Ultra-parabolic equations with rough coefficients. Entropy solutions and strong pre-compactness property

E.Yu. Panov [1]

Abstract

Under some non-degeneracy condition we show that sequences of entropy solutions of a semi-linear ultra-parabolic equation are strongly pre-compact in the general case of a Caratheodory flux vector and a diffusion matrix. The proofs are based on localization principles for the parabolic H-measures corresponding to sequences of measure-valued functions.

1 Introduction

Let Ω be an open subset of $\mathbb{R}^n$. In the domain Ω we consider the semi-linear ultra-parabolic equation

$$\mathrm{div}\varphi(x,u) - D^2 \cdot B(x,u) + \psi(x,u) = 0, \tag{1}$$

where $D^2 \cdot B(x,u) = \partial^2_{x_i x_j} b_{ij}(x,u)$, $u = u(x)$ (we use the conventional rule of summation over repeated indexes), $B(x,u) = \{b_{ij}(x,u)\}_{i,j=1}^n$ is a symmetric matrix. We shall assume that the components of this matrix are Caratheodory functions: $b_{ij}(x,u) \in L^2_{loc}(\Omega, C(\mathbb{R}))$, $i,j = 1,\dots,n$. This means that $b_{ij}(x,u)$ are measurable with respect to x, continuous with respect to u, and $\max\limits_{|u|\le M} |b_{ij}(x,u)| \in L^2_{loc}(\Omega)$ $\forall M > 0$. In this case the parabolicity of (1) is understood in the following sense

$$\forall x \in \Omega, u_1, u_2 \in \mathbb{R},\ u_1 > u_2 \quad B(x,u_1) - B(x,u_2) \ge 0, \tag{2}$$

that is, $\forall \xi \in \mathbb{R}^n$ $(B(x,u_1) - B(x,u_2))\xi \cdot \xi \ge 0$ (here $u \cdot v$ denotes the scalar product of vectors $u, v \in \mathbb{R}^n$). We shall also assume that the matrix $B(x,u)$ is degenerated on a linear subspace $X \subset \mathbb{R}^n$, that is, for all $\xi \in X$ the function $B(x,u)\xi \cdot \xi$ does not depend on u: $B(x,u)\xi \cdot \xi = C(x)$. Hence, (1) is a semi-linear ultra-parabolic equation.

Concerning the convective terms, we suppose that $\varphi(x,u) = (\varphi_1(x,u),\dots,\varphi_n(x,u)) \in L^2_{loc}(\Omega, C(\mathbb{R},\mathbb{R}^n))$ is a Caratheodory vector. We also assume that for any $p \in \mathbb{R}$ the distribution

$$\mathrm{div}_x \varphi(x,p) - D^2_x \cdot B(x,p) = \gamma_p \in \mathrm{M}_{loc}(\Omega), \tag{3}$$

where $\mathrm{M}_{loc}(\Omega)$ is the space of locally finite Borel measures on Ω with the standard locally convex topology generated by semi-norms $p_\Phi(\mu) = \mathrm{Var}\,(\Phi\mu)$, $\Phi = \Phi(x) \in$

[1] The work was supported by the Russian Foundation for Basic Research (grant No. 09-01-00490-a) and DFG project No. 436 RUS 113/895/0-1. This article was written as part of the the international research program on Nonlinear Partial Differential Equations at the Centre for Advanced Study at the Norwegian Academy of Science and Letters in Oslo during the academic year 2008–09.

$C_0(\Omega)$. The function $\psi(x,u)$ is assumed to be a Caratheodory function on $\Omega\times\mathbb{R}$: $\psi(x,u)\in L^1_{loc}(\Omega,C(\mathbb{R}))$.

Let $\gamma_p=\gamma_p^r+\gamma_p^s$ be the decomposition of the measure γ_p into the sum of regular and singular measures, so that $\gamma_p^r=\omega_p(x)dx$, $\omega_p(x)\in L^1_{loc}(\Omega)$, and γ_p^s is a singular measure (supported on a set of zero Lebesgue measure). We denote by $|\gamma_p^s|$ the variation of the measure γ_p^s, which is a non-negative locally finite Borel measure on Ω. Denote, as usual, $\operatorname{sign} u=\begin{cases} 1 & , \quad u>0, \\ -1 & , \quad u<0, \\ 0 & , \quad u=0. \end{cases}$

Now, we introduce a notion of entropy solution of (1).

Definition 1. A measurable function $u(x)$ on Ω is called an entropy solution of equation (1) if $\varphi_i(x,u(x)),b_{ij}(x,u(x)),\psi(x,u(x))\in L^1_{loc}(\Omega)$, $i,j=1,\dots,n$, and for all $p\in\mathbb{R}$ the Kruzhkov-type entropy inequality (see [10]) holds

$$\begin{aligned} &\operatorname{div}[\operatorname{sign}(u(x)-p)(\varphi(x,u(x))-\varphi(x,p))]- \\ &D^2\cdot(\operatorname{sign}(u(x)-p)(B(x,u(x))-B(x,p)))+ \\ &\operatorname{sign}(u(x)-p)[\omega_p(x)+\psi(x,u(x))]-|\gamma_p^s|\le 0 \end{aligned} \tag{4}$$

in the sense of distributions on Ω (in the space $\mathcal{D}'(\Omega)$); that is, for all non-negative functions $f(x)\in C_0^\infty(\Omega)$

$$\int_\Omega \operatorname{sign}(u(x)-p)[(\varphi(x,u(x))-\varphi(x,p))\cdot\nabla f(x)+(B(x,u(x))-B(x,p))\cdot D^2 f -$$
$$(\omega_p(x)+\psi(x,u(x)))f(x)]dx+\int_\Omega f(x)d|\gamma_p^s|(x)\ge 0.$$

We use the notation $D^2 f$ for the matrix $\{\partial^2_{x_ix_j}f\}_{i,j=1}^n$ and $P\cdot Q=\operatorname{Tr}PQ=\sum_{i,j=1}^n p_{ij}q_{ij}$ denotes scalar product of symmetric matrices $P=\{p_{ij}\}_{i,j=1}^n$, $Q=\{q_{ij}\}_{i,j=1}^n$. In particular,

$$(B(x,u(x))-B(x,p))\cdot D^2 f=(b_{ij}(x,u)-b_{ij}(x,p))\partial^2_{x_ix_j}f.$$

In the case when the second-order term is absent ($B(x,u)\equiv 0$) our definition extends the notion of the entropy solution for first-order balance laws introduced for the case of one space variable in [7, 8]. If $\varphi(x,u),B(x,u)$ are smooth, and the strong ellipticity condition $A(x,u)=B'_u(x,u)\ge\varepsilon E$, $\varepsilon>0$ is satisfied then weak (variational) solutions of (1) are entropy solutions as well. This fact will be demonstrated in last Section 5.2 (as a part of the proof of Theorem 2).

We also notice that we do not require $u(x)$ to be a distributional solution of (1). If $u(x)\in L^\infty(\Omega)$ and $\gamma_p^s=0$ for all $p\in\mathbb{R}$ then any entropy solution $u(x)$ satisfies (1) in $\mathcal{D}'(\Omega)$, i.e. $u(x)$ is a distributional solution of (1). Indeed, this follows from (4) with $p=\pm\|u\|_\infty$. But, generally, entropy solutions are not distributional ones, even in the case when the singular measures γ_p^s are absent. For instance, as is easily verified,

$u(x) = \operatorname{sign} x|x|^{-1/2}$ is an entropy solution of the first-order equation $(xu^2)_x = 0$ on the line $\Omega = \mathbb{R}$, but it does not satisfy this equation in $\mathcal{D}'(\mathbb{R})$.

We assume that equation (1) is non-degenerate in the sense of the following definition.

Definition 2. Equation (1) is said to be *non-degenerate* if for almost all $x \in \Omega$ for all $\tilde{\xi} \in X$, $\bar{\xi} \in X^\perp$ such that $\tilde{\xi} \neq 0$, $\bar{\xi} \neq 0$ the function $\lambda \to \tilde{\xi} \cdot \varphi(x, \lambda)$, $\lambda \to B(x, \lambda)\bar{\xi} \cdot \bar{\xi}$ are not constant on non-degenerate intervals.

In this paper, we shall establish the strong pre-compactness property for sequences of entropy solutions. This result generalizes the previous results of [12, 13, 14, 15, 16] to the case of ultra-parabolic equations.

Theorem 1. *Suppose that u_k, $k \in \mathbb{N}$ is a sequence of entropy solutions of non-degenerate equation (1) such that $|\varphi(x, u_k(x))| + |\psi(x, u_k(x))| + |B(x, u_k(x))| + m(u_k(x))$ is bounded in $L^1_{loc}(\Omega)$, where $m(u)$ is a nonnegative super-linear function (i.e. $m(u)/u \to \infty$ as $u \to \infty$). Then there exists a subsequence of u_k, which converges in $L^1_{loc}(\Omega)$ to some entropy solution $u(x)$.*

We use here and everywhere below the notation $|B|$ for the Euclidean norm of a symmetric matrix B, that is $|B|^2 = B \cdot B = \mathrm{Tr} B^2$.

More generally, we establish the strong pre-compactness of approximate sequences $u_k(x)$ for non-degenerate equation (1). The only assumption we need is that the sequences of distributions

$$\operatorname{div}\varphi(x, s_{a,b}(u_k(x))) - D^2 \cdot B(x, s_{a,b}(u_k(x)))$$

are pre-compact in the anisotropic Sobolev space $W^{-1,-2}_{d,loc}(\Omega)$ for some $d > 1$ and each $a, b \in \mathbb{R}$, $a < b$ where $s_{a,b}(u) = \max(a, \min(u, b))$ are cut-off functions, and the space $W^{-1,-2}_{d,loc}(\Omega)$ will be specified below, in Section 4. We do not require here that condition (3) is satisfied.

Remark that the non-degeneracy condition is essential for the statement of Theorem 1. For example, assume that (1) has the form $\operatorname{div}\varphi(u) - D^2 \cdot B(u) = 0$ and $\xi \cdot \varphi(u) = \mathrm{const}$ on the segment $[a, b]$ with $\xi \in X$, $\xi \neq 0$ then the sequence $u_k(x) = [a + b + (b - a)\sin(k\xi \cdot x)]/2$ of entropy solutions does not contain strongly convergent subsequences.

We also stress that for sequences of distributional solutions (without additional entropy constraints) the statement of Theorem 1 does not hold. For example, the sequence $u_k = \operatorname{sign} \sin kx$ consists of distributional solutions for the Burgers equation $u_t + (u^2)_x = 0$ (as well as for the corresponding stationary equation $(u^2)_x = 0$) and converges only weakly, while the non-degeneracy condition is evidently satisfied.

Theorem 1 will be proved in the last section. The proof is based on general localization properties for parabolic H-measures corresponding to bounded sequences of measure-valued functions. It also follows from these properties the strong convergence of various approximate solutions for equation (1).

We describe below one useful approximation procedure. For simplicity we assume that $\psi(x, u) \equiv 0$. Let $\zeta(s) \in C_0^\infty(\mathbb{R})$ be a non-negative function with support in the segment $[-1,1]$ such that $\int \zeta(s)ds = 1$. We set $\zeta_m(s) = m\zeta(ms)$ for $m \in \mathbb{N}$, $\alpha_m(y) = \prod_{i=1}^{n} \zeta_m(y_i)$, $y \in \mathbb{R}^n$, so that the sequence α_m is an approximate unity on $\mathbb{R}^n$. We introduce the averaged functions

$$\bar{\varphi}_m(x,u) = (\varphi(\cdot,u) * \alpha_m)(x) = \int_{\mathbb{R}^n} \varphi(x-y,u)\alpha_m(y)dy,$$
$$\bar{B}_m(x,u) = (B(\cdot,u) * \alpha_m)(x) = \int_{\mathbb{R}^n} B(x-y,u)\alpha_m(y)dy.$$

Then, by known properties of averaged functions, $\bar{\varphi}_m(x,u) \in C^\infty(\Omega, C(\mathbb{R},\mathbb{R}^n))$, $\bar{B}_m(x,u) \in C^\infty(\Omega, C(\mathbb{R}, Sym_n))$, where Sym_n denotes the space of symmetric matrices of order n, $\bar{\gamma}_m(x,p) \doteq \mathrm{div}_x \bar{\varphi}_m(x,p) - D_x^2 \cdot \bar{B}_m(x,p) \in C^\infty(\Omega, C(\mathbb{R}))$ for all $p \in \mathbb{R}$, and

$$\bar{\varphi}_m(x,\cdot) \underset{m\to\infty}{\to} \varphi(x,\cdot) \ \text{ in } L^2_{loc}(\Omega, C(\mathbb{R},\mathbb{R}^n)), \tag{5}$$
$$\bar{B}_m(x,\cdot) \underset{m\to\infty}{\to} B(x,\cdot) \ \text{ in } L^2_{loc}(\Omega, C(\mathbb{R}, Sym_n)), \tag{6}$$
$$\bar{\gamma}_m(x,p) \underset{m\to\infty}{\to} \gamma_p \text{ weakly in } M_{loc}(\Omega). \tag{7}$$

Then, recall that $\gamma_p = \gamma_p^r + \gamma_p^s$, where $\gamma_p^r = \omega_p(x)dx$ and therefore

$$\bar{\gamma}_m(x,p) = (\gamma_p * \alpha_m)(x) = \bar{\gamma}^r_{mp} + \bar{\gamma}^s_{mp},$$

where $\bar{\gamma}^r_{mp} = \omega_p * \alpha_m, \bar{\gamma}^s_{mp} = \gamma_p^s * \alpha_m \in C^\infty(\Omega)$, and

$$\bar{\gamma}^r_{mp} \underset{m\to\infty}{\to} \omega_p \ \text{ in } L^1_{loc}(\Omega), \tag{8}$$
$$|\bar{\gamma}^s_{mp}| \le |\gamma^s_p| * \alpha_m \underset{m\to\infty}{\to} |\gamma^s_p| \ \text{ weakly in } M_{loc}(\Omega). \tag{9}$$

Now we average the vector $\bar{\varphi}_m$ and the matrix $\bar{B}_m$ with respect to the variable u, introducing for $l \in \mathbb{N}$ the functions

$$\bar{\varphi}_{m,l}(x,u) = (\bar{\varphi}_m(x,\cdot) * \zeta_l)(u) = \int \bar{\varphi}_m(x,u-v)\zeta_l(v)dv,$$
$$\bar{B}_{m,l}(x,u) = (\bar{B}_m(x,\cdot) * \zeta_l)(x) = \int \bar{B}_m(x,u-v)\zeta_l(v)dv.$$

Clearly, $\bar{\varphi}_{m,l}(x,u) \in C^\infty(\Omega\times\mathbb{R},\mathbb{R}^n)$, $\bar{B}_{m,l}(x,u) \in C^\infty(\Omega\times\mathbb{R}, Sym_n)$ and for each fixed $m \in \mathbb{N}$

$$\bar{\varphi}_{m,l}(x,u) \underset{l\to\infty}{\to} \bar{\varphi}_m(x,u), \ \ \bar{B}_{m,l}(x,u) \underset{l\to\infty}{\to} \bar{B}_m(x,u),$$
$$\mathrm{div}_x \bar{\varphi}_{m,l}(x,u) - D_x^2 \cdot \bar{B}_{m,l}(x,u) = \bar{\gamma}_m(x,\cdot) * \zeta_l(u) \underset{l\to\infty}{\to} \bar{\gamma}_m(x,u)$$

uniformly on compact subset of $\Omega\times\mathbb{R}$. These relations allow to choose an increasing sequence $l=l_m$ in such a way that for $\varphi_m(x,u)\doteq\bar\varphi_{m,l_m}(x,u)$, $B_m(x,u)\doteq\bar B_{m,l_m}(x,u)+\varepsilon_m uE$, where a sequence $\varepsilon_m>0$, $\varepsilon_m \underset{m\to\infty}{\to} 0$, and E is the unit matrix, we have

$$\varphi_m(x,u)-\bar\varphi_m(x,u)\underset{m\to\infty}{\to}0,\ B_m(x,u)-\bar B_m(x,u)\underset{m\to\infty}{\to}0, \tag{10}$$

$$(\mathrm{div}_x\varphi_m(x,u)-D_x^2\cdot B_m(x,u))-(\mathrm{div}_x\bar\varphi_m(x,u)-D_x^2\cdot\bar B_m(x,u))\underset{m\to\infty}{\to}0 \tag{11}$$

uniformly on compact subset of $\Omega\times\mathbb{R}$. It follows from relations (10), (5), (6) that

$$\varphi_m(x,\cdot)\underset{m\to\infty}{\to}\varphi(x,\cdot)\ \text{ in } L^2_{loc}(\Omega,C(\mathbb{R},\mathbb{R}^n)), \tag{12}$$

$$B_m(x,\cdot)\underset{m\to\infty}{\to}B(x,\cdot)\ \text{ in } L^2_{loc}(\Omega,C(\mathbb{R},Sym_n)). \tag{13}$$

Now, observe that $\gamma_m(x,p)\doteq\mathrm{div}_x\varphi_m(x,p)-D_x^2\cdot B_m(x,p)=\gamma^r_{mp}+\bar\gamma^s_{mp}$, where

$$\gamma^r_{mp}\doteq(\mathrm{div}_x\varphi_m(x,p)-D_x^2\cdot B_m(x,p))- \\ (\mathrm{div}_x\bar\varphi_m(x,p)-D_x^2\cdot\bar B_m(x,p))+\bar\gamma^r_{mp}\underset{m\to\infty}{\to}\omega_p(x)\text{ in } L^1_{loc}(\Omega) \tag{14}$$

in accordance with (11), (8).

Further, from relation (9) it follows that for each $f(x)\in C_0(\Omega)$, $f(x)\ge 0$

$$\overline{\lim_{m\to\infty}}\int_\Omega f(x)|\bar\gamma^s_{mp}(x)|dx\le\int_\Omega f(x)d|\gamma^s_p|(x). \tag{15}$$

Remark that, as follows from the assumption (2) and the choice of our approximations,

$$A_m(x,u)=(B_m)'_u(x,u)\ge\varepsilon_m E,\ \text{ and } (A_m(x,u)-\varepsilon_m E)\xi=0\ \forall\xi\in X.$$

Let K be a compact subset of Ω, $M>0$. We introduce the sequence

$$I_m(K,M)=1+\int_K\int_{-M}^{M}|\mathrm{div}_x\varphi_m(x,p)-D_x^2\cdot B_m(x,p)|dpdx.$$

Generally, the sequence $I_m(K,M)$ may tend to infinity as $m\to\infty$. Obviously, this sequence does not depend on ε_m, which allows to choose the sequence $\varepsilon_m>0$ in such a way that

$$\varepsilon_m I_m(K,M)\underset{m\to\infty}{\to}0 \tag{16}$$

for each $M>0$ and each compact $K\subset\Omega$.

Now, we consider the approximate equation

$$\mathrm{div}\varphi_m(x,u)-D^2\cdot B_m(x,u)=\mathrm{div}[\tilde\varphi_m(x,u)-A_m(x,u)\nabla u]=0, \tag{17}$$

where $\tilde\varphi_m(x,u)$ is a vector with coordinates

$$\tilde\varphi_{mi}(x,u)=\varphi_{mi}(x,u)-\partial_{x_j}(B_m)_{ij}(x,u),\ i=1,\dots,n,$$

where $\varphi_{mi}(x,u)$, $(B_m)_{ij}(x,u)$, $i,j=1,\dots,n$, are components of the vectors $\varphi_m(x,u)$ and the matrix $B_m(x,u)$, respectively.

We suppose that $u=u_m(x)$ is a bounded weak solution of elliptic equation(17) (for instance, we can take $u=u_m(x)$ being a weak solution to the Dirichlet problem with a bounded data at $\partial\Omega$). This means (see [11][Chapter 4]) that $u\in L^\infty(\Omega)\cap W^1_{2,loc}(\Omega)$, where $W^1_{2,loc}(\Omega)$ is the Sobolev space consisting of functions whose generalized derivatives lay in $L^2_{loc}(\Omega)$, and the following standard integral identity is satisfied: $\forall f=f(x)\in C_0^1(\Omega)$.

$$\int_\Omega [\tilde{\varphi}_m(x,u(x))-A_m(x,u(x))\nabla u(x)]\cdot\nabla f(x)dx=0. \tag{18}$$

We also assume that the sequence u_m is bounded in $L^\infty(\Omega)$. Under the above assumptions we establish the strong convergence of the approximations.

Theorem 2. *Suppose that equation (1) is non-degenerate. Then the sequence* $u_m(x) \underset{m\to\infty}{\to} u(x)$ *in* $L^1_{loc}(\Omega)$, *where* $u=u(x)$ *is an entropy and a distributional solution of (1).*

Remark that Theorem 2 allows to establish the existence of entropy solutions of boundary value problems for equation (1) (as well as initial or initial boundary value problems for evolutionary equations of the kind (1)).

For example, in [16] we use approximations and the strong pre-compactness property in order to prove the existence of entropy solutions to the Cauchy problem for an evolutionary hyperbolic equation with discontinuous multidimensional flux. This extends results of [9], where the two-dimensional case is treated by the compensated compactness method.

In the next section 2 we describe the main concepts, in particular the concept of measure-valued functions. In sections 3,4 we introduce a notion of H-measure and prove the localization property. Finally, in the last section 5, these results are applied to prove our main Theorems 1,2.

2 Main concepts

Recall (see [3, 4, 20]) that a *measure-valued* function on Ω is a weakly measurable map $x\to\nu_x$ of the set Ω into the space of probability Borel measures with compact support in $\mathbb{R}$. The weak measurability of ν_x means that for each continuous function $f(\lambda)$ the function $x\to\int f(\lambda)d\nu_x(\lambda)$ is Lebesgue-measurable on Ω.

Remark 1. If ν_x is a measure-valued function then, as was shown in [13], the functions $\int g(\lambda)d\nu_x(\lambda)$ are measurable in Ω for all bounded Borel functions $g(\lambda)$. More generally, if $f(x,\lambda)$ is a Caratheodory function and $g(\lambda)$ is a bounded Borel function then the function $\int f(x,\lambda)g(\lambda)d\nu_x(\lambda)$ is measurable. This follows from the fact that any Caratheodory function is strongly measurable as a map $x\to f(x,\cdot)\in C(\mathbb{R})$ (see [6], Chapter 2) and, therefore, is a pointwise limit of step functions $f_m(x,\lambda)=$

$\sum\limits_i g_{mi}(x)h_{mi}(\lambda)$ with measurable functions $g_{mi}(x)$ and continuous $h_{mi}(\lambda)$ so that for $x \in \Omega$ $f_m(x,\cdot) \underset{m\to\infty}{\to} f(x,\cdot)$ in $C(\mathbb{R})$.

A measure-valued function ν_x is said to be bounded if there exists $M > 0$ such that $\operatorname{supp} \nu_x \subset [-M, M]$ for almost all $x \in \Omega$. We denote by $\|\nu_x\|_\infty$ the smallest value of M with this property.

Finally, measure-valued functions of the form $\nu_x(\lambda) = \delta(\lambda - u(x))$, where $\delta(\lambda - u)$ is the Dirac measure concentrated at u are said to be *regular*; we identify them with the corresponding functions $u(x)$. Thus, the set $MV(\Omega)$ of bounded measure-valued functions on Ω contains the space $L^\infty(\Omega)$. Note that for a regular measure-valued function $\nu_x(\lambda) = \delta(\lambda - u(x))$ the value $\|\nu_x\|_\infty = \|u\|_\infty$. Extending the concept of boundedness in $L^\infty(\Omega)$ to measure-valued functions, we shall say that a subset A of $MV(\Omega)$ is *bounded* if $\sup_{\nu_x\in A} \|\nu_x\|_\infty < \infty$.

Below we define the weak and the strong convergence of sequences of measure-valued functions.

Definition 3. Let $\nu_x^k \in MV(\Omega)$, $k \in \mathbb{N}$, and let $\nu_x \in MV(\Omega)$. Then
1) the sequence ν_x^k converges weakly to ν_x if for each $f(\lambda) \in C(\mathbb{R})$,

$$\int f(\lambda)d\nu_x^k(\lambda) \underset{k\to\infty}{\to} \int f(\lambda)d\nu_x(\lambda) \ \text{ weakly-}* \text{ in } \ L^\infty(\Omega);$$

2) the sequence ν_x^k converges to ν_x *strongly* if for each $f(\lambda) \in C(\mathbb{R})$,

$$\int f(\lambda)d\nu_x^k(\lambda) \underset{k\to\infty}{\to} \int f(\lambda)d\nu_x(\lambda) \ \text{ in } L^1_{loc}(\Omega).$$

The next result was proved in [20] for regular functions ν_x^k. The proof can be easily extended to the general case, as was done in [13].

Theorem T. *Let ν_x^k, $k \in \mathbb{N}$ be a bounded sequence of measure-valued functions. Then there exist a subsequence $\nu_x^r = \nu_x^k$, $k = k_r$, and a measure-valued function $\nu_x \in MV(\Omega)$ such that $\nu_x^r \to \nu_x$ weakly as $r \to \infty$.*

Theorem T shows that bounded sets of measure-valued functions are weak1y pre-compact. If $u_k(x) \in L^\infty(\Omega)$ is a bounded sequence, treated as a sequence of regular measure valued functions, and $u_k(x)$ weakly converges to a measure valued function ν_x then ν_x is regular, $\nu_x(\lambda) = \delta(\lambda - u(x))$, if and only if $u_k(x) \to u(x)$ in $L^1_{loc}(\Omega)$ (see [20]). Obviously, if $u_k(x)$ converges to ν_x strongly then $u_k(x) \to u(x) = \int \lambda d\nu_x(\lambda)$ in $L^1_{loc}(\Omega)$ and then $\nu_x(\lambda) = \delta(\lambda - u(x))$.

We shall study the strong pre-compactness property using Tartar's techniques of H-measures.

Let $F(u)(\xi) = \displaystyle\int e^{-2\pi i\xi\cdot x}u(x)dx$, $\xi \in \mathbb{R}^n$, be the Fourier transform extended as unitary operator on the space $u(x) \in L^2(\mathbb{R}^n)$, and let $S = S^{n-1} = \{\ \xi \in \mathbb{R}^n \ \mid\ |\xi| = 1\ \}$ be the unit sphere in $\mathbb{R}^n$. Denote by $u \to \overline{u}$, $u \in \mathbb{C}$, the complex conjugation.

The concept of H-measure corresponding to some sequence of vector-valued functions bounded in $L^2(\Omega)$ was introduced by L. Tartar [21] and P. Gerárd [5] on the basis of the following result. For $l \in \mathbb{N}$ let $U_k(x) = \big(U_k^1(x), \dots, U_k^l(x)\big) \in L^2(\Omega, \mathbb{R}^l)$ be a sequence weakly convergent to the zero vector.

Proposition 1. [see [21], Theorem 1.1] *There exists a family of complex Borel measures $\mu = \{\mu^{ij}\}_{i,j=1}^l$ in $\Omega \times S$ and a subsequence $U_r(x) = U_k(x)$, $k = k_r$, such that*

$$\langle \mu^{ij}, \Phi_1(x)\overline{\Phi_2(x)}\psi(\xi)\rangle = \lim_{r\to\infty} \int_{\mathbb{R}^n} F(U_r^i\Phi_1)(\xi)\overline{F(U_r^j\Phi_2)(\xi)}\psi\left(\frac{\xi}{|\xi|}\right) d\xi \tag{19}$$

for all $\Phi_1(x), \Phi_2(x) \in C_0(\Omega)$ and $\psi(\xi) \in C(S)$.

The family $\mu = \{\mu^{ij}\}_{i,j=1}^l$ is called an H-measure corresponding to $U_r(x)$.

Recently in [1] the new concept of parabolic H-measures was introduced. Here we present the more general variant of this concept. Suppose that $X \subset \mathbb{R}^n$ is a linear subspace, $X^\perp$ is its orthogonal complement, P_1, P_2 are orthogonal projections on X, $X^\perp$, respectively. We denote for $\xi \in \mathbb{R}^n$ $\tilde{\xi} = P_1\xi$, $\bar{\xi} = P_2\xi$, so that $\tilde{\xi} \in X$, $\bar{\xi} \in X^\perp$, $\xi = \tilde{\xi} + \bar{\xi}$. Let $S_X = \{ \ \xi \in \mathbb{R}^n \ | \ |\tilde{\xi}|^2 + |\bar{\xi}|^4 = 1 \ \}$. Then S_X is a compact smooth manifold of codimension 1, in the case when $X = \{0\}$ or $X = \mathbb{R}^n$ it coincides with the unit sphere $S = \{\xi \in \mathbb{R}^n \ | \ |\xi| = 1 \ \}$. Let us define the projection $\pi_X : \mathbb{R}^n \setminus \{0\} \to S_X$:

$$\pi_X(\xi) = \frac{\tilde{\xi}}{(|\tilde{\xi}|^2 + |\bar{\xi}|^4)^{1/2}} + \frac{\bar{\xi}}{(|\tilde{\xi}|^2 + |\bar{\xi}|^4)^{1/4}}.$$

Remark that in the case when $X = \{0\}$ or $X = \mathbb{R}^n$ $\pi_X(\xi) = \xi/|\xi|$. We denote $p(\xi) = (|\tilde{\xi}|^2 + |\bar{\xi}|^4)^{1/4}$. The following useful property of the projection holds.

Lemma 1. *Let $\xi, \eta \in \mathbb{R}^n$, $\max(p(\xi), p(\eta)) \ge 1$. Then*

$$|\pi_X(\xi) - \pi_X(\eta)| \le \frac{6|\xi - \eta|}{\max(p(\xi), p(\eta))}.$$

Proof. We define for $\xi \in \mathbb{R}^n$, $\alpha > 0$ $\xi_\alpha = \alpha^2\tilde{\xi} + \alpha\bar{\xi}$. Observe that for all $\alpha > 0$ $\pi_X(\xi_\alpha) = \pi_X(\xi)$. Without lose of generality we may suppose that $p(\xi) \ge p(\eta)$, and in particular $p(\xi) \ge 1$. Remark that $\pi_X(\xi) = \xi_\alpha$, $\pi_X(\eta) = \eta_\beta$, where $\alpha = 1/p(\xi)$, $\beta = 1/p(\eta)$. Therefore,

$$\begin{aligned} |\pi_X(\xi) - \pi_X(\eta)| = |\xi_\alpha - \eta_\beta| \le |\xi_\alpha - \eta_\alpha| + |\eta_\alpha - \eta_\beta| \le \\ \left(\alpha^4|\tilde{\xi} - \tilde{\eta}|^2 + \alpha^2|\bar{\xi} - \bar{\eta}|^2\right)^{1/2} + \left((\beta^2 - \alpha^2)^2|\tilde{\eta}|^2 + (\beta - \alpha)^2|\bar{\eta}|^2\right)^{1/2} \le \\ \alpha|\xi - \eta| + (\beta - \alpha)\left((\beta + \alpha)^2|\tilde{\eta}|^2 + |\bar{\eta}|^2\right)^{1/2}. \end{aligned} \tag{20}$$

Here we take into account that $\alpha \le 1$ and therefore $\alpha^4 \le \alpha^2$. Since

$$(\beta + \alpha)^2 \le 4\beta^2 = 4(|\tilde{\eta}|^2 + |\bar{\eta}|^4)^{-1/2} \le 4/|\tilde{\eta}|,$$

we have the estimate

$$(\beta + \alpha)^2|\tilde{\eta}|^2 + |\bar{\eta}|^2 \le 4(|\tilde{\eta}| + |\bar{\eta}|^2) \le 4\left(2(|\tilde{\eta}|^2 + |\bar{\eta}|^4)\right)^{1/2} \le 6(p(\eta))^2. \tag{21}$$

Concerning the term $\beta-\alpha$, we estimate it as follows

$$\beta-\alpha=\frac{p(\xi)-p(\eta)}{p(\xi)p(\eta)}=\frac{|\tilde{\xi}|^2-|\tilde{\eta}|^2+|\bar{\xi}|^4-|\bar{\eta}|^4}{p(\xi)p(\eta)(p(\xi)+p(\eta))((p(\xi))^2+(p(\eta))^2)}\leq$$
$$\frac{(|\tilde{\xi}|+|\tilde{\eta}|)|\tilde{\xi}-\tilde{\eta}|+(|\bar{\xi}|+|\bar{\eta}|)(|\bar{\xi}|^2+|\bar{\eta}|^2)|\bar{\xi}-\bar{\eta}|}{p(\xi)p(\eta)(p(\xi)+p(\eta))((p(\xi))^2+(p(\eta))^2)}\leq$$
$$\frac{|\tilde{\xi}|+|\tilde{\eta}|+(|\bar{\xi}|+|\bar{\eta}|)(|\bar{\xi}|^2+|\bar{\eta}|^2)}{p(\xi)p(\eta)(p(\xi)+p(\eta))((p(\xi))^2+(p(\eta))^2)}|\xi-\eta|\leq$$
$$\frac{(p(\xi))^2+(p(\eta))^2+(p(\xi)+p(\eta))((p(\xi))^2+(p(\eta))^2)}{p(\xi)p(\eta)(p(\xi)+p(\eta))((p(\xi))^2+(p(\eta))^2)}|\xi-\eta|\leq$$
$$\frac{1+p(\xi)+p(\eta)}{p(\xi)+p(\eta)}\frac{|\xi-\eta|}{p(\xi)p(\eta)}\leq\frac{2|\xi-\eta|}{p(\xi)p(\eta)}. \tag{22}$$

Here we use that $\tilde{\xi}\leq(p(\xi))^2$, $\bar{\xi}\leq p(\xi)$, $\tilde{\eta}\leq(p(\eta))^2$, $\bar{\eta}\leq p(\eta)$, and that $p(\xi)+p(\eta)\geq 1$. Now it follows from (20), (21), (22) that

$$|\pi_X(\xi)-\pi_X(\eta)|\leq\frac{|\xi-\eta|}{p(\xi)}+\frac{2\sqrt{6}|\xi-\eta|}{p(\xi)}\leq\frac{6|\xi-\eta|}{p(\xi)}=\frac{6|\xi-\eta|}{\max(p(\xi),p(\eta))},$$

as was to be proved.□

Let $b(x)\in C_0(\mathbb{R}^n)$, $a(z)\in C(S_X)$. Then we can define pseudo-differential operators $\mathcal{B},\mathcal{A}$ with symbols $b(x)$, $a(\pi_X(\xi))$, respectively. These operators are multiplication operators $\mathcal{B}u(x)=b(x)u(x)$, $F(\mathcal{A}u)(\xi)=a(\pi_X(\xi))F(u)(\xi)$. Obviously, the operators $\mathcal{B},\mathcal{A}$ are well-defined and bounded in L^2. As was proved in [21], in the case when $S_X=S$, $\pi_X(\xi)=\xi/|\xi|$ the commutator $[\mathcal{A},\mathcal{B}]=\mathcal{A}\mathcal{B}-\mathcal{B}\mathcal{A}$ is a compact operator. Using the assertion of Lemma 1 one can easily extend this result for the general case (in the case $\dim X=1$ this was done in [1]). For completeness we give below the details for the general setting.

Lemma 2. *The operator $[\mathcal{A},\mathcal{B}]$ is compact in L^2.*

Proof. We can find sequences $a_k(z)\in C^\infty(S_X)$, $b_k(x)\in C^\infty(\mathbb{R}^n)$, $k\in\mathbb{N}$ with the following properties: $F(b_k)(\xi)\in C_0^\infty(\mathbb{R}^n)$, and as $k\to\infty$ $a_k(z)\to a(z)$, $b_k(x)\to b(x)$ uniformly on S_X, $\mathbb{R}^n$, respectively. Then the sequences of the operators $\mathcal{A}_k$, $\mathcal{B}_k$ with symbols $a_k(\pi_X(\xi))$, $b_k(x)$ converge as $k\to\infty$ to the operators $\mathcal{A}$, $\mathcal{B}$, respectively (in the operator norm). Therefore, $[\mathcal{A}_k,\mathcal{B}_k]\underset{k\to\infty}{\to}[\mathcal{A},\mathcal{B}]$ and it is sufficient to prove that the operators $[\mathcal{A}_k,\mathcal{B}_k]$ are compact for all $k\in\mathbb{N}$ (then $[\mathcal{A},\mathcal{B}]$ is a compact operator as a limit of compact operators). Let $u=u(x)\in L^2(\mathbb{R}^n)$. Then by the known property $F(bu)(\xi)=F(b)*F(u)(\xi)=\int F(b)(\xi-\eta)F(u)(\eta)d\eta$,

$$F([\mathcal{A}_k,\mathcal{B}_k]u)(\xi)=F(\mathcal{A}_k\mathcal{B}_ku)(\xi)-F(\mathcal{B}_k\mathcal{A}_ku)(\xi)=$$
$$a_k(\pi_X(\xi))F(b_ku)(\xi)-F(b_k\mathcal{A}_ku)(\xi)=$$
$$\int_{\mathbb{R}^n}(a_k(\pi_X(\xi))-a_k(\pi_X(\eta)))F(b_k)(\xi-\eta)F(u)(\eta)d\eta.$$

We have to prove that the integral operator $Kv(\xi) = \int_{\mathbb{R}^n} k(\xi,\eta)v(\eta)d\eta$ with the kernel $k(\xi,\eta) = (a_k(\pi_X(\xi)) - a_k(\pi_X(\eta)))F(b_k)(\xi-\eta)$ is compact on $L^2(\mathbb{R}^n)$.

Since $a_k \in C^\infty(S_X)$ then by Lemma 1

$$|a_k(\pi_X(\xi)) - a_k(\pi_X(\eta))| \le C\frac{|\xi-\eta|}{\max(p(\xi),p(\eta))}$$

for $\max(p(\xi),p(\eta)) \ge 1$, where $C = \text{const}$. Thus for all $\xi,\eta \in \mathbb{R}^n$ such that $\max(p(\xi),p(\eta)) > m > 1$

$$|a_k(\pi_X(\xi)) - a_k(\pi_X(\eta))| \le \frac{C}{m}|\xi-\eta|. \tag{23}$$

Let $\chi_m(\xi,\eta)$ be the indicator function of the set $\{\ (\xi,\eta) \in \mathbb{R}^{2n} \mid \max(p(\xi),p(\eta)) \le m\ \}$, and

$$k_m(\xi,\eta) = \chi(\xi,\eta)(a_k(\pi_X(\xi)) - a_k(\pi_X(\eta)))F(b_k)(\xi-\eta),$$
$$r_m(\xi,\eta) = (1-\chi(\xi,\eta))(a_k(\pi_X(\xi)) - a_k(\pi_X(\eta)))F(b_k)(\xi-\eta).$$

Then $k(\xi,\eta) = k_m(\xi,\eta) + r_m(\xi,\eta)$ and $K = K_m + R_m$, where K_m, R_m are integral operators with the kernels $k_m(\xi,\eta)$, $r_m(\xi,\eta)$, respectively. Since the function $k_m(\xi,\eta)$ is bounded and compactly supported then the operator K_m is a Hilbert-Schmidt operator, which is compact. On the other hand, in view of (23)

$$|R_m v(\xi)| \le \frac{C}{m}\int_{\mathbb{R}^n} |(\xi-\eta)F(b_k)(\xi-\eta)||v(\eta)|d\eta = [|\xi F(b_k)| * |v|](\xi)$$

and, by the Young inequality, for every $v \in L^2(\mathbb{R}^n)$

$$\|R_m v\|_2 \le \frac{C}{m}\|\xi F(b_k)\|_1\|v\|_2.$$

Therefore, $\|R_m\| \le \text{const}/m$ and $R_m \to 0$ as $m \to \infty$. We conclude that $K_m \to K$ and therefore K is a compact operator, as a limit of compact operators. This complete the proof. $\square$

The parabolic H-measure μ^{ij}, $i,j = 1,\dots,l$ corresponding to a subspace $X \subset \mathbb{R}^n$ and a sequence $U_r(x) \in L^2(\Omega,\mathbb{R}^l)$ is defined on $\Omega \times S_X$ by the relation similar to (19): $\forall \Phi_1(x), \Phi_2(x) \in C_0(\Omega)$, $\psi(\xi) \in C(S_X)$

$$\langle \mu^{ij}, \Phi_1(x)\overline{\Phi_2(x)}\psi(\xi)\rangle = \lim_{r\to\infty}\int_{\mathbb{R}^n} F(\Phi_1 U_r^i)(\xi)\overline{F(\Phi_2 U_r^j)(\xi)}\psi(\pi_X(\xi))d\xi. \tag{24}$$

The existence of the H-measure μ^{ij} is proved exactly in the same way as in [21], with using the statement of Lemma 2. This H-measure satisfies the same properties as the "usual" H-measure μ^{pq} (corresponding to the case $X = \{0\}$ or $X = \mathbb{R}^n$).

The concept of H-measure was extended in [13] (see also [14, 15]) to the case of "continuous" indexes i,j. The similar extension can be also established for parabolic H-measures. We study the properties of such H-measures in the next section.

3 H-measures corresponding to bounded sequences of measure-valued functions

Let $\nu_x^k \in MV(\Omega)$ be a bounded sequence of measure-valued functions weakly convergent to a measure-valued function $\nu_x^0 \in MV(\Omega)$. For $x \in \Omega$ and $p \in \mathbb{R}$ we introduce the distribution functions

$$u_k(x,p) = \nu_x^k((p,+\infty)), \quad u_0(x,p) = \nu_x^0((p,+\infty)).$$

Then, as mentioned in Remark 1, for $k \in \mathbb{N} \cup \{0\}$ and $p \in \mathbb{R}$ the functions $u_k(x,p)$ are measurable in $x \in \Omega$; thus, $u_k(x,p) \in L^\infty(\Omega)$ and $0 \le u_k(x,p) \le 1$. Let

$$E = E(\nu_x^0) = \left\{ \ p_0 \in \mathbb{R} \ \mid \ u_0(x,p) \underset{p \to p_0}{\to} u_0(x,p_0) \ \text{ in } L^1_{loc}(\Omega) \ \right\}.$$

We have the following result, whose proof can be found in [13].

Lemma 3. *The complement $\bar{E} = \mathbb{R} \setminus E$ is at most countable and if $p \in E$ then $u_k(x,p) \underset{k\to\infty}{\to} u_0(x,p)$ weakly-$*$ in $L^\infty(\Omega)$.*

Let $U_k^p(x) = u_k(x,p) - u_0(x,p)$. Then, by Lemma 3, $U_k^p(x) \to 0$ as $k \to \infty$ weakly-$*$ in $L^\infty(\Omega)$ for $p \in E$. Let X be a linear subspace of $\mathbb{R}^n$. The next result, similar to Proposition 1, was also established in [13] in the case $X = \mathbb{R}^n$. The general case of arbitrary X is proved exactly in the same way.

Proposition 2. *1) There exists a family of locally finite complex Borel measures $\{\mu^{pq}\}_{p,q\in E}$ in $\Omega \times S_X$ and a subsequence $U_r(x) = \{U_r^p(x)\}_{p\in E}$, $U_r^p(x) = U_k^p(x)$, $k = k_r$ such that for all $\Phi_1(x), \Phi_2(x) \in C_0(\Omega)$ and $\psi(\xi) \in C(S_X)$*

$$\langle \mu^{pq}, \Phi_1(x)\overline{\Phi_2(x)}\psi(\xi)\rangle = \lim_{r\to\infty} \int_{\mathbb{R}^n} F(\Phi_1 U_r^p)(\xi)\overline{F(\Phi_2 U_r^q)(\xi)}\psi(\pi_X(\xi))d\xi. \tag{25}$$

2) The correspondence $(p,q) \to \mu^{pq}$ is a continuous map from $E \times E$ into the space $\mathrm{M}_{loc}(\Omega \times S)$.

We call the family of measures $\{\mu^{pq}\}_{p,q\in E}$ the H-measure corresponding to the subsequence $\nu_x^r = \nu_x^k$, $k = k_r$.

Remark 2. We can replace the function $\psi(\pi_X(\xi))$ in relation (25) (and in (24)) to a function $\tilde{\psi}(\xi) \in C(\mathbb{R}^n)$, which equals $\psi(\pi_X(\xi))$ for large $|\xi|$. Indeed, since $U_r^q \underset{r\to\infty}{\to} 0$ weakly-$*$ in $L^\infty(\Omega)$ we have $F(\Phi_2 U_r^q)(\xi) \underset{r\to\infty}{\to} 0$ point-wise and in $L^2_{loc}(\mathbb{R}^n)$ (in view of the bound $|F(\Phi_2 U_r^q)(\xi)| \le \|\Phi_2 U_r^q\|_1 \le \text{const}$). Taking into account that the function $\chi(\xi) = \tilde{\psi}(\xi) - \psi(\pi_X(\xi))$ is bounded and has a compact support, we conclude that

$$\overline{F(\Phi_2 U_r^q)(\xi)}\chi(\xi) \underset{r\to\infty}{\to} 0 \ \text{ in } L^2(\mathbb{R}^n).$$

This implies that

$$\lim_{r\to\infty} \int_{\mathbb{R}^n} F(\Phi_1 U_r^p)(\xi)\overline{F(\Phi_2 U_r^q)(\xi)}\chi(\xi)d\xi = 0.$$

Therefore

$$\lim_{r\to\infty}\int_{\mathbb{R}^n} F(\Phi_1 U_r^p)(\xi)\overline{F(\Phi_2 U_r^q)(\xi)}\tilde{\psi}(\xi)d\xi =$$

$$\lim_{r\to\infty}\int_{\mathbb{R}^n} F(\Phi_1 U_r^p)(\xi)\overline{F(\Phi_2 U_r^q)(\xi)}\psi(\pi_X(\xi))d\xi = \langle \mu^{pq}, \Phi_1(x)\overline{\Phi_2(x)}\psi(\xi)\rangle,$$

as required.

We point out the following important properties of an H-measure.

Lemma 4. *(i) $\mu^{pp} \geq 0$ for each $p \in E$; (ii) $\mu^{pq} = \overline{\mu^{qp}}$ for all $p, q \in E$; (iii) for $p_1, \dots, p_l \in E$ and $g_1, \dots, g_l \in C_0(\Omega \times S_X)$ the matrix $A = a_{ij} = \langle \mu^{p_i p_j}, g_i \overline{g_j}\rangle$, $i, j = 1, \dots, l$ is Hermitian and positive-definite.*

Proof. We prove (iii). First let the functions $g_i = g_i(x, \xi)$ be finite sums of functions of the form $\Phi(x)\psi(\xi)$, where $\Phi(x) \in C_0(\Omega)$ and $\psi(\xi) \in C(S_X)$. Then it follows from (25) that

$$a_{ij} = \lim_{r\to\infty}\int_{\mathbb{R}^n} H_r^i(\xi)\overline{H_r^j(\xi)}d\xi, \tag{26}$$

where $H_r^i(\xi) = F(g_i(\cdot, \pi_X(\xi))U_r^{p_i})(\xi)$. Hence, setting $g_i(x, \xi) = g(x, \xi) = \sum\limits_{k=1}^{m} \Phi_k(x)\psi_k(\xi)$, we obtain

$$H_r^i(\xi) = \sum_{k=1}^{m} F(\Phi_k U_r^{p_i})(\xi)\psi_k(\pi_X(\xi)).$$

It immediately follows from (26) that $a_{ji} = \overline{a_{ij}}$, $i, j = 1, \dots, l$, which shows that A is a Hermitian matrix. Further, for $\alpha_1, \dots, \alpha_l \in \mathbb{C}$ we have

$$\sum_{i,j=1}^{l} a_{ij}\alpha_i\overline{\alpha_j} = \lim_{r\to\infty}\int_{\mathbb{R}^n} |H_r(\xi)|^2 d\xi \geq 0, \quad H_r(\xi) = \sum_{i=1}^{l} H_r^i(\xi)\alpha_i,$$

which means that A is positive-definite.

In the general case when $g_i \in C_0(\Omega \times S_X)$ one carries out the proof of (iii) by approximating the functions g_i, $i = 1, \dots, l$ in the uniform norm by finite sums of functions of the form $\Phi(x)\psi(\xi)$.

Assertions (i) and (ii) are easy consequences of (iii). Indeed, setting $l = 1$, $p_1 = p$ and $g_1 = g$, we obtain the relation $\langle \mu^{pp}, |g|^2\rangle \geq 0$, which holds for all $g \in C_0(\Omega \times S_X)$, thus showing that μ^{pp} is real and non-negative. To prove (ii) we represent an arbitrary function $g = g(x, \xi)$ with compact support in the form $g = g_1\overline{g_2}$. Let $l = 2$, $p_1 = p$ and $p_2 = q$. In view of (iii),

$$\langle \mu^{pq}, g\rangle = \langle \mu^{pq}, g_1\overline{g_2}\rangle = \overline{\langle \mu^{qp}, g_2\overline{g_1}\rangle} = \overline{\langle \mu^{qp}, \overline{g}\rangle} = \langle \overline{\mu^{qp}}, g\rangle$$

and $\mu^{pq} = \overline{\mu^{qp}}$. The proof is complete.□

We consider now a countable dense index subset $D \subset E$.

Proposition 3 (cf. [15]). *There exists a family of complex finite Borel measures μ_x^{pq} in S_X with $p,q \in D$, $x \in \Omega'$ where Ω' is a subset of Ω of full measure, such that $\mu^{pq} = \mu_x^{pq} dx$, that is, for all $\Phi(x,\xi) \in C_0(\Omega \times S_X)$ the function*

$$x \to \langle \mu_x^{pq}(\xi), \Phi(x,\xi) \rangle = \int_{S_X} \Phi(x,\xi) d\mu_x^{pq}(\xi)$$

is Lebesgue-measurable on Ω, bounded, and

$$\langle \mu^{pq}, \Phi(x,\xi) \rangle = \int_\Omega \langle \mu_x^{pq}(\xi), \Phi(x,\xi) \rangle dx.$$

Moreover, $\operatorname{Var} \mu_x^{pq} \le 1$ *for all* $p,q \in D$.

Proof. We claim that $\mathrm{pr}_\Omega |\mu^{pq}| \le \mathrm{meas}$ for $p,q \in E$, where meas is the Lebesgue measure on Ω. Assume first that $p = q$. By Lemma 4, the measure μ^{pp} is non-negative. Next, in view of relation (25) with $\Phi_1(x) = \Phi_2(x) = \Phi(x) \in C_0(\Omega)$ and $\psi(\xi) \equiv 1$,

$$\langle \mu^{pp}, |\Phi(x)|^2 \rangle = \lim_{r\to\infty} \int_{\mathbb{R}^n} F(\Phi U_r^p)(\xi) \overline{F(\Phi U_r^p)(\xi)} d\xi =$$
$$\lim_{r\to\infty} \int_\Omega |U_r^p(x)|^2 |\Phi(x)|^2 dx \le \int_\Omega |\Phi(x)|^2 dx$$

(we use here Plancherel's equality and the estimate $|U_r^p(x)| \le 1$). Thus, we see that that $\mathrm{pr}_\Omega \mu^{pp} \le \mathrm{meas}$.

Let $p,q \in E$, A be a bounded open subset of Ω, and $g = g(x,\xi) \in C_0(A \times S_X)$, $|g| \le 1$. Let also $g_1 = g/\sqrt{|g|}$ (we set $g_1 = 0$ for $g = 0$) and $g_2 = \sqrt{|g|}$. Then $g_1, g_2 \in C_0(A \times S_X)$, $g = g_1 \overline{g_2}$, $|g_1|^2 = |g_2|^2 = |g|$ and the matrix

$$\begin{pmatrix} \langle \mu^{pp}, |g| \rangle & \langle \mu^{pq}, g \rangle \\ \overline{\langle \mu^{pq}, g \rangle} & \langle \mu^{qq}, |g| \rangle \end{pmatrix}$$

is positive-definite by Lemma 4; in particular,

$$|\langle \mu^{pq}, g \rangle| \le (\langle \mu^{pp}, |g| \rangle \langle \mu^{qq}, |g| \rangle)^{1/2} \le (\mu^{pp}(A \times S_X) \mu^{qq}(A \times S))^{1/2} \le \mathrm{meas}(A).$$

We take into account the inequalities $\mathrm{pr}_\Omega \mu^{pp} \le \mathrm{meas}$ and $\mathrm{pr}_\Omega \mu^{qq} \le \mathrm{meas}$ to obtain the last estimate. Since g can be an arbitrary function in $C_0(A \times S_X)$, $|g| \le 1$, we obtain the inequality $|\mu^{pq}|(A \times S_X) \le \mathrm{meas}(A)$. The measure μ^{pq} is regular, therefore this estimate holds for all Borel subsets A of Ω and

$$\mathrm{pr}_\Omega |\mu^{pq}| \le \mathrm{meas}\,. \tag{27}$$

It follows from (27) that for all $\psi(\xi) \in C(S_X)$ we have

$$|\mathrm{pr}_\Omega\, (\psi(\xi) \mu^{pq}(x,\xi))\,| \le \|\psi\|_\infty \cdot \mathrm{pr}_\Omega |\mu^{pq}| \le \|\psi\|_\infty \cdot \mathrm{meas}\,. \tag{28}$$

In view of (28) the measures $\mathrm{pr}_\Omega(\psi(\xi)\mu^{pq}(x,\xi))$ are absolutely continuous with respect to the Lebesgue measure, and the Radon-Nikodym theorem shows that

$$\mathrm{pr}_\Omega\, (\psi(\xi)\mu^{pq}(x,\xi)) = h_\psi^{pq}(x) \cdot \mathrm{meas},$$

where the densities $h_\psi^{pq}(x)$ are measurable on Ω and, as seen from (28),

$$\|h_\psi^{pq}(x)\|_\infty \le \|\psi\|_\infty. \tag{29}$$

We now choose a non-negative function $K(x) \in C_0^\infty(\mathbb{R}^n)$ with support in the unit ball such that $\int K(x)dx = 1$ and set $K_m(x) = m^n K(mx)$ for $m \in \mathbb{N}$. Clearly, the sequence of K_m converges in $\mathcal{D}'(\mathbb{R}^n)$ to the Dirac δ-function (that is, this sequence is an approximate unity).

Let $\operatorname{B}\lim\limits_{m\to\infty} c_m$ be a generalized Banach limit on the space l_∞ of bounded sequences $c = \{c_m\}_{m\in\mathbb{N}}$, i.e. $L(c) = \operatorname{B}\lim\limits_{m\to\infty} c_m$ is a linear functional on l_∞ with the property:

$$\varliminf_{m\to\infty} c_m \le L(c) \le \varlimsup_{m\to\infty} c_m$$

(in particular for convergent sequences $c = \{c_m\}$ $L(c) = \lim\limits_{m\to\infty} c_m$). For complex sequences $c_m = a_m + ib_m$ the Banach limits is defined by complexification: $\operatorname{B}\lim\limits_{m\to\infty} c_m = L(a) + iL(b)$, where $a = \{a_m\}$, $b = \{b_m\}$ are real and imaginary parts of the sequence $c = \{c_m\}$, respectively. Modifying the densities $h_\psi^{pq}(x)$ on subsets of measure zero, for instance, replacing them by the functions

$$\operatorname{B}\lim_{m\to\infty} \int_\Omega h_\psi^{pq}(y) K_m(x-y)dy$$

(obviously, the value $h_\psi^{pq}(x)$ does not change for any Lebesgue point x of the function h_ψ^{pq}), we shall assume that for all $x \in \Omega$

$$h_\psi^{pq}(x) = \operatorname{B}\lim_{m\to\infty} \int_\Omega h_\psi^{pq}(y) K_m(x-y)dy. \tag{30}$$

Let Ω' be the set of common Lebesgue points of the functions $h_\psi^{pq}(x)$, $u_0(x,p) = \nu_x^0((p,+\infty))$, and $u_0^-(x,p) = \nu_x^0([p,+\infty)) = \lim\limits_{q\to p-} u_0(x,q)$, where $p,q \in D$ and ψ belongs to F, some countable dense subset of $C(S_X)$. The family of (p,q,ψ) is countable, therefore Ω' is of full measure.

The dependence of h_ψ^{pq} on ψ, regarded as a map from $C(S_X)$ into $L^\infty(\Omega)$, is clearly linear and continuous (in view of (29)), therefore it follows from the density of F in $C(S_X)$ that $x \in \Omega'$ is a Lebesgue point of the functions $h_\psi^{pq}(x)$ for all $\psi(\xi) \in C(S_X)$ and $p,q \in D$ (here we also take (30) into account).

For $p,q \in D$ and $x \in \Omega'$ the equality $l(\psi) = h_\psi^{pq}(x)$ defines a continuous linear functional in $C(S_X)$; moreover, $\|l\| \le 1$ in view of (29). By the Riesz-Markov theorem this functional can be defined by integration with respect to some complex Borel measure $\mu_x^{pq}(\xi)$ in S_X and $\operatorname{Var}\mu_x^{pq} = \|l\| \le 1$. Hence

$$h_\psi^{pq}(x) = \langle \mu_x^{pq}(\xi), \psi \rangle = \int_{S_X} \psi(\xi) d\mu_x^{pq}(\xi) \tag{31}$$

for all $\psi(\xi) \in C(S_X)$.

Equality (31) shows that the functions $x \to \int_S \psi(\xi) d\mu_x^{pq}(\xi)$ are bounded and measurable for all $\psi(\xi) \in C(S_X)$. Next, for $\Phi(x) \in C_0(\Omega)$ and $\psi(\xi) \in C(S_X)$ we have

$$\int_\Omega \left(\int_{S_X} \Phi(x)\psi(\xi) d\mu_x^{pq}(\xi) \right) dx = \int_\Omega \Phi(x) h_\psi^{pq}(x) dx =$$
$$\int_\Omega \Phi(x) d\mathrm{pr}_\Omega \left(\psi(\xi)\mu^{pq} \right) = \int_{\Omega \times S_X} \Phi(x)\psi(\xi) d\mu^{pq}(x, \xi). \qquad (32)$$

Approximating an arbitrary function $\Phi(x, \xi) \in C_0(\Omega \times S_X)$ in the uniform norm by linear combinations of functions of the form $\Phi(x)\psi(\xi)$, we derive from (32) that the integral $\int_{S_X} \Phi(x, \xi) d\mu_x^{pq}(\xi)$ is Lebesgue-measurable with respect to $x \in \Omega$, bounded, and

$$\int_\Omega \left(\int_{S_X} \Phi(x, \xi) d\mu_x^{pq}(\xi) \right) dx = \int_{\Omega \times S_X} \Phi(x, \xi) d\mu^{pq}(x, \xi),$$

that is, $\mu^{pq} = \mu_x^{pq} dx$. Recall that $\operatorname{Var} \mu_x^{pq} \le 1$.□

The assumption that $x \in \Omega'$ are Lebesgue points of the functions $u_0(x, p)$, $u_0^-(x, p)$ for all $p \in D$ will be used later. Observe that since $p \in D \subset E$ is a continuity point of the map $p \to u_0(x, p)$ in $L^1_{loc}(\Omega)$ then $u_0^-(x, p) = u_0(x, p)$ a.e. in Ω. By the construction $x \in \Omega'$ is a common Lebesgue point of the functions $u_0(x, p)$, $u_0^-(x, p)$, therefore

$$\nu_x^0(\{p\}) = u_0^-(x, p) - u_0(x, p) = 0 \quad \forall p \in D. \qquad (33)$$

Remark 3. a) Since the H-measure is absolutely continuous with respect to x-variables identity (25) is satisfied for $\Phi_1(x), \Phi_2(x) \in L^2(\Omega)$. Indeed, by Proposition 3 we can rewrite this identity in the form: $\forall \Phi_1(x), \Phi_2(x) \in C_0(\Omega)$, $\psi(\xi) \in C(S_X)$

$$\int_\Omega \Phi_1(x)\overline{\Phi_2(x)} \langle \psi(\xi), \mu_x^{pq}(\xi) \rangle dx = \lim_{r \to \infty} \int_{\mathbb{R}^n} F(\Phi_1 U_r^p)(\xi) \overline{F(\Phi_2 U_r^q)(\xi)} \psi(\pi_X(\xi)) d\xi. \quad (34)$$

Both sides of this identity are continuous with respect to $(\Phi_1(x), \Phi_2(x))$ in $L^2(\Omega) \times L^2(\Omega)$ and since $C_0(\Omega)$ is dense in $L^2(\Omega)$ we conclude that (34) is satisfied for each $\Phi_1(x), \Phi_2(x) \in L^2(\Omega)$;

b) if $x \in \Omega'$ is a Lebesgue point of a function $\Phi(x) \in L^2(\Omega)$ then

$$\Phi(x) \langle \mu_x^{pq}, \psi(\xi) \rangle = \lim_{m \to \infty} \lim_{r \to \infty} \int_{\mathbb{R}^n} F(\Phi \Phi_m U_r^p)(\xi) \overline{F(\Phi_m U_r^q)(\xi)} \psi(\pi_X(\xi)) d\xi \qquad (35)$$

for all $\psi(\xi) \in C(S_X)$, where $(\Phi \Phi_m U_r^p)(y) = \Phi(y)\Phi_m(x - y) U_r^p(y)$ and $(\Phi_m U_r^q)(y) = \Phi_m(x - y) U_r^q(y)$.

Indeed, it follows from (34) that

$$\lim_{r \to \infty} \int_{\mathbb{R}^n} F(\Phi \Phi_m U_r^p)(\xi) \overline{F(\Phi_m U_r^q)(\xi)} \psi(\pi_X(\xi)) d\xi = \int_\Omega h_\psi^{pq}(y) \Phi(y) K_m(x - y) dy. \quad (36)$$

Now, since $x \in \Omega'$ is a Lebesgue point of the functions $h_\psi^{pq}(y)$ and $\Phi(y)$, and the function $h_\psi^{pq}(y)$ is bounded, x is also a Lebesgue point for the product of these functions. Therefore,

$$\lim_{m\to\infty} \int_\Omega h_\psi^{pq}(y)\Phi(y)K_m(x-y)dy = \Phi(x)h_\psi^{pq}(x) = \Phi(x)\langle \mu_x^{pq}, \psi(\xi)\rangle,$$

and (35) follows from (36) in the limit as $m \to \infty$;

c) for $x \in \Omega'$ and each family $p_i \in D$, $\psi_i(\xi) \in C(S_X)$, $i = 1, \dots, l$ the matrix $\langle \mu_x^{p_i p_j}, \psi_i \overline{\psi_j}\rangle$, $i, j = 1, \dots, l$ is positive definite. Indeed, as follows from Lemma 4(iii), for $\alpha_1, \dots, \alpha_l \in \mathbb{C}$

$$\sum_{i,j=1}^{l} \langle \mu_x^{p_i p_j}, \psi_i \overline{\psi_j}\rangle \alpha_i \overline{\alpha_j} = \lim_{m\to\infty} \sum_{i,j=1}^{l} \langle \mu^{p_i p_j}(y,\xi), \Phi_m(x-y)\psi_i(\xi)\overline{\Phi_m(x-y)\psi_j(\xi)}\rangle \alpha_i \overline{\alpha_j} \ge 0.$$

Taking in the above property $l = 2$, $p_1 = p$, $p_2 = q$, $\psi_1(\xi) = \psi(\xi)/\sqrt{|\psi(\xi)|}$ ($\psi_1 = 0$ for $\psi = 0$) and $\psi_2(\xi) = \sqrt{|\psi(\xi)|}$, $\psi(\xi) \in C(S_X)$, we obtain, as in the proof of Proposition 3, that the matrix $\begin{pmatrix} \langle \mu_x^{pp}, |\psi|\rangle & \langle \mu_x^{pq}, \psi\rangle \\ \overline{\langle \mu_x^{pq}, \psi\rangle} & \langle \mu_x^{qq}, |\psi|\rangle \end{pmatrix}$ is positive definite. In particular,

$$|\langle \mu_x^{pq}, \psi\rangle| \le (\langle \mu_x^{pp}, |\psi|\rangle \cdot \langle \mu_x^{qq}, |\psi|\rangle)^{1/2}$$

and this easily implies that for any Borel set $A \subset S_X$

$$|\mu_x^{pq}|(A) \le (\mu_x^{pp}(A)\mu_x^{qq}(A))^{1/2}. \tag{37}$$

We denote by $\theta(\lambda)$ the Heaviside function:

$$\theta(\lambda) = \begin{cases} 1, & \lambda > 0, \\ 0, & \lambda \le 0. \end{cases}$$

Below we shall frequently use the following simple estimate

Lemma 5. *Let $p_0, p \in D$, $\chi(\lambda) = \theta(\lambda - p_0) - \theta(\lambda - p)$, $V_r(y) = \int |\chi(\lambda)| d(\nu_y^r(\lambda) + \nu_y^0(\lambda))$, $\Phi(y) \in L^2(\Omega)$, $x \in \Omega'$ is a Lebesgue point of $(\Phi(y))^2$. Then*

$$\overline{\lim_{m\to\infty}}\ \overline{\lim_{r\to\infty}} \|\Phi_m(x-y)\Phi(y)V_r(y)\|_2 \le 2|\Phi(x)||u_0(x,p_0) - u_0(x,p)|^{1/2} \underset{p\to p_0}{\to} 0.$$

Proof. It is clear that

$$V_r(y) = |u_r(y,p_0) - u_r(y,p) + u_0(y,p_0) - u_0(y,p)| =$$
$$\operatorname{sign}(p - p_0)(u_r(y,p_0) - u_r(y,p) + u_0(y,p_0) - u_0(y,p)) \le 2$$

and, in particular, $(V_r(y))^2 \le 2V_r(y)$. Therefore,

$$\|\Phi_m(x-y)\Phi(y)V_r(y)\|_2^2 \le$$
$$2\operatorname{sign}(p - p_0) \int (\Phi(y))^2 K_m(x-y)(u_r(y,p_0) - u_r(y,p) + u_0(y,p_0) - u_0(y,p))dy.$$

Since $p_0, p \in D \subset E$, $u_r(y,p_0) - u_r(y,p) \to u_0(y,p_0) - u_0(y,p)$ as $r \to \infty$ weakly-$*$ in $L^\infty(\Omega)$ and we derive from the above inequality that

$$\overline{\lim_{r\to\infty}} \|\Phi_m(x-y)\Phi(y)V_r(y)\|_2^2 \le 4\,\mathrm{sign}(p-p_0)\int (\Phi(y))^2 K_m(x-y)(u_0(y,p_0)-u_0(y,p))dy.$$

Now, passing to the limit as $m \to \infty$ and taking into account that $x \in \Omega'$ is a Lebesgue point of the bounded function $u_0(y,p_0) - u_0(y,p)$ as well as the function $(\Phi(y))^2$ (therefore, x is a Lebesgue point of the product of these functions), we find

$$\overline{\lim_{m\to\infty}}\ \overline{\lim_{r\to\infty}} \|\Phi_m(x-y)\Phi(y)V_r(y)\|_2^2 \le 4(\Phi(x))^2|u_0(x,p_0) - u_0(x,p)|.$$

This implies the required relation

$$\overline{\lim_{m\to\infty}}\ \overline{\lim_{r\to\infty}} \|\Phi_m(x-y)\Phi(y)V_r(y)\|_2 \le 2|\Phi(x)||u_0(x,p_0) - u_0(x,p)|^{1/2}.$$

To complete the proof it only remains to observe that, in view of (33), $\nu_x^0(\{p_0\}) = 0$ and therefore $u_0(x,p) \to u_0(x,p_0)$ as $p \to p_0$.□

The following statement is rather well-known.

Lemma 6. *Let $U_r(x)$ be a sequence bounded in $L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ and weakly convergent to zero, $a(\xi)$ be a bounded function on $\mathbb{R}^n$ such that $a(\xi) \to 0$ as $|\xi| \to \infty$. Then $a(\xi)F(U_r)(\xi) \underset{r\to\infty}{\to} 0$ in $L^2(\mathbb{R}^n)$.*

Proof. First, observe that by the assumption $a(\xi) \to 0$ at infinity for any $\varepsilon > 0$ we can choose $R > 0$ such that $|a(\xi)| < \varepsilon$ for $|\xi| > R$. Then

$$\int_{|\xi|>R} |a(\xi)|^2|F(U_r)(\xi)|^2 d\xi \le \varepsilon^2\|F(U_r)\|_2 = \varepsilon^2\|U_r\|_2 \le C\varepsilon^2, \tag{38}$$

where $C = \sup_{r\in\mathbb{N}} \|U_r\|_2$ is a constant independent of r.

Further, by our assumption $U_r \to 0$ as $r \to \infty$ weakly in L^1. This implies that $F(U_r)(\xi) \to 0$ point-wise as $r \to \infty$. Moreover, $|F(U_r)(\xi)| \le \|U_r\|_1 \le \text{const}$. Hence, using the Lebesgue dominated convergence theorem, we find that

$$\int_{|\xi|\le R} |a(\xi)|^2|F(U_r)(\xi)|^2 d\xi \to 0 \tag{39}$$

as $r \to \infty$. It follows from (38), (39) that

$$\overline{\lim_{r\to\infty}} \int_{\mathbb{R}^n} |a(\xi)|^2|F(U_r)(\xi)|^2 d\xi \le C\varepsilon^2.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that

$$\lim_{r\to\infty} \int_{\mathbb{R}^n} |a(\xi)|^2|F(U_r)(\xi)|^2 d\xi = 0,$$

that is, $a(\xi)F(U_r)(\xi) \underset{r\to\infty}{\to} 0$ in $L^2(\mathbb{R}^n)$. The proof is complete.□

We now fix $x \in \Omega'$, $p_0, p \in D$. Let $L(p) \subset \mathbb{R}^n$ be the smallest linear subspace containing $\operatorname{supp} \mu_x^{pp_0}$, and $L = L(p_0)$.

As follows from (37), $\operatorname{supp} \mu_x^{pp_0} \subset \operatorname{supp} \mu_x^{p_0p_0}$ and therefore $L(p) \subset L$.

Suppose that $f(y, \lambda)$ is a Caratheodory vector-function on $\Omega \times \mathbb{R}$ such that $f(y, \lambda) \in L^2_{loc}(\Omega, C(\mathbb{R}, \mathbb{R}^n))$, that is,

$$\forall M > 0 \quad \|f(x, \cdot)\|_{M,\infty} = \max_{|\lambda| \le M} |f(x, \lambda)| \le \alpha_M(x) \in L^2_{loc}(\Omega). \tag{40}$$

Since the space $C(\mathbb{R}, \mathbb{R}^n)$ is separable with respect to the standard locally convex topology generated by seminorms $\|\cdot\|_{M,\infty}$, then, by the Pettis theorem (see [6], Chapter 3), the map $x \to F(x) = f(x, \cdot) \in C(\mathbb{R}, \mathbb{R}^n)$ is strongly measurable and in view of estimate (40) we see that $F(x) \in L^2_{loc}(\Omega, C(\mathbb{R}, \mathbb{R}^n))$, $|F(x)|^2 \in L^1_{loc}(\Omega, C(\mathbb{R}))$. In particular (see [6], Chapter 3), the set Ω_f of common Lebesgue points of the maps $F(x), |F(x)|^2$ has full measure. For $x \in \Omega_f$ we have

$$\forall M > 0 \ \lim_{m\to\infty} \int K_m(x-y)\|F(x) - F(y)\|_{M,\infty} dy = 0,$$
$$\lim_{m\to\infty} \int K_m(x-y)\||F(x)|^2 - |F(y)|^2\|_{M,\infty} dy = 0.$$

Since, evidently,

$$\|F(x) - F(y)\|^2_{M,\infty} \le 2\|F(x) - F(y)\|_{M,\infty}\|F(x)\|_{M,\infty} + \||F(x)|^2 - |F(y)|^2\|_{M,\infty},$$

it follows from the above limit relations that for $x \in \Omega_f$

$$\lim_{m\to\infty} \int K_m(x-y)\|F(x) - F(y)\|^2_{M,\infty} dy = 0 \quad \forall M > 0. \tag{41}$$

Clearly, each $x \in \Omega_f$ is a Lebesgue point of all functions $x \to f(x, \lambda)$, $\lambda \in \mathbb{R}$. Let $\gamma_x^r = \nu_x^r - \nu_x^0$. Suppose that $x \in \Omega \cap \Omega_f$, $p_0 \in D$, $\chi(\lambda) = \theta(\lambda - p_1) - \theta(\lambda - p_2)$, where $p_1, p_2 \in D$. For a vector-function $h(y, \lambda)$ on $\Omega \times \mathbb{R}$, which is Borel and locally bounded with respect to the second variable, we denote $I_r(h)(y) = \int h(y, \lambda) d\gamma_y^r(\lambda)$. In view of the strong measurability of $F(x)$ and (40) we see that $I_r = I_r(f \cdot \chi)(y) \in L^2_{loc}(\Omega)$ (see Remark 1). We also denote by $\tilde{L}$, $\bar{L}$ the spaces obtained by orthogonal projections of L on the subspaces X, $X^\perp$, respectively: $\tilde{L} = P_1(L)$, $\bar{L} = P_2(L)$.

Proposition 4. *Assume that $f(x, \lambda) \in \tilde{L}^\perp$, and $\rho(\xi) \in C^\infty(\mathbb{R}^n)$ is a function such that $0 \le \rho(\xi) \le 1$ and $\rho(\xi) = 0$ for $|\tilde{\xi}|^2 + |\bar{\xi}|^4 \le 1$, $\rho(\xi) = 1$ for $|\tilde{\xi}|^2 + |\bar{\xi}|^4 \ge 2$. Then $\forall \psi(\xi) \in C(S_X)$*

$$\lim_{m\to\infty} \lim_{r\to\infty} \int_{\mathbb{R}^n} \frac{\rho(\xi)\xi \cdot F(\Phi_m I_r(f \cdot \chi))(\xi)}{(|\tilde{\xi}|^2 + |\bar{\xi}|^4)^{1/2}} \overline{F(\Phi_m U_r^{p_0})(\xi)} \psi(\pi_X(\xi)) d\xi = 0.$$

Here $\Phi_m = \Phi_m(x - y) = \sqrt{K_m(x - y)}$ and $I_r(f \cdot \chi)$ are supposed to be functions of the variable $y \in \Omega$.

Proof. Note that

$$|I_r(y)| \le \int |f(y,\lambda)||\chi(\lambda)|d|\gamma_y^r|(\lambda) \le 2\alpha_M(y), \tag{42}$$

where $M = \sup \|\nu_x^r\|_\infty$. Let us first show that for each $m \in \mathbb{N}$

$$\lim_{r\to\infty} \int_{\mathbb{R}^n} \frac{\rho(\xi)\bar{\xi}\cdot F(\Phi_m I_r)(\xi)}{(|\tilde{\xi}|^2+|\bar{\xi}|^4)^{1/2}} \overline{F(\Phi_m U_r^{p_0})(\xi)}\psi(\pi_X(\xi))d\xi = 0. \tag{43}$$

For that, it is sufficient to demonstrate that

$$\frac{\rho(\xi)|\bar{\xi}|}{(|\tilde{\xi}|^2+|\bar{\xi}|^4)^{1/2}}|F(\Phi_m I_r)(\xi)| \underset{r\to\infty}{\to} 0 \text{ in } L^2(\mathbb{R}^n). \tag{44}$$

Remark that the sequence $\Phi_m I_r(y)$, $r \in \mathbb{N}$ is bounded in $L^2(\mathbb{R}^n)$ and in $L^1(\mathbb{R}^n)$ (since $\operatorname{supp}\Phi_m$ is compact) and weakly converges to zero (in view of the weak convergence $\nu_x^r \underset{r\to\infty}{\to} \nu_x^0$). Hence, (44) follows from Lemma 6. We only need to demonstrate that the function

$$a(\xi) = \frac{\rho(\xi)|\bar{\xi}|}{(|\tilde{\xi}|^2+|\bar{\xi}|^4)^{1/2}}$$

satisfies the assumptions of this Lemma. First, we show that $a(\xi) \le 1$. Indeed, for $|\tilde{\xi}|^2+|\bar{\xi}|^4 \le 1$ the value $\rho(\xi) = 0$ while in the case $|\tilde{\xi}|^2+|\bar{\xi}|^4 > 1$ we have $\frac{\rho(\xi)|\bar{\xi}|}{(|\tilde{\xi}|^2+|\bar{\xi}|^4)^{1/2}} \le \min(|\bar{\xi}|, 1/|\bar{\xi}|) \le 1$.

Then, observe that for $|\tilde{\xi}|^2+|\bar{\xi}|^4 \ge R^4 > 0$

$$a(\xi) \le \frac{|\bar{\xi}|}{(|\tilde{\xi}|^2+|\bar{\xi}|^4)^{1/2}} \le (|\tilde{\xi}|^2+|\bar{\xi}|^4)^{-1/4} \le R^{-1}.$$

Therefore, $a(\xi) \to 0$ as $|\xi| \to \infty$. Thus, assumptions of Lemma 6 are satisfied and by Lemma 6 we conclude that (44), (43) hold.

In view of (43),

$$\lim_{m\to\infty}\lim_{r\to\infty} \int_{\mathbb{R}^n} \frac{\rho(\xi)\xi\cdot F(\Phi_m I_r)(\xi)}{(|\tilde{\xi}|^2+|\bar{\xi}|^4)^{1/2}} \overline{F(\Phi_m U_r^{p_0})(\xi)}\psi(\pi_X(\xi))d\xi =$$
$$\lim_{m\to\infty}\lim_{r\to\infty} \int_{\mathbb{R}^n} \frac{\rho(\xi)\tilde{\xi}\cdot F(\Phi_m I_r)(\xi)}{(|\tilde{\xi}|^2+|\bar{\xi}|^4)^{1/2}} \overline{F(\Phi_m U_r^{p_0})(\xi)}\psi(\pi_X(\xi))d\xi. \tag{45}$$

Let $g(\lambda) = f(x,\lambda)$, $I_r' = I_r(g\chi)(y) = \int g(\lambda)\chi(\lambda)d\gamma_y^r(\lambda)$, $M = \sup \|\nu_y^r\|_\infty$. Then

$$|I_r - I_r'| \le \int |f(y,\lambda) - f(x,\lambda)|d|\gamma_y^r|(\lambda) \le 2\|F(y) - F(x)\|_{M,\infty}.$$

This and the Plancherel identity imply that

$$\left|\int_{\mathbb{R}^n} \frac{\rho(\xi)\tilde{\xi}\cdot F(\Phi_m(I_r-I'_r))(\xi)}{(|\tilde{\xi}|^2+|\bar{\xi}|^4)^{1/2}}\overline{F(\Phi_m U_r^{p_0})(\xi)}\psi(\pi_X(\xi))d\xi\right| \le$$

$$\|\psi\|_\infty\|F(\Phi_m(I_r-I'_r))\|_2\|F(\Phi_m U_r^{p_0})\|_2 \le \|\psi\|_\infty\|\Phi_m(I_r-I'_r)\|_2 \le$$

$$2\|\psi\|_\infty\left(\int_{\mathbb{R}^n} K_m(x-y)\|F(y)-F(x)\|^2_{M,\infty}\right)^{1/2}.$$

It follows from the above estimate and (41) that

$$\lim_{m\to\infty}\lim_{r\to\infty}\left|\int_{\mathbb{R}^n} \frac{\rho(\xi)\tilde{\xi}\cdot F(\Phi_m I_r)(\xi)}{(|\tilde{\xi}|^2+|\bar{\xi}|^4)^{1/2}}\overline{F(\Phi_m U_r^{p_0})(\xi)}\psi(\pi_X(\xi))d\xi - \right.$$

$$\left.\int_{\mathbb{R}^n} \frac{\rho(\xi)\tilde{\xi}\cdot F(\Phi_m I'_r)(\xi)}{(|\tilde{\xi}|^2+|\bar{\xi}|^4)^{1/2}}\overline{F(\Phi_m U_r^{p_0})(\xi)}\psi(\pi_X(\xi))d\xi\right| \le$$

$$\lim_{m\to\infty}\lim_{r\to\infty}\left|\int_{\mathbb{R}^n} \frac{\rho(\xi)\tilde{\xi}\cdot F(\Phi_m(I_r-I'_r))(\xi)}{(|\tilde{\xi}|^2+|\bar{\xi}|^4)^{1/2}}\overline{F(\Phi_m U_r^{p_0})(\xi)}\psi(\pi_X(\xi))d\xi\right| = 0$$

and, in view of this relation and (45), it is sufficient to prove that

$$\lim_{m\to\infty}\lim_{r\to\infty}\left|\int_{\mathbb{R}^n} \frac{\rho(\xi)\tilde{\xi}\cdot F(\Phi_m I'_r)(\xi)}{(|\tilde{\xi}|^2+|\bar{\xi}|^4)^{1/2}}\overline{F(\Phi_m U_r^{p_0})(\xi)}\psi(\pi_X(\xi))d\xi\right| = 0. \tag{46}$$

The vector-function $g(\lambda)$ is continuous and does not depend on y. Therefore for any $\varepsilon>0$ there exists a vector-valued function $h(\lambda)$ of the form $h(\lambda)=\sum\limits_{i=1}^k v_i\theta(\lambda-p_i)$, where $v_i\in\tilde{L}^\perp$ and $p_i\in D$ such that $\|g\cdot\chi-h\|_\infty\le\varepsilon$ on $\mathbb{R}$.

Using again the Plancherel's identity and the fact that

$$|I'_r-I_r(h)| = \left|\int(g\cdot\chi-h)(\lambda)d\gamma_y^r(\lambda)\right| \le \int|(g\cdot\chi-h)(\lambda)|d|\gamma_y^r|(\lambda)\le 2\varepsilon,$$

we obtain

$$\left|\int_{\mathbb{R}^n} \frac{\rho(\xi)\tilde{\xi}\cdot F(\Phi_m I'_r)(\xi)}{(|\tilde{\xi}|^2+|\bar{\xi}|^4)^{1/2}}\overline{F(\Phi_m U_r^{p_0})(\xi)}\psi(\pi_X(\xi))d\xi - \right.$$

$$\left.\int_{\mathbb{R}^n} \frac{\rho(\xi)\tilde{\xi}\cdot F(\Phi_m I_r(h))(\xi)}{(|\tilde{\xi}|^2+|\bar{\xi}|^4)^{1/2}}\overline{F(\Phi_m U_r^{p_0})(\xi)}\psi(\pi_X(\xi))d\xi\right| \le$$

$$\|\psi\|_\infty\|\Phi_m I_r(g\cdot\chi-h)\|_2 \le 2\varepsilon\|\psi\|_\infty\|\Phi_m\|_2 = 2\varepsilon\|\psi\|_\infty. \tag{47}$$

Since

$$I_r(h)(y) = \int\left(\sum_{i=1}^k v_i\theta(\lambda-p_i)\right)d\gamma_y^r(\lambda) = \sum_{i=1}^k v_i U_r^{p_i}(y),$$

it follows from (35) the limit relation

$$\lim_{m\to\infty}\lim_{r\to\infty}\int_{\mathbb{R}^n}\frac{\rho(\xi)\tilde{\xi}\cdot F(\Phi_m I_r(h))(\xi)}{(|\tilde{\xi}|^2+|\bar{\xi}|^4)^{1/2}}\overline{F(\Phi_m U_r^{p_0})(\xi)}\psi(\pi_X(\xi))d\xi = \sum_{i=1}^k\langle\mu_x^{p_ip_0},(v_i\cdot\tilde{\xi})\psi(\xi)\rangle. \quad (48)$$

Here we also take Remark 2 into account. Since $\rho(\xi)\psi(\pi_X(\xi))=\psi(\pi_X(\xi))$ for large $|\xi|$ then, by this Remark, for $i=1,\dots,k$

$$\lim_{r\to\infty}\int_{\mathbb{R}^n}\frac{\rho(\xi)\tilde{\xi}\cdot v_iF(\Phi_m U_r^{p_i})(\xi)}{(|\tilde{\xi}|^2+|\bar{\xi}|^4)^{1/2}}\overline{F(\Phi_m U_r^{p_0})(\xi)}\psi(\pi_X(\xi))d\xi =$$
$$\lim_{r\to\infty}\int_{\mathbb{R}^n}\frac{\tilde{\xi}\cdot v_iF(\Phi_m U_r^{p_i})(\xi)}{(|\tilde{\xi}|^2+|\bar{\xi}|^4)^{1/2}}\overline{F(\Phi_m U_r^{p_0})(\xi)}\psi(\pi_X(\xi))d\xi =$$
$$\langle\mu^{p_ip_0}(y,\xi),K_m(x-y)(v_i\cdot\tilde{\xi})\psi(\xi)\rangle.$$

Now observe that $\operatorname{supp}\mu_x^{p_ip_0}\subset L(p_0)=L$, and for each $\xi\in L$ $v_i\cdot\tilde{\xi}=0$ because $\tilde{\xi}\in\tilde{L}$ while $v_i\perp\tilde{L}$. Hence $\sum_{i=1}^k\langle\mu_x^{p_ip_0},(v_i\cdot\tilde{\xi})\psi(\xi)\rangle=0$, and it follows from (48) that

$$\lim_{m\to\infty}\lim_{r\to\infty}\int_{\mathbb{R}^n}\frac{\rho(\xi)\tilde{\xi}\cdot F(\Phi_m I_r(h))(\xi)}{(|\tilde{\xi}|^2+|\bar{\xi}|^4)^{1/2}}\overline{F(\Phi_m U_r^{p_0})(\xi)}\psi(\pi_X(\xi))d\xi = 0.$$

This relation together with (47) yields

$$\overline{\lim_{m\to\infty}}\,\overline{\lim_{r\to\infty}}\left|\int_{\mathbb{R}^n}\frac{\rho(\xi)\tilde{\xi}\cdot F(\Phi_m I_r')(\xi)}{(|\tilde{\xi}|^2+|\bar{\xi}|^4)^{1/2}}\overline{F(\Phi_m U_r^{p_0})(\xi)}\psi(\pi_X(\xi))d\xi\right|\le 2\varepsilon\|\psi\|_\infty,$$

and since $\varepsilon>0$ is arbitrary we claim that (46) holds. This completes the proof. □

Let $Q(x,\lambda)$ be a Caratheodory matrix-valued function, which ranges in the space Sym_n of symmetric matrices of order n such that $Q(x,\lambda)\in L^2_{loc}(\Omega,C(\mathbb{R},Sym_n))$.

Denote Ω_Q the set of full measure consisting of common Lebesgue points of the maps $x\to G(x)=Q(x,\cdot)\in C(\mathbb{R},Sym_n)$, $x\to|G(x)|^2\in C(\mathbb{R})$. As can be easily verified, for $x\in\Omega_Q$ the following relation similar to (41) holds

$$\lim_{m\to\infty}\int K_m(x-y)\|G(x)-G(y)\|^2_{M,\infty}dy=0\quad\forall M>0. \quad (49)$$

Let $x\in\Omega'\cap\Omega_Q$, $p_0,p_1,p_2\in D$, $\chi(\lambda)=\theta(\lambda-p_1)-\theta(\lambda-p_2)$, and let $J_r(y)=J_r(Q)(y)=\int\chi(\lambda)Q(y,\lambda)d\gamma_y^r(\lambda)$, $\rho(\xi)$ be a function as in Proposition 4. Also assume that for $L=L(p_0)$

$$Q(x,\lambda)\xi\cdot\xi=0\quad\forall\xi\in\bar{L}=P_2(L)$$

(recall that P_2 is the orthogonal projection onto $X^\perp$).

Proposition 5. *Under the above notations for each* $\psi(\xi) \in C(S_X)$

$$\lim_{m\to\infty}\lim_{r\to\infty}\int_{\mathbb{R}^n} \frac{\rho(\xi)F(\Phi_m J_r)(\xi)\bar\xi\cdot\bar\xi}{(|\tilde\xi|^2+|\bar\xi|^2)^{1/2}}\overline{F(\Phi_m U_r^{p_0})(\xi)}\psi(\pi_X(\xi))d\xi = 0. \tag{50}$$

Proof. Denote $\tilde Q(\lambda) = Q(x,\lambda)$ (here x is the fixed above point), $\tilde J_r(y) = \int \chi(\lambda)\tilde Q(\lambda)d\gamma_y^r(\lambda)$. Then

$$|J_r - \tilde J_r| \le \int |Q(y,\lambda)-Q(x,\lambda)|d|\gamma_y^r|(\lambda) \le 2\|G(y)-G(x)\|_{M,\infty}$$

where $M = \sup\|\nu_x^r\|_\infty$. This and the Plancherel identity imply that

$$\left|\int_{\mathbb{R}^n} \frac{\rho(\xi)F(\Phi_m(J_r-\tilde J_r))(\xi)\bar\xi\cdot\bar\xi}{(|\tilde\xi|^2+|\bar\xi|^4)^{1/2}}\overline{F(\Phi_m U_r^{p_0})(\xi)}\psi(\pi_X(\xi))d\xi\right| \le$$

$$\|\psi\|_\infty\|F(\Phi_m(J_r-\tilde J_r))\|_2\|F(\Phi_m U_r^{p_0})\|_2 \le \|\psi\|_\infty\|\Phi_m(J_r-\tilde J_r)\|_2 \le$$

$$2\|\psi\|_\infty\left(\int_{\mathbb{R}^n} K_m(x-y)\|G(y)-G(x)\|^2_{M,\infty}\right)^{1/2}.$$

It follows from the above estimate and (49) that

$$\lim_{m\to\infty}\lim_{r\to\infty}\left|\int_{\mathbb{R}^n} \frac{\rho(\xi)F(\Phi_m J_r)(\xi)\bar\xi\cdot\bar\xi}{(|\tilde\xi|^2+|\bar\xi|^4)^{1/2}}\overline{F(\Phi_m U_r^{p_0})(\xi)}\psi(\pi_X(\xi))d\xi - \right.$$

$$\left.\int_{\mathbb{R}^n} \frac{\rho(\xi)F(\Phi_m \tilde J_r)(\xi)\bar\xi\cdot\bar\xi}{(|\tilde\xi|^2+|\bar\xi|^4)^{1/2}}\overline{F(\Phi_m U_r^{p_0})(\xi)}\psi(\pi_X(\xi))d\xi\right| \le$$

$$\lim_{m\to\infty}\lim_{r\to\infty}\left|\int_{\mathbb{R}^n} \frac{\rho(\xi)F(\Phi_m(J_r-\tilde J_r))(\xi)\bar\xi\cdot\bar\xi}{(|\tilde\xi|^2+|\bar\xi|^4)^{1/2}}\overline{F(\Phi_m U_r^{p_0})(\xi)}\psi(\pi_X(\xi))d\xi\right| = 0$$

and, in view of this relation we have to prove that

$$\lim_{m\to\infty}\lim_{r\to\infty}\int_{\mathbb{R}^n} \frac{\rho(\xi)F(\Phi_m \tilde J_r)(\xi)\bar\xi\cdot\bar\xi}{(|\tilde\xi|^2+|\bar\xi|^4)^{1/2}}\overline{F(\Phi_m U_r^{p_0})(\xi)}\psi(\pi_X(\xi))d\xi = 0. \tag{51}$$

Introduce the linear space Y of symmetric matrices A, satisfying the property $A\xi\cdot\xi = 0$ for $\xi \in \bar L$. Since the matrix-valued function $\tilde Q(\lambda)$ ranges in Y and does not depend on y for every $\varepsilon > 0$ one can find a step function $H(\lambda) = \sum_{i=1}^k \theta(\lambda - p_i)Q_i$, where $p_i \in D$, $Q_i \in Y$ for each $i = 1,\dots,k$ such that $|\chi(\lambda)\tilde Q(\lambda) - H(\lambda)| < \varepsilon$ for all $\lambda \in \mathbb{R}$. We denote $J_r'(y) = \int H(\lambda)d\gamma_y^r(\lambda)$ and observe that

$$J_r'(y) = \sum_{i=1}^k U_r^{p_i}(y)Q_i, \tag{52}$$

$$|\tilde J_r(y) - J_r'(y)| \le \int |\tilde Q(\lambda) - H(\lambda)||\chi(\lambda)|d|\gamma_y^r|(\lambda) \le 2\varepsilon. \tag{53}$$

We also remark that

$$\left|\frac{F(\Phi_m(\tilde{J}_r - J'_r))(\xi)\bar{\xi}\cdot\bar{\xi}}{(|\tilde{\xi}|^2+|\bar{\xi}|^4)^{1/2}}\right| \le |F(\Phi_m(\tilde{J}_r - J'_r))(\xi)||\bar{\xi}|^2/(|\tilde{\xi}|^2+|\bar{\xi}|^4)^{1/2} \le |F(\Phi_m(\tilde{J}_r - J'_r))(\xi)|.$$

The latter estimate and (53) imply that

$$\left|\int_{\mathbb{R}^n} \frac{\rho(\xi)F(\Phi_m\tilde{J}_r)(\xi)\bar{\xi}\cdot\bar{\xi}}{(|\tilde{\xi}|^2+|\bar{\xi}|^4)^{1/2}}\overline{F(\Phi_m U_r^{p_0})(\xi)}\psi(\pi_X(\xi))d\xi - \right.$$
$$\left.\int_{\mathbb{R}^n} \frac{\rho(\xi)F(\Phi_m J'_r)(\xi)\bar{\xi}\cdot\bar{\xi}}{(|\tilde{\xi}|^2+|\bar{\xi}|^4)^{1/2}}\overline{F(\Phi_m U_r^{p_0})(\xi)}\psi(\pi_X(\xi))d\xi\right| =$$
$$\left|\int_{\mathbb{R}^n} \frac{\rho(\xi)F(\Phi_m(\tilde{J}_r - J'_r))(\xi)\bar{\xi}\cdot\bar{\xi}}{(|\tilde{\xi}|^2+|\bar{\xi}|^4)^{1/2}}\overline{F(\Phi_m U_r^{p_0})(\xi)}\psi(\pi_X(\xi))d\xi\right| \le$$
$$\|\psi\|_\infty\|F(\Phi_m(\tilde{J}_r - J'_r))\|_2\|F(\Phi_m U_r^{p_0})\|_2 = \|\psi\|_\infty\|\Phi_m(\tilde{J}_r - J'_r)\|_2\|\Phi_m U_r^{p_0}\|_2 \le$$
$$\|\psi\|_\infty\|\Phi_m(\tilde{J}_r - J'_r)\|_2 = \|\psi\|_\infty\left(\int_{\mathbb{R}^n} K_m(x-y)|\tilde{J}_r(y) - J'_r(y)|^2 dy\right)^{1/2} \le 2\varepsilon\|\psi\|_\infty. \quad (54)$$

We also use that $|U_r^{p_0}| \le 1$ and therefore $\|\Phi_m U_r^{p_0}\|_2 \le 1$. In view of (52)

$$\int_{\mathbb{R}^n} \frac{\rho(\xi)F(\Phi_m J'_r)(\xi)\bar{\xi}\cdot\bar{\xi}}{(|\tilde{\xi}|^2+|\bar{\xi}|^4)^{1/2}}\overline{F(\Phi_m U_r^{p_0})(\xi)}\psi(\pi_X(\xi))d\xi =$$
$$\sum_{i=1}^k \int_{\mathbb{R}^n} \frac{\rho(\xi)F(\Phi_m U_r^{p_i})(\xi)Q_i\bar{\xi}\cdot\bar{\xi}}{(|\tilde{\xi}|^2+|\bar{\xi}|^4)^{1/2}}\overline{F(\Phi_m U_r^{p_0})(\xi)}\psi(\pi_X(\xi))d\xi,$$

and by relation (35) and Remark 2 we find

$$\lim_{m\to\infty}\lim_{r\to\infty}\int_{\mathbb{R}^n} \frac{\rho(\xi)F(\Phi_m J'_r)(\xi)\bar{\xi}\cdot\bar{\xi}}{(|\tilde{\xi}|^2+|\bar{\xi}|^4)^{1/2}}\overline{F(\Phi_m U_r^{p_0})(\xi)}\psi(\pi_X(\xi))d\xi =$$
$$\sum_{i=1}^k \langle \mu_x^{p_ip_0}\psi(\xi)Q_i\bar{\xi}\cdot\bar{\xi}\rangle = 0 \quad (55)$$

because $\operatorname{supp}\mu_x^{p_ip_0} \subset L$ and therefore $Q_i\bar{\xi}\cdot\bar{\xi} = 0$ on $\operatorname{supp}\mu_x^{p_ip_0}$ (recall that $Q_i\bar{\xi}\cdot\bar{\xi} = 0$ for $\bar{\xi} \in \bar{L}$).

By (54), (55) we obtain the relation

$$\lim_{m\to\infty}\lim_{r\to\infty}\left|\int_{\mathbb{R}^n} \frac{\rho(\xi)F(\Phi_m\tilde{J}_r)(\xi)\bar{\xi}\cdot\bar{\xi}}{(|\tilde{\xi}|^2+|\bar{\xi}|^4)^{1/2}}\overline{F(\Phi_m U_r^{p_0})(\xi)}\psi(\pi_X(\xi))d\xi\right| \le 2\varepsilon\|\psi\|_\infty$$

and since $\varepsilon > 0$ is arbitrary we conclude that (51) holds. The proof is complete.□

In the sequel we will need the following simple result.

Lemma 7. *Let* $\{ \xi_k | \ k = 1, \dots, l \} \subset L$ *be a basis in* L. *Then there exists a positive constant* C *such that for every* $v \in \mathbb{R}^n$, $Q \in Sym_n$

$$|v_1| + |Q_1| \le C \max_{k=1,\dots,l} |iv \cdot \tilde{\xi}_k + Q\bar{\xi}_k \cdot \bar{\xi}_k|,$$

where $v_1 = \tilde{P}v$, $Q_1 = \bar{P}Q\bar{P}$, $\tilde{P}$, $\bar{P}$ *are orthogonal projections on the spaces* $\tilde{L}$, $\bar{L}$, *respectively, and* $i = \sqrt{-1}$.

Proof. We introduce the linear spaces $\bar{S} = \{ Q \in Sym_n \mid Q = \bar{P}Q\bar{P} \}$, $H = \tilde{L} \oplus \bar{S}$ and remark that $p(v, Q) = \max\limits_{k=1,\dots,l} |iv \cdot \tilde{\xi}_k + Q\bar{\xi}_k \cdot \bar{\xi}_k|$ is a norm in H. Indeed, it is clear that p is a seminorm. To prove that p is a norm, suppose that $p(v, Q) = 0$. Then $v \cdot \tilde{\xi}_k = Q\bar{\xi}_k \cdot \bar{\xi}_k = 0$ and since vectors $\tilde{\xi}_k$, $\bar{\xi}_k$ generate spaces $\tilde{L}$, $\bar{L}$, respectively, we claim that $v\tilde{\xi} = 0$ for all $\xi \in \tilde{L}$ and $Q\xi \cdot \xi = 0$ for all $\xi \in \bar{L}$. Since $v \in \tilde{L}$ we see that $v = 0$. Further, since $Q \in \bar{S}$ we find that for every $\xi \in \mathbb{R}^n$

$$Q\xi \cdot \xi = \bar{P}Q\bar{P}\xi \cdot \xi = Q\bar{P}\xi \cdot \bar{P}\xi = 0,$$

and we conclude that $Q = 0$. It is well-known that any two norms in finite-dimensional space are equivalent. Applying this property to the norms $p(v, Q)$ and $p_1(v, Q) = |v| + |Q|$ and using the relations

$$v \cdot \tilde{\xi}_k = v_1 \cdot \tilde{\xi}_k, \ Q\bar{\xi}_k \cdot \bar{\xi}_k = Q\bar{P}\bar{\xi}_k \cdot \bar{P}\bar{\xi}_k = Q_1\bar{\xi}_k \cdot \bar{\xi}_k, \ k = 1, \dots l,$$

we find that for some constant $C > 0$

$$|v_1| + |Q_1| \le C \max_{k=1,\dots,l} |iv_1 \cdot \tilde{\xi}_k + Q_1\bar{\xi}_k \cdot \bar{\xi}_k| = C \max_{k=1,\dots,l} |iv \cdot \tilde{\xi}_k + Q\bar{\xi}_k \cdot \bar{\xi}_k|,$$

as was to be proved. $\square$

Corollary 1. There exist functions $\psi_k(\xi) \in C(S_X)$, $k = 1, \dots, l = \dim L$ and a constant $C > 0$ such that, in the notations of Lemma 7, for all $v \in \mathbb{R}^n$, $Q \in Sym_n$ such that $Q \ge 0$

$$|v_1| + |Q_1| \le C \max_{k=1,\dots,l} |\langle \mu_x^{p_0p_0}, (iv \cdot \tilde{\xi} + Q\bar{\xi} \cdot \bar{\xi})\psi_k(\xi) \rangle|. \tag{56}$$

Proof. Remark that the measure $\mu_x^{p_0p_0} \ge 0$. If $\mu_x^{p_0p_0} = 0$ then the both parts of equality (56) equal zero, and this equality is evidently satisfied. Thus, suppose that $\mu_x^{p_0p_0}(S_X) > 0$. Since L is a linear span of $\operatorname{supp} \mu_x^{p_0p_0}$, we can choose functions $\psi_k(\xi) \in C(S_X)$, $k = 1, \dots, l$ such that $\psi_k(\xi) \ge 0$, $\int \psi_k(\xi) d\mu_x^{p_0p_0} = 1$ for all $k = 1, \dots, l$, and the family $\xi_k = \int \xi \psi_k(\xi) d\mu_x^{p_0p_0}$, $k = 1, \dots, l$ is a basis in L. By Lemma 7 there exists a constant $C > 0$ such that for all $v \in \mathbb{R}^n$, $Q \in Sym_n$

$$|v_1| + |Q_1| \le C \max_{k=1,\dots,l} |iv \cdot \tilde{\xi}_k + Q\bar{\xi}_k \cdot \bar{\xi}_k|, \tag{57}$$

where $v_1 = \tilde{P}v$, $Q_1 = \bar{P}Q\bar{P}$. Now, we observe that

$$\tilde{\xi}_k = \int \tilde{\xi}\psi_k(\xi) d\mu_x^{p_0p_0}(\xi), \ \bar{\xi}_k = \int \bar{\xi}\psi_k(\xi) d\mu_x^{p_0p_0}(\xi).$$

Therefore,

$$v \cdot \tilde{\xi}_k = \int v \cdot \tilde{\xi}\psi_k(\xi) d\mu_x^{p_0 p_0}(\xi),$$

and if $Q \geq 0$ then

$$Q\bar{\xi}_k \cdot \bar{\xi}_k = Q \int \bar{\xi}\psi_k(\xi) d\mu_x^{p_0 p_0}(\xi) \cdot \int \bar{\xi}\psi_k(\xi) d\mu_x^{p_0 p_0}(\xi) \leq \int Q\bar{\xi} \cdot \bar{\xi}\psi_k(\xi) d\mu_x^{p_0 p_0}(\xi)$$

by Jensen's inequality applied to the convex function $\xi \to Q\bar{\xi} \cdot \bar{\xi}$. In view of the above relation, (56) readily follows from (57) (we also take into account that for real a the function $f(x) = |ia + x|$ increases on $[0, +\infty)$). The proof is complete.□

4 Localization principle and strong precompactness of bounded sequences of measure-valued functions

In this Section we need some results about Fourier multipliers in spaces L^d, $d > 1$. Recall that a function $a(\xi) \in L^\infty(\mathbb{R}^n)$ is a Fourier multiplier in L^d if the pseudo-differential operator $\mathcal{A}$ with the symbol $a(\xi)$, defined as $F(\mathcal{A}u)(\xi) = a(\xi)F(u)(\xi)$, $u = u(x) \in L^2(\mathbb{R}^n) \cap L^d(\mathbb{R}^n)$ can be extended as a bounded operator on $L^d(\mathbb{R}^n)$, that is

$$\|\mathcal{A}u\|_d \leq C\|u\|_d \ \forall u \in L^2(\mathbb{R}^n) \cap L^d(\mathbb{R}^n), \quad C = \text{const}.$$

We denote by M_d the space of Fourier multipliers in L^d. We also denote

$$\dot{\mathbb{R}}^n = (\mathbb{R} \setminus \{0\})^n = \{ \ \xi = (\xi_1, \ldots, \xi_n) \mid \prod_{k=1}^n \xi_k \neq 0 \ \}.$$

The following statement readily follows from the known Marcinkiewicz multiplier theorem (see [19][Chapter 4]).

Theorem 3. *Suppose that $a(\xi) \in C^n(\dot{\mathbb{R}}^n)$ be a function such that for some constant C*

$$|\xi^\alpha D^\alpha a(\xi)| \leq C \quad \forall \xi \in \dot{\mathbb{R}}^n \tag{58}$$

for every multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$ such that $|\alpha| = \alpha_1 + \cdots + \alpha_n \leq n$. Then $a(\xi) \in M_d$ for all $d > 1$.

Here we use the standard notations $\xi^\alpha = \prod_{i=1}^n (\xi_i)^{\alpha_i}$, $D^\alpha = \displaystyle\prod_{i=1}^n \left(\frac{\partial}{\partial \xi_i}\right)^{\alpha_i}$. Actually (see [19]), it is sufficient to require that (58) is satisfied for multi-indexes α such that $\alpha_i \in \{0, 1\}$, $i = 1, \ldots, n$.

We also need the following simple lemma.

Lemma 8. *Let $h(y, z) \in C^n((\mathbb{R}^l \times \mathbb{R}^{n-l}) \setminus \{0\})$ be such that for some $k \in \mathbb{N}$, $\gamma \in \mathbb{R}$*

$$\forall t > 0 \quad h(t^k y, tz) = t^\gamma h(y, z). \tag{59}$$

Then there exists a constant $C > 0$ such that for each multi-indexes $\alpha = (\alpha_1, \dots, \alpha_l)$, $\beta = (\beta_1, \dots, \beta_{n-l})$, $|\alpha| + |\beta| \le n$ and all $y \in \mathbb{R}^l$ $z \in \mathbb{R}^{n-l}$, $y, z \ne 0$

$$|D_y^\alpha D_z^\beta h(y,z)| \le C(|y|^2 + |z|^{2k})^{\frac{\gamma}{2k}} |y|^{-|\alpha|} |z|^{-|\beta|}.$$

Proof. In view of (59) for all $t > 0$

$$D_y^\alpha D_z^\beta h(y,z) = t^{k|\alpha| + |\beta| - |\gamma|} (D_y^\alpha D_z^\beta h)(t^k y, tz).$$

Taking $t = (|y|^2 + |z|^{2k})^{-\frac{1}{2k}}$ in this relation, we arrive at

$$D_y^\alpha D_z^\beta h(y,z) = (|y|^2 + |z|^{2k})^{\frac{\gamma - k|\alpha| - |\beta|}{2k}} (D_y^\alpha D_z^\beta h)(y', z'), \tag{60}$$

where $y' = t^k y$, $z' = tz$, so that $|y'|^2 + |z'|^{2k} = 1$. Since the set of such (y', z') is a compact subset of $\mathbb{R}^n \setminus \{0\}$ the derivatives $(D_y^\alpha D_z^\beta h)(y', z')$, $|\alpha| + |\beta| \le n$, are bounded, and relation (60) implies that for some constant $C > 0$

$$|D_y^\alpha D_z^\beta h(y,z)| \le C(|y|^2 + |z|^{2k})^{\frac{\gamma}{2k}} (|y|^2 + |z|^{2k})^{-|\alpha|/2} (y|^2 + |z|^{2k})^{-|\beta|/(2k)} \le \\ C(|y|^2 + |z|^{2k})^{\frac{\gamma}{2k}} |y|^{-|\alpha|} |z|^{-|\beta|}$$

for all $y, z \ne 0$. The proof is complete. □

Now we can prove that some useful for us functions are Fourier multipliers. Namely, assume that X is a linear subspace of $\mathbb{R}^n$, and $\pi_X : \mathbb{R}^n \to S_X$ be the projection defined in Section 2.

Proposition 6. *The following functions are multipliers in spaces L^d for all $d > 1$:*
(i) $a_1(\xi) = \psi(\pi_X(\xi))$ where $\psi \in C^n(S_X)$;
(ii) $a_2(\xi) = \rho(\xi)(1 + |\tilde{\xi}|^2 + |\bar{\xi}|^4)^{1/2} (|\tilde{\xi}|^2 + |\bar{\xi}|^4)^{-1/2}$, where $\rho(\xi) \in C^\infty(\mathbb{R}^n)$ is a function with the properties indicated in Proposition 4, namely: $0 \le \rho(\xi) \le 1$, $\rho(\xi) = 0$ for $|\tilde{\xi}|^2 + |\bar{\xi}|^4 \le 1$, $\rho(\xi) = 1$ for $|\tilde{\xi}|^2 + |\bar{\xi}|^4 \ge 2$;
(iii) $a_3(\xi) = (1 + |\xi|^2)^{1/2} (1 + |\tilde{\xi}|^2 + |\bar{\xi}|^4)^{-1/2}$;
(iv) $a_4(\xi) = (1 + |\tilde{\xi}|^2 + |\bar{\xi}|^4)^{1/2} (1 + |\xi|^2)^{-1}$.

Proof. Since the space M_d is invariant under non-degenerate linear transformations of the variables ξ (see [2][Chapter 6]) then we can assume that $X = \mathbb{R}^l = \{\xi \in \mathbb{R}^n \mid \xi = (y_1, \dots, y_l, 0, \dots, 0)\}$ while $X^\perp = \{\xi \in \mathbb{R}^n \mid \xi = (0, \dots, 0, z_1, \dots, z_{n-l})\}$. Since $\pi_X(t^2 y, tz) = \pi_X(y, z)$ for $t > 0$, $y \in X$, $z \in X^\perp$ then $h = a_1(\xi) = \psi(\pi_X(\xi))$ satisfies the assumptions of Lemma 8 with $k = 2$, $\gamma = 0$. By this Lemma for each multi-indexes α, β, $|\alpha| + |\beta| \le n$

$$|y|^{|\alpha|} |z|^{|\beta|} |D_y^\alpha D_z^\beta a_1(y,z)| \le C = \text{const}.$$

This, in particular, implies that assumption (58) of Theorem 3 is satisfied. By this Theorem we conclude that $a_1(\xi) \in M_d$ for each $d > 1$.

To prove that $a_2(\xi) \in M_d$ we introduce the function $h_1(s, y, z) = (s^2 + |y|^2 + |z|^4)^{1/2}$, $s \in \mathbb{R}$. This function satisfies the assumptions of Lemma 8 with y replaced by $(s, y) \in \mathbb{R}^{l+1}$, and $k = \gamma = 2$. By this Lemma

$$|D_y^\alpha D_z^\beta h_1(s,y,z)| \le C(s^2 + |y|^2 + |z|^4)^{1/2} |y|^{-|\alpha|} |z|^{-|\beta|}, \ C = \text{const}.$$

Taking $s = 1$ in this relation, we arrive at the estimate

$$|D_y^\alpha D_z^\beta h_1(1, y, z)| \leq C(1 + |y|^2 + |z|^4)^{1/2}|y|^{-|\alpha|}|z|^{-|\beta|},$$

and by the Leibnitz formula we obtain that for each multi-indexes α, β such that $|\alpha| + |\beta| \leq n$

$$|D_y^\alpha D_z^\beta \rho(y, z)h_1(1, y, z)| \leq C_1(1 + |y|^2 + |z|^4)^{1/2}|y|^{-|\alpha|}|z|^{-|\beta|}, \ C_1 = \text{const} \tag{61}$$

(we use that $\rho(y, z) = 1$ for $|y|^2 + |z|^4 \geq 2$). Let $h_2(y, z) = (|y|^2 + |z|^4)^{-1/2}$. This function satisfies (59) with $k = 2$, $\gamma = -2$. By Lemma 8 for some constant C_2 and every multi-indexes α, β such that $|\alpha| + |\beta| \leq n$

$$|D_y^\alpha D_z^\beta h_2(y, z)| \leq C_2(|y|^2 + |z|^4)^{-1/2}|y|^{-|\alpha|}|z|^{-|\beta|}. \tag{62}$$

By the Leibnitz formula we derive from (61), (62) the estimates

$$\begin{aligned} |D_y^\alpha D_z^\beta \rho(y, z)h_1(1, y, z)h_2(y, z)| \leq \\ C_3(1 + |y|^2 + |z|^4)^{1/2}(|y|^2 + |z|^4)^{-1/2}|y|^{-|\alpha|}|z|^{-|\beta|} \leq 2C_3|y|^{-|\alpha|}|z|^{-|\beta|} \end{aligned} \tag{63}$$

in the domain $|y|^2 + |z|^4 \geq 1$, here $|\alpha| + |\beta| \leq n$, $C_3 = \text{const}$. In view of (63) we conclude that in this domain for each α, β, $|\alpha| + |\beta| \leq n$

$$|y|^{|\alpha|}|z|^{|\beta|}|D_y^\alpha D_z^\beta a_2(y, z)| \leq \text{const}.$$

Since $a_2(y, z) = 0$ for $|y|^2 + |z|^4 < 1$ we see that the requirements of Theorem 3 are satisfied. Therefore, $a_2(\xi) \in M_d$ for all $d > 1$.

Now we introduce the functions $h_1(s, y, z) = (s^2 + |y|^2 + |z|^2)^{1/2}$, $h_2(s, y, z) = (s^2 + |y|^2 + |z|^2)^{-1}$, $h_3(s, y, z) = (s^2 + |y|^2 + |z|^4)^{-1/2}$, $h_4(s, y, z) = (s^2 + |y|^2 + |z|^4)^{1/2}$, $s \in \mathbb{R}$, $y \in X = \mathbb{R}^l$, $z \in X^\perp$. These functions satisfy (59) where y is replaced by $(s, y) \in \mathbb{R}^{l+1}$ with the parameters $k = \gamma = 1$; $k = 1, \gamma = -2$; $k = 2, \gamma = -2$; $k = \gamma = 2$, respectively. By Lemma 8 we find that for each α, β, $|\alpha| + |\beta| \leq n$

$$\begin{aligned} |y|^{|\alpha|}|z|^{|\beta|}|D_y^\alpha D_z^\beta h_1(1, y, z)| \leq C(1 + |y|^2 + |z|^2)^{1/2}, \\ |y|^{|\alpha|}|z|^{|\beta|}|D_y^\alpha D_z^\beta h_2(1, y, z)| \leq C(1 + |y|^2 + |z|^2)^{-1}, \\ |y|^{|\alpha|}|z|^{|\beta|}|D_y^\alpha D_z^\beta h_3(1, y, z)| \leq C(1 + |y|^2 + |z|^4)^{-1/2}, \\ |y|^{|\alpha|}|z|^{|\beta|}|D_y^\alpha D_z^\beta h_4(1, y, z)| \leq C(1 + |y|^2 + |z|^4)^{1/2}, \end{aligned}$$

where $C = \text{const}$. Since $a_3(\xi) = h_1(1, y, z)h_3(1, y, z)$, $a_4(\xi) = h_2(1, y, z)h_4(1, y, z)$ where $y = \tilde{\xi}$, $z = \bar{\xi}$ then, using again the Leibnitz formula, we derive the estimates: for some constant C

$$\begin{aligned} |y|^{|\alpha|}|z|^{|\beta|}|D_y^\alpha D_z^\beta a_3(y, z)| \leq C(1 + |y|^2 + |z|^2)^{1/2}(1 + |y|^2 + |z|^4)^{-1/2} \leq 2C, \\ |y|^{|\alpha|}|z|^{|\beta|}|D_y^\alpha D_z^\beta a_4(y, z)| \leq C(1 + |y|^2 + |z|^2)^{-1}(1 + |y|^2 + |z|^4)^{1/2} \leq 2C. \end{aligned}$$

Here we take into account the following simple inequalities

$$\frac{1+|y|^2+|z|^2}{1+|y|^2+|z|^4} = \frac{1+|y|^2}{1+|y|^2+|z|^4} + \frac{|z|^2}{1+|y|^2+|z|^4} \le 1 + \min(|z|^2, |z|^{-2}) \le 2,$$

$$\frac{(1+|y|^2+|z|^4)^{1/2}}{1+|y|^2+|z|^2} \le \frac{(1+|y|^2)^{1/2}}{1+|y|^2+|z|^2} + \frac{|z|^2}{1+|y|^2+|z|^2} \le 2.$$

In view of Theorem 3, we conclude that $a_3(\xi), a_4(\xi) \in M_d$ for each $d > 1$. The proof is now complete.□

We define the anisotropic Sobolev space $W_d^{-1,-2}$ consisting of distributions $u(x)$ such that $(1+|\tilde{\xi}|^2+|\bar{\xi}|^4)^{-1/2}F(u)(\xi) = F(v)(\xi)$, $v = v(x) \in L^d(\mathbb{R}^n)$. This is a Banach space with the norm $\|u\| = \|v\|_d$. The following proposition claims that this space lays between the spaces W_d^{-1} and W_d^{-2}.

Proposition 7. *For each $d > 1$ $W_d^{-1} \subset W_d^{-1,-2} \subset W_d^{-2}$ and the both embeddings are continuous.*

Proof. Let $u \in W_d^{-1}$. This means that $(1+|\xi|^2)^{-1/2}F(u)(\xi) = F(w)(\xi)$, $w = w(x) \in L^d(\mathbb{R}^n)$. By Proposition 6(iii) $a_3(\xi) = (1+|\xi|^2)^{1/2}(1+|\tilde{\xi}|^2+|\bar{\xi}|^4)^{-1/2} \in M_d$. Therefore,

$$(1+|\tilde{\xi}|^2+|\bar{\xi}|^4)^{-1/2}F(u)(\xi) = a_3(\xi)F(w)(\xi) = F(v)(\xi), \ v(x) \in L^d(\mathbb{R}^n),$$

that is, $u \in W_d^{-1,-2}$. We claim that $W_d^{-1} \subset W_d^{-1,-2}$. Since $\|v\|_d \le C\|w\|_d$, $C = \mathrm{const}$ this embedding is continuous.

Now suppose that $u \in W_d^{-1,-2}$. Then $(1+|\tilde{\xi}|^2+|\bar{\xi}|^4)^{-1/2}F(u)(\xi) = F(v)(\xi)$, $v = v(x) \in L^d(\mathbb{R}^n)$. By Proposition 6(iv) $a_4(\xi) = (1+|\tilde{\xi}|^2+|\bar{\xi}|^4)^{1/2}(1+|\xi|^2)^{-1} \in M_d$, and

$$(1+|\xi|^2)^{-1}F(u)(\xi) = a_4(\xi)F(v)(\xi) = F(w)(\xi), \ w \in L^d(\mathbb{R}^n).$$

This means that $u \in W_d^{-2}$. We established that $W_d^{-1,-2} \subset W_d^{-2}$. The continuity of this embedding follows from the estimate $\|w\|_d \le C\|v\|_d$, $C = \mathrm{const}$. The proof is complete.□

We also introduce the local space $W_{d,loc}^{-1,-2}(\Omega)$ consisting of distributions $u(x)$ such that $uf(x)$ belongs to $W_d^{-1,-2}$ for all $f(x) \in C_0^\infty(\Omega)$. The space $W_{d,loc}^{-1,-2}(\Omega)$ is a locally convex space with the topology generated by the family of semi-norms $u \to \|uf\|_{W_d^{-1,-2}}$, $f(x) \in C_0^\infty(\Omega)$. Analogously we define the spaces $W_{d,loc}^{-1}(\Omega)$, $W_{d,loc}^{-2}(\Omega)$. As readily follows from Proposition 7, $W_{d,loc}^{-1} \subset W_{d,loc}^{-1,-2} \subset W_{d,loc}^{-2}$ and these embeddings are continuous.

Now we consider the bounded sequence of measure valued functions $\nu_x^k \in \mathrm{MV}(\Omega)$ and suppose that for some $d > 1$ and each $a, b \in \mathbb{R}$, $a < b$ the sequence of distributions

$$\operatorname{div} \int \varphi(x, s_{a,b}(\lambda))d\nu_x^k(\lambda) - D^2 \cdot \int B(x, s_{a,b}(\lambda))d\nu_x^k(\lambda) \text{ is pre-compact in } W_{d,loc}^{-1,-2}(\Omega). \tag{64}$$

Here $s_{a,b}(u) = \max(a, \min(u, b))$ is the cut-off function.

We choose the subsequence $\nu_x^r = \nu_x^k$, $k = k_r$ weakly convergent to a bounded measure-valued function ν_x^0 such that the parabolic H-measure $\mu^{pq} = \mu_X^{pq}$, $p, q \in E$ is well defined. By Proposition 3 this H-measure can be represented in the form $\mu^{pq} = \mu_x^{pq} dx$, $p, q \in D$, $x \in \Omega'$, where $\Omega' \subset \Omega$ is a set of full measure indicated in the proof of Proposition 3. Define the set of full measure Ω_φ consisting of common Lebesgue points of the maps $x \to F(x) = \varphi(x, \cdot) \in C(\mathbb{R}, \mathbb{R}^n)$, $x \to |F(x)|^2 = |\varphi(x, \cdot)| \in C(\mathbb{R})$. Similarly, we define the set Ω_B of common Lebesgue points of the maps $x \to G(x) = B(x, \cdot) \in C(\mathbb{R}, Sym_n)$, $x \to |G(x)|^2 \in C(\mathbb{R})$. Clearly, the set $\Omega'' = \Omega' \cap \Omega_\varphi \cap \Omega_B$ has full Lebesgue measure. We fix $x \in \Omega''$.

Under the above assumptions we have the following localization principle

Theorem 4. *Let L be a linear span of* $\operatorname{supp} \mu_x^{p_0 p_0}$. *Then there exists $\delta > 0$ such that*

$$(\varphi(x, \lambda) - \varphi(x, p_0)) \cdot \tilde{\xi} = 0, \ (B(x, \lambda) - B(x, p_0))\bar{\xi} \cdot \bar{\xi} = 0$$

for all $\xi \in L$, $\lambda \in [p_0, p_0 + \delta]$.

Proof. As follows from (64) and the weak convergence $\nu_y^r \to \nu_y^0$,

$$\mathcal{L}_p^r(y) = \operatorname{div}_y \int \varphi(y, s_{p_0,p}(\lambda)) d\gamma_y^r(\lambda) - D^2 \cdot \int B(y, s_{p_0,p}(\lambda)) d\gamma_y^r(\lambda) \underset{r\to\infty}{\to} 0 \text{ in } W_{d,loc}^{-1,-2}(\Omega), \tag{65}$$

where $\gamma_y^r = \nu_y^r - \nu_y^0$. As is easy to compute,

$$\begin{aligned}\varphi(y, s_{p_0,p}(\lambda)) &= \varphi(y, p_0) + (\varphi(y, p) - \varphi(y, p_0))\theta(\lambda - p_0) - (\varphi(y, p) - \varphi(y, \lambda))\chi(\lambda),\\ B(y, s_{p_0,p}(\lambda)) &= B(y, p_0) + (B(y, p) - B(y, p_0))\theta(\lambda - p_0) - (B(y, p) - B(y, \lambda))\chi(\lambda)\end{aligned}$$

where $\chi(\lambda) = \theta(\lambda - p_0) - \theta(\lambda - p)$ is the indicator function of the interval $(p_0, p]$. Therefore, $\mathcal{L}_p^r = \operatorname{div}_y(P_r(y)) - D^2 \cdot Q_r(y)$ where the vector $P_r(y)$ and the matrix $Q_r(y) = \{(Q_r)_{kl}(y)\}_{kl=1}^n$ are as follows (notice that $\int d\gamma_y^r(\lambda) = 0$):

$$\begin{aligned}P_r(y) = \int (\varphi(y, p) - \varphi(y, p_0))\theta(\lambda - p_0) d\gamma_y^r(\lambda) - \\ \int (\varphi(y, p) - \varphi(y, \lambda))\chi(\lambda) d\gamma_y^r(\lambda) = \\ U_r^{p_0}(y)(\varphi(y, p) - \varphi(y, p_0)) - \int (\varphi(y, p) - \varphi(y, \lambda))\chi(\lambda) d\gamma_y^r(\lambda);\end{aligned} \tag{66}$$

$$Q_r(y) = U_r^{p_0}(y)(B(y, p) - B(y, p_0)) - \int (B(y, p) - B(y, \lambda))\chi(\lambda) d\gamma_y^r(\lambda). \tag{67}$$

In particular, it follows from (67) that $X \subset \ker Q_r$.

For $\Phi(y) \in C_0^\infty(\Omega)$ we consider the sequence

$$\begin{aligned}L_r = \operatorname{div}_y(\Phi(y) P_r(y)) + 2\operatorname{div}(Q_r(y) \nabla \Phi(y)) - D^2 \cdot (\Phi(y) Q_r(y)) = \\ = \operatorname{div}_y(\Phi(y) P_r(y)) + 2(\Phi_{y_l}(Q_r)_{kl}(y))_{y_k} - \partial^2_{y_k y_l}(\Phi(y)(Q_r)_{kl}(y)) = \\ \Phi(y)\mathcal{L}_p^r(y) + P_r(y) \cdot \nabla \Phi(y) + D^2 \Phi(y) \cdot Q_r(y).\end{aligned}$$

Since the sequence $P_r(y)\cdot\nabla\Phi(y)+D^2\Phi(y)\cdot Q_r(y)$ is bounded in L^2 and weakly converges to zero as $r\to\infty$, this sequence converges to zero in $W_d^{-1}\subset W_d^{-1,-2}$ (we can suppose that $d\le 2$). Besides, in view of (65), $\Phi(y)\mathcal{L}_p^r(y)\underset{r\to\infty}{\to}0$ in $W_d^{-1,-2}$ as well, and we claim that $L_r\underset{r\to\infty}{\to}0$ in $W_d^{-1,-2}$. Introduce the vector $G_r(y,\lambda)=2Q_r(y)\nabla\Phi(y)$ with components $(G_r)_k(y)=2\Phi_{y_l}(Q_r)_{kl}(y)$, $k=1,\dots,n$. Then the distributions L_r can be represented in the form $L_r=\operatorname{div}_y(\Phi P_r+G_r)-D^2\cdot(\Phi Q_r)$. Hence,

$$\operatorname{div}_y(\Phi P_r+G_r)-D^2\cdot(\Phi Q_r)\underset{r\to\infty}{\to}0 \ \text{ in } W_d^{-1,-2}.$$

Applying the Fourier transformation to this relation and then multiplying by $\rho(\xi)(|\tilde\xi|^2+|\bar\xi|^4)^{-1/2}$, we arrive at

$$\frac{\rho(\xi)(2\pi i\xi\cdot F(\Phi P_r+G_r)(\xi)+4\pi^2F(\Phi Q_r)(\xi)\xi\cdot\xi)}{(|\tilde\xi|^2+|\bar\xi|^4)^{1/2}}=F(l_r)(\xi),\ l_r\underset{r\to\infty}{\to}0 \ \text{ in } L^d(\mathbb{R}^n) \tag{68}$$

(the function $\rho(\xi)$ is indicated in Proposition 4). Indeed, (68) follows from the representation

$$\rho(\xi)(|\tilde\xi|^2+|\bar\xi|^4)^{-1/2}=\frac{\rho(\xi)(1+|\tilde\xi|^2+|\bar\xi|^4)^{1/2}}{(|\tilde\xi|^2+|\bar\xi|^4)^{1/2}}(1+|\tilde\xi|^2+|\bar\xi|^4)^{-1/2},$$

the statement of Proposition 6(ii) and the definition of $W_d^{-1,-2}$. Let $\psi(\xi)\in C^n(S_X)$. Then by Proposition 6(i) we see that the sequence $F(\Phi U_r^{p_0})(\xi)\overline{\psi(\pi_X(\xi))}=F(h_r)$, where h_r is bounded in $L^{d'}(\mathbb{R}^n)$, $d'=d/(d-1)$. This and (68) imply the relation

$$\int_{\mathbb{R}^n}\frac{\rho(\xi)(2\pi i\xi\cdot F(\Phi P_r+G_r)(\xi)+4\pi^2F(\Phi Q_r)(\xi)\xi\cdot\xi)}{(|\tilde\xi|^2+|\bar\xi|^4)^{1/2}}\overline{F(\Phi U_r^{p_0})(\xi)}\psi(\pi_X(\xi))d\xi=$$
$$\int_{\mathbb{R}^n}l_r(x)\overline{h_r(x)}dx\underset{r\to\infty}{\to}0. \tag{69}$$

Now, we remark that the sequences $\Phi(y)P_r(y)$ and $G_r(y)$ are bounded in $L^2\cap L^1$ and weakly converge to zero. By Lemma 6 we have

$$\frac{\rho(\xi)\bar\xi\cdot F(\Phi P_r+G_r)(\xi)}{(|\tilde\xi|^2+|\bar\xi|^4)^{1/2}}\underset{r\to\infty}{\to}0 \ \text{ in } L^2(\mathbb{R}^n) \tag{70}$$

because

$$a(\xi)=\frac{\rho(\xi)|\bar\xi|}{(|\tilde\xi|^2+|\bar\xi|^4)^{1/2}}\le\frac{|\bar\xi|}{(|\tilde\xi|^2+|\bar\xi|^4)^{1/4}}\le 1$$

and evidently $a(\xi)\to 0$ as $|\xi|\to\infty$. Besides,

$$\tilde\xi\cdot F(G_r)(\xi)=2\int_{\mathbb{R}^n}e^{-2\pi i\xi\cdot y}(Q_r)_{kl}(y)\Phi_{y_l}(y)\tilde\xi_k dy=$$
$$\int_{\mathbb{R}^n}e^{-2\pi i\xi\cdot y}Q_r(y)\tilde\xi\cdot\nabla\Phi(y)dy=0, \tag{71}$$

$$F(\Phi Q_r)(\xi)\tilde\xi=\int_{\mathbb{R}^n}e^{-2\pi i\xi\cdot y}\Phi(y)Q_r(y)\tilde\xi dy=0 \tag{72}$$

since $\tilde{\xi} \in X \subset \ker Q_r$. Taking into account relations (70), (71), (72), and the boundedness of the sequence $F(\Phi U_r^{p_0})(\xi)$ in $L^2(\mathbb{R}^n)$, we derive from (69) that

$$\int_{\mathbb{R}^n} \frac{\rho(\xi)(2\pi i\tilde{\xi} \cdot F(\Phi P_r)(\xi) + 4\pi^2 F(\Phi Q_r)(\xi)\bar{\xi} \cdot \bar{\xi})}{(|\tilde{\xi}|^2 + |\bar{\xi}|^4)^{1/2}} \overline{F(\Phi U_r^{p_0})(\xi)}\psi(\pi_X(\xi))d\xi \underset{r\to\infty}{\to} 0. \quad (73)$$

Taking into account representations (66), (67) we can rewrite the last relation as follows

$$\lim_{r\to\infty} \left\{ \int_{\mathbb{R}^n} \frac{\rho(\xi)(2\pi i\tilde{\xi} \cdot F(\Phi U_r^{p_0} f)(\xi) + 4\pi^2 F(\Phi U_r^{p_0} H)(\xi)\bar{\xi} \cdot \bar{\xi})}{(|\tilde{\xi}|^2 + |\bar{\xi}|^4)^{1/2}} \times \right.$$
$$\overline{F(\Phi U_r^{p_0})(\xi)}\psi(\pi_X(\xi))d\xi -$$
$$\left. \int_{\mathbb{R}^n} \frac{\rho(\xi)(i\tilde{\xi} \cdot F(\Phi V_r^p)(\xi) + F(\Phi G_r^p)(\xi)\bar{\xi} \cdot \bar{\xi})}{(|\tilde{\xi}|^2 + |\bar{\xi}|^4)^{1/2}} \overline{F(\Phi U_r^{p_0})(\xi)}\psi(\pi_X(\xi))d\xi \right\} = 0, \quad (74)$$

where

$$f(y) = \varphi(y,p) - \varphi(y,p_0),\ V_r^p(y) = 2\pi \int (\varphi(y,p) - \varphi(y,\lambda))\chi(\lambda)d\gamma_y^r(\lambda) \in \mathbb{R}^n,$$
$$H(y) = B(y,p) - B(y,p_0),\ G_r^p(y) = 4\pi^2 \int (B(y,p) - B(y,\lambda))\chi(\lambda)d\gamma_y^r(\lambda) \in Sym_n.$$

In (74) we set $\Phi(y) = \Phi_m(x-y)$, where the functions Φ_m were defined in section 3 in the proof of Proposition 3, and pass to the limit as $m \to \infty$. By Remark 3 (see equality (35)) we obtain

$$\lim_{m\to\infty} \lim_{r\to\infty} \int_{\mathbb{R}^n} \frac{\rho(\xi)(2\pi i\tilde{\xi} \cdot F(\Phi_m U_r^{p_0} f)(\xi) + 4\pi^2 F(\Phi_m U_r^{p_0} H)(\xi)\bar{\xi} \cdot \bar{\xi})}{(|\tilde{\xi}|^2 + |\bar{\xi}|^4)^{1/2}} \times$$
$$\overline{F(\Phi_m U_r^{p_0})(\xi)}\psi(\pi_X(\xi))d\xi = \langle \mu_x^{p_0p_0}, (2\pi i\tilde{\xi} \cdot f(x) + 4\pi^2 H(x)\bar{\xi} \cdot \bar{\xi})\psi(\xi)\rangle,$$

therefore

$$\langle \mu_x^{p_0p_0}, (2\pi i\tilde{\xi} \cdot f(x) + 4\pi^2 H(x)\bar{\xi} \cdot \bar{\xi})\psi(\xi)\rangle =$$
$$\lim_{m\to\infty} \lim_{r\to\infty} \int_{\mathbb{R}^n} \frac{\rho(\xi)(i\tilde{\xi} \cdot F(\Phi_m V_r^p)(\xi) + F(\Phi_m G_r^p)(\xi)\bar{\xi} \cdot \bar{\xi})}{(|\tilde{\xi}|^2 + |\bar{\xi}|^4)^{1/2}} \overline{F(\Phi_m U_r^{p_0})(\xi)}\psi(\pi_X(\xi))d\xi. \quad (75)$$

Since the space $C^n(S_X)$ is dense in $C(S_X)$, it is clear that (75) holds for each $\psi(\xi) \in C(S_X)$. Let $g(y,\lambda) = \tilde{P}\varphi(x,\lambda)$, $B_1(y,\lambda) = \bar{P}B(y,\lambda)\bar{P}$, where $\tilde{P}$, $\bar{P}$ are operators of orthogonal projections on the spaces $\tilde{L} = P_1(L)$, $\bar{L} = P_2(L)$, respectively, L being the linear span of $\operatorname{supp} \mu_x^{p_0p_0}$ (see the notations of section 3). Obviously,

$$\langle \mu_x^{p_0p_0}, (2\pi i\tilde{\xi} \cdot f(x) + 4\pi^2 H(x)\bar{\xi} \cdot \bar{\xi})\psi(\xi)\rangle =$$
$$\langle \mu_x^{p_0p_0}, (2\pi i\tilde{\xi} \cdot (g(x,p) - g(x,p_0)) + 4\pi^2(B_1(x,p) - B_1(x,p_0))\bar{\xi} \cdot \bar{\xi})\psi(\xi)\rangle. \quad (76)$$

We denote $h(y,\lambda)=\varphi(y,\lambda)-g(y,\lambda)$, $B_2(y,\lambda)=B(y,\lambda)-B_1(y,\lambda)$,

$$V_{r1}^p(y)=2\pi\int(g(y,p)-g(y,\lambda))\chi(\lambda)d\gamma_y^r(\lambda),$$
$$V_{r2}^p(y)=2\pi\int(h(y,p)-h(y,\lambda))\chi(\lambda)d\gamma_y^r(\lambda),$$
$$G_{r1}^p(y)=4\pi^2\int(B_1(y,p)-B_1(y,\lambda))\chi(\lambda)d\gamma_y^r(\lambda),$$
$$G_{r2}^p(y)=4\pi^2\int(B_2(y,p)-B_2(y,\lambda))\chi(\lambda)d\gamma_y^r(\lambda).$$

In the notations of Propositions 4,5 $V_{r2}^p(y)=I_r(f\chi)(y)$ with $f(y,\lambda)=2\pi(h(y,p)-h(y,\lambda))$, $G_{r2}^p(y)=J_r(Q)(y)$ with $Q(y,\lambda)=4\pi^2(B_2(y,p)-B_2(y,\lambda))$. Since $\xi\cdot f(y,\lambda)=0$ for all $\xi\in\tilde{L}$, $Q(y,\lambda)\xi=0$ for all $\xi\in\bar{L}$ then by Propositions 4,5

$$\lim_{m\to\infty}\lim_{r\to\infty}\int_{\mathbb{R}^n}\frac{\rho(\xi)(i\tilde{\xi}\cdot F(\Phi_m V_{r2}^p)(\xi)+F(\Phi_m G_{r2}^p)(\xi)\bar{\xi}\cdot\bar{\xi})}{(|\tilde{\xi}|^2+|\bar{\xi}|^4)^{1/2}}\overline{F(\Phi_m U_r^{p_0})(\xi)}\psi(\pi_X(\xi))d\xi=0$$

and (75) acquires the form

$$\langle\mu_x^{p_0p_0},(2\pi i\tilde{\xi}\cdot(g(x,p)-g(x,p_0))+4\pi^2(B_1(x,p)-B_1(x,p_0))\bar{\xi}\cdot\bar{\xi})\psi(\xi)\rangle=$$
$$\lim_{m\to\infty}\lim_{r\to\infty}\int_{\mathbb{R}^n}\frac{\rho(\xi)(i\tilde{\xi}\cdot F(\Phi_m V_{r1}^p)(\xi)+F(\Phi_m G_{r1}^p)(\xi)\bar{\xi}\cdot\bar{\xi})}{(|\tilde{\xi}|^2+|\bar{\xi}|^4)^{1/2}}\overline{F(\Phi_m U_r^{p_0})(\xi)}\psi(\pi_X(\xi))d\xi.\quad(77)$$

Here we also use relation (76). Now we observe that

$$\left|\frac{\rho(\xi)(i\tilde{\xi}\cdot F(\Phi_m V_{r1}^p)(\xi)+F(\Phi_m G_{r1}^p)(\xi)\bar{\xi}\cdot\bar{\xi})}{(|\tilde{\xi}|^2+|\bar{\xi}|^4)^{1/2}}\right|\le$$
$$|F(\Phi_m V_{r1}^p)(\xi)|+|F(\Phi_m G_{r1}^p)(\xi)|,$$

and therefore

$$\left\|\frac{\rho(\xi)(i\tilde{\xi}\cdot F(\Phi_m V_{r1}^p)(\xi)+F(\Phi_m G_{r1}^p)(\xi)\bar{\xi}\cdot\bar{\xi})}{(|\tilde{\xi}|^2+|\bar{\xi}|^4)^{1/2}}\right\|_2\le$$
$$\|F(\Phi_m V_{r1}^p)\|_2+\|F(\Phi_m G_{r1}^p)\|_2=\|\Phi_m V_{r1}^p\|_2+\|\Phi_m G_{r1}^p\|_2\quad(78)$$

by Plancherel's equality. Since $|U_r^{p_0}|\le 1$ then

$$\|F(\Phi_m U_r^{p_0})\|_2=\|\Phi_m U_r^{p_0}\|_2\le 1,$$

and we derive from (77) with the help of Buniakovskii inequality and (78) that

$$\left|\langle\mu_x^{p_0p_0},(2\pi i\tilde{\xi}\cdot(g(x,p)-g(x,p_0))+4\pi^2(B_1(p)-B_1(p_0))\bar{\xi}\cdot\bar{\xi})\psi(\xi)\rangle\right|\le$$
$$\|\psi\|_\infty\varlimsup_{m\to\infty}\varlimsup_{r\to\infty}\left(\|\Phi_m V_{r1}^p\|_2+\|\Phi_m G_{r1}^p\|_2\right).\quad(79)$$

Next, for $M_p(y) = \max\limits_{\lambda\in[p_0,p]} |g(y,p) - g(y,\lambda)|$

$$|V^p_{r1}(y)| \le 2\pi M_p(y) \int \chi(\lambda) d\left(\nu^r_y(\lambda) + \nu^0_y(\lambda)\right) =$$
$$2\pi M_p(y)(u_r(y,p_0) - u_r(y,p) + u_0(y,p_0) - u_0(y,p))$$

and by Lemma 5

$$\varlimsup_{m\to\infty} \varlimsup_{r\to\infty} \|\tilde{V}^p_{r1}\Phi_m\|_2 \le 4\pi M_p(x)(u_0(x,p_0) - u_0(x,p))^{1/2}. \tag{80}$$

Here we bear in mind that x is a Lebesgue point of the function $(M_p(y))^2$ (which easily follows from the fact that $x \in \Omega_\varphi$ is a Lebesgue point of the maps $y \to \varphi(y,\cdot)$, $y \to |\varphi(y,\cdot)|^2$ into the spaces $C(\mathbb{R},\mathbb{R}^n)$, $C(\mathbb{R})$, respectively). Further, the matrix $0 \le B_1(y,p) - B_1(y,\lambda) \le B_1(y,p) - B_1(y,p_0)$ for each $\lambda \in [p_0,p]$ (since the matrix $B_1(y,\lambda) - B_1(y,p_0)$ is positive definite). This implies the corresponding inequality for the Euclidean norms $|B_1(y,p) - B_1(y,\lambda)| \le |B_1(y,p) - B_1(y,p_0)|$. Therefore,

$$|G^p_{r1}(y)| \le 4\pi^2 \int |B_1(y,p) - B_1(y,\lambda)| \chi(\lambda) d\left(\nu^r_y(\lambda) + \nu^0_y(\lambda)\right) \le$$
$$4\pi^2 |B_1(y,p) - B_1(y,p_0)| (u_r(y,p_0) - u_r(y,p) + u_0(y,p_0) - u_0(y,p)).$$

By Lemma 5 again we claim that

$$\varlimsup_{m\to\infty} \varlimsup_{r\to\infty} \|G^p_{r1}\Phi_m\|_2 \le 8\pi^2 |B_1(x,p) - B_1(x,p_0)| (u_0(x,p_0) - u_0(x,p))^{1/2}. \tag{81}$$

In view of (80), (81) we derive from (79) that

$$\left|\langle \mu^{p_0p_0}_x, (2\pi i\tilde{\xi} \cdot (g(x,p) - g(x,p_0)) + 4\pi^2(B_1(x,p) - B_1(x,p_0))\bar{\xi} \cdot \bar{\xi})\psi(\xi)\rangle\right| \le$$
$$c\|\psi\|_\infty (M_p(x) + |B_1(x,p) - B_1(x,p_0)|)\omega(p), \tag{82}$$

where $c = \text{const}$ and $\omega(p) = (u_0(x,p_0) - u_0(x,p))^{1/2} \underset{p\to p_0}{\to} 0$ (remind that $p_0 \in D$ is a continuity point of the function $p \to u_0(x,p)$ for $x \in \Omega'$). Next, by Corollary 1, we can choose functions $\psi_k(\xi) \in C(S_X)$, $k = 1,\ldots,l$ such that for some positive constant C

$$|g(x,p) - g(x,p_0)| + |B_1(x,p) - B_1(x,p_0)| \le$$
$$C \max_{k=1,\ldots,l} \left|\langle \mu^{p_0p_0}_x, (i\tilde{\xi} \cdot (g(x,p) - g(x,p_0)) + (B_1(x,p) - B_1(x,p_0))\bar{\xi} \cdot \bar{\xi})\psi_k(\xi)\rangle\right|.$$

Then, in view of (82), we find

$$|g(x,p) - g(x,p_0)| + |B_1(x,p) - B_1(x,p_0)| \le c(M_p(x) + |B_1(x,p) - B_1(x,p_0)|)\omega(p), \tag{83}$$

where c is a positive constant.

We choose $\delta > 0$ such that $2c\omega(p) \le \varepsilon < 1$ for all $p \in [p_0, p_0 + \delta]$. Then by (83) for all $p \in [p_0, p_0 + \delta] \cap D$

$$\begin{aligned} &|g(x,p) - g(x,p_0)| + |B_1(x,p) - B_1(x,p_0)| \le \\ &\frac{\varepsilon}{2}\left(\max_{\lambda\in[p_0,p]} |g(x,p) - g(x,\lambda)| + |B_1(x,p) - B_1(x,p_0)|\right), \end{aligned} \tag{84}$$

and since $g(x,p)$, $B_1(x,p)$ are continuous with respect to p and the set D is dense, the estimate (84) holds for all $p \in [p_0, p_0 + \delta]$.

Now we claim that $g(x,p) = g(x,p_0)$, $B_1(x,p) = B_1(x,p_0)$ for $p \in [p_0, p_0 + \delta]$. Indeed, assume that for $p' \in [p_0, p_0 + \delta]$

$$|g(x,p') - g(x,p_0)| = \max_{\lambda\in[p_0,p_0+\delta]} |g(x,\lambda) - g(x,p_0)|.$$

Then for $\lambda \in [p_0, p']$ we have

$$\begin{aligned} &|g(x,p') - g(x,\lambda)| \le |g(x,\lambda) - g(x,p_0)| + \\ &|g(x,p') - g(x,p_0)| \le 2|g(x,p') - g(x,p_0)| \end{aligned}$$

and

$$\max_{\lambda\in[p_0,p']} |g(x,p') - g(x,\lambda)| \le 2|g(x,p') - g(x,p_0)|.$$

We derive from (84) with $p = p'$ that

$$|g(x,p')-g(x,p_0)|+|B_1(x,p')-B_1(x,p_0)| \le \varepsilon(|g(x,p')-g(x,p_0)|+|B_1(x,p')-B_1(x,p_0)|),$$

and since $\varepsilon < 1$, this implies that

$$|g(x,p') - g(x,p_0)| = \max_{\lambda\in[p_0,p_0+\delta]} |g(x,\lambda) - g(x,p_0)| = 0.$$

This means that $g(x,\lambda) = g(x,p_0)$ for $\lambda \in [p_0, p_0 + \delta]$. Then, (84) acquires the form

$$|B_1(x,p) - B_1(x,p_0)| \le \frac{\varepsilon}{2}|B_1(x,p) - B_1(x,p_0)|, \quad \varepsilon < 1.$$

Hence $B_1(x,p) = B_1(x,p_0)$ for every $p \in [p_0, p_0 + \delta]$. By the definition of $B_1(x,p)$ we see that $(B(x,p) - B(x,p_0))\bar{P} = 0$, that is $\bar{L} \subset \ker(B(x,p) - B(x,p_0))$ and $(B(x,p) - B(x,p_0))\bar{\xi} = 0$ for all $p \in [p_0, p_0 + \delta]$, $\xi \in L$. The relation $\tilde{P}(\varphi(x,\lambda) - \varphi(x,p_0)) = g(x,\lambda) - g(x,p_0) = 0$ on $[p_0, p_0 + \delta]$ implies that for all $\xi \in L$ $(\varphi(x,\lambda) - \varphi(x,p_0)) \cdot \tilde{\xi} = 0$ on the segment $\lambda \in [p_0, p_0 + \delta]$. The proof is complete.$\square$

Under the non-degeneracy condition, indicated in Definition 2, Theorem 4 yields the following result.

Theorem 5. *Suppose that the non-degeneracy condition is satisfied. Then any sequence ν_x^k weakly converging as $k \to \infty$ to ν_x^0 and satisfying (64) strongly converges to ν_x^0.*

Proof. Let $\nu_x^r = \nu_x^k$, $k = k_r$, be a subsequence such that the parabolic H-measure $\{\tilde{\mu}^{pq}\}_{p,q\in E}$, corresponding to the subspace X, is well defined. This H-measure admits the representation $\mu^{pq} = \mu_x^{pq}dx$ and, as directly follows from the assertion of Theorem 4 and non-degeneracy condition in Definition 2, $\mu_x^{p_0p_0} = 0$ for a.e. $x \in \Omega$. Therefore, $\mu^{p_0p_0} = \mu_x^{p_0p_0}dx = 0$. Since an arbitrary $p_0 \in E$ can be included in the set D we conclude that $\tilde{\mu}^{pp} = 0$ for all $p \in E$. By relation (25) with $\psi \equiv 1$ we see that

$$u_r(x,p) \to u_0(x,p) \ \text{ in } \ L^2_{loc}(\Omega)$$

as $r \to \infty$. Indeed, it follows from the definition of an H-measure and Plancherel's equality that

$$\lim_{r\to\infty} \|U_r^p\Phi\|_2^2 = \langle \mu^{pp}, |\Phi(x)|^2\rangle = 0$$

for all $\Phi(x) \in C_0(\Omega)$ and $p \in E$. Thus, for $p \in E$ we have

$$\int \theta(\lambda - p)d\nu_x^r(\lambda) \underset{r\to\infty}{\to} \int \theta(\lambda - p)d\nu_x^0(\lambda) \ \text{ in } L^2_{loc}(\Omega). \tag{85}$$

Any continuous function can be uniformly approximated on any compact subset by finite linear combinations of functions $\lambda \to \theta(\lambda - p)$, $p \in E$. Hence, it follows from (85) that for all $f(\lambda) \in C(\mathbb{R})$ we have

$$\int f(\lambda)d\nu_x^r(\lambda) \underset{r\to\infty}{\to} \int f(\lambda)d\nu_x^0(\lambda) \ \text{ in } L^2_{loc}(\Omega),$$

and therefore also in $L^1_{loc}(\Omega)$, that is, the subsequence ν_x^r strongly converges to ν_x^0. Finally, for each admissible choice of the subsequence ν_x^r the limit measure-valued function is uniquely defined, therefore the original sequence ν_x^k is also strongly convergent to ν_x^0. The proof is complete.$\square$

Taking into account Theorem T, one can give another formulation of Theorem 5: each bounded sequence of measure-valued functions satisfying (64) is pre-compact in the sense of strong convergence. Observe that in the regular case $\nu_x^k(\lambda) = \delta(\lambda - u_k(x))$ condition (64) has the form: for some $d > 1$ and each $a, b \in \mathbb{R}$, $a < b$

$$\operatorname{div}\varphi(x, s_{a,b}(u_k(x))) - D^2 \cdot B(x, s_{a,b}(u_k(x))) \ \text{ is pre-compact in } W^{-1,-2}_{d,loc}(\Omega). \tag{86}$$

In this case Theorem 5 yields the following

Corollary 2. *Under the non-degeneracy condition, each bounded sequence $u_k(x) \in L^\infty(\Omega)$ satisfying (86) contains a subsequence convergent in $L^1_{loc}(\Omega)$.*

Proof. We only need to note that if the sequence $u_k(x)$ converges to a measure-valued function ν_x^0 strongly in $MV(\Omega)$, then by the definition of strong convergence

$$u_k(x) \underset{k\to\infty}{\to} u_0(x) = \int \lambda d\nu_x^0(\lambda) \ \text{ in } L^1_{loc}(\Omega)$$

(which also shows that $\nu_x^0(\lambda) = \delta(\lambda - u_0(x))$ is regular in Ω).$\square$

The statements of Theorems 4 and 5 remain true for sequences of unbounded measure-valued (or usual) functions. For the proof we should apply the cut-off functions $s_{a,b}(u) = \max(a, \min(u, b))$, $a, b \in \mathbb{R}$ and derive that bounded sequences of measure-valued functions $s^*_{a,b}\nu^k_x$ ($s^*_{a,b}\nu^k_x$ is the image of ν^k_x under the map $s_{a,b}$) satisfy (64). Then, under the non-degeneracy condition, we obtain the strong pre-compactness property for these sequences.

For instance, consider the sequence $u_k(x)$, $k \in \mathbb{N}$ of measurable functions on Ω. Suppose that condition (86) and the non-degeneracy condition hold. Let $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$, $v_k = s_{\alpha,\beta}(u_k) = \max(\alpha, \min(u_k, \beta))$. Then $v_k = v_k(x)$ is a bounded sequence in $L^\infty(\Omega)$ and for each $a, b \in \mathbb{R}$, $a < b$

$$\begin{gathered}\mathrm{div}\varphi(x, s_{a,b}(v_k(x))) - D^2 \cdot B(x, s_{a,b}(v_k(x))) = \\ \mathrm{div}\varphi(x, s_{a',b'}(u_k(x))) - D^2 \cdot B(x, s_{a',b'}(u_k(x)))\end{gathered}$$

where $a' = s_{a,b}(\alpha)$, $b' = s_{a,b}(\beta)$. It follows from this identity and (86) that the sequence $\mathrm{div}\varphi(x, s_{a,b}(v_k(x))) - D^2 \cdot B(x, s_{a,b}(v_k(x)))$ is pre-compact in $H^{-1,-2}_{d,loc}(\Omega)$. By Corollary 2 the sequences $v_k(x) = s_{\alpha,\beta}(u_k)$ are pre-compact in $L^1_{loc}(\Omega)$ for every $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$. Using the standard diagonal extraction, we can choose a subsequence $u_r(x) = u_{k_r}(x)$ such that for each $m \in \mathbb{N}$ the sequence $s_{-m,m}(u_r)$ converges as $r \to \infty$ to some function $w_m(x)$ in $L^1_{loc}(\Omega)$. Obviously, a.e. in Ω

$$|w_m(x)| \le m, \quad \text{and} \quad w_m(x) = s_{-m,m}(w_l(x)) \ \forall l > m.$$

This allows to define a unique (up to equality a.e.) measurable function $u(x) \in \mathbb{R} \cup \{\pm\infty\}$ such that $w_m(x) = s_{-m,m}(u(x))$ a.e. on Ω. If $a, b \in \mathbb{R}$, $a < b$ then for $m > \max(|a|, |b|)$

$$\begin{gathered}s_{a,b}(u_r) = s_{a,b}(s_{-m,m}(u_r)) \underset{r\to\infty}{\to} s_{a,b}(w_m) = \\ s_{a,b}(s_{-m,m}(u)) = s_{a,b}(u) \ \text{ in } L^1_{loc}(\Omega).\end{gathered}$$

In fact, we proved the following general statement.

Theorem 6. *Suppose that the sequence of measurable functions $u_k(x)$ satisfies (86) and the nondegeneracy condition holds. Then*

a) there exists a measurable function $u(x) \in \mathbb{R} \cup \{\pm\infty\}$ such that, after extraction of a subsequence u_r, $r \in \mathbb{N}$, $s_{a,b}(u_r) \to s_{a,b}(u)$ as $r \to \infty$ in $L^1_{loc}(\Omega)$ $\forall a, b \in \mathbb{R}$, $a < b$.

b) If, in addition, the following estimates are satisfied

$$\int_K m(u_k(x))dx \le C_K, \tag{87}$$

for each compact set $K \subset \Omega$, where $m(u)$ is a positive Borel function, such that $m(u)/u \underset{u\to\infty}{\to} \infty$, then $u(x) \in L^1_{loc}(\Omega)$ and $u_r \to u$ in $L^1_{loc}(\Omega)$ as $r \to \infty$.

Proof. We only need to prove b). Observe that, extracting a subsequence, if necessary, we can assume that $s_{-m,m}(u_r) \to s_{-m,m}(u)$ as $m \to \infty$ a.e. in Ω for every

$m \in \mathbb{N}$. This implies that $u_r \to u$ a.e. in Ω and by Fatou lemma it follows from (87) that

$$\int_K m(u(x))dx \le C_K.$$

In particular, $u(x) \in L^1_{loc}(\Omega)$. Now, fix a compact $K \subset \Omega$ and $\varepsilon > 0$. By the assumption $m(u)/u \underset{u\to\infty}{\to} \infty$ we can choose $l \in \mathbb{N}$ such that $|u|/m(u) \le \varepsilon/(2C_K)$ for $|u| > l$. Then

$$\int_K |u_r(x) - u(x)|dx \le \int_K |s_{-l,l}(u_r(x)) - s_{-l,l}(u(x))|dx +$$
$$\int_K |u_r(x)|\theta(|u_r(x)| - l)dx + \int_K |u(x)|\theta(|u(x)| - l)dx$$
$$\le \int_K |s_{-l,l}(u_r(x)) - s_{-l,l}(u(x))|dx +$$
$$\frac{\varepsilon}{2C_K}\left(\int_K m(u_r(x))dx + \int_K m(u(x))dx\right) \le$$
$$\int_K |s_{-l,l}(u_r(x)) - s_{-l,l}(u(x))|dx + \varepsilon.$$

This implies that $\overline{\lim\limits_{r\to\infty}} \int_K |u_r(x)-u(x)|dx \le \varepsilon$ and since $\varepsilon > 0$ is arbitrary we conclude that $\lim\limits_{r\to\infty} \int_K |u_r(x)-u(x)|dx = 0$ for any compact $K \subset \Omega$, i.e. $u_r \to u$ in $L^1_{loc}(\Omega)$. The proof is complete.□

5 Proofs of Theorems 1,2

We need the following simple

Lemma 9. *Suppose $u = u(x)$ is an entropy solution of (1). Then for all $a, b \in \mathbb{R}$, $a < b$*

$$\mathrm{div}\varphi(x, s_{a,b}(u)) - D^2 \cdot B(x, s_{a,b}(u)) = \zeta_{a,b} \quad \text{in } \mathcal{D}'(\Omega), \tag{88}$$

where $\zeta_{a,b} \in \mathrm{M}_{loc}(\Omega)$. Moreover, for each compact set $K \subset \Omega$ we have $\mathrm{Var}\,\zeta_{a,b}(K) \le C(K, a, b, I)$, where $I = I(x) = |\varphi(x, u(x))| + |\psi(x, u(x))| + |B(x, u(x))| \in L^1_{loc}(\Omega)$ and the map $I \to C(K, a, b, I)$ is bounded on bounded sets in $L^1_{loc}(\Omega)$.

Proof. By the known representation property for non-negative distributions we derive from (4) that

$$\mathrm{div}[\mathrm{sign}(u(x) - p)(\varphi(x, u(x))] - \varphi(x, p)) - D^2 \cdot [\mathrm{sign}(u(x) - p)(B(x, u(x)) - B(x, p))]$$
$$+ \mathrm{sign}(u(x) - p)[\omega_p(x) + \psi(x, u(x))] - |\gamma_p^s| = -\kappa_p \ \text{ in } \mathcal{D}'(\Omega),$$

where $\kappa_p \in \mathrm{M}_{loc}(\Omega)$, $\kappa_p \ge 0$. Further, for a compact set $K \subset \Omega$ we choose a non-negative function $f_K(x) \in C_0^\infty(\Omega)$, which equals 1 on K. Then we have the estimate

$$\kappa_p(K) \le \int f_K(x)d\kappa_p(x) = \int_\Omega [\mathrm{sign}(u(x) - p)(\varphi(x, u(x)) - \varphi(x, p)) \cdot \nabla f_K(x) +$$

$$\mathrm{sign}(u(x)-p)(B(x,u(x))-B(x,p))\cdot D^2 f_K(x)-$$
$$\mathrm{sign}(u(x)-p)(\omega_p(x)+\psi(x,u(x)))f_K(x)]dx+\int_\Omega f_K(x)d|\gamma_p^s|(x)\le$$
$$A(K,p,I)=\int_\Omega [I(x)\max(|f_K(x)|,|\nabla f_K(x)|,|D^2 f_K(x)|)+|\varphi(x,p)|\cdot|\nabla f_K(x)|+$$
$$|B(x,p)|\cdot|D^2 f_K(x)|+|\omega_p(x)|f_K(x)]dx+\int_\Omega f_K(x)d|\gamma_p^s|(x).$$

Hence,

$$\begin{aligned}&\mathrm{div}[\mathrm{sign}(u(x)-p)(\varphi(x,u(x))-\varphi(x,p))]-\\ &D^2\cdot[\mathrm{sign}(u(x)-p)(B(x,u(x))-B(x,p))]=\zeta_p,\end{aligned} \tag{89}$$

where

$$\zeta_p=|\gamma_p^s|-\kappa_p-\mathrm{sign}(u(x)-p)[\omega_p(x)+\psi(x,u(x))]\in \mathrm{M}_{loc}(\Pi).$$

In particular, taking into account the equality $|\gamma_p^s|+|\omega_p(x)|dx=|\gamma_p|$, we obtain the estimates for the measures ζ_p: $|\zeta_p|\le\kappa_p+|\gamma_p|+|\psi(x,u(x))|dx$.

Further, notice that

$$\begin{aligned}\varphi(x,s_{a,b}(u))&=(\varphi(x,a)+\varphi(x,b))/2+\\ &(\mathrm{sign}(u-a)(\varphi(x,u)-\varphi(x,a))-\mathrm{sign}(u-b)(\varphi(x,u)-\varphi(x,b)))/2;\\ B(x,s_{a,b}(u))&=(B(x,a)+B(x,b))/2+(\mathrm{sign}(u-a)(B(x,u)-B(x,a))-\\ &\mathrm{sign}(u-b)(B(x,u)-B(x,b)))/2,\end{aligned}$$

and it follows from (89) that relation (88) holds with $\zeta_{a,b}=(\zeta_a-\zeta_b+\gamma_a+\gamma_b)/2$. Moreover, we have

$$\mathrm{Var}\,\zeta_{a,b}(K)\le C(K,a,b,I)=(A(K,a,I)+A(K,b,I))/2+$$
$$|\gamma_a|(K)+|\gamma_b|(K)+\int_K|\psi(x,u(x))|dx.$$

To complete the proof, it remains to note that for fixed K,a,b the constant $C(K,a,b,I)$ is bounded on bounded sets of $I(x)\in L^1_{loc}(\Omega)$.□

5.1 Proof of Theorem 1.

Taking into account that the sequence $I_k(x)=|\varphi(x,u_k(x))|+|\psi(x,u_k(x))|+|B(x,u_k(x))|$ is bounded in $L^1_{loc}(\Omega)$, we derive from Lemma 9 that for all $a,b\in\mathbb{R}$

$$\mathrm{div}\varphi(x,s_{a,b}(u_k))-D^2\cdot B(x,s_{a,b}(u_k))=\zeta^k_{a,b}\ \text{ in } \mathcal{D}'(\Omega),$$

where $\zeta^k_{a,b}$ is a bounded sequence in $\mathrm{M}_{loc}(\Omega)$. Since $\mathrm{M}_{loc}(\Omega)$ is compactly embedded in $W^{-1}_{d,loc}(\Omega)$ for each $d\in[1,n/(n-1))$ then, taking into account the statement of Proposition 7, we see that condition (86) is satisfied. By our assumption condition

(87) is also satisfied. By Theorem 6 we conclude that some subsequence u_r converges as $r \to \infty$ to a limit function u in $L^1_{loc}(\Omega)$. Extracting a subsequence if necessary, we can assume that $u_r \underset{r\to\infty}{\to} u$ a.e. in Ω. Passing to the limit as $r \to \infty$ in relation (4) with $u = u_r$, we claim that the limit function $u = u(x)$ satisfies this relation for all p such that the level set $u^{-1}(p)$ has zero measure (then $\operatorname{sign}(u_r - p) \to \operatorname{sign}(u - p)$ as $r \to \infty$ a.e. in Ω). Since the set P of such p has full measure and, therefore, is dense, for an arbitrary $p \in \mathbb{R}$ we can choose sequences $p_r^- < p < p_r^+$, $p_r^\pm \in P$, $r \in \mathbb{N}$ convergent to p. Summing relations (4) with $p = p_r^-$ and $p = p_r^+$ and passing to the limit as $r \to \infty$, with the help of the point-wise relation $\operatorname{sign}(u - p_r^-) + \operatorname{sign}(u - p_r^+) \underset{r\to\infty}{\to} 2\operatorname{sign}(u - p)$, we obtain that (4) holds for all $p \in \mathbb{R}$, i.e. $u(x)$ is an entropy solution of (1).$\Box$

5.2 Proof of Theorem 2.

To simplify the notations, we temporarily drop the index m in equation (17), and stress that the flux vector $\tilde{\varphi}(x, u)$ and the diffusion matrix $A(x, u)$ in this equation are smooth.

First we show that a weak solution $u = u(x)$ of equation (17) is an entropy solution in the sense of Definition 1. For this observe that in relation (18) we can choose test functions $f(x) \in W_2^1(\Omega)$, which have compact supports in Ω. In particular, for $\eta(u) \in C^2(\mathbb{R})$, $f = f(x) \in C_0^\infty(\Omega)$ the function $\eta'(u)f$, $u = u(x)$ is an admissible test function, and we derive from (18) that

$$
\begin{aligned}
0 = -\int_\Omega [\tilde{\varphi}(x,u)\nabla\eta'(u)f - A(x,u)\nabla(u)\cdot\nabla\eta'(u)f]\,dx = \\
\int_\Omega [(\operatorname{div}\tilde{\varphi}(x,u))\eta'(u)f + \eta''(u)fA(x,u)\nabla u\cdot\nabla u + \\
A(x,u)\eta'(u)\nabla u\cdot\nabla f]dx.
\end{aligned} \tag{90}
$$

Introduce the vector $\tilde{q}(x, u)$ such that $\tilde{q}'_u(x, u) = \eta'(u)\tilde{\varphi}'_u(x, u)$. This vector is determined by the above equality up to an additive constant $c = c(x)$. We also introduce the symmetric matrix $Q(x, u)$ defined, up to an additive matrix constant $C(x)$, by the equality $Q'_u(x, u) = \eta'(u)A(x, u) = \eta'(u)B'_u(x, u)$.

Now we can transform the terms $(\operatorname{div}\tilde{\varphi}(x, u))\eta'(u)f$, $A(x, u)\eta'(u)\nabla u\cdot\nabla f$ as follows

$$
\begin{aligned}
(\operatorname{div}\tilde{\varphi}(x,u))\eta'(u)f = (\operatorname{div}_x\tilde{\varphi}(x,u) + \tilde{\varphi}'_u(x,u)\cdot\nabla u)\eta'(u)f = \\
(\eta'(u)\operatorname{div}_x\tilde{\varphi}(x,u))f + (\tilde{q}'_u(x,u)\cdot\nabla u)f = \\
f\operatorname{div}\tilde{q}(x,u) + (\eta'(u)\operatorname{div}_x\tilde{\varphi}(x,u) - \operatorname{div}_x\tilde{q}(x,u))f; \\
A(x,u)\eta'(u)\nabla u\cdot\nabla f = Q'_u(x,u)\nabla u\cdot\nabla f = (Q_{ij})'_u(x,u)u_{x_j}f_{x_i} = \\
(Q_{ij}(x,u))_{x_j}f_{x_i} - (Q_{ij})_{x_j}(x,u)f_{x_i}
\end{aligned}
$$

(here Q_{ij}, $i, j = 1, \ldots, n$ being components of the matrix Q). Putting these equalities into (90) and integrating by parts, we obtain that

$$
\int_\Omega [q(x,u)\cdot\nabla f + (\operatorname{div}_x\tilde{q}(x,u) - \eta'(u)\operatorname{div}_x\tilde{\varphi}(x,u))f +
$$

$$Q(x,u)\cdot D^2 f - \eta''(u) f A(x,u)\nabla u\cdot\nabla u\big]\,dx = 0, \quad (91)$$

where $q(x,u)$ is a vector with components $q_i(x,u) = \tilde{q}_i(x,u) + (Q_{ij})_{x_j}(x,u)$. Observe that

$$(q_i)'_u(x,u) = (\tilde{q}_i)'_u(x,u) + \partial_{x_j}(Q_{ij})'_u(x,u) = \eta'(u)(\tilde{\varphi}_i + \partial_{x_j} b_{ij})'_u(x,u) = \eta'(u)(\varphi_i)'_u(x,u),$$

that is, $q'_u(x,u) = \eta'(u)\varphi'_u(x,u)$.

We shall assume that $\eta''(u)$ has a compact support in $\mathbb{R}$. Let $R>0$ be such that $\operatorname{supp}\eta''(u)\subset(-R,R)$ and $L = (\eta'(-R)+\eta'(R))/2$ (evidently, L does not depend on R). Then we can choose $\tilde{q}(x,u)$ in the following way

$$\tilde{q}(x,u) = \frac{1}{2}\int \operatorname{sign}(u-p)(\tilde{\varphi}(x,u)-\tilde{\varphi}(x,p))d\eta'(p) + L\tilde{\varphi}(x,u). \quad (92)$$

Indeed, taking $R>|u|$ and integrating by parts, we obtain the equality

$$\begin{gathered}
\int \operatorname{sign}(u-p)(\tilde{\varphi}(x,u)-\tilde{\varphi}(x,p))d\eta'(p) = \\
\int_{-R}^{R} \operatorname{sign}(u-p)(\tilde{\varphi}(x,u)-\tilde{\varphi}(x,p))d\eta'(p) = \\
\int_{-R}^{u}(\tilde{\varphi}(x,u)-\tilde{\varphi}(x,p))d\eta'(p) - \int_{u}^{R}(\tilde{\varphi}(x,u)-\tilde{\varphi}(x,p))d\eta'(p) = \\
\int_{-R}^{u}\tilde{\varphi}'_u(x,p)\eta'(p)dp - \int_{u}^{R}\tilde{\varphi}'_u(x,p)\eta'(p)dp - \\
2L\tilde{\varphi}(x,u) + \tilde{\varphi}(x,-R)\eta'(-R) + \tilde{\varphi}(x,R)\eta'(R).
\end{gathered}$$

We see that, up to a function which does not depend on u,

$$\begin{gathered}
\frac{1}{2}\int \operatorname{sign}(u-p)(\tilde{\varphi}(x,u)-\tilde{\varphi}(x,p))d\eta'(p) + L\tilde{\varphi}(x,u) = \\
\frac{1}{2}\left(\int_{-R}^{u}\tilde{\varphi}'_u(x,p)\eta'(p)dp - \int_{u}^{R}\tilde{\varphi}'_u(x,p)\eta'(p)dp\right)
\end{gathered}$$

and therefore

$$\frac{\partial}{\partial u}\left(\frac{1}{2}\int \operatorname{sign}(u-p)(\tilde{\varphi}(x,u)-\tilde{\varphi}(x,p))d\eta'(p) + L\tilde{\varphi}(x,u)\right) = \eta'(u)\tilde{\varphi}'_u(x,u),$$

as required. In the similar way we find that, up to an additive matrix constant,

$$Q(x,u) = \frac{1}{2}\int \operatorname{sign}(u-p)(B(x,u)-B(x,p))d\eta'(p) + LB(x,u). \quad (93)$$

It follows from (92), (93) that

$$q(x,u) = \frac{1}{2}\int \operatorname{sign}(u-p)(\varphi(x,u)-\varphi(x,p))d\eta'(p) + L\varphi(x,u). \quad (94)$$

Further, the function $\eta'(u)\mathrm{div}_x\tilde{\varphi}(x,u) - \mathrm{div}_x\tilde{q}(x,u)$ admits the representation

$$\eta'(u)\mathrm{div}_x\tilde{\varphi}(x,u) - \mathrm{div}_x\tilde{q}(x,u) = \frac{1}{2}\int \mathrm{sign}(u-p)\mathrm{div}_x\tilde{\varphi}(x,p)d\eta'(p). \tag{95}$$

Indeed, in view of (92), we see that for sufficiently large R

$$\begin{aligned}2\tilde{q}(x,u) = \int_{-R}^{u}(\tilde{\varphi}(x,u)-\tilde{\varphi}(x,p))d\eta'(p) - \int_{u}^{R}(\tilde{\varphi}(x,u)-\tilde{\varphi}(x,p))d\eta'(p) + 2L\tilde{\varphi}(x,u) = \\ \tilde{\varphi}(x,u)(\eta'(u)-\eta'(-R)) - \int_{-R}^{u}\tilde{\varphi}(x,p)d\eta'(p) - \tilde{\varphi}(x,u)(\eta'(R)-\eta'(u)) + \\ \int_{u}^{R}\tilde{\varphi}(x,p)d\eta'(p) + 2L\tilde{\varphi}(x,u) = 2\eta'(u)\tilde{\varphi}(x,u) - \int \mathrm{sign}(u-p)\tilde{\varphi}(x,p)d\eta'(p),\end{aligned}$$

where we use the equality $2L = \eta'(R)+\eta'(-R)$. Applying the operator div_x to the above equality, we arrive at (95).

Now, we suppose that $\eta''(u) \ge 0$. We transform (91), using equalities (92), (93), (95) and the identity

$$\int_\Omega \{\varphi(x,u)\cdot\nabla f + B(x,u)\cdot D^2 f\}dx = 0, \tag{96}$$

following from (91) with $\eta(u) \equiv u$ (then $\tilde{q}(x,u) = \tilde{\varphi}(x,u)$, $Q(x,u) = B(x,u)$, and $q(x,u) = \varphi(x,u)$). We find that for each $f = f(x) \in C_0^\infty(\Omega)$, $f \ge 0$

$$\begin{aligned}\int\int_\Omega \mathrm{sign}(u-p)\{(\varphi(x,u)-\varphi(x,p))\cdot\nabla f - f\mathrm{div}_x\tilde{\varphi}(x,p) + \\ (B(x,u)-B(x,p))\cdot D^2 f\}\eta''(p)dxdp = 2\int_\Omega \eta''(u) f A(x,u)\nabla u\cdot\nabla u \ge 0\end{aligned}$$

and since $\eta''(p)$ is an arbitrary finite continuous non-negative function on $\mathbb{R}$ we arrive at

$$\begin{aligned}I(p) \doteq \int_\Omega \mathrm{sign}(u-p)\{(\varphi(x,u)-\varphi(x,p))\cdot\nabla f - f\mathrm{div}_x\tilde{\varphi}(x,p) + \\ (B(x,u)-B(x,p))\cdot D^2 f)\}dx \ge 0\end{aligned} \tag{97}$$

for all $p \in P$, where the set P consists of points p such that the level set $u^{-1}(p)$ has null Lebesgue measure. We use the fact that the function $I(p)$ is continuous at any point of P. In view of (97) for all $p \in P$

$$\begin{aligned}\mathrm{div}[\mathrm{sign}(u-p)(\varphi(x,u)-\varphi(x,p))] + \\ \mathrm{sign}(u-p)\mathrm{div}_x\tilde{\varphi}(x,p) - D^2\cdot[\mathrm{sign}(u-p)(B(x,u)-B(x,p))] \le 0\end{aligned} \tag{98}$$

in $\mathcal{D}'(\Omega)$. Since the set P has full measure and therefore is dense, for an arbitrary $p \in \mathbb{R}$ we can choose sequences $p_r^- < p < p_r^+$, $p_r^\pm \in P$, $r \in \mathbb{N}$ convergent to p. Taking a sum of relations (98) with $p = p_r^-$ and $p = p_r^+$ and passing to the limit as $r \to \infty$, in view of the point-wise relation $\operatorname{sign}(u - p_r^-) + \operatorname{sign}(u - p_r^+) \underset{r\to\infty}{\to} 2\operatorname{sign}(u - p)$, we obtain that (98) holds for all $p \in \mathbb{R}$. Taking into account that

$$\operatorname{div}_x\tilde{\varphi}(x,p) = \operatorname{div}_x\varphi(x,p) - D_x^2 \cdot B(x,p),$$

we conclude that $u(x)$ is an entropy solution of (17).

We also need a-priori estimate of ∇u. Choose $M \ge \|u\|_\infty$ and a function $\eta(u) \in C_0^2(\mathbb{R})$ such that $\eta(u) = u^2/2$ on the segment $[-M, M]$ and $\operatorname{supp}\eta(u) \in [-M-1, M+1]$. Then for $u = u(x)$ $\eta''(u) = 1$ a.e. in Ω and we derive from (91) that for each $f = f(x) \in C_0^\infty(\Omega)$, $f \ge 0$

$$\int_\Omega fA(x,u)\nabla u \cdot \nabla u dx \le \left| \int_\Omega \left[q(x,u) \cdot \nabla f + (\operatorname{div}_x\tilde{q}(x,u) - \eta'(u)\operatorname{div}_x\tilde{\varphi}(x,u))f + Q(x,u) \cdot D^2 f \right] dx \right|. \quad (99)$$

It follows from (94), (93), (95) that

$$|q(x,u)| \le C \max_{|u|\le M+1} |\varphi(x,u)|, \quad |Q(x,u)| \le C \max_{|u|\le M+1} |B(x,u)|,$$
$$|\operatorname{div}_x\tilde{q}(x,u) - \eta'(u)\operatorname{div}_x\tilde{\varphi}(x,u)| \le C \int_{-M-1}^{M+1} |\operatorname{div}_x\tilde{\varphi}(x,p)| dp,$$

where C is the constant depending only on the fixed function η. Putting these estimates into (99), we get

$$\int_\Omega fA(x,u)\nabla u \cdot \nabla u dx \le C \int_\Omega \{ \max_{|u|\le M+1} |\varphi(x,u)||\nabla f| + \max_{|u|\le M+1} |B(x,u)||D^2 f| \} dx + C \int_\Omega \int_{-M-1}^{M+1} |\operatorname{div}_x\tilde{\varphi}(x,p)| f(x) dp dx. \quad (100)$$

Now we recall that $\varphi(x,u) = \varphi_m(x,u)$, $B(x,u) = B_m(x,u)$, $m \in \mathbb{N}$. By our assumptions these sequences converge as $m \to \infty$ in $L^2_{loc}(\Omega, C(\mathbb{R}))$ and in $C^1(\mathbb{R}, Sym_n)$, respectively. Therefore, the sequence

$$\int_\Omega \{ \max_{|u|\le M+1} |\varphi_m(x,u)||\nabla f| + \max_{|u|\le M+1} |B_m(x,u)||D^2 f| \} dx$$

is bounded by a constant depending only on f. Here we take $M \ge \sup_m \|u_m\|_\infty$. It follows from estimate (100) and the condition $A_m \ge \varepsilon_m E$ that

$$\varepsilon_m \int_\Omega |\nabla u_m|^2 f(x) dx \le \int_\Omega fA_m(x,u_m)\nabla u_m \cdot \nabla u_m dx \le C_f I_m(K, M+1), \quad (101)$$

with $K = \operatorname{supp} f$, where the sequence

$$I_m(K,M) = 1 + \int_K \int_{-M}^{M} |\mathrm{div}_x \tilde{\varphi}_m(x,p)| dp dx =$$
$$1 + \int_K \int_{-M}^{M} |\mathrm{div}_x \varphi_m(x,p) - D_x^2 \cdot B_m(x,p)| dp dx$$

was mentioned in Introduction. It follows from (101) and condition (16) that

$$(\varepsilon_m)^2 \int_\Omega |\nabla u_m|^2 f(x) dx \le C_f \varepsilon_m I_m(K, M+1) \underset{m\to\infty}{\to} 0 \tag{102}$$

for all M, K and f.

Now we take $a, b \in \mathbb{R}$, $a < b$. Let us demonstrate that the sequence

$$L_m = \mathrm{div}\varphi(x, s_{a,b}(u_m)) - D^2 \cdot B(x, s_{a,b}(u_m))$$

is pre-compact in $W_{d,loc}^{-1,-2}$ with some $d > 1$. For that, recall that $u_m(x)$ is an entropy solution of (17) and by Lemma 9 (also see the proof of this Lemma)

$$\mathrm{div}\varphi_m(x, s_{a,b}(u_m)) - D^2 \cdot B_m(x, s_{a,b}(u_m)) = \xi_m$$

where ξ_m is a bounded sequence in the space $\mathrm{M}_{loc}(\Omega)$, which is compactly embedded in $W_{d,loc}^{-1}(\Omega)$ for each $d \in [1, n/(n-1))$. Further, we have $L_m = L_{1m} + L_{2m} + \xi_m$, where

$$L_{1m} = \mathrm{div}(\varphi(x, s_{a,b}(u_m)) - \varphi_m(x, s_{a,b}(u_m))),$$
$$L_{2m} = D^2 \cdot (B_m(x, s_{a,b}(u_m)) - B(x, s_{a,b}(u_m)))\,.$$

In view of the estimate

$$|\varphi(x, s_{a,b}(u_m)) - \varphi_m(x, s_{a,b}(u_m))| \le \max_{|u|\le M} |\varphi_m(x,u) - \varphi(x,u)|$$

and the condition $\varphi_m(x,u) \underset{m\to\infty}{\to} \varphi(x,u)$ in $L^2_{loc}(\Omega, C(\mathbb{R}, \mathbb{R}^m))$ we have

$$\varphi(x, s_{a,b}(u_m)) - \varphi_m(x, s_{a,b}(u_m)) \underset{m\to\infty}{\to} 0 \ \text{ in } L^2_{loc}(\Omega).$$

Hence $L_{1m} \to 0$ in $W_{2,loc}^{-1}(\Omega)$. Concerning the sequence L_{2m}, we represent it as follows

$$\begin{aligned} L_{2m} = D^2 \cdot (B_m(x, s_{a,b}(u_m)) - B_m(x,p) - B(x, s_{a,b}(u_m)) + B(x,p)) - \\ (\mathrm{div}_x \varphi_m(x,p) - D^2 \cdot B_m(x,p)) + (\mathrm{div}_x \varphi(x,p) - D^2 \cdot B(x,p)) + \\ \mathrm{div}_x(\varphi_m(x,p) - \varphi(x,p)) = D^2 \cdot R_m - \gamma_p^m + \gamma_p + \mathrm{div}_x(\varphi_m(x,p) - \varphi(x,p)) \end{aligned} \tag{103}$$

where p is some fixed value, $R_m = R_m(x) = B_m(x, s_{a,b}(u_m)) - B_m(x,p) - B(x, s_{a,b}(u_m)) + B(x,p) \underset{m\to\infty}{\to} 0$ in $L^2_{loc}(\Omega, C(\mathbb{R}, Sym_n))$. The latter implies that for each $\Phi(x) \in C_0^\infty(\Omega)$ $\Phi(x)R_m(x) \underset{m\to\infty}{\to} 0$ in $L^2(\Omega, C(\mathbb{R}, Sym_n))$. Observe that by the structure of our approximations $(B_m(x,u) - B_m(x,p) - \varepsilon_m(u-p)E)\xi = 0$ for all $\xi \in X$. Therefore, $(R_m(x) - \varepsilon_m(s_{a,b}(u_m) - p)E)\tilde{\xi} = 0$ for $\tilde{\xi} = P_1\xi$, $\xi \in \mathbb{R}^n$, and the matrix $H_m(x) = R_m(x) - \varepsilon_m(s_{a,b}(u_m) - p)E$ satisfies the property

$$F(\Phi H_m)(\xi)\xi\cdot\xi = \int_{\mathbb{R}^n} e^{-2\pi i\xi\cdot x}\Phi(x)H_m(x)\xi\cdot\xi dx =$$
$$\int_{\mathbb{R}^n} e^{-2\pi i\xi\cdot x}\Phi(x)H_m(x)\bar{\xi}\cdot\bar{\xi}dx = F(\Phi H_m)(\xi)\bar{\xi}\cdot\bar{\xi}.$$

This implies that

$$\left|(1+|\tilde{\xi}|^2+|\bar{\xi}|^4)^{-1/2}F(D^2\cdot\Phi H_m)(\xi)\right| =$$
$$4\pi^2\left|(1+|\tilde{\xi}|^2+|\bar{\xi}|^4)^{-1/2}F(\Phi H_m)(\xi)\bar{\xi}\cdot\bar{\xi}\right| \le |F(\Phi H_m)(\xi)|$$

and by Plancherel's equality

$$\|(1+|\tilde{\xi}|^2+|\bar{\xi}|^4)^{-1/2}F(D^2\cdot\Phi H_m)(\xi)\|_2 \le \|F(\Phi H_m)(\xi)\|_2 = \|\Phi H_m\|_2 \underset{m\to\infty}{\to} 0$$

in $L^2(\mathbb{R}^n)$. This means that

$$D^2\cdot\Phi H_m \underset{m\to\infty}{\to} 0 \ \text{ in } W_2^{-1,-2}. \tag{104}$$

Since
$$\Phi D^2\cdot H_m = D^2\cdot\Phi H_m - 2\mathrm{div}(H_m\nabla\Phi) + H_m\cdot D^2\Phi \ \text{ in } \mathcal{D}'(\mathbb{R}^n)$$
and, evidently, the sequences $\mathrm{div}(H_m\nabla\Phi)$, $H_m\cdot D^2\Phi$ converges to zero as $m\to\infty$ in $W_2^{-1}\subset W_2^{-1,-2}$ then by (104) we claim that $\Phi D^2\cdot H_m \underset{m\to\infty}{\to} 0$ in $W_2^{-1,-2}$. Here $\Phi(x)\in C_0^\infty(\Omega)$ is arbitrary and therefore

$$D^2\cdot H_m \underset{m\to\infty}{\to} 0 \ \text{ in } W_{2,loc}^{-1,-2}(\Omega). \tag{105}$$

Further, using the chain rule, we find

$$\varepsilon_m D^2\cdot(s_{a,b}(u_m)-p)E = \varepsilon_m\mathrm{div}\nabla s_{a,b}(u_m) = \varepsilon_m\mathrm{div}[\chi(u_m)\nabla u_m],$$

$\chi(u)$ is the indicator function of the segment $[a,b]$. In view of estimate (102) $\varepsilon_m\nabla u_m \to 0$ in $L^2_{loc}(\Omega)$ as $m\to\infty$. Hence

$$\varepsilon_m D^2\cdot(s_{a,b}(u_m)-p)E \underset{m\to\infty}{\to} 0 \ \text{ in } W_{2,loc}^{-1}(\Omega)\subset W_{2,loc}^{-1,-2}(\Omega).$$

This and (105) imply that

$$D^2\cdot R_m \underset{m\to\infty}{\to} 0 \ \text{ in } W_{2,loc}^{-1,-2}(\Omega). \tag{106}$$

It is clear that

$$\mathrm{div}_x(\varphi_m(x,p)-\varphi(x,p)) \underset{m\to\infty}{\to} 0 \ \text{ in } W_{2,loc}^{-1}(\Omega)\subset W_{2,loc}^{-1,-2}(\Omega). \tag{107}$$

Finally, by (7) the sequence $\gamma_p^m - \gamma_p \to 0$ weakly in $\mathrm{M}_{loc}(\Omega)$. Therefore, for $d \in (1, n/(n-1))$

$$\gamma_p^m - \gamma_p \underset{m\to\infty}{\to} 0 \text{ in } W_{d,loc}^{-1}(\Omega) \subset W_{d,loc}^{-1,-2}(\Omega). \tag{108}$$

In view of (106), (107), (108) we derive from representation (103) that $L_{2m} \to 0$ as $m \to \infty$ in $W_{d,loc}^{-1,-2}(\Omega)$ for some $d > 1$. As was demonstrated above, the same limit relations fulfill for sequences L_{1m}, ξ_m and we conclude that $L_m = L_{1m} + L_{2m} + \xi_m$ is pre-compact in $W_{d,loc}^{-1,-2}(\Omega)$ with some $d > 1$. Hence, assumption (86) is satisfied. By Corollary 2 we see that the sequence u_m converges in $L_{loc}^1(\Omega)$ to some function $u = u(x) \in L^\infty(\Omega)$. Obviously, $\|u\|_\infty \le M$. It only remains to demonstrate that u is an entropy and distributional solution of (1). By relation (97) for each $p \in \mathbb{R}$, $f = f(x) \in C_0^\infty(\Omega)$, $f \ge 0$

$$\int_\Omega \operatorname{sign}(u_m - p)\{(\varphi_m(x, u_m) - \varphi_m(x, p)) \cdot \nabla f - f\gamma_p^m(x) +$$
$$(B_m(x, u_m) - B_m(x, p)) \cdot D^2 f\} dx \ge 0$$

where

$$\gamma_p^m(x) = \operatorname{div}_x \tilde{\varphi}_m(x, p) = \operatorname{div}_x \varphi_m(x, p) - D^2 \cdot B_m(x, p).$$

Since $\gamma_p^m(x) = \gamma_{pr}^m(x) + \bar{\gamma}_{ps}^m(x)$ (see Introduction) the above relation implies that

$$\int_\Omega \{\operatorname{sign}(u_m - p)[(\varphi_m(x, u_m) - \varphi_m(x, p)) \cdot \nabla f - f\gamma_{pr}^m(x) +$$
$$(B_m(x, u_m) - B_m(x, p)) \cdot D^2 f] + f|\bar{\gamma}_{ps}^m(x)|\} dx \ge 0. \tag{109}$$

Passing to a subsequence, we may assume that $u_m(x) \to u(x)$ as $m \to \infty$ a.e. in Ω. Then, in view of (12), (13),

$$\operatorname{sign}(u_m - p)(\varphi_m(x, u_m) - \varphi_m(x, p)) \underset{m\to\infty}{\to} \operatorname{sign}(u - p)(\varphi(x, u) - \varphi(x, p)),$$
$$\operatorname{sign}(u_m - p)(B_m(x, u_m) - B_m(x, p)) \underset{m\to\infty}{\to} \operatorname{sign}(u - p)(B(x, u) - B(x, p)),$$
$$\operatorname{sign}(u_m - p) \underset{m\to\infty}{\to} \operatorname{sign}(u - p)$$

a.e. in Ω and, as a consequence, in $L_{loc}^1(\Omega)$. The latter relation holds for $p \in \mathbb{R}$ such that the level set $u^{-1}(p)$ has zero Lebesgue measure. Besides, by our assumptions (see relations (14), (15)) $\gamma_{pr}^m(x) \underset{m\to\infty}{\to} \omega_p(x)$ in $L_{loc}^1(\Omega)$, $\overline{\lim\limits_{m\to\infty}} \int_\Omega f(x)|\bar{\gamma}_{mp}^s(x)|dx \le \int_\Omega f(x)d|\gamma_p^s|(x)$. Taking into account the above limit relations, we can pass to the limit in (109) and obtain that

$$\int_\Omega \operatorname{sign}(u - p)\{(\varphi(x, u_m) - \varphi(x, p)) \cdot \nabla f - f\omega_p(x) +$$
$$(B(x, u) - B(x, p)) \cdot D^2 f\} dx + \int_\Omega f(x) d|\gamma_p^s|(x) \ge 0 \tag{110}$$

for all $p \in \mathbb{R}$ such that the level set $u^{-1}(p)$ has zero Lebesgue measure. Repeating the arguments concluding the proof of Theorem 1, we obtain that (110) holds for all $p \in \mathbb{R}$, i.e. $u(x)$ is an entropy solution of (17). Finally, passing to the limit as $m \to \infty$ in relation (96)

$$\int_\Omega \{\varphi_m(x,u_m) \cdot \nabla f + B_m(x,u_m) \cdot D^2 f\} dx = 0,$$

we obtain that for all $f = f(x) \in C_0^\infty(\Omega)$

$$\int_\Omega \{\varphi(x,u) \cdot \nabla f + B(x,u) \cdot D^2 f\} dx = 0.$$

Hence, $u = u(x)$ is a distributional solution of (1). This completes the proof of Theorem 2. $\square$

Remark in conclusion that the strong pre-compactness property for equations of Graetz-Nusselt type

$$\operatorname{div}(\varphi(x,u) - A(x)\nabla g(u)) + \psi(x,u) = 0$$

was studied in [18, 17]. In particular, Theorems 1,2 was proved in [17] for such the equation under the less restrictive non-degeneracy requirement:

for a.e. $x \in \Omega$ for all $\xi \in \mathbb{R}^n$, $\xi \neq 0$, the functions $u \to \xi \cdot \varphi(x,u)$, $u \to g(u)A(x)\xi \cdot \xi$ are not constant **simultaneously** *on non-degenerate intervals.*

References

[1] N. Antonić, M. Lazar, H-measures and variants applied to parabolic equations, J. Math. Anal. Appl. 343(2008) 207–225.

[2] J. Bergh, J. Löfström, Interpolation spaces. An introduction, Springer-Verlag, Berlin, 1976.

[3] B. Dacorogna, Weak continuity and weak lower semicontinuity of nonlinear functionals, Springer-Verlag, Berlin, 1982.

[4] R.J. DiPerna, Measure-valued solutions to conservation laws, Arch. Ration. Mech. Anal. 88(1985) 223–270.

[5] P. Gerárd, Microlocal defect measures, Comm. Partial Diff. Equat. 16(1991) 1761–1794.

[6] E. Hille, R.S. Phillips, Functional analysis and semi-groups, Amer. Math. Soc., Providence, RI, 1957.

[7] K.H. Karlsen, N.H. Risebro, J.D. Towers, L^1 stability for entropy solutions of nonlinear degenerate parabolic connection-diffusion equations with discontinuous coefficients, Skr. K. Nor. Vid. Selsk. 3(2003) 1–14.

[8] K.H. Karlsen, J.D. Towers, Convergence of the Lax-Friedrichs scheme and stability for conservation laws with a discontionuous space-time dependent flux, Chin. Ann. Math. 25B:3(2004) 287–318.

[9] K.H. Karlsen, M. Rascle, E. Tadmor, On the existence and compactness of a two-dimensional resonant system of conservation laws, Commun. Math. Sci. 5:2(2007) 253–265.

[10] S.N. Kruzhkov, First order quasilinear equations in several independent variables, Mat. Sb., 81:2(1970) 228–255; English transl. in Math. USSR Sb. 10:2(1970) 217–243.

[11] O.A. Ladyzhenskaya, N.N. Ural'tseva, Linear and quasilinear elliptic equations, 2-nd ed., Nauka, Moscow, 1973.

[12] P.L. Lions, B. Perthame, E. Tadmor, A kinetic formulation of multidimensional scalar conservation laws and related equations, J. Amer. Math. Soc. 7:1(1994) 169–191.

[13] E.Yu. Panov, On sequences of measure-valued solutions of first-order quasilinear equations, Mat. Sb. 185:2(1994) 87–106; English transl. in Russian Acad. Sci. Sb. Math. 81:1(1995) 211–227.

[14] E.Yu. Panov, On strong precompactness of bounded sets of measure valued solutions for a first order quasilinear equation, Mat. Sb. 186:5(1995) 103–114; Engl. transl. in Russian Acad. Sci. Sb. Math. 186:5(1995) 729–740.

[15] E.Yu. Panov, Property of strong precompactness for bounded sets of measure valued solutions of a first-order quasilinear equation, Mat. Sb. 190:3(1999) 109–128; Engl. transl. in Russian Acad. Sci. Sb. Math. 190:3(1999) 427–446.

[16] E.Yu. Panov, Existence and strong pre-compactness properties for entropy solutions of a first-order quasilinear equation with discontinuous flux, Arch. Ration. Mech. Anal. 2009. Published online: 24 February 2009

[17] E.Yu. Panov, On the strong pre-compactness property for entropy solutions of an ultra-parabolic equation with discontinuous flux [preprint], cited on 12 July 2008, http://www.math.ntnu.no/conservation/2008/022.html

[18] S.A. Sazhenkov, The genuinely nonlinear Graetz-Nusselt ultraparabolic equation, Sibirsk. Mat. Zh. 47:2(2006) 431–454; Engl. transl. in. Siberian Math. J. 47:2(2006) 355–375.

[19] E.M. Stein, Singular integrals and differentiability properties of functions, Princeton Mathematical Series, No. 30, Princeton University Press, Princeton, N.J., 1970.

[20] L. Tartar, Compensated compactness and applications to partial differential equations, in: Nonlinear analysis and mechanics: Heriot. Watt Symposium, vol. 4 (Edinburgh 1979), Res. Notes in Math. 39(1979) 136–212.

[21] L. Tartar, H-measures, a new approach for studying homogenisation, oscillations and concentration effects in partial differential equations, Proc. Roy. Soc. Edinburgh Sect. A 115:3-4(1990) 193–230.