}^d D^2_{uu}\left[U_\rho(T_{l,\sigma}(u_\varepsilon)-c)\right]\partial_{x_i}u_\varepsilon\partial^2_{x_ix_i}u_\varepsilon\Big|dxdt$$

$$\leq \sum_{i=1}^d\left(\frac{\delta}{\sigma}+\frac{\delta}{\rho}\right)\left(\iint_{\mathbf{R}^+\times\mathbf{R}^d}|\partial_{x_i}u_\varepsilon|^2dxdt\right)^{1/2}\left(\iint_{\mathbf{R}^+\times\mathbf{R}^d}|\partial^2_{x_ix_i}u_\varepsilon|^2dxdt\right)^{1/2}$$

$$\leq \left(\frac{\delta}{\varepsilon^2\sigma}+\frac{\delta}{\varepsilon^2\rho}\right)\sum_{i=1}^d\left(\varepsilon\int_{\mathbf{R}^+\times\mathbf{R}^d}|\partial_{x_i}u_\varepsilon|^2dxdt\right)^{1/2}\left(\varepsilon^3\int_{\mathbf{R}^+\times\mathbf{R}^d}|\partial^2_{x_ix_i}u_\varepsilon|^2dxdt\right)^{1/2}$$

$$\leq C\left(\frac{\delta}{\varepsilon^2\sigma}+\frac{\delta}{\varepsilon^2\rho}\right)\left(\frac{\varepsilon}{\varrho}+\frac{1}{(\beta(\varrho))^2}\right)\leq\tilde{C},$$

for some constants C and $\tilde{C}$. The second estimate holds since $U''_\rho\leq\frac{1}{\rho}$ and $T''_{l,\sigma}\leq\frac{1}{\sigma}$ implying $|D_{uu}[U_\rho(T_{l,\sigma}(u_\varepsilon)-c)]|\leq\left(\frac{1}{\sigma}+\frac{1}{\rho}\right)$. Therefore, $\Gamma_{6,\varepsilon}\in\mathcal{M}_{\mathrm{loc,B}}$.

Finally, we will prove that

$$\Gamma_{7\varepsilon} = -\varepsilon U''_\rho(T_{l,\sigma}(u_\varepsilon)-c)(T'_{l,\sigma}(u_\varepsilon))^2|\nabla u_\varepsilon|^2\in\mathcal{M}_{\mathrm{loc,B}}.$$

First, notice that $\mathrm{supp}U''_\rho=(0,\rho)$, and therefore:

$$U''_\rho(T_{l,\sigma}(u_\varepsilon)-c)\neq 0 \ \text{ for } \ c\leq T_{l,\sigma}(u_\varepsilon)\leq c+\rho. \tag{41}$$

Then, assume initially that $c\geq l$. In that case $U''_\rho(T_{l,\sigma}(u_\varepsilon)-c)\neq 0$ only if $u_\varepsilon\geq l$. But, then $T'_{l,\sigma}(u_\varepsilon)=0$ and thus $\Gamma_{7\varepsilon}\equiv 0\in\mathcal{M}_{\mathrm{loc,B}}$.

Now, assume that $c<l$. In this case, we can assume that $c+\rho<l-\sigma$ since we can choose ρ and σ arbitrary small. Therefore, from the definition of $\Gamma_{7\varepsilon}$ and (41) it follows that we can assume $T_{l,\sigma}(u_\varepsilon)=u_\varepsilon$. Thus,

$$\Gamma_{7\varepsilon}=-\varepsilon U''_\rho(u_\varepsilon-c)|\nabla u_\varepsilon|^2\in\mathcal{M}_{\mathrm{loc,B}},$$

according to (41) and (32) (we put there $l=c$).

Collecting the previous items, due to the properties of $\Gamma_{i,\varepsilon}$, $i=1,\dots,7$, it follows from (33) that

$$\partial_t\theta(T_l(u_\varepsilon)-c)(T_l(u_\varepsilon)-c)$$
$$+\operatorname{div}_x\theta(T_l(u_\varepsilon)-c)(f(t,x,T_l(u_\varepsilon))-f(t,x,c))\in\mathcal{M}_{\mathrm{loc,B}}+H^{-1}_{\mathrm{loc,c}}.$$

Therefore, we see that (ii) is satisfied and we can use Murat's lemma to conclude that

$$\partial_t\theta(T_l(u_\varepsilon)-c)(T_l(u_\varepsilon)-c)$$
$$+\operatorname{div}_x\theta(T_l(u_\varepsilon)-c)(f(t,x,T_l(u_\varepsilon))-f(t,x,c))\in H^{-1}_{\mathrm{loc,c}}.$$

Thus we conclude that the conditions of Theorem 4 are satisfied, and we find that for every $l > 0$ the sequence $(T_l(u_\varepsilon))_{\varepsilon>0}$ is precompact in $L^1_{\rm loc}(\mathbf{R}^+ \times \mathbf{R})$.

Since the sequence $(u_\varepsilon)_{\varepsilon>0}$ is uniformly bounded in $L^2(\mathbf{R}^+ \times \mathbf{R}^d)$, from [4, Lemma 7], we conclude that $(u_\varepsilon)_{\varepsilon>0}$ is precompact in $L^1_{\rm loc}(\mathbf{R}^+ \times \mathbf{R}^d)$. ❑

4. The one-dimensional case

We will analyze the convergence of the sequence $(u_\varepsilon)_{\varepsilon>0}$ of solutions to (14), (15) in the one dimensional case. Unlike the situation we had in the previous section, we shall assume that the flux is continuously differentiable in the u variable. This will enable us to optimize the ratio δ/ε^2. We will work under the following assumptions on the flux $f = f(t,x,u)$ denoted **(H4')**:

(H4a') For the flux $f = f(t,x,u)$ we assume that $f \in C^1(\mathbf{R}; BV(\mathbf{R}^+ \times \mathbf{R}_x)) \cap L^\infty(\mathbf{R} \times \mathbf{R}^+ \times \mathbf{R}_x)$ and $\partial_u f \in L^\infty(\mathbf{R} \times \mathbf{R}^+ \times \mathbf{R}_x)$.

(H4b') There exists a sequence $(f_\varrho)_{\varrho>0}$ defined on $\mathbf{R}^+ \times \mathbf{R} \times \mathbf{R}$, smooth in $(t,x) \in \mathbf{R}^+ \times \mathbf{R}$ and continuously differentiable in $u \in \mathbf{R}$, satisfying for some $p > 2$:

$$\lim_{\varrho\to 0} \max_{z\in\mathbf{R}} ||f_\varrho(t,x,z) - f(t,x,z)||_{L^p_{\rm loc}(\mathbf{R}^+\times\mathbf{R})} = 0,$$

$$|\partial_x f_\varrho(t,x,u)| \leq \frac{\mu_1(t,x)}{1+|u|^{1+\alpha}}, \quad \varrho^3 |\partial_x f_\varrho(t,x,u)|^2 \leq \mu_2(t,x),$$

$$|\partial^2_{xu} f_\varrho(t,x,u)| \leq \frac{\mu_3(t,x)}{1+|u|^{1+\alpha}},$$

$$|\partial_u f(t,x,u)| \leq C,$$

where $\mu_i \in \mathcal{M}(\mathbf{R}^+ \times \mathbf{R})$, $i = 1,2,3$, are bounded measures (and, accordingly, the above inequalities involving μ_i, $i = 1,2,3$, are understood in the sense of measures).

Under these assumptions we will prove the following:

- Without assuming non-degeneracy of the flux, the sequence $(u_\varepsilon)_{\varepsilon>0}$ converges along a subsequence to a solution of (10)–(11) in the distributional sense when $\delta = \mathcal{O}(\varepsilon^2)$ and $\varrho = \mathcal{O}(\varepsilon)$ (less stringent assumptions than in the multidimensional case).
- If, in addition, we assume $f \in C^2(\mathbf{R}; BV(\mathbf{R}^+ \times \mathbf{R}_x)) \cap L^\infty(\mathbf{R} \times \mathbf{R}^+ \times \mathbf{R}_x)$, and that f is genuinely nonlinear in the sense of (44), the sequence $(u_\varepsilon)_{\varepsilon>0}$ of solutions to problem (14)–(15) is strongly precompact in $L^1_{\rm loc}(\mathbf{R}^+ \times \mathbf{R})$ when $\delta = \mathcal{O}(\varepsilon^2)$.

Remark 6. The proof relies on a priori inequalities (17) and (18). Notice that in the inequality (18) we can take $\beta(\varrho) = 1$ due to (H4a').

We shall need the fundamental theorem of Young measures.

Theorem 7. [18] *Assume that the sequence (u_{ε_k}) is uniformly bounded in $L^\infty(\mathbf{R}^+; L^p(\mathbf{R}^d)) \cap L^r(\mathbf{R}^+ \times \mathbf{R}^d)$, $p, r \geq 1$. Then, there exists a subsequence (not relabeled) (u_{ε_k}) and a sequence of probability measures*

$$\nu_{(t,x)} \in \mathcal{M}(\mathbf{R}), \quad (t,x) \in \mathbf{R}^+ \times \mathbf{R}^d$$

such that the limit

$$\bar{g}(t,x) := \lim_{k\to\infty} g(t,x,u_{\varepsilon_k}(t,x))$$

exists in the distributional sense for all g measurable with respect to $(t,x) \in \mathbf{R}^+ \times \mathbf{R}^d$, continuous in $u \in \mathbf{R}$ and satisfying uniformly in (t,x):

$$|g(t,x,u)| \leq C(1+|u|^q)$$

for constants C, M and q such that $0 \leq q < p$. The limit is represented by the expectation value

$$\bar{g}(t,x) = \int_{\mathbf{R}^+ \times \mathbf{R}^d} g(t,x,\lambda) d\nu_{(t,x)}(\lambda),$$

for almost all points $(t,x) \in \mathbf{R}^+ \times \mathbf{R}^d$.

We refer to such a sequence of measures $\nu = (\nu_{(t,x)})$ as the Young measure associated to sequence $(u_{\varepsilon_k})_{k\in\mathbf{N}}$.

Furthermore,

$$u_{\varepsilon_k} \to u \ \text{ in } \ L^r_{\mathrm{loc}}(\mathbf{R}^+ \times \mathbf{R}^d), \ 1 \leq r < p$$

if and only if

$$\nu_y = \delta_{u(y)} \ \ a.e.$$

Before we continue, we need to recall the celebrated Div-Curl lemma.

Lemma 8 (Div-Curl)**.** *Let $Q \subset \mathbf{R}^2$ be a bounded domain. Suppose*

$$\begin{aligned} v^1_\varepsilon \rightharpoonup \overline{v}^1, \quad v^2_\varepsilon \rightharpoonup \overline{v}^2, \\ w^1_\varepsilon \rightharpoonup \overline{w}^1, \quad w^2_\varepsilon \rightharpoonup \overline{w}^2, \end{aligned}$$

in $L^2(Q)$ as $\varepsilon \downarrow 0$. Suppose also that the two sequences $\left\{\mathrm{div}\left(v^1_\varepsilon, v^2_\varepsilon\right)\right\}_{\varepsilon>0}$ and $\left\{\mathrm{curl}\left(w^1_\varepsilon, w^2_\varepsilon\right)\right\}_{\varepsilon>0}$ lie in a (common) compact subset of $H^{-1}_{\mathrm{loc}}(Q)$, where $\mathrm{div}\left(v^1_\varepsilon, v^2_\varepsilon\right) = \partial_{x_1} v^1_\varepsilon + \partial_{x_2} v^2_\varepsilon$ and $\mathrm{curl}\left(w^1_\varepsilon, w^2_\varepsilon\right) = \partial_{x_1} w^2_\varepsilon - \partial_{x_2} w^1_\varepsilon$. Then along a subsequence

$$\left(v^1_\varepsilon, v^2_\varepsilon\right) \cdot \left(w^1_\varepsilon, w^2_\varepsilon\right) \to \left(\overline{v}^1, \overline{v}^2\right) \cdot \left(\overline{w}^1, \overline{w}^2\right) \quad \text{in } \mathcal{D}'(Q) \text{ as } \varepsilon \downarrow 0.$$

Lemma 9. *Assume that $(u_\varepsilon)_{\varepsilon>0} \in L^2(\mathbf{R}^+ \times \mathbf{R})$ weakly converges in $L^2(\mathbf{R}^+ \times \mathbf{R})$ to a function $u \in L^2(\mathbf{R}^+ \times \mathbf{R})$. Assume that $\eta(t,x,\lambda)$, $(t,x,\lambda) \in \mathbf{R}^+ \times \mathbf{R}^2$ is a function such that $\eta \in C^2(\mathbf{R}_\lambda; L^\infty \cap BV(\mathbf{R}^+_t \times \mathbf{R}_x))$.*

By η_n we denote the truncation of the function η:

$$\eta_n(t,x,\lambda) = \begin{cases} \eta(t,x,\lambda), & |\lambda| < n, \\ 0, & |\lambda| > 2n \end{cases}, \quad (t,x) \in \mathbf{R}^+ \times \mathbf{R}, \tag{42}$$

and $q_n(t,x,\lambda)$ the corresponding entropy flux.

If for every $n \in \mathbf{N}$ we have

$$\mathrm{div}(\eta_n(u_\varepsilon), q_n(t,x,u_\varepsilon)) \in H^{-1}_{loc,c}(\mathbf{R}^+ \times \mathbf{R}). \tag{43}$$

then the limit function u is a weak solution to (2).

Furthermore, if the flux function $f = f(t,x,\lambda)$ is twice differentiable with respect to λ, and it is genuinely nonlinear, i.e., for every $(t,x) \in \mathbf{R}^+ \times \mathbf{R}^d$ the mapping

$$\mathbf{R} \ni \lambda \mapsto \partial_\lambda f(t,x,\lambda) \ \text{is non-constant} \tag{44}$$

on non-degenerate intervals, then $(u_\varepsilon)_{\varepsilon>0}$ strongly converges to u in $L^1_{\mathrm{loc}}(\mathbf{R}^+ \times \mathbf{R})$.

Proof: We shall apply the method of compensated compactness as in [21].

First notice that according to Theorem 7 there exists a subsequence $(u_{\varepsilon_k}) \subset (u_\varepsilon)$ and a sequence of probability measures

$$\nu_{(t,x)} \in \mathcal{M}(\mathbf{R}), \quad (t,x) \in \mathbf{R}^+ \times \mathbf{R}$$

such that the limit

$$\bar{g}(t,x) := \lim_{k\to\infty} g(t,x,u_{\varepsilon_k}(t,x))$$

exists in the distributional sense for all g measurable with respect to $(t,x) \in \mathbf{R}^+\times\mathbf{R}$, continuous in $u \in \mathbf{R}$ and satisfying uniformly in (t,x):

$$|g(t,x,u)| \le C(1+|u|^q)$$

for constants C, M and q such that $0 \le q < p$, and is represented by the expectation value

$$\bar{g}(t,x) = \int_{\mathbf{R}^+\times\mathbf{R}} g(t,x,\lambda) d\nu_{(t,x)}(\lambda),$$

for almost all points $(t,x) \in \mathbf{R}^+ \times \mathbf{R}$. From this, due to (H4), we conclude that for the flux function $f(t,x,v)$ we have

$$\lim_{k\to\infty} f(t,x,u_{\varepsilon_k}(t,x)) = \int_{\mathbf{R}^+\times\mathbf{R}} f(t,x,\lambda) d\nu_{(t,x)}(\lambda).$$

To continue, notice that

$$u(t,x) = \int \lambda \, d\nu_{(t,x)}(\lambda). \tag{45}$$

Take $\eta(u) = I(u) = u$ in (42), and consider the vector fields $(I_n(u_\varepsilon), f_n(t,x,u_\varepsilon))$ where $f_n(t,x,u_\varepsilon) = I_n'(v)\partial_\lambda f(t,x,u_\varepsilon)$, and $(-\psi_n(t,x,u_\varepsilon), \phi_n(u_\varepsilon))$, where $\phi \in C^1(\mathbf{R})$ is an arbitrary entropy, and ψ_n is the entropy flux corresponding to ϕ_n. Here I_n and ϕ_n denote the smooth truncation functions of I and ϕ, respectively, cf. (42).

According to (43) we can apply the Div-Curl lemma on the given vector fields. Hence, we get after letting $\varepsilon \to 0$ along a subsequence:

$$\begin{aligned}
&\int \big(I_n(\lambda)\psi_n(t,x,\lambda) - \phi_n(\lambda) f_n(t,x,\lambda)\big) d\nu_{(t,x)}(\lambda) \\
&\qquad = \int \big(\bar{u}_n(t,x)\psi_n(t,x,\lambda) - \bar{f}_n(t,x)\phi_n(\lambda)\big) d\nu_{(t,x)}(\lambda),
\end{aligned} \tag{46}$$

where

$$\bar{f}_n(t,x) = \int f_n(t,x,\lambda) d\nu_{(t,x)}(\lambda), \quad \bar{u}_n(t,x) = \int I_n(\lambda) d\nu_{(t,x)}(\lambda).$$

Then, put $\phi(\lambda) = |\lambda - u(t,x)|$. Notice that for $|\lambda| < n$ we have $\psi_n(t,x,\lambda) = \operatorname{sgn}(\lambda - u(t,x))(f(t,x,\lambda) - f(t,x,u(t,x)))$. Therefore, we have from (46):

$$\begin{aligned}
&\int_{-n}^{n} \big(\lambda \operatorname{sgn}(\lambda - u(t,x))(f(t,x,\lambda) - f(t,x,u(t,x))) - |u(t,x)-\lambda| f(t,x,\lambda)\big) d\nu_{(t,x)}(\lambda) \\
&- \int_{-n}^{n} \big(u(t,x) \operatorname{sgn}(\lambda - u(t,x))(f(t,x,\lambda) - f(t,x,u(t,x))) - |u(t,x)-\lambda| \bar{f}_n\big) d\nu_{(t,x)}(\lambda) \\
&= -\left(\int_{-\infty}^{-n} + \int_{n}^{\infty}\right) \big(I_n(\lambda)\psi_n(t,x,\lambda) - \phi_n(\lambda) f_n(t,x,\lambda)\big) d\nu_{(t,x)}(\lambda)
\end{aligned}$$

$$+\left(\int_{-\infty}^{-n}+\int_{n}^{\infty}\right)\big(u(t,x)\psi_n(t,x,\lambda)-\bar f_n\phi_n(\lambda)\big)d\nu_{(t,x)}(\lambda)$$
$$+\left(\int_{-\infty}^{-n}+\int_{n}^{\infty}\right)(I_n(\lambda)-\lambda)d\nu_{(t,x)}(\lambda)\int\psi_n(t,x,\lambda)d\nu_{(t,x)}(\lambda). \tag{47}$$

It is clear that for every fixed $(t,x)\in\mathbf{R}^+\times\mathbf{R}^d$ the right-hand side of (47) tends to zero as $n\to\infty$ implying:

$$\int\big(\lambda\,\mathrm{sgn}(\lambda-u(t,x))(f(t,x,\lambda)-f(t,x,u(t,x)))-|u(t,x)-\lambda|f(t,x,\lambda)\big)d\nu_{(t,x)}(\lambda)$$
$$-\int\big(u(t,x)\mathrm{sgn}(\lambda-u(t,x))(f(t,x,\lambda)-f(t,x,u(t,x)))$$
$$-|u(t,x)-\lambda|\bar f(t,x)\big)d\nu_{(t,x)}(\lambda)=0.$$

Now a standard procedure gives (see, e.g., [10, Remark 2.3])

$$(f(t,x,u(t,x))-\bar f(t,x))\int|\lambda-u(t,x)|d\nu_{(t,x)}(\lambda)=0, \tag{48}$$

where $\bar f(t,x)=\int f(t,x,\lambda)d\nu_{(t,x)}(\lambda)$. From here it follows that u is a weak solution to (10). This concludes the first part of the lemma. For the details of the procedure one should consult, e.g., [21].

Now, assume that $f\in C^2(\mathbf{R};BV(\mathbf{R}^+\times\mathbf{R}_x))\cap L^\infty(\mathbf{R}\times\mathbf{R}^+\times\mathbf{R}_x)$, and that it is genuinely nonlinear in the sense of (44).

Then, take arbitrary $\eta_1(t,x,u)\in C^1((\mathbf{R};L^\infty\cap BV(\mathbf{R}_t^+\times\mathbf{R}_x)))$ and $\eta_2\in C^1(\mathbf{R})$; thus $\partial_u\eta_1$ depends explicitly on (t,x), while $D_u\eta_2$ does not. Denote by $\eta_{1,n}$ and $\eta_{2,n}$ the appropriate smooth truncations, cf. (42), and by $q_{1,n}$ and $q_{2,n}$ the corresponding entropy fluxes, that is,

$$q_{1,n}(t,x,\lambda)=\int^\lambda\partial_z\eta_{1,n}(t,x,z)\partial_z f(t,x,z)dz,$$
$$q_{1,n}(t,x,\lambda)=\int^\lambda\partial_z\eta_{2,n}(z)\partial_z f(t,x,z)dz.$$

Due to (43) and the Div-Curl lemma the following commutation relation holds:

$$\int_{\mathbf{R}}\left(\eta_{1,n}(t,x,\lambda)q_{2,n}(t,x,\lambda)-\eta_2(\lambda)q_{1,n}(t,x,\lambda)\right)d\nu_{(t,x)} \tag{49}$$
$$=\int_{\mathbf{R}}\eta_{1,n}(t,x,\lambda)d\nu_{(t,x)}\int_{\mathbf{R}}q_{2,n}(t,x,\lambda)d\nu_{(t,x)}$$
$$-\int_{\mathbf{R}}\eta_{2,n}(\lambda)d\nu_{(t,x)}\int_{\mathbf{R}}q_{1,n}(t,x,\lambda)d\nu_{(t,x)}.$$

Letting $n\to\infty$ as in (47), we get:

$$\int_{\mathbf{R}}\left(\eta_1(t,x,\lambda)q_2(t,x,\lambda)-\eta_2(\lambda)q_1(t,x,\lambda)\right)d\nu_{(t,x)} \tag{50}$$
$$=\int_{\mathbf{R}}\eta_1(t,x,\lambda)d\nu_{(t,x)}\int_{\mathbf{R}}q_2(t,x,\lambda)d\nu_{(t,x)}-\int_{\mathbf{R}}\eta_2(\lambda)d\nu_{(t,x)}\int_{\mathbf{R}}q_1(t,x,\lambda)d\nu_{(t,x)}.$$

Then, following [10], we insert in (50):

$$\eta_1(t,x,\lambda)=f(t,x,\lambda)-f(t,x,u(t,x)),\qquad q_1(t,x,\lambda)=\int_{u(t,x)}^\lambda(\partial_v f(t,x,v))^2dv,$$

$$\eta_2(\lambda) = \lambda - u(t,x), \qquad\qquad q_2(t,x,\lambda) = f(t,x,\lambda) - f(t,x,u(t,x)),$$

which yields the following relation:

$$\left(\int_{\mathbf{R}} (f(t,x,\lambda) - f(t,x,u(t,x)))d\nu_{(t,x)}\right)^2 \tag{51}$$
$$+ \int_{\mathbf{R}} \left((\lambda - u(t,x)) \int_{u(t,x)}^{\lambda} (\partial_\varrho f(t,x,\varrho))^2 d\varrho - (f(t,x,\lambda) - f(t,x,u))^2 \right) d\nu_{(t,x)}(\lambda) = 0.$$

By the Cauchy–Schwarz inequality

$$\begin{aligned}(f(t,x,\lambda) - f(t,x,u))^2 &= \left(\int_{u(t,x)}^{\lambda} \partial_\varrho f(t,x,\varrho) d\varrho \right)^2 \\ &\le (\lambda - u(t,x)) \int_{u(t,x)}^{\lambda} [\partial_\varrho f(t,x,\varrho)]^2 d\varrho d\nu_{(t,x)}(\lambda),\end{aligned}$$

with equality only if $f_\varrho(t,x,\varrho)$ is constant for all ϱ between $u(t,x)$ and λ. Still, this is not possible according to the genuine nonlinearity condition (44). Thus, from this and (51) we conclude that

$$(\lambda - u(t,x)) \int_{u(t,x)}^{\lambda} [\partial_\varrho f(t,x,\varrho)]^2 d\varrho d\nu_{(t,x)}(\lambda) = 0,$$

i.e., that $\nu_{(t,x)} = \delta_{u(t,x)}$ a.e. on $\mathbf{R}^+ \times \mathbf{R}$ implying strong L^1_{loc} convergence of $(u_\varepsilon)_{\varepsilon>0}$ along a subsequence (see Theorem 7). ❑

Now we are ready to prove the main theorem of the section:

Theorem 10. *Assume that*

$$\delta = \delta(\varepsilon) = \mathcal{O}(\varepsilon^2), \quad \varrho = \mathcal{O}(\varepsilon), \quad \varepsilon \to 0, \tag{52}$$

and $u_0 \in H^1(\mathbf{R})$.

Assume that the flux function f from equation (10) with $d = 1$ satisfies (H4'). Assume also that the function b from (14) satisfies (H1) and (H2). Then a subsequence of solutions $(u_{\varepsilon_k}) \subset (u_\varepsilon)$ to problem (14)–(15) converges in the sense of distributions to a weak solution of problem (10)–(11).

If the flux function $f \in C^2(\mathbf{R}; BV(\mathbf{R}^+ \times \mathbf{R}_x)) \cap L^\infty(\mathbf{R} \times \mathbf{R}^+ \times \mathbf{R}_x)$, and if it is genuinely nonlinear in the sense of (44) then a subsequence of solutions $(u_{\varepsilon_k}) \subset (u_\varepsilon)$ to problem (14)–(15) converges strongly in $L^1(\mathbf{R}^+ \times \mathbf{R})$ to a weak solution of (10)–(11).

Proof: Assume that $\eta(t,x,\lambda)$, $(t,x,\lambda) \in \mathbf{R}^+ \times \mathbf{R}^2$ is a function such that $\eta \in C^2(\mathbf{R}; L^\infty \cap BV(\mathbf{R}^+_t \times \mathbf{R}_x))$. As usual, denote by η_n the truncation given by (42), and let the entropy flux corresponding to η_n and f be:

$$q_n(t,x,u) = \int^u \partial_v \eta_n(t,x,v) \partial_v f(t,x,v) dv. \tag{53}$$

According to Lemma 9, it is enough to prove that for every fixed $n \in \mathbf{N}$ the expression $\mathrm{div}(\eta_n(t,x,u_\varepsilon(t,x)), q_n(t,x,u_\varepsilon(t,x)))$ is precompact in $H^{-1}_{\mathrm{loc}}(\mathbf{R}^+ \times \mathbf{R})$.

In order to prove the latter, take the following mollifier $\eta_{n,\varepsilon}(t,x,u) = \eta_n(\cdot,\cdot,u) \star \frac{1}{\varepsilon^{1/2}} \omega(\frac{t}{\varepsilon^{1/4}}) \omega(\frac{x}{\varepsilon^{1/4}})$, where ω is a nonnegative real function with unit mass. Denote the entropy flux corresponding to η_n and f by:

$$q_{n,\varepsilon}(t,x,u)=\int^{u}\partial_v\eta_{n,\varepsilon}(t,x,v)\partial_v f_\varrho(t,x,v)dv. \tag{54}$$

Recall that here (and in the sequel) we assume that $\varrho=\mathcal{O}(\varepsilon)$. Actually, we can take $\varrho=\varepsilon$ without loss of generality.

Notice that according to the assumptions on η and the choice of the mollifier $\eta_{n,\varepsilon}$ we have:

$$\begin{aligned}|\partial_t\eta_{n,\varepsilon}(t,x,u)|,\ |\partial_x\eta_{n,\varepsilon}(t,x,u)|,\ |\partial_{xv}\eta_{n,\varepsilon}(t,x,u)|\leq\mu(t,x),\\ |\partial_x\eta_{n,\varepsilon}(t,x,u)|^2,\ |\partial^2_{xv}\eta_{n,\varepsilon}(t,x,u)|^2\leq\frac{\mu(t,x)}{\varepsilon},\end{aligned} \tag{55}$$

for a locally bounded Radon measure $\mu\in\mathcal{M}(\mathbf{R}^+\times\mathbf{R})$.

Then, apply equation (24) with S replaced by $\eta_{n,\varepsilon}$. We find

$$\begin{aligned}&D_t\eta_n(t,x,u_\varepsilon)+D_xq_n(t,x,u_\varepsilon)\\
&\quad=\int^{u_\varepsilon}\left(\partial^2_{xv}f_\varrho(t,x,v)\partial_v\eta_{n,\varepsilon}(t,x,v)+\partial_vf_\varrho(t,x,v)\partial^2_{xv}\eta_{n,\varepsilon}(t,x,v)\right)dv\\
&\quad-\partial_v\eta_{n,\varepsilon}(t,x,u_\varepsilon)\partial_xf_\varrho(t,x,u_\varepsilon)-\partial_t\eta_{n,\varepsilon}(t,x,u_\varepsilon)\\
&\quad+\varepsilon D_x(\partial_v\eta_{n,\varepsilon}(t,x,u_\varepsilon)b(\partial_xu_\varepsilon))-\varepsilon\partial^2_{vv}\eta_{n,\varepsilon}(t,x,u_\varepsilon)b(\partial_xu_\varepsilon)\partial_xu_\varepsilon\\
&\quad-\varepsilon\partial_xb(u_\varepsilon)\partial^2_{xv}\eta_{n,\varepsilon}(t,x,u_\varepsilon)-\delta\partial^2_{xx}u_\varepsilon\partial^2_{xv}\eta_{n,\varepsilon}(t,x,u_\varepsilon)\\
&\quad+\delta D_x(\partial_v\eta_{n,\varepsilon}(t,x,u_\varepsilon)\partial^2_{xx}u_\varepsilon)-\frac{\delta}{2}\partial^2_{vv}\eta_{n,\varepsilon}(t,x,u_\varepsilon)D_x(\partial_xu_\varepsilon)^2\\
&\quad+D_x(-q_{n,\varepsilon}(t,x,u_\varepsilon)+q_n(t,x,u_\varepsilon))\\
&\quad+D_t(-\eta_{n,\varepsilon}(t,x,u_\varepsilon)+\eta_n(t,x,u_\varepsilon)).\end{aligned} \tag{56}$$

Now, we apply a similar procedure as in the multidimensional case.

Combining (H4b') and (55) we get for a constant C_1 depending only on η_n

$$\begin{aligned}|\int^{u_\varepsilon}\Big(\partial^2_{xv}f_\varrho(t,x,v)\partial_v\eta_n(t,x,v)+\partial_vf_\varrho(t,x,v)\partial_v\eta_n(t,x,v)\Big)dv|\\ \leq C_1(\mu_3(t,x)+\mu(t,x)),\end{aligned} \tag{57}$$

implying boundedness in the sense of measures.

Similarly, for a constant C_2:

$$\begin{aligned}|-\partial_v\eta_{n,\varepsilon}(t,x,u_\varepsilon)\partial_xf_\varrho(t,x,u_\varepsilon)-\partial_t\eta_{n,\varepsilon}(t,x,u_\varepsilon)|\\ \leq C_2(\mu_1(t,x)+\mu(t,x)),\end{aligned} \tag{58}$$

implying boundedness in the sense of measures.

Then, combining (55) with (17) and (18) we infer (see estimation of $\Gamma_{6\varepsilon}$):

$$-\varepsilon\partial_xb(u_\varepsilon)\partial^2_{xv}\eta_{n,\varepsilon}(t,x,u_\varepsilon)-\delta\partial^2_{xx}u_\varepsilon\partial^2_{xv}\eta_{n,\varepsilon}(t,x,u_\varepsilon) \tag{59}$$

is bounded in $\mathcal{M}(\mathbf{R}^+\times\mathbf{R})$.

Next,

$$D_x\left(\varepsilon\partial_v\eta_n(t,x,u_\varepsilon)b(\partial_xu_\varepsilon)+\delta\partial_v\eta_n(t,x,u_{\varepsilon_k})\partial^2_{xx}u_{\varepsilon_k}\right) \tag{60}$$

is precompact in $H^{-1}(\mathbf{R}^+\times\mathbf{R})$ since $|\eta_n'|<C$, $\delta=\mathcal{O}(\varepsilon^2)$, $\varrho=\mathcal{O}(\varepsilon)$, and from (17) and (18) (see also Remark 6) we have

$$\varepsilon b(\partial_xu_\varepsilon)+\delta\partial^2_{xx}u_\varepsilon\to0 \text{ as } \varepsilon\to0$$

in $L^2(\mathbf{R}^+\times\mathbf{R})$.

Similarly, by (17) and (18) (see estimation of $\Gamma_{6\varepsilon}$ again):

$$\varepsilon\partial_{vv}\eta_{n,\varepsilon}(t,x,u_\varepsilon)b(\partial_x u_\varepsilon)\partial_x u_\varepsilon + \frac{\delta}{2}\partial_{vv}\eta_{n,\varepsilon}(t,x,u_\varepsilon)D_x(\partial_x u_\varepsilon)^2 \tag{61}$$

is bounded in $\mathcal{M}(\mathbf{R}^+\times\mathbf{R})$.

Next, due to (H4b') and the definitions of $q_{n,\varepsilon}$ and q_n:

$$\begin{aligned}&|q_{n,\varepsilon}(t,x,u_\varepsilon) - q_n(t,x,u_\varepsilon)|\\ &\quad\le 4nC \max_{-2n<v<2n}|f_\varrho(t,x,v) - f(t,x,v)| \to 0 \text{ in } L^2_{\rm loc}(\mathbf{R}^+\times\mathbf{R}) \text{ as } \varepsilon\to 0\end{aligned}$$

for arbitrary $p>0$ and a constant $C>0$, implying precompactness in $H^{-1}_{\rm loc}$ of the sequence

$$D_x(q_{n,\varrho}(t,x,u_\varepsilon) - q_n(t,x,u_\varepsilon)). \tag{62}$$

Similarly, it is easy to see that

$$\max_{-2n<v<2n}(-\eta_{n,\varepsilon}(t,x,u_\varepsilon)+\eta_n(t,x,u_\varepsilon)) \to 0 \ \text{ in } \ L^2(\mathbf{R}^+\times\mathbf{R}),$$

and thus

$$D_t(-\eta_{n,\varepsilon}(t,x,u_\varepsilon)+\eta_n(t,x,u_\varepsilon)) \in H^{-1}_c(\mathbf{R}^+\times\mathbf{R}). \tag{63}$$

From (57)–(63) and the fact that $(\eta_n(t,x,u_\varepsilon), q_n(t,x,u_\varepsilon)) \in L^\infty(\mathbf{R}^+\times\mathbf{R})$, we conclude using Murat's lemma that

$$\operatorname{div}(\eta_n(t,x,u_\varepsilon), q_n(t,x,u_\varepsilon)) \in H^{-1}_{loc,c}(\mathbf{R}^+\times\mathbf{R}). \tag{64}$$

Finally, relying on Lemma 9 we conclude the theorem.

❏

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