ON A STOCHASTIC FIRST ORDER HYPERBOLIC EQUATION IN A BOUNDED DOMAIN

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Abstract

In this paper, we are interested in the stochastic perturbation of a first order hyperbolic equation of nonlinear type. In order to illustrate our purposes, we have chosen a scalar conservation law in a bounded domain with homogeneous Dirichlet condition on the boundary. Using the concept of measure-valued solutions and Kruzhkov's entropy formulation, a result of existence and uniqueness of the entropy solution is given.
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1 Introduction

In this paper, we are interested in the formal stochastic partial differential equation of first order nonlinear hyperbolic type:

$$du - \operatorname{div}(\mathbf{f}(u))dt = hdw \quad \text{ in } \Omega \times D \times]0, T[, \tag{1}$$

with an initial condition u_0 and homogeneous "Dirichlet" boundary condition.

In the sequel, one assumes that D is a bounded Lipschitz domain of $\mathbb{R}^d$, that T is a positive number, $Q =]0, T[\times D$ and that $W = \{w_t, \mathcal{F}_t; 0 \leq t \leq T\}$ denotes a standard adapted one-dimensional continuous Brownian motion, defined on some probability space $(\Omega, \mathcal{F}, P)$, with the property that $w_0 = 0$ (cf. I. Karatzas*et al.* [20] for example). This assumption on W is made for convenience. Our aim is to adapt known methods of first-order nonlinear PDE to noise perturbed ones. For more general noise, one can consider cylindrical Wiener processes on separable Hilbert spaces (cf. G. Da Prato and J. Zabczyk[12]) or space-time noise.

On the one hand, remind that, even in the deterministic case, the weak solution to such a problem is not unique in general. One needs to introduce

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the notion of entropy solution in order to discriminate the physical solution. Moreover, weak and entropy solutions are not smooth enough allowing for trace properties. Trace has to be understood in a weak way. Moreover, it is not possible to impose the Dirichlet condition on the whole boundary of D, but only on a free set: the one corresponding to entering characteristics (without exhaustiveness, see for example C. Bardos *et al.*[5], J. Carrillo *et al.*[9], F. Otto[22], E.Y. Panov[24], G. Vallet[27] and A. Vasseur[29]).
On the other hand, the stochastic perturbation will not simplify the situation.

Many papers on the viscous Burgers type stochastic problem (*i.e.* usually in 1-D with $f(x) = x^2$ and a Laplacian) can be found in the literature, where, usually, the stochastic convolution is used. Let us mention, without exhaustiveness, G. Da Prato, A. Debussche and R. Temam[11] , G. Da Prato *et al.*[12], W. Grecksch and C. Tudor[17] or I. Gyöngy and D. Nualart[18].

Few papers exist concerning the stochastic perturbation of nonlinear first order hyperbolic problems. Most of them are interested in the Cauchy problem in the 1-D case. Let us cite the paper of H. Holden *et al.*[19] where an operator splitting method is proposed to prove the existence of a weak solution to the Cauchy problem

$$du + f(u)_x dt = g(u)dw \qquad \text{in } \mathbb{R}.$$

The convergence is obtained by using path-wise arguments.
In the paper of E. Weinan, K. Khanin, A. Mazel and Y. Sinai[30], the authors are interested in the invariant measures for the Burgers equation

$$du + \frac{1}{2}(u^2)_x = (\sum_{k\geq 0} F_k(x)dw_k)_x$$

with a periodic assumption in space. The existence and uniqueness of a stochastic entropy[1] solution is proved thanks to a Hopf-Lax type formula for the corresponding Hamilton-Jacobi equation. A parabolic perturbation problem approach is considered, too, based on the Hopf-Cole transformation.
In the paper of J. H. Kim[21], a method of compensated-compactness is presented to prove the existence of a stochastic weak entropy solution to the Cauchy problem

$$du + \varphi(u)_x dt = g(t,x)dw \qquad \text{in } \mathbb{R}.$$

Then, a Kruzhkov-type method is used to prove the uniqueness.
J. Feng and D. Nualart[15] propose to extend the above-mentioned result to the Cauchy problem in $\mathbb{R}^d$:

$$du + divF(u) = \int_{z\in Z} \sigma(.,u,z)dw(t,z),$$

where the right-hand side depends on u. For this reason, a notion of strong entropy solution has to be introduced in order to prove the uniqueness of the

[1] in the sense of P. D. Lax and O. A. Oleinik.

solution. The existence result is achieved in the $1-D$ case.
Some other papers can be found in the references cited therein.

In our main result, we propose a result of existence and uniqueness of the stochastic entropy solution to Problem (1). A multi-dimensional bounded domain with homogeneous Dirichlet conditions is considered. A method of artificial viscosity is proposed to prove the existence of a solution. The compactness property is based on the theory of Young measure solutions and the trace formulation is based on the one proposed by J. Carrillo[7]. An adaptation of the classical method of Kruzhkov is proposed to prove the uniqueness of the entropy measure-valued solution. The existence of such a solution follows as usual from the theorem of Prohorov for Young measures.

After giving the assumptions on the data and the definition of an entropy solution, we devote a section to the existence of an entropy measure-valued solution in the sense of Young measures. The uniqueness of the entropy measure-valued solution is proved by using the doubling-variable method of Kruzhkov in a following section. Then, the result of existence of the entropy solution comes from the properties of Young measures connected to weak convergence. The last section constitutes a basic reminder on Young measures.

As mentioned by J. U. Kim[21] for example, the equation has to be understood in the following way:

$$\partial_t[u - \int_0^t h dw(s)] - \operatorname{div}(\mathbf{f}(u)) = 0,$$

where $\int_0^t h dw(s)$ denotes the Itô integration of h.

Let us assume that

. $\mathbf{f} = (f_1, .., f_d) : \mathbb{R} \to \mathbb{R}^d$ is a Lipschitz-continuous function[2], as well as $\mathbf{f'}$, and $f_i(0) = 0,\ \forall i = 1, ..., d$.

. $h \in L^2[0, T, H_0^1(D)]$. Note that h is the restriction to Q of the function $\bar{h}$ of $L^2[\mathbb{R}, H^1(\mathbb{R}^d)]$ by considering that $\bar{h}(t, x) = 0$ if $(t, x) \notin Q$.

Our aim is to prove a result of existence and uniqueness of the stochastic entropy solution to the above-mentioned problem. Let us fix in what sense such a solution is understood.

Notations. In the sequel, for any bounded Lipschitz $G \subset \mathbb{R}^k$, one denotes by $H^1(G)$ the usual Sobolev space and by $H_0^1(G)$ the space of Sobolev functions with null trace on the boundary of G. Remind that $H_0^1(G)$ is also the closure in $H^1(G)$ of the distribution space $\mathcal{D}(G)$: the space of $C^\infty(\mathbb{R}^k)$ with compact support in G. Then, one denotes by $H^{-1}(G)$ the dual space of $H_0^1(G)$ (see for example R. A. Adams[1] or L. C. Evans and R. Gariepy[13]). In general, if G is not assumed to be an open set ($G = \bar{D}$ or $[0, T[\times D)$, $\mathcal{D}(G)$ denotes

[2]Some information are given in section 3.4 about locally-Lipschitz f.

the restriction to G of $\mathcal{D}(\mathbb{R}^k)$ functions u such that support$(u)\cap G$ is compact. Then, $\mathcal{D}^+(G)$ will denote any non-negative element of $\mathcal{D}(G)$.

For convenience, for any function u of $L^2(\Omega\times Q)$, any real k and any function φ in $H^1(Q)$, denote by:

1. $\mathcal{K}(t,x,.) = \int_0^t h(s,x)dw(s)$ and $\Lambda = u - \mathcal{K}$.

2. $\mathrm{sgn}_0^+(x) = 1$ if $x > 0$, 0 else; $x^+ = x\mathrm{sgn}_0^+(x)$; $F^+(a,b) = [\mathbf{f}(a) - \mathbf{f}(b)]\mathrm{sgn}_0^+(a-b)$. Note, in particular, that F^+ is a Lipschitz-continuous function.

3. dP-a.s. in Ω, denote by μ_k^+ the distribution in $\mathbb{R}^{d+1}$,

$$\begin{aligned}\varphi \mapsto \mu_k^+(\varphi) &= \int_{\{u>\mathcal{K}+k\}} \Big\{(u-\mathcal{K}-k)\partial_t\varphi - [\mathbf{f}(u)-\mathbf{f}(\mathcal{K}+k)].\nabla\varphi\Big\}\,dxdt \\ &\quad + \int_{\{u>\mathcal{K}+k\}} \varphi\,\mathrm{div}\,\mathbf{f}[\mathcal{K}+k]\,dxdt + \int_D (u_0-k)^+\varphi(0,.)\,dx. \\ &= \int_Q \Big\{(\Lambda-k)^+\partial_t\varphi - F^+(\mathcal{K}+\Lambda,\mathcal{K}+k).\nabla\varphi\Big\}\,dxdt \\ &\quad + \int_Q \varphi\mathrm{sgn}_0^+(\Lambda-k)\,\mathrm{div}\,\mathbf{f}[\mathcal{K}+k]\,dxdt + \int_D (u_0-k)^+\varphi(0,.)\,dx.\end{aligned}$$

4. dP-a.s. in Ω, denote by μ_k^- the distribution in $\mathbb{R}^{d+1}$,

$$\begin{aligned}\varphi \mapsto \mu_k^-(\varphi) &= \int_{\{u<\mathcal{K}+k\}} \Big\{(\mathcal{K}+k-u)\partial_t\varphi - [\mathbf{f}(\mathcal{K}+k)-\mathbf{f}(u)].\nabla\varphi\Big\}\,dxdt \\ &\quad - \int_{\{u<\mathcal{K}+k\}} \varphi\,\mathrm{div}\,\mathbf{f}[\mathcal{K}+k]\,dxdt + \int_D (k-u_0)^+\varphi(0,.)\,dx \\ &= \int_Q \Big\{(k-\Lambda)^+\partial_t\varphi - F^+(\mathcal{K}+k,\mathcal{K}+\Lambda).\nabla\varphi\Big\}\,dxdt \\ &\quad - \int_Q \varphi\mathrm{sgn}_0^+(k-\Lambda)\,\mathrm{div}\,\mathbf{f}[\mathcal{K}+k]\,dxdt + \int_D (k-u_0)^+\varphi(0,.)\,dx.\end{aligned}$$

□

Then, one would say that

Definition 1. *Any function u of $L^2(\Omega\times Q)$, adapted to the filtration $\mathcal{F}_t$ as an $L^2(D)$-valued function, is an entropy solution if*
i) For any $(k,\varphi)\in\mathbb{R}\times H^1(Q)$ such that $k\geq 0$ and $\varphi\geq 0$, and for any $(k,\varphi)\in\mathbb{R}\times[H^1(Q)\cap L^2(0,T;H_0^1(D))]$ such that $\varphi\geq 0$,

$$0 \leq \mu_k^+(\varphi), \quad dP-a.s.$$

ii) For any $(k,\varphi)\in\mathbb{R}\times H^1(Q)$ such that $k\leq 0$ and $\varphi\geq 0$, and for any $(k,\varphi)\in\mathbb{R}\times[H^1(Q)\cap L^2(0,T;H_0^1(D))]$ such that $\varphi\geq 0$,

$$0 \leq \mu_k^-(\varphi), \quad dP-a.s.$$

For technical reasons, one also need to consider a generalised notion of entropy solution. In fact, in a first step, we will only prove the existence of a Young measure-valued solution. Then, thanks to a result of uniqueness, we are able to deduce the existence of an entropy solution in the sense of Definition 1.

Definition 2. *Any function u of $L^2(\Omega\times Q\times]0,1[)$, adapted to the filtration $\mathcal{F}_t$ as an $L^2(D)$-valued function, is a Young measured-valued entropy solution if*
i) For any $(k,\varphi)\in\mathbb{R}\times H^1(Q)$ such that $k\geq 0$ and $\varphi\geq 0$, and
for any $(k,\varphi)\in\mathbb{R}\times[H^1(Q)\cap L^2(0,T;H^1_0(D))]$ such that $\varphi\geq 0$,

$$0 \quad\leq\quad \int_0^1 \mu_k^+(\varphi)d\alpha, \quad dP-a.s.$$

ii) For any $(k,\varphi)\in\mathbb{R}\times H^1(Q)$ such that $k\leq 0$ and $\varphi\geq 0$, and
for any $(k,\varphi)\in\mathbb{R}\times[H^1(Q)\cap L^2(0,T;H^1_0(D))]$ such that $\varphi\geq 0$,

$$0 \quad\leq\quad \int_0^1 \mu_k^-(\varphi)d\alpha, \quad dP-a.s.$$

Note that in this definition the measures μ_k^+, μ_k^- also depend on α because u does.

Therefore, immediate consequences are:

Remark 1. *Consider u an entropy solution and $\mathcal{A}$ a countable dense sub-family of $H^1_+(Q)$, the set of all the non-negative elements of $H^1(Q)$. Then, $\tilde\Omega\subset\Omega$ exists such that $P(\Omega\backslash\tilde\Omega)=0$ and, for any $\omega\in\tilde\Omega$: $\forall k\in\mathbb{Q}^+$, $\forall\varphi\in\mathcal{A}$, $0\leq\mu_k^+(\varphi)$.*
Since μ_k^+ is a $H^1(Q)$-continuous function, for any $\omega\in\tilde\Omega$:
$\forall k\in\mathbb{Q}^+$, $\forall\varphi\in H^1(Q)$, $0\leq\mu_k^+(\varphi)$.
Since $k\mapsto sgn_0^+(\Lambda(\omega)-k)$ is a right-continuous function, by approximating any positive number by an upper-sequence of rational numbers, for any $\omega\in\tilde\Omega$:
$\forall k\geq 0$, $\forall\varphi\in H^1(Q)$, $0\leq\mu_k^+(\varphi)$.

Since this remark holds similarly for any of the assertions of the above definition, it can be said that any function u of $L^2(\Omega\times Q)$, adapted to the filtration $\mathcal{F}_t$ as an $L^2(D)$-valued function, is an entropy solution if, $dP-a.s.$,

$$\forall(k,\varphi)\in\Big[\mathbb{R}^+\times H^1_+(Q)\Big]\cup\Big[\mathbb{R}\times[H^1_+(Q)\cap L^2(0,T;H^1_0(D))]\Big]: \qquad 0\leq\mu_k^+(\varphi),$$
$$\forall(k,\varphi)\in\Big[\mathbb{R}^-\times H^1_+(Q)\Big]\cup\Big[\mathbb{R}\times[H^1_+(Q)\cap L^2(0,T;H^1_0(D))]\Big]: \qquad 0\leq\mu_k^-(\varphi).$$

Remark 2. *dP-a.s., for any real k, $\mu_k^\pm$ are non-negative Radon measures on Q.*
Moreover, dP-a.s., for any non-negative k, $|\mu_{\pm k}^\pm|=\mu_{\pm k}^\pm(1)<+\infty$ and $\mu_{\pm k}^\pm$ are bounded non-negative Radon measures on Q.

Let us also mention

Remark 3. *Any entropy solution is a.s. a weak solution, too.*

Following J. Carrillo *et al.*[9], dP-a.s, for any positive $\varphi \in \mathcal{D}^+([0,T[\times D)$, note that

$$\begin{aligned}
\mu_k^+(\varphi) = \quad & \int_Q \Big\{(u-\mathcal{K})\partial_t\varphi - \mathbf{f}(u).\nabla\varphi\Big\}\,dxdt + \int_D u_0\varphi(0,.)\,dx & (:= I_1)\\
- & \int_Q \Big\{k\partial_t\varphi - \mathbf{f}(\mathcal{K}+k).\nabla\varphi - \varphi\operatorname{div}\mathbf{f}[\mathcal{K}+k]\Big\}\,dxdt - \int_D k\varphi(0,.)\,dx & (:= I_2)\\
- & \int_{\{u\le\mathcal{K}+k\}} \Big\{(u-\mathcal{K}-k)\partial_t\varphi - [\mathbf{f}(u)-\mathbf{f}(\mathcal{K}+k)].\nabla\varphi\Big\}\,dxdt & (:= I_3)\\
- & \int_{\{u\le\mathcal{K}+k\}} \varphi\operatorname{div}\mathbf{f}[\mathcal{K}+k]\,dxdt + \int_D (u_0-k)^-\varphi(0,.)\,dx & (:= I_4)
\end{aligned}$$

If $k<0$, then

$$\begin{aligned}
|I_3| &\le \int_{\{u\le\mathcal{K}+k\}} \Big\{(|u|+|\mathcal{K}|)|\partial_t\varphi| + \sum_{i=1}^d \int_u^{\mathcal{K}} |f_i'(s)|ds|\partial_{x_i}\varphi|\Big\}\,dxdt \underset{k\to-\infty}{\to} 0,\\
|I_4| &\le \int_{\{u\le\mathcal{K}+k\}} [||\mathbf{f}''||_\infty(|u|+|\mathcal{K}|) + |\mathbf{f}'(u)|]|\nabla\mathcal{K}||\varphi|\,dxdt + \int_{\{u_0\le k\}} |u_0||\varphi(0,.)|\,dx.
\end{aligned}$$

Then, I_4 tends to 0 with k to $-\infty$ and, since $I_2=0$, one concludes that for any positive $\varphi \in \mathcal{D}([0,T[\times D)$,

$$0 \le \int_Q \Big\{(u-\mathcal{K})\partial_t\varphi - \mathbf{f}(u).\nabla\varphi\Big\}\,dxdt + \int_D u_0\varphi(0,.)\,dx.$$

Since the opposite inequality can be proved by using μ_k^- for large values of k, u is a solution in the sense of distributions.

Remark 4. *The unique solution obtained in this paper satisfies the initial condition in the following sense:*

$$\underset{t\to0^+}{ess\lim}\, E\int_D |\Lambda - u_0|dx = 0.$$

Indeed, by the existence proof, the solution u will be in $L^\infty(0,T,L^2(\Omega\times D))$. Therefore, following F. Otto[22] (see also G. Vallet[27]), if one considers any real k and any non-negative β in $H_0^1(D)$, then, for any non-negative α in $H^1(0,T)$, one has that

$$\begin{aligned}
0 \le E\mu_k^+(\alpha\otimes\beta) \;&=\; \int_0^T \Big\{\alpha'\int_D E\Big[(\Lambda-k)^+\Big]\beta - \alpha\int_D E\Big[F^+(\mathcal{K}+\Lambda,\mathcal{K}+k)\Big].\nabla\beta dx\Big\}dt\\
&\quad + \int_0^T \alpha\int_D \beta E\Big[\operatorname{sgn}_0^+(\Lambda-k)\operatorname{div}\mathbf{f}[\mathcal{K}+k]\Big]dxdt + \alpha(0)\int_D (u_0-k)^+\beta dx\\
&=\; \int_0^T [\alpha'(t)A_{k,\beta}(t) + \alpha(t)B_{k,\beta}(t)]dt + \alpha(0)C_{k,\beta}.
\end{aligned}$$

Therefore, $E\mu^+_{k,\beta} : \alpha \in \mathcal{D}^+(\mathbb{R}) \mapsto \int_0^T [\alpha'(t)A_{k,\beta}(t) + \alpha(t)B_{k,\beta}(t)]dt + \alpha(0)C_{k,\beta}$ is a positive Radon measure on $\mathbb{R}$. Its restriction to $]0,T[$, denoted by $E\mu^+_{k,\beta,\lfloor]0,T[}$, is a positive bounded Radon measure on $]0,T[$ and

$$\begin{aligned}|E\mu^+_{k,\beta,\lfloor]0,T[}| &\leq E\mu^+_{k,\beta}(1) = \int_0^T B_{k,\beta}(t)dt + C_{k,\beta}\\ &\leq C(T,\mathbf{f})||\Lambda - k||_{L^2(\Omega\times Q)}||\nabla\beta||_{(L^2(D))^d} + C(\mathbf{f})||\beta||_{L^2(D)}\\ &\quad +||u_0 - k||_{L^2(D)}||\beta||_{L^2(D)}.\end{aligned}$$

In particular, $\psi : t \mapsto A_{k,\beta}(t) - \int_0^t B_{k,\beta}(s)ds$ is a non-increasing function of bounded variation on $[0,T]$. Thus, $\psi(0^+) = \operatorname*{ess\,lim}_{t\to 0^+} \psi(t)$ exists and

$$\psi(0^+) = \lim_{n\to\infty} n\int_0^{1/n} \psi(t)dt = \lim_{n\to\infty}\int_0^T \alpha_n'\psi(t)dt,$$

where $\alpha_n(t) = \min(nt,1)^+$.
Since $\lim_{t\to 0^+} \int_0^t B_{k,\beta}(s)ds = 0$, $A_{k,\beta}(0^+) = \operatorname*{ess\,lim}_{t\to 0^+} A_{k,\beta}(t) = \psi(0^+)$ and

$$\begin{aligned}0 \leq A_{k,\beta}(0^+) &= \lim_{n\to\infty}\int_0^T (\alpha_n - 1)'[A_{k,\beta}(t) - \int_0^t B_{k,\beta}(s)ds]dt\\ &= \lim_{n\to\infty} -\int_0^T [(1-\alpha_n)'A_{k,\beta}(t) + B_{k,\beta}(t)(1-\alpha_n)]dt,\\ &= \lim_{n\to\infty} \big[-\mu^+_{k,\beta}(1-\alpha_n)\big] + C_{k,\beta} \leq \int_D (u_0-k)^+\beta dx.\end{aligned}$$

Thanks to the hypothesis on u, uniformly with respect to t, $\beta \mapsto A_{k,\beta}(t)$ is a continuous linear function on $L^2(D)$, a density argument leads to the existence, for any real k and any non-negative β in $L^2(D)$, of $\operatorname*{ess\,lim}_{t\to 0^+} A_{k,\beta}(t) = A_{k,\beta}(0^+)$ with, moreover, $A_{k,\beta}(0^+) \leq \int_D (u_0-k)^+\beta dx$.
In order to keep essential limits, consider k in $\mathbb{Q}$. Then, if $w_n = \sum_{i=0}^n k_i 1_{B_i}$ is a simple function with k_i in $\mathbb{Q}$, one gets that

$$A_{w_n,\beta}(t) = E\int_D (\Lambda(t)-w_n)^+\beta dx = \sum_{i=0}^n E\int_D (\Lambda(t)-k_i)^+\beta 1_{B_i}dx = \sum_{i=0}^n A_{k_i,\beta}1_{B_i}(t),$$

and $\operatorname*{ess\,lim}_{t\to 0^+} A_{w,\beta}(t)$ exists with moreover $A_{w,\beta}(0^+) \leq \int_D (u_0-w)^+\beta dx$, for any non-negative β in $L^2(D)$ and any $\mathbb{Q}$-valued simple function w.
As any w of $L^2(D)$ is a limit in $L^2(D)$ of a sequence of such simple functions and since for w and $\hat{w}$ in $L^2(D)$, $|A_{w,\beta}(t) - A_{\hat{w},\beta}(t)| \leq ||w-\hat{w}||_{L^2(D)}||\beta||_{L^2}$, independently of t, the same argument of density leads to:
$\operatorname*{ess\,lim}_{t\to 0^+} A_{w,\beta}(t)$ exists with moreover $A_{w,\beta}(0^+) \leq \int_D (u_0-w)^+\beta dx$, for any non-negative β in $L^2(D)$ and any w in $L^2(D)$.

Now, for $w = u_0$ and $\beta = 1$, this leads to: $\operatorname*{ess\,lim}_{t\to 0^+} E\int_D (\Lambda - u_0)^+ dx = 0$.

A similar reasoning with the second inequality of the definition of a solution would yields: $\operatorname*{ess\,lim}_{t\to 0^+} E\int_D (u_0-\Lambda)^+dx = 0$ and thus $\operatorname*{ess\,lim}_{t\to 0^+} E\int_D |\Lambda - u_0|dx = 0$.

2 Existence of a solution

The aim of this section is to give a result on existence of a measure-valued entropy solution to the problem. The technique is based on the notion of narrow convergence of Young measures (or entropy processes) (cf. Appendix). Then, thanks to the uniqueness result of the next section, one is able to prove that the measure-valued solution is an entropy weak solution and that the sequence of approximation proposed to prove the existence of the solution converges in L^p for any $p < 2$.

Let us set, in the sequel of this section, for any positive integer n, u_n the unique weak solution to the stochastic viscous parabolic equation:

$$\partial_t[u_n - \mathcal{K}] - \frac{1}{n}\Delta u_n - \operatorname{div}(\mathbf{f}(u_n)) = 0;$$

i.e., u_n exists in $L^2(\Omega\times]0,T[; H_0^1(D))$, adapted to the filtration $\mathcal{F}_t$ as an $L^2(D)$-valued function, with moreover $\partial_t[u - \mathcal{K}] \in L^2[]0,T[\times\Omega; H^{-1}(D)]$ and, a.s. in Ω, a.e. in $]0,T[$, for any v in $H_0^1(D)$,

$$< \partial_t[u_n - \mathcal{K}], v >_{H^{-1}(D),H_0^1(D)} + \int_D \frac{1}{n}\nabla u_n.\nabla v + \mathbf{f}(u_n).\nabla v \, dx = 0. \qquad (2)$$

We admit such a result and refer *e.g.* to G. Da Prato *et al.*[12], W. Grecksch*et al.*[17] or G. Vallet[28] for further information on the viscous stochastic parabolic equation.

Then, thanks to the stochastic energy equality (see for example W. Grecksch[17] Th. 3.4 p.42), the following estimate holds:

$$\begin{aligned} & ||u_n(t)||^2_{L^2(D)} + 2\int_0^t \int_D [\frac{1}{n}|\nabla u_n|^2 + \mathbf{f}(u_n).\nabla u_n] \, dxds \\ = \; & ||u(0)||^2_{L^2(D)} + 2\int_0^t \int_D u_n h \, dxdw_s + \int_0^t \int_D h^2 \, dxds. \end{aligned}$$

Since $\int_0^t \int_D \mathbf{f}(u_n).\nabla u_n \, dxds = 0$, one gets that

Proposition 1. *There exists a positive constant C such that,*

$$\forall n \in \mathbb{N}^*, \quad ||u_n||^2_{L^\infty[0,T;L^2(\Omega\times D)]} + \frac{1}{n}||u_n||^2_{L^2[],0,T[\times\Omega;H_0^1(D)]} \leq C.$$

In particular, u_n is a bounded sequence in $L^2(]0,T[\times\Omega\times D)$ and the associated Young measure sequence τ_n converges (up to a sub-sequence still indexed in the same way) narrowly to an entropy process denoted by u (see the Appendix).

Consider η, a non-decreasing Lipschitz-continuous function satisfying the assumptions that $\operatorname{supp}\eta'$ is compact and $\eta(0) = 0$, k an integer and φ a positive element of $\mathcal{D}(\bar{Q})$ such that a.s. in Ω and a.e. in $]0,T[$, $v = \eta(u_n - \mathcal{K} - k)\varphi$ belongs to $H_0^1(D)$. Therefore, v is an admissible test-function in (2).

i) Thanks to the chain rule lemma of Alt -Bamberger - Luckhaus - Mignot (see A. Bamberger[4] and H. W. Alt *et al.*[2]) based on convex inequalities, if Ψ denotes the primitive of η such that $\Psi(0)=0$, one has that

$$\begin{aligned}
I_{1,\eta} &= \int_0^T < \partial_t[u_n-\mathcal{K}], \eta(u_n-\mathcal{K}-k)\varphi >_{H^{-1}(D),H_0^1(D)} dt \\
&= \int_D \Psi[u_n(T)-\mathcal{K}(T)-k]\varphi(T) - \Psi[u(0)-k]\varphi(0)\,dx - \int_Q \Psi[u_n-\mathcal{K}-k]\partial_t\varphi\,dxdt \\
&\geq -\int_D \Psi[u(0)-k]\varphi(0)\,dx - \int_Q \Psi[u_n-\mathcal{K}-k]\partial_t\varphi\,dxdt;
\end{aligned}$$

ii) Concerning the viscous term, one gets that

$$\begin{aligned}
I_{2,\eta} &= \frac{1}{n}\int_Q \nabla u_n.\nabla[\eta(u_n-\mathcal{K}-k)\varphi]\,dxdt \\
&= \frac{1}{n}\int_Q \varphi\eta'(u_n-\mathcal{K}-k)|\nabla[u_n-\mathcal{K}]|^2\,dxdt + \frac{1}{n}\int_Q \eta(u_n-\mathcal{K}-k)\nabla u_n.\nabla\varphi\,dxdt \\
&\quad +\frac{1}{n}\int_Q \eta'(u_n-\mathcal{K}-k)\varphi\nabla\mathcal{K}.\nabla[u_n-\mathcal{K}]\,dxdt \\
&\geq \frac{1}{n}\int_Q \eta'(u_n-\mathcal{K}-k)\varphi\nabla\mathcal{K}.\nabla[u_n-\mathcal{K}]\,dxdt + \frac{1}{n}\int_Q \eta(u_n-\mathcal{K}-k)\nabla u_n.\nabla\varphi\,dxdt;
\end{aligned}$$

iii) Then, for the flux term, the Gauss-Green formulae and the chain rule (since η' has a compact support) lead to

$$\begin{aligned}
&I_{3,\eta} = \int_Q \mathbf{f}(u_n).\nabla[\eta(u_n-\mathcal{K}-k)\varphi]\,dxdt \\
&= \int_Q [\mathbf{f}(u_n)-\mathbf{f}(\mathcal{K}+k)].\nabla[\eta(u_n-\mathcal{K}-k)\varphi]\,dxdt - \int_Q \mathrm{div}\mathbf{f}(\mathcal{K}+k)\eta(u_n-\mathcal{K}-k)\varphi\,dxdt \\
&= \int_Q \eta'(u_n-\mathcal{K}-k)\varphi[\mathbf{f}(u_n)-\mathbf{f}(\mathcal{K}+k)].\nabla[u_n-\mathcal{K}]\,dxdt \\
&+ \int_Q \eta(u_n-\mathcal{K}-k)[\mathbf{f}(u_n)-\mathbf{f}(\mathcal{K}+k)].\nabla\varphi\,dxdt - \int_Q \eta(u_n-\mathcal{K}-k)\varphi\mathbf{f}'(\mathcal{K}+k).\nabla\mathcal{K}\,dxdt.
\end{aligned}$$

Let us note that

$$\begin{aligned}
&\mathrm{div}\{\int_k^{u_n-\mathcal{K}} [\mathbf{f}(r+\mathcal{K})-\mathbf{f}(\mathcal{K}+k)]\eta'(r-k)\,dr\} \\
=\ & \eta'(u_n-\mathcal{K}-k)[\mathbf{f}(u_n)-\mathbf{f}(\mathcal{K}+k)].\nabla(u_n-\mathcal{K}) \\
&\qquad\qquad + \int_k^{u_n-\mathcal{K}} \mathrm{div}[\mathbf{f}(r+\mathcal{K})-\mathbf{f}(\mathcal{K}+k)]\eta'(r-k)\,dr \\
=\ & \eta'(u_n-\mathcal{K}-k)[\mathbf{f}(u_n)-\mathbf{f}(\mathcal{K}+k)].\nabla(u_n-\mathcal{K}) \\
&\qquad\qquad + \int_k^{u_n-\mathcal{K}} [\mathbf{f}'(r+\mathcal{K})-\mathbf{f}'(\mathcal{K}+k)].\nabla\mathcal{K}\eta'(r-k)\,dr.
\end{aligned}$$

Thus, it yields

$$\begin{aligned}
I_{3,\eta} &= \int_Q \eta(u_n-\mathcal{K}-k)[\mathbf{f}(u_n)-\mathbf{f}(\mathcal{K}+k)].\nabla\varphi\,dxdt \\
&\quad -\int_Q \eta(u_n-\mathcal{K}-k)\varphi\mathbf{f'}(\mathcal{K}+k).\nabla\mathcal{K}\,dxdt \\
&\quad -\int_Q \varphi[\int_k^{u_n-\mathcal{K}} \eta'(r-k)[\mathbf{f'}(r+\mathcal{K})-\mathbf{f'}(\mathcal{K}+k)].\nabla\mathcal{K}\,dr]\,dxdt \\
&\quad +\int_{]0,T[\times\partial D}[\int_k^0 [\mathbf{f}(r)-\mathbf{f}(k)]\eta'(r-k)\,dr]\varphi\,d\sigma dt \\
&\quad -\int_Q [\int_k^{u_n-\mathcal{K}} [\mathbf{f}(r+\mathcal{K})-\mathbf{f}(\mathcal{K}+k)]\eta'(r-k)\,dr].\nabla\varphi\,dxdt. \qquad (3)
\end{aligned}$$

Since η is a non-decreasing Lipschitz-continuous function with $\operatorname{supp}\eta'$ compact, Ψ is a Lipschitz-continuous function and, for any $A\in\mathcal{F}$, $\Psi[u_n-\mathcal{K}-k]\partial_t\varphi 1_A$ is uniformly integrable. Then, (see Appendix) one concludes that

$$\begin{aligned}
&\liminf_{n\to\infty} E[I_{1,\eta}1_A] \\
\geq\ & -E[1_A\int_{Q\times]0,1[}\Psi[u(\alpha)-\mathcal{K}-k]\partial_t\varphi\,dxdtd\alpha]-E[1_A\int_D\Psi[u(0)-k]\varphi(0)\,dx].
\end{aligned}$$

As $\frac{1}{\sqrt{n}}||u_n||_{L^2(]0,T[\times\Omega;H_0^1(D))}$ is bounded and since η and η' are bounded functions, the following result holds:

$$\liminf_{n\to\infty} E[1_A I_{2,\eta}]\geq 0.$$

As η is a bounded Lipschitz-continuous function with $\operatorname{supp}\eta'$ compact, $\mathbf{f}$ is a Lipschitz-continuous function (for the first term of $I_{3,\eta}$) and **f'** is a Lipschitz-continuous function, the integrands involved in the first three terms of $EI_{3,\eta}1_A$ are uniformly integrable and the convergence in the sense of Young measures holds. Noting that the fourth term is independent of n, one needs to take care of the last one. As $\mathbf{f}$ is not a bounded function, the uniform integrability of the integrand is ensured by the hypothesis of compact support for η'.

Conclusion: testing (2) with $v=\eta(u_n-\mathcal{K}-k)\varphi$, estimating all terms as above, yields for any $A\in\mathcal{F}$,

$$\begin{aligned}
0 \geq\ & -E[1_A\int_{Q\times]0,1[}\Psi[u(\alpha)-\mathcal{K}-k]\partial_t\varphi\,dxdtd\alpha]-E[1_A\int_D\Psi[u(0)-k]\varphi(0)\,dx] \\
&+E[1_A\int_{Q\times]0,1[}\eta(u(\alpha)-\mathcal{K}-k)[\mathbf{f}(u(\alpha))-\mathbf{f}(\mathcal{K}+k)].\nabla\varphi\,dxdtd\alpha] \\
&-E[1_A\int_{Q\times]0,1[}\eta(u(\alpha)-\mathcal{K}-k)\varphi\mathbf{f'}(\mathcal{K}+k).\nabla\mathcal{K}\,dxdtd\alpha] \\
&-E[1_A\int_{Q\times]0,1[}\varphi[\int_k^{u(\alpha)-\mathcal{K}}\eta'(r-k)[\mathbf{f'}(r+\mathcal{K})-\mathbf{f'}(\mathcal{K}+k)].\nabla\mathcal{K}\,dr]\,dxdtd\alpha]
\end{aligned}$$

$$
\begin{aligned}
& +E[1_A \int_{]0,T[\times\partial D}[\int_k^0 [\mathbf{f}(r)-\mathbf{f}(k)]\eta'(r-k)\,dr]\varphi\,d\sigma dt] \\
& -E[1_A \int_{Q\times]0,1[}[\int_k^{u(\alpha)-\mathcal{K}} [\mathbf{f}(r+\mathcal{K})-\mathbf{f}(\mathcal{K}+k)]\eta'(r-k)\,dr].\nabla\varphi\,dxdtd\alpha] \\
= \quad & J_1+J_2+J_3+J_4+J_5+J_6.
\end{aligned}
$$

Assume now that $\eta(x)=\eta_\epsilon(x)=\min(1,\frac{x^+}{\epsilon})$. Then, in order to be compatible with the trace assumption for $\eta_\epsilon(u_n-\mathcal{K}-k)\varphi$, $\varphi\in\mathcal{D}(\bar{Q})$ if $k\geq 0$, $\varphi\in\mathcal{D}([0,T]\times D)$ otherwise.

Obviously,

$$
\begin{aligned}
\lim_{\epsilon\to 0} J_1 &= -E[1_A \int_{Q\times]0,1[}[u(\alpha)-\mathcal{K}-k]^+\partial_t\varphi\,dxdtd\alpha]-E[1_A\int_D [u(0)-k]^+\varphi(0)\,dx], \\
\lim_{\epsilon\to 0} J_2 &= E[1_A \int_{Q\times]0,1[}\operatorname{sgn}_0^+(u(\alpha)-\mathcal{K}-k)[\mathbf{f}(u(\alpha))-\mathbf{f}(\mathcal{K}+k)].\nabla\varphi\,dxdtd\alpha], \\
\lim_{\epsilon\to 0} J_3 &= -E[1_A \int_{Q\times]0,1[}\operatorname{sgn}_0^+(u(\alpha)-\mathcal{K}-k)\varphi\mathbf{f}'(\mathcal{K}+k).\nabla\mathcal{K}\,dxdtd\alpha], \\
\lim_{\epsilon\to 0} J_4 &= 0 \text{ since } \mathbf{f}' \text{ is a Lipschitz-continuous function}, \\
\lim_{\epsilon\to 0} J_5 &= 0.
\end{aligned}
$$

$$
\text{Then, } J_6 = -E[1_A \int_{Q\times]0,1[}[\int_0^{u(\alpha)-\mathcal{K}-k}\int_0^r \mathbf{f}'(\mathcal{K}+k+\sigma)d\sigma\eta'(r)\,dr].\nabla\varphi\,dxdtd\alpha]
$$

vanishes as ϵ goes to 0^+ thanks to the hypothesis on $\mathbf{f}'$ and one gets that:

i) For any $(k,\varphi)\in\mathbb{R}\times H^1(]0,T[\times D)$ such that $k\geq 0$ and $\varphi\geq 0$, and for any $(k,\varphi)\in\mathbb{R}\times[H^1(]0,T[\times D)]\cap L^2(0,T;H_0^1(D))$ such that $\varphi\geq 0$,

$$
\begin{aligned}
0 \leq & \int_{\{u>\mathcal{K}+k\}}\Big\{(u-\mathcal{K}-k)\partial_t\varphi-[\mathbf{f}(u)-\mathbf{f}(\mathcal{K}+k)].\nabla\varphi\Big\}\,dxdtd\alpha \\
& +\int_{\{u>\mathcal{K}+k\}}\varphi\operatorname{div}\mathbf{f}[\mathcal{K}+k]\,dxdtd\alpha+\int_D(u_0-k)^+\varphi(0,.)\,dx, \quad dP-\text{a.s.}
\end{aligned}
$$

i.e.

$$
\begin{aligned}
0 \leq & \int_{Q\times]0,1[}\Big\{(u-\mathcal{K}-k)^+\partial_t\varphi-F^+(u,\mathcal{K}+k).\nabla\varphi\Big\}\,dxdtd\alpha \\
& +\int_{Q\times]0,1[}\operatorname{sgn}_0^+(u-\mathcal{K}-k)\varphi\operatorname{div}\mathbf{f}[\mathcal{K}+k]\,dxdtd\alpha+\int_D(u_0-k)^+\varphi(0,.)\,dx,\ dP-\text{a.s.}
\end{aligned}
$$

where $F^+(x,y)=\mathbf{f}(x)-\mathbf{f}(y)$ if $x>y$ and 0 else.
In the same way, one can prove

ii) For any $(k,\varphi)\in\mathbb{R}\times H^1(]0,T[\times D)$ such that $k\leq 0$ and $\varphi\geq 0$, and for any $(k,\varphi)\in\mathbb{R}\times[H^1(]0,T[\times D)]\cap L^2(0,T;H^1_0(D))$ such that $\varphi\geq 0$,

$$\begin{aligned}0 \quad \leq \quad & \int_{\{u<\mathcal{K}+k\}}\Big\{(\mathcal{K}+k-u)\partial_t\varphi-[\mathbf{f}(\mathcal{K}+k)-\mathbf{f}(u)].\nabla\varphi\Big\}\,dxdtd\alpha \\ & -\int_{\{u<\mathcal{K}+k\}}\varphi\operatorname{div}\mathbf{f}[\mathcal{K}+k]\,dxdtd\alpha+\int_D(k-u_0)^+\varphi(0,.)\,dx, \quad dP-\text{a.s.}\end{aligned}$$

i.e.

$$\begin{aligned}0 \quad \leq \quad & \int_{Q\times]0,1[}\Big\{(\mathcal{K}+k-u)^+\partial_t\varphi-F^+(\mathcal{K}+k,u).\nabla\varphi\Big\}\,dxdtd\alpha \\ & -\int_{Q\times]0,1[}\operatorname{sgn}_0^+(\mathcal{K}+k-u)\varphi\operatorname{div}\mathbf{f}[\mathcal{K}+k]\,dxdtd\alpha+\int_D(k-u_0)^+\varphi(0,.)\,dx,\ dP-\text{a.s.}\end{aligned}$$

This proves that an entropy measure-valued solution exists.

One needs to use the uniqueness result to conclude that this Young measure is associated to a function that should be the unique entropy solution. Moreover, u belongs to $L^\infty(0,T,L^2(\Omega\times D))$ and the strong convergence in L^p would be obtained too, for any $p\in[1,2[$.

Remark: Note that, for any $(k,\varphi)\in\mathbb{R}\times[H^1(]0,T[\times D)]\cap L^2(0,T;H^1_0(D))$ such that $\varphi\geq 0$, we also have

$$\begin{aligned}0 \quad \leq \quad & \int_{Q\times]0,1[}[|u-\mathcal{K}-k|\partial_t\varphi-F(u,\mathcal{K}+k).\nabla\varphi]\,dxdtd\alpha \\ & +\int_{Q\times]0,1[}\operatorname{sgn}_0(u-\mathcal{K}-k)\varphi\operatorname{div}\mathbf{f}[\mathcal{K}+k]\,dxdtd\alpha+\int_D|u_0-k|\varphi(0,.)\,dx,\ dP-\text{a.s.}\end{aligned}$$

where $F(x,y)=\operatorname{sgn}_0(x-y)[\mathbf{f}(x)-\mathbf{f}(y)]$ and $\operatorname{sgn}_0(x)=0$ if $x=0$ and $\frac{x}{|x|}$ else.

3 Uniqueness

Let us denote by u_1 and u_2 two admissible Young measure-valued solutions associated to two initial conditions $u_{1,0}$ and $u_{2,0}$.

3.1 Interior inequality

Consider φ in $\mathcal{D}^+([0,T]\times D)$ and $G(t,x,s,y)=\varphi(s,y)\rho_n(x-y)\rho_l(s-t)$ where ρ_n and ρ_l denote the usual mollifier sequences in $\mathbb{R}^d$ and $\mathbb{R}$, respectively, with $\operatorname{supp}\rho_l\subset[-\frac{2}{l},0]$. We assume moreover that n and l are large enough for G to belong to $\mathcal{D}([0,T]\times D\times]0,T]\times D)$.

Proposition 2. *For any positive φ in $H^1(Q) \cap L^2(0,T,H_0^1(D))$,*

$$
\begin{aligned}
0 \quad \leq \quad & E\int_{Q\times]0,1[^2} \Big(u_1(t,x,\alpha) - u_2(t,x,\beta)\Big)^+ \partial_t\varphi \, dxdtd\alpha d\beta \\
& -E\int_{Q\times]0,1[^2} F^+\Big(u_1(t,x,\alpha), u_2(t,x,\beta)\Big).\nabla\varphi \, dxdtd\alpha d\beta \\
& +\int_D (u_{1,0} - u_{2,0})^+\varphi(0)\, dx.
\end{aligned}
$$

For convenience set $p = (t,x,\alpha)$, $q = (s,y,\beta)$, $\Lambda = u_1 - \mathcal{K}$, $\hat{\Lambda} = u_2 - \mathcal{K}$. Since u_1 is a solution, for $k = \hat{\Lambda}(q)$, the following inequality holds $dP-$a.s.:

$$
\begin{aligned}
0 \quad \leq \quad & \int_{Q^2\times]0,1[^2} \Big(\Lambda(p) - \hat{\Lambda}(q)\Big)^+ \partial_t G \, dpdq \\
& -\int_{Q^2\times]0,1[^2} F^+\Big(\mathcal{K}(t,x) + \Lambda(p), \mathcal{K}(t,x) + \hat{\Lambda}(q)\Big).\nabla_x G \, dpdq \\
& +\int_{Q^2\times]0,1[^2} G \, \mathrm{sgn}_0^+\Big(\Lambda(p) - \hat{\Lambda}(q)\Big)\mathbf{f}'[\mathcal{K}(t,x) + \hat{\Lambda}(q)].\nabla_x\mathcal{K}(t,x)\, dpdq.
\end{aligned}
$$

Similarly, since u_2 is a solution, for $k = \Lambda(p)$, one has $dP-$a.s.:

$$
\begin{aligned}
0 \quad \leq \quad & \int_{Q^2\times]0,1[^2} \Big(\Lambda(p) - \hat{\Lambda}(q)\Big)^+ \partial_s G \, dpdq \\
& -\int_{Q^2\times]0,1[^2} F^+\Big(\mathcal{K}(s,y) + \Lambda(p), \mathcal{K}(s,y) + \hat{\Lambda}(q)\Big).\nabla_y G \, dpdq \\
& -\int_{Q^2\times]0,1[^2} G \, \mathrm{sgn}_0^+\Big(\Lambda(p) - \hat{\Lambda}(q)\Big)\mathbf{f}'[\mathcal{K}(s,y) + \Lambda(p)].\nabla_y\mathcal{K}(s,y)\, dpdq \\
& +\int_{Q\times]0,1[\times D} (\Lambda(p) - u_{2,0}(y))^+\varphi(0,y)\rho_n(x-y)\rho_l(-t) dpdy.
\end{aligned}
$$

Summing up the preceding two inequalities, we obtain

$$
\begin{aligned}
0 \quad \leq \quad & \int_{Q^2\times]0,1[^2} \Big(\Lambda(p) - \hat{\Lambda}(q)\Big)^+ (\partial_t + \partial_s) G \, dpdq \\
& -\int_{Q^2\times]0,1[^2} F^+\Big(\mathcal{K}(t,x) + \Lambda(p), \mathcal{K}(t,x) + \hat{\Lambda}(q)\Big).\nabla_x G \, dpdq \\
& -\int_{Q^2\times]0,1[^2} F^+\Big(\mathcal{K}(s,y) + \Lambda(p), \mathcal{K}(s,y) + \hat{\Lambda}(q)\Big).\nabla_y G \, dpdq \\
& +\int_{\{\Lambda(p) > \hat{\Lambda}(q)\}} G\mathbf{f}'[\mathcal{K}(t,x) + \hat{\Lambda}(q)].\nabla_x\mathcal{K}(t,x)\, dpdq \\
& -\int_{\{\Lambda(p) > \hat{\Lambda}(q)\}} G\mathbf{f}'[\mathcal{K}(s,y) + \Lambda(p)].\nabla_y\mathcal{K}(s,y)\, dpdq \\
& +\int_{Q\times]0,1[\times D} (\Lambda(p) - u_{2,0}(y))^+\varphi(0,y)\rho_n(x-y)\rho_l(-t) dpdy, \quad dP-\text{a.s.}
\end{aligned}
$$

For convenience, denote by $A = \{\Lambda(p) > \hat{\Lambda}(q)\}$. Then we can rewrite the preceding inequality as

$$\begin{aligned}
0 \leq & \int_{Q^2\times]0,1[^2} \Big(\Lambda(p) - \hat{\Lambda}(q)\Big)^+ (\partial_t + \partial_s) G \, dpdq \\
& - \int_A \Big(\mathbf{f}[\mathcal{K}(t,x) + \Lambda(p)] - \mathbf{f}[\mathcal{K}(t,x) + \hat{\Lambda}(q)]\Big).\nabla_x G \, dpdq \\
& - \int_A \Big(\mathbf{f}[\mathcal{K}(s,y) + \Lambda(p)] - \mathbf{f}[\mathcal{K}(s,y) + \hat{\Lambda}(q)]\Big).\nabla_y G \, dpdq \\
& + \int_A G\mathbf{f}'[\mathcal{K}(t,x) + \hat{\Lambda}(q)].\nabla_x \mathcal{K}(t,x) \, dpdq - \int_A G\mathbf{f}'[\mathcal{K}(s,y) + \Lambda(p)].\nabla_y \mathcal{K}(s,y) \, dpdq \\
& + \int_{Q\times]0,1[\times D} (\Lambda(p) - u_{2,0}(y))^+ \varphi(0,y)\rho_n(x-y)\rho_l(-t) dpdy, \quad dP-\text{a.s.},
\end{aligned}$$

which yields

$$\begin{aligned}
0 \leq & \int_{Q^2\times]0,1[^2} \Big(\Lambda(p) - \hat{\Lambda}(q)\Big)^+ \partial_s \varphi(s,y) \, \rho_n(x-y)\rho_l(s-t) \, dpdq \\
& - \int_{Q^2\times]0,1[^2} F^+[\mathcal{K}(s,y) + \Lambda(p), \mathcal{K}(s,y) + \hat{\Lambda}(q)].\nabla_y \varphi(s,y) \, \rho_n(x-y)\rho_l(s-t) \, dpdq \\
& - \int_A \Big(\mathbf{f}[\mathcal{K}(t,x) + \Lambda(p)] - \mathbf{f}[\mathcal{K}(t,x) + \hat{\Lambda}(q)]\Big).\nabla_x \rho_n(x-y) \, \varphi(s,y)\rho_l(s-t) \, dpdq \\
& + \int_A \Big(\mathbf{f}[\mathcal{K}(s,y) + \Lambda(p)] - \mathbf{f}[\mathcal{K}(s,y) + \hat{\Lambda}(q)]\Big).\nabla_x \rho_n(x-y) \, \varphi(s,y)\rho_l(s-t) \, dpdq \\
& + \int_A G\mathbf{f}'[\mathcal{K}(t,x) + \hat{\Lambda}(q)].\nabla_x \mathcal{K}(t,x) \, dpdq - \int_A G\mathbf{f}'[\mathcal{K}(s,y) + \Lambda(p)].\nabla_y \mathcal{K}(s,y) \, dpdq \\
& + \int_{Q\times]0,1[\times D} (\Lambda(p) - u_{2,0}(y))^+ \varphi(0,y)\rho_n(x-y)\rho_l(-t) dpdy, \quad dP-\text{a.s.}
\end{aligned}$$

i.e.

$$\begin{aligned}
0 \leq & \int_{Q^2\times]0,1[^2} \Big(\Lambda(p) - \hat{\Lambda}(q)\Big)^+ \partial_s \varphi(s,y) \, \rho_n(x-y)\rho_l(s-t) \, dpdq \\
& - \int_{Q^2\times]0,1[^2} F^+[\mathcal{K}(s,y) + \Lambda(p), \mathcal{K}(s,y) + \hat{\Lambda}(q)].\nabla_y \varphi(s,y) \, \rho_n(x-y)\rho_l(s-t) \, dpdq \\
& + \int_A \Big(\mathbf{f}[\mathcal{K}(s,y) + \Lambda(p)] - \mathbf{f}[\mathcal{K}(t,x) + \Lambda(p)]\Big).\nabla_x \rho_n(x-y) \, \varphi(s,y)\rho_l(s-t) \, dpdq \\
& + \int_A \Big(\mathbf{f}[\mathcal{K}(t,x) + \hat{\Lambda}(q)] - \mathbf{f}[\mathcal{K}(s,y) + \hat{\Lambda}(q)]\Big).\nabla_x \rho_n(x-y) \, \varphi(s,y)\rho_l(s-t) \, dpdq \\
& + \int_A G\mathbf{f}'[\mathcal{K}(t,x) + \hat{\Lambda}(q)].\nabla_x \mathcal{K}(t,x) \, dpdq - \int_A G\mathbf{f}'[\mathcal{K}(s,y) + \Lambda(p)].\nabla_y \mathcal{K}(s,y) \, dpdq \\
& + \int_{Q\times]0,1[\times D} (\Lambda(p) - u_{2,0}(y))^+ \varphi(0,y)\rho_n(x-y)\rho_l(-t) dpdy, \quad dP-\text{a.s.} \\
= \; & I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7.
\end{aligned}$$

Thanks to the properties of Lebesgue sets, the following convergence holds:

$$\begin{aligned}\lim_{n\to\infty}\lim_{l\to\infty} E(I_1+I_2) = \quad & E\int_{Q\times]0,1[^2}\Big(u_1(t,x,\alpha)-u_2(t,x,\beta)\Big)^+\partial_t\varphi\,dxdtd\alpha d\beta\\ & -E\int_{Q\times]0,1[^2} F^+\Big(u_1(t,x,\alpha),u_2(t,x,\beta)\Big).\nabla_x\varphi\,dxdtd\alpha d\beta.\end{aligned}$$

Note that

$$\begin{aligned}I_3 \quad &= \quad \int_A \Big(\int_0^1 \mathbf{f}'[u_1(p)+\sigma(\mathcal{K}(s,y)-\mathcal{K}(t,x))]\,d\sigma\Big).\nabla_x\rho_n(x-y)(\mathcal{K}(s,y)-\mathcal{K}(t,x))\times\\ &\qquad\qquad \times\varphi(s,y)\rho_l(s-t)\,dpdq\\ &= \quad \int_A \Big(\int_0^1 \int_{\Lambda(p)+\mathcal{K}(s,y)}^{u_1(p)+\sigma(\mathcal{K}(s,y)-\mathcal{K}(t,x))} \mathbf{f}''[\eta]\,d\eta\Big).\nabla_x\rho_n(x-y)(\mathcal{K}(s,y)-\mathcal{K}(t,x))\times\\ &\qquad\qquad \times\varphi(s,y)\rho_l(s-t)\,dpdq\\ &+ \quad \int_A \mathbf{f}'[\mathcal{K}(s,y)+\Lambda(p)].\nabla_x\rho_n(x-y)(\mathcal{K}(s,y)-\mathcal{K}(t,x))\,\varphi(s,y)\rho_l(s-t)\,dpdq\\ &= \quad I_{3,1}+I_{3,2}.\end{aligned}$$

Since $\mathbf{f}'$ is assumed to be a Lipschitz-continuous function, one has the following estimate

$$|I_{3,1}| \quad\leq\quad ||\mathbf{f}''||_\infty \sum_{i=1}^d \int_A |\partial_{x_i}\rho_n|(x-y)(\mathcal{K}(s,y)-\mathcal{K}(t,x))^2\,\varphi(s,y)\rho_l(s-t)\,dpdq.$$

Lemma 1. $\limsup_{n\to\infty}\limsup_{l\to\infty}|EI_{3,1}|\leq 0.$

Proof. Starting from the above inequality, using the classical properties of the Ito integral, we deduce that

$$\begin{aligned}|EI_{3,1}| \quad &\leq \quad ||\mathbf{f}''||_\infty \sum_{i=1}^d E\int_{Q^2} |\partial_{x_i}\rho_n|(x-y)(\mathcal{K}(s,y)-\mathcal{K}(t,x))^2\,\varphi(s,y)\rho_l(s-t)\,dxdtdyds\\ &\leq \quad C\sum_{i=1}^d E\int_{Q^2}|\partial_{x_i}\rho_n|(x-y)(\mathcal{K}(s,y)-\mathcal{K}(t,y))^2\,\rho_l(s-t)\,dxdtdyds\\ &\qquad +C\sum_{i=1}^d E\int_{Q^2}|\partial_{x_i}\rho_n|(x-y)(\mathcal{K}(t,y)-\mathcal{K}(t,x))^2\,\rho_l(s-t)\,dxdtdyds\\ &\leq \quad C\sum_{i=1}^d \int_{Q^2}|\partial_{x_i}\rho_n|(x-y)\,dx\int_s^t h(\sigma,y)^2\,d\sigma\rho_l(s-t)\,dtdyds\\ &\qquad +C\sum_{i=1}^d \int_{D^2\times]0,T[}|\partial_{x_i}\rho_n|(x-y)\int_0^t[h(\sigma,y)-h(\sigma,x)]^2\,d\sigma\,dxdtdy\end{aligned}$$

Thus, one has that

$$
\begin{aligned}
|EI_{3,1}| &\leq C(n)\int_{D\times\mathbb{R}^2}\int_{t+r}^{t}\bar{h}(\sigma,y)^2\,d\sigma\rho_l(r)\,dydtdr \\
&\quad +C\sum_{i=1}^{d}\int_{\mathbb{R}^{2d}\times]0,T[}|\partial_{x_i}\rho_n|(z)\int_0^t[\bar{h}(\sigma,x+z)-\bar{h}(\sigma,x)]^2\,d\sigma\,dxdzdt \\
&\leq C(n)\int_{\mathbb{R}}\int_r^0\int_{\mathbb{R}}\int_D\bar{h}(t+\sigma,y)^2\,dydt\;d\sigma\rho_l(r)\,dr \\
&\quad +C\sum_{i=1}^{d}\int_{\mathbb{R}^{2d}\times]0,T[}|\partial_{x_i}\rho_n|(z)\int_0^t[\bar{h}(\sigma,x+z)-\bar{h}(\sigma,x)]^2\,d\sigma\;dxdtdz \\
&\leq C(n)||\bar{h}||^2_{L^2(Q)}\int_{\mathbb{R}}\int_r^0 d\sigma\rho_l(r)\,dr \\
&\quad +C\sum_{i=1}^{d}\int_{\mathbb{R}^d}|\partial_{x_i}\rho_n|(z)\int_{\mathbb{R}^d\times]0,T[}\int_0^t[\bar{h}(\sigma,x+z)-\bar{h}(\sigma,x)]^2\,d\sigma\,dxdtdz \\
&\leq \frac{C(n)}{l}||\bar{h}||^2_{L^2(Q)}+C\sum_{i=1}^{d}\int_{\mathbb{R}^d}|\partial_{x_i}\rho_n|(z)||z||^2||\bar{h}||^2_{L^2(\mathbb{R};H^1(\mathbb{R}^d))}\,dz,
\end{aligned}
$$

and the assertion of the Lemma follows. □

In a similar way,

$$
\begin{aligned}
I_4 &= \int_A\Big(\int_0^1\mathbf{f}'[u_2(q)+\sigma(\mathcal{K}(t,x)-\mathcal{K}(s,y))]\,d\sigma\Big).\nabla_x\rho_n(x-y)(\mathcal{K}(t,x)-\mathcal{K}(s,y))\times \\
&\qquad \times\varphi(s,y)\rho_l(s-t)\,dpdq \\
&= \int_A\Big(\int_0^1\int_{\mathcal{K}(t,x)+\hat{\Lambda}(q)}^{u_2(q)+\sigma(\mathcal{K}(t,x)-\mathcal{K}(s,y))}\mathbf{f}''[\eta]\,d\eta\Big).\nabla_x\rho_n(x-y)(\mathcal{K}(t,x)-\mathcal{K}(s,y))\times \\
&\qquad \times\varphi(s,y)\rho_l(s-t)\,dpdq \\
&+ \int_A\mathbf{f}'[\mathcal{K}(t,x)+\hat{\Lambda}(q)].\nabla_x\rho_n(x-y)(\mathcal{K}(t,x)-\mathcal{K}(s,y))\,\varphi(s,y)\rho_l(s-t)\,dpdq \\
&= I_{4,1}+I_{4,2}.
\end{aligned}
$$

Since $\mathbf{f}'$ is assumed to be a Lipschitz-continuous function, one has the following estimate

$$
|I_{4,1}| \leq ||\mathbf{f}''||_\infty\sum_{i=1}^{d}\int_A|\partial_{x_i}\rho_n|(x-y)(\mathcal{K}(s,y)-\mathcal{K}(t,x))^2\,\varphi(s,y)\rho_l(s-t)\,dpdq
$$

and, in the same way as for $EI_{3,1}$, we can prove

Lemma 2. $\limsup_{n\to\infty}\limsup_{l\to\infty}|EI_{4,1}|\leq 0$.

Next, note that,

$$
\begin{aligned}
I_{3,2}+I_6 &= \int_A \varphi(s,y)\rho_l(s-t)\mathbf{f}'[\mathcal{K}(s,y)+\Lambda(p)]\times \\
&\qquad \times[\nabla_x\rho_n(x-y)(\mathcal{K}(s,y)-\mathcal{K}(t,x))-\nabla_y\mathcal{K}(s,y)\rho_n(x-y)]\,dpdq \\
&= -\int_A \varphi(s,y)\rho_l(s-t)\mathbf{f}'[\mathcal{K}(s,y)+\Lambda(p)].\nabla_y[\rho_n(x-y)(\mathcal{K}(s,y)-\mathcal{K}(t,x))]\,dpdq
\end{aligned}
$$

and that

$$
\begin{aligned}
I_{4,2}+I_5 &= \int_A \varphi(s,y)\rho_l(s-t)\mathbf{f}'[\mathcal{K}(t,x)+\hat{\Lambda}(q)]\times \\
&\qquad \times[\nabla_x\rho_n(x-y)(\mathcal{K}(t,x)-\mathcal{K}(s,y))+\nabla_x\mathcal{K}(t,x)\rho_n(x-y)]\,dpdq \\
&= \int_A \varphi(s,y)\rho_l(s-t)\mathbf{f}'[\mathcal{K}(t,x)+\hat{\Lambda}(q)].\nabla_x[\rho_n(x-y)(\mathcal{K}(t,x)-\mathcal{K}(s,y))]\,dpdq.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
& I_{3,2}+I_6+I_{4,2}+I_5 \\
={}& \int_A \varphi(s,y)\rho_l(s-t)\mathbf{f}'[\mathcal{K}(t,x)+\hat{\Lambda}(q)].\nabla_x[\rho_n(x-y)(\mathcal{K}(t,x)-\mathcal{K}(s,y))]\,dpdq \\
& -\int_A \varphi(s,y)\rho_l(s-t)\mathbf{f}'[\mathcal{K}(s,y)+\Lambda(p)].\nabla_y[\rho_n(x-y)(\mathcal{K}(s,y)-\mathcal{K}(t,x))]\,dpdq \\
={}& \int_A \varphi(s,y)\rho_l(s-t)\mathbf{f}'[\mathcal{K}(t,x)+\hat{\Lambda}(q)].\nabla_x[\rho_n(x-y)(\mathcal{K}(t,x)-\mathcal{K}(s,y))]\,dpdq \\
& +\int_A \varphi(s,y)\rho_l(s-t)\mathbf{f}'[\mathcal{K}(s,y)+\Lambda(p)].\nabla_y[\rho_n(x-y)(\mathcal{K}(t,x)-\mathcal{K}(s,y))]\,dpdq \\
={}& \int_A \varphi(s,y)\rho_l(s-t)\mathbf{f}'[\mathcal{K}(t,x)+\hat{\Lambda}(q)]\times \\
& \times\Big[\nabla_y[\rho_n(x-y)(\mathcal{K}(t,x)-\mathcal{K}(s,y))]+\nabla_x[\rho_n(x-y)(\mathcal{K}(t,x)-\mathcal{K}(s,y))]\Big]\,dpdq \\
& +\int_A \varphi(s,y)\rho_l(s-t)\Big[\mathbf{f}'[\mathcal{K}(s,y)+\Lambda(p)]-\mathbf{f}'[\mathcal{K}(t,x)+\hat{\Lambda}(q)]\Big]\times \\
& \qquad\qquad \times\nabla_y[\rho_n(x-y)(\mathcal{K}(t,x)-\mathcal{K}(s,y))]\,dpdq \\
={}& J_1+J_2
\end{aligned}
$$

Note that

$$
\begin{aligned}
& |J_1| \\
={}& |\int_A \varphi(s,y)\rho_l(s-t)\mathbf{f}'[\mathcal{K}(t,x)+\hat{\Lambda}(q)].\Big[\nabla_x\mathcal{K}(t,x)-\nabla_y\mathcal{K}(s,y)\Big]\rho_n(x-y)\,dpdq| \\
\le{}& C\int_A \varphi(s,y)\rho_l(s-t)\Big[|\mathcal{K}(t,x)+\hat{\Lambda}(q)|+1\Big]|\nabla_x\mathcal{K}(t,x)-\nabla_y\mathcal{K}(s,y)|\rho_n(x-y)\,dpdq
\end{aligned}
$$

since $\mathbf{f}'$ is a Lipschitz-continuous function.

Lemma 3. $\limsup_{n\to\infty}\limsup_{l\to\infty}|EJ_1|\le 0$

Proof. Note that one has:

$$\begin{aligned}
|EJ_1| &\leq CE\int_A \rho_l(s-t)|\nabla_x\mathcal{K}(t,x)-\nabla_x\mathcal{K}(s,x)|\Big[|\mathcal{K}(t,x)+\hat{\Lambda}(q)|+1\Big]\rho_n(x-y)\,dpdq \\
&\quad +CE\int_A |\nabla_x\mathcal{K}(s,x)-\nabla_y\mathcal{K}(s,y)|\Big[|\mathcal{K}(t,x)+\hat{\Lambda}(q)|+1\Big]^2\rho_n(x-y)\rho_l(s-t)\,dpdq \\
&\leq C[\int_{D\times]0,T[^2}\rho_l(s-t)\int_s^t|\nabla_x h(\sigma,x)|^2\,d\sigma\,dxdtds]^{\frac12}\times \\
&\qquad\qquad \times[E\int_A\rho_l(s-t)\rho_n(x-y)\Big[|\mathcal{K}(t,x)+\hat{\Lambda}(q)|+1\Big]^2\,dpdq]^{\frac12} \\
&\quad +C[\int_{D^2\times]0,T[}\int_0^s|\nabla_x h(\sigma,x)-\nabla_y h(\sigma,y)|^2\,d\sigma\rho_n(x-y)\,dxdyds]^{\frac12}\times \\
&\qquad\qquad \times[E\int_A\rho_l(s-t)\rho_n(x-y)\Big[|\mathcal{K}(t,x)+\hat{\Lambda}(q)|+1\Big]\,dpdq]^{\frac12} \\
&\leq C[\int_{D\times\mathbb{R}^2}\rho_l(r)\int_{t+r}^t|\nabla_x\bar{h}(\sigma,x)|^2\,d\sigma\,dxdtdr]^{\frac12} \\
&\quad +C[\int_{\mathbb{R}^{2d}\times]0,T[}\int_0^s|\nabla_x\bar{h}(\sigma,x)-\nabla_y\bar{h}(\sigma,x+z)|^2\,d\sigma\rho_n(z)\,dxdzds]^{\frac12} \\
&\leq C[\int_{D\times\mathbb{R}^2}\rho_l(r)\int_r^0|\nabla_x\bar{h}(t+\sigma,x)|^2\,d\sigma\,dxdtdr]^{\frac12} \\
&\quad +C[\int_{\mathbb{R}^d}\rho_n(z)\int_{]0,T[}\int_{\mathbb{R}^d}\int_0^s|\nabla_x\bar{h}(\sigma,x)-\nabla_x\bar{h}(\sigma,x+z)|^2\,d\sigma\,dxdsdz]^{\frac12} \\
&\leq \frac{C}{\sqrt{l}}||\bar{h}||_{L^2(\mathbb{R},H^1(\mathbb{R}^d))}+C[\int_{\mathbb{R}^d}||\nabla\bar{h}(.,.)-\nabla\bar{h}(.,.+z)||^2_{L^2(0,T,L^2(\mathbb{R}^d))}\rho_n(z)\,dz]^{\frac12}.
\end{aligned}$$

Therefore, $\limsup_{n\to\infty}\limsup_{l\to\infty}|EJ_1|\leq 0$ thanks to the continuity of translations in Lebesgue spaces. $\square$

Moreover,

$$\begin{aligned}
J_2 &= \int_A\varphi(s,y)\rho_l(s-t)\Big[\mathbf{f}'[\mathcal{K}(s,y)+\Lambda(p)]-\mathbf{f}'[\mathcal{K}(t,x)+\hat{\Lambda}(q)]\Big]\times \\
&\qquad\qquad \times\nabla_y[\rho_n(x-y)(\mathcal{K}(t,x)-\mathcal{K}(s,y))]\,dpdq \\
&= \int_A\varphi(s,y)\rho_l(s-t)\Big[\mathbf{f}'[u_1(p)+\mathcal{K}(s,y)-\mathcal{K}(t,x)]-\mathbf{f}'[u_1(p)+\mathcal{K}(t,y)-\mathcal{K}(t,x)]\Big]\times \\
&\qquad\qquad \times\nabla_y[\rho_n(x-y)(\mathcal{K}(t,x)-\mathcal{K}(s,y))]\,dpdq \\
&\quad +\int_A\varphi(s,y)\rho_l(s-t)\Big[\mathbf{f}'[u_1(p)+\mathcal{K}(t,y)-\mathcal{K}(t,x)]-\mathbf{f}'[u_2(q)+\mathcal{K}(t,x)-\mathcal{K}(s,x)]\Big]\times \\
&\qquad\qquad \times\nabla_y[\rho_n(x-y)(\mathcal{K}(t,x)-\mathcal{K}(s,y))]\,dpdq \\
&\quad +\int_A\varphi(s,y)\rho_l(s-t)\Big[\mathbf{f}'[u_2(q)+\mathcal{K}(t,x)-\mathcal{K}(s,x)]-\mathbf{f}'[u_2(q)+\mathcal{K}(t,x)-\mathcal{K}(s,y)]\Big]\times \\
&\qquad\qquad \times\nabla_y[\rho_n(x-y)(\mathcal{K}(t,x)-\mathcal{K}(s,y))]\,dpdq \\
&= J_{2,1}+J_{2,2}+J_{2,3}.
\end{aligned}$$

Then, one has that

$$
\begin{aligned}
&|J_{2,1}| \\
&\leq ||\mathbf{f}''||_\infty \int_{Q^2} \varphi(s,y)\rho_l(s-t)|\mathcal{K}(s,y)-\mathcal{K}(t,y)|\,|\nabla_y[\rho_n(x-y)(\mathcal{K}(t,x)-\mathcal{K}(s,y))]|\,dxdtdyds \\
&\leq ||\mathbf{f}''||_\infty \int_{Q^2} \varphi(s,y)\rho_l(s-t)|\mathcal{K}(s,y)-\mathcal{K}(t,y)||\mathcal{K}(t,x)-\mathcal{K}(s,y)||\nabla_y\rho_n(x-y)|\,dxdtdyds \\
&\quad +||\mathbf{f}''||_\infty \int_{Q^2} \varphi(s,y)\rho_l(s-t)|\mathcal{K}(s,y)-\mathcal{K}(t,y)|\rho_n(x-y)|\nabla_y\mathcal{K}(s,y)|\,dxdtdyds.
\end{aligned}
$$

Thus,

Lemma 4. $\limsup_{n\to\infty}\limsup_{l\to\infty} |EJ_{2,1}| \leq 0$

Proof. Indeed,

$$
\begin{aligned}
0 &\leq E\int_{Q^2} \varphi(s,y)\rho_l(s-t)|\mathcal{K}(s,y)-\mathcal{K}(t,y)||\mathcal{K}(t,x)-\mathcal{K}(s,y)||\nabla\rho_n(x-y)|\,dxdtdyds \\
&\leq \frac{1}{2}E\int_{Q^2} \varphi(s,y)\rho_l(s-t)|\mathcal{K}(s,y)-\mathcal{K}(t,y)|^2|\nabla\rho_n(x-y)|\,dxdtdyds \\
&\quad +\frac{1}{2}E\int_{Q^2} \varphi(s,y)\rho_l(s-t)|\mathcal{K}(t,x)-\mathcal{K}(s,y)|^2|\nabla\rho_n(x-y)|\,dxdtdyds,
\end{aligned}
$$

and the terms on the right tend to 0 as first l then n tend to $+\infty$, as has been shown already in the study of integral $I_{3,1}$. Moreover,

$$
\begin{aligned}
&E\int_{Q^2} \varphi(s,y)\rho_l(s-t)|\mathcal{K}(s,y)-\mathcal{K}(t,y)|\rho_n(x-y)|\nabla_y\mathcal{K}(s,y)|\,dxdtdyds \\
\leq\ &[E\int_{Q^2} \varphi(s,y)\rho_l(s-t)|\mathcal{K}(s,y)-\mathcal{K}(t,y)|^2\rho_n(x-y)\,dxdtdyds]^{\frac{1}{2}} \times \\
&\times[E\int_{Q^2} \varphi(s,y)\rho_l(s-t)\rho_n(x-y)|\nabla_y\mathcal{K}(s,y)|^2\,dxdtdyds]^{\frac{1}{2}} \\
\leq\ &[E\int_{Q^2} \varphi(s,y)\rho_l(s-t)|\mathcal{K}(s,y)-\mathcal{K}(t,y)|^2\rho_n(x-y)\,dxdtdyds]^{\frac{1}{2}}C||\mathcal{K}(s,y)||_{L^2(\Omega\times Q)}
\end{aligned}
$$

a term of the same nature as the one studied already in connection with EJ_1.
□

Next, observe that

$$\begin{aligned}
|J_{2,3}| &\leq ||\mathbf{f}''||_\infty \int_{Q^2} \varphi(s,y)\rho_l(s-t)|\mathcal{K}(s,y)-\mathcal{K}(s,x)|\,|\nabla_y[\rho_n(x-y)(\mathcal{K}(t,x)-\mathcal{K}(s,y))]| \\
&\qquad dxdtdyds \\
&= ||\mathbf{f}''||_\infty \int_{Q^2} \varphi(s,y)\rho_l(s-t)|\mathcal{K}(s,y)-\mathcal{K}(s,x)||\mathcal{K}(t,x)-\mathcal{K}(s,y)||\nabla_y\rho_n(x-y)| \\
&\qquad dxdtdyds \\
&\quad +||\mathbf{f}''||_\infty \int_{Q^2} \varphi(s,y)\rho_l(s-t)|\mathcal{K}(s,y)-\mathcal{K}(s,x)|\rho_n(x-y)|\nabla_y\mathcal{K}(s,y)|\,dxdtdyds,
\end{aligned}$$

and thus, we can prove

Lemma 5. $\limsup_{n\to\infty}\limsup_{l\to\infty} |EJ_{2,3}| \leq 0$

Proof. Indeed,

$$\begin{aligned}
0 &\leq E\int_{Q^2} \varphi(s,y)\rho_l(s-t)|\mathcal{K}(s,y)-\mathcal{K}(s,x)||\mathcal{K}(t,x)-\mathcal{K}(s,y)||\nabla_y\rho_n(x-y)|\,dxdtdyds \\
&\leq \frac{1}{2}E\int_{Q^2} \varphi(s,y)\rho_l(s-t)|\mathcal{K}(s,y)-\mathcal{K}(s,x)|^2|\nabla_y\rho_n(x-y)|\,dxdtdyds \\
&\quad +\frac{1}{2}E\int_{Q^2} \varphi(s,y)\rho_l(s-t)|\mathcal{K}(t,x)-\mathcal{K}(s,y)|^2|\nabla_y\rho_n(x-y)|\,dxdtdyds,
\end{aligned}$$

whose limits have been studied in the treatment of integral $I_{3,1}$.
Moreover,

$$\begin{aligned}
& E\int_{Q^2} \varphi(s,y)\rho_l(s-t)|\mathcal{K}(s,y)-\mathcal{K}(s,x)|\rho_n(x-y)|\nabla_y\mathcal{K}(s,y)|\,dxdtdyds \\
\leq\; & [E\int_{Q^2} \varphi(s,y)\rho_l(s-t)|\mathcal{K}(s,y)-\mathcal{K}(s,x)|^2\rho_n(x-y)\,dxdtdyds]^{\frac{1}{2}} \\
& \times[E\int_{Q^2} \varphi(s,y)\rho_l(s-t)\rho_n(x-y)|\nabla_y\mathcal{K}(s,y)|^2\,dxdtdyds]^{\frac{1}{2}} \\
\leq\; & [E\int_{Q^2} \varphi(s,y)\rho_l(s-t)|\mathcal{K}(s,y)-\mathcal{K}(s,x)|^2\rho_n(x-y)\,dxdtdyds]^{\frac{1}{2}} c(\varphi)||\mathcal{K}(s,y)||_{L^2(\Omega\times Q)}
\end{aligned}$$

whose limit is similar to the one studied in the treatment of integral EJ_1. $\square$

Since $G(t,x,.,.) \in \mathcal{D}([0,T[\times D)$, by Gauss-Green, one has that

$$\begin{aligned}
J_{2,2} &= \int_{(Q\times]0,1[)^2} F'^{+}\Big[u_1(p)+\mathcal{K}(t,y)-\mathcal{K}(t,x), u_2(q)+\mathcal{K}(t,y)-\mathcal{K}(s,x)\Big] \times \\
&\qquad\qquad \times\varphi(s,y)\rho_l(s-t)\nabla_y[\rho_n(x-y)(\mathcal{K}(t,x)-\mathcal{K}(s,y))]\, dpdq \\
&= \int_{(Q\times]0,1[)^2} \varphi(s,y)\rho_l(s-t)\nabla_y[\rho_n(x-y)(\mathcal{K}(t,x)-\mathcal{K}(s,y))] \times \\
&\qquad \times\Big\{F'^{+}\Big[u_1(p)+\mathcal{K}(t,y)-\mathcal{K}(t,x), u_2(q)+\mathcal{K}(t,y)-\mathcal{K}(s,x)\Big] \\
&\qquad\qquad -F'^{+}\Big[u_1(p), u_2(t,x,\beta)+\mathcal{K}(t,x)-\mathcal{K}(s,x)\Big]\Big\}\, dpdq \\
&\quad -\int_{(Q\times]0,1[)^2} \rho_l(s-t)[\rho_n(x-y)(\mathcal{K}(t,x)-\mathcal{K}(s,y))]\nabla_y\varphi(s,y) \times \\
&\qquad\qquad \times F'^{+}\Big[u_1(p), u_2(t,x,\beta)+\mathcal{K}(t,x)-\mathcal{K}(s,x)\Big]\, dpdq
\end{aligned}$$

Since $\mathbf{f}'$ and F'^{+} are Lipschitz-continuous functions, with a Lipschitz-constant C depending on $||\mathbf{f}''||_\infty$,

$$\begin{aligned}
|J_{2,2}| &\leq C\int_{Q^2\times]0,1[} \varphi(s,y)\rho_l(s-t)|\nabla_y[\rho_n(x-y)(\mathcal{K}(t,x)-\mathcal{K}(s,y))]| \times \\
&\qquad \times\Big\{2|\mathcal{K}(t,y)-\mathcal{K}(t,x)| + |u_2(q)-u_2(t,x,\beta)|\Big\}\, dxdtdydsd\beta \\
&\quad +C\int_{(Q\times]0,1[)^2} \rho_l(s-t)\rho_n(x-y)|\mathcal{K}(t,x)-\mathcal{K}(s,y)| \times \\
&\qquad \times|u_1(p)-u_2(t,x,\beta)-\mathcal{K}(t,x)+\mathcal{K}(s,x)|\, dpdq.
\end{aligned}$$

Now, we can prove

Lemma 6. $\limsup_{n\to\infty}\limsup_{l\to\infty} |EJ_{2,2}| \leq 0$

Proof. On the one hand, the last integral vanishes since it is lower than

$$\begin{aligned}
&C[\int_{Q^2} \rho_l(s-t)\rho_n(x-y)|\mathcal{K}(t,x)-\mathcal{K}(s,y)|^2\, dxdtdyds]^{\frac{1}{2}} \times \\
&\times[\int_{(Q\times]0,1[)^2} \rho_l(s-t)\rho_n(x-y)||u_1(p)-u_2(t,x,\beta)-\mathcal{K}(t,x)+\mathcal{K}(s,x)|^2\, dpdq]^{\frac{1}{2}}.
\end{aligned}$$

On the other hand, the first part of the first integral is similar to the one already studied with $J_{2,3}$, so we concentrate on

$$\begin{aligned}
&E\int_{Q^2\times]0,1[} \varphi(s,y)\rho_l(s-t)|\nabla_y[\rho_n(x-y)(\mathcal{K}(t,x)-\mathcal{K}(s,y))]||u_2(q)-u_2(t,x,\beta)|\, dxdtdq \\
\leq\ &E\int_{Q^2\times]0,1[} \varphi(s,y)\rho_l(s-t)|\nabla_y\rho_n(x-y)||\mathcal{K}(t,x)-\mathcal{K}(s,y))||u_2(q)-u_2(t,x,\beta)|\, dxdtdq \\
&+E\int_{Q^2\times]0,1[} \varphi(s,y)\rho_l(s-t)\rho_n(x-y)|\nabla_y\mathcal{K}(s,y)||u_2(q)-u_2(t,x,\beta)|\, dxdtdq \qquad (4)
\end{aligned}$$

As to the first term on the right, since $|\nabla\rho_n(x)| = \frac{2n^2|x|}{(|nx|^2-1)^2}\rho_n(x)$, one gets that

$$\begin{aligned}
& E\int_{Q^2\times]0,1[}\varphi(s,y)\rho_l(s-t)|\nabla_y\rho_n(x-y)||\mathcal{K}(t,x)-\mathcal{K}(s,y))||u_2(q)-u_2(t,x,\beta)|\,dxdtdq \\
\leq\ & 4[E\int_{Q^2\times]0,1[}\varphi(s,y)\rho_l(s-t)\rho_n(x-y)|u_2(q)-u_2(t,x,\beta)|^2\,dxdtdq]^{\frac{1}{2}}\times \\
& \times\Big\{[E\int_{Q^2}\varphi(s,y)\rho_l(s-t)\frac{n^4|x-y|^2}{(|n(x-y)|^2-1)^4}\rho_n(x-y)|\mathcal{K}(t,x)-\mathcal{K}(s,x))|^2\,dxdtdyds]^{\frac{1}{2}} \\
& +[E\int_{Q^2}\varphi(s,y)\rho_l(s-t)\frac{n^4|x-y|^2}{(|n(x-y)|^2-1)^4}\rho_n(x-y)|\mathcal{K}(s,x)-\mathcal{K}(s,y))|^2\,dxdtdyds]^{\frac{1}{2}}\Big\} \\
:=\ & 4A\cdot B
\end{aligned}$$

Note that, if one still denotes by u_2 the same function extended by 0 outside Q, we have

$$A \leq [E\int_{\mathbb{R}^{d+1}}\rho_l(r)\rho_n(z)\int_{\mathbb{R}^{d+1}}|u_2(t+r,x+z,\beta)-u_2(t,x,\beta)|^2\,dxdtdzdrd\beta]^{\frac{1}{2}},$$

which tends to 0 thanks to the continuity of translations in the Lebesgue spaces. Let us prove that B is bounded. In order to do so, note that

$$\begin{aligned}
B \leq\ & C[\int_{Q^2}\rho_l(s-t)\frac{n^4|x-y|^2}{(|n(x-y)|^2-1)^4}\rho_n(x-y)\int_s^t h^2(\sigma,x)\,d\sigma\,dxdtdyds]^{\frac{1}{2}} \\
& +C[\int_{Q^2}\rho_l(s-t)\frac{n^4|x-y|^2}{(|n(x-y)|^2-1)^4}\rho_n(x-y)\int_0^s|h(\sigma,x)-h(\sigma,y))|^2\,d\sigma\,dxdtdyds]^{\frac{1}{2}} \\
\leq\ & C[\int_{D\times\mathbb{R}^2}\rho_l(r)\int_D\frac{n^4|x-y|^2}{(|n(x-y)|^2-1)^4}\rho_n(x-y)dy\int_{t+r}^t\bar{h}^2(\sigma,x)\,d\sigma\,dxdtdr]^{\frac{1}{2}} \\
& +C[\int_{\mathbb{R}^{2d}\times]0,T[}\frac{n^4|z|^2}{(|nz|^2-1)^4}\rho_n(z)\int_0^s|\bar{h}(\sigma,x)-\bar{h}(\sigma,x+z))|^2\,d\sigma\,dxdzds]^{\frac{1}{2}} \\
\leq\ & n^2C\int_{D\times\mathbb{R}^2}\rho_l(r)\int_{t+r}^t\bar{h}^2(\sigma,x)\,d\sigma\,dxdt\,dr \\
& +C\int_{\mathbb{R}^d}\frac{n^4|z|^2}{(|nz|^2-1)^4}\rho_n(z)\int_{\mathbb{R}^d\times]0,T[}\int_0^s(\bar{h}(\sigma,x)-\bar{h}(\sigma,x+z))^2\,d\sigma\,dxds\,dz \\
\leq\ & n^2C\int_{D\times\mathbb{R}^2}\rho_l(r)\int_r^0\bar{h}^2(t+\sigma,x)\,d\sigma\,dxdt\,dr \\
& +C\int_{\mathbb{R}^d}\frac{n^4|z|^2}{(|nz|^2-1)^4}\rho_n(z)\int_{]0,T[}\int_0^s||\bar{h}(\sigma)||^2_{H^1(\mathbb{R}^d)}|z|^2\,d\sigma\,ds\,dz \\
\leq\ & \frac{n^2C}{l}||\bar{h}||_{L^2(\mathbb{R}^{d+1})}+C\int_{\mathbb{R}^d}\frac{n^4|z|^4}{(|nz|^2-1)^4}\rho_n(z)\,dz||\bar{h}||^2_{L^2(0,T;H^1(\mathbb{R}^d))}.
\end{aligned}$$

Therefore, $\limsup_{l\to\infty} B$ becomes uniformly bounded with respect to n and the result holds.

As to the second term on the right of (4), if one still denotes by u_2 the same function extended by 0 outside Q, one has that

$$
\begin{aligned}
& E\int_{Q^2\times]0,1[}\varphi(s,y)\rho_l(s-t)\rho_n(x-y)|\nabla_y\mathcal{K}(s,y)||u_2(q)-u_2(t,x,\beta)|\,dxdtdydsd\beta \\
\leq\ & [E\int_{Q^2}\varphi(s,y)\rho_l(s-t)\rho_n(x-y)|\nabla_y\mathcal{K}(s,y)|^2\,dxdtdyds]^{\frac{1}{2}}\times \\
& \times[E\int_{Q^2\times]0,1[}\varphi(s,y)\rho_l(s-t)\rho_n(x-y)|u_2(q)-u_2(t,x,\beta)|^2\,dxdtdydsd\beta]^{\frac{1}{2}}. \\
\leq\ & C||\mathcal{K}||^2_{L^2(\Omega\times Q)}\times \\
& \times[E\int_{\mathbb{R}^{d+1}}\rho_l(r)\rho_n(z)\int_{\mathbb{R}^{d+1}\times]0,1[}|u_2(t+r,x+z,\beta)-u_2(t,x,\beta)|^2\,dxdtd\beta\,dzdr]^{\frac{1}{2}}.
\end{aligned}
$$

Again, the result follows from the continuity of translations in the Lebesgue spaces. $\square$

Finally, let us show

Lemma 7. $\limsup_{n\to\infty}\limsup_{l\to\infty}|EI_7|\leq\int_D(u_{1,0}-u_{2,0})^+\varphi(0,x)dx$

Proof. Denote by $\phi(t,x,y)=\int_t^T\rho_l(-r)dr\rho_n(x-y)\varphi(0,y)=\int_{\inf(t,\frac{2}{l})}^{\frac{2}{l}}\rho_l(-r)dr\rho_n(x-y)\varphi(0,y)$. Since it is a non-negative function of $\mathcal{D}([0,T[\times D)$ for any y in D as soon as n is large enough, and as u_1 is a solution, one gets that

$$
\begin{aligned}
& \int_{D\times Q\times]0,1[}\Big\{(\Lambda-u_{2,0}(y))^+\rho_l(-t)\rho_n(x-y)\varphi(0,y)\Big\}\,dpdy \\
& \leq\int_{D\times Q\times]0,1[}\int_{\inf(t,\frac{2}{l})}^{\frac{2}{l}}\rho_l(-r)dr\Big\{-F^+(\mathcal{K}+\Lambda,\mathcal{K}+u_{2,0}(y)).\nabla\rho_n(x-y)\varphi(0,y)\Big\}\,dpdy \\
& +\int_{D\times Q\times]0,1[}\int_{\inf(t,\frac{2}{l})}^{\frac{2}{l}}\rho_l(-r)dr\rho_n(x-y)\varphi(0,y)\mathrm{sgn}_0^+(\Lambda-u_{2,0}(y))\,\mathrm{div}\,\mathbf{f}[\mathcal{K}+u_{2,0}(y)]\,dpdy \\
& +\int_{D^2}(u_{1,0}(x)-u_{2,0}(y))^+\int_0^{\frac{2}{l}}\rho_l(-r)dr\rho_n(x-y)\varphi(0,y)\,dxdy.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
& \limsup_{n\to\infty}\limsup_{l\to\infty}E\int_{D\times Q\times]0,1[}\Big\{(\Lambda-u_{2,0}(y))^+\rho_l(-t)\rho_n(x-y)\varphi(0,y)\Big\}\,dpdy \\
\leq\ & \int_D(u_{1,0}-u_{2,0})^+\varphi(0,.)\,dx
\end{aligned}
$$

$\square$

3.2 Global inequality

Proposition 3. *For any positive φ in $H^1(Q)$,*

$$\begin{aligned} 0 &\leq E\int_{Q\times]0,1[^2}\Big(u_1(t,x,\alpha)-u_2(t,x,\beta)\Big)^+\partial_t\varphi\,dxdtd\alpha d\beta \\ &\quad -E\int_{Q\times]0,1[^2}F^+\Big(u_1(t,x,\alpha),u_2(t,x,\beta)\Big).\nabla\varphi\,dxdtd\alpha d\beta+\int_D(u_{1,0}-u_{2,0})^+\varphi(0)\,dx. \end{aligned}$$

Following J. Carrillo[6, 7, 8], choose a partition of unity subordinate to a covering of $\bar{D}$ by balls B_i, $i=0,...,k$ satisfying $B_0\cap\partial D=\emptyset$, and, for $i>0$, $B_i\subset B_i'$ with $B_i'\cap\partial D$ part of a Lipschitz graph.

Consider φ in $\mathcal{D}^+([0,T[\times\mathbb{R}^d)$ with supp$\varphi\subset B:=B_i$ for some $i>0$.

Moreover, we choose a sequence of mollifiers ρ_l in $\mathbb{R}$ with supp$\rho_l\subset]-2/l,0[$ and a sequence of mollifiers ρ_n in $\mathbb{R}^d$ such that $y\mapsto\rho_n(y-x)\in\mathcal{D}(D)$ for all $x\in B$[3], $\sigma_n(y)=\int_D\rho_n(y-x)dx$ is an increasing sequence for $y\in B$, and $\sigma_n(y)=1$ for any $y\in B$ such that $d(y,\mathbb{R}^d\backslash D)>c/n$ (with $c=C(i)$ depending on B). Denote $G(t,x,s,y)=\varphi(s,y)\rho_n(y-x)\rho_l(s-t)$.

Note that, for l, n sufficiently large, $(t,x)\mapsto G(.,.,s,y)\in\mathcal{D}(]0,T[\times\bar{D})$ for any $(s,y)\in Q$, and $(s,y)\mapsto G(t,x,.,.)\in\mathcal{D}([0,T[\times D)$ for any $(t,x)\in Q$. Moreover, the function

$$G_n(s,y)=\int_Q G(t,x,s,y)dxdt=\varphi(s,y)\int_D\rho_n(y-x)dx\int_{]0,T[}\rho_l(s-t)dt=\varphi(s,y)\sigma_n(y),$$

satisfies: $G_n\in\mathcal{D}([0,T[\times D),\quad 0\leq G_n\leq G_{n+1}\leq\varphi$.

Therefore, a non-negative Borel function ψ exists such that the monotonically increasing sequence G_n converges to ψ everywhere in B and $0\leq\psi\leq\varphi$.

For convenience set $p=(t,x,\alpha)$, $q=(s,y,\beta)$, $\Lambda=u_1-\mathcal{K}$ and $\hat{\Lambda}=u_2-\mathcal{K}$.

Since $k=\hat{\Lambda}^+(q)\geq 0$, using that u_1 is a solution and $G(t=0)=0$, $dP-$a.s.

[3]For every $i=1,..,k$, depending on the local representation of the boundary of D in B_i as the graph of a Lipschitz function, we can construct a vector $\eta_i\in\mathbb{R}^d$ such that the translated sequence of mollifiers $\rho_n(x-y)=\bar{\rho}_n(x-y-\frac{1}{n}\eta_i)$ satisfies that $y\mapsto\bar{\rho}_n(x-y-\frac{1}{n}\eta_i)\in\mathcal{D}(D)$ for all $x\in B=B_i$, where $\bar{\rho}_n$ denotes the standard mollifier sequence, see J. Carrillo[6] or V. Girault[16]

leads to

$$\begin{aligned}
0 \quad \leq & \quad \int_{Q^2\times]0,1[^2} \Big(\Lambda(p) - \hat\Lambda^+(q)\Big)^+ \partial_t G \, dpdq \\
& - \int_{Q^2\times]0,1[^2} F^+\Big(\mathcal{K}(t,x)+\Lambda(p), \mathcal{K}(t,x)+\hat\Lambda^+(q)\Big).\nabla_x G \, dpdq \\
& + \int_{Q^2\times]0,1[^2} G \,\mathrm{sgn}_0^+\Big(\Lambda(p) - \hat\Lambda^+(q)\Big)\mathbf{f}'[\mathcal{K}(t,x)+\hat\Lambda^+(q)].\nabla_x\mathcal{K}(t,x)\, dpdq \\
= & \quad \int_{Q^2\times]0,1[^2} \Big(\Lambda^+(p) - \hat\Lambda^+(q)\Big)^+ \partial_t G \, dpdq \\
& - \int_{Q^2\times]0,1[^2} F^+\Big(\mathcal{K}(t,x)+\Lambda^+(p), \mathcal{K}(t,x)+\hat\Lambda^+(q)\Big).\nabla_x G \, dpdq \\
& + \int_{Q^2\times]0,1[^2} G \,\mathrm{sgn}_0^+\Big(\Lambda^+(p) - \hat\Lambda^+(q)\Big)\mathbf{f}'[\mathcal{K}(t,x)+\hat\Lambda^+(q)].\nabla_x\mathcal{K}(t,x)\, dpdq.
\end{aligned}$$

Using that u_2 is also a solution, with $k = \Lambda(p)^+$ and $y \mapsto G(t,s,x,y) \in \mathcal{D}(D)$, one gets, $dP-$a.s.,

$$\begin{aligned}
0 \quad \leq & \quad \int_{Q^2\times]0,1[^2} \Big(\Lambda^+(p) - \hat\Lambda(q)\Big)^+ \partial_s G \, dpdq + \int_{Q\times]0,1[}\int_D (\Lambda^+(p) - u_{2,0}(y))^+ G(t,x,0,y)dydp \\
& - \int_{Q^2\times]0,1[^2} F^+\Big(\mathcal{K}(s,y)+\Lambda^+(p), \mathcal{K}(s,y)+\hat\Lambda(q)\Big).\nabla_y G \, dpdq \\
& - \int_{Q^2\times]0,1[^2} G \,\mathrm{sgn}_0^+\Big(\Lambda^+(p) - \hat\Lambda(q)\Big)\mathbf{f}'[\mathcal{K}(s,y)+\Lambda^+(p)].\nabla_y\mathcal{K}(s,y)\, dpdq,
\end{aligned}$$

Next note

Lemma 8. *For any real a, b and c,*

$$\begin{aligned}
(a^+ - b)^+ &= (a-b^+)^+ + (-b)^+ = (a^+ - b^+)^+ + (-b)^+, \\
F^+(c+a^+, c+b) &= F^+(c+a^+, c+b^+) + F^+(c, c+b). \\
\mathbf{f}'(c+a^+)\mathbb{I}_{\{b<a^+\}} &= \mathbf{f}'(c)\mathbb{I}_{\{b<0\}} + \mathbf{f}'(c+a^+)\mathbb{I}_{\{b^+<a^+\}} - \mathbf{f}'(c)\mathbb{I}_{\{b<0<a\}}.
\end{aligned}$$

Therefore,

$$
\begin{aligned}
0 \;\leq\; & \int_{Q^2\times]0,1[^2} \Big\{ \Big(\Lambda^+(p)-\hat\Lambda^+(q)\Big)^+ \partial_s G - F^+\Big(\mathcal{K}(s,y)+\Lambda^+(p),\mathcal{K}(s,y)+\hat\Lambda^+(q)\Big).\nabla_y G\Big\}\, dpdq \\
& - \int_{Q^2\times]0,1[^2} G\,\mathrm{sgn}_0^+\Big(\Lambda^+(p)-\hat\Lambda(q)\Big)\mathbf{f}'[\mathcal{K}(s,y)+\Lambda^+(p)].\nabla_y\mathcal{K}(s,y)\,dpdq \\
& + \int_{Q\times]0,1[}\int_D (\Lambda^+(p)-u_{2,0}(y))^+ G(t,x,0,y)dydp \\
& + \int_{Q^2\times]0,1[^2} \Big\{ \Big(0-\hat\Lambda(q)\Big)^+ \partial_s G - F^+\Big(\mathcal{K}(s,y)+0,\mathcal{K}(s,y)+\hat\Lambda(q)\Big).\nabla_y G\Big\}\, dpdq \\
=\; & \int_{Q^2\times]0,1[^2} \Big\{ \Big(\Lambda^+(p)-\hat\Lambda^+(q)\Big)^+ \partial_s G - F^+\Big(\mathcal{K}(s,y)+\Lambda^+(p),\mathcal{K}(s,y)+\hat\Lambda^+(q)\Big).\nabla_y G\Big\}\, dpdq \\
& - \int_{Q^2\times]0,1[^2} G\,\mathrm{sgn}_0^+\Big(\Lambda^+(p)-\hat\Lambda^+(q)\Big)\mathbf{f}'[\mathcal{K}(s,y)+\Lambda^+(p)].\nabla_y\mathcal{K}(s,y)\,dpdq \\
& + \int_{Q\times]0,1[}\int_D (\Lambda^+(p)-u_{2,0}(y))^+ G(t,x,0,y)dydp \\
& + \int_{Q^2\times]0,1[^2} \Big\{ \Big(0-\hat\Lambda(q)\Big)^+ \partial_s G - F^+\Big(\mathcal{K}(s,y)+0,\mathcal{K}(s,y)+\hat\Lambda(q)\Big).\nabla_y G\Big\}\, dpdq \\
& - \int_{Q^2\times]0,1[^2} G\,\mathrm{sgn}_0^+\Big(0-\hat\Lambda(q)\Big)\mathbf{f}'[\mathcal{K}(s,y)].\nabla_y\mathcal{K}(s,y)\,dpdq \\
& + \int_{Q^2\times]0,1[^2} G\,\mathrm{sgn}_0^+\Big(\Lambda(p)\Big)\mathrm{sgn}_0^+\Big(0-\hat\Lambda(q)\Big)\mathbf{f}'[\mathcal{K}(s,y)].\nabla_y\mathcal{K}(s,y)\,dpdq \\
=\; & \int_{Q^2\times]0,1[^2} \Big\{ \Big(\Lambda^+(p)-\hat\Lambda^+(q)\Big)^+ \partial_s G - F^+\Big(\mathcal{K}(s,y)+\Lambda^+(p),\mathcal{K}(s,y)+\hat\Lambda^+(q)\Big).\nabla_y G\Big\}\, dpdq \\
& - \int_{Q^2\times]0,1[^2} G\,\mathrm{sgn}_0^+\Big(\Lambda^+(p)-\hat\Lambda^+(q)\Big)\mathbf{f}'[\mathcal{K}(s,y)+\Lambda^+(p)].\nabla\mathcal{K}(s,y)\,dpdq \\
& + \int_{Q\times]0,1[} \Big\{ \Big(0-\hat\Lambda(q)\Big)^+ \partial_s G_n - F^+\Big(\mathcal{K}(s,y)+0,\mathcal{K}(s,y)+\hat\Lambda(q)\Big).\nabla_y G_n\Big\}\, dq \\
& - \int_{Q\times]0,1[} G_n\mathrm{sgn}_0^+\Big(0-\hat\Lambda(q)\Big)\mathbf{f}'[\mathcal{K}(s,y)].\nabla_y\mathcal{K}(s,y)\,dq \\
& + \int_{\{\hat\Lambda<0<\Lambda\}} G\mathbf{f}'[\mathcal{K}(s,y)].\nabla_y\mathcal{K}(s,y)\,dpdq \\
& + \int_{Q\times]0,1[}\int_D (\Lambda^+(p)-u_{2,0}(y))^+ G(t,x,0,y)dydp \\
=\; & \int_{Q^2\times]0,1[^2} \Big\{ \Big(\Lambda^+(p)-\hat\Lambda^+(q)\Big)^+ \partial_s G - F^+\Big(\mathcal{K}(s,y)+\Lambda^+(p),\mathcal{K}(s,y)+\hat\Lambda^+(q)\Big).\nabla_y G\Big\}\, dpdq \\
& - \int_{Q^2\times]0,1[^2} G\,\mathrm{sgn}_0^+\Big(\Lambda^+(p)-\hat\Lambda^+(q)\Big)\mathbf{f}'[\mathcal{K}(s,y)+\Lambda^+(p)].\nabla_y\mathcal{K}(s,y)\,dpdq \\
& + \int_{Q\times]0,1[}\int_D [(\Lambda^+(p)-u_{2,0}(y))^+ - (0-u_{2,0}(y))^+]G(t,x,0,y)dydp \\
& + \int_0^1 <\hat\mu_0^-, G_n> d\beta + \int_{\{\hat\Lambda<0<\Lambda\}} G\mathbf{f}'[\mathcal{K}(s,y)].\nabla_y\mathcal{K}(s,y)\,dpdq.
\end{aligned}
$$

Therefore, with the first inequality, one gets that

$$\begin{aligned}
0 \ \leq\ & \int_{Q^2\times]0,1[^2} \Big\{ \Big(\Lambda^+(p) - \hat{\Lambda}^+(q)\Big)^+ (\partial_s + \partial_t) G \Big\}\, dpdq \\
& - \int_{Q^2\times]0,1[^2} \Big\{ F^+\Big(\mathcal{K}(s,y) + \Lambda^+(p), \mathcal{K}(s,y) + \hat{\Lambda}^+(q)\Big).\nabla_y G \\
& \qquad\qquad + F^+\Big(\mathcal{K}(t,x) + \Lambda^+(p), \mathcal{K}(t,x) + \hat{\Lambda}^+(q)\Big).\nabla_x G \Big\}\, dpdq \\
& + \int_{Q^2\times]0,1[^2} G\, \mathrm{sgn}_0^+\Big(\Lambda^+(p) - \hat{\Lambda}^+(q)\Big) \Big[\mathbf{f}'[\mathcal{K}(t,x) + \hat{\Lambda}^+(q)].\nabla_x \mathcal{K}(t,x) \\
& \qquad\qquad - \mathbf{f}'[\mathcal{K}(s,y) + \Lambda^+(p)].\nabla_y \mathcal{K}(s,y)\Big]\, dpdq \\
& + \int_0^1 < \hat{\mu}_0^-, G_n > d\beta + \int_{Q\times]0,1[} \int_D [(\Lambda^+(p) - u_{2,0}^+(y))^+ G(t,x,0,y) dydp \\
& + \int_{\{\hat{\Lambda}<0<\Lambda\}} G \mathbf{f}'[\mathcal{K}(s,y)].\nabla_y \mathcal{K}(s,y)\, dpdq.
\end{aligned}$$

i.e.

$$\begin{aligned}
0 \ \leq\ & \int_{Q^2\times]0,1[^2} \Big\{ \Big(\Lambda^+(p) - \hat{\Lambda}^+(q)\Big)^+ \partial_s \varphi(s,y) \rho_n(y-x) \rho_l(s-t) \Big\}\, dpdq \\
- & \int_{Q^2\times]0,1[^2} F^+\Big(\mathcal{K}(s,y) + \Lambda^+(p), \mathcal{K}(s,y) + \hat{\Lambda}^+(q)\Big).\nabla_y \varphi(s,y) \rho_n(y-x) \rho_l(s-t)\, dpdq \\
- & \int_{Q^2\times]0,1[^2} F^+\Big(\mathcal{K}(s,y) + \Lambda^+(p), \mathcal{K}(s,y) + \hat{\Lambda}^+(q)\Big) \varphi(s,y) \nabla_y \rho_n(y-x) \rho_l(s-t)\, dpdq \\
+ & \int_{Q^2\times]0,1[^2} F^+\Big(\mathcal{K}(t,x) + \Lambda^+(p), \mathcal{K}(t,x) + \hat{\Lambda}^+(q)\Big) \varphi(s,y) \nabla_y \rho_n(y-x) \rho_l(s-t)\, dpdq \\
& + \int_{Q^2\times]0,1[^2} G\, \mathrm{sgn}_0^+\Big(\Lambda^+(p) - \hat{\Lambda}^+(q)\Big) \Big[\mathbf{f}'[\mathcal{K}(t,x) + \hat{\Lambda}^+(q)].\nabla_x \mathcal{K}(t,x) \\
& \qquad\qquad - \mathbf{f}'[\mathcal{K}(s,y) + \Lambda^+(p)].\nabla_y \mathcal{K}(s,y)\Big]\, dpdq \\
& + \int_0^1 < \hat{\mu}_0^-, G_n > d\beta + \int_{Q\times]0,1[} \int_D [(\Lambda^+(p) - u_{2,0}^+(y))^+ G(t,x,0,y) dydp \\
& + \int_{\{\hat{\Lambda}<0<\Lambda\}} G \mathbf{f}'[\mathcal{K}(s,y)].\nabla_y \mathcal{K}(s,y)\, dpdq.
\end{aligned}$$

1. First, the Lebesgue set properties ensure that

$$\begin{aligned}
& \lim_{n\to\infty} \lim_{l\to\infty} \Big[E \int_{Q^2\times]0,1[^2} \Big\{ \Big(\Lambda^+(p) - \hat{\Lambda}^+(q)\Big)^+ \partial_s \varphi(s,y) \rho_n(y-x) \rho_l(s-t) \Big\}\, dpdq \\
& - E \int_{Q^2\times]0,1[^2} F^+\Big(\mathcal{K}(s,y) + \Lambda^+(p), \mathcal{K}(s,y) + \hat{\Lambda}^+(q)\Big).\nabla_y \varphi(s,y) \rho_n(y-x) \rho_l(s-t)\, dpdq \Big] \\
& = \ E \int_{Q\times]0,1[^2} \Big\{ (\Lambda^+ - \hat{\Lambda}^+)^+ \partial_t \varphi - F^+\Big(\mathcal{K} + \Lambda^+, \mathcal{K} + \hat{\Lambda}^+\Big).\nabla \varphi \Big\}\, dpd\beta.
\end{aligned}$$

2. The expectation of the third and the fourth terms vanish following the same arguments as the one proposed in the previous section.

3. Then, since $\hat{\mu}_0^-$ is a Radon measure in $\mathbb{R}^{d+1}$, the theorem of monotone convergence ensures that

$$\lim_{n\to\infty}\lim_{l\to\infty}\int_0^1 < \hat{\mu}_0^-, G_n > d\beta = \int_0^1 < \hat{\mu}_0^-, \psi > d\beta.$$

4. Denote by $\phi(t,x,y) = \int_t^T \rho_l(-r)dr\rho_n(y-x)\varphi(0,y) = \int_{\inf(t,\frac{2}{l})}^{\frac{2}{l}} \rho_l(-r)dr\rho_n(y-x)\varphi(0,y)$. Since ϕ is a non-negative function of $\mathcal{D}([0,T[\times D)$ for any y in D as soon as n is large enough, and as u_1 is a solution, one gets that

$$\begin{aligned}
&\int_{D\times Q\times]0,1[}\Big\{(\Lambda^+ - u_{2,0}^+(y))^+\rho_l(-t)\rho_n(y-x)\varphi(0,y)\Big\}\,dpdy\\
=\;&\int_{D\times Q\times]0,1[}\Big\{(\Lambda - u_{2,0}^+(y))^+\rho_l(-t)\rho_n(y-x)\varphi(0,y)\Big\}\,dpdy\\
\leq\;&\int_{D\times Q\times]0,1[}\int_{\inf(t,\frac{2}{l})}^{\frac{2}{l}}\rho_l(-r)dr\Big\{-F^+(\mathcal{K}+\Lambda,\mathcal{K}+u_{2,0}^+(y)).\nabla_y\rho_n(y-x)\varphi(0,y)\Big\}\,dpdy\\
+&\int_{D\times Q\times]0,1[}\int_{\inf(t,\frac{2}{l})}^{\frac{2}{l}}\rho_l(-r)dr\rho_n(y-x)\varphi(0,y)\mathrm{sgn}_0^+(\Lambda - u_{2,0}^+(y))\,\mathrm{div}\,\mathbf{f}[\mathcal{K}+u_{2,0}(y)]\,dpdy\\
&+\int_{D^2}(u_{1,0}(x)-u_{2,0}^+(y))^+\int_0^{\frac{2}{l}}\rho_l(-r)dr\rho_n(y-x)\varphi(0,y)\,dxdy.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\limsup_{n\to\infty}\limsup_{l\to\infty} E\int_{D\times Q\times]0,1[}\Big\{(\Lambda^+ - u_{2,0}^+(y))^+\rho_l(-t)\rho_n(y-x)\varphi(0,y)\Big\}\,dpdy\\
\leq\;&\int_D(u_{1,0}-u_{2,0}^+)^+\varphi(0,.)\,dx = \int_D(u_{1,0}^+-u_{2,0}^+)^+\varphi(0,.)\,dx
\end{aligned}$$

5. $\displaystyle\int_{\{\hat{\Lambda}<0<\Lambda\}} G\mathbf{f}'[\mathcal{K}(s,y)].\nabla_y\mathcal{K}(s,y)\,dpdq$

$$\begin{aligned}
&= \int_{Q\times]0,1[}\mathbb{I}_{\{\hat{\Lambda}(q)<0\}}\mathbf{f}'[\mathcal{K}(s,y)].\nabla_y\mathcal{K}(s,y)\varphi(s,y)\int_{Q\times]0,1[}\mathbb{I}_{\{\Lambda(p)>0\}}\rho_l(s-t)\rho_n(y-x)\,dpdq\\
&\quad\underset{l\to\infty,n\to\infty}{\longrightarrow}\int_{Q\times]0,1[^2}\mathbb{I}_{\{\hat{\Lambda}<0<\Lambda\}}\mathbf{f}'[\mathcal{K}].\nabla\mathcal{K}\varphi\,dpd\beta,
\end{aligned}$$

Therefore we may conclude

$$\begin{aligned}
0\;\leq\;& E\int_{Q\times]0,1[^2}\Big\{(\Lambda^+ - \hat{\Lambda}^+)^+\partial_t\varphi - F^+\Big(\mathcal{K}+\Lambda^+,\mathcal{K}+\hat{\Lambda}^+\Big).\nabla\varphi\Big\}\,dpd\beta\\
&+\int_0^1 < \hat{\mu}_0^-, \psi > d\beta + E\int_{\{\hat{\Lambda}<0<\Lambda\}}\mathbf{f}'[\mathcal{K}].\nabla\mathcal{K}\varphi\,dpd\beta + \int_D(u_{1,0}^+ - u_{2,0}^+)^+\varphi(0,.)\,dx.
\end{aligned}$$

Now, repeating the same arguments with u_1 replaced by $-u_2$, u_2 by $-u_1$, $\mathbf{f}$ by $-\mathbf{f}(-.)$, h by $-h$ and for the initial conditions $-u_{2,0}$ and $-u_{1,0}$, lead to the inequality:

$$\begin{aligned}0 \quad \leq \quad & E\int_{Q\times]0,1[^2} \Big(\hat{\Lambda}^- - \Lambda^-\Big)^+ \partial_t\varphi - F^+\Big(\mathcal{K}-\Lambda^-, \mathcal{K}-\hat{\Lambda}^-\Big).\nabla\varphi \, dpd\beta \\ & + \int_0^1 < \mu_0^+, \psi > d\alpha - E\int_{\{\hat{\Lambda}<0<\Lambda\}} G\mathbf{f}'[\mathcal{K}].\nabla\mathcal{K}\, dpd\beta + \int_D (u_{2,0}^- - u_{1,0}^-)^+\varphi(0,.)\, dx.\end{aligned}$$

Summing up these two inequalities, one gets that

$$\begin{aligned}0 \quad \leq \quad & E\int_{Q\times]0,1[^2} \Big\{(\Lambda-\hat{\Lambda})^+\partial_t\varphi - F^+\Big(\mathcal{K}+\Lambda, \mathcal{K}+\hat{\Lambda}\Big).\nabla\varphi\Big\}\, dpd\beta \\ & + \int_0^1 < \hat{\mu}_0^-, \psi > d\beta + \int_0^1 < \mu_0^+, \psi > d\alpha + \int_D (u_{1,0} - u_{2,0})^+\varphi(0,.)\, dx.\end{aligned}$$

Remind that $(\varphi\sigma_m)_m \subset \mathcal{D}([0,T[\times D)$ with $\varphi\sigma_n = \varphi\sigma_m\sigma_n$ for m large enough.

Thus, on the one hand, thanks to the proposition 2, one has that

$$\begin{aligned}0 \quad \leq \quad & E\int_{Q\times]0,1[^2} \Big(u_1(t,x,\alpha) - u_2(t,x,\beta)\Big)^+ \partial_t\varphi\sigma_m \, dxdtd\alpha d\beta \\ & -E\int_{Q\times]0,1[^2} F^+\Big(u_1(t,x,\alpha), u_2(t,x,\beta)\Big).\nabla[\varphi\sigma_m]\, dxdtd\alpha d\beta.\end{aligned}$$

On the other hand, one has

$$\begin{aligned}0 \quad \leq \quad & E\int_{Q\times]0,1[^2} \Big\{(u_1(t,x,\alpha) - u_2(t,x,\beta))^+\partial_t\varphi(1-\sigma_m) - \\ & \qquad\qquad F^+\Big(u_1(t,x,\alpha), u_2(t,x,\beta)\Big).\nabla[\varphi(1-\sigma_m)]\Big\}\, dpd\beta \\ & + \int_0^1 < \hat{\mu}_0^-, \psi(1-\sigma_m) > d\beta + \int_0^1 < \mu_0^+, \psi(1-\sigma_m) > d\alpha \\ & \qquad\qquad + \int_D (u_{1,0} - u_{2,0})^+\varphi(0,x)(1-\sigma_m)\, dx.\end{aligned}$$

Thus, for any n,

$$\begin{aligned}0 \quad \leq \quad & E\int_{Q\times]0,1[^2} \Big\{(u_1(t,x,\alpha) - u_2(t,x,\beta))^+\partial_t\varphi \\ & \qquad\qquad -F^+\Big(u_1(t,x,\alpha), u_2(t,x,\beta)\Big).\nabla\varphi\Big\}\, dpd\beta \\ & + \int_0^1 < \hat{\mu}_0^-, \psi(1-\sigma_m) > d\beta + \int_0^1 < \mu_0^+, \psi(1-\sigma_m) > d\alpha \\ & \qquad\qquad + \int_D (u_{1,0} - u_{2,0})^+\varphi(0,x)(1-\sigma_m)\, dx.\end{aligned}$$

Since $G_n(1-\sigma_m) = \varphi\sigma_n - \varphi\sigma_m\sigma_n = 0$ for m large,

$$\begin{aligned}
&\int_0^1 < \hat{\mu}_0^-, \psi(1-\sigma_m) > d\beta \\
= \quad & \lim_{n\to\infty} \int_{Q\times]0,1[} \Big\{ \Big(0 - \hat{\Lambda}(q)\Big)^+ \partial_s G_n(1-\sigma_m) \\
& \qquad\qquad - F^+\Big(\mathcal{K}(s,y) + 0, \mathcal{K}(s,y) + \hat{\Lambda}(q)\Big).\nabla_y[G_n(1-\sigma_m)]\Big\}\, dq \\
& - \int_{Q\times]0,1[} G_n(1-\sigma_m)\mathrm{sgn}_0^+\Big(0 - \hat{\Lambda}(q)\Big)\mathbf{f}'[\mathcal{K}(s,y)].\nabla_y\mathcal{K}(s,y)\, dq \\
& \qquad\qquad + \int_D (-u_0)^+ G_n(0,x)(1-\sigma_m)dx \\
= \quad & 0.
\end{aligned}$$

Then, using the partition of unity, the result holds.

3.3 Uniqueness of the measure-valued solution, existence of the solution

Proposition 4. *The measure-valued solution is unique. Moreover, it is the unique entropy solution.*

Proof. Since for any positive φ in $H^1(Q)$,

$$\begin{aligned}
0 \quad \leq \quad & E\int_{Q\times]0,1[^2} \Big(u_1(t,x,\alpha) - u_2(t,x,\beta)\Big)^+ \partial_t\varphi\, dxdtd\alpha d\beta \\
& -E\int_{Q\times]0,1[^2} F^+\Big(u_1(t,x,\alpha), u_2(t,x,\beta)\Big).\nabla\varphi\, dxdtd\alpha d\beta + \int_D (u_{1,0} - u_{2,0})^+\varphi(0)\, dx,
\end{aligned}$$

if $u_{1,0} = u_{2,0}$ and $\varphi(t,x) = T-t$, one gets that

$$0 \quad \geq \quad E\int_{Q\times]0,1[^2} \Big(u_1(t,x,\alpha) - u_2(t,x,\beta)\Big)^+ dxdtd\alpha d\beta,$$

and, by permutation of the solutions,

$$0 \quad \geq \quad E\int_{Q\times]0,1[^2} \Big(u_2(t,x,\alpha) - u_1(t,x,\beta)\Big)^+ dxdtd\alpha d\beta.$$

Therefore, on the one hand, the uniqueness of the measure-valued solution is proved and, on the other hand, $u_1(t,x,\alpha) = u_2(t,x,\beta)$ for a.e α and β ensures that the solution does not depend on α or β. $\square$

Proposition 5. *Moreover, entropy solutions satisfy a comparison and a contraction principle:*

1. *If $u_{1,0} \leq u_{2,0}$ then $u_1 \leq u_2$ a.e. on Q, a.s. on Ω.*

2. $E \int_Q |u_1 - u_2| \, dxdt \leq \int_D |u_{1,0} - u_{2,0}| \, dx.$

Proof. The first part of the proposition is proved in the same way as Proposition 4.

For any positive φ in $H^1(0,T)$ with $\varphi(T) = 0$, one has that

$$\begin{aligned} 0 &\leq \int_0^T E \int_D (u_1 - u_2)^+ \, dx \varphi' \, dt + \int_D (u_{1,0} - u_{2,0})^+ \varphi(0) \, dx \\ &= \int_0^T E \int_D [(u_1 - u_2)^+ - (u_{1,0} - u_{2,0})^+] \, dx \varphi' \, dt, \end{aligned}$$

and the second assertion follows. □

3.4 A remark about locally Lipschitz f

Assume in this section that $\mathbf{f}$ is merely a locally Lipschitz-continuous function with a Lipschitz-continuous derivative $\mathbf{f}'$. Then, one has in particular:

1. $\exists C(f) > 0$, $\forall x \in \mathbb{R}$, $|f_i(x)| \leq C(f)(x^2 + 1)$ for any $i \in \{1, .., d\}$.
2. By Sobolev embedding, $\forall w \in H^1(D)$, $\mathbf{f}(w) \in [L^p(D)]^d$ for some $p > 1$.
3. By truncation arguments, $\forall w \in H^1(D)$, $\mathbf{f}'(w)\nabla w \in L^p(D)$ for some $p > 1$ and the chain rule holds: $\operatorname{div} \mathbf{f}(w) = \mathbf{f}'(w)\nabla w$.

In this case, the definition of a solution has to be slightly modified in order to give sense to the integrals: the test-functions φ need to belong to $\mathcal{D}(\overline{Q})$ instead of $H^1(Q)$ or to $\mathcal{D}([0,T] \times D)$ instead of $L^2(0,T; H_0^1(D))$.

The result of uniqueness holds in the same way with such $\mathbf{f}$, as well as the main part of the demonstrations of the existence section. If one assumes again the existence of the solution to the viscous problem (2), it remains to prove the property of uniform integrability of the sequence $(\mathbf{f}(u_n))$ needed when one passes to the limit in the first term of $I_{3,\eta}$ in (3). The aim of the following lemma is to propose two different possible assumptions for that.

Lemma 9. *If one of the following assumptions holds:*

$\mathbf{H}_3$ $\exists \delta \in]0, 2[$, $\exists C > 0$, *such that* $\forall x \in \mathbb{R}$, $|f(x)| \leq C(|x|^\delta + 1)$
or

$\mathbf{H}_4$ $\exists p_0 > 2$, $h \in L^{p_0}(0, T; H^1(D))$ *and* $u_0 \in L^{p_0}(D)$

then, $(f(u_n))$ *is uniformly integrable.*

Proof. If H_3 is assumed, the sequence is uniformly integrable since it is bounded in $L^{\frac{2}{\delta}}(]0, T[\times \Omega \times D)$ with $\frac{2}{\delta} > 1$.

If H_4 is assumed, thanks to the Sobolev embedding, there exists $2 < p \leq p_0$ such that $h \in L^p(Q)$ and $u_0 \in L^p(D)$. Then, for any positive M, Ito's formula leads to

$$E \int_D \phi_M(u_n(t))\, dx \leq \frac{1}{2} E \int_0^t \int_D h^2 \phi_M''(u_n(s))\, dxds + E \int_D \phi_M(u_0)\, dx$$

where $\phi_M(t) = p(p-1) \int_0^{|t|} \int_0^r \inf(M, \sigma^{p-2}) d\sigma dr$. Thus, one gets that

$$\begin{aligned}
& E \int_D \phi_M(u_n(t))\, dx \\
\leq\ & \frac{1}{p} E \int_0^t \int_D h^p\, dxds + \frac{p-2}{2p} E \int_0^t \int_D \inf(M^{\frac{p}{p-2}}, |u_n|^p(s))\, dxds + \int_D |u_0|^p\, dx \\
\leq\ & C + CE \int_0^t \int_D \phi_M(u_n(s))\, dxds.
\end{aligned}$$

Then, the lemma of Gronwall and the theorem of Beppo Levi ensure that (u_n) is a bounded sequence in $L^\infty(]0,T[; L^p(\Omega \times D))$. Since for any real x, $|f(x)| \leq C(f)(x^2+1)$, the result holds. □

4 A basic reminder of Young measures

In this section we recall some basic facts on Young measures and refer to E. J. Balder[3], Ch. Castaing *et al.*[10], R. Eymard *et al.*[14], E. Y. Panov[23], M. Saadoune *et al.*[25] and M. Valadier[26] for more information.

Consider the space $L^1(\Omega, \mu, \mathbb{R})$ where $(\Omega, \mathcal{F}, \mu)$ is a measure space with a positive bounded measure μ.
For u in $L^1(\Omega, \mu, \mathbb{R})$, the Young measure associated with u is τ_u, the measure on $\Omega \times \mathbb{R}$ image of μ by $x \mapsto (x, u(x))$.
A general Young measure τ is a positive measure on $\Omega \times \mathbb{R}$ such that, for any A in $\mathcal{F}$, $\tau(A \times \mathbb{R}) = \mu(A)$.
A Young measure τ is described by its disintegration which is the unique family of probabilities on $\mathbb{R}$, $(d\tau_x)_{x\in\Omega}$, such that for any $\tau-$measurable function ψ ,

$$x \mapsto \int_{\mathbb{R}} \psi(x,\lambda)\, d\tau_x(\lambda) \text{ is } \mu-\text{measurable on } \Omega \text{ and}$$

$$\text{if } \psi \geq 0,\ \int_{\Omega\times\mathbb{R}} \psi\, d\tau = \int_\Omega \int_{\mathbb{R}} \psi(x,\lambda)\, d\tau_x(\lambda)\, \mu(dx)\ .$$

Therefore, if $\tau = \tau_u$ is the Young measure associated with the above function u, then $\tau_x = \delta_{u(x)}$, the Dirac mass at $u(x)$.

Another way to define Young measures on $\Omega \times \mathbb{R}$ is to consider the notion of entropy process proposed by Th. Gallouët[14] or E. Yu. Panov[23]. For a

Young measure τ on $\Omega\times\mathbb{R}$ and F_x the repartition function of τ_x, one considers the function u, defined in $\Omega\times]0,1[$ by :

$$u(x,\alpha)=\inf\{t\in\mathbb{R}, F_x(t)>\alpha\}.$$

It is a $\mu\times\mathcal{L}$ measurable function on $\Omega\times]0,1[$ and for any positive Carathéodory function ψ,

$$\int_{\Omega\times\mathbb{R}}\psi\ d\tau=\int_\Omega\int_\mathbb{R}\psi(x,\lambda)\ d\tau_x(\lambda)\ \mu(dx)=\int_\Omega\int_0^1\psi(x,u(x,\alpha))\ d\alpha\ \mu(dx).$$

A sequence of Young measure $(\tau^n)_n$ is said to converge narrowly towards τ if $\int_{\Omega\times\mathbb{R}}\psi\ d\tau^n$ converges towards $\int_{\Omega\times\mathbb{R}}\psi\ d\tau$ for all bounded Carathéodory function ψ.

Consider now $(u_n)_n\subset L^1(\Omega,\mu,\mathbb{R})$ and denote by τ^n the associated Young measures.
If the sequence $(u_n)_n$ is assumed to be bounded in $L^1(\Omega)$, the theorem of Prohorov for Young measures (E. J. Balder[3], M. Saadoune *et al.*[25] and M. Valadier[26]) ensures that a sub-sequence $(\tau^{n_k})_k$ of $(\tau^n)_n$ and a Young measure τ exist such that τ^{n_k} converges narrowly towards τ.
Moreover:
i) for μ-a.e. x in Ω, $supp(d\tau_x)\subset\cap_{p=1}^\infty\overline{\cup_{n\geq p}\{u_n(x)\}}$
ii) for any Carathéodory function ψ such that the sequence of functions $\{\psi(.,u_n(.))\}_n$ is uniformly integrable,

$$\int_\Omega\psi(x,u_n(x))\ \mu(dx)\to\int_{\Omega\times\mathbb{R}}\psi(x,\lambda)\ d\tau$$

(if the sequence $(u_n)_n$ is uniformly integrable, the above convergence still holds if one assumes that $|\psi(x,\lambda)|\leq\alpha(x)+k|\lambda|$ where $k\geq 0$ and $\alpha\in L^1(\Omega)$).
iii) for any measurable function ψ, l.s.c. with respect to its second variable and such that $\{\psi(.,u_n(.))^-\}_n$ is uniformly integrable,

$$\liminf_{n\to\infty}\int_\Omega\psi(x,u_n(x))\ \mu(dx)\geq\int_{\Omega\times\mathbb{R}}\psi(x,\lambda)\ d\tau.$$

As a consequence, if u_n converges weakly to some u in L^1, it converges strongly to u in L^1, if and only if τ^n converges narrowly to τ_u.

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