Singularly perturbed ODEs and profiles for stationary symmetric Euler and Navier-Stokes shocks

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Abstract

We construct stationary solutions to the non-barotropic, compressible Euler and Navier-Stokes equations in several space dimensions with spherical or cylindrical symmetry. The equation of state is assumed to satisfy standard monotonicity and convexity assumptions. For given Dirichlet data on a sphere or a cylinder we first construct smooth and radially symmetric solutions to the Euler equations in an exterior domain. On the other hand, stationary smooth solutions in an interior domain necessarily become sonic and cannot be continued beyond a critical inner radius. We then use these solutions to construct entropy-satisfying shocks for the Euler equations in the region between two concentric spheres (or cylinders).

Next we construct smooth solutions w^ε to the Navier-Stokes system converging to the previously constructed Euler shocks in the small viscosity limit $\varepsilon \to 0$. The viscous solutions are obtained by a new technique for constructing solutions to a class of two-point boundary problems with a fast transition region. The construction is explicit in the sense that it produces high order expansions in powers of ε for w^ε, and the coefficients in the expansion satisfy simple, explicit ODEs, which are *linear* except in the case of the leading term. The solutions to the Euler equations described above provide the slowly varying contribution to the leading term in the expansion.

The approach developed here is applicable to a variety of singular perturbation problems, including the construction of heteroclinic orbits with fast transitions. For example, a variant of our method is used in [W] to give a new construction of detonation profiles for the reactive Navier-Stokes equations.

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1 Introduction

This article extends to non-barotropic flows the constructions for the barotropic case in the companion paper [EJW] of stationary solutions to the Euler and Navier-Stokes systems with spherical or cylindrical symmetry. The physical setup is the same: we consider the domain between two concentric spheres or cylinders $r = a$ and $r = b$, where $a < b$, into which a compressible fluid is injected with a prescribed constant density ρ_a, velocity $\mathbf{U}_a$ and temperature θ_a at the inner boundary $r = a$. Depending on how fast the fluid is allowed to exit at the outer boundary, fluid may or may not accumulate in the interior and a shock may or may not form. Similarly, one can consider the case where fluid is injected radially at the outer boundary, or the cases where spheres are replaced by cylinders.

1.1 Euler shocks.

In order to build stationary shock solutions in the spherically symmetric (SS) case, we first construct *inner solutions*, that is, smooth solutions defined everywhere in the exterior $r \geq a$ of a sphere $r = a$ with data (ρ_a, u_a, θ_a) prescribed at the inner boundary, and *outer solutions*, which are smooth and defined inside $r = b$ when data (ρ_b, u_b, θ_b) is prescribed at the outer boundary. We find that inner solutions remain subsonic (resp., supersonic) everywhere if they are subsonic (resp., supersonic) at $r = a$. A similar result holds for outer solutions, with the difference that there is a critical inner radius at which the flow becomes sonic and beyond which the stationary solution cannot be extended. In the cylindrically symmetric (CS) case we allow swirling flows with nonzero angular (v) and axial (w) components. However, only the *radial* Mach number is relevant for classifying solutions and for determining the critical radius in the case of outer solutions. The main results on inner and outer solutions are summarized in Propositions 2.2 and 2.3.

In section 3 we show how to build symmetric, entropy-satisfying shock solutions to the Euler equations by using the inner or outer solutions from Section 2. These results are summarized in Theorems 3.1 and 3.2. The last part of section 3 addresses the following issue: Taking a, b, and data at $r = a$ as fixed, formulate necessary and sufficient conditions on the flow variables at $r = b$ that guarantee existence of a stationary, weak solution of the barotropic Euler equations with these boundary values, and which contains a single shock at *some* location $\bar{r} \in (a, b)$. We formulate the answer in Theorem 3.3 in terms of possible values for the density at $r = b$.

Remark 1.1. *(1) In the inviscid case the specific entropy is constant throughout smooth regions. In this sense the situation reduces to the barotropic case already considered in [EJW], and we use part of that analysis to establish existence of smooth and stationary profiles for the full system. On the other hand, the Rankine-Hugoniot relations are genuinely different in the full case we consider here and this necessitates a separate analysis of the inviscid shock solutions.*

(2) The case of smooth, inviscid flow without swirl of an ideal polytropic gas in a cone was analyzed in [CF], pp. 377-380. Below we extend the analysis to more general equations of state, and in the CS case we also consider flows with swirl.

(3) Chen and Glimm [CG1] - [CG2] performed a detailed local analysis of stationary shocks for isentropic *flow. In these works the shock solutions serve as building blocks in a Godunov type scheme. A similar analysis does not seem to have been carried out for the full system.*

(4) The spectral stability of the inviscid symmetric shocks constructed in section 3 has been analyzed in [Cos].

1.2 Navier-Stokes shocks.

The goal of sections 4 and 5 is to construct smooth Navier-Stokes solutions converging to the previously constructed Euler shocks in the small viscosity limit. We focus now and in those sections on the spherically symmetric case with prescribed supersonic inflow at $r = a$ and subsonic outflow at $r = b$. (The same arguments treat the cylindrically symmetric case as we explain in Remark 5.19.) We assume we are given an inviscid shock taking values (ρ_a, u_a, θ_a) at $r = a$ and (ρ_b, u_b, θ_b) at $r = b$, and we seek solutions to the second-order viscous equations on $[a, b]$, which assume these boundary values for each fixed viscosity ϵ, and which converge (in an appropriate sense) to the given inviscid shock as $\epsilon \to 0$.

It is clear that smooth viscous solutions converging in any reasonable sense to a discontinuous, inviscid shock will have to exhibit a fast "shock-layer" transition region near the shock. As viscosity $\varepsilon \to 0$ this region becomes thinner as the transition becomes faster. In the case where the limiting inviscid shock is planar, the inviscid profile consists of two constant states; for curved shocks the inviscid profile is nonconstant, and this greatly complicates the construction of viscous profiles converging to the inviscid profile as $\varepsilon \to 0$.

Classical two-point boundary theory [H, BSW, DH, K] appears unsuitable for dealing with problems in which such fast interior transitions occur. In this paper as in [EJW], where the original viscous system in the spherically symmetric case was 2×2, the construction of viscous solutions is based on a new approach using conjugations (to separate slow and fast variables and, crucially, to *remove* fast variables), a splitting of boundary conditions between right and left endpoints, and matching arguments. We review that approach below, point out the new difficulties associated with larger systems like the non-barotropic Navier-Stokes equations, and briefly describe the new features introduced here to deal with those difficulties.

To anchor the discussion we first give a statement of our main result for the non-barotropic SS case when the underlying inviscid shock is built from inner solutions (see Proposition 3.1); it is an immediate consequence of Theorem 5.18. The stationary SS Euler equations are stated at the beginning of section 2. The corresponding Navier-Stokes equations are given in (4.1)-(4.4) (see also (1.27)-(1.31)).

Theorem 1.1. *Let $U^0(r)$, $r \in [a,b]$, be a piecewise C^1, stationary, spherically symmetric shock solution to the nonbarotropic, spherically symmetric Euler equations with supersonic inflow at $r = a$ and shock surface at $r = \overline{r}$. Assume the viscosity coefficients in the Navier-Stokes equations satisfy*

$$\nu = \varepsilon\underline{\nu},\ \mu = \varepsilon\underline{\mu},\ \kappa = \varepsilon\underline{\kappa}. \tag{1.1}$$

We make the standard thermodynamic assumptions stated in section 1.3. Then there is a family of C^1 solutions w^ε of the stationary SS Navier-Stokes equations satisfying for any $\beta > 0$, some $\varepsilon_0 > 0$, and $0 < \varepsilon \leq \varepsilon_0$:

$$\begin{aligned}
&\lim_{\varepsilon \to 0} w^\varepsilon(r) = U^0(r) \text{ in } L^p([a,b]),\ 1 \leq p < \infty \\
&\lim_{\varepsilon \to 0} w^\varepsilon(r) = U^0(r) \text{ in } L^\infty([a,b] \cap \{|r - \overline{r}| \geq \beta\}), \\
&w^\varepsilon(a) = U^0(a) + O(\varepsilon),\ \ w^\varepsilon(b) = U^0(b).
\end{aligned} \tag{1.2}$$

The solutions w^ε have a high order expansion of the form (1.4), (1.6), *where the profiles U^j, V^j defining the approximate solution $\tilde{w}^\varepsilon$ satisfy explicit and, for $j \geq 1$,* linear *profile equations.*

Similar results hold for the CS case as explained in Remark 5.19, and for the case where the inviscid shock is built from outer solutions (Proposition 3.2).

In section 4 after the change of variables $s = r - \overline{r}$, where $r = \overline{r}$ ($a < \overline{r} < b$) is the inviscid shock location, we reformulate the stationary Navier-Stokes equations as a second-order, 3×3 *transmission problem* on the bounded interval $[a - \overline{r}, b - \overline{r}]$. The unknowns are $w = (\rho, u, \theta) = (\rho_\pm(s), u_\pm(s), \theta_\pm(s))$ in $\pm s \geq 0$ and transmission conditions at $s = 0$ are given by

$$[\rho, u, \theta] = 0,\ [u_s, \theta_s] = 0, \tag{1.3}$$

where, for example, $[u] := u_+(0) - u_-(0)$. It is natural to adopt the transmission formulation since different analyses are needed in the regions $s \leq 0$ and $s \geq 0$ as a consequence of the fact that the flow changes from supersonic to subsonic at $s = 0$. The equation (4.8)(a) implies that solutions on $\pm s \geq 0$ satisfying the matching conditions (1.3) are actually at least C^2 on $[a - \overline{r}, b - \overline{r}]$. As explained below we shall later need to further subdivide $[a - \overline{r}, b - \overline{r}]$ and impose transmission conditions at $s = \pm\delta$ for some small enough $\delta > 0$. Boundary conditions for w are now imposed at $s = a - \overline{r}$ and $s = b - \overline{r}$ as in (4.9).

Writing $w_\pm = (w^1, w^2) := (\rho, (u, \theta))$ (and suppressing ε and some $\pm$), in Proposition 4.2 we construct high-order *approximate* solutions to the transmission problem,

$$\tilde{w}^\epsilon(s) = \left(\mathcal{U}^0(s,z) + \epsilon\mathcal{U}^1(s,z) + \cdots + \epsilon^M \mathcal{U}^M(s,z)\right)|_{z=\frac{s}{\epsilon}}, \tag{1.4}$$

where $\mathcal{U}^j(s,z) = U^j(s) + V^j(z)$, $V^j(z) \to 0$ exponentially fast as $z \to \pm\infty$, and $U^0(s)$ is the given inviscid shock. The functions $\tilde{w}^\varepsilon$ satisfy the Navier-Stokes transmission problem to order $O(\varepsilon^M)$ and clearly satisfy, for small $\delta > 0$,

$$\begin{aligned} &\tilde{w}^\epsilon(s) \to U^0(s) \text{ in } L^\infty(|s| \geq \delta) \text{ as } \varepsilon \to 0, \text{ while} \\ &\tilde{w}^\epsilon(s) \to U^0(s) \text{ in } L^p(|s| \leq \delta), 1 \leq p < \infty. \end{aligned} \tag{1.5}$$

Observe that the terms $V^j(\frac{s}{\epsilon})$ describe the fast transition in the viscous solutions that occurs near the inviscid shock front at $s = 0$.

Next we look for an exact solution $w^\epsilon(s)$ to the transmission problem that is close to the approximate solution. We seek w^ϵ in the form

$$w^\epsilon(s) = \tilde{w}^\epsilon(s) + \epsilon^L v^\epsilon(s),\ 1 \leq L < M, \tag{1.6}$$

where the v^ϵ satisfy an appropriate error problem (4.17) and turn out to be uniformly bounded in $L^\infty[a - \overline{r}, b - \overline{r}]$ as $\epsilon \to 0$. The second-order 3×3 problem for $v^\epsilon = (v^1, v^2)$ is written as a 5×5 first-order system for $V = (v^1, v^2, \epsilon v^2_s)$ (see (4.31)):

$$\begin{aligned} &(a)\ V_s = \frac{1}{\epsilon} GV + F \text{ on } [a - \overline{r}, b - \overline{r}] \\ &(b)\ [V] = 0 \text{ on } s = 0 \\ &(c)\ (v^1, v^2) = \overline{v} \text{ at } s = b - \overline{r}. \end{aligned} \tag{1.7}$$

where $\overline{v}$ is chosen so that $w^\varepsilon(b - \overline{r}) = \tilde{w}^\varepsilon(b - \overline{r}) + \varepsilon^L \overline{v} = (\rho_b, u_b, \theta_b)$. Note that $\overline{v}$ depends on ε, but here and elsewhere we shall often suppress this dependence in the notation.

There are two main obstacles to obtaining uniformly bounded solutions to (1.7) as $\epsilon \to 0$. The first is that the entries of the matrix $G = G(\tilde{w}^\epsilon + \epsilon^L v)$ are functions $g^{ij}(\frac{s}{\epsilon}, q^\epsilon(s))$ that undergo fast transitions near $s = 0$. The eigenvalues of G therefore exhibit similar behavior. If those eigenvalues, which are of size $O(1)$ for s near 0, had real parts that were of fixed sign and bounded away from zero, the factor of $\frac{1}{\epsilon}$ in front of G would not pose a serious problem. However, the fast transitions make those eigenvalues impossible to analyze directly. Moreover, they are expected to change sign near $s = 0$ as s and ε vary. Thus, the factor $\frac{1}{\epsilon}$ is a serious obstacle to obtaining uniform L^∞ estimates on $[a - \overline{r}, b - \overline{r}]$ that are independent of ε.

The second obstacle is the need to smoothly piece together the part of the solution in $|s| \geq \delta > 0$ that changes slowly and takes on prescribed boundary values at $s = b - \overline{r}$, with the part of the solution in $|s| \leq \delta$ that undergoes a fast transition. The matching is complicated by the need to use different conjugators in different subintervals, and also by the need to "split" boundary conditions between right and left endpoints in certain subintervals. Here the larger systems like non-barotropic Navier-Stokes (or even barotropic NS in the cylindrically symmetric case "with swirl") present new difficulties.

The matrix G in (1.7) can be written

$$G = G(z,q)|_{z=\frac{s}{\epsilon}, q=q^\epsilon(s)}, \tag{1.8}$$

where, roughly speaking, the first argument describes fast behavior, and the second argument slow behavior. A serious problem is that it is essentially impossible to understand in detail how the eigenvalues of $G(z,q)$ vary with z in any bounded neighborhood of $z = 0$. However, the exponential decay of $V^0(z)$ to 0 as $z \to \pm\infty$ implies that there exist limiting matrices $G(\pm\infty, q)$ to which $G(z,q)$ converges exponentially fast as $z \to \pm\infty$:

$$|G(z,q) - G(\pm\infty, q)| \leq Ce^{-\kappa|z|}, \text{ for some } C > 0, \kappa > 0. \tag{1.9}$$

We deal with the first of the obstacles described above (large, fast-varying eigenvalues that may cross the imaginary axis) by using a conjugation argument first introduced in [MZ], and also used in later papers such as [GMWZ1, GMWZ2], that allows us to replace the matrix $G(z,q)$ by $G(\pm\infty, q)$ when analyzing (1.7) on the fast transition subinterval $|s| \leq \delta$. This is possible because of the exponential decay in (1.9). For $\frac{s}{\varepsilon} = z \in [0,\infty]$, for example, one constructs a matrix $T_+(z,q)$, uniformly bounded along with its inverse, such that

$$\partial_z T_+(z,q) = G(z,q)T_+(z,q) - T_+(z,q)G(+\infty, q) \tag{1.10}$$

(see Lemma 5.4 and Remark 5.6). An immediate consequence of (1.10) is that $V(z)$ satisfies

$$d_z V = G(z,q)V + K \text{ on } [0,\infty] \tag{1.11}$$

for some function K if and only if W defined by $V = T_+ W$ satisfies

$$d_z W = G(+\infty, q)W + T_+^{-1} K \text{ on } [0,\infty]. \tag{1.12}$$

This reduction to studying (1.12) with its removal of the fast scale in G greatly simplifies the analysis of eigenvalues and the construction of solutions; the price is that the intervention of the conjugator T_+ complicates the problem of satisfying boundary and transmission conditions.

One observes readily using Lemma 4.7 that three of the eigenvalues of $G(\pm\infty, q^\epsilon(s))$ are $O(\epsilon)$, while the remaining two, denoted $\lambda_\pm$, are $O(1)$. (Note: the subscript on $\lambda_\pm$ refers to the $\pm$ appearing in the quadratic formula, *not* to the sign of s.) In Proposition 4.10 we show that in $s \leq 0$, $\lambda_\pm$ are both positive and bounded away from 0, while

$$\lambda_+(q^\varepsilon(s)) \geq c_1 > 0 \text{ and } \lambda_-(q^\varepsilon(s)) \leq c_2 < 0 \text{ in } s \geq 0. \tag{1.13}$$

As shown in Proposition 4.10, the change of sign of λ_- at $s=0$ reflects the transition from supersonic to subsonic flow across the inviscid shock. A second and more straightforward conjugation can then be used to reduce G to the block forms

$$G_{B\pm}(q^\epsilon(s)) = \begin{pmatrix} O(\epsilon) & 0 & 0 \\ 0 & \lambda_+(q^\epsilon(s)) + O(\epsilon) & 0 \\ 0 & 0 & \lambda_-(q^\epsilon(s)) + O(\epsilon) \end{pmatrix} \text{ on } \{|s| \leq \delta\} \cap \{\pm s \geq 0\}, \tag{1.14}$$

as in Proposition 5.5, where the $O(\varepsilon)$ block is 3×3. Observe that on $|s| \geq \delta$, $V^0(\frac{s}{\epsilon})$ is already negligible for ϵ small, so in that region the G matrix in (1.7) can be conjugated directly to the form (1.14) without a preliminary conjugation to remove the fast scale. We note that in the barotropic case the $O(\varepsilon)$ block was 2×2, while the lower right block was 1×1 and changed sign at $s=0$.

We deal with the second obstacle (matching slow and fast transition regions) by splitting the transmission problem (1.7) into four separate boundary problems labelled I, II, III, and IV on the subintervals $[a-\overline{r}, -\delta]$, $[-\delta, 0]$, $[0, \delta]$, and $[\delta, b-\overline{r}]$ respectively, and solving the problems in the order IV to I. Each subproblem has its own conjugator, S_1,...,S_4 (which depends on the unknown V). Define the unknown $\mathcal{V}_j \in \mathbb{R}^5$ for the $j-$th conjugated problem by the equation $V = S_j \mathcal{V}_j$, where $\mathcal{V}_j = (\nu^*, \nu_+^3, \nu_-^3)$ with respective components corresponding to the blocks of $G_{B\pm}$ (the $\pm$ subscript on G_B distinguishes $\pm s \geq 0$; also, $\nu^* = (\nu^1, \nu^2)$ with $\nu^1 \in \mathbb{R}$, $\nu^2 \in \mathbb{R}^2$). For example, in place of (1.7)(a) the problem satisfied by $\mathcal{V}_4$ is

$$\partial_s \mathcal{V}_4 = \frac{1}{\varepsilon} G_{B+} \mathcal{V}_4 - (S_4^{-1} \partial_s S_4) \mathcal{V}_4 + S_4^{-1} F \text{ on } [\delta, b-\overline{r}]. \tag{1.15}$$

For each of the four subproblems we prescribe boundary conditions in terms of unknown parameters that are later determined so that (1.7)(c) holds and so that $[V] = 0$ at the joining points $s = \delta$, 0, and $-\delta$ (see, for example, (5.30)). The ν^* component, which corresponds to the $O(\varepsilon)$ block in (1.14), can be prescribed at either the right or left endpoint in any given subproblem. In every case we choose the right endpoint. In problems I and II the $\nu_\pm^3$ components, which correspond to eigenvalues that are $O(1)$ and strictly positive, *must* be prescribed at the right endpoint in each case; prescription at the left endpoint would yield a solution that blows up as $\varepsilon \to 0$ like $e^{\frac{c}{\varepsilon}}$ for some $c > 0$. For a simple example of this consider the scalar ODE

$$\frac{dy}{dt} = \frac{a}{\varepsilon} y \text{ on } [0,1], \tag{1.16}$$

where $a > 0$ is constant. Imposing a boundary condition at $t = 0$ yields a solution $y_0 e^{\frac{at}{\varepsilon}}$ that blows up as $\varepsilon \to 0$, while a boundary condition at $t = 1$ gives the well-behaved solution $y_1 e^{\frac{a(t-1)}{\varepsilon}}$.

In problems III and IV we see from (1.13) that ν_+^3 must be prescribed at the right endpoint, while ν_-^3 must be prescribed at the left endpoint. Thus, in problems III and IV we have a splitting of boundary conditions between the two endpoints for the modes that correspond to $O(1)$ eigenvalues. In the barotropic SS case, where there is only a single $O(1)$ eigenvalue, there could be no such splitting.

The fact that the modes $\nu^3_\pm$ must now be prescribed at different endpoints in Problems III and IV creates new difficulties for the matching problem at $s = \delta$. These are dealt with in sections 5.1-5.4, where we show that the matching of solutions can be accomplished by fixed point arguments (Lemma 5.14 and Proposition 5.15), provided the conjugators S_3 and S_4 are themselves constructed to match to order ε^L, $L \geq 1$, at $s = \delta$:

$$S_3 - S_4 = O(\varepsilon^L) \text{ at } s = \delta. \tag{1.17}$$

This procedure based on matching conjugators is simpler and more widely applicable than the type of matching argument used in [EJW], which involved showing the existence of points of intersection of geometric structures in parameter space (a curve and a surface in $\mathbb{R}^3$ in the barotropic SS case). Indeed, matching conjugators are used again in [W] to construct the fast transition region in detonation profiles for the reactive Navier-Stokes equations.

The matching of solutions at $s = 0$ and $s = -\delta$ is relatively easy, since in problems I and II *all* components of $\mathcal{V}_j$ can be prescribed at the right endpoint. Thus, boundary data for $\mathcal{V}_2$ at $s = 0$ can be determined from $V_3(0)$, and boundary data for $\mathcal{V}_1(-\delta)$ can be determined from $V_2(-\delta)$.

Remark 1.2. *(1) In the barotropic CS case with swirl when $v \neq 0$ and $w \neq 0$, the matrices $G_{B\pm}$ are 7×7 with a 3×3 lower right block. That block has 3 strictly positive eigenvalues of size $O(1)$ in $s \leq 0$, but 2 positive and 1 negative in $s \geq 0$. This case (and the non-barotropic CS case) can be treated by the same arguments we use for the non-barotropic SS case. The barotropic CS case without swirl ($v = 0$ and $w = 0$) is similar to the barotropic SS case and was treated in [EJW].*

(2) The convergence of viscous shocks to piecewise smooth inviscid shocks has been studied in the one (space) dimensional case by, for example, [GX] and in the multidimensional case by [GMWZ1, GMWZ2]. The mathematical problems studied in those papers are quite different from the one considered here. The cited papers consider nonstationary shocks which exist only on a finite *time interval, whereas we study stationary shocks which of course exist for all time. The viscous and inviscid problems considered in the nonstationary case are, respectively, parabolic (or partially parabolic) and hyperbolic PDEs on unbounded spatial domains, while our viscous and inviscid problems reduce under the symmetry assumption to ODEs on a bounded spatial interval interval. There is no way to derive our results from the earlier nonstationary ones.*

(3) Our viscous problem (4.7), (4.9) *is a second-order two-point boundary problem, yet standard two-point methods like those based on comparison theorems, upper and lower solutions, and shooting methods [BSW, DH, K] appear unsuitable for constructing solutions involving fast interior transitions, like the shock layers in our viscous solutions, that must be smoothly matched to outer, slowly varying inviscid solutions.*

Geometric singular perturbation theory (or "Fenichel theory") has been used by several authors (e.g., [J, GS]) to construct solutions that match slow with fast transitional behavior and connect equilibrium points *or, more generally, invariant manifolds of ODEs. Our viscous solutions exhibit both slow and fast behavior, but since, for example, the endstates (ρ_a, u_a, θ_a), (ρ_b, u_b, θ_b) are not equilibria, we do not see how to apply Fenichel theory to construct the viscous solutions being sought here. In any case we believe that the direct and self-contained approach developed here and further in [W] is of interest in its own right.*

1.3 Equations and assumptions

The full (non-barotropic) compressible Navier-Stokes equations express the conservation of mass and the balance of momentum and of energy. In Eulerian coordinates the equations in $\mathbb{R}^3$ take the form

$$\begin{aligned}
\rho_t + \operatorname{div}(\rho \mathbf{U}) &= 0 && (1.18)\\
(\rho \mathbf{U}^i)_t + \operatorname{div}(\rho \mathbf{U}^i \mathbf{U}) + p_{x_i} &= \mu \Delta \mathbf{U}^i + (\lambda+\mu)\operatorname{div} \mathbf{U}_{x_i}\,, \qquad i = 1,\, 2,\, 3, && (1.19)\\
(\rho E)_t + \operatorname{div}\big((\rho E + p)\mathbf{U}\big) &= \Delta\big(\kappa\theta + \tfrac{1}{2}\mu|\mathbf{U}|^2\big)\\
&\quad + \mu \operatorname{div}((\nabla \mathbf{U})\mathbf{U}) + \lambda \operatorname{div}\big((\operatorname{div}\mathbf{U})\mathbf{U}\big)\,. && (1.20)
\end{aligned}$$

Here $x \in \mathbb{R}^3$ is the spatial coordinate, $t > 0$ is time, and $\rho, \mathbf{U} = (\mathbf{U}^1, \mathbf{U}^2, \mathbf{U}^3)$, and θ are the density, velocity, and temperature, respectively. The specific total energy is then $E = \frac{1}{2}|\mathbf{U}|^2 + e$, where e the specific internal energy. Denoting the specific volume by $\tau \equiv \frac{1}{\rho}$ and specific entropy by S, we assume that a complete equation of state $e = e(\tau, S)$ is prescribed. Temperature θ and pressure p are then defined via the fundamental relation

$$de = \theta dS - p d\tau\,, \qquad \text{or} \qquad \theta = e_S\,, \qquad p = -e_\tau\,, \tag{1.21}$$

both of which are non-negative quantities. The function $e(\tau, S)$ is required to be positive, smooth and convex. In particular we may choose ρ and θ as the thermodynamical unknown quantities. We will make the additional assumption that pressure increases with entropy at fixed volume:

$$p_S(\tau, S) > 0\,. \tag{1.22}$$

We make use of this last assumption in Section 3 when we consider e as a function of τ and p. Following [CF] we use the following notation for pressure as a function of density and specific entropy:

$$p := f(\rho, S)\,. \tag{1.23}$$

where f is a smooth map which according to the assumptions above is increasing in both ρ and S. In addition we require convexity in ρ:

$$f_\rho > 0\,, \qquad f_{\rho\rho} \geq 0\,, \qquad \text{and} \qquad f_S > 0\,. \tag{1.24}$$

Finally we assume that

$$e \to 0\,, \quad p/\rho \to 0\,, \quad c \to 0 \qquad\qquad \text{as } \rho \to 0, \tag{1.25}$$

and that

$$c \to \infty \qquad\qquad \text{as } \rho \to \infty, \tag{1.26}$$

where $c = \sqrt{f_\rho}$ is the local sound speed.

The transport coefficients μ, λ, κ in (1.19)-(1.20) are assumed to be positive constants, and $\nabla\mathbf{U}$ denotes the Jacobian of the velocity vector with respect to the space variables. Finally, the compressible Euler equations are obtained by setting $\mu = \lambda = \kappa = 0$.

For spherical (cylindrical) symmetric flow the density, velocities, and temperature depend only on time and the radial distance to the origin (x_3-axis). We refer to these as the spherically symmetric (SS) and the cylindrically symmetric (CS) cases, respectively. We let (u, v, w) be the velocity components in either spherical or cylindrical coordinates. We set $r = |x|$ in the SS case, while $r = \sqrt{x_1^2 + x_2^2}$ in the CS case. In either case, with a slight abuse of notation we write $\rho(x,t) = \rho(r,t)$, *etc.* Thus,

$$\begin{aligned}
\mathbf{U}(x,t) &= \frac{u(r,t)}{r}x\,, \qquad v = w \equiv 0 \qquad \text{(SS case)} \\
\mathbf{U}(x,t) &= \frac{u(r,t)}{r}(x_1, x_2, 0) + \frac{v(r,t)}{r}(-x_2, x_1, 0) + w(r,t)(0,0,1) \qquad \text{(CS case).}
\end{aligned}$$

The equations (1.18)-(1.20) reduce to (see [RJ])

$$\rho_t + (\rho u)_\xi = 0 \tag{1.27}$$

$$(\rho u)_t + (\rho u^2)_\xi - \frac{\rho v^2}{r} + p_r - \nu u_{\xi r} = 0 \tag{1.28}$$

$$(\rho v)_t + (\rho u v)_\xi + \frac{\rho u v}{r} - \mu v_{\xi r} = 0 \tag{1.29}$$

$$(\rho w)_t + (\rho u w)_\xi - \mu w_{r\xi} = 0 \tag{1.30}$$

$$(\rho e)_t + (\rho u e)_\xi + p u_\xi - \kappa \theta_{r\xi} - Q = 0, \tag{1.31}$$

where $\partial_\xi = \partial_r + m/r$, $\nu := \lambda + 2\mu$, $m = 1$ in the CS case, $m = 2$ in the SS case, and

$$Q = \nu(u_\xi)^2 + \mu(v_\xi)^2 + \mu(w_r)^2 - \frac{2\mu m}{r}(v^2)_r - \frac{2\mu m}{r^m}(r^{m-1}u^2)_r\,. \tag{1.32}$$

Note that the operators ∂_ξ and ∂_r do not commute; in an expression like $u_{\xi r}$ the operator ∂_ξ is applied first.

2 Stationary solutions of the non-barotropic Euler equations

ODE system for spherically/cylindrically symmetric flow We treat simultaneously the SS and CS cases in domains which are bounded by concentric and fixed spheres or cylinders with radii $b > a > 0$. The stationary Euler equations for SS or CS flow reduce to the ODE system

$$\frac{d(\rho u r^m)}{dr} = 0 \tag{2.1}$$

$$\rho u \frac{du}{dr} - \rho \frac{v^2}{r} + \frac{dp}{dr} = 0 \tag{2.2}$$

$$r\frac{dv}{dr} + v = 0 \tag{2.3}$$

$$\frac{dw}{dr} = 0 \tag{2.4}$$

$$\frac{d}{dr}\left[(\rho E + p)u r^m\right] = 0\,, \tag{2.5}$$

where we recall that $v = w \equiv 0$ in the SS-case. The Rankine-Hugoniot conditions across a stationary discontinuity reduce to

$$[\rho u] = 0\,, \qquad \left[p + \rho u^2\right] = 0\,, \qquad [\rho u v] = 0\,, \qquad [\rho u w] = 0\,, \qquad [(\rho E + p)u] = 0\,. \tag{2.6}$$

In regions where the flow is smooth the changes in the state of the gas are adiabatic and we have $\dot{S} = 0$, see [CF] p. 16. In stationary flow this reduces to

$$\frac{dS}{dr} = 0\,, \tag{2.7}$$

such that the entropy takes on the same constant value throughout any smooth region.

2.1 Inner solutions for spherically/cylindrically symmetric flow

We prescribe Dirichlet data $\rho_a > 0$, $\theta_a > 0$, $u_a \neq 0$, and v_a, w_a on the inner boundary $r = a$, and seek a smooth, stationary solution to the Euler equations in the region $r \geq a$. We refer to this as an *inner solution*. The data at the inner boundary determine the value $S = S_a$ of the specific entropy at $r = a$. By equation (2.7) the specific entropy takes the same constant value throughout the domain of definition of the inner solution. According to (1.23) we can thus suppress the dependence of pressure on entropy when discussing the existence of an inner solution, and write

$$p = f(\rho) \equiv f(\rho, S_a)\,.$$

This effectively reduces the problem of existence of an inner solution to the corresponding question for *barotropic* flow. This was treated in detail in the earlier work [EJW] under the hypotheses that pressure $p = P(\rho)$ is an increasing, convex function of density with $\lim_{\rho\downarrow 0} P'(\rho) = 0$. In the present context these assumptions are satisfied due to $(1.24)_1$, $(1.24)_2$, and $(1.25)_3$, respectively.

For convenience we briefly outline the arguments from [EJW] for existence of inner solutions. Equation (2.3) yields $r\,v \equiv D_a := v_a a$. Substituting into (2.2) and integrating once, we obtain

$$u^2 + v^2 + \Pi(\rho) \equiv u_a^2 + v_a^2 =: V_a^2\,, \tag{2.8}$$

where we have defined the function

$$\Pi(\rho) = \Pi(\rho, \rho_a, S_a) := \int_{\rho_a}^{\rho} \frac{2 f_\rho(\rho, S_a)}{\sigma}\, d\sigma\,. \tag{2.9}$$

From (2.1) we get $\rho u r^m \equiv C_a := \rho_a u_a a^m$. Together with $rv \equiv D_a$ and (2.8) this shows that the denisty $\rho = \rho(r)$ along the profile is given implicitly as the solution of the algebraic equation

$$\frac{1}{r^m} = \frac{\rho^2}{C_a^2 + \rho^2 D_a^2 \delta_{m,1}} \left[V_a^2 - \Pi(\rho)\right] =: \Psi(\rho, \rho_a, u_a, v_a, S_a)\,. \tag{2.10}$$

(The Kronecker delta $\delta_{m,1}$ is used in order to treat both SS and CS flow at the same time.) In [EJW] it was shown that, as a consequence of our assumptions on the pressure function $p = f(\rho, S)$, the function $\Psi(\rho) = \Psi(\rho, \rho_a, u_a, v_a, S_a)$ has the form as in Figure 1, and that (2.10) defines an inner solution $\rho(r)$ for all $r \geq a$. The radial velocity along the profile is then given by $u(r) = \frac{C_a}{r^m \rho(r)}$.

Remark 2.1. *Note that the energy equation (2.5) has been applied in finding $\rho(r)$ and $u(r)$. In the analysis above we use that the specific entropy remains constant along the flow, a fact that depends on (2.5).*

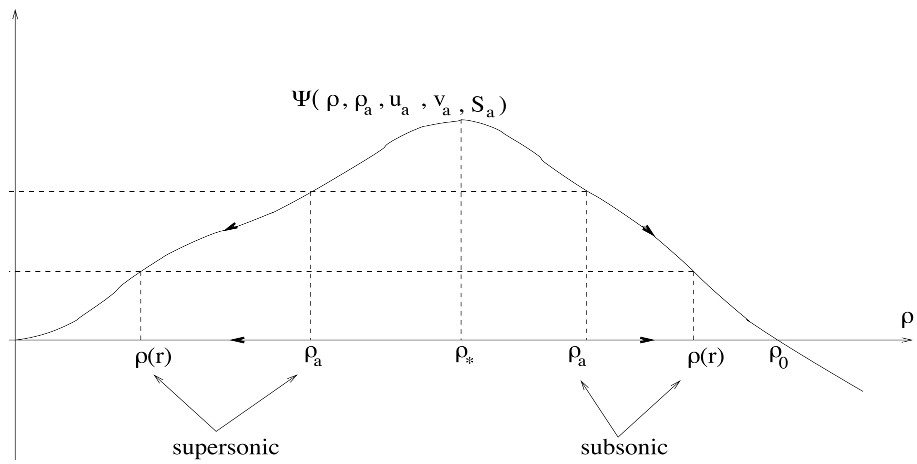

Figure 1: Inner solutions. The function $\Psi(\rho, \rho_a, u_a, v_a, S_a)$. Arrows indicate direction as r increases from $r = a$.

Finally, the internal energy along the profile is obtained by substituting $\rho u r^m \equiv C_a$ into (2.5) and integrating. This yields Bernoulli's identity for steady flow:

$$\frac{u^2+v^2}{2} + e + \frac{p}{\rho} \equiv B_a\,, \qquad \text{where } B_a := \frac{V_a^2}{2} + e_a + \frac{p_a}{\rho_a}\,. \tag{2.11}$$

We refer to [EJW] for the analysis of the sonicity of the constructed solution. The conclusion is that if the flow is supersonic (subsonic) at $r = a$, then it becomes increasingly so as r increases from a: the radial Mach-number $M = \frac{|u|}{c}$ increases (decreases) as r increases.

We may also analyze the radial velocity by deriving an ODE for $u(r)$. From (2.2) we first calculate

$$\frac{dp}{dr} = f_\rho \frac{d\rho}{dr} = c^2\,\frac{d\rho}{dr}\,,$$

where we have used that the entropy is constant, $S \equiv S_a$. We then use $\rho(r)u(r)r^m \equiv C_a$ to compute $\frac{d\rho}{dr}$ and substitute into (2.2) to obtain

$$\frac{du}{dr} = \frac{u(v^2+mc^2)}{r(u^2-c^2)}\,, \tag{2.12}$$

where (suppressing the dependence of f on S) $c^2 = f_\rho\left(\frac{C_a}{ur^m}\right)$ and $v = \frac{D_a}{r}$. We observe from (2.11) and $e > 0$ that the velocity is uniformly bounded along the flow. The four possible velocity profiles of an inner solution, corresponding to whether $u_a \gtrless 0$ and $|u_a| \gtrless c_a$, are sketched in Figure 2 (for $r \geq a$). Finally, let us see that

$$\lim_{r\uparrow\infty} u^2 = \begin{cases} 2B_a & \text{for supersonic flow,} \\ 0 & \text{for subsonic flow.} \end{cases} \tag{2.13}$$

For supersonic flow we have (see Figure 1) that $\rho \downarrow 0$ as $r \uparrow \infty$. Assumptions $(1.25)_{1,2}$ then give $e + p/\rho \to 0$, and Bernoulli's identity yields $u^2 \to 2B_a$. For subsonic flow ρ is bounded away from zero, whence $\rho(r)u(r)r^m \equiv C_a$ shows that $|u| \downarrow 0$ as $r \uparrow \infty$. This establishes (2.13). We summarize the results in:

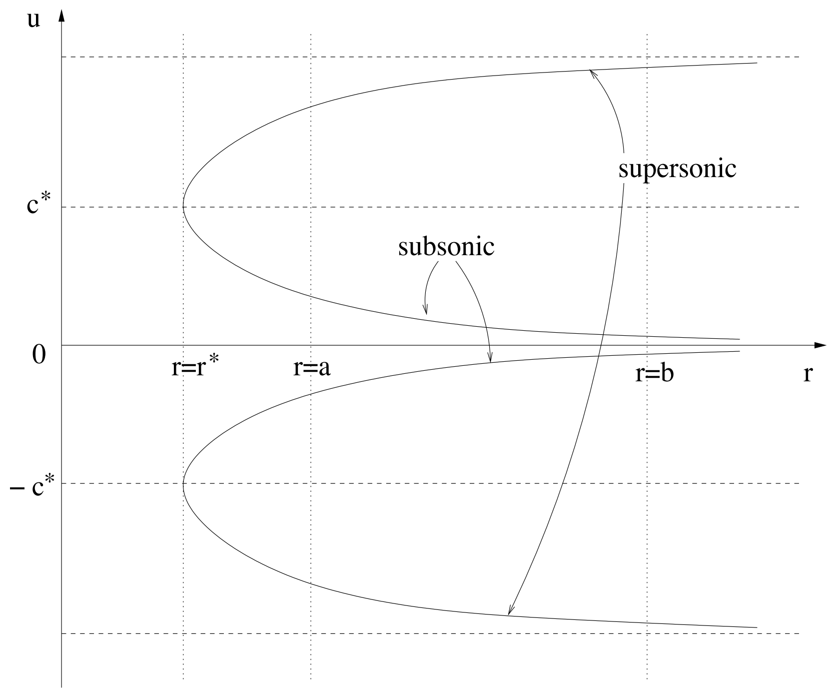

Figure 2: Radial velocity and critical inner radius in stationary solutions

Proposition 2.2. (Existence of spherically/cylindrically symmetric inner solutions.) *Consider the full, stationary Euler equations with spherical/cylindrical symmetry (2.1) - (2.5) in the exterior of a sphere/cylinder with radius $a > 0$, and with prescribed Dirichlet data $\rho_a > 0$, $u_a \neq 0$, v_a, w_a, $\theta_a > 0$ at $r = a$. Assuming the data are radially non-sonic, then (2.1) -(2.5) have a unique solution defined for all $r \geq a$. The resulting flow is strictly supersonic (subsonic) with increasing (decreasing) Mach number as r increases, if and only if it is strictly supersonic (subsonic) at the inner boundary $r = a$. See Figure 1 and Figure 2.*

2.2 Outer solutions for spherically/cylindrically symmetric flow

Next we consider Dirichlet data $\rho_b > 0$, $\theta_b > 0$, $u_b \neq 0$, v_b, w_b on the outer boundary $r = b$, and we seek a smooth stationary solution to the Euler equations in the region $r \leq b$. We refer to this as an *outer solution*. To obtain a solution on a non-trivial interval we assume that the data are radially non-sonic. As indicated in Figure 2 there is then a critical radius $r^* < b$ where the flow becomes sonic and beyond which the solution cannot be extended. Arguing as in the case of inner solutions shows that

$$\rho u r^m \equiv C_b\,, \qquad rv \equiv D_b\,, \qquad w \equiv w_b\,, \qquad \frac{u^2+v^2}{2} + e + \frac{p}{\rho} \equiv B_b\,. \tag{2.14}$$

Arguing as above (see [EJW] for details) we get that the density profile again is given by an algebraic equation of the form (2.10) (with a replaced by b). Letting r decrease from $r = b$ it is clear from Figure 1 that there is a finite and strictly positive inner radius r^* beyond which the stationary solution can not be defined. The situation for the velocity profile is given in Figure 2. The analysis in [EJW] shows that the flow becomes sonic at the critical radius. Summarizing we have:

Proposition 2.3. (Existence of spherically/cylindrically symmetric outer solution) *Consider the full, stationary Euler equations with spherical/cylindrical symmetry (2.1) - (2.5) in the interior of a sphere with radius $b > 0$, and with prescribed Dirichlet data $\rho_b > 0$, $u_b \neq 0$, v_b, w_b, $\theta_b > 0$ at $r = b$. Assuming the data are radially non-sonic, there is a critical radius $r^* \in (0, b)$ such that (2.1) - (2.5) have a unique solution defined for all $r \in (r^*, b)$. The resulting flow is strictly supersonic (subsonic) with decreasing (increasing) Mach number as r decreases on the interval (r^*, b) if and only if it is strictly supersonic (subsonic) at the outer boundary $r = b$. The flow becomes sonic at $r = r^*$ and cannot be extended as a stationary solution inside this radius. See Figure 1 and Figure 2.*

3 Stationary solutions of the Euler equations with shocks

Next we use the inner and outer solutions from above to construct symmetric *weak* solutions with a single stationary, admissible shock located at any intermediate location $\bar{r} \in (a, b)$.

3.1 Shocks built from inner solutions

Consider Dirichlet data as in Proposition 2.2 given on the inner boundary $r = a$. We fix an outer boundary at $r = b > a$ and choose an intermediate radius $\bar{r} \in (a, b)$ which will be the shock location. We next describe the assumptions on the equation of state that guarantee existence and uniqueness of an admissibility shock. As the Rankine-Hugoniot conditions are identical with those of *planar* shocks (oblique, in the case of swirl) we refer to [CF] for the details of the arguments.

Let the specific volume be denoted $\tau = 1/\rho$ and set $\rho(\bar{r}\pm) = \rho_\pm$, etc. For (τ_-, p_-) fixed, the points of intersection (different from (τ_-, p_-)) between these two curves

$$-k^2 = \frac{p - p_-}{\tau - \tau_-}\,, \qquad k := \rho_- u_- = \rho_+ u_+ \qquad \text{(Rayleigh line)} \tag{3.1}$$

and

$$H(\tau, p) := e(\tau, p) - e(\tau_-, p_-) + \tfrac{1}{2}(\tau - \tau_-)(p + p_-) = 0\,, \qquad \text{(Hugoniot curve)} \tag{3.2}$$

provide the possible states at $\bar{r}+$ that satisfy the Rankine-Hugoniot conditions (2.6). (e is now considered as a function of τ and p). We make the following standard assumptions (see [CF] p. 140) on the Hugoniot curve $\{H(\tau, p) = 0\}$:

A1. The pressure along the Hugoniot curve increases monotonically from 0 to $+\infty$ as the specific volume decreases from a maximal value $\tau_{max} \leq \infty$ to a minimal value $\tau_{min} \geq 0$. We denote the pressure along the Hugoniot curve by $p = G(\tau)$.

A2. Any straight line through (τ_-, p_-) which intersects the τ-axis at a point with $\tau \leq \tau_{max}$ intersects the Hugoniot curve at a unique point (different from (τ_-, p_-)).

As admissibility criteria we request that the entropy of a fluid particle should increase as it passes through a shock. It is demonstrated in [CF] that A1 and A2, in conjunction with assumptions (1.24), imply that admissible shocks are such that fluid particles are compressed, and pass from supersonic to subsonic flow, as they traverse the shock surface.

Consider data at $r = a$ that are strictly supersonic (subsonic). By the earlier analysis there is a unique flow satisfying the stationary and symmetric Euler equations on $r \in [a, \bar{r})$, and it is supersonic (subsonic) also at $r = \bar{r}-$. It follows that the flow must be directed outward (inward), i.e. $u_a > 0$ ($u_a < 0$). Under our assumptions the data ρ_+, e_+, u_+, v_+ and w_+ at $\bar{r}+$ are uniquely determined by the Rankine-Hugoniot conditions, and corresponds to strictly subsonic (supersonic) flow. Again there is a unique subsonic (supersonic) flow satisfying the stationary and symmetric Euler equations in $r \in [\bar{r}, b]$. This yields a stationary solution on $[a, b]$ with a single, entropy admissible shock at $\bar{r}$.

Proposition 3.1. (Stationary symmetric shocks built from inner solutions) *Consider the full, stationary Euler equations with spherical (or cylindrical) symmetry in the domain between two concentric spheres (cylinders) with radii $a < b$ with prescribed Dirichlet data $\rho_a > 0$, $\theta_a > 0$, u_a, v_a and w_a at $r = a$. Assume that the flow is radially non-sonic at $r = a$, and fix any $\bar{r} \in (a, b)$.*

Then there exists a unique weak admissible solution with a single shock located at $\bar{r}$ if and only if, either, the flow is radially supersonic at $r = a$ and directed into the domain (i.e. $u_a > 0$), or the flow is radially subsonic at $r = a$ and directed out of the domain (i.e. $u_a < 0$). In the former case the flow is (radially) supersonic in $(a, \bar{r})$ and (radially) subsonic in $(\bar{r}, b)$, while the opposite holds in the latter case.

3.2 Shocks built from outer solutions

Given data $\rho_b > 0$, u_b, v_b, w_b, and $\theta_b > 0$ at the outer boundary $r = b$. The construction of stationary, symmetric solutions for $r < b$ with an admissible shock at some location $\bar{r} < b$ is similar to above, the only restriction being that we need to place the shock at a location $\bar{r}$ where the flow is defined. That is, provided we choose $\bar{r} \in (r_1^*, b)$, where r_1^* is the critical radius at which the flow constructed by starting at $r = b$ and solving inward, becomes radially sonic. Assuming this, our assumptions on the equation of state and the Hugoniot curve guarantee unique values of the solution at $\bar{r}-$, and we can solve inward until we reach the critical radius $r_2^* > 0$ corresponding to these values.

Proposition 3.2. (Stationary symmetric shocks built from outer solutions) *Consider the full, stationary Euler equations with spherical (or cylindrical) symmetry in the domain between two concentric spheres (cylinders) with radii $a < b$ with prescribed Dirichlet data $\rho_b > 0$, u_b, v_b, w_b, and $\theta_b > 0$ at $r = b$. Assume that the flow is radially non-sonic at $r = b$ and given any $\bar{r} \in (r_1^*, b)$ and $a \in (r_2^*, \bar{r})$.*

Then there exists a unique weak admissible solution, defined on $[a, b]$ and with a single shock located at $\bar{r}$ if and only if the flow is radially supersonic at $r = b$ and directed into the domain (i.e. $u_b < 0$), or the flow is radially subsonic at $r = b$ and directed out of the domain (i.e. $u_b > 0$). In the former case the flow is radially supersonic at $r = b$ and radially subsonic at $r = a$, while the opposite holds in the latter case.

3.3 When can a shock solution be found?

We next consider the possibility of finding shock solutions for given boundary data. We consider the following question: given ρ_a, u_a, v_a, w_a, θ_a at the inner boundary, what are the possible states that can be reached at $r = b$ through an admissible, stationary shock located at some intermediate $\bar{r} \in (a, b)$?

From the earlier analysis we know that $\rho u r^m \equiv C_a$, $rv \equiv D_a$, and $w \equiv w_a$ along any stationary solution (smooth or not). Thus, a necessary condition for the existence of a shock solution with "final" data ρ_b, u_b, v_b, w_b, θ_b at $r = b$ is that

$$\rho_b u_b = \frac{C_a}{b^m}\,, \qquad v_b = \frac{D_a}{b}\,, \qquad w_b = w_a\,. \tag{3.3}$$

We choose to work with the density as the primary unknown so that the issue becomes: what final densities ρ_b can be attained for a solution with a shock at some $\bar{r} \in (a, b)$. For concreteness we consider the case with (radially) supersonic inflow at $r = a$, that is, $u_a > 0$ and $u_a^2 > c_a^2$.

To see how the final density ρ_b depends on the shock location $\bar{r}$ we find it convenient to use the ODE satisfied by $\rho(r)$ (instead of the algebraic relation (2.10)). We first observe that this ODE takes the same form in the two intervals $(a, \bar{r})$ and $(\bar{r}, b)$, and it is independent of $\bar{r}$. Indeed, from (2.12) and $\rho u r^m \equiv C_a$ it follows that

$$\frac{d\rho}{dr} = \frac{\rho(v^2 + mu^2)}{r(c^2 - u^2)}\,. \tag{3.4}$$

From the earlier analysis we know that the flow remains (radially) subsonic for all $r > \bar{r}$, whence (3.4) is a well-behaved ODE with unique solutions. Thus, if $\rho_1(r)$, $\rho_2(r)$ are two smooth solutions with $\rho_1(s) > \rho_2(s)$ for some $s > \bar{r}$, then necessarily $\rho_1(r) > \rho_2(r)$ for all $r > s$.

We can use this to infer how ρ_b varies with the shock location $\bar{r}$. Specifically we will show that an increase in the shock location $\bar{r}$ implies a lower ending value for the density at $r = b$, see Figure 3. Let $\rho_1(r)$ denote the solution to (3.4) for $r > \bar{r}$ whose "data" at $\bar{r}+$ is $\tilde{\rho}(\bar{r})$ = the density immediately on the outside of the shock. By uniqueness of solutions to (3.4), an increase in $\bar{r}$ implies a lower ending value for the density at $r = b$ provided that

$$\tilde{\rho}'(\bar{r}) \;<\; \rho_1'(\bar{r}+)\,. \tag{3.5}$$

To verify this inequality we calculate the left-hand side in (3.5) from the Rankine-Hugoniot relation, while the right-hand side is given by (3.4). As in Section 3 it is convenient to regard the internal energy as a function of specific volume and pressure: $e = e(\tau, p)$. We start from the Rankine-Hugoniot relations

$$e(\tilde{\tau}, \tilde{p}) - e(\bar{\tau}, \bar{p}) + \tfrac{1}{2}(\tilde{\tau} - \bar{\tau})(\tilde{p} + \bar{p}) = 0\,, \qquad \tilde{p} - \bar{p} + k^2(\tilde{\tau} - \bar{\tau}) = 0\,,$$

where $k = \tilde{\rho}\tilde{u} = \bar{\rho}\bar{u} = C_a/\bar{r}^m$, and bars (tildes) denote evaluation immediately on the inside (outside) of the shock. Differentiating both relations with respect to the shock location $\bar{r}$ yields two linear equations for $\tilde{\tau}' - \bar{\tau}'$ and $\tilde{p}' - \bar{p}'$ (where $' = \frac{d}{d\bar{r}}$), and we get

$$\tilde{\tau}' - \bar{\tau}' = \frac{1}{\Delta}\Big\{\big[\bar{e}_\tau - \tilde{e}_\tau\big]\bar{\tau}' + \big[(\bar{e}_p - \tilde{e}_p) + (\bar{\tau} - \tilde{\tau})\big]\bar{p}' + \frac{2mk^2}{\bar{r}}(\bar{\tau} - \tilde{\tau})\big[\tilde{e}_p + \tfrac{1}{2}(\tilde{\tau} - \bar{\tau})\big]\Big\}\,,$$

where

$$\Delta = \big[\tilde{e}_\tau + \tfrac{1}{2}(\tilde{p}+\bar{p})\big] - k^2\big[\tilde{e}_p + \tfrac{1}{2}(\tilde{\tau}-\bar{\tau})\big] = \frac{\tilde{c}^2-\tilde{u}^2}{\tilde{\tau}^2}\tilde{e}_p\,.$$

Using that $\bar{\tau}' = -\bar{\tau}^2\bar{\rho}'$ and $\tilde{\tau}' = -\tilde{\tau}^2\tilde{\rho}'$, we obtain

$$\tilde{\rho}' = \frac{\bar{\tau}^2}{\tilde{\tau}^2}\bar{\rho}' - \frac{1}{\tilde{\tau}^2\Delta}\Big\{\big[\bar{e}_\tau - \tilde{e}_\tau\big]\bar{\tau}' + \big[(\bar{e}_p - \tilde{e}_p) + (\bar{\tau}-\tilde{\tau})\big]\bar{p}' + \frac{2mk^2}{\bar{r}}(\bar{\tau}-\tilde{\tau})\big[\tilde{e}_p + \tfrac{1}{2}(\tilde{\tau}-\bar{\tau})\big]\Big\}\,,$$

Next we use that $\bar{p}' = \bar{c}^2\bar{\rho}'$, collect terms multiplying $\bar{\rho}'$, and use that $e_\tau = -p + c^2 e_p/\tau^2$. Rearranging gives

$$\tilde{\rho}' = \left\{\frac{\bar{\tau}^2}{\tilde{\tau}^2} + \frac{\bar{\tau}^2}{(\tilde{c}^2-\tilde{u}^2)\tilde{e}_p}\left[\Big(\frac{\bar{c}^2}{\bar{\tau}^2} - \frac{\tilde{c}^2}{\tilde{\tau}^2}\Big)\tilde{e}_p - \frac{\bar{c}^2-\bar{u}^2}{\bar{\tau}^2}(\bar{\tau}-\tilde{\tau})\right]\right\}\bar{\rho}' - \frac{2mk^2(\bar{\tau}-\tilde{\tau})}{\bar{r}\tilde{\tau}^2\Delta}\big[\tilde{e}_p + \tfrac{1}{2}(\tilde{\tau}-\bar{\tau})\big]\,.$$

We want to show that this last expression is majorized by $\rho_1'(\bar{r}+)$, which is given by (3.4):

$$\rho_1'(\bar{r}+) = \frac{(\bar{v}^2 + m\tilde{u}^2)}{\tilde{\tau}\bar{r}(\tilde{c}^2-\tilde{u}^2)}\,.$$

We substitute the two expressions into (3.5) and rearrange. In doing so we use the fact that the flow is subsonic at $\bar{r}+$ and also that $\tilde{e}_p > 0$. This last inequality follows from our assumption that pressure increases with increasing entropy for fixed density. Finally we use the expression for $\bar{\rho}'$ given by (3.4). Collecting terms that multiply $\tilde{e}_p$ we conclude that (3.5) holds if and only if

$$\begin{aligned}(\bar{v}^2 + m\bar{u}^2)(\tilde{\tau}-\bar{\tau}) \;&+\; mk^2\bar{\tau}(\bar{\tau}-\tilde{\tau})^2 \\ &< \left[2mk^2\bar{\tau}(\bar{\tau}-\tilde{\tau}) + (\bar{v}^2 + m\tilde{u}^2) - \frac{(\bar{v}^2+m\bar{u}^2)}{(\bar{c}^2-\bar{u}^2)}\frac{(\tilde{c}^2\tilde{\tau}^2 - \tilde{u}^2\bar{\tau}^2)}{\tilde{\tau}^2}\right]\tilde{e}_p\,.\end{aligned}$$

The left hand side equals $(\tilde{\tau}-\bar{\tau})(\bar{v}^2 + mk^2\bar{\tau}\tilde{\tau})$, which is negative since the shock is compressive, while the right hand side simplifies to $\big(\bar{v}^2 + mk^2\bar{\tau}\tilde{\tau}\big)(\bar{\tau}-\tilde{\tau})\tilde{e}_p/\tilde{\tau}$, which is positive.

It follows that the minimal value for the density at $r = b$ is attained by placing the shock at $\bar{r} = b-$, while the maximal value is attained by placing the shock at $\bar{r} = a+$. We summarize our findings in:

Theorem 3.3. (Possible shocks for outward symmetric flow) *Consider the stationary, symmetric, non-barotropic Euler equations (2.1) - (2.5). Consider radii $a < b$ and data ρ_a, u_a, v_a, w_a, θ_a corresponding to supersonic inflow at $r = a$ (i.e. $u_a^2 > c_a^2$, $u_a > 0$).*

Then there is a finite interval $(\rho_{b,\min}, \rho_{b,\max})$ of ρ_b-values that can be reached from the data at $r = a$ through a stationary, compressive shock located at some location $\bar{r} \in (a, b)$. The limiting values $\rho_{b,\min}$, $\rho_{b,\max}$ depend on a, ρ_a, u_a, v_a, θ_a and b, and there is a one-to-one correspondence between ρ_b values in $(\rho_{b,\min}, \rho_{b,\max})$ and shock locations in (a, b).

4 Exact Navier-Stokes solutions converging to Euler shocks: preliminaries

In these last two sections we show how to construct exact smooth solutions of the Navier-Stokes equations that converge in the small viscosity limit to the inviscid shocks constructed earlier. We will focus on the spherically symmetric case, where the unknowns are (ρ, u, θ). The cylindrically symmetric case can be treated in the same way and is discussed in Remark 5.19.

Defining $w = (w^1, w^2) := (\rho, (u, \theta))$ and

$$\nu = \underline{\nu}\varepsilon, \quad \mu = \underline{\mu}\varepsilon, \quad \kappa = \underline{\kappa}\varepsilon, \tag{4.1}$$

we can write the viscous equations (1.27)-(1.31) (note: (1.29) and (1.30) are now absent and $v = 0$ in (1.28)) in the stationary case as

$$d_r f(w) + g(w, r) - \varepsilon h(w^2, w^2_r, w^2_{rr}, r) = 0 \text{ on } [a, b], \tag{4.2}$$

where

$$\begin{aligned}
&(a)\ f(w) := \begin{pmatrix} \rho u \\ \rho u^2 + p \\ (\rho E + p)u \end{pmatrix}, \quad g(w, r) = \begin{pmatrix} \frac{2\rho u}{r} \\ \frac{2\rho u^2}{r} \\ \frac{2(\rho E + p)u}{r} \end{pmatrix} \\
&(b)\ h = \begin{pmatrix} 0 \\ \underline{\nu} u_{rr} \\ \underline{\nu} u u_{rr} + \underline{\kappa}\theta_{rr} \end{pmatrix} + \begin{pmatrix} 0 \\ \underline{\nu}\left(\frac{2u_r}{r} - \frac{2u}{r^2}\right) \\ \underline{\nu}(u_r)^2 + (6\underline{\nu} - 8\underline{\mu})\frac{uu_r}{r} + (2\underline{\nu} - 4\underline{\mu})\frac{u^2}{r^2} + \underline{\kappa}\frac{2\theta_r}{r} \end{pmatrix}
\end{aligned} \tag{4.3}$$

and

$$p = p(\rho, \theta), \ E = e(\rho, \theta) + \frac{u^2}{2}. \tag{4.4}$$

We are given (from Proposition 3.1) a stationary inviscid shock solution that we now denote

$$U^0(r) = (\rho^0(r), u^0(r), \theta^0(r)) \tag{4.5}$$

with supersonic inflow at $r = a$, shock surface $r = \overline{r} \in (a, b)$, and taking the values $w_a = (\rho_a, u_a, \theta_a)$ at $r = a$ and $w_b = (\rho_b, u_b, \theta_b)$ at $r = b$. Recall from Proposition 3.1 that

$$\begin{aligned}
&u^0(r) > 0 \text{ and supersonic in } [a, \overline{r}] \\
&u^0(r) > 0 \text{ and subsonic in } [\overline{r}, b].
\end{aligned} \tag{4.6}$$

Setting $s = r - \overline{r}$ and $w^*(s) = w(s + \overline{r})$, we obtain a viscous problem equivalent to (4.2) on $[a - \overline{r}, b - \overline{r}]$ with shock surface at $s = 0$ now (dropping $*$):

$$d_s f(w) + g(w, s) - \varepsilon(h(w^2, w^2_s, w^2_{ss}, s) = 0 \text{ on } [a - \overline{r}, b - \overline{r}]. \tag{4.7}$$

Observe that $w^\varepsilon(s)$ is a smooth solution to (4.7) if and only if $w^\varepsilon_\pm(s) := w^\varepsilon_{|\pm s \geq 0}$ satisfies the transmission problem

$$\begin{aligned}
&(a)\ d_s f(w) + g(w, s) - \varepsilon h(w^2, w^2_s, w^2_{ss}, s) = 0 \text{ on } [a - \overline{r}, b - \overline{r}] \cap \{\pm s \geq 0\} \\
&(b)\ [w] = 0, \ [w^2_s] = 0 \text{ on } s = 0;
\end{aligned} \tag{4.8}$$

since (4.8) (a), (b) imply $[w^1] = 0$ and also higher regularity of w. We construct w^ε to satisfy boundary conditions

$$w^\varepsilon(b - \overline{r}) = w_b,\ w^\varepsilon(a - \overline{r}) = w_a + O(\varepsilon), \tag{4.9}$$

where w_a, w_b are the endstates of the inviscid shock U^0 constructed earlier.

The equation (4.8) and the lines that precede it illustrate our frequent practice of introducing a function like $w^\varepsilon_\pm$, and then suppressing the ε or $\pm$ shortly thereafter. Whenever we describe a function as "smooth", we always mean at least C^2.

4.1 Approximate solution to the viscous problem

The problem (4.7) with boundary conditions (4.9) is a two-point boundary problem. However, since we seek a solution with a fast transition region in the interior of its domain (near $s = 0$), standard two-point methods do not apply [H, BSW, DH, K]. As a first step we construct high-order approximate solutions that converge to U^0 as $\varepsilon \to 0$. These solutions are approximate in the sense that they satisfy (4.8),(4.9) with errors that are small in L^∞ for ε small. More precisely, we define these to be functions $\tilde{w}^\varepsilon$ with the properties given below in Proposition 4.2. The approximate solution will be modified to obtain an exact solution in sections 4 and 5.

Definition 4.1 (Spaces)**.** *1. For $k \in \mathbb{N}$ let C^k_p (the subscript indicates "piecewise") be the set of functions $U(s)$ on $[a - \overline{r}, b - \overline{r}]$ such that the restrictions $U_\pm$ belong to $C^k\left([a - \overline{r}, b - \overline{r}] \cap \{\pm s \geq 0\}\right)$.*

2. Let $\tilde{C}^k_p$ be the set of functions $V(z)$ on $\mathbb{R}$ such that the restrictions $V_\pm$ belong to $C^k(\pm z \geq 0)$ and satisfy, for some $\beta > 0$,

$$\left|\left(\frac{d}{dz}\right)^j V(z)\right| \leq C_j e^{-\beta|z|} \text{ for } j \leq k. \tag{4.10}$$

Proposition 4.2 (Approximate solutions)**.** *Let k and $M \geq 1$ be integers with $k \geq M + 2$. Let the functions f, g, and h be as defined in (4.3). Assume that e and p satisfy the assumptions of section (1.3) and are C^k functions of their arguments. Let $U^0(s) \in C^k_p$ be a stationary inviscid shock on $[a - \overline{r}, b - \overline{r}]$ with supersonic inflow at $a - \overline{r}$, shock surface at $s = 0$, and taking the values w_a, w_b at $s = a - \overline{r}$ and $s = b - \overline{r}$ respectively. With $w = (w^1, w^2) := (\rho, (u, \theta))$, write the interior equation (4.8)(a) as $\mathcal{E}(w) = 0$. Then one can construct an approximate solution of (4.8) of the form*

$$\tilde{w}^\epsilon(s) = \left(\mathcal{U}^0(s, z) + \epsilon \mathcal{U}^1(s, z) + \cdots + \epsilon^M \mathcal{U}^M(s, z)\right)|_{z = \frac{s}{\epsilon}}. \tag{4.11}$$

Denoting the left side of (4.8)(a) by $\mathcal{E}(w)$, we can write the transmission problem satisfied by $\tilde{w}^\varepsilon$ as

$$\begin{aligned}
&\mathcal{E}(\tilde{w}) = \varepsilon^M R^{M,\varepsilon} \text{ on } [a - \overline{r}, b - \overline{r}] \cap \{\pm s \geq 0\} \\
&[\tilde{w}] = 0, [\tilde{w}^2_s] = 0 \text{ on } s = 0 \\
&\tilde{w}(a - \overline{r}) = w_a + O(\varepsilon),\ \tilde{w}(b - \overline{r}) = w_b + O\left(e^{-\frac{\beta}{\varepsilon}}\right) \text{ for some } \beta > 0.
\end{aligned} \tag{4.12}$$

Here $\mathcal{U}^j(s,z) = U^j(s) + V^j(z)$, with $U^0(s)$ the given inviscid shock, and

$$\begin{aligned} &U^j \in C_p^{k-j} \\ &V^0 \in \tilde{C}_p^k, \ \ V^j \in \tilde{C}_p^{k-1} \text{ for } j \geq 1, \end{aligned} \tag{4.13}$$

and there exist constants M_j such that

$$|(\epsilon\partial_s)^j R^{M,\epsilon}|_{C_p^0} \leq M_j, \text{ for } j \leq k - M - 2. \tag{4.14}$$

The profiles U^j, V^j defining the approximate solution $\tilde{w}^\varepsilon$ satisfy explicit and, for $j \geq 1$, linear *profile equations. The fast part of the leading profile $\mathcal{U}^0_\pm(0,z) = U^0_\pm(0) + V^0_\pm(z)$ has $\mathcal{U}^0_\pm(0,z)$ matching smoothly across $z = 0$ and is the profile constructed by Gilbarg in [Gi].*

Proof. The proof of this Proposition is essentially the same as that of Proposition 6.6 of [EJW]. Although [EJW] is concerned mainly with the barotropic case, the proof of Proposition 6.6 of that paper carries over essentially unchanged. The existence of the fast part, $\mathcal{U}^0(0,z)$, of the leading profile in (4.11) follows directly from an argument (in a slightly different context) due to Gilbarg [Gi]. Indeed, the equation satisfied by $\mathcal{U}^0(0,z)$,

$$\partial_z f(\mathcal{U}^0) - \partial_z(\underline{B}(\mathcal{U}^0)\partial_z\mathcal{U}^0) = 0, \quad \text{where } \underline{B}(w) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \underline{\nu} & 0 \\ 0 & \underline{\nu}u & \underline{\kappa} \end{pmatrix}, \tag{4.15}$$

is equivalent to the one studied by Gilbarg; our convexity assumptions on p and e allow his argument to be applied. The slow part of the leading profile, $\mathcal{U}^0(s,z) - \mathcal{U}^0(0,z)$ is given by $U^0(s) - U^0(0)$, where $U^0(s)$ is the inviscid shock constructed earlier. With the leading profile thereby constructed, the essential condition that needs to be satisfied in order to obtain the higher profiles is a transversality condition: the orbit $\mathcal{U}^0(0,z)$ must be transverse in the sense that the stable and unstable manifolds of the rest points it connects, namely $U^0(0+)$ and $U^0(0-)$, must intersect transversally. This condition is known to be satisfied by the Gilbarg profile. We refer to the proof of [EJW], Proposition 6.6, for the explicit profile equations, as we shall not have need for them below. The higher order profile equations are obtained from linearizations of (4.15) and the Euler equations about the leading profiles. □

We remark that the proof of Proposition 4.2 is a simplified version of an argument first given in given in [GW].

4.2 Error problem

We look for an exact solution to the transmission problem (4.8) in the form

$$w^\varepsilon = \tilde{w}^\varepsilon + \varepsilon^L v^\varepsilon, \ \ 1 \leq L < M. \tag{4.16}$$

Here L, M are any integers satisfying $1 \le L < M$. Writing $A = d_w f$, subtracting $\mathcal{E}(w^\varepsilon) - \mathcal{E}(\tilde{w}^\varepsilon)$, and cancelling ε^L, we obtain the *error problem* for v^ε:

$$\begin{aligned}
&(a)\ A(\tilde{w} + \varepsilon^L v) d_s v + \left(v \cdot \int_0^1 \partial_w A(\tilde{w} + \sigma\varepsilon^L v) d\sigma \right) d_s \tilde{w} + v \cdot \int_0^1 \partial_w g(\tilde{w} + \sigma\varepsilon^L v, s) d\sigma \\
&\qquad - \varepsilon\ \frac{h(w^2, w_s^2, w_{ss}^2, s) - h(\tilde{w}^2, \tilde{w}_s^2, \tilde{w}_{ss}^2, s)}{\varepsilon^L} = -\varepsilon^{M-L} R^{M,\varepsilon} \\
&(b)\ [v] = 0,\ [v_s^2] = 0 \text{ on } s = 0 \\
&(c)\ v(b - \overline{r}) = \varepsilon^{-L}\left(w_b - \tilde{w}(b - \overline{r})\right) = O\left(e^{-\frac{\beta}{\varepsilon}}\right).
\end{aligned} \tag{4.17}$$

Let us write $v = (v^1, v^2) = (\rho, (u, \theta))$ and $\tilde{w} = (\tilde{\rho}, \tilde{u}, \tilde{\theta})$ (again suppressing epsilons). The quotient $D(v, v_s, v_{ss})$ multiplying $-\varepsilon$ in (4.17)(a) is given by

$$\begin{aligned}
&D = D_1 v_{ss} + D_2(v, v_s) + D_3 v := \\
&\begin{pmatrix} 0 \\ \underline{\nu} u_{ss} \\ \underline{\nu}\tilde{u} u_{ss} + \underline{\kappa}\theta_{ss} \end{pmatrix} + \begin{pmatrix} 0 \\ \underline{\nu}\left(\frac{2u_s}{s+\overline{r}} - \frac{2u}{(s+\overline{r})^2}\right) \\ \underline{\nu}\tilde{u}_s u_s + (6\underline{\nu} - 8\underline{\mu})(\frac{\tilde{u}_s u}{s+\overline{r}}) + H(\tilde{u}, v^2, v_s^2, \varepsilon) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \underline{\nu}\tilde{u}_{ss} u \end{pmatrix}.
\end{aligned} \tag{4.18}$$

Here H is a sum of terms

$$\begin{aligned}
&(a)\ \left(c_1(s)\tilde{u} + c_2(s)\varepsilon^L u_s + c_3(s)\varepsilon^L u\right) u_s := C_a(\tilde{u}, \varepsilon^L u, \varepsilon^L u_s, s) u_s,\ \ c_4(s)\theta_s \text{ and} \\
&(b)\ \left(c_5(s)\tilde{u} + c_6(s)\varepsilon^L u\right) u := C_b(\tilde{u}, \varepsilon^L u, s) u,
\end{aligned} \tag{4.19}$$

where for every $j = 1, ..., 6$, $c_j(s)$ is a smooth function bounded away from 0 on $[a - \overline{r}, b - \overline{r}]$. The important term in εD is $\varepsilon D_1 v_{ss}$. Our later analysis will show that $\varepsilon D_2(v, v_s)$ and even $\varepsilon D_3 v$ are in a sense negligible.

Next we rewrite (4.17)(a) as

$$A d_s v + \frac{1}{\varepsilon} B v + C v - \varepsilon D(v, v_s, v_{ss}) = -\varepsilon^{M-L} R^M, \tag{4.20}$$

where, for example,

$$B v := \varepsilon \left(v \cdot \int_0^1 \partial_w A(\tilde{w} + \sigma\varepsilon^L v) d\sigma \right) d_s \tilde{w}. \tag{4.21}$$

Incorporating terms from $D_2(v, v_s)$ into A and C we can rewrite (4.20) as

$$\mathcal{A} d_s v + \frac{1}{\varepsilon} B v + \mathcal{C} v - \varepsilon D_1 v_{ss} - \varepsilon D_3 v = -\varepsilon^{M-L} R^M, \tag{4.22}$$

where now

$$\mathcal{A} := A - \varepsilon \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \underline{\nu}\tilde{u}_s & 0 \end{pmatrix} - \varepsilon \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{2\underline{\nu}}{s+\overline{r}} & 0 \\ 0 & C_a(\tilde{u}, \varepsilon^L u, \varepsilon^L u_s, s) & c_4(s) \end{pmatrix} := A - \varepsilon \mathbb{A}_1 - \varepsilon \mathbb{A}_2 \text{ and} \tag{4.23}$$

$$\mathcal{C} := C - \varepsilon \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{(6\underline{\nu}-8\underline{\mu})\tilde{u}_s}{s+\overline{r}} & 0 \end{pmatrix} - \varepsilon \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\frac{2\underline{\nu}}{(s+\overline{r})^2} & 0 \\ 0 & C_b(\tilde{u}, \varepsilon^L u, s) & 0 \end{pmatrix} := C - \varepsilon\mathbb{C}_1 - \varepsilon\mathbb{C}_2. \tag{4.24}$$

Remark 4.3. *(1) Let us write $f^\varepsilon(s) = O(\varepsilon^k)$ to mean that*

$$|f^\varepsilon(s)| \le C\varepsilon^k \text{ uniformly on } [a-\overline{r}, b-\overline{r}]. \tag{4.25}$$

Clearly, as long as $(\varepsilon^L u, \varepsilon^L u_s)$ remains bounded, we have

$$\varepsilon\mathbb{A}_2 = O(\varepsilon) \text{ and } \varepsilon\mathbb{C}_2 = O(\varepsilon), \tag{4.26}$$

but the same is not true for $\varepsilon\mathbb{A}_1$, $\varepsilon\mathbb{C}_1$. In treating the latter terms we will later use the fact, a consequence of (4.11)*, that*

$$\varepsilon\tilde{w}_s = V_z^0(\frac{s}{\varepsilon}) + O(\varepsilon), \text{ where } V_z^0(z) = O(e^{-\beta|z|}) \text{ for some } \beta > 0. \tag{4.27}$$

(2) We make the temporary assumption *that*

$$(v, \varepsilon v_s^2) = O(1) \text{ for } \varepsilon \in (0,1]. \tag{4.28}$$

This assumption is justified later by the estimates (5.40)*.*

4.3 First-order system

Setting $E := B + \epsilon\mathcal{C} - \varepsilon^2 D_3$ (note: E is uniformly bounded for ε small) and writing

$$D_1 v_{ss} = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{D} \end{pmatrix} \begin{pmatrix} v_{ss}^1 \\ v_{ss}^2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \underline{\nu} & 0 \\ 0 & \underline{\nu}\tilde{u} & \underline{\kappa} \end{pmatrix} \begin{pmatrix} \rho_{ss} \\ u_{ss} \\ \theta_{ss} \end{pmatrix}, \tag{4.29}$$

we next split the matrix equation (4.22) into components:

$$\begin{aligned} &\mathcal{A}^{11} d_s v^1 + \mathcal{A}^{12} d_s v^2 + \frac{1}{\epsilon} E^{11} v^1 + \frac{1}{\epsilon} E^{12} v^2 = -\epsilon^{M-L} R^{M,1} \\ &\mathcal{A}^{21} d_s v^1 + \mathcal{A}^{22} d_s v^2 + \frac{1}{\epsilon} E^{21} v^1 + \frac{1}{\epsilon} E^{22} v^2 - \epsilon\mathbb{D} v_{ss}^2 = -\epsilon^{M-L} R^{M,2}. \end{aligned} \tag{4.30}$$

Define $V = (v^1, v^2, v^3)^t$, where $v^3 = \epsilon v_s^2$, and rewrite (4.17) as a 5×5 first-order transmission problem on $[a-\overline{r}, b-\overline{r}]$:

$$\begin{aligned} &d_s V = \frac{1}{\epsilon} G V + F, \\ &[V] = 0 \text{ on } s = 0, \\ &\begin{pmatrix} v^1 \\ v^2 \end{pmatrix} (b-\overline{r}) = \epsilon^{-L}\left((w_b - \tilde{w}(b-\overline{r})\right) := \overline{v}, \end{aligned} \tag{4.31}$$

where

$$G = \begin{pmatrix} g^{11} & g^{12} & g^{13} \\ 0 & 0 & 1 \\ g^{31} & g^{32} & g^{33} \end{pmatrix} \quad F = \begin{pmatrix} -(\mathcal{A}^{11})^{-1}\epsilon^{M-L}R^{M,1} \\ 0 \\ \mathbb{D}^{-1}\left(\epsilon^{M-L}R^{M,2} - \mathcal{A}^{21}(\mathcal{A}^{11})^{-1}\epsilon^{M-L}R^{M,1}\right) \end{pmatrix}, \text{ with} \tag{4.32}$$

$$\begin{aligned} &g^{11} = -(\mathcal{A}^{11})^{-1}E^{11}, \quad g^{12} = -(\mathcal{A}^{11})^{-1}E^{12}, \quad g^{13} = -(\mathcal{A}^{11})^{-1}\mathcal{A}^{12} \\ &g^{31} = \mathbb{D}^{-1}\left(E^{21} + \mathcal{A}^{21}g^{11}\right), \quad g^{32} = \mathbb{D}^{-1}\left(E^{22} + \mathcal{A}^{21}g^{12}\right), \\ &g^{33} = \mathbb{D}^{-1}\left(\mathcal{A}^{22} + \mathcal{A}^{21}g^{13}\right) = \mathbb{D}^{-1}\left(\mathcal{A}^{22} - \mathcal{A}^{21}(\mathcal{A}^{11})^{-1}\mathcal{A}^{12}\right). \end{aligned} \tag{4.33}$$

Notation 4.4. *Suppressing the subscript* $\pm$, *we define* $q^\epsilon(s) = (q^{\epsilon,1}, \dots, q^{\epsilon,7})$ *and* $Q^\varepsilon(s) = (Q^{\varepsilon,1}, Q^{\varepsilon,2}, Q^{\varepsilon,3})$ *by*

1. $q^{\epsilon,1}(s) = U^0(0)$.

2. $q^{\epsilon,2}(s) = (U^0(s) - U^0(0)) + \epsilon\left(U^1(s) + V^1(\frac{s}{\epsilon})\right) + \cdots + \epsilon^M\left(U^M(s) + V^M(\frac{s}{\epsilon})\right)$.

3. $q^{\epsilon,3}(s) = \epsilon^L v^\epsilon$, *where we suppose that* $v^\epsilon(s)$ *is bounded in* C^j_p *for some* $j \geq 1$.

4. $d_s\tilde{w} = \frac{1}{\epsilon}\left(d_zV^0(z)|_{z=\frac{s}{\epsilon}} + q^{\epsilon,4}(s)\right)$ *(hence,* $q^{\epsilon,4}(s) = O(\epsilon)$ *uniformly on* $[a-\overline{r}, b-\overline{r}]$*).*

5. $q^{\epsilon,5}(s) = \varepsilon\mathcal{C}$.

6. $q^{\epsilon,6}(s) = -\varepsilon\mathbb{A}_2$.

7. $d^2_s\tilde{w} = \frac{1}{\varepsilon^2}\left(d^2_zV^0(z)|_{z=\frac{s}{\varepsilon}} + q^{\varepsilon,7}(s)\right)$ *(so* $q^{\varepsilon,7}(s) = O(\varepsilon)$ *uniformly on* $[a-\overline{r}, b-\overline{r}]$*).*

8. $Q^{\varepsilon,1}(s) = V^0(z)_{|z=\frac{s}{\varepsilon}}$.

9. $Q^{\varepsilon,2}(s) = d_zV^0(z)_{|z=\frac{s}{\varepsilon}}$.

10. $Q^{\varepsilon,3}(s) = d^2_zV^0(z)_{|z=\frac{s}{\varepsilon}}$.

11. When $\epsilon \downarrow 0$ *observe that* $q^\epsilon(s) \to q^0(s) := (U^0(0), U^0(s) - U^0(0), 0)$ *in* C^0_p. *Moreover, for* $|s| \geq \delta > 0$,

$$|Q^\varepsilon(s)| \leq Ce^{-\beta/\varepsilon} \text{ for some } \beta > 0. \tag{4.34}$$

12. The symbol ε_0 *will always denote some sufficiently small positive number.*

Suppressing some epsilons and evaluations at s, we have

$$\begin{aligned} &(a)\ \tilde{w}(s) = V^0(z)|_{z=\frac{s}{\epsilon}} + q^1 + q^2 = Q^1 + q^1 + q^2 \\ &(b)\ w(s) = V^0(z)|_{z=\frac{s}{\epsilon}} + q^1 + q^2 + q^3 = Q^1 + q^1 + q^2 + q^3 \\ &(c) d_s\tilde{w}(s) = \frac{1}{\varepsilon}(Q^2 + q^4);\ d^2_s\tilde{w}(s) = \frac{1}{\varepsilon}(Q^3 + q^7). \end{aligned} \tag{4.35}$$

In the obvious way, we may now regard the matrix coefficients appearing in (4.29)-(4.33) *either* as defining corresponding functions of (z, q), *or* as defining corresponding functions of (Q, q). For example, we write with slight abuse,

$$\begin{aligned} &A(\tilde{w} + \epsilon^L v) = A(z,q)|_{z=\frac{s}{\epsilon}, q=q^\epsilon(s)} = A(Q,q)|_{Q=Q^\varepsilon(s), q=q^\epsilon(s)} \\ &E = E(z,q)|_{z=\frac{s}{\epsilon}, q=q^\epsilon(s)} = E(Q,q)|_{Q=Q^\varepsilon(s), q=q^\varepsilon(s)}, \\ &G = G(z,q)|_{z=\frac{s}{\epsilon}, q=q^\epsilon(s)} = G(Q,q)|_{Q=Q^\varepsilon(s), q=q^\varepsilon(s)}. \end{aligned} \tag{4.36}$$

More precisely, one could write for example,

$$\begin{aligned} E(z,q) &= \mathcal{E}_1(V^0(z)+q^1+q^2+q^3)(d_zV^0(z)+q^4)+q^5+\mathcal{E}_2(d_z^2V^0(z)+q^7) \\ E(Q,q) &= \mathcal{E}_1(Q^1+q^1+q^2+q^3)(Q^2+q^4)+q^5+\mathcal{E}_2(Q^3+q^7). \end{aligned} \tag{4.37}$$

for obvious functions $\mathcal{E}_1$, $\mathcal{E}_2$. Note that z-dependence in the above functions of (z,q) enters *only* through V^0, d_zV^0, or $d_z^2V^0$.

Definition 4.5. *We will solve the transmission problem* (4.31) *on* $[a-\overline{r}, b-\overline{r}]$ *by breaking up the interval into four subintervals* $[a-\overline{r}, -\delta]$, $[-\delta, 0]$, $[0,\delta]$, *and* $[\delta, b-\overline{r}]$, *and solving separate problems, labelled I, II, III, and IV, respectively, on these subintervals. Here* $\delta > 0$ *is a small parameter to be chosen later. Thus, for example, Problem IV is*

$$d_sV = \frac{1}{\varepsilon}GV + F \ \text{ on } [\delta, b-\overline{r}] \tag{4.38}$$

with boundary conditions split between the endpoints that are prescribed for a conjugated variable $\mathcal{V}$, *instead of* V, *in terms of initially undetermined parameters* $P = (p^*, p^3_+, p^3_-) \in \mathbb{R}^5$ *(see* (5.30)*). Problem III is defined similarly on* $[0,\delta]$ *and the (split) boundary conditions there are specified in terms of parameters* $\Pi = (\pi^*, \pi^3_+, \pi^3_-)$ *(see* (5.48)*). The problems will be solved in the order IV, ..., I. The parameters entering problems IV and III are chosen in section 5.4 so that the boundary condition on* V *at* $b-\overline{r}$ *in* (4.31) *is satisfied and so that the transmission condition* $[V] = 0$ *holds at* $s = \delta$. *Once the solutions* V_4 *and* V_3 *to problems IV and III are completely determined, the boundary condition for problem II is chosen simply to be* $V_2(0) = V_3(0)$. *Problem I is handled similarly with a boundary condition imposed at* $s = -\delta$.

Remark 4.6. *In problems II and III on* $[-\delta, 0]$ *and* $[0,\delta]$ *respectively, we shall regard* G *as a function of* (z,q). *In problems I and IV on* $[a-\overline{r}, -\delta]$*and* $[\delta, b-\overline{r}]$ *respectively, we shall regard* G *as a function of* (Q,q).

We will use the next Lemma to conjugate G to simpler forms.

Lemma 4.7. *Let* $G(\infty, q)$ *be the matrix obtained from* $G(z,q)$ *by setting* $V^0(z) = 0$ *and* $d_zV^0(z) = 0$ *in all coefficients. The matrices* $G(\infty,q)$ *and* $G(Q,q)$ *satisfy*

$$\begin{aligned} G(Q,q) &= \begin{pmatrix} O(Q^2+q^4,q^5,Q^3+q^7) & O(Q^2+q^4,q^5,Q^3+q^7) & g^{13}(Q,q) \\ 0 & 0 & 1 \\ O(Q^2+q^4,q^5,Q^3+q^7) & O(Q^2+q^4,q^5,Q^3+q^7) & g^{33}(Q,q) \end{pmatrix} \\ G(\infty,q) &= \begin{pmatrix} O(q^4,q^5,q^7) & O(q^4,q^5,q^7) & g^{13}(\infty,q) \\ 0 & 0 & 1 \\ O(q^4,q^5,q^7) & O(q^4,q^5,q^7) & g^{33}(\infty,q) \end{pmatrix}. \end{aligned} \tag{4.39}$$

Here we write $f(Q,q) = O(Q^2+q^4, q^5, Q^3+q^7)$, *for example, if* $|f(Q,q)| \le C|(Q^2 + q^4, q^5, Q^3+q^7)|$.

Proof. Consider, for example,

$$g^{31}(Q,q) = \mathbb{D}^{-1}(E^{21} - \mathcal{A}^{21}(\mathcal{A}^{11})^{-1}E^{11}). \tag{4.40}$$

The estimate $g^{31}(Q,q) = O(Q^2+q^4, q^5, Q^3+q^7)$ follows immediately from (4.37), since $\mathcal{E}_1$, $\mathcal{E}_2$ are smooth functions and (Q,q) varies in a bounded, closed ball in $(Q,q)-$space. The estimate $g^{31}(\infty,q) = O(q^4,q^5,q^7)$ follows by setting $Q=0$.

$\square$

Remark 4.8. *(1) From the definitions of $Q^\varepsilon(s)$ and $q^\varepsilon(s)$ in Notation 4.4, we deduce immediately from Lemma 4.7*

$$\begin{aligned} &(a)\ G(\infty, q^\varepsilon(s)) = \begin{pmatrix} O(\varepsilon) & O(\varepsilon) & g^{13} \\ 0 & 0 & 1 \\ O(\varepsilon) & O(\varepsilon) & g^{33} \end{pmatrix} \ \textit{on}\ [a-\overline{r}, b-\overline{r}] \\ &(b)\ G(Q^\varepsilon(s), q^\varepsilon(s)) = \begin{pmatrix} O(\varepsilon) & O(\varepsilon) & g^{13} \\ 0 & 0 & 1 \\ O(\varepsilon) & O(\varepsilon) & g^{33} \end{pmatrix} \ \textit{on}\ [a-\overline{r}, b-\overline{r}] \cap \{|s| \geq \delta > 0\}. \end{aligned} \tag{4.41}$$

(2)On $|s| \leq \delta$ the nonzero entries of $G(Q^\varepsilon(s), q^\varepsilon(s))$ are all $O(1)$.

4.4 Eigenvalues of g^{33}.

The matrix A in (4.17) is evaluated at

$$\tilde{w} + \varepsilon^L v = U^0(s) + V^0(\frac{s}{\varepsilon}) + O(\varepsilon). \tag{4.42}$$

Using the definition of $\mathcal{A}$ and recalling Remark 4.3(2), we see that if $\delta > 0$, we have

$$\tilde{w}_\pm + \varepsilon^L v_\pm = U^0_\pm + O(\varepsilon) \text{ on } [a-\overline{r}, b-\overline{r}] \cap \pm\{s \geq \delta\}. \tag{4.43}$$

We proceed to compute the eigenvalues of the 2×2 matrix $g^{33}(U^0_\pm(s))$, which play a critical role in the later analysis. Here by $g^{33}(U^0_\pm(s))$ we denote, with slight abuse, the matrix computed using the formula in (4.33), but with the $\mathcal{A}^{ij}$ replaced by $A^{ij}(U^0_\pm)$.

Set $U^0 = (\rho_0, u_0, \theta_0)$. Direct computation using (4.33), (4.29), and the expression for $f(w)$ in (4.3) yields

$$g^{33}(U^0) = \begin{pmatrix} (\rho_0 u_0 - p_\rho \frac{\rho_0}{u_0})/\underline{\nu} & p_\theta/\underline{\nu} \\ \frac{1}{\underline{\kappa}}(p - \rho_0^2 e_\rho) & \frac{\rho_0 e_\theta u_0}{\underline{\kappa}} \end{pmatrix}. \tag{4.44}$$

The characteristic polynomial of $g^{33}(U^0)$ is (here we drop the subscript 0 on ρ_0, u_0, θ_0)

$$\begin{aligned} &\lambda^2 + b\lambda + d := \\ &\quad \lambda^2 - \lambda\left[\frac{\rho e_\theta u}{\underline{\kappa}} + \frac{1}{\underline{\nu}}\left(\rho u - p_\rho \frac{\rho}{u}\right)\right] + \frac{1}{\underline{\nu\kappa}}\left[(\rho e_\theta u)(\rho u - p_\rho \frac{\rho}{u}) - (p - \rho^2 e_\rho)p_\theta\right]. \end{aligned} \tag{4.45}$$

Thus, the eigenvalues of $g^{33}(U^0)$ are

$$\begin{aligned} \lambda_\pm &= \frac{-b \pm \sqrt{b^2-4d}}{2}, \quad \text{where } -b = \frac{\rho e_\theta u}{\underline{\kappa}} + \frac{1}{\underline{\nu}}\left(\rho u - p_\rho \frac{\rho}{u}\right) \text{ and} \\ b^2 - 4d &= \left[\frac{\rho e_\theta u}{\underline{\kappa}} - \frac{1}{\underline{\nu}}\left(\rho u - p_\rho \frac{\rho}{u}\right)\right]^2 + \frac{4}{\underline{\nu\kappa}}(p - \rho^2 e_\rho)p_\theta. \end{aligned} \tag{4.46}$$

Notation 4.9. *In the equations* (4.2) *we have taken pressure p, internal energy e, and entropy S to be functions of (ρ,θ). Let $\hat{p}(\rho,S)$ and $\hat{e}(\rho,S)$ be such that*

$$p(\rho,\theta)=\hat{p}(\rho,S(\rho,\theta)),\ e(\rho,\theta)=\hat{e}(\rho,S(\rho,\theta)). \tag{4.47}$$

We will sometimes write $\hat{e}(\rho,S)=\hat{e}(\tau,S)$, $e(\rho,\theta)=e(\tau,\theta)$, etc., where $\tau=1/\rho$.

The following Proposition is proved in the next section:

Proposition 4.10. *Let $\hat{e}(\tau,S)$ be as described in section 1.3 and assume as before (see* (1.24)*)*

$$\hat{p}_\rho>0,\ \hat{p}_S>0. \tag{4.48}$$

(a) If $u>c:=\sqrt{\hat{p}_\rho}$ (u supersonic), then both eigenvalues $\lambda_\pm$ of $g^{33}(\rho,u,\theta)$ are positive.
(b)If $0<u<c$ (u subsonic), we have $\lambda_+>0$ and $\lambda_-<0$.

Remark 4.11. *Since $e_\theta=\hat{e}_S/\hat{\theta}_S=\hat{e}_S/\hat{e}_{SS}$, a quotient of positive quantities, we have $e_\theta>0$.*

4.5 Thermodynamic relations

We first express the sound speed in terms of the functions $p(\rho,\theta)$ and $e(\rho,\theta)$.

Lemma 4.12. *Under the assumptions of Proposition 4.10, we have*

$$c^2=p_\rho+\frac{p_\theta}{e_\theta}\left(\frac{p}{\rho^2}-e_\rho\right). \tag{4.49}$$

Proof. We have

$$\hat{p}(\rho,S)=p(\rho,\hat{\theta}(\rho,S)). \tag{4.50}$$

Thus,

$$c^2=\hat{p}_\rho=p_\rho+p_\theta\hat{\theta}_\rho. \tag{4.51}$$

On the other hand

$$\hat{e}(\rho,S)=e(\rho,\hat{\theta}(\rho,S)) \text{ and thus } \hat{e}_\rho=e_\rho+e_\theta\hat{\theta}_\rho. \tag{4.52}$$

The first law of thermodynamics,

$$d\hat{e}=\hat{\theta}dS-\hat{p}d\tau \text{ or equivalently } \hat{e}_S=\hat{\theta},\ \hat{e}_\tau=-\hat{p}, \tag{4.53}$$

shows that $\hat{e}_\rho=\frac{p}{\rho^2}$. Solving for $\hat{\theta}_\rho$ in (4.52) and substituting in (4.51) gives (4.49).

□

Lemma 4.13. *Under the assumptions of Proposition 4.10, we have*
(a) $p_\theta>0$ and
(b)$\frac{1}{\theta}\left(e_\rho-\frac{p}{\rho^2}\right)=S_\rho<0$.

Proof. **1.** From (4.47) we have

$$e_\theta = \hat{e}_S S_\theta. \tag{4.54}$$

Since $e_\theta > 0$ and (from (4.53)) $\hat{e}_S = \hat{\theta} > 0$ we have $S_\theta > 0$. Thus,

$$p_\theta = \hat{p}_S S_\theta > 0. \tag{4.55}$$

2. From (4.47) and (4.53) we have

$$\begin{aligned} &e_\tau = \hat{e}_\tau + \hat{e}_S S_\tau = -p + \theta S_\tau, \text{ hence} \\ &\rho^2 e_\rho = p + \theta \rho^2 S_\rho, \end{aligned} \tag{4.56}$$

which gives the equality in part (b). To determine the sign we note

$$\hat{e}_S(\rho, S(\rho,\theta)) = \theta, \text{ so } \hat{e}_{S\tau}(-\frac{1}{\rho^2}) + \hat{e}_{SS} S_\rho = 0. \tag{4.57}$$

Since $\hat{e}_{SS} > 0$, it follows that S_ρ has the same sign as $\hat{e}_{S\tau} = -\hat{p}_S < 0$.

□

Proof of Proposition 4.10. We refer to the formulas for $\lambda_\pm$ and $b^2 - 4d$ in (4.46). Assuming $u^2 > c^2$, we have from Lemmas 4.12 and 4.13

$$u^2 - p_\rho > \frac{p_\theta}{e_\theta}(\frac{p}{\rho^2} - e_\rho) > 0. \tag{4.58}$$

Thus, both terms in the sum defining $-b$ are positive and we have

$$-b > 0. \tag{4.59}$$

Clearly, Lemma 4.13 implies

$$b^2 - 4d > 0. \tag{4.60}$$

From (4.45) and the formula for c^2 in Lemma 4.12 we see that

$$\pm(u^2 - c^2) > 0 \Leftrightarrow \pm d > 0. \tag{4.61}$$

From (4.59)-(4.61) we immediately deduce

$$u^2 > c^2 \Rightarrow \lambda_\pm > 0. \tag{4.62}$$

On the other hand if $u^2 < c^2$ (4.61) implies

$$b^2 - 4d > b^2, \text{ so } \lambda_+ > 0 \text{ and } \lambda_- < 0. \tag{4.63}$$

□

5 Exact Navier-Stokes solutions: solution of the error problem

5.1 Conjugators

We first construct a conjugator that will be used for solving problem IV. We define a contractible neighborhood $\mathcal{Q}_\rho^+$ in $(Q,q)-$space to be the Cartesian product

$$\mathcal{Q}_\rho^+ = B_\rho \times \mathcal{T}^+ \tag{5.1}$$

of an open ball B_ρ of radius ρ centered at 0 in $(Q, q^3, \dots, q^7)-$space with a small tubular neighborhood $\mathcal{T}^+$ of the curve $\{\big(U_+^0(0), U_+^0(s) - U_+^0(0)\big) : s \in [0, b-\overline{r}]\}$ in $(q^1, q^2)-$space.

Remark 5.1. *Observe that for any $\rho > 0$ and $0 < \delta < b - \overline{r}$, we have*

$$(Q^\varepsilon(s), q^\varepsilon(s)) \in \mathcal{Q}_\rho^+ \text{ for } s \in [\delta, b-\overline{r}] \text{ and } \varepsilon \text{ sufficiently small.} \tag{5.2}$$

Proposition 5.2. *For $\mathcal{Q}_\rho^+$ as in (5.1) small enough, there exists a smooth invertible 5×5 matrix $S(Q,q)$ defined on $\mathcal{Q}_\rho^+$ such that*

$$S^{-1}G(Q,q)S = \begin{pmatrix} H & 0 \\ 0 & P \end{pmatrix} := G_B, \tag{5.3}$$

where H is 3×3, P is 2×2, and

$$\begin{aligned} &H(Q,q) = O(Q, q^3, \dots, q^7), \\ &P(Q,q) = \begin{pmatrix} \lambda_+(Q,q) + O(Q,q^3,\dots,q^7) & 0 \\ 0 & \lambda_-(Q,q) + O(Q,q^3,\dots,q^7) \end{pmatrix}. \end{aligned} \tag{5.4}$$

The matrix $S(Q,q)$ has the form

$$S = \begin{pmatrix} S^{11} & S^{12} \\ O(Q,q^3,\dots,q^7) & S^{22} \end{pmatrix}, \tag{5.5}$$

with S^{11} and S^{22} invertible on $\overline{\mathcal{Q}}_\rho^+$ and of size 3×3 and 2×2 respectively.

Proof. **1.** Recall that the matrices A^{ij} appearing in the definition of $g^{33}(Q,q)$ are evaluated at $Q^1 + q^1 + q^2 + q^3$. Thus, in view of the formula (4.39) for $G(Q,q)$ and Proposition 4.10, we see that for $(Q,q) \in \mathcal{Q}_\rho^+$, $G(Q,q)$ has 3 eigenvalues close to 0 and 2 eigenvalues close to those of g^{33}. Moreover, for such (Q,q) there exist constants c_1, c_2 independent of (Q,q) satisfying

$$\lambda_+(Q,q) > c_1 > 0 > c_2 > \lambda_-(Q,q). \tag{5.6}$$

2. Using the above described spectral separation for G, we construct smooth projectors $P_i(Q,q)$, $i = 1,2$ onto invariant spaces of $G(Q,q)$:

$$P_i(Q,q) = \frac{1}{2\pi i}\int_{\Gamma_i} (\xi - G(Q,q))^{-1} d\xi, \tag{5.7}$$

where Γ_i, $i = 1, 2$ are closed contours in the complex $\xi-$plane enclosing the origin and the eigenvalues of g^{33}, respectively.

3. The contractibility of $\mathcal{Q}^+_\rho$ allows us to choose smoothly varying bases $\mathcal{B}_1(Q, q)$, $\mathcal{B}_2(Q, q)$ for the images of $P_1(Q, q)$ and $P_2(Q, q)$ respectively. Taking the first 3 columns of $S_a(Q, q)$ to be the elements of $\mathcal{B}_1(Q, q)$ and the last two columns the elements of $\mathcal{B}_2(Q, q)$, we obtain a matrix S_a that conjugates G to a block form like (5.3), but where instead of P we have

$$\tilde{P}(Q, q) = g^{33}(Q, q) + O(Q, q^3, \dots, q^7). \tag{5.8}$$

4. At $(Q, q^3, \dots, q^7) = 0$ the span of the first 3 columns of S_a, span $S_{a,I}$, is

$$\ker G(0, q^1, q^2, 0) = \mathbb{R}^3 \times \{0\}. \tag{5.9}$$

Thus, we must have

$$S_{a,I}(Q, q) = \begin{pmatrix} S_a^{11} \\ S_a^{21} \end{pmatrix}, \tag{5.10}$$

where S_a^{11} is 3×3 and invertible and S_a^{21} vanishes at $(Q, q^3, \dots, q^7)$. Since S_a is invertible and S_a^{21} vanishes at $(Q, q^3, \dots, q^7)$, we conclude that S_a^{22} is invertible for ρ small enough.

Let $e_j(Q, q)$, $j = 1, 2$ be smoothly varying eigenvectors of $\tilde{P}(Q, q)$. If we conjugate the matrix with blocks H and $\tilde{P}$ by

$$S_b = \begin{pmatrix} I & 0 \\ 0 & s_2 \end{pmatrix} \tag{5.11}$$

where I is 3×3 and the last two columns of S_b have the form $(0, e_j(Q, q))$, we obtain the form G_B as in (5.3), (5.4). Thus, the conjugator is $S := S_a S_b$ has all the stated properties. $\square$

Remark 5.3. *Since we will use the conjugator S in solving problem IV, let us denote it by $S_4(Q, q)$. Clearly, one can construct in the same way a conjugator $S_1(Q, q)$ for problem I on a set*

$$\mathcal{Q}^-_\rho = B_\rho \times \mathcal{T}^-, \tag{5.12}$$

where $\mathcal{T}^-$ is now a small tubular neighborhood of the curve $\{(U^0_-(0), U^0_-(s) - U^0_-(0)) : s \in [a - \bar{r}, 0]\}$ in $(q^1, q^2)-$space. The conjugator S_1 has the properties (5.3)-(5.5)*, except that now in place of* (5.6) *we have*

$$\lambda_+(Q, q) > c_3 > \lambda_-(Q, q) > c_4 > 0 \textit{ for } (Q, q) \in \mathcal{Q}^-_\rho, \tag{5.13}$$

for some constants c_3, c_4 independent of (Q, q).

We recall from Remark 4.8 that, since the fast variable $z = \frac{s}{\varepsilon}$ is sometimes close to zero in regions II and III, the matrix $G(Q^\varepsilon(s), q^\varepsilon(s))$ does *not* have the form (4.41)(b) in those regions; in fact, all nonzero entries are $O(1)$. The following Lemma provides a conjugator that "removes" the fast variable z in those regions and allows us to make use of (4.41)(a) instead.

Lemma 5.4 (See [MZ], Lemma 2.6)**.** *Let $U^0_\pm(0)$ be the endstates of the given inviscid shock, $U^0(s)$. There are neighborhoods $\mathcal{Q}_\pm$ of $(U^0_\pm(0), 0, \dots, 0)$ in $q = (q^1, \dots, q^7)-$space and matrices $T_\pm(z, q)$ defined and C^1 on $\{\pm z \geq 0\} \times \mathcal{Q}_\pm$ satisfying:*

(a) $T_\pm$ and $(T_\pm)^{-1}$ are uniformly bounded and there is a $\beta > 0$ such that for $q \in \mathcal{Q}_\pm$ and $|\alpha| \leq 1$,

$$|\partial^\alpha_{z,q}\,(T_\pm(z, q) - Id)\,| = O(e^{-\beta|z|}) \text{ on } \pm z \geq 0; \tag{5.14}$$

(b) $T_\pm$ satisfies the matrix differential equation on $\pm z \geq 0$

$$\partial_z T_\pm(z, q) = G(z, q)T_\pm(z, q) - T_\pm(z, q)G(\pm\infty, q). \tag{5.15}$$

$T_\pm(z, q)$ can be chosen to have the same regularity as $G(z, q)$.

An immediate corollary is that $V_\pm(z)$ satisfies

$$d_z V_\pm = G(z, q)V + f_\pm \text{ on } \pm z \geq 0 \tag{5.16}$$

if and only if $\mathcal{V}_\pm := (T_\pm)^{-1}V_\pm$ satisfies

$$d_z \mathcal{V}_\pm = G(\pm\infty, q)\mathcal{V}_\pm + (T_\pm)^{-1} f_\pm \text{ on } \pm z \geq 0. \tag{5.17}$$

Proposition 5.5. *There exist smooth invertible 5×5 matrices $S_\pm(q)$ defined on $\mathcal{Q}_\pm$ such that*

$$S^{-1}_\pm G(\pm\infty, q)S_\pm = \begin{pmatrix} H_\pm(q) & 0 \\ 0 & P_\pm(q) \end{pmatrix} := G_{B\pm}, \tag{5.18}$$

where $H_\pm$ is 3×3, $P_\pm$ is 2×2, and

$$H_\pm(q) = O(q^3, \dots, q^7),\ P_\pm(q) = \begin{pmatrix} \lambda_+(q) + O(q^3, \dots, q^7) & 0 \\ 0 & \lambda_-(q) + O(q^3, \dots, q^7) \end{pmatrix}. \tag{5.19}$$

The matrices $S_\pm(q)$ have the form

$$S_\pm(q) = \begin{pmatrix} S^{11}_\pm & S^{12}_\pm \\ O(q^3, \dots, q^7) & S^{22}_\pm \end{pmatrix}, \tag{5.20}$$

with $S^{11}_\pm$ and $S^{22}_\pm$ invertible on $\overline{\mathcal{Q}}_\pm$ and of size 3×3 and 2×2 respectively.

Proof. Fix $\rho > 0$. Observe that for $\mathcal{Q}_\pm$ small enough, we have

$$\{(Q, q) = (0, q) : q \in \mathcal{Q}_\pm\} \subset \mathcal{Q}^\pm_\rho. \tag{5.21}$$

Thus, the matrices $H_\pm(q)$, $P_\pm(q)$, and $S_\pm(q)$ in Proposition 5.5 can be obtained from the corresponding matrices in Proposition 5.2 just by setting $Q = 0$ in the latter matrices.

$\square$

Remark 5.6. *(1) For a given choice of $\mathcal{Q}_\pm$, we have for small enough positive constants δ and ε_0:*

$$q^\varepsilon_\pm(s) \in \mathcal{Q}_\pm \text{ for } |s| \leq \delta,\ 0 < \varepsilon \leq \varepsilon_0. \tag{5.22}$$

We now fix such a δ and ε_0; we shall possibly need to reduce ε_0 further in what follows. In addition we choose $Q^\pm_\rho$ and $\mathcal{Q}_\pm$ so that

$$(Q, q) \in Q^\pm_\rho \text{ for } q \in \mathcal{Q}_\pm \text{ and } |Q| \text{ small.} \tag{5.23}$$

(2) Setting $S_3(z, q) := T_+(z, q)S_+(q)$ for $q \in \mathcal{Q}_+$ and using (5.15) *and* (5.18)*, we note that $V_+(z)$ satisfies* (5.16) *on $z \geq 0$ if and only if $\mathcal{V}_+ := (S_3)^{-1}V_+$ satisfies*

$$d_z\mathcal{V}_+ = G_{B+}(q)\mathcal{V}_+ + (S_3)^{-1}f \text{ on } z \geq 0. \tag{5.24}$$

The conjugator $S_3(z, q)$ is used in solving Problem III. A conjugator $S_2(z, q)$ for Problem II is defined similarly.

(3) Observe that for s near 0, we know nothing about the behavior of $\lambda_\pm(Q^\varepsilon(s), q^\varepsilon(s))$, since $Q^\varepsilon(s)$ is of size $O(1)$. In particular, these eigenvalues may be nonreal for such s. However, for ε small, we know that $\lambda_\pm(q^\varepsilon)$ are real and bounded away from 0 on $[a-\overline{r}, b-\overline{r}]$.

The following Lemma and its Corollary are used to match the solutions of Problems III and IV at $s = \delta$.

Lemma 5.7. *(1) For $\beta > 0$ as in* (5.14)*, $(z, q) \in [0, \infty) \times \mathcal{Q}_+$, and $(Q, q) \in \mathcal{Q}^+_\rho$ we have*

$$S_3(z, q) - S_4(Q, q) = O(Q) + O(e^{-\beta|z|}). \tag{5.25}$$

Proof. Using the definition of $S_+(q)$ from the proof of Proposition 5.5, we have

$$S_3(z, q) = T_+(z, q)S_+(q) = \left(I + O(e^{-\beta|z|})\right) S_4(0, q), \text{ so} \tag{5.26}$$

$$S_3(z, q) - S_4(Q, q) = (S_4(0, q) - S_4(Q, q)) + O(e^{-\beta|z|})S_4(0, q). \tag{5.27}$$

□

Since $V^0(z)$ and $V^0_z(z)$ decay to 0 like $e^{-\beta|z|}$, as an immediate corollary we obtain

Corollary 5.8. *For $\delta > 0$ as fixed in Remark 5.6(1) and ε_0 small enough we have*

$$S_3\left(\frac{\delta}{\varepsilon}, q^\varepsilon(\delta)\right) - S_4(Q^\varepsilon(\delta), q^\varepsilon(\delta)) = O\left(e^{-\beta\frac{\delta}{\varepsilon}}\right) \text{ for } 0 < \varepsilon \leq \varepsilon_0. \tag{5.28}$$

5.2 Problem IV on $[\delta, b-\overline{r}]$

We return now to the task of solving the error transmission problem (4.31) on $[a-\overline{r}, b-\overline{r}]$ by solving in order Problems IV, III, II, and I. To solve Problem IV:

$$\begin{aligned} &d_s V = \frac{1}{\epsilon} GV + F \text{ on } [\delta, b-\overline{r}] \\ &\begin{pmatrix} v^1 \\ v^2 \end{pmatrix} (b-\overline{r}) = \overline{v} = O(e^{-\frac{\beta}{\epsilon}}),\ \beta > 0,\ \text{(as in (4.12))}, \end{aligned} \tag{5.29}$$

we first study the conjugated problem for the unknown $\mathcal{V} = (\nu^*, \nu^3) \in \mathbb{R}^5$, $\nu^* \in \mathbb{R}^3$, $\nu^3 = (\nu^3_+, \nu^3_-) \in \mathbb{R}^2$, defined by $V = S_4(Q^\varepsilon(s), q^\varepsilon(s))\mathcal{V}$:

$$\begin{aligned} &d_s \mathcal{V} = \frac{1}{\epsilon} G_B \mathcal{V} + S_4^{-1} F - (S_4^{-1} \partial_s S_4)\mathcal{V} \text{ on } [\delta, b-\overline{r}] \\ &\begin{pmatrix} \nu^* \\ \nu^3_+ \end{pmatrix} (b-\overline{r}) = \begin{pmatrix} p^* \\ p^3_+ \end{pmatrix} \in \mathbb{R}^4,\ \nu^3_-(\delta) = p^3_- \in \mathbb{R}, \end{aligned} \tag{5.30}$$

where $P = (p^*, p^3_+, p^3_-)$ are parameters to be chosen. The decomposition $\mathcal{V} = (\nu^*, \nu^3_+, \nu^3_-)$ corresponds to the block form of G_B described in (5.3), (5.4).

As explained in the Introduction, we have to prescribe ν^3_+ at the right endpoint and ν^3_- at the left endpoint of $[\delta, b-\overline{r}]$ in order to avoid exponential blowup of those components as $\varepsilon \to 0$.

Remark 5.9. *(1) Consider the restrictions to $|s| \geq \delta > 0$ of the functions of (s, ϵ) given by $Q^\varepsilon(s)$ and $q^{\epsilon,j}(s)$, $j \neq 3, 5, 6$. The exponential decay of $V^0(z)$ and $V^0_z(z)$ implies that these restrictions extend to $\{|s| \geq \delta\} \times [0, \epsilon_0]$ with the same regularity in (s, ϵ) that they have on $\{|s| \geq \delta\} \times (0, \epsilon_0]$.*

(2) Recall that G_B and S_4 both depend on $q^{\epsilon,3}$, $q^{\varepsilon,5}$ and $q^{\varepsilon,6}$, all of which depend in turn on $\epsilon^L v$, and hence on the unknown V. The form of the functional dependence of S_4 on (Q, q) was determined, independently of V, in Proposition 5.2. A simple contraction mapping argument shows that, for $\mathcal{V}$ in a bounded set of $\mathbb{R}^5$, the equation

$$V - S_4(..., \epsilon^L V, ...)\mathcal{V} = 0 \tag{5.31}$$

uniquely determines $V = V(s, \epsilon, \mathcal{V})$. The contraction argument and part (1) of this Remark show that $V(s, \varepsilon, \mathcal{V})$ is uniformly continuous in $(s, \varepsilon, \mathcal{V})$, for $s \in [\delta, b-\overline{r}]$, $\varepsilon \in (0, \varepsilon_0]$, and $\mathcal{V}$ in a bounded set. The regularity of this map may be determined from the known regularity of S_4 by the implicit function theorem. Under the assumptions of Proposition 4.2, the regularity of $V(s, \varepsilon, \mathcal{V})$ is at least C^{k-M-1} ($d_s U^M$ occurs in the E^{ij}). By substituting $V(s, \varepsilon, \mathcal{V})$ into (5.30) we obtain a well-defined nonlinear problem for the unknown $\mathcal{V}$.

(3) Using the function $V(s, \varepsilon, \mathcal{V}$ defined above, we will sometimes write with slight abuse

$$V = S_4(\varepsilon^L \mathcal{V})\mathcal{V}, \tag{5.32}$$

when studying Problem IV.

Proposition 5.10. *We make the same regularity assumptions as in Proposition 4.2. For fixed $R > 0$ and parameters $P = (p^*, p^3)$ satisfying $|p^*, p^3| \leq R$, there exists an $\epsilon_0 > 0$ such that for $0 < \epsilon \leq \epsilon_0$, the problem (5.30) has a solution on $[\delta, b - \overline{r}]$, $\mathcal{V}(s, \epsilon, P)$, that is uniformly bounded with respect to (s, ε, P). The function is C^1 in P uniformly with respect to (s, ε) as above, and is* locally *C^1 in its arguments. In addition we have*

$$\nu_+^3(\delta, \varepsilon, P) = O(\epsilon) \text{ and } \nu_-^3(b - \overline{r}, \varepsilon, P) = O(\varepsilon). \tag{5.33}$$

Proof. The proof is slight modification of the proof of Proposition 7.5 in [EJW], so we shall just note the changes needed. We sometimes suppress dependence of solutions on (ε, P) in the notation.

1. Rewrite equations. Setting $\mathcal{V} = (\nu^*, \nu^3)$, using the properties of G_B and $(Q^\varepsilon(s), q^\varepsilon(s))$ and notation similar to (5.32), we rewrite (5.30) as

$$\begin{aligned}
&d_s\nu^* = B_1(\epsilon^L\mathcal{V})\nu^* + B_2(\epsilon^L\mathcal{V})\nu^3 + \epsilon^{M-L}H^*(\epsilon^L\mathcal{V})\\
&d_s\nu_+^3 = \frac{1}{\epsilon}\lambda_+(\epsilon^L\mathcal{V})\nu_+^3 + B_{3+}(\epsilon^L\mathcal{V})\mathcal{V} + \epsilon^{M-L}H_+^3(\epsilon^L\mathcal{V}),\\
&d_s\nu_-^3 = \frac{1}{\epsilon}\lambda_-(\epsilon^L\mathcal{V})\nu_+^3 + B_{3-}(\epsilon^L\mathcal{V})\mathcal{V} + \epsilon^{M-L}H_-^3(\epsilon^L\mathcal{V}),\\
&\nu^*(b-\overline{r}) = p^*,\ \ \nu_+^3(b-\overline{r}) = p_+^3,\ \ \nu_-^3(\delta) = p_-^3,
\end{aligned} \tag{5.34}$$

where $\lambda_\pm$ are as in Proposition 5.2, $H = (H^*, H^3) := S_4^{-1}F$, the matrices B_j are uniformly bounded with respect to ϵ, and we've suppressed the dependence of the coefficients on all arguments except $\epsilon^L\mathcal{V}$.

2. Iteration scheme. The scheme is not quite standard so we write it explicitly:

$$\begin{aligned}
&(a)\ d_s\nu_{n+1}^* = B_1(\epsilon^L\mathcal{V}_n)\nu_{n+1}^* + B_2(\epsilon^L\mathcal{V}_n)\nu_{n+1}^3 + \epsilon^{M-L}H^*(\epsilon^L\mathcal{V}_n)\\
&(b)\ d_s\nu_{+,n+1}^3 = \frac{1}{\epsilon}\lambda_+(\epsilon^L\mathcal{V}_n)\nu_{+,n+1}^3 + B_{3+}(\epsilon^L\mathcal{V}_n)\mathcal{V}_n + \epsilon^{M-L}H_+^3(\epsilon^L\mathcal{V}_n),\\
&(c)\ d_s\nu_{-,n+1}^3 = \frac{1}{\epsilon}\lambda_-(\epsilon^L\mathcal{V}_n)\nu_{-,n+1}^3 + B_{3-}(\epsilon^L\mathcal{V}_n)\mathcal{V}_n + \epsilon^{M-L}H_-^3(\epsilon^L\mathcal{V}_n),\\
&\nu_{n+1}^*(b-\overline{r}) = p^*,\ \ \nu_{+,n+1}^3(b-\overline{r}) = p_+^3,\ \ \nu_{-,n+1}^3(\delta) = p_-^3,
\end{aligned} \tag{5.35}$$

Observe that ν_{n+1}^3 occurs in (5.35)(a), but ν_{n+1}^* does not occur in (5.35)(b),(c).

3. Estimates. For the moment we assume $|\mathcal{V}_n|_\infty \leq K$ for all n. Denoting $L^\infty([\delta, b-\overline{r}])$ norms by $|\cdot|_\infty$ we have

$$\begin{aligned}
&(a)\ |\nu_{n+1}^*|_\infty \leq C_1(|\nu_{n+1}^3|_\infty + \epsilon^{M-L}) + |p^*|\\
&(b)\ |\nu_{\pm,n+1}^3|_\infty \leq C_2\epsilon(|\mathcal{V}_n|_\infty + \epsilon^{M-L}) + |p_\pm^3|
\end{aligned} \tag{5.36}$$

where the constants C_1, C_2 may be chosen independently of K for $0 < \epsilon \leq \epsilon_0$ provided $\epsilon_0 = \epsilon_0(K)$ is small enough. Since the coefficients of (5.35)(a) are uniformly bounded with respect to ϵ, the first estimate is standard ([CL], Chpt. 1, Thm. 2.1).

To prove (5.36)(b)$_-$ for $s \geq \delta$ set

$$\lambda_-(\epsilon^L\mathcal{V}_n) := b(s),\ p(s) := \int_\delta^s b(\sigma)d\sigma, \text{ and } f(s) := B_{3-}(\epsilon^L\mathcal{V}_n)\mathcal{V}_n + \epsilon^{M-L}H_-^3. \tag{5.37}$$

We have

$$\nu^3_{-,n+1}(s) = \int_\delta^s e^{\frac{p(s)-p(t)}{\epsilon}} f(t)dt + e^{\frac{p(s)}{\epsilon}} p^3_- := A + B. \tag{5.38}$$

Since $b(s) \le \alpha < 0$ on $[\delta, b-\overline{r}]$ (by (5.6)), we obtain

$$\begin{aligned} &(a)\ |A| \le |f|_\infty \int_\delta^s e^{\frac{\alpha}{\epsilon}(s-t)} dt \le |f|_\infty \frac{\epsilon}{|\alpha|} \\ &(b)\ |B| \le e^{\int_\delta^s \frac{\alpha}{\epsilon} d\sigma} p^3_- = e^{\frac{\alpha}{\epsilon}(s-\delta)} p^3_-. \end{aligned} \tag{5.39}$$

This gives (5.36)(b)$_-$. The “ + ” case is similar.

4. Induction step. To initialize take $\nu^*_0 = p^*$, $\nu^3_0 = p^3$. With C_1 as in (5.36) we will show that for ϵ small enough,

$$\begin{aligned} &|\nu^*_n|_\infty \le C_1(|p^3| + 1) + |p^*| + 1 \\ &|\nu^3_n|_\infty \le |p^3| + 1 \end{aligned} \tag{5.40}$$

for all n. Indeed, assuming (5.40) for a given n, the estimate (5.36)(b) implies

$$|\nu^3_{n+1}|_\infty \le C_2\epsilon\left(\left(C_1(|p^3|+1) + |p^*| + 1 + |p^3| + 1\right) + \epsilon^{M-L}\right) + |p^3| \le |p^3| + 1 \tag{5.41}$$

for ϵ small. Estimate (5.36)(a) then gives

$$|\nu^*_{n+1}|_\infty \le C_1(|p^3| + 1 + \epsilon^{M-L}) + |p^*| \le C_1(|p^3|+1) + |p^*| + 1 \tag{5.42}$$

for ϵ small.

5. Conclusion. One now uses the estimates (5.40) to show by a standard contraction mapping argument that the $\mathcal{V}_n$ converge in $L^\infty(s,\varepsilon,P)$ (see [EJW], Proposition 7.5, for extra detail.) The limit function is thus uniformly bounded and continuous in P uniformly with respect to (s,ε). The uniform C^1 dependence on P with respect to (s,ε) is proved by differentiating the scheme (5.35) with respect to P to obtain a similar scheme for $\dot{\mathcal{V}}_n := \partial_P \mathcal{V}_n$. Estimates similar to (5.40) then hold for the iterates $\dot{\mathcal{V}}_n$. The local C^1 regularity of $\mathcal{V}$ in its arguments is classical. The property

$$\nu^3_-(b-\overline{r}) = O(\varepsilon) \tag{5.43}$$

now follows from (5.38) and (5.39). The property $\nu^3_+(\delta) = O(\varepsilon)$ is proved similarly.

□

Next we must show that the parameters P can be chosen so that the boundary condition in (5.29) can be satisfied.

Proposition 5.11. *We make the same regularity assumptions as in Proposition 4.2. For fixed $R > 0$ and parameters $p^3 = (p^3_+, p^3_-)$ satisfying $|p^3| \le R$, there exists an $\epsilon_0 > 0$ such that for $0 < \epsilon \le \epsilon_0$, the problem (5.29) has a solution on $[\delta, b-\overline{r}]$, $V_4(s,\epsilon,p^3)$, with the same regularity (except that p^3 replaces P now) as $\mathcal{V}(s,\varepsilon,P)$ in Proposition 5.10.*

Proof. Let us write the solution to (5.30) as $\mathcal{V}_4(s,\varepsilon,P)=(\nu^*,\nu^3)$. For $\overline{v}$ as in (5.29) we must choose the parameters $P=(p^*,p^3)$ in (5.30) so that

$$V_4(b-\overline{r},\varepsilon,P)=S_4(\varepsilon^L\mathcal{V}_4)\mathcal{V}_4(b-\overline{r},\varepsilon,P) \tag{5.44}$$

satisfies $V_4^*(b-\overline{r},\varepsilon,P)=\overline{v}$. In other words

$$\overline{v}=S_4^{11}(\varepsilon^L\mathcal{V}_4)p^*+S_4^{12}(\varepsilon^L\mathcal{V}_4)\begin{pmatrix}p_+^3\\ \nu_-^3(b-\overline{r},\varepsilon,p^*,p^3)\end{pmatrix}. \tag{5.45}$$

Since S_4^{11} is uniformly invertible on $[\delta,b-\overline{r}]$ and $\nu_-^3(b-\overline{r},\varepsilon,p^*,p^3)=O(\varepsilon)$, we can solve for $p^*=p^*(p^3)$ in (5.45) by a contraction mapping argument. The implicit function theorem shows that this map is C^1. Finally, we set

$$V_4(s,\varepsilon,p^3)=S_4(\varepsilon^L\mathcal{V}_4)\mathcal{V}_4(s,\varepsilon,p^*(p^3),p^3). \tag{5.46}$$

$\square$

5.3 Problem III on $[0,\delta]$.

For $\delta>0$ chosen as in Remark 5.6(1) we will produce a solution to Problem III

$$d_sV=\frac{1}{\varepsilon}GV+F \text{ on } [0,\delta] \tag{5.47}$$

by first studying the conjugated problem for the unknown $\mathcal{V}$ defined by $V=S_3(\frac{s}{\varepsilon},q^\varepsilon(s))\mathcal{V}$:

$$\begin{aligned}&d_s\mathcal{V}=\frac{1}{\varepsilon}G_{B+}\mathcal{V}-S_3^{-1}(\partial_qS_3\cdot\partial_sq^\varepsilon)\mathcal{V}+S_3^{-1}F \text{ on } [0,\delta],\\ &\begin{pmatrix}\nu^*\\ \nu_+^3\end{pmatrix}(\delta)=\begin{pmatrix}\pi^*\\ \pi_+^3\end{pmatrix}\in\mathbb{R}^4,\ \nu_-^3(0)=\pi_-^3\in\mathbb{R},\end{aligned} \tag{5.48}$$

for parameters $\Pi=(\pi^*,\pi^3)\in\mathbb{R}^5$ to be chosen.

Remark 5.12. *Parallel to our definition of the unknown $\mathcal{V}$ in Problem IV, we now define $\mathcal{V}$ for Problem III by*

$$V=S_3\left(\frac{s}{\varepsilon},q^\varepsilon(s)\right)\mathcal{V} \tag{5.49}$$

for S_3 as in Remark 5.6; recall that q^ε has ε^LV dependence. As before a contraction mapping argument shows that for $\mathcal{V}$ in a bounded set, (5.49) determines a unique function $V=V(s,\epsilon,\mathcal{V})$ that is uniformly bounded for $s\in[0,\delta]$, $\varepsilon\in(0,\varepsilon_0]$. Although part (1) of Remark 5.9 does not apply on $[0,\delta]$, we obtain that $V(s,\varepsilon,\mathcal{V})$ is continuous in $\mathcal{V}$ uniformly for (s,ε) as above, with the same higher local *regularity in $(s,\varepsilon,\mathcal{V})$ as before, namely C^{k-M-1}. By differentiating (5.49) with respect to $\mathcal{V}$ (after substituting $V(s,\varepsilon,\mathcal{V})$ in both sides) and solving for $\partial_{\mathcal{V}}V$, we see that $V(s,\varepsilon,\mathcal{V})$ is C^1 in $\mathcal{V}$ uniformly with respect to (s,ε) as above.*

Proposition 5.13. *We make the same regularity assumptions as in Proposition 4.2. For fixed $R > 0$ and $|\pi^*, \pi^3| \leq R$, there exists an $\epsilon_0 > 0$ such that for $0 < \epsilon \leq \epsilon_0$, the problem (5.48) has a solution on $[0, \delta]$, $\mathcal{V}(s, \epsilon, \Pi)$, that is uniformly bounded with respect to (s, ε, Π), C^1 in Π uniformly with respect to (s, ϵ), and locally C^{k-M-1} in (s, ε, Π). In addition we have*

$$\nu_+^3(0, \varepsilon, \Pi) = O(\epsilon) \text{ and } \nu_-^3(\delta, \varepsilon, \Pi) = O(\varepsilon). \tag{5.50}$$

Denote this solution by $\mathcal{V}_3(s, \varepsilon, \Pi)$. The function $V_3(s, \varepsilon, \Pi) := S_3(\varepsilon^L \mathcal{V}_3)\mathcal{V}_3$ is then a solution of (5.47) with the same regularity as $\mathcal{V}_3$.

Proof. Recall from the proof of Proposition 5.5 that for $\lambda_\pm(q)$ as in (5.19) and $\lambda_\pm(Q, q)$ as in (5.4), we have

$$\lambda_\pm(q) = \lambda_\pm(0, q), \text{ and thus } \lambda_+(q) > c_1 > 0 > c_2 > \lambda_-(q) \tag{5.51}$$

by (5.6) for $q \in \mathcal{Q}_+$. Although the family of functions on $[0, \delta]$,

$$\{\partial_s q^\varepsilon : \varepsilon \in (0, 1]\}, \tag{5.52}$$

is not equicontinuous, we do have $\partial_s q^\epsilon = O(1)$. Together with (5.18) and (5.19), this means that (5.48) is the same type of problem as (5.30). The regularity of the change of variables $V = V(s, \varepsilon, \mathcal{V})$ described in Remark 5.12 allows us to repeat the proof of Proposition 5.10 to obtain V_3 and $\mathcal{V}_3$ with the stated regularity.

□

5.4 Matching exact solutions at $s = \delta$.

We have $\mathcal{V}_3 = (\nu_3^*, \nu_3^3) = (\nu_3^*, \nu_{3+}^3, \nu_{3-}^3)$ and $\mathcal{V}_4 = (\nu_4^*, \nu_4^3) = (\nu_4^*, \nu_{4+}^3, \nu_{4-}^3)$. In order for the solutions $V_3(s, \varepsilon, \Pi) = S_3(\varepsilon^L \mathcal{V}_3)\mathcal{V}_3$ and $V_4(s, \varepsilon, p^3) = S_4(\varepsilon^L \mathcal{V}_4)\mathcal{V}_4$ to match at $s = \delta$, the following two equations must hold:

$$\begin{gathered} S_3^{11}\left(\varepsilon^L \mathcal{V}_3(\delta, \varepsilon, \Pi)\right)\pi^* + S_3^{12}(\varepsilon^L \mathcal{V}_3)\begin{pmatrix} \pi_+^3 \\ \nu_{3-}^3(\delta, \varepsilon, \Pi) \end{pmatrix} = \\ S_4^{11}\left(\varepsilon^L \mathcal{V}_4(\delta, \varepsilon, p^3)\right)\nu_4^*(\delta, \varepsilon, p^3) + S_4^{12}(\varepsilon^L \mathcal{V}_4)\begin{pmatrix} \nu_{4+}^3(\delta, \varepsilon, p^3) \\ p_-^3 \end{pmatrix}, \text{ and} \end{gathered} \tag{5.53}$$

$$\begin{gathered} S_3^{21}\left(\varepsilon^L \mathcal{V}_3(\delta, \varepsilon, \Pi)\right)\pi^* + S_3^{22}(\varepsilon^L \mathcal{V}_3)\begin{pmatrix} \pi_+^3 \\ \nu_{3-}^3(\delta, \varepsilon, \Pi) \end{pmatrix} = \\ S_4^{21}\left(\varepsilon^L \mathcal{V}_4(\delta, \varepsilon, p^3)\right)\nu_4^*(\delta, \varepsilon, p^3) + S_4^{22}(\varepsilon^L \mathcal{V}_4)\begin{pmatrix} \nu_{4+}^3(\delta, \varepsilon, p^3) \\ p_-^3 \end{pmatrix} \end{gathered} \tag{5.54}$$

The undetermined parameters in these equations are Π and p^3. In the next Lemma we use equation (5.54) to solve for (π_+^3, p_-^3) in terms of the other parameters.

Lemma 5.14. *For fixed $R > 0$ and $|\pi^*, \pi_-, p^3_+| \leq R$ there exists an $\varepsilon_0 > 0$ and a function*

$$(\pi^*, \pi^3_-, p^3_+, \varepsilon) \to \begin{pmatrix} \pi^3_+ \\ p^3_- \end{pmatrix} (\pi^*, \pi^3_-, p^3_+, \varepsilon) := \mathcal{M}(\pi^*, \pi^3_-, p^3_+, \varepsilon). \tag{5.55}$$

such that for $0 < \varepsilon \leq \varepsilon_0$ the equation (5.54) holds for values of the parameters Π, p^3 satisfying

$$(\Pi, p^3_+, p^3_-) = \left(\pi^*, \pi^3_+(\pi^*, \pi^3_-, p^3_+, \varepsilon), \pi^3_-, p^3_+, p^3_-(\pi^*, \pi^3_-, p^3_+, \varepsilon)\right). \tag{5.56}$$

The function $\mathcal{M}$ is C^1 in (π^*, π^3_-, p^3_+) uniformly for $0 < \varepsilon \leq \varepsilon_0$, and $\mathcal{M} = O(\varepsilon)$.

Proof. **1.** Recall that $S_3 = S_3(\frac{s}{\varepsilon}, q^\varepsilon(s))$ and $S_4 = S_4(Q^\varepsilon(s), q^\varepsilon(s))$, where the components $q^{\varepsilon,j}$, $j = 3, 5, 6$ depend on $\varepsilon^L V$. The functions $q^\varepsilon(s)$ appearing in S_3 and S_4 are defined on $[0, \delta]$ and $[\delta, b - \overline{r}]$ respectively, but even at $s = \delta$ they need not agree since $v_3(\delta, \varepsilon, \Pi) \neq v_4(\delta, \varepsilon, p^3)$ unless Π and p^3 are chosen carefully. Using subscripts 3 and 4 to distinguish the functions q^ε, we have

$$q^\varepsilon_3(\delta) - q^\varepsilon_4(\delta) = O(\varepsilon^L). \tag{5.57}$$

Corollary 5.8 implies

$$S_3\left(\frac{\delta}{\varepsilon}, q^\varepsilon_3(\delta)\right) - S_4(Q^\varepsilon(\delta), q^\varepsilon_3(\delta)) = O\left(e^{-\beta\frac{\delta}{\varepsilon}}\right) \text{ for } 0 < \varepsilon \leq \varepsilon_0. \tag{5.58}$$

Along with (5.57) this implies

$$S_3\left(\frac{\delta}{\varepsilon}, q^\varepsilon_3(\delta)\right) - S_4(Q^\varepsilon(\delta), q^\varepsilon_4(\delta)) = O(\varepsilon^L) \text{ for } 0 < \varepsilon \leq \varepsilon_0. \tag{5.59}$$

In particular, the uniformly invertible matrices S^{22}_3 and S^{22}_4 appearing in (5.54) satisfy

$$S^{22}_3 - S^{22}_4 = O(\varepsilon^L) \text{ and thus } (S^{22}_3)^{-1} S^{22}_4 = I + O(\varepsilon^L). \tag{5.60}$$

2. Applying $(S^{22}_3)^{-1}$ to equation (5.54), using (5.60), and rearranging terms we obtain:

$$\begin{pmatrix} \pi^3_+ \\ p^3_- \end{pmatrix} = \begin{pmatrix} -\left[(S^{22}_3)^{-1} S^{21}_3 \pi^*\right]_+ + \left[(S^{22}_3)^{-1} S^{21}_4 \nu^*_4(\delta, \varepsilon, p^3)\right]_+ + \nu^3_{4+}(\delta, \varepsilon, p^3) + \left[O(\varepsilon^L) \begin{pmatrix} \nu^3_{4+} \\ p^3_- \end{pmatrix}\right]_+ \\ \left[(S^{22}_3)^{-1} S^{21}_3 \pi^*\right]_- + \nu^3_{3-}(\delta, \varepsilon, \Pi) - \left[(S^{22}_3)^{-1} S^{21}_4 \nu^*_4(\delta, \varepsilon, p^3)\right]_- - \left[O(\varepsilon^L) \begin{pmatrix} \nu^3_{4+} \\ p^3_- \end{pmatrix}\right]_- \end{pmatrix}. \tag{5.61}$$

3. Fixed point argument. For a given $(\pi^*, \pi^3_-, p^3_+, \varepsilon)$ we define a map

$$T(\pi^3_+, p^3_-) = \text{ (the right side of (5.61))}, \tag{5.62}$$

recalling that $\Pi-$dependence occurs in every function with subscript 3, and p^3-dependence in every function with subscript 4. The right side of each equation in (5.61) has four terms, and each of those terms is $O(\varepsilon)$. To see that we use (5.33), (5.50), and the fact that

$$S^{21}_3 = O(\varepsilon), \ S^{21}_4 = O(\varepsilon), \tag{5.63}$$

which follows from (5.20) and (5.5) and the definition of $(Q^\varepsilon, q^\varepsilon)$. Thus, for ε small enough T defines a contraction on sufficiently small (but still radius $O(1)$), closed $(\pi^3_+, p^3_-)-$balls. The fixed points, which depend as T does on $(\pi^*, \pi^3_-, p^3_+, \varepsilon)$, define the map $\mathcal{M}$ in (5.55), and clearly $\mathcal{M} = O(\varepsilon)$.

4. Regularity. The known regularity of the functions appearing in (5.54), together with the proof of the contraction mapping theorem, imply that $\mathcal{M}(\pi^*, \pi^3_-, p^3_+, \varepsilon)$ is C^0 in (π^*, π^3_-, p^3_+) uniformly for $0 < \varepsilon < \varepsilon_0$. The *local* regularity in $(\pi^*, \pi^3_-, p^3_+, \varepsilon)$ for $\varepsilon > 0$ is at least C^{k-M-1} by the implicit function theorem. By differentiating (5.61) with respect to (π^*, π^3_-, p^3_+) after substituting $\mathcal{M}$ in both sides, and then solving for $\partial_{\pi^*,\pi^3_-,p^3_+}\mathcal{M}$, we see that $\partial_{\pi^*,\pi^3_-,p^3_+}\mathcal{M}$ is C^0 in (π^*, π^3_-, p^3_+) uniformly for $0 < \varepsilon \le \varepsilon_0$. □

Proposition 5.15. *For fixed $R > 0$ and $|\pi_-, p^3_+| \le R$ there exists an $\varepsilon_0 > 0$ and a function*

$$(\pi^3_-, p^3_+, \varepsilon) \to \begin{pmatrix} \pi^* \\ \pi^3_+ \\ p^3_- \end{pmatrix} (\pi^3_-, p^3_+, \varepsilon) := \mathcal{N}(\pi^3_-, p^3_+, \varepsilon). \tag{5.64}$$

such that for $0 < \varepsilon \le \varepsilon_0$ the equations (5.53) and (5.54) both hold for values of the parameters Π, p^3 satisfying

$$(\Pi, p^3_+, p^3_-) = \big(\pi^*(\pi^3_-, p^3_+, \varepsilon), \pi^3_+(\pi^3_-, p^3_+, \varepsilon), \pi^3_-, p^3_+, p^3_-(\pi^3_-, p^3_+, \varepsilon)\big) := \mathbb{N}(\pi^3_-, p^3_+, \varepsilon). \tag{5.65}$$

The functions $\mathcal{N}$ and $\mathbb{N}$ are C^1 in (π^3_-, p^3_+) uniformly for $0 < \varepsilon \le \varepsilon_0$.

Proof. Substitute $\mathcal{M}(\pi^*, \pi^3_-, p^3_+, \varepsilon)$ into both sides of (5.53) to obtain an equation where the only undetermined parameters are (π^*, π^3_-, p^3_+), and then apply $(S^{11}_3)^{-1}$ to both sides of the resulting equation. The term on the far left is now simply π^*, and in the remaining terms π^* dependence occurs only in the arguments:

$$\varepsilon^L \mathcal{V}_3,\ \pi^3_+,\ p^3_-,\ \nu^3_{3-}(\delta, \varepsilon, \Pi). \tag{5.66}$$

Since $\mathcal{M} = O(\varepsilon)$ and $\nu^3_{3-}(\delta, \varepsilon, \Pi) = O(\varepsilon)$, we apply now the same fixed point and regularity arguments as in the proof of Lemma 5.14 to obtain $\pi^*(\pi^3_-, p^3_+, \varepsilon)$. The other components of $\mathcal{N}(\pi^3_-, p^3_+, \varepsilon)$ are then given by $\mathcal{M}(\pi^*(\pi^3_-, p^3_+, \varepsilon), \pi^3_-, p^3_+, \varepsilon)$.

□

The next Corollary summarizes what we have shown so far.

Corollary 5.16. *Let $V_3(s, \varepsilon, \Pi)$ and $V_4(s, \varepsilon, p^3)$ be the functions defined in Propositions 5.13 and 5.11 respectively, and let π^*, π^3_+, and p^3_- be the components of the function $\mathcal{N}$ in (5.64).*

For fixed $R > 0$ and $|\pi^3_-, p^3_+| \le R$, there exists an $\varepsilon_0 > 0$ such that the function defined by

$$V_+(s, \pi^3_-, p^3_+, \varepsilon) = \begin{cases} V_3(s, \varepsilon, \pi^*(\pi^3_-, p^3_+, \varepsilon), \pi^3_+(\pi^3_-, p^3_+, \varepsilon), \pi^3_-),\ s \in [0, \delta] \\ V_4(s, \varepsilon, p^3_+, p^3_-(\pi^3_-, p^3_+, \varepsilon)),\ s \in [\delta, b - \overline{r}] \end{cases} \tag{5.67}$$

is an exact solution to the error problem (4.31) on $[0, b - \overline{r}]$, uniformly bounded with respect to $\varepsilon \in (0, \varepsilon_0]$, and C^1 in (π^3_-, p^3_+) uniformly with respect to (s, ε) as above.

5.5 Problems II and I.

Let $V_+(s, \pi_-^3, p_+^3, \epsilon)$ be the solution to the error equation on $[0, b - \overline{r}]$ defined in Corollary 5.16. Continuing to work our way from right to left, we can write Problem II as

$$\begin{aligned} &d_s V_2 = \frac{1}{\epsilon} G V_2 + F \text{ on } [-\delta, 0] \\ &V_2(0) = V_+(0, \pi_-^3, p_+^3, \epsilon), \end{aligned} \tag{5.68}$$

and Problem I as

$$\begin{aligned} &d_s V_1 = \frac{1}{\epsilon} G V_1 + F \text{ on } [a - \overline{r}, -\delta] \\ &V_1(-\delta) = V_2(-\delta). \end{aligned} \tag{5.69}$$

Clearly, the boundary condition in (5.68) is chosen so that the transmission condition $[V] = 0$ in (4.31) holds, and the boundary condition in (5.69) is chosen so that V_1 and V_2 match smoothly at $s = -\delta$.

The solution of problems II and I is quite similar to that of problems III and IV respectively, but now the argument is much simpler because $\lambda_\pm$ are both strictly positive in region I and, *after* removal of the fast variable $\frac{s}{\varepsilon}$ using Lemma 5.4, are both strictly positive also in region II (recall, for example, Propositions 5.5 and 4.10). Thus, there is no need to split the boundary conditions or to introduce extra parameters as in problems III and IV. The conjugators S_2 (Remark 5.6) and S_1 (Remark 5.3) are used for problems II and I respectively. In problem II the conjugated unknown is $\mathcal{V}_2$ defined by $V_2 = S_2 \mathcal{V}_2$, and boundary data for $\mathcal{V}_2$ is determined directly from $V_+(0, \pi_-^3, p_+^3, \epsilon)$ and prescribed at the right endpoint $s = 0$ of region II. Similarly, boundary data for problem I is prescribed at the right endpoint $s = -\delta$ of region I.

The rapid variation and unknown behavior of $\lambda_\pm(Q^\varepsilon, q^\varepsilon)$ for s near 0 in region II present the same difficulty in problem II as in problem III, but as before we handle that by using the conjugator S_2 to remove $z = \frac{s}{\varepsilon}$ dependence: $\lambda_\pm(q^\varepsilon)$ remain strictly bounded away from 0 and positive on $[-\delta, 0]$.

After conjugation problems II and I are both solved by iteration schemes like the one used earlier, where the iterates satisfy estimates like (5.36). Summarizing, we have proved:

Proposition 5.17. *Let $V_+(s, \pi_-^3, p_+^3, \epsilon)$ be the solution to the error equation on $[0, b - \overline{r}]$ defined in Corollary 5.16, and let $V_2(s, \epsilon)$, $V_1(s, \epsilon)$ be the solutions to problems II and I above. For fixed $R > 0$ and $|\pi_-^3, p_+^3| \le R$, there exists an $\varepsilon_0 > 0$ such that the function defined by*

$$V(s, \pi_-^3, p_+^3, \epsilon) = \begin{cases} V_+(s, \pi_-^3, p_+^3, \epsilon), s \in [0, b - \overline{r}] \\ V_2(s, \epsilon), s \in [-\delta, 0] \\ V_1(s, \epsilon), s \in [a - \overline{r}, -\delta] \end{cases} \tag{5.70}$$

is an exact solution to the error problem (4.31) *on $[a - \overline{r}, b - \overline{r}]$. The function V is uniformly bounded for $\varepsilon \in (0, \varepsilon_0]$ and is C^1 in (π_-^3, p_+^3) uniformly with respect to (s, ε) as above.*

This finishes the proof of the main result of this section:

Theorem 5.18. *We make the standard thermodynamic assumptions on e and p of section 1.3.*

For $R > 0$ and ε_0 as in Proposition 5.17 let $V(s, \pi^3_-, p^3_+, \varepsilon) = \big(v^*(s,\epsilon), v^3(s,\epsilon)\big)$ be the function defined there, and set $w^\epsilon(s) = \tilde{w} + \epsilon^L v^*$, where $\tilde{w}$ is the approximate solution constructed in Proposition 4.2. Then w^ϵ is an exact solution to the transmission problem (4.17) for $0 < \epsilon \leq \epsilon_0$, with $w^\epsilon(b - \overline{r}) = (\rho_b, u_b, \theta_b)$ the outflow data at $r = b$ for the original inviscid shock. In particular, we have for any $\beta > 0$:

$$\begin{aligned} &\lim_{\epsilon\to 0} w^\epsilon(s) = U^0(s) \text{ in } L^p\left([a-\overline{r}, b-\overline{r}]\right), 1 \leq p < \infty \\ &\lim_{\epsilon\to 0} w^\epsilon(s) = U^0(s) \text{ in } L^\infty\left([a-\overline{r}, b-\overline{r}] \cap \{|s| \geq \beta\}\right), \end{aligned} \tag{5.71}$$

where $U^0_\pm(s)$ is the original inviscid shock with discontinuity at $s = 0$.

Remark 5.19. *We have stated Theorem 5.18 for non-barotropic SS shocks with supersonic inflow at $r = a$. The same results for* barotropic *SS shocks and for* barotropic *CS shocks with supersonic inflow at $r = a$ in the case when angular (v) and axial (w) velocity components are both zero were proved by different arguments in [EJW].*

The analogue of Theorem 5.18 in the barotropic CS case when either $v \neq 0$ or $w \neq 0$ can be proved just like Theorem 5.18. For example, when both v and w are nonzero, the matrix G is 7×7, while $g^{33}(U^0(s))$ is 3×3 with 3 positive eigenvalues in $s \leq 0$, but 2 positive eigenvalues and 1 negative in $s \geq 0$. Boundary conditions for the conjugated variables $\mathcal{V}_3$ and $\mathcal{V}_4$ are now split just as in (5.48) and (5.30), with the 6 components corresponding to positive eigenvalues prescribed at the right endpoint of their respective domains, and the single component corresponding to the negative eigenvalue prescribed at the left endpoint.

The analogue of Theorem 5.18 in the non-barotropic CS case can also be proved just like Theorem 5.18. For example, when v and w are both nonzero, the variables V and $\mathcal{V}$ take values in $\mathbb{R}^9$ and the matrix $g^{33}(U^0(s))$ is 4×4 with 4 positive eigenvalues in $s \leq 0$, but 3 positive eigenvalues and 1 negative in $s \geq 0$. Boundary conditions are split in problems III and IV, but not in problems I and II, just as before.

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