

# SYSTEMS OF HYPERBOLIC CONSERVATION LAWS WITH PRESCRIBED EIGENCURVES

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ABSTRACT. We study the problem of constructing systems of hyperbolic conservation laws in one space dimension with prescribed eigencurves, i.e. the eigenvector fields of the Jacobian of the flux are given. We formulate this as a typically overdetermined system of equations for the eigenvalues-to-be. Equivalent formulations in terms of differential and algebraic-differential equations are considered. The resulting equations are then analyzed using appropriate integrability theorems (Frobenius, Darboux and Cartan-Kähler). We give a complete analysis of the possible scenarios, including examples, for systems of three equations. As an application we characterize conservative systems with the same eigencurves as the Euler system for 1-dimensional compressible gas dynamics. The case of general rich systems of any size (i.e. when the given eigenvector fields are pairwise in involution; this includes all systems of two equations) is completely resolved and we consider various examples in this class.

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## 1. INTRODUCTION

Consider a system of  $n$  conservation laws in one space dimension written in canonical form,

$$(1.1) \quad u_t + f(u)_x = 0.$$

Here the unknown state  $u = u(t, x) \in \mathbb{R}^n$  is assumed to vary over some open subset  $\Omega \subset \mathbb{R}^n$  and the flux  $f = (f^1, \dots, f^n)^T$  is a nonlinear map from  $\Omega$  into  $\mathbb{R}^n$ . The eigenvalues and eigenvectors of the Jacobian matrix  $Df(u)$  provide information that is used to solve the Cauchy problem for (1.1). In particular, the geometric properties of the integral curves of eigenvector fields of  $Df$  play a key role. Together with the so-called Hugoniot curves (see below for definition) these form *wave curves* in  $u$ -space that are used to build solutions to (1.1).

There is a well developed theory for the Cauchy problem for a large class of systems (1.1) in the near equilibrium regime  $u(0, x) \approx \text{constant}$ . Glimm [17] established global-in-time existence of weak solutions for data with sufficiently small total variation. By now there are several methods available to obtain these solutions: Glimm's original random choice method [17], Liu's deterministic version [29], the wave-front tracking schemes of DiPerna [14], Bressan [5] and Risebro [33], and the wave tracing methods of Bianchini and Bressan [4], [2], [3].

Single equations and systems of two equations enjoy particular properties that yield global existence beyond the perturbative regime, [26], [18], [30], [31], [40], [15], [8]. On the other hand there is no general existence result available for "large" data when the system has three or more equations (e.g. the Euler system for compressible gas dynamics). Indeed, examples of blowup in total variation and/or sup norm are known even for genuinely nonlinear and strictly hyperbolic systems, [24],

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[44], [21], [22], [1]. These examples show that the global behavior of the wave curves in  $u$ -space is related to finite time blow up, [44], [21], [22], [1]. It is of interest to have more examples of this type. In particular one would like to know if a physical system can display similar behavior<sup>1</sup>. In searching for systems whose wave curves have special properties one is naturally lead to ask what freedom one has in *prescribing* such curves.

In this paper we consider the situation where we are given a frame of  $n$  linearly independent vector fields and their integral curves. We then ask if there are any systems of the form (1.1) with the property that the given vector fields are the eigenvector fields of  $Df(u)$ .

It turns out that there is always a one-parameter family of trivial solutions, and that there are cases where no non-trivial solutions exist. More generally we are interested in knowing “how many” solutions there are, in terms of how many constants and functions appear in a general solution. The basic tool for this sort of questions is provided by the integrability theorems for overdetermined systems of PDEs such as Frobenius, Darboux and Cartan-Kähler theorems. The latter requires re-writing the system of PDE’s as an exterior differential systems (EDSs) [7], [20].

The case of  $2 \times 2$ -systems was treated in [11]; it is also covered by the analysis of rich systems in Section 4. On the other hand we shall see that already for systems with three equations there are several possible scenarios. Before giving a precise formulation of the problem we review some relevant background material.

**Notation:** We denote the  $(i, j)$ -entry (i.e., the element in the  $i$ th row and the  $j$ th column) of an  $m \times n$ -matrix  $A$  by  $A_j^i$ . Superscript  $T$  denotes transpose. Summation convention is *not* used.

**1.1. Conservation laws in one space dimension.** We consider *hyperbolic* systems of conservation laws in one spatial dimension (1.1), i.e. the Jacobian matrix  $Df(u)$  is diagonalizable over  $\mathbb{R}$  at each state  $u \in \Omega$ . The system is *strictly* hyperbolic in  $\Omega$  provided the eigenvalues  $\lambda^i(u)$  of  $Df(u)$  are real and distinct:

$$(1.2) \quad \lambda^1(u) < \dots < \lambda^n(u), \quad \forall u \in \Omega.$$

Let us for now fix a choice of the associated right and left eigenvectors  $R_i(u)$  and  $L^i(u)$  of  $Df(u)$ . These are considered as column and row vectors of functions, respectively, and we write

$$R_i(u) = [R_i^1(u), \dots, R_i^n(u)]^T, \quad L^i(u) = [L_1^i(u), \dots, L_n^i(u)].$$

We refer to the  $R_i(u)$  as the *eigenfields* and their integral curves in  $u$ -space as *eigencurves*. Diagonalizing  $Df$  we have

$$(1.3) \quad Df(u) = R(u) \Lambda(u) L(u),$$

where

$$R(u) = [R_1(u) \mid \dots \mid R_n(u)], \quad \Lambda(u) = \text{diag}[\lambda^1(u) \dots \lambda^n(u)],$$

and

$$L(u) = R(u)^{-1} = \begin{bmatrix} L^1(u) \\ \vdots \\ L^n(u) \end{bmatrix}.$$

Note that in setting  $L(u) = R(u)^{-1}$  we have introduced the normalization  $R_i(u) \cdot L^j(u) = \delta_i^j$  (Kronecker delta).

Next consider the initial value problem for (1.1) where the data consist of two constant states separated by a jump at  $x = 0$ ,

$$(1.4) \quad u_0(x) = \begin{cases} u_-, & x < 0 \\ u_+, & x > 0. \end{cases}$$

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<sup>1</sup>This refers to the strictly hyperbolic regime. It is well-known that failure of strict hyperbolicity can cause singular behavior (blowup in total variation or  $L^\infty$ ), even in physical systems. See [41], [37] and references therein.

This is the so-called Riemann problem and its solution serves as a building block for more general solutions, [27], [17]. For a sufficiently smooth flux  $f$  whose characteristic fields are genuinely nonlinear or linearly degenerate in the sense of Lax [27], it is well known that through every strictly hyperbolic state  $\bar{u}$  there exist  $n$  locally defined and  $C^2$  smooth *wave curves*. These curves collectively provide self-similar solutions to Riemann problems; for details see [6], [12], [38]. Each wave curve is locally made up of two components with second order contact at the base point  $\bar{u}$ : the rarefaction states that are part of the eigencurves, and the shock states that are part of the Hugoniot locus  $\{u \in \Omega \mid \exists s \in \mathbb{R} : f(u) - f(\bar{u}) = s(u - \bar{u})\}$ . The geometry of these curves in state space thus provide information about the solutions to (1.1).

**1.2. Connections on frame bundles.** Given an  $n$ -dimensional smooth manifold  $M$  we let  $\mathcal{X}(M)$  and  $\mathcal{X}^*(M)$  denote the set of smooth vector fields and differential 1-forms on  $M$ , respectively. A *frame*  $\{r_1, \dots, r_n\}$  is a set of vector fields which span the tangent space  $T_p M$  at each point  $p \in M$ . A *coframe*  $\{\ell^1, \dots, \ell^n\}$  is a set of  $n$  differential 1-forms which span the cotangent space  $T_p^* M$  at each point  $p \in M$ . The coframe and frame are *dual* if  $\ell^i(r_j) = \delta_j^i$  (Kronecker delta). If  $u^1, \dots, u^n$  are local coordinate functions on  $M$ , then  $\{\frac{\partial}{\partial u^1}, \dots, \frac{\partial}{\partial u^n}\}$  is the corresponding local *coordinate frame*, while  $\{du^1, \dots, du^n\}$  is the dual local *coordinate coframe*. For a given frame  $\{r_1, \dots, r_n\}$  the *structure coefficients*  $c_{ij}^k$  are defined through

$$(1.5) \quad [r_i, r_j] = \sum_{k=1}^n c_{ij}^k r_k,$$

and the dual coframe has related structure equations given by

$$(1.6) \quad d\ell^k = - \sum_{i < j} c_{ij}^k \ell^i \wedge \ell^j.$$

It can be shown that there exist coordinate functions  $w^1, \dots, w^n$  on  $\Omega$  such that  $r_i = \frac{\partial}{\partial w^i}$ ,  $i = 1, \dots, n$ , if and only if  $r_1, \dots, r_n$  commute, i.e. all structure coefficients are zero. Next, an *affine connection*  $\nabla$  on  $M$  is an  $\mathbb{R}$ -bilinear map

$$\mathcal{X}(M) \times \mathcal{X}(M) \rightarrow \mathcal{X}(M) \quad (X, Y) \mapsto \nabla_X Y$$

such that for any smooth function  $f$  on  $M$

$$(1.7) \quad \nabla_{fX} Y = f \nabla_X Y, \quad \nabla_X (fY) = (Xf)Y + f \nabla_X Y.$$

By  $\mathbb{R}$ -bilinearity and (1.7) the connection is uniquely defined by prescribing it on a frame:

$$\nabla_{r_i} r_j = \sum_{k=1}^n \Gamma_{ij}^k r_k,$$

where the smooth coefficients  $\Gamma_{ij}^k$  are called *connection components*, or *Christoffel symbols*, relative to the frame  $\{r_1, \dots, r_n\}$ . Any choice of a frame and  $n^3$  functions  $\Gamma_{ij}^k$ ,  $i, j, k = 1, \dots, n$ , defines an affine connection on  $M$ . A change of frame induces a change of the connection components, and this change is not tensorial. E.g., a connection with zero components relative to a coordinate frame, may have non-zero components relative to a non-coordinate frame.

Given a frame  $\{r_1, \dots, r_n\}$  with associated Christoffel symbols  $\Gamma_{ij}^k$  and dual frame  $\{\ell^1, \dots, \ell^n\}$ , we define the *connection 1-forms*  $\mu_i^j$  by

$$(1.8) \quad \mu_i^j := \sum_{k=1}^n \Gamma_{ki}^j \ell^k.$$

In turn, these are used to define two important tensor-fields: the *torsion* 2-forms

$$(1.9) \quad \mathbf{T}^i := d\ell^i + \sum_{k=1}^n \mu_k^i \wedge \ell^k = \sum_{k < m} T_{km}^i \ell^k \wedge \ell^m, \quad i = 1, \dots, n,$$

and the *curvature* 2-forms

$$(1.10) \quad \mathbf{R}_i^j := d\mu_i^j + \sum_{k=1}^n \mu_k^j \wedge \mu_i^k = \sum_{k < m} R_{ikm}^j \ell^k \wedge \ell^m.$$

Here

$$(1.11) \quad T_{km}^i = \Gamma_{km}^i - \Gamma_{mk}^i - c_{km}^i$$

$$(1.12) \quad R_{ikm}^j = r_k(\Gamma_{mi}^j) - r_m(\Gamma_{ki}^j) + \sum_{s=1}^n (\Gamma_{ks}^j \Gamma_{mi}^s - \Gamma_{ms}^j \Gamma_{ki}^s - c_{km}^s \Gamma_{si}^j)$$

are components of the torsion and curvature tensors respectively, and these *do* change tensorially under a change of frame. We can write equations (1.9) and (1.10) in the compact matrix form

$$(1.13) \quad \mathbf{T} = d\ell + \mu \wedge \ell, \quad \mathbf{R} = d\mu + \mu \wedge \mu$$

where  $\ell = (\ell^1, \dots, \ell^n)^T$ ,  $\mathbf{T} = (\mathbf{T}^1, \dots, \mathbf{T}^n)^T$ , and  $\mathbf{R}$  and  $\mu$  are the matrices with components  $\mathbf{R}_i^j$  and  $\mu_i^j$  respectively. The connection is called *symmetric* if the torsion form is identically zero and it is called *flat* if the curvature form is identically zero. Equivalently:

$$(1.14) \quad d\ell = -\mu \wedge \ell \quad (\text{Symmetry}), \quad d\mu = -\mu \wedge \mu \quad (\text{Flatness}).$$

In terms of Christoffel symbols and structure coefficients this is equivalent to

$$(1.15) \quad c_{km}^i = \Gamma_{km}^i - \Gamma_{mk}^i \quad (\text{Symmetry})$$

and

$$(1.16) \quad r_m(\Gamma_{ki}^j) - r_k(\Gamma_{mi}^j) = \sum_{s=1}^n (\Gamma_{ks}^j \Gamma_{mi}^s - \Gamma_{ms}^j \Gamma_{ki}^s - c_{km}^s \Gamma_{si}^j) \quad (\text{Flatness}).$$

One can also show that a connection  $\nabla$  is symmetric and flat if and only if in a neighborhood of each point there exist coordinate functions  $u^1, \dots, u^n$  with the property that the Christoffel symbols relative to the coordinate frame are zero:  $\nabla_{\frac{\partial}{\partial u^i}} \frac{\partial}{\partial u^j} = 0$  for all  $i, j = 1, \dots, n$ .

**1.3. Formulation of the problem.** Returning to (1.1) we next provide a precise statement of the “inverse” problem of constructing flux functions  $f$  whose geometric properties are given. There are various ways to formulate such problems. One could prescribe families of curves which are then required to be the Hugoniot loci, the eigencurves, or the wave curves for a system of conservation laws (1.1). One might also consider giving combinations of these. The most direct formulation is obtained by prescribing the eigenfields (equivalently, the eigencurves), and this is what we do here.

We will be working locally near a fixed base point  $\bar{u}$  in an open set  $\Omega \subset \mathbb{R}^n$ , with  $\Omega$  smoothly contractible to a point. Throughout  $u = (u^1, \dots, u^n)$  will denote a fixed system of coordinates on a neighborhood of  $\bar{u}$ . We assume that we are given  $n$  linearly independent column  $n$ -vectors  $R_i(u)$  (the eigenfields to be) on  $\Omega$ , and we define

$$(1.17) \quad R(u) := [R_1(u) \mid \dots \mid R_n(u)] = (R_j^i(u))_{i,j}, \quad L(u) := R(u)^{-1} =: \begin{bmatrix} L^1(u) \\ \vdots \\ L^n(u) \end{bmatrix} = (L_j^i(u))_{i,j}.$$

The problem we consider may be formulated as follows:

**Problem 1.** Consider an open subset  $\Omega \subset \mathbb{R}^n$  equipped with a coordinate system  $u = (u^1, \dots, u^n)$ . Given a point  $\bar{u} \in \Omega$  together with a (sufficiently smooth) frame  $\{R_1(u), \dots, R_n(u)\}$  on  $\Omega$ .

Then: find  $n$  real functions  $\lambda^1(u), \dots, \lambda^n(u)$  defined on a neighborhood  $\mathcal{U} \subset \Omega$  of  $\bar{u}$  such that, with  $\Lambda(u) := \text{diag}[\lambda^1(u) \dots \lambda^n(u)]$ , the matrix

$$(1.18) \quad A(u) := R(u)\Lambda(u)L(u)$$

is the Jacobian matrix with respect to  $u$  of some map  $f : \mathcal{U} \rightarrow \mathbb{R}^n$ . We are further interested in how large the set of solutions is, i.e. how many arbitrary constants and functions appear in a general solution  $\lambda^1(u), \dots, \lambda^n(u)$ .

A solution  $\lambda(u) = (\lambda^1(u), \dots, \lambda^n(u))$  to Problem 1 generates a flux  $f$  and an associated system of conservation laws (1.1) in which  $u^1, \dots, u^n$  are the *conserved quantities*. We note that we do not impose strict hyperbolicity for solutions to our problem; indeed, we shall see that there are cases where two or more eigenvalues  $\lambda^i$  must necessarily coalesce.

Below we derive a system of partial differential equations (PDEs) for the eigenvalues  $\lambda^i$ , where the coefficients are expressed in terms of the components of  $R_i$  and  $L^i$ . The system may be formulated in different ways, all of which we refer to as the  $\lambda$ -system. Typically this will be an overdetermined system of linear, variable coefficients PDEs. It will turn out that the most useful formulation of the  $\lambda$ -system is as an *algebraic-differential* system; see Section 2.

**Remark 1.1.** Let us clarify the coordinate dependence in Problem 1. The property of being a Jacobian with respect to a system of coordinates is not invariant under most changes of coordinates. For this reason we need to fix the coordinates  $u^i$  at the outset. Of course, having started with a particular choice of coordinates, it may be that the resulting  $\lambda$ -system takes a simpler form when formulated in other coordinates. This is the case for so-called rich systems where the “good” coordinates are the Riemann invariants (see Definition 1 below and Section 4).

It may also be that a change of coordinates happens to preserve Jacobians. This occurs for the Euler system of compressible flow when written in Lagrangian variables (Example 5.1). In this case the nonlinear change of dependent variables (specific volume, velocity, total energy)  $\mapsto$  (specific volume, velocity, entropy) preserves the conservative form of the system. As far as Problem 1 is concerned the two forms are equivalent.

**Remark 1.2.** Besides its intrinsic interest Problem 1 is motivated by possible applications where one searches for systems (1.1) with particular geometric properties, and associated properties (e.g. blowup) of solutions to Cauchy problems. In our setup we thus consider the “geometry” (the eigenfields) as given, and we look for the scalar eigenvalue fields. Alternatively one could prescribe the eigenvalues and search for eigenfields  $R_i(u)$  that renders  $A(u)$  in (1.18) a Jacobian matrix.

A special class of systems is given by so-called *rich* systems. These are systems equipped with a coordinate system of *Riemann invariants*. For definitions, and the fact that richness can be expressed in terms of the eigenfields, we refer to [35], and Section 7.3 in [12]. Starting from a given frame it is convenient to make the following, equivalent, definition:

**Definition 1.** The frame  $\{R_1(u), \dots, R_n(u)\}$  of linearly independent vector fields is said to be rich if each pair of vector fields is in involution:

$$(1.19) \quad [R_i, R_j] \in \text{span}\{R_i, R_j\} \quad \text{for all } 1 \leq i, j \leq n.$$

While it is natural to consider the class of rich systems in connection with Problem 1 we shall see that richness does not imply any uniformity with respect to how many constants and functions are needed to specify a general solution.

**Remark 1.3.** The class of symmetrizable systems plays a central role in the general theory of conservation laws, [16], [19]. These are the systems (1.1) that admit a convex “entropy”  $\eta : \Omega \rightarrow \mathbb{R}$ ,

with an associated entropy-flux  $q : \Omega \rightarrow \mathbb{R}$ , such that an additional conservation law

$$\eta(u)_t + q(u)_x = 0$$

holds whenever  $u$  is a smooth solution of (1.1). Equivalently (at least in the strictly hyperbolic regime), the Hessian  $D^2\eta$  should define an inner product with respect to which the given eigenvectors  $R_i$  are orthogonal ( $R_j^T D^2\eta R_i = 0$  for  $i \neq j$ ; see [12] for details). Differently from richness, symmetrizability is not expressed directly in terms of the eigenfields alone. Problem 1 for symmetrizable systems will be taken up elsewhere.

**1.4. Related works.** Problem 1 was addressed by Dafermos [11] for  $2 \times 2$ -systems in several space dimensions under the requirement that the Jacobians in the various spatial directions commute. Commutativity implies that the Jacobians have the same eigenfields. In [11] it was shown how to construct such systems for any pair of linearly independent vector fields. The case of one-dimensional  $2 \times 2$ -systems is covered by the analysis of rich systems in Section 4.

In his geometric analysis of systems of conservation laws Sévenec [36] provides a characterization of those quasilinear systems

$$v_t + A(v)v_x = 0$$

that can be transformed to conservative form (1.1) by a suitable change of variables  $u = \phi(v)$ . In particular, the characterization involves a version of what we refer to as the  $\lambda$ -system.

The class of rich systems has been studied by many authors. Conlon and Liu [10] considered rich systems in connection with entropy criteria and showed that such systems are endowed with large families of entropies. From a different perspective the same class of systems were studied by Tsarev [42, 43]. Sévenec [36] showed that the eigenvalues of strictly hyperbolic, rich systems must satisfy certain restrictive conditions (see Proposition 5 in [36]). Serre [35] has performed a comprehensive analysis of rich systems, including building of entropies, commuting families of systems, and construction of rich systems. In Section 4.2 we analyze rich systems in relation to Problem 1.

**1.5. Outline and summary of results.** It turns out that the complete solution of Problem 1 for arbitrary  $n$  is quite complicated. In this paper we provide a complete solution for the case  $n = 3$ , as well as for rich systems for any  $n$ . (This latter class covers the case  $n = 2$ .) We also give a list of examples that illustrate the various types of solutions.

In Section 2 we begin by noting some properties the  $\lambda$ -system that follow directly from the formulation of Problem 1. We then formulate three equivalent versions of the  $\lambda$ -system, including an algebraic-differential system, and we record the extreme cases with minimal and maximal number of algebraic constraints. A few general facts are collected in Proposition 2.2.

Our solution of the  $n = 3$  case in Section 3 reveals that the solution set of the  $\lambda$ -system depends on the number of independent algebraic equations, as well as on the number of  $\lambda^i$  that appear in these equations. For  $n \geq 4$  there seems to be no easy way to analyze completely the resulting cases. We provide a complete breakdown of the possible scenarios when  $n = 3$ . The algebraic part of the  $\lambda$ -system now contains two, one, or no independent algebraic equations. The two extreme cases fall into either the trivial or rich categories. In the case of one algebraic equation the size of the solution set depends further on the number of  $\lambda^i$  involved in the algebraic constraint. The Frobenius integrability theorem can be applied when all three  $\lambda^i$  appear in the algebraic equation, and in this case the general solution depends on two constants. The only other possibility is that exactly two  $\lambda^i$  occur in the algebraic equation. This case may be analyzed by using the Cartan-Kähler integrability theorem, and the general solution now depends on one function of one variable and one constant.

We solve Problem 1 for the class of rich systems of any dimension in Section 4. The subclass of rich systems without algebraic constraints are analyzed in Section 4.2, while Section 4.3 treats the

more involved case of rich systems where the eigenvalues are related algebraically. In both cases the  $\lambda$ -system can be analyzed using an integrability theorem of Darboux (Theorem 4.1).

For completeness we include statements of the various integrability theorems we apply. Concerning smoothness assumptions of the given frame we need to require analyticity when we apply the Cartan-Kähler theorem. On the other hand, for the cases that use the theorems of Frobenius and Darboux we only need to require  $C^2$  smoothness of the given frame.

Section 5 collects several examples that are of interest in themselves or that illustrate the different cases treated in Section 3 and Section 4. We start by considering the various solutions of Problem 1 for the case where the given eigenfields are those of the Euler system describing one-dimensional compressible fluid flow (Example 5.1). Depending on the prescribed pressure function these eigenfields may form either a rich or a non-rich frame. We proceed with several more examples of non-rich frames on  $\mathbb{R}^3$  illustrating various scenarios treated in Section 3 and an example of a frame on  $\mathbb{R}^4$  with only trivial solutions for the  $\lambda$ -system. We then give a set of examples of rich frames whose  $\lambda$ -systems do not impose algebraic constraints on eigenfunctions. This includes a frame on  $\mathbb{R}^2$ , a rich orthogonal frame on  $\mathbb{R}^3$  and a constant frame on  $\mathbb{R}^n$ . We finally give two examples of rich frames on  $\mathbb{R}^3$  whose  $\lambda$ -systems impose certain algebraic constraints on eigenfunctions. We have written a set of procedures<sup>2</sup> in MAPLE to obtain and analyze the  $\lambda$ -system for a given frame, and used it to construct some of the above examples.

Finally, our analysis indicates that a solution of Problem 1 for general systems with  $n \geq 4$  is rather involved, with a large number of different subcases.

## 2. THE $\lambda$ -SYSTEM

**2.1. Trivial solutions and scalings.** Problem 1 always has a one-parameter family of solutions given by

$$\lambda^1(u) = \cdots = \lambda^n(u) \equiv \hat{\lambda},$$

where  $\hat{\lambda}$  is any real constant. We refer to these as *trivial* solutions. The resulting system (1.1) is linearly degenerate in all families (i.e.  $\nabla \lambda_i(u) \cdot R_i(u) \equiv 0$ ) and any map  $f(u) = \hat{\lambda}u + \hat{u}$ , where  $\hat{u} \in \mathbb{R}^n$ , is a corresponding flux. While not of interest themselves, the trivial solutions show that any compatibility condition associated with the  $\lambda$ -system will not rule out existence of solutions altogether. A trivial solution can be added to any solution of the  $\lambda$ -system to give another solution of the same system. For later reference we record the following related result:

**Proposition 2.1.** *If  $\lambda^1(u) = \cdots = \lambda^n(u)$  is a solution to Problem 1, then their common value is a constant.*

*Proof.* If  $\lambda^1(u) = \cdots = \lambda^n(u) = \hat{\lambda}(u)$  then  $A(u) = \Lambda(u) = \hat{\lambda}(u)I_{n \times n}$ . For this to be a Jacobian of a map  $f : \mathcal{U} \rightarrow \mathbb{R}^n$  we must have that

$$\forall i, j = 1, \dots, n : \quad \frac{\partial f_i}{\partial u^j}(u) = \delta_{ij} \hat{\lambda}(u),$$

whence  $\hat{\lambda}(u)$  is a function of  $u^i$  alone for each  $i$ , and thus constant. □

In formulating Problem 1 we may use any (non-vanishing) scalings of the eigenfields  $R_i(u)$ . That is, given smooth functions  $\alpha^j : \Omega \rightarrow \mathbb{R} \setminus \{0\}$ ,  $j = 1, \dots, n$ , we may set  $\tilde{R}_j(u) = \alpha^j(u)R_j(u)$ , together with the inversely scaled left eigenfields  $\tilde{L}_j(u) := \alpha^j(u)^{-1}L_j(u)$ . Letting  $\tilde{R} := R\alpha$ ,  $\tilde{L} := \alpha^{-1}L$ , where  $\alpha(u) = \text{diag}[\alpha_1(u) \dots \alpha_n(u)]$ , we get that  $R(u)\Lambda(u)L(u)$  is a Jacobian if and only if  $\tilde{R}(u)\Lambda(u)\tilde{L}(u)$  is a Jacobian. E.g., in the case of rich systems the simplest form of the  $\lambda$ -system is obtained by using the versions that makes the matrix  $L$  a Jacobian with respect to the  $u$  variable (see Section 4).

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<sup>2</sup>MAPLE code is available at <http://www.math.ncsu.edu/~iakogan/mapleHTML/lambda-system.html>

## 2.2. Formulating the $\lambda$ -system.

2.2.1. *Direct formulation.* We have that a matrix  $A(u) = (A_j^i(u))_{i,j}$ , is a Jacobian with respect to  $u$ -coordinates if and only if

$$(2.1) \quad \partial_k A_j^i(u) = \partial_j A_k^i(u) \quad \text{for all } i, j, k = 1, \dots, n \text{ with } j < k,$$

where  $\partial_i$  denotes partial differentiation with respect to  $u_i$ . In the case when  $A(u)$  is given by (1.18) we set

$$C_{mj}^i(u) := R_m^i(u) L_j^m(u) \quad (\text{no summation}),$$

and (2.1) may be written

$$(2.2) \quad \sum_{m=1}^n \left[ C_{mj}^i \partial_k \lambda^m - C_{mk}^i \partial_j \lambda^m + \lambda^m (\partial_k C_{mj}^i - \partial_j C_{mk}^i) \right] = 0,$$

where  $i, j, k \in \{1, \dots, n\}$  with  $j < k$ . This version of the  $\lambda$ -system provides a homogeneous, variable coefficient system of  $\frac{n^2(n-1)}{2}$  linear PDEs for  $n$  unknowns. For  $n \geq 3$  it is thus typically an overdetermined system of PDEs.

2.2.2. *Formulations using 1-forms.* A simpler formulation of the  $\lambda$ -system is obtained by using differential forms to express the condition that  $A(u)$  in Problem 1 is a Jacobian. By Poincaré's lemma (recall that  $\Omega$  is assumed smoothly contractible to a point) we have

$$(2.3) \quad A(u) \text{ is a Jacobian matrix w.r.t. } u \quad \iff \quad dA(u) \wedge du = 0,$$

where the  $d$ -operator is applied component-wise. Applying the product rule, condition (2.3) is thus equivalent to

$$(2.4) \quad \{L(dR)\Lambda + d\Lambda - \Lambda L(dR)\} \wedge Ldu = 0,$$

or

$$(2.5) \quad (d\Lambda) \wedge (Ldu) = \{\Lambda(LdR) - (LdR)\Lambda\} \wedge (Ldu),$$

where we have used that  $L = R^{-1}$ . (Clearly (2.5) is satisfied if  $d\Lambda = \Lambda(LdR) - (LdR)\Lambda$ ; however, the associated solutions are exactly the trivial solutions  $(\lambda_1, \dots, \lambda_n) \equiv (\hat{\lambda}, \dots, \hat{\lambda})$ ,  $\hat{\lambda} \in \mathbb{R}$ .)

The system (2.5) is an equation for  $n$ -vectors of 2-forms. We proceed to write out the system in  $u$ -coordinates by applying (2.5) to the pair of vector fields  $(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j})$ . A direct calculation yields:

$$(2.6) \quad L_j^i(\partial_l \lambda^i) - L_l^i(\partial_j \lambda^i) = \sum_{m \neq i} B_{lj}^{mi}(\lambda^m - \lambda^i), \quad \text{for each } i \text{ and for all } 1 \leq l < j \leq n,$$

where  $B_{lj}^{mi} := \{L_j^m(\partial_l L^i) - L_l^m(\partial_j L^i)\} \cdot R_m$ . Not surprisingly this is again a homogeneous system of  $\frac{n^2(n-1)}{2}$  linear PDEs for  $n$  unknowns. Thus, in this formulation of the  $\lambda$ -system the equations involving derivatives of  $\lambda^i$  are such that, first, no other eigenvalue appears differentiated, and second, at most two derivatives of  $\lambda^i$  occur in each equation.

To obtain a formulation of the  $\lambda$ -system which explicitly records *algebraic* constraints on the eigenvalues  $\lambda^i$ , we introduce the following notation. Whereas we treat  $R_i(u)$  and  $L^i(u)$  as arrays of functions we let  $r_i(u)$ ,  $\ell^i(u)$  denote the corresponding vector fields (differential operators) and differential 1-forms:

$$(2.7) \quad r_i(u) := \sum_{m=1}^n R_m^i(u) \frac{\partial}{\partial u^m} \Big|_u, \quad \ell^i(u) := \sum_{m=1}^n L_m^i(u) du^m \Big|_u.$$



Since  $R(u)$  is assumed invertible on  $\Omega$ , the vector fields  $r_i(u)$ ,  $i = 1, \dots, n$ , provide a frame on  $\Omega$ , and the 1-forms  $\ell^i(u)$ ,  $i = 1, \dots, n$ , provide the dual coframe on  $\Omega$  (see Section 1.2). We define the  $n$ -vector of 1-forms  $\ell$  by

$$\ell := \begin{bmatrix} \ell^1 \\ \vdots \\ \ell^n \end{bmatrix} = Ldu,$$

and introduce the following coefficients

$$(2.8) \quad \Gamma_{ij}^k := L^k(DR_j)R_i,$$

where  $D$  denotes Jacobian with respect to  $u$ . A direct computation shows that they are, in fact, the Christoffel symbols (connection components) of the flat and symmetric connection  $\nabla_{\frac{\partial}{\partial u^i}} \frac{\partial}{\partial u^j} = 0$  computed relative to the frame  $\{r_1, \dots, r_n\}$ . In other words, the covariant derivatives of the frame vector fields are  $\nabla_{r_i} r_j = \sum_{k=1}^n \Gamma_{ij}^k r_k$ . We note that the symmetry condition (1.15) implies

$$(2.9) \quad [r_i, r_j] = \sum_{k=1}^n c_{ij}^k r_k = \sum_{k=1}^n (\Gamma_{ij}^k - \Gamma_{ji}^k) r_k.$$

This shows that if  $\{r_1, \dots, r_n\}$  is rich, then  $\Gamma_{ij}^k = \Gamma_{ji}^k$  whenever  $k \notin \{i, j\}$ .

Another direct calculation shows that the  $(k, j)$ -entry of the matrix  $\mu := R^{-1}dR = LdR$  of 1-forms is given by the connection forms

$$(2.10) \quad \mu_j^k := (LdR)_j^k = \sum_{i=1}^n \Gamma_{ij}^k \ell^i.$$

Thus (2.5) reads

$$(2.11) \quad (d\Lambda) \wedge \ell = \{\Lambda\mu - \mu\Lambda\} \wedge \ell.$$

Again (2.11) is an equation of  $n$ -vectors of 2-forms. Applying each component to pairs of frame vector fields  $(r_i, r_j)$ ,  $i, j = 1, \dots, n$ , we obtain an equivalent form of the  $\lambda$ -system as a differential-algebraic system:

$$(2.12) \quad r_i(\lambda^j) = \Gamma_{ji}^j(\lambda^i - \lambda^j) \quad \text{for } i \neq j,$$

$$(2.13) \quad (\lambda^i - \lambda^k)\Gamma_{ji}^k = (\lambda^j - \lambda^k)\Gamma_{ij}^k \quad \text{for } i < j, i \neq k, j \neq k,$$

where there are no summations. (2.12) gives  $n(n-1)$  linear, homogeneous PDEs, while (2.13) gives  $\frac{n(n-1)(n-2)}{2}$  algebraic relations. We observe that (2.13) is symmetric in  $i$  and  $j$ , and that *all* coefficients  $\Gamma_{ij}^k$  with  $i \neq j \neq k \neq i$  appear in (2.13). This form of the  $\lambda$ -system explicitly records the algebraic relations inherent in (2.2) and (2.6).

**Remark 2.1.** *The equations (2.12)-(2.13) appear in Sévenec's characterization of quasilinear systems that admit a conservative form, see [36]. For a given quasilinear system*

$$v_t + A(v)v_x = 0, \quad A(v) \in \mathbb{R}^{n \times n},$$

*Sévenec shows that there is a coordinate system in which the system is conservative if and only if there exists a flat and symmetric affine connection  $\nabla$  such that its Christoffel symbols and the eigenvalues of  $A(u)$  satisfy (2.12)-(2.13). The same system appears in Tsarev [43].*

**Remark 2.2.** *Note that condition (2.11) is equivalent to*

$$(2.14) \quad \Theta := (d\Lambda - \Lambda\mu + \mu\Lambda) \wedge \ell = 0,$$

where  $\Theta$  is a differential form on a  $2n$ -dimensional manifold with coordinates  $u^1, \dots, u^n, \lambda^1, \dots, \lambda^n$ . Moreover, the ideal  $\mathcal{I}$  algebraically generated by  $\Theta$  in the ring of differential forms is a differential ideal, i.e.  $d\omega \in \mathcal{I}$  for all  $\omega \in \mathcal{I}$ . Indeed, from the flatness and symmetry properties of the connection:

$$d\mu = -\mu \wedge \mu \quad \text{and} \quad dl = -\mu \wedge l,$$

it follows that

$$\begin{aligned} d\Theta &= (-d\Lambda \wedge \mu - \Lambda(d\mu) + (d\mu)\Lambda - \mu \wedge d\Lambda) \wedge \ell - (d\Lambda - \Lambda\mu + \mu\Lambda) \wedge dl \\ &= (-d\Lambda \wedge \mu + \Lambda(\mu \wedge \mu) - (\mu \wedge \mu)\Lambda - \mu \wedge d\Lambda) \wedge \ell + (d\Lambda - \Lambda\mu + \mu\Lambda) \wedge \mu \wedge \ell \\ &= (-(\mu \wedge \mu)\Lambda - \mu \wedge d\Lambda) \wedge \ell + (\mu\Lambda) \wedge \mu \wedge \ell \\ &= \mu \wedge (-\mu\Lambda - d\Lambda + \Lambda\mu) \wedge \ell = -\mu \wedge \Theta \equiv 0 \pmod{\Theta}. \end{aligned}$$

Solving the  $\lambda$ -systems is therefore equivalent to finding  $n$ -dimensional integral sub-manifolds  $\lambda^i = F^i(u^1, \dots, u^n)$ ,  $i = 1, \dots, n$ , of  $\mathcal{I}$ .

**2.3. Observations about the rank of the algebraic sub-system (2.13).** We next consider the algebraic constraints (2.13), which is a system of  $\frac{n(n-1)(n-2)}{2}$  linear equations. Choosing the variables to be the differences

$$x^k := \lambda^k - \lambda^1, \quad k = 2, \dots, n,$$

we rewrite (2.13) in matrix form as

$$(2.15) \quad Nx = 0.$$

Here  $x$  is the  $(n-1)$ -vector  $(x^2, \dots, x^n)^T$  and  $N$  is a certain  $\frac{n(n-1)(n-2)}{2} \times (n-1)$ -matrix whose entries are given in terms of the  $\Gamma_{ij}^k$ . It is easily checked that each entry of  $N$  is either zero, a single Christoffel symbol ( $\pm\Gamma_{ij}^k$ ), or a difference of such ( $\Gamma_{ij}^k - \Gamma_{ji}^k$ ). Furthermore,  $N = 0$  if and only if  $\Gamma_{ij}^k = 0$  for all choices  $i \neq j \neq k \neq i$ .

The number of independent algebraic constraints is given by  $\text{rank}(N)$ . It is convenient to use this as a first, rough classification. However,  $\text{rank}(N)$  does not characterize the solutions to the  $\lambda$ -system (in terms of number of constants and functions in the general solution). This is clear already from the cases of maximal and minimal rank. We briefly consider these extreme cases:

- $\text{rank}(N) = 0$ . In this case  $N = 0$  and there are no algebraic constraints imposed on the eigenvalues. This occurs if and only if  $\Gamma_{ij}^k = 0$  for all choices of  $i \neq j \neq k \neq i$ . By (2.9) this implies that  $[r_i, r_j] \in \text{span}\{r_i, r_j\}$ , i.e. we are in the rich case:

$$\text{rank}(N) = 0 \quad \Rightarrow \quad \{r_1, \dots, r_n\} \text{ is rich.}$$

The case  $\text{rank}(N) = 0$  will be analyzed in more detail in Section 4.2. We will show that any frame on  $\mathbb{R}^2$ , any constant frame on  $\mathbb{R}^n$ , as well as any rich orthogonal frame on  $\mathbb{R}^n$ , falls in this category. Corresponding examples are given in Section 5.3.

The question of whether richness implies  $\text{rank}(N) = 0$  is somewhat subtle. Namely, if we were given a rich and strictly hyperbolic system (1.1), then indeed  $\text{rank}(N) = 0$ . (See [12] p. 185 for the proof that  $\Gamma_{ij}^k = L^k(DR_j)R_i = 0$  for all choices of  $i \neq j \neq k \neq i$  in this case.) However, with Problem 1 we are starting from given vector fields  $r_i$  without insisting on strict hyperbolicity. Example 5.9 and Example 5.10 show that it is possible to prescribe a collection of vector fields that form a rich family, without  $\text{rank}(N)$  being zero. Furthermore, Example 5.9 shows that the associated  $\lambda$ -system for frames of this type may admit nontrivial solutions. Thus:

$$\{r_1, \dots, r_n\} \text{ is rich} \not\Rightarrow \text{rank}(N) = 0.$$

In Section 4.3 we prove, however, that the  $\lambda$ -system associated with a rich,  $\text{rank}(N) > 0$  frame allows no strictly hyperbolic solutions.

- $\text{rank}(N) = n - 1$ . In this case the only solution to (2.15) is  $x = 0$ , that is, all  $\lambda^i$  are equal. According to Proposition 2.1 it follows that all eigenvalues are equal to a common *constant*:

$$\text{rank}(N) = n - 1 \quad \Rightarrow \quad \lambda\text{-system has only trivial solutions.}$$

In particular, if the  $\lambda$ -system admits a strictly hyperbolic solution, then necessarily  $\text{rank}(N) < n - 1$ . It is a non-obvious fact that there *are* cases where all solutions to (2.12)-(2.13) are trivial; explicit examples are provided by Examples 5.2, 5.4, 5.5 and 5.10.

On the other hand we observe that the condition  $\text{rank}(N) = n - 1$  does not characterize the cases where the  $\lambda$ -system (2.12)-(2.13) has only trivial solutions. In other words, it may be that the only solutions to (2.12)-(2.13) are the trivial ones, while  $\text{rank}(N) < n - 1$ ; for a concrete example see Example 5.2 and 5.4. Thus:

$$\lambda\text{-system has only trivial solutions} \quad \not\Rightarrow \quad \text{rank}(N) = n - 1.$$

We summarize our findings:

**Proposition 2.2.** *Consider Problem 1 for a given frame  $\{R_1(u), \dots, R_n(u)\}$ . Then the  $\lambda$ -system may be formulated as an algebraic-differential system (2.12)-(2.13) for the eigenvalues  $\lambda^i$ . Absence of the algebraic constraints (2.13) implies that the frame is rich, but not vice versa. A maximal number of  $n - 1$  independent algebraic constraints (2.13) implies that the  $\lambda$ -system admits only trivial solutions, but not vice versa.*

### 3. SYSTEMS OF THREE EQUATIONS

In this section we present a complete breakdown of the possible solutions of Problem 1 for a given frame of vector-fields in  $\mathbb{R}^3$ . When  $n = 3$  the  $\lambda$ -system (2.12)-(2.13) consists of six linear PDEs and three linear algebraic equations. For concreteness we record these; the PDEs are

$$(3.1) \quad r_1(\lambda^2) = \Gamma_{21}^2(\lambda^1 - \lambda^2)$$

$$(3.2) \quad r_1(\lambda^3) = \Gamma_{31}^3(\lambda^1 - \lambda^3)$$

$$(3.3) \quad r_2(\lambda^1) = \Gamma_{12}^1(\lambda^2 - \lambda^1)$$

$$(3.4) \quad r_2(\lambda^3) = \Gamma_{32}^3(\lambda^2 - \lambda^3)$$

$$(3.5) \quad r_3(\lambda^1) = \Gamma_{13}^1(\lambda^3 - \lambda^1)$$

$$(3.6) \quad r_3(\lambda^2) = \Gamma_{23}^2(\lambda^3 - \lambda^2),$$

while the algebraic constraints may be written as

$$(3.7) \quad Nx = \begin{bmatrix} \Gamma_{32}^1 & -\Gamma_{23}^1 \\ (\Gamma_{31}^2 - \Gamma_{13}^2) & \Gamma_{13}^2 \\ \Gamma_{12}^3 & (\Gamma_{21}^3 - \Gamma_{12}^3) \end{bmatrix} \begin{bmatrix} x^2 \\ x^3 \end{bmatrix} = 0,$$

where  $x^2 = \lambda^2 - \lambda^1$  and  $x^3 = \lambda^3 - \lambda^1$ . There are three possibilities depending on  $\text{rank}(N)$ :

I:  $\text{rank}(N) = 0$ . There are no algebraic constraints; the eigenvectors are pairwise in involution, and any corresponding system of conservation laws (1.1) is rich. The analysis in Section 4.2, which applies to systems of any size, demonstrates that the  $\lambda$ -system always has many non-trivial (in particular, many strictly hyperbolic) solutions in this case.

II:  $\text{rank}(N) = 1$ . In this case (3.7) imposes a single linear relationship among the eigenvalues. This case is more involved and there are several possibilities in terms of how many constants and functions determine a general solution. The analysis is detailed in Section 3.1 below.

III:  $\text{rank}(N) = 2$ . In this case there are only trivial solutions  $\lambda^1 = \lambda^2 = \lambda^3 \equiv \text{constant}$ .

Section 5 provides examples for each type of behavior.

**3.1. Case II: a single algebraic relation.** Using (1.15) we rewrite the algebraic relations (3.7):

$$(3.8) \quad c_{32}^1 \lambda^1 = \Gamma_{32}^1 \lambda^2 - \Gamma_{23}^1 \lambda^3,$$

$$(3.9) \quad c_{31}^2 \lambda^2 = \Gamma_{31}^2 \lambda^1 - \Gamma_{13}^2 \lambda^3,$$

$$(3.10) \quad c_{21}^3 \lambda^3 = \Gamma_{21}^3 \lambda^1 - \Gamma_{12}^3 \lambda^2.$$

By assumption the rank of the system (3.7) is 1, whence the three equations are all equivalent to a single non-trivial algebraic condition (unique up to non-vanishing scalings)

$$(3.11) \quad \alpha_1 \lambda^1 + \alpha_2 \lambda^2 + \alpha_3 \lambda^3 = 0,$$

where necessarily

$$(3.12) \quad \alpha_3 = -(\alpha_1 + \alpha_2).$$

There are therefore two sub-cases to consider:

- **IIa:** all three  $\lambda^i$  appear in (3.11) with non-zero coefficients,
- **IIb:** only two of three  $\lambda^i$  are involved in (3.11) with non-zero coefficients.

In either case it may be that the only solutions are trivial. To analyze non-trivial solutions we employ the Frobenius integrability theorem in case IIa, while case IIb requires the more general Cartan-Kähler integrability theorem. It will turn out that the number of non-trivial solutions differ in the two situations.

**3.2. Subcase IIa: All three  $\lambda^i$  appear in the unique algebraic relation.** We first recall the relevant formulation of the Frobenius integrability theorem.

**Definition 2.** Let  $\mathcal{U} \subset \mathbb{R}^m$ ,  $\mathcal{V} \subset \mathbb{R}^n$  be open sets, and assume that  $g_j^i : \mathcal{U} \times \mathcal{V} \rightarrow \mathbb{R}$ ,  $1 \leq j \leq m$ ,  $1 \leq i \leq n$ , are smooth functions and  $\{Y_1, \dots, Y_m\}$  is a frame on  $\mathcal{U}$ . Then the first order PDE system

$$(3.13) \quad Y_j(v^i) = g_j^i(x, v(x)) \quad \text{for the unknown } n\text{-vector } v(x) = (v^1(x), \dots, v^n(x)),$$

is called a Frobenius system (in  $n$  unknowns on  $\mathcal{U} \times \mathcal{V}$ ).

That is, a Frobenius system prescribes *all* first derivatives of *all* the unknowns, see [39].

**Theorem 3.1** (Frobenius Integrability Theorem - Frame Version). *Suppose (3.13) is a Frobenius system in  $n$  unknowns on  $\mathcal{U} \times \mathcal{V}$  that satisfies the following integrability conditions as identities in  $(x, v) \in \mathcal{U} \times \mathcal{V}$ :*

$$(3.14) \quad \hat{Y}_k(g_j^i) - \hat{Y}_j(g_k^i) = \sum_{l=1}^m \alpha_{kj}^l g_l^i \quad \text{for } 1 \leq j < k \leq m, 1 \leq i \leq n.$$

Here the structure coefficients  $\alpha_{kj}^l$ ,  $1 \leq j, k, l \leq m$ , are given by  $[Y_k, Y_j] = \sum_{l=1}^m \alpha_{kj}^l Y_l$ , and  $\hat{Y}_k(g_j^i)$  denote the total derivatives obtained by using the equations in (3.13):  $\hat{Y}_k(g_j^i) := Y_k(g_j^i) + (\nabla_u g_j^i) \cdot g_k$ . Then, for a fixed point  $(\bar{x}, \bar{v}) \in \mathcal{U} \times \mathcal{V}$  (and under suitable smoothness conditions on  $g_j^i$ ,  $Y_k$ ), the system (3.13) has a unique local solution  $v(x)$  defined for  $x$  near  $\bar{x}$ . Furthermore, these solutions foliate a neighborhood of  $(\bar{x}, \bar{v})$  as  $\bar{v}$  varies over  $\mathcal{V}$ , and the general solution to (3.13) depends on  $n$  constants.

In the case IIa it turns out that the  $\lambda$ -system can be rewritten as a Frobenius system and we have:

**Theorem 3.2.** *Assume  $n = 3$  and that the  $\lambda$ -system contains a single algebraic constraint (3.11) ( $\text{rank}(N) = 1$ ). Consider the case IIa where all three  $\lambda^i$  appear with non-vanishing coefficients in (3.11). Then, after elimination of one of the unknowns, the PDEs in the  $\lambda$ -system, reduce to a Frobenius system for two unknown functions of three variables. If the corresponding compatibility*

conditions (3.14) are satisfied as identities then the general solution to the  $\lambda$ -system (3.1)-(3.6), (3.7) depends on two constants. Otherwise, the only solutions are trivial solutions. Finally, both situations can occur.

*Proof.* We return to (3.11), where we assume that all three  $\lambda^i$  are non-vanishing. By relabeling indices if necessary we may assume that (3.11) is proportional to (3.8), and hence  $c_{32}^1 \neq 0$ ,  $\Gamma_{32}^1 \neq 0$  and  $\Gamma_{23}^1 \neq 0$ . We solve (3.8) for  $\lambda^1$ :

$$(3.15) \quad \lambda^1 = \frac{1}{c_{32}^1}(\Gamma_{32}^1 \lambda^2 - \Gamma_{23}^1 \lambda^3),$$

and use this to eliminate  $\lambda_1$  in the differential equations (3.1)-(3.6). By using repeatedly that  $c_{ij}^k = \Gamma_{ij}^k - \Gamma_{ji}^k$ , such that  $r_m \left( \frac{\Gamma_{ij}^k}{c_{ij}^k} \right) = r_m \left( \frac{\Gamma_{ji}^k}{c_{ij}^k} \right)$  for all  $i, j, k, m$ , we obtain the PDEs

$$(3.16) \quad r_1(\lambda^2) = \frac{\Gamma_{21}^2 \Gamma_{23}^1}{c_{32}^1}(\lambda^2 - \lambda^3)$$

$$(3.17) \quad r_1(\lambda^3) = \frac{\Gamma_{31}^3 \Gamma_{32}^1}{c_{32}^1}(\lambda^2 - \lambda^3),$$

$$(3.18) \quad r_2 \left( \frac{\Gamma_{32}^1}{c_{32}^1} \right) (\lambda^2 - \lambda^3) + \frac{\Gamma_{32}^1}{c_{32}^1} r_2(\lambda^2) - \frac{\Gamma_{23}^1}{c_{32}^1} r_2(\lambda^3) = \frac{\Gamma_{12}^1 \Gamma_{23}^1}{c_{32}^1}(\lambda^3 - \lambda^2)$$

$$(3.19) \quad r_2(\lambda^3) = \Gamma_{32}^3(\lambda^2 - \lambda^3),$$

$$(3.20) \quad r_3 \left( \frac{\Gamma_{32}^1}{c_{32}^1} \right) (\lambda^2 - \lambda^3) + \frac{\Gamma_{32}^1}{c_{32}^1} r_3(\lambda^2) - \frac{\Gamma_{23}^1}{c_{32}^1} r_3(\lambda^3) = \frac{\Gamma_{13}^1 \Gamma_{32}^1}{c_{32}^1}(\lambda^3 - \lambda^2)$$

$$(3.21) \quad r_3(\lambda^2) = \Gamma_{23}^2(\lambda^3 - \lambda^2),$$

Since  $\Gamma_{32}^1, \Gamma_{23}^1 \neq 0$ , we can solve (3.18) and (3.20) for  $r_2(\lambda^2)$  and  $r_3(\lambda^3)$  by using (3.19) and (3.21):

$$(3.22) \quad \begin{aligned} r_1(\lambda^2) &= \frac{\Gamma_{21}^2 \Gamma_{23}^1}{c_{32}^1}(\lambda^2 - \lambda^3), & r_1(\lambda^3) &= \frac{\Gamma_{31}^3 \Gamma_{32}^1}{c_{32}^1}(\lambda^2 - \lambda^3), \\ r_2(\lambda^2) &= \left[ \frac{\Gamma_{23}^1}{\Gamma_{32}^1}(\Gamma_{32}^3 - \Gamma_{12}^1) - \frac{c_{32}^1}{\Gamma_{32}^1} r_2 \left( \frac{\Gamma_{32}^1}{c_{32}^1} \right) \right] (\lambda^2 - \lambda^3), & r_2(\lambda^3) &= \Gamma_{32}^3(\lambda^2 - \lambda^3), \\ r_3(\lambda^3) &= \left[ \frac{\Gamma_{32}^1}{\Gamma_{23}^1}(\Gamma_{13}^1 - \Gamma_{23}^2) + \frac{c_{32}^1}{\Gamma_{23}^1} r_3 \left( \frac{\Gamma_{23}^1}{c_{32}^1} \right) \right] (\lambda^2 - \lambda^3), & r_3(\lambda^2) &= -\Gamma_{23}^2(\lambda^2 - \lambda^3). \end{aligned}$$

This system specifies the derivatives of the two unknown functions  $\lambda^2$  and  $\lambda^3$  along all three vector fields  $r_1, r_2$  and  $r_3$ . Hence the system is of Frobenius type. For simplicity we write the system as

$$(3.23) \quad r_i(\lambda^s) = \phi_i^s(u)(\lambda^2 - \lambda^3) \quad \text{for } i = 1, 2, 3 \text{ and } s = 2, 3,$$

where  $\phi_i^s$  are known functions of  $\Gamma$ 's, given by the right-hand sides in (3.22). According to the Frobenius Integrability Theorem this system is integrable provided

$$(3.24) \quad [r_i, r_j] = \sum_{k=1}^3 c_{ij}^k r_k, \quad \text{for } 1 \leq i < j \leq 3,$$

where  $c_{ij}^k = \Gamma_{ij}^k - \Gamma_{ji}^k$  and the left-hand side is computed by using the equations (3.23). A calculation reduces (3.24) to:

$$(3.25) \quad \left[ r_i(\phi_j^s) - r_j(\phi_i^s) + \phi_j^s(\phi_i^2 - \phi_i^3) - \phi_i^s(\phi_j^2 - \phi_j^3) \right] (\lambda^2 - \lambda^3) = \left[ \sum_{k=1}^3 c_{ij}^k \phi_k^s \right] (\lambda^2 - \lambda^3),$$

where  $1 \leq i < j \leq 3$  and  $s = 2, 3$ .

These conditions are satisfied if  $\lambda^2 = \lambda^3$ , in which case the system (3.22) implies that  $\lambda^2 = \lambda^3$  is a constant. Equation (3.15) then shows that  $\lambda^1 = \lambda^2$ , and Proposition 2.1 implies that the solution in this case is trivial:  $\lambda^1 = \lambda^2 = \lambda^3 \equiv \text{constant}$ .

For a non-trivial solution to exist the following six conditions must hold:

$$(3.26) \quad r_i(\phi_j^2) - r_j(\phi_i^2) = \phi_j^2 \phi_i^3 - \phi_i^2 \phi_j^3 + \sum_{k=1}^3 c_{ij}^k \phi_k^2 \quad 1 \leq i < j \leq 3,$$

$$(3.27) \quad r_i(\phi_j^3) - r_j(\phi_i^3) = \phi_j^3 \phi_i^2 - \phi_i^3 \phi_j^2 + \sum_{k=1}^3 c_{ij}^k \phi_k^3 \quad 1 \leq i < j \leq 3.$$

Example 5.1 and Example 5.2 show that these compatibility conditions may or may not be satisfied: they must be checked for each case individually. If the compatibility conditions are met then, according to the Frobenius Theorem, the general solution to the  $\lambda$ -system depends on two constants.  $\square$

**3.3. Subcase IIb: Exactly two  $\lambda^i$  appear in the unique algebraic relation.** This case is more involved than IIa: the  $\lambda$ -system does not reduce to a Frobenius system and the non-trivial solutions must be analyzed by using the more general Cartan-Kähler theorem. This requires a reformulation of the  $\lambda$ -system as an Exterior Differential System (EDS). Using the terminology and notation of [7] and [20] we give the following formulation of Cartan-Kähler theorem (see Theorem 7.3.3 in [20] and discussion on p. 87 in [7]). In order to apply this theorem we need to assume that the given frame is analytic.

**Theorem 3.3.** (*Cartan-Kähler Integrability Theorem*). *Let  $E_k$ ,  $0 \leq k \leq n$ , be a flag of integral elements at a point  $p$  for an analytic EDS on an  $(n+s)$ -dimensional manifold, with  $\dim E_k = k$ , and such that  $E_k$  is Kähler regular. Then there exists a smooth  $n$ -dimensional integral manifold  $S$  whose tangent space at  $p$  is  $E_n$ . Furthermore, let  $H(E_k)$  be the polar spaces of  $E_k$  and  $c_k = \text{codim } H(E_k)$  for  $0 \leq k \leq n-1$ , with  $c_n = \text{codim } E_n$ . Let  $x = (x^1, \dots, x^n)$  and  $y = (y^1, \dots, y^s)$  be local coordinates around  $p$ , chosen so that  $E_k$  is spanned by  $\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^k}$ ,  $E_n$  is annihilated by  $dy^1, \dots, dy^n$ , and  $H(E_k)$  is annihilated by  $dy^1, \dots, dy^{c_k}$ , for  $0 \leq k \leq n-1$ . Then  $S$  is defined in the coordinates  $(x, y)$  by analytic equations of the form  $y^\alpha = F^\alpha(x^1, \dots, x^n)$ ,  $\alpha = 1, \dots, s$ .*

*More precisely, letting  $\bar{x}^k = x^k(p)$ , then in a neighborhood of  $p$  the integral manifold  $S$  is uniquely determined by the following initial data:*

$$(3.28) \quad \begin{aligned} f^\alpha &:= F^\alpha(\bar{x}^1, \bar{x}^2, \dots, \bar{x}^n) && \text{for } 0 < \alpha \leq c_0, \\ f^\alpha(x^1) &:= F^\alpha(x^1, \bar{x}^2, \dots, \bar{x}^n) && \text{for } c_0 < \alpha \leq c_1, \\ f^\alpha(x^1, x^2) &:= F^\alpha(x^1, x^2, \bar{x}^3, \dots, \bar{x}^n) && \text{for } c_1 < \alpha \leq c_2, \\ &\vdots \\ f^\alpha(x^1, x^2, \dots, x^n) &:= F^\alpha(x^1, x^2, \dots, x^n) && \text{for } c_{n-1} < \alpha \leq c_n. \end{aligned}$$

*Namely, for each  $k = 0, \dots, n$  and  $\alpha$  such that  $c_{k-1} < \alpha \leq c_k$ , let  $f^\alpha(x^1, \dots, x^k)$  be an arbitrary analytic function of  $k$  variables, such that  $|f^\alpha(x^1, \dots, x^k) - y^\alpha(p)|$  is sufficiently small in a neighborhood of  $(\bar{x}^1, \dots, \bar{x}^k)$  (where we define  $c_{-1} := 0$  and by a function of zero variables we mean a constant). Then there exists a unique analytic integral manifold  $y^\alpha = F^\alpha(x^1, \dots, x^n)$ ,  $\alpha = 1, \dots, s$  that satisfies the initial conditions (3.28).*

We now return to the  $\lambda$ -system. To structure the presentation we first relabel indices (if necessary) and make the assumption

$$(A) \quad \text{the unique algebraic relation (3.11) does not involve } \lambda^1, \text{ i.e. } \alpha_1 = 0.$$

**Theorem 3.4.** *Assume  $n = 3$  and that the  $\lambda$ -system contains a single algebraic constraint (3.11) ( $\text{rank}(N) = 1$ ). Consider the case I**b** where exactly two  $\lambda^i$  appear with non-vanishing coefficients in (3.11), and assume without loss of generality a labeling such that the assumption (A) holds.*

*If  $\Gamma_{31}^3 \neq \Gamma_{21}^2$  then the only solutions to the  $\lambda$ -system are trivial solutions  $\lambda^1 = \lambda^2 = \lambda^3 = \text{constant}$ , while if  $\Gamma_{31}^3 = \Gamma_{21}^2$  then the general solution depends on one arbitrary function of one variable and one arbitrary constant. Finally, both situations can occur.*

*Proof.* The algebraic relation (3.11) is equivalent to (3.8)-(3.10), and from (A) it follows that

$$(3.29) \quad c_{32}^1 = \Gamma_{32}^1 - \Gamma_{23}^1 = 0, \quad \Gamma_{31}^2 = 0, \quad \text{and} \quad \Gamma_{21}^3 = 0.$$

Thus the algebraic relations (3.8)-(3.10) reduce to:

$$\begin{aligned} \Gamma_{32}^1(\lambda^2 - \lambda^3) &= 0 \\ \Gamma_{13}^2(\lambda^2 - \lambda^3) &= 0 \\ \Gamma_{12}^3(\lambda^2 - \lambda^3) &= 0. \end{aligned}$$

By assumption this system has rank 1 such that the algebraic equations in the  $\lambda$ -system are satisfied if and only if  $\lambda^3 = \lambda^2$ . Using this relation to eliminate  $\lambda^3$  in the PDE system (3.1)-(3.6) we obtain:

$$(3.30) \quad r_1(\lambda^2) = \Gamma_{21}^2(\lambda^1 - \lambda^2)$$

$$(3.31) \quad r_1(\lambda^2) = \Gamma_{31}^3(\lambda^1 - \lambda^2)$$

$$(3.32) \quad r_2(\lambda^1) = \Gamma_{12}^1(\lambda^2 - \lambda^1)$$

$$(3.33) \quad r_2(\lambda^2) = 0$$

$$(3.34) \quad r_3(\lambda^1) = \Gamma_{13}^1(\lambda^2 - \lambda^1)$$

$$(3.35) \quad r_3(\lambda^2) = 0.$$

It follows immediately from the first two equations that if  $\Gamma_{31}^3 \neq \Gamma_{21}^2$ , then  $\lambda^1 = \lambda^2 = \lambda^3$ , and thus, by Proposition 2.1, the only solutions are trivial:  $\lambda^1 = \lambda^2 = \lambda^3 \equiv \text{constant}$ . On the other hand, if  $\lambda^3 = \lambda^2$ , such that the algebraic equations in the  $\lambda$ -system are satisfied, and in addition

$$(3.36) \quad \Gamma_{31}^3 = \Gamma_{21}^2,$$

then the  $\lambda$ -system reduces to the sub-system (3.31)-(3.35) of five PDEs on  $\mathbb{R}^3$  for two unknowns  $\lambda^1$  and  $\lambda^2$ . As before,  $\lambda^1 = \lambda^2 (= \lambda^3 = \text{constant})$  provides a trivial solution to the  $\lambda$ -system by Proposition 2.1. Example 5.4 shows that such  $\lambda$ -systems exist.

To analyze non-trivial solutions we now assume, in addition to (3.29) and (3.36), that  $\lambda^1 \neq \lambda^2 = \lambda^3$ . We note that the sub-system (3.31)-(3.35) is *not* of Frobenius type, and instead we apply the Cartan-Kähler theorem. This requires some preliminary calculations.

We start by verifying the integrability conditions corresponding to equality of all mixed 2nd derivatives that *can* be computed from (3.31)-(3.35):  $[r_3, r_2](\lambda^1) = \sum c_{32}^k r_k(\lambda^1)$ ,  $[r_2, r_1](\lambda^2) = c_{21}^k r_k(\lambda^2)$ ,  $[r_3, r_1](\lambda^2) = \sum c_{31}^k r_k(\lambda^2)$ , and  $[r_3, r_2](\lambda^2) = \sum c_{32}^k r_k(\lambda^2)$ , where the summations are over  $k = 1, 2, 3$ . Taking into account the assumptions  $\lambda^1 \neq \lambda^2$  and  $c_{23}^1 = 0$ , these reduce to:

$$(3.37) \quad r_3(\Gamma_{12}^1) - r_2(\Gamma_{13}^1) = c_{32}^2 \Gamma_{12}^1 + c_{32}^3 \Gamma_{13}^1$$

$$(3.38) \quad r_2(\Gamma_{21}^2) = \Gamma_{21}^2 \Gamma_{21}^1$$

$$(3.39) \quad r_3(\Gamma_{21}^2) = \Gamma_{21}^2 \Gamma_{31}^1.$$

**Lemma 3.5.** *Due to symmetry (1.15) and flatness (1.16), and the assumptions (3.29) and (3.36), the compatibility conditions (3.37), (3.38), (3.39) are all satisfied as identities.*

*Proof of Lemma 3.5.* We only include the most involved case of (3.37). Equation (1.16) with  $m = 3$ ,  $k = 1$ ,  $i = 2$ , and  $j = 1$ , after simplifications due to (1.15), reads:

$$r_3(\Gamma_{12}^1) - r_1(\Gamma_{32}^1) = (\Gamma_{11}^1 \Gamma_{32}^1 - \Gamma_{13}^1 \Gamma_{12}^1) + (\Gamma_{12}^1 \Gamma_{32}^2 - \Gamma_{32}^1 \Gamma_{12}^2 - c_{13}^2 \Gamma_{22}^1) + (\Gamma_{13}^1 \Gamma_{32}^3 - \Gamma_{33}^1 \Gamma_{12}^3 - c_{13}^3 \Gamma_{32}^1).$$

Similarly, equation (1.16) with  $m = 2$ ,  $k = 1$ ,  $i = 3$ ,  $j = 1$  gives:

$$r_2(\Gamma_{13}^1) - r_1(\Gamma_{23}^1) = (\Gamma_{11}^1\Gamma_{23}^1 - \Gamma_{12}^1\Gamma_{13}^1) + (\Gamma_{12}^1\Gamma_{23}^2 - \Gamma_{22}^1\Gamma_{13}^2 - c_{12}^2\Gamma_{23}^1) + (\Gamma_{13}^1\Gamma_{23}^3 - \Gamma_{23}^1\Gamma_{13}^3 - c_{12}^3\Gamma_{33}^1).$$

Subtracting these and keeping in mind (1.15) and assumptions (3.29) and (3.36), we get

$$\begin{aligned} r_3(\Gamma_{12}^1) - r_2(\Gamma_{13}^1) &= (\Gamma_{12}^1\Gamma_{32}^2 - \Gamma_{23}^1\Gamma_{12}^2 - c_{13}^2\Gamma_{22}^1 + \Gamma_{13}^1\Gamma_{32}^3 - \Gamma_{33}^1\Gamma_{12}^3 - c_{13}^3\Gamma_{23}^1) \\ &\quad - (\Gamma_{12}^1\Gamma_{23}^2 - \Gamma_{22}^1\Gamma_{13}^2 - c_{12}^2\Gamma_{23}^1 + \Gamma_{13}^1\Gamma_{23}^3 - \Gamma_{23}^1\Gamma_{13}^3 - c_{12}^3\Gamma_{33}^1) \\ &= \Gamma_{12}^1c_{32}^2 - \Gamma_{23}^1\Gamma_{21}^2 + \Gamma_{22}^1\Gamma_{31}^2 + \Gamma_{13}^1c_{32}^3 - \Gamma_{33}^1\Gamma_{21}^3 + \Gamma_{23}^1\Gamma_{31}^3 \\ &= \Gamma_{12}^1c_{32}^2 + \Gamma_{13}^1c_{32}^3. \end{aligned}$$

which is (3.37). Similar calculations verify (3.38) and (3.39).  $\square$

Returning to the proof of Proposition 3.4 we proceed to analyze the EDS  $\mathcal{I}$  associated to the PDE system (3.31)-(3.35), which is differentially generated by the 1-forms

$$\begin{aligned} \theta^1 &= d\lambda^1 - s\ell^1 - \Gamma_{12}^1(\lambda^2 - \lambda^1)\ell^2 - \Gamma_{13}^1(\lambda^2 - \lambda^1)\ell^3 \\ \theta^2 &= d\lambda^2 - \Gamma_{21}^2(\lambda^1 - \lambda^2)\ell^1, \end{aligned}$$

on  $M := \mathbb{R}^6$  with coordinates  $u^1, u^2, u^3, \lambda^1, \lambda^2$ , and  $s$ , where  $s$  represents  $r_1(\lambda^1)$ . Lemma 3.5 amounts to the fact that the essential torsion of  $\mathcal{I}$  vanishes identically. A direct computation (making use of Lemma 3.5,  $c_{32}^1 = 0$  and  $\Gamma_{31}^3 = \Gamma_{21}^2$ ) shows that

$$\begin{aligned} d\theta^1 &\equiv \pi \wedge \ell^1 \pmod{\{\theta^1, \theta^2\}} \\ d\theta^2 &\equiv 0 \pmod{\{\theta^1, \theta^2\}}, \end{aligned}$$

where

$$\begin{aligned} \pi &= -ds + [-s(c_{12}^1 + \Gamma_{12}^1) + (\lambda^2 - \lambda^1)(r_1(\Gamma_{12}^1) - \Gamma_{12}^1\Gamma_{12}^2 - \Gamma_{13}^1c_{12}^3)]\ell^2 \\ &\quad + [-s(c_{13}^1 + \Gamma_{13}^1) + (\lambda^2 - \lambda^1)(r_1(\Gamma_{13}^1) - \Gamma_{13}^1\Gamma_{13}^3 - \Gamma_{12}^1c_{13}^2)]\ell^3 \\ &=: -ds + A_2\ell^2 + A_3\ell^3. \end{aligned} \tag{3.40}$$

Thus, the 1-forms  $\ell^1, \ell^2, \ell^3, d\lambda^1, d\lambda^2, \pi$  provide a coframe on  $\mathbb{R}^6$ , and the EDS  $\mathcal{I}$  is *algebraically* generated by  $\theta^1, \theta^2$ , and  $\pi \wedge \ell^1$ .

The remaining parts of the proof consist in: (1) describing the variety of 3-dimensional integral elements of  $\mathcal{I}$  that satisfy an independence condition, (2) choosing a flag of integral elements, and computing corresponding polar spaces, and (3) applying the Cartan test and the Cartan-Kähler theorem to determine the set of solutions.

(1) *The variety of integral elements:* let  $E_3 \in G_3(TM|_{\mathbf{p}})$  be an integral element of  $\mathcal{I}$  at the arbitrary point  $\mathbf{p} \in M \cong \mathbb{R}^6$ , where  $G_3(TM|_{\mathbf{p}})$  denotes the Grassmannian manifold of 3-dimensional subspaces of the tangent space  $T_{\mathbf{p}}M$ . The 3-plane  $E_3$  is required to satisfy the independence condition

$$\ell^1 \wedge \ell^2 \wedge \ell^3|_{E_3} \neq 0. \tag{3.41}$$

We let  $\{e_1, e_2, e_3\}$  be a basis of  $E_3$  such that

$$\ell^i(e_j) = \delta_{ij} \quad i, j = 1, 2, 3. \tag{3.42}$$

The following are then necessary and sufficient conditions for  $E_3$  to be an integral element of  $\mathcal{I}$ :

$$\theta^1(e_i) = 0, \quad \theta^2(e_i) = 0 \quad \text{and} \quad \pi \wedge \ell^1(e_i, e_j) = 0, \quad \text{where } i, j = 1, 2, 3.$$

Due to conditions (3.42) these are equivalent to

$$\theta^1(e_i) = 0, \quad \theta^2(e_i) = 0, \quad \text{for } i, j = 1, 2, 3 \quad \text{and} \quad \pi(e_2) = 0, \quad \pi(e_3) = 0. \tag{3.43}$$



Thus the variety  $\mathcal{V}_3(\mathcal{I}_{\mathbf{p}}) \subset G_3(TM|_{\mathbf{p}})$  of 3-dimensional integral elements is defined by 8 independent linear equations (3.43). We conclude that

$$\text{codim } \mathcal{V}_3(\mathcal{I}_{\mathbf{p}}) = 8, \text{ and so } \dim \mathcal{V}_3(\mathcal{I}_{\mathbf{p}}) = \dim G_3(TM|_{\mathbf{p}}) - \text{codim } \mathcal{V}_3(\mathcal{I}_{\mathbf{p}}) = 9 - 8 = 1.$$

Explicitly we can write

$$e_i = r_i + p_i^1 \frac{\partial}{\partial \lambda^1} + p_i^2 \frac{\partial}{\partial \lambda^2} + a_i \frac{\partial}{\partial s}, \quad \text{for } i = 1, 2, 3,$$

where  $p_i^1, p_i^2, a_i, i = 1, 2, 3$ , are coordinate functions on an open subset of  $G_3(TM|_{\mathbf{p}})$  on which the independence condition (3.41) is satisfied. The conditions (3.43) imply that  $\mathcal{V}_3(\mathcal{I})$  is defined by the equations

$$(3.44) \quad p_1^1 = s, \quad p_2^1 = \Gamma_{12}^1(\lambda^2 - \lambda^1), \quad p_3^1 = \Gamma_{13}^1(\lambda^2 - \lambda^1), \quad p_1^2 = \Gamma_{21}^2(\lambda^1 - \lambda^2),$$

and

$$(3.45) \quad p_2^2 = 0, \quad p_3^2 = 0, \quad a_2 = A_2, \quad a_3 = A_3,$$

where  $A_2$  and  $A_3$  are defined in (3.40). The variety  $\mathcal{V}_3(\mathcal{I})$  is therefore parametrized by a unique coordinate function  $a_1$ . Equations (3.44) are of constant rank at every point  $\mathbf{p} \in M$ , and therefore all integral elements are ordinary at every point of  $M$ .

(2) *Flag of integral elements:* By setting the arbitrary parameter  $a_1 = 0$  we specify a *particular* integral plane  $E_3 = \text{span}\{e_1, e_2, e_3\}$ , where

$$\begin{aligned} e_1 &= r_1 + s \frac{\partial}{\partial \lambda^1} + \Gamma_{21}^2(\lambda^1 - \lambda^2) \frac{\partial}{\partial \lambda^2}, \\ e_2 &= r_2 + \Gamma_{12}^1(\lambda^2 - \lambda^1) \frac{\partial}{\partial \lambda^1} + A_2 \frac{\partial}{\partial s}, \\ e_3 &= r_3 + \Gamma_{13}^1(\lambda^2 - \lambda^1) \frac{\partial}{\partial \lambda^1} + A_3 \frac{\partial}{\partial s}. \end{aligned}$$

We define a flag of integral elements:

$$(3.46) \quad E_0 = \{0\} \subset E_1 = \text{span}\{e_1\} \subset E_2 = \text{span}\{e_1, e_2\} \subset E_3 = \text{span}\{e_1, e_2, e_3\},$$

A calculation, using that  $\pi(e_1) = 0$ ,  $\ell^1(e_1) = 1$ , and  $\ell^1(e_2) = 0$ , shows that the corresponding polar spaces are

$$\begin{aligned} H(E_0) &= \{\theta^1, \theta^2\}^\perp, \\ H(E_1) &= \{v \in H(E_0) | \pi \wedge \ell^1(e_1, v) = 0\} = \{\theta^1, \theta^2, \pi\}^\perp \\ H(E_2) &= \{v \in H(E_0) | \pi \wedge \ell^1(e_1, v) = 0 \text{ and } \pi \wedge \ell^1(e_2, v) = 0\} = \{\theta^1, \theta^2, \pi\}^\perp. \end{aligned}$$

Therefore

$$c_0 := \text{codim } H(E_0) = 2, \quad c_1 := \text{codim } H(E_1) = 3, \quad c_2 := \text{codim } H(E_2) = 3.$$

(3) *Applying Cartan test and the Cartan-Kähler theorem:* Since  $c_0 + c_1 + c_2 = \text{codim } \mathcal{V}_3(\mathcal{I}) = 8$  the equality in the Cartan's Test for involutivity holds. Therefore each  $E_k$  in the flag (3.46) is Kähler regular (see Theorem 7.4.1 [20]).

According to the Cartan-Kähler Theorem 3.3 the general solution depends on 2 constants that prescribe the values of  $\lambda^1$  and  $\lambda^2 = \lambda^3$  at an initial point  $\bar{u}$  and one arbitrary function of one variable that prescribes the directional derivative  $s = r_1(\lambda^1)$  along a curve. This arbitrary function absorbs the arbitrary constant that prescribe the value of  $\lambda^1$  at  $\bar{u}$  and thus effectively the general solution depends on one arbitrary function of one variable and one arbitrary constant. The  $\lambda$ -system in Examples 5.3 has solutions of this type. This concludes the proof of Proposition 3.4.  $\square$

## 4. RICH SYSTEMS

In this section we consider Problem 1 for rich systems where we are given a coordinate system  $u$  and a frame  $\{R_1(u), \dots, R_n(u)\}$  satisfying Definition 1. This corresponds to searching for systems (1.1) whose eigencurves are the coordinate curves of some system of coordinates  $w^1(u), \dots, w^n(u)$  (the Riemann invariants) on  $\mathbb{R}^n$ . This leads to a slight reformulation of Problem 1, and a theorem of Darboux provides a complete solution for any dimension  $n$ .

The reason why general rich systems can be completely analyzed in this way is the fact that all the algebraic constraints in this case either are absent, or always impose equality of pairs of eigenvalues. This is in contrast to general (non-rich) systems for which the algebraic conditions may be more complicated; see Section 3.1.

**4.1. The  $\lambda$ -system in Riemann invariants.** We recall that richness may be formulated in several equivalent ways. (For the setting where a system (1.1) is given, see Chapter 12 in [35] or Sections 7.3-7.4 in [12].) In particular, it follows from Definition 1 and the Frobenius theorem that the frame  $\{R_1(u), \dots, R_n(u)\}$  is rich if and only if there is a change of coordinates

$$u \mapsto \rho(u) = (w^1(u), \dots, w^n(u)) \quad \text{with} \quad \nabla w^i(u) \cdot R_j(u) \begin{cases} = 0 & \text{if } i \neq j, \\ \neq 0 & \text{if } i = j. \end{cases}$$

The  $w$ -coordinates are referred to as associated *Riemann invariants*. These are not unique and we assume that we have fixed one choice of the map  $w = \rho(u)$ . In this case we may scale the given vector fields  $R_i(u)$  according to the normalization

$$(4.1) \quad \nabla w^i(u) \cdot R_j(u) \equiv \delta_j^i,$$

which we assume throughout this section. A calculation (see [12] Section 7.3) then shows that these scalings render  $\{R_1(u), \dots, R_n(u)\}$  a *commutative* frame: all structure coefficients  $c_{jk}^i$  in (1.5) vanish. Furthermore, according to the normalization (4.1), the corresponding left eigenvectors  $L^i(u)$  in (1.17)<sub>2</sub> are given by  $L^i(u) = \nabla w^i(u)$ , such that the matrix  $L(u)$  is a Jacobian with respect to  $u$ :  $Ldu = dw$ .

In  $w$ -coordinates the  $\lambda$ -system (2.12) - (2.13) becomes

$$(4.2) \quad \partial_i \kappa^j = Z_{ji}^j(\kappa^i - \kappa^j) \quad \text{for } 1 \leq i \neq j \leq n, \quad \left(\partial_i = \frac{\partial}{\partial w^i}\right)$$

$$(4.3) \quad Z_{ij}^k(\kappa^j - \kappa^i) = 0 \quad \text{for } 1 \leq k \neq i < j \neq k \leq n,$$

where

$$(4.4) \quad \kappa^i(w) := \lambda^i \circ \rho^{-1}(w) \quad \text{and} \quad Z_{ij}^k(w) := \Gamma_{ij}^k \circ \rho^{-1}(w).$$

Symmetry (1.15) and flatness (1.16) of the connection  $\nabla$  imply the following properties of the Christoffel symbols  $Z_{ij}^k(w)$ :

$$(4.5) \quad Z_{ij}^k = Z_{ji}^k \quad (\text{symmetry}),$$

$$(4.6) \quad \partial_m(Z_{ik}^j) - \partial_k(Z_{im}^j) = \sum_{t=1}^n (Z_{ik}^j Z_{im}^t - Z_{im}^j Z_{ik}^t), \quad \forall i, j, k, m \quad (\text{flatness}).$$

Problem 1 in the rich case thus takes the following form:

**Problem 2.** (*Rich frame*) *With the same notation as in Section 1.3, assume that the given frame  $\{R_1(u), \dots, R_n(u)\}$  is rich, and let  $w = (w^1, \dots, w^n)$  be associated Riemann invariants. Assume the normalization (4.1), and define the connection coefficients  $Z_{ij}^k$  by (4.4). Then, determine the set of solutions  $\kappa^1(w), \dots, \kappa^n(w)$  of the  $\lambda$ -system in Riemann invariants (4.2)-(4.3).*

Below we give a complete answer to this problem by applying a theorem of Darboux. For completeness we include a precise statement of this result.

**Theorem 4.1.** (Darboux [13]) *Given a system of first order PDEs with dependent variables  $v = (v^1, \dots, v^m)$  and independent variables  $x = (x^1, \dots, x^n)$ . Assume that each equation is of the form*

$$(4.7) \quad \frac{\partial v^i}{\partial x^h} = f_{ih}(v, x),$$

where the given maps  $v \mapsto f_{ih}(v, x)$  are  $C^1$ -smooth, uniformly in  $x$  in a neighborhood of a given point  $x_0 = (x_0^1, \dots, x_0^n)$ . Assume that the system prescribes compatible second order mixed derivatives in the following sense:

- (C) *Whenever  $v^i$  is such that two different first derivatives  $\frac{\partial v^i}{\partial x^h}$  and  $\frac{\partial v^i}{\partial x^k}$  are given by the system (4.7), the equation*

$$\frac{\partial}{\partial x^k}(f_{ih}(v, x)) = \frac{\partial}{\partial x^h}(f_{ik}(v, x))$$

*(after taking the total derivative on each side), contains only first order derivatives which are prescribed by (4.7), and substitution of (4.7) for the first derivatives that appear gives an identity in  $x$  and  $v$ .*

Suppose a dependent variable  $v^i$  appears differentiated in (4.7) with respect to  $x^{h_1}, \dots, x^{h_p}$ . Then, letting  $\tilde{x}$  denote the remaining independent variables, we prescribe a real-valued  $C^1$ -smooth function  $\varphi^i(\tilde{x})$  and require that

$$(4.8) \quad v^i(x^1, \dots, x^n) \Big|_{x^{h_1}=x_0^{h_1}, \dots, x^{h_p}=x_0^{h_p}} = \varphi^i(\tilde{x}).$$

We make such an assignment of data for each dependent variable that appears differentiated in (4.7). Then, under the compatibility condition (C), the problem (4.7) - (4.8) has a unique local  $C^2$ -solution.

**Remark 4.1.** *The initial data (4.8) provide the values of each dependent variable in the directions where its derivatives are not prescribed by the system (4.7). In particular, if all partial derivatives of a variable  $v^j$  are prescribed by (4.7), then  $\varphi^j(\tilde{x})$  is taken to be a constant.*

Our formulation of Theorem 4.1 covers several situations of both over- and under-determined systems. In [13] the different cases are stated in three separate theorems, one of which is the PDE version of the Frobenius theorem (all first partials of all dependent variables are prescribed). The theorem is proved by a suitable Picard iteration and requires only  $C^1$ -regularity.

**4.2. Rich systems with no algebraic constraints.** We first consider the situation when  $Z_{ij}^k = 0$  for all triples of distinct  $i, j, k$ , such that the algebraic constraints (4.3) are trivially satisfied, i.e.,  $\text{rank}(N) = 0$ . We verify that the compatibility conditions (C) in Theorem 4.1 are satisfied in this case, and that the general solution of the system (4.2) depends on  $n$  arbitrary functions of one variable. In particular, there are many strictly hyperbolic solutions of Problem 2 in this case.

We also show that if the given vector fields  $R_i$  (in addition to forming a rich frame) are orthogonal, then the corresponding  $\lambda$ -system necessarily belongs to this unconstrained case. Two concrete examples of rich and algebraically unconstrained systems are the class of  $2 \times 2$ -systems, and the class of  $n \times n$ -systems with constant eigenfields, see Examples 5.6, 5.7, and 5.8.

**Theorem 4.2.** *Given a  $C^1$ -smooth, rich frame  $\{R_1(u), \dots, R_n(u)\}$  in a neighborhood of  $\bar{w} \in \mathbb{R}^n$ . Let  $\rho(u) = (w^1(u), \dots, w^n(u))$  be associated Riemann invariants and assume the normalization (4.1). Let the connection coefficients  $Z_{ij}^k$  be defined by (4.4) and assume that  $Z_{ij}^k = 0$  whenever  $i \neq j \neq k \neq i$ . Then, for given functions  $\varphi_i$ ,  $i = 1, \dots, n$ , of one variable, there is a unique local solution  $\kappa^1(w), \dots, \kappa^n(w)$  to the  $\lambda$ -system (4.2) with*

$$\kappa^i(\bar{w}^1, \dots, \bar{w}^{i-1}, w^i, \bar{w}^{i+1}, \dots, \bar{w}^n) = \varphi_i(w^i).$$

*Proof.* The  $\lambda$ -system (4.2) consists of  $n(n-1)$  PDEs that prescribe, for each  $j = 1, \dots, n$ , all first partials of  $\kappa^j$  except  $\partial_j \kappa^j$ . The conclusion of the theorem thus follows from Darboux's Theorem 4.1 provided the system (4.2) satisfies the compatibility conditions (C). That is, for each  $j = 1, \dots, n$  the equalities

$$\partial_k \partial_m \kappa^j = \partial_m \partial_k \kappa^j$$

should hold as identities for all  $k \neq j$ ,  $m \neq j$  and  $m \neq k$ , when the first derivatives of  $\kappa$ 's are substituted for from (4.2). This leads to the compatibility conditions

$$(4.9) \quad \begin{aligned} (\partial_m Z_{jk}^j - \partial_k Z_{jm}^j) \kappa^j &+ (Z_{jm}^j Z_{mk}^m + Z_{jk}^j Z_{km}^k - Z_{jm}^j Z_{jk}^j - \partial_m Z_{jk}^j) \kappa^k \\ &- (Z_{jm}^j Z_{mk}^m + Z_{jk}^j Z_{km}^k - Z_{jk}^j Z_{jm}^j - \partial_k Z_{jm}^j) \kappa^m \equiv 0, \end{aligned}$$

for all distinct  $j, m, k$ . We verify these conditions by showing that the coefficients of  $\kappa^j$ ,  $\kappa^m$  and  $\kappa^k$  vanish identically due to (4.5) and the flatness condition (4.6). We first substitute  $i = j$  in (4.6) to obtain

$$\partial_m Z_{jk}^j - \partial_k Z_{jm}^j = \sum_{t=1}^n (Z_{tk}^j Z_{jm}^t - Z_{tm}^j Z_{jk}^t),$$

which vanishes since  $k, m, j$  are distinct and since, by assumption,  $Z_{i_2 i_3}^{i_1} \equiv 0$  for distinct  $i_1, i_2, i_3$ . This shows that the coefficient of  $\kappa^j$  in (4.9) is zero. The arguments for  $\kappa^k$  and  $\kappa^m$  are similar and we only consider the coefficient of  $\kappa^k$ . Interchanging  $k$  and  $i$  in (4.6) and setting  $i = j$  yields

$$\partial_m Z_{jk}^j - \partial_j Z_{mk}^j = \sum_{t=1}^n (Z_{jt}^j Z_{mk}^t - Z_{mt}^j Z_{jk}^t).$$

Again, using (4.5), that  $Z_{i_2 i_3}^{i_1} \equiv 0$  for distinct  $i_1, i_2, i_3$ , and that  $k, m, j$  are distinct, we obtain

$$\partial_m Z_{jk}^j = Z_{jm}^j Z_{mk}^m + Z_{jk}^j Z_{mk}^k - Z_{mj}^j Z_{jk}^j.$$

□

**4.2.1. Rich, orthogonal frames.** Assume now that the given rich frame  $\{R_1(u), \dots, R_n(u)\}$  has the additional property that it is orthogonal:  $R_i \cdot R_j = 0$  if  $i \neq j$ . That is, we search for systems (1.1) whose eigencurves are the coordinate curves of an *orthogonal coordinate system*  $(w^1(u), \dots, w^n(u))$  on  $\mathbb{R}^n$ . In this case we show that the connection components  $Z_{ij}^k$  necessarily vanish whenever  $i, j, k$  are distinct, such that Theorem 4.2 applies.

We define a matrix  $S(w)$ , whose components  $S_j^i(w) := R_j^i \circ \rho^{-1}(w)$  are the pull-backs of components of the matrix  $R$  under  $\rho^{-1}$ . The connections one-forms (see (1.8) and (2.10)) are then given by :

$$(4.10) \quad \mu_i^j := \sum_{k=1}^n Z_{ki}^j dw^k = (S^{-1} dS)_i^j.$$

By orthogonality we have that

$$(4.11) \quad S^T S = \text{diag}[|S_1|^2 \dots |S_n|^2] \quad \text{such that} \quad S^{-1} = \text{diag}[|S_1|^{-2} \dots |S_n|^{-2}] S^T.$$

It follows that the connection matrix  $\mu$  is given by

$$\mu = S^{-1} dS = \text{diag}[|S_1|^{-2} \dots |S_n|^{-2}] S^T dS.$$

Differentiation of (4.11) gives

$$d(S^T)S + S^T dS = \text{diag}[d|S_1|^2 \dots d|S_n|^2],$$

such that

$$\mu^T = \text{diag} \left[ \frac{d|S_1|^2}{|S_1|^2} \cdots \frac{d|S_n|^2}{|S_n|^2} \right] - \text{diag} [|S_1|^2 \cdots |S_n|^2] \mu \text{diag} [|S_1|^{-2} \cdots |S_n|^{-2}].$$

Component-wise we thus have

$$(4.12) \quad \mu_j^i = -\frac{|S_j|^2}{|S_i|^2} \mu_i^j \quad \forall j \neq i \quad \text{and} \quad \mu_i^i = \frac{1}{2|S_i|^2} d|S_i|^2.$$

From (4.10) and (4.12) it now follows that

$$Z_{jl}^i = -\frac{|S_j|^2}{|S_i|^2} Z_{il}^j \quad \forall l, \forall j \neq i,$$

By symmetry in the lower indices (4.5) we get, for indices  $i \neq j \neq l \neq i$ , that

$$(4.13) \quad Z_{jl}^i = -\frac{|S_j|^2}{|S_i|^2} Z_{il}^j = -\frac{|S_j|^2}{|S_i|^2} Z_{li}^j = \frac{|S_l|^2}{|S_i|^2} Z_{ji}^l,$$

while, at the same time,

$$(4.14) \quad Z_{jl}^i = Z_{lj}^i = -\frac{|S_l|^2}{|S_i|^2} Z_{ij}^l = -\frac{|S_l|^2}{|S_i|^2} Z_{ji}^l.$$

It follows from (4.13) and (4.14) that  $Z_{jl}^i = 0$  whenever all three indices are distinct, such that Theorem 4.2 applies. We conclude that for rich, orthogonal frames the solution of Problem 2 depends on  $n$  functions of one variable.

**4.3. Rich systems with algebraic constraints.** We next consider the more involved situation of rich systems with non-trivial algebraic constraints (4.3) such that  $\text{rank}(N) > 0$ : there exist distinct  $i, j, k$  such that  $Z_{ij}^k \neq 0$ . The algebraic relations (4.3) then impose the equality  $\kappa^i = \kappa^j$ , such that there are multiplicity conditions on the eigenvalues. (In particular, there are no strictly hyperbolic systems in this “rich &  $\text{rank}(N) > 0$ ” case.) This makes the analysis of the  $\lambda$ -system more involved.

The analysis is further complicated by the fact that there may be additional equalities among eigenvalues that are imposed by the PDEs (4.2). We claim that, after taking all of these relations into account, the  $\lambda$ -system (4.2)-(4.3) reduces to a first order system of PDEs to which the Darboux theorem (Theorem 4.1) can be applied. We have:

**Theorem 4.3.** *Given a  $C^1$ -smooth, rich frame  $\{R_1(u), \dots, R_n(u)\}$  in a neighborhood of  $\bar{w} \in \mathbb{R}^n$ . Let  $(w^1(u), \dots, w^n(u))$  be associated Riemann invariants and assume the normalization (4.1). Let the connection coefficients  $Z_{ij}^k$  be defined by (4.4), and assume that there exists at least one triple of distinct  $i, j, k$  with  $Z_{ij}^k \neq 0$ .*

*Then the  $\lambda$ -system (4.2)-(4.3) imposes multiplicity conditions on the eigenvalues in the following sense. There are disjoint subsets  $A_1, \dots, A_{s_0} \subset \{1, \dots, n\}$  ( $s_0 \geq 1$ ) of cardinality two or more, and such that (4.2)-(4.3) impose the equality  $\kappa^i = \kappa^j$  if and only if  $i, j \in A_\alpha$  for some  $\alpha \in \{1, \dots, s_0\}$ . Let  $l = \sum_{\alpha=1}^{s_0} |A_\alpha| \leq n$  and  $s_1 = n - l$ . By relabeling indices we may assume that  $\{1, \dots, n\} \setminus \bigcup_{\alpha=1}^{s_0} A_\alpha = \{1, \dots, s_1\}$ .*

*Then, given  $s_1$  functions of one variable  $\varphi^1, \dots, \varphi^{s_1}$ , and  $s_0$  constants  $\hat{h}^1, \dots, \hat{h}^{s_0}$ , there is a unique local solution  $\kappa^1, \dots, \kappa^n$  of the  $\lambda$ -system (4.2)-(4.3) such that*

$$\begin{aligned} \kappa^i(\bar{w}^1, \dots, \bar{w}^{i-1}, w^i, \bar{w}^{i+1} \dots \bar{w}^n) &= \varphi_i(w^i) && \text{if } i = 1, \dots, s_1, \\ \kappa^i(\bar{w}) &= \hat{h}^\alpha && \text{if } i \in A_\alpha \text{ for some } \alpha = 1, \dots, s_0. \end{aligned}$$

Moreover,  $\partial_i \kappa^j(w) = 0$  whenever  $i, j \in A_\alpha$  for some  $\alpha = 1, \dots, s_0$ . Here  $s_1$  is the maximal number of simple eigenvalues, while  $s_0$  is the maximal number of non-simple eigenvalues, in a solution of the  $\lambda$ -system.

This result is a direct consequence of the following two lemmas and Theorem 4.1. The first lemma details how the index sets  $A_\alpha$  are defined, and how to write the  $\lambda$ -system (4.2)-(4.3) as a system of only PDEs.

**Lemma 4.4.** *With the same assumptions as in Proposition 4.3, the  $\lambda$ -system (4.2) - (4.3) can be re-written as a first order PDE system without algebraic constraints. More precisely, there are integers  $s_0, s_1$ , and disjoint subsets  $A_1, \dots, A_{s_0} \subset \{1, \dots, n\}$  as described in Proposition 4.3 such that the following holds. The re-written system involves  $s_1 + s_0$  unknowns, denoted  $\kappa^1(w), \dots, \kappa^{s_1}(w), h^1(w), \dots, h^{s_0}(w)$ , and has the following form:*

$$(4.15) \text{ for } j = 1, \dots, s_1 : \quad \partial_i \kappa^j = \begin{cases} Z_{ji}^j(\kappa^i - \kappa^j) & \text{if } 1 \leq i \neq j \leq s_1 \\ Z_{ji}^j(h^\alpha - \kappa^j) & \text{if } i \in A_\alpha \text{ for some } \alpha = 1, \dots, s_0, \end{cases}$$

$$(4.16) \text{ for } \alpha = 1, \dots, s_0 : \quad \partial_i h^\alpha = \begin{cases} W_i^\alpha(\kappa^i - h^\alpha) & \text{if } i = 1, \dots, s_1 \\ W_i^\alpha(h^\beta - h^\alpha) & \text{if } i \in A_\beta \text{ and } 1 \leq \beta \neq \alpha \leq s_0 \\ 0 & \text{if } i \in A_\alpha. \end{cases}$$

Here the coefficients  $W_i^\alpha$  are defined by  $W_i^\alpha := Z_{ji}^j$  if  $j \in A_\alpha$  and  $i \notin A_\alpha$ , and these are well-defined by the properties of the sets  $A_\alpha$ .

*Proof.* We construct the index sets  $A_1, \dots, A_{s_0}$  by first considering the algebraic conditions (4.3), and then taking into account the differential equations (4.2).

Each of the algebraic relations in (4.3) is either trivial (“ $0 = 0$ ”) or non-trivial, imposing equality of two eigenvalues. More precisely, if  $1 \leq i \neq j \leq n$  are such that there exists  $k \notin \{i, j\}$  with  $Z_{ij}^k \neq 0$ , then necessarily  $\kappa^i = \kappa^j$ . By assumption there is at least one such non-trivial algebraic relation. Grouping together the indices  $i$  of the unknowns  $\kappa^i$  that must be identical according to these relations, we obtain a certain number  $\tilde{s}_0 \geq 1$  of disjoint index sets  $\tilde{A}_1, \dots, \tilde{A}_{\tilde{s}_0} \subset \{1, \dots, n\}$ : two distinct  $j_1, j_2$  belong to the same  $\tilde{A}_\alpha$  if and only if one of the relations in (4.3) imposes the equality  $\kappa^{j_1} = \kappa^{j_2}$ . Clearly, the cardinality of each  $\tilde{A}_\alpha$  is at least two. For  $\alpha = 1, \dots, \tilde{s}_0$  we introduce the unknowns  $\tilde{h}^\alpha$  by setting

$$\tilde{h}^\alpha(w) := \kappa^j(w) \quad \forall j \in \tilde{A}_\alpha,$$

which is well-defined by the definition of  $\tilde{A}_\alpha$ . Also let

$$(4.17) \quad \tilde{A} := \bigcup_{\alpha=1}^{\tilde{s}_0} \tilde{A}_\alpha \quad \tilde{s}_1 := n - |\tilde{A}|.$$

If necessary we relabel the unknowns such that  $\kappa^1, \dots, \kappa^{\tilde{s}_1}$  denote the unknowns for which the algebraic relations (4.3) impose no multiplicity constraint.

We next turn to the PDE system (4.2). For each pair  $(i, j)$ , with  $i \neq j$ , either none, exactly one, or both of  $i$  and  $j$  belong to  $\tilde{A}$ . We list the possible cases together with the corresponding form of

the PDEs in each case:

$$(4.18) \quad \left\{ \begin{array}{ll} \text{(a)} & 1 \leq i \neq j \leq \tilde{s}_1 : \quad \partial_i \kappa^j = Z_{ji}^j(\kappa^i - \kappa^j) \\ \text{(b)} & \exists \alpha : i \in \tilde{A}_\alpha \text{ and } 1 \leq j \leq \tilde{s}_1 : \quad \partial_i \kappa^j = Z_{ji}^j(\tilde{h}^\alpha - \kappa^j) \\ \text{(c)} & \exists \beta : j \in \tilde{A}_\beta \text{ and } 1 \leq i \leq \tilde{s}_1 : \quad \partial_i \tilde{h}^\beta = Z_{ji}^j(\kappa^i - \tilde{h}^\beta) \\ \text{(d)} & \exists \gamma \neq \delta : i \in \tilde{A}_\gamma \text{ and } j \in \tilde{A}_\delta : \quad \partial_i \tilde{h}^\delta = Z_{ji}^j(\tilde{h}^\gamma - \tilde{h}^\delta) \\ \text{(e)} & \exists \epsilon : i, j \in \tilde{A}_\epsilon : \quad \partial_i \tilde{h}^\epsilon = 0. \end{array} \right.$$

At this stage we have used all non-trivial algebraic relations imposed by the algebraic part (4.3) of the  $\lambda$ -system. The issue now is that the cases (c) and (d) may impose further *algebraic* conditions since their left-hand sides are independent of the index  $j$ :

- in case (c): for a given pair  $(i, j)$  with  $j \in \tilde{A}_\beta$  and  $1 \leq i \leq \tilde{s}_1$ , unless all the coefficients  $Z_{ki}^k$  coincide as  $k$  ranges over  $\tilde{A}_\beta$ , we must impose that  $\kappa^i = \tilde{h}^\beta$ . If so we add  $i$  to the index set  $\tilde{A}_\beta$  and replace the  $|\tilde{A}_\beta|$  PDEs  $\partial_i \tilde{h}^\beta = Z_{ji}^j(\kappa^i - \tilde{h}^\beta)$  in (c) by the single PDE  $\partial_i \tilde{h}^\beta = 0$ . At the same time we substitute  $\tilde{h}^\beta$  for  $\kappa^i$  in the remaining PDEs in (a)-(c) in which  $\kappa^i$  appears.

On the other hand, if  $Z_{ki}^k = Z_{ji}^j$  for all  $k \in \tilde{A}_\beta$ , then we define  $\tilde{W}_i^\beta := Z_{ji}^j$ , and replace the same  $|\tilde{A}_\beta|$  PDEs by the single PDE  $\partial_i \tilde{h}^\beta = \tilde{W}_i^\beta(\kappa^i - \tilde{h}^\beta)$ .

- in case (d): for a given pair  $(i, j)$  with  $i \in \tilde{A}_\gamma$ ,  $j \in \tilde{A}_\delta$ , and  $\gamma \neq \delta$ , unless the coefficients  $Z_{ki}^k$  all coincide as  $k$  ranges over  $\tilde{A}_\delta$ , we must impose that  $\tilde{h}^\delta = \tilde{h}^\gamma$ . We then merge the index sets  $\tilde{A}_\gamma$  and  $\tilde{A}_\delta$  and replace the  $|\tilde{A}_\delta|$  PDEs  $\partial_i \tilde{h}^\delta = Z_{ji}^j(\tilde{h}^\gamma - \tilde{h}^\delta)$  in (d) by the single PDE  $\partial_i \tilde{h}^\delta = 0$ . At the same time we substitute  $\tilde{h}^\delta$  for  $\tilde{h}^\gamma$  in the remaining PDEs in (b)-(e) in which  $\tilde{h}^\gamma$  appears.

On the other hand, if  $Z_{ki}^k = Z_{ji}^j$  for all  $k \in \tilde{A}_\delta$ , then we define  $\tilde{W}_i^\delta := Z_{ji}^j$  and replace the same  $|\tilde{A}_\delta|$  PDEs by the single PDE  $\partial_i \tilde{h}^\delta = \tilde{W}_i^\delta(\tilde{h}^\gamma - \tilde{h}^\delta)$ .

We continue this process of identifying unknowns  $\kappa^i$  and  $\tilde{h}^\alpha$  that must necessarily coincide (enlarging and merging index sets), until no further reduction is possible. Since each reduction consists in setting unknowns equal to each other, and since setting all unknowns equal gives the trivial solutions to the  $\lambda$ -system (Proposition 2.1), it follows that no contradiction will be obtained in this manner. Also, as there is a finite number of unknowns, the process must terminate after finitely many reductions.

At this point we have obtained a certain number  $s_0 \geq 1$  of disjoint index sets  $A_1, \dots, A_{s_0} \subset \{1, \dots, n\}$  such that two distinct  $j_1, j_2$  belong to  $A_\alpha$  if and only if (4.2) and (4.3) impose the equality  $\kappa^{j_1} = \kappa^{j_2}$  (through the reduction process described above). By assumption no further algebraic reduction is possible, and it follows that the sets  $A_\alpha$  have the following properties:

$$(4.19) \quad Z_{j_1 i}^{j_1} = Z_{j_2 i}^{j_2} =: W_i^\alpha \quad \text{whenever } j_1, j_2 \in A_\alpha \text{ and } i \notin A_\alpha,$$

and

$$(4.20) \quad Z_{ij}^k = 0 \quad \text{whenever } i \in A_\alpha, j \notin A_\alpha \text{ and } k \notin \{i, j\}.$$

We also have that

$$(4.21) \quad Z_{ij}^k = 0 \quad \text{whenever } 1 \leq i \neq j \leq s_1 \text{ and } k \notin \{i, j\}.$$

Each of the sets  $\tilde{A}_\beta$ ,  $\beta = 1, \dots, \tilde{s}_0$ , is contained in one of the sets  $A_\alpha$ ,  $\alpha = 1, \dots, s_0$ , whence the latter have cardinality  $|A_\alpha| \geq 2$ . Setting

$$A := \bigcup_{\alpha=1}^{s_0} A_\alpha, \quad \text{and} \quad s_1 := n - |A|,$$

it follows that  $s_0$  is the maximal number of non-simple eigenvalues, and  $s_1$  is the maximal number of simple eigenvalues, in a solution of the  $\lambda$ -system. If necessary we relabel the indices such that  $\kappa^1, \dots, \kappa^{s_1}$  denote the unknowns for which (4.2)-(4.3) impose no multiplicity constraints, and we introduce unknowns

$$(4.22) \quad h^\alpha(w) := \kappa^j(w) \quad \text{for } j \in A_\alpha, \alpha = 1, \dots, s_0.$$

Similar to above, for each pair  $(i, j)$  with  $i \neq j$ , either none, exactly one, or both of  $i$  and  $j$  belong to  $A$ , and again there are five possible cases of the PDEs in (4.2). We thus obtain the same type of system as in (4.18) except with  $\tilde{s}_1$  replaced by  $s_1$  and  $\tilde{A}_\alpha$ 's replaced by  $A_\alpha$ 's. Using (4.19) we obtain the system (4.15)-(4.16). Finally, from the last equation in (4.16) it follows that  $h^\alpha$  is independent of the coordinates  $w^i$  for  $i \in A_\alpha$ :  $\partial_i \kappa^j(w) = 0$  for all  $i, j \in A_\alpha$ .  $\square$

**Remark 4.2.** We note that if  $s_0 = 1$  and  $|A_1| = n$ , i.e.  $\kappa^1 = \dots = \kappa^n = h^1$ , then (4.15)-(4.16) degenerate into the trivial system  $\frac{\partial h^1}{\partial w^i} = 0$ ,  $i = 1, \dots, n$ . In this case the only solutions to the original problem are the trivial ones:  $\kappa^1(w) = \dots = \kappa^n(w) \equiv \hat{\kappa} \in \mathbb{R}$ . If  $l = n$ , then  $s_1 = 0$  and the system (4.15) is empty. In this case all eigenvalues  $\kappa^i$  necessarily has multiplicity greater than one.

The PDE system (4.15) - (4.16) is in the form required by Theorem 4.1, and we proceed to verify the compatibility conditions (C) in this case.

**Lemma 4.5.** *With the same assumptions as in Lemma 4.4, the compatibility conditions (C) in Theorem 4.1 are satisfied for the rewritten  $\lambda$ -system (4.15)-(4.16).*

*Proof.* Compatibility conditions for (4.15): For  $j = 1, \dots, s_1$ , we need to verify that

$$(4.23) \quad \partial_k \partial_m \kappa^j = \partial_m \partial_k \kappa^j \quad \text{for } k \neq j, m \neq j \text{ and } m \neq k.$$

There are two main cases depending on whether  $m$  and  $k$  belong to a common  $A_\alpha$  or not, and we first treat the latter.

• *Case 1.*  $\nexists \alpha$  such that  $\{m, k\} \subset A_\alpha$ . There are several sub-cases according to whether  $m$  or  $k$  belong to  $A$ . However, in each case the compatibility condition takes the form:

$$(4.24) \quad \begin{aligned} & \left( \partial_m Z_{jk}^j - \partial_k Z_{jm}^j \right) \kappa^j + \left( Z_{jm}^j Z_{mk}^m + Z_{jk}^j Z_{km}^k - Z_{jm}^j Z_{jk}^j - \partial_m Z_{jk}^j \right) \kappa^k \\ & - \left( Z_{jm}^j Z_{mk}^m + Z_{jk}^j Z_{km}^k - Z_{jk}^j Z_{jm}^j - \partial_k Z_{jm}^j \right) \kappa^m \equiv 0, \end{aligned}$$

with the provision that

$$\kappa^m = h^\alpha, \quad Z_{mk}^m = W_k^\alpha \quad \text{if } m \in A_\alpha \text{ for some } 1 \leq \alpha \leq s_0,$$

and

$$\kappa^k = h^\beta, \quad Z_{km}^k = W_m^\beta \quad \text{if } k \in A_\beta \text{ for some } 1 \leq \beta \leq s_0.$$

We verify that (4.24) is satisfied by showing that coefficients of  $\kappa^j$ ,  $\kappa^m$  and  $\kappa^k$  vanish identically due to the flatness condition (4.6). We first substitute  $i = j$  in (4.6) to obtain

$$\partial_m (Z_{jk}^j) - \partial_k (Z_{jm}^j) = \sum_{t=1}^n (Z_{kt}^j Z_{mj}^t - Z_{mt}^j Z_{kj}^t).$$



By assumption (4.20), for distinct  $k, m, j$  with  $j \notin A$ , we have

$$\begin{aligned} Z_{mj}^t &= 0 && \text{unless } t = m \text{ or } t = j \\ Z_{kj}^t &= 0 && \text{unless } t = k \text{ or } t = j, \end{aligned}$$

such that

$$(4.25) \quad \partial_m(Z_{jk}^j) - \partial_k(Z_{jm}^j) = Z_{mk}^j(Z_{mj}^m - Z_{kj}^k).$$

By assumption  $\nexists \alpha$  such that  $\{m, k\} \subset A_\alpha$ , whence  $Z_{mk}^j = 0$  due to assumption (4.20). Thus

$$(4.26) \quad \partial_m(Z_{jk}^j) - \partial_k(Z_{jm}^j) = 0,$$

such that the coefficient of  $\kappa^j$  in (4.24) vanishes.

Next consider the coefficient of  $\kappa^k$  in (4.24). Interchange  $k$  and  $i$  in (4.6) and set  $i := j$  to get

$$(4.27) \quad \partial_m(Z_{jk}^j) - \partial_j(Z_{mk}^j) = \sum_t (Z_{jt}^j Z_{mk}^t - Z_{mt}^j Z_{jk}^t).$$

Again, by assumption  $\nexists \alpha$  such that  $\{m, k\} \subset A_\alpha$ , and (4.20) gives

$$(4.28) \quad \begin{cases} Z_{mk}^j = 0 \\ Z_{mk}^t = 0 & \text{unless } t = m \text{ or } t = k \\ Z_{jk}^t = 0 & \text{unless } t = j \text{ or } t = k. \end{cases}$$

Thus

$$\partial_m Z_{jk}^j = Z_{jm}^j Z_{mk}^m + Z_{jk}^j Z_{km}^k - Z_{jm}^j Z_{jk}^j - Z_{mk}^j Z_{kj}^k = Z_{jm}^j Z_{mk}^m + Z_{jk}^j Z_{km}^k - Z_{jm}^j Z_{jk}^j,$$

such that the coefficient of  $\kappa^k$  in (4.24) vanishes. The coefficient of  $\kappa^m$  is treated similarly.

- *Case 2.*  $\exists \alpha$  such that  $\{m, k\} \subset A_\alpha$ . In this case the compatibility conditions (4.23) read

$$[\partial_m(Z_{jk}^j) - \partial_k(Z_{jm}^j)](h^\alpha - \kappa^j) \equiv 0.$$

We claim that the first factor vanishes, i.e., (4.26) is valid in this case as well. Indeed, the derivation of (4.25) is valid in this case as well. By assumption  $\exists \alpha$  such that  $\{m, k\} \subset A_\alpha$ , such that  $Z_{mj}^m = Z_{kj}^k = Z_j^\alpha$  due to assumption (4.19). Thus the compatibility conditions for (4.15) are met.

*Compatibility conditions for (4.16):* For  $\alpha = 1, \dots, s_0$  we need to verify that

$$(4.29) \quad \partial_k \partial_m h^\alpha = \partial_m \partial_k h^\alpha \quad \text{for } m \neq k.$$

For a fixed  $\alpha$  there are several cases depending on whether both, exactly one, or neither of  $m$  and  $k$  belong to  $A_\alpha$ .

- *Case 1.* If both  $m$  and  $k$  belong to  $A_\alpha$  then (4.29) amounts to the trivial condition  $0 = 0$ .

• *Case 2.* Without loss of generality we assume that  $m \in A_\alpha \not\subset k$ . The compatibility condition (4.29) then reduces to

$$(4.30) \quad 0 = \partial_m \partial_k h^\alpha = \partial_m (W_k^\alpha (\kappa^k - h^\alpha)) = [\partial_m (W_k^\alpha) - W_k^\alpha Z_{km}^k] (\kappa^k - h^\alpha).$$

Since  $|A_\alpha| > 1$  there is a  $j \neq m \in A_\alpha$ , such that  $W_k^\alpha = Z_{jk}^j$ . We will verify (4.30) by showing that the first factor on the right-hand side vanishes:

$$(4.31) \quad \partial_m (W_k^\alpha) - W_k^\alpha Z_{km}^k = \partial_m (Z_{jk}^j) - Z_{jk}^j Z_{km}^k = 0.$$

Observing that (4.20) again yields (4.28), we see that (4.27) gives

$$\begin{aligned} \partial_m (Z_{jk}^j) &= Z_{jk}^j Z_{km}^k + Z_{jm}^j Z_{mk}^m - Z_{jm}^j Z_{jk}^j \\ &= Z_{jk}^j Z_{km}^k + Z_{jm}^j (Z_{mk}^m - Z_{jk}^j) = Z_{jk}^j Z_{km}^k, \end{aligned}$$

where we have used that  $Z_{mk}^m = Z_{jk}^j = W_k^\alpha$ , since  $j, m \in A_\alpha$ . This verifies (4.31).

• *Case 3.* If neither  $m$  nor  $k$  belong to  $A_\alpha$  then the compatibility condition (4.29) reduces to

$$(4.32) \quad \partial_m [W_k^\alpha (\kappa^k - h^\alpha)] = \partial_k [W_m^\alpha (\kappa^m - h^\alpha)],$$

where  $W_k^\alpha = Z_{jk}^j$  and  $W_m^\alpha = Z_{jm}^j$ , for all  $j \in A_\alpha$ , and with the provision that

$$\kappa^k = h^\beta \quad \text{if } k \in A_\beta \ (\beta \neq \alpha) \quad \text{and} \quad \kappa^m = h^\gamma \quad \text{if } m \in A_\gamma \ (\gamma \neq \alpha).$$

There are three sub-cases depending on whether both, exactly one, or neither of  $m$  nor  $k$  belong to some  $A_\beta$  with  $\beta \neq \alpha$ . To verify the identity (4.32) we use the same techniques as above: flatness (4.6), and (4.19)-(4.21). The details are similar to the previous cases and are left out.  $\square$

Combining Lemmas 4.4 - 4.5 with Theorem 4.1 completes the proof of Proposition 4.3.

## 5. EXAMPLES

### 5.1. The Euler system for 1-dimensional compressible flow.

**Example 5.1.** *The Euler system for 1-dimensional compressible flow in Lagrangian variables is:*

$$(5.1) \quad v_t - u_x = 0$$

$$(5.2) \quad u_t + p_x = 0$$

$$(5.3) \quad E_t + (up)_x = 0,$$

where  $v, u, p$  are the specific volume, velocity, and pressure, respectively, and  $E = \epsilon + \frac{u^2}{2}$  is the total specific energy, where  $\epsilon$  denotes the specific internal energy. The thermodynamic variables are related through Gibbs' relation  $d\epsilon = TdS - pdv$ , where  $T$  is temperature and  $S$  is specific entropy. Using this, the Euler system may be rewritten as

$$(5.4) \quad v_t - u_x = 0$$

$$(5.5) \quad u_t + p_x = 0$$

$$(5.6) \quad S_t = 0.$$

The system is closed by prescribing a pressure function  $p = p(v, S) > 0$ , and we make the standard assumption that  $p_v(v, S) < 0$ . For a given a pressure function the eigenvalues of (5.4)-(5.6) are:

$$\lambda^1 = -\sqrt{-p_v}, \quad \lambda^2 \equiv 0, \quad \lambda^3 = \sqrt{-p_v},$$

with corresponding right and left eigenvectors (in  $(v, u, S)$ -space and normalized according to  $R_i \cdot L^j \equiv \delta_i^j$ ):

$$(5.7) \quad R_1 = [1, \sqrt{-p_v}, 0]^T, \quad R_2 = [-p_S, 0, p_v]^T, \quad R_3 = [1, -\sqrt{-p_v}, 0]^T,$$

and

$$(5.8) \quad L^1 = \frac{1}{2} \left[ 1, \frac{1}{\sqrt{-p_v}}, \frac{p_S}{p_v} \right], \quad L^2 = \left[ 0, 0, \frac{1}{p_v} \right], \quad L^3 = \frac{1}{2} \left[ 1, -\frac{1}{\sqrt{-p_v}}, \frac{p_S}{p_v} \right].$$

The fact that the eigenfields depend only on the pressure allows us to pose the ‘‘inverse’’ problem:

**Problem 3.** (Problem 1 for the Euler system) *For a given pressure function  $p = p(v, S) > 0$ , with  $p_v < 0$ , consider the frame given by (5.7). Then: determine the class of conservative systems with these as eigenfields by solving the associated  $\lambda$ -system (2.12)-(2.13) for the eigenvalues  $\lambda^1, \lambda^2, \lambda^3$ .*

**Remark 5.1.** *As explained in Remark 1.1 the two forms (5.1)-(5.3) and (5.4)-(5.6) of the Euler system are equivalent as far as Problem 3 is concerned. However, they are not equivalent at the level of computing weak solutions to Cauchy problems [38].*

We shall see that there are two distinct cases depending on whether the frame is rich or not. From (5.7), (5.8), and (2.8) we obtain

$$\Gamma_{21}^2 = \Gamma_{23}^2 = 0, \quad \Gamma_{31}^3 = \Gamma_{13}^1 = -\frac{p_{vv}}{4p_v}, \quad \Gamma_{12}^1 = \Gamma_{32}^3 = -\frac{p_v}{2} \left( \frac{p_S}{p_v} \right)_v.$$

Thus the PDEs (2.12) in the  $\lambda$ -system are given by

$$(5.9) \quad r_1(\lambda^2) = 0$$

$$(5.10) \quad r_1(\lambda^3) = \frac{p_{vv}}{4p_v}(\lambda^3 - \lambda^1)$$

$$(5.11) \quad r_2(\lambda^1) = \frac{p_v}{2} \left( \frac{p_S}{p_v} \right)_v (\lambda^1 - \lambda^2)$$

$$(5.12) \quad r_2(\lambda^3) = \frac{p_v}{2} \left( \frac{p_S}{p_v} \right)_v (\lambda^3 - \lambda^2)$$

$$(5.13) \quad r_3(\lambda^1) = \frac{p_{vv}}{4p_v}(\lambda^1 - \lambda^3)$$

$$(5.14) \quad r_3(\lambda^2) = 0.$$

where  $r_1, r_2, r_3$  are the derivations corresponding to  $R_1, R_2, R_3$ . The coefficients appearing in the algebraic equations (2.13) of the  $\lambda$ -system are:

$$\Gamma_{32}^1 = \Gamma_{12}^3 = -\frac{p_v}{2} \left( \frac{p_S}{p_v} \right)_v, \quad \Gamma_{23}^1 = \Gamma_{21}^3 = -\frac{p_v}{4} \left( \frac{p_S}{p_v} \right)_v, \quad \Gamma_{31}^2 = \Gamma_{13}^2 = 0.$$

Using these we see that the matrix  $N$  in (3.7) is given by

$$(5.15) \quad N = \frac{p_v}{4} \left( \frac{p_S}{p_v} \right)_v \begin{bmatrix} -2 & 1 \\ 0 & 0 \\ -2 & 1 \end{bmatrix}.$$

By assumption  $p_v < 0$ , and we thus consider the two cases:

- (a)  $\left( \frac{p_S}{p_v} \right)_v \equiv 0$ :  $\text{rank}(N) = 0$  and there are no algebraic constraints among the eigenvalues. Proposition 2.2 and Theorem 4.2 show that the Euler system is rich in this case, and that the general solution to Problem 3 depends on three functions of one variable.
- (b)  $\left( \frac{p_S}{p_v} \right)_v \neq 0$ :  $\text{rank}(N) = 1$  and there is a single algebraic relation among the eigenvalues:

$$(5.16) \quad \lambda^1 + \lambda^3 = 2\lambda^2.$$

For case (a) the pressure must be of the form  $p(v, S) = \Pi(\xi)$ , where  $\xi = v + F(S)$  and  $F$  is a function of  $S$  alone. It follows from (5.9) and (5.14) that  $\lambda^2$  is any function of  $S$  alone, and from (5.11) and (5.12) it follows that  $\lambda^1 = A(\xi, u)$  and  $\lambda^3 = B(\xi, u)$ . The functions  $A$  and  $B$  should then be determined from the remaining equations (5.10) and (5.13):

$$A_\xi - \sqrt{-\Pi'(\xi)} A_u = \alpha(B - A), \quad B_\xi + \sqrt{-\Pi'(\xi)} B_u = \alpha(A - B),$$

where  $\alpha = -\frac{p_{vv}}{4p_v} = -\frac{\Pi''(\xi)}{4\Pi'(\xi)}$ . These equations must be solved case-by-case. However, by deriving a second order equation for  $A$  alone, which in turn will determine  $B$ , we see that solving for  $A$  and  $B$  requires two functions of one variable. Together with  $\lambda^2(S)$  these are the three functions that determine a general solution in case (a).

We proceed to explicitly solve the PDE system in the more interesting second case (b). Adding (5.11) and (5.12), and using (5.16), gives  $r_2(\lambda^1 + \lambda^3) = 0$ . Applying (5.16) again yields  $r_2(\lambda^2) = 0$ . Together with (5.9) and (5.14) this shows that

$$(5.17) \quad \lambda^2 \equiv \bar{\lambda} \quad (\text{constant}).$$

To continue it is convenient to express the coordinate frame  $\{\partial_v, \partial_u, \partial_S\}$  in terms of the  $r$ -frame:

$$(5.18) \quad \partial_v = \frac{1}{2}(r_1 + r_3), \quad \partial_u = \frac{1}{2\sqrt{-p_v}}(r_1 - r_3), \quad \partial_S = \frac{1}{p_v}r_2 + \frac{p_S}{2p_v}(r_1 + r_3).$$

A direct calculation, using (5.18) and (5.16), now shows that  $\Delta := (\lambda^3 - \lambda^1)$  satisfies  $\partial_u \Delta = 0$  and

$$\partial_v \Delta = \left(\frac{p_{vv}}{2p_v}\right)\Delta, \quad \partial_S \Delta = \left(\frac{p_{vS}}{2p_v}\right)\Delta.$$

Integration with respect to  $v$  and  $S$ , respectively, shows that

$$\Delta(v, S) = \lambda^3 - \lambda^1 = D\sqrt{-p_v}$$

for a constant  $D$ . Invoking (5.16) a last time (with  $\lambda^2 \equiv \bar{\lambda}$ ), we conclude that the general solution of the  $\lambda$ -system for non-rich gas dynamics (case (b)) depends on two constants  $\bar{\lambda}$ ,  $C$  according to:

$$\lambda^1 = \bar{\lambda} - C\sqrt{-p_v}, \quad \lambda^2 \equiv \bar{\lambda}, \quad \lambda^3 = \bar{\lambda} + C\sqrt{-p_v}.$$

**5.2. Additional examples with non-rich frames.** The case of non-rich gas dynamics in Example 5.1 provides an example of Subcase IIa in Section 3.1 in which the compatibility conditions (3.26)-(3.27) for the Frobenius system (3.22) hold identically. The following Example 5.2 of a system in the same category, but with only trivial solutions, shows that these compatibility conditions do not necessarily hold as identities and must be checked in each case. This verifies the last claim in Proposition 3.2.

**Example 5.2.**  $n = 3$ , NON-RICH SYSTEM WITH A SINGLE ALGEBRAIC RELATION OF TYPE IIa AND ONLY TRIVIAL SOLUTIONS. We prescribe the frame  $\{R_1, R_2, R_3\}$  on  $\mathbb{R}^3$  by

$$R_1 = [0, 0, 1]^T, \quad R_2 = [0, 1, u^1]^T, \quad R_3 = [u^3, 0, 1]^T.$$

The system (3.1)-(3.6) takes form

$$\begin{aligned} \partial_3 \lambda^2 &= 0, \\ \partial_3 \lambda^3 &= 0, \\ \partial_2 \lambda^1 + u^1 \partial_3 \lambda^1 &= 0, \\ \partial_2 \lambda^3 + u^1 \partial_3 \lambda^3 &= 0, \\ u^3 \partial_1 \lambda^1 + \partial_3 \lambda^1 + \frac{1}{u^3}(\lambda^3 - \lambda^1) &= 0, \\ u^3 \partial_1 \lambda^2 + \partial_3 \lambda^2 &= 0. \end{aligned}$$

In this case the matrix  $N$  is given by

$$N = \begin{bmatrix} u^3 & \frac{u^1}{u^3} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and has rank 1. It corresponds to the unique algebraic relation that involves all three eigenvalues:

$$(u^3)^2(\lambda^2 - \lambda^1) + u^1(\lambda^3 - \lambda^1) = 0.$$

The only solution of the above system of differential and algebraic conditions is trivial  $\lambda^1 = \lambda^2 = \lambda^3 = \text{const}$ .

Next we provide examples of a non-rich 3-frame with a single algebraic relation of type IIb. The examples show that in accordance with Proposition 3.4 the solution of the  $\lambda$ -system in this case either depends on one arbitrary function of one variable and one arbitrary constant, or is trivial.

**Example 5.3.**  $n = 3$ , NON-RICH SYSTEM WITH A SINGLE ALGEBRAIC RELATION OF TYPE IIb AND NON TRIVIAL SOLUTIONS. Consider a system with eigenvectors:

$$R_1 = (0, 1, 0)^T, \quad R_2 = (1, 0, 0)^T, \quad R_3 = (u^2, u^3, 1)^T.$$

Since  $[r_1, r_3] = r_2$ , these vector-fields are not pair-wise in involution, and therefore the system is not rich. The  $\lambda$ -system implies a unique algebraic relations  $\lambda^2 = \lambda^3$ , which involves only two of the  $\lambda$ 's. We are thus in Subcase IIb in Section 3.3. A computation show that  $\Gamma_{31}^3 = 0$  and  $\Gamma_{21}^2 = 0$  are

equal and thus according to Proposition 3.4 we can expect a non-trivial general solution depending on one arbitrary function of one variable and one arbitrary constant. Indeed, taking into account the algebraic relation the differential equations become:

$$\begin{aligned} r_2(\lambda^1) &= \partial_1 \lambda^1 = 0, \\ r_3(\lambda^1) &= u^2 \partial_1 \lambda^1 + u^3 \partial_2 \lambda^1 + \partial_3 \lambda^1 = 0, \\ r_1(\lambda^2) &= \partial_2 \lambda^2 = 0, \\ r_2(\lambda^2) &= \partial_1 \lambda^2 = 0, \\ r_3(\lambda^2) &= u^2 \partial_1 \lambda^2 + u^3 \partial_2 \lambda^2 + \partial_3 \lambda^2 = 0. \end{aligned}$$

The general solution of this system is  $\lambda^1 = \varphi(u_3^2 - 2u_2)$ ,  $\lambda^2 = \lambda^3 \equiv \bar{\lambda}$ , where  $\bar{\lambda}$  is an arbitrary constant and  $\varphi$  is an arbitrary function of one variable.

**Example 5.4.**  $n = 3$ , NON-RICH SYSTEM WITH A SINGLE ALGEBRAIC RELATION OF TYPE IIb AND ONLY TRIVIAL SOLUTIONS. Consider a system with eigenvectors:

$$R_1 = (u^1 + u^3, 1, 0)^T, \quad R_2 = (1, 0, 0)^T, \quad R_3 = (u^2, u^3, -u^2)^T.$$

Since  $[r_1, r_3] = -\frac{u^3}{u^2} r^1 + \frac{(u^1+u^3)u^3}{u^2} r^2 + \frac{1}{u^2} r^3$  these vector-fields are not pair-wise in involution, and therefore the system is not rich. The  $\lambda$ -system implies a unique algebraic relations  $\lambda^2 = \lambda^3$ , which involves only two of the  $\lambda$ 's. We are thus in Subcase IIb in Section 3.3. A computation shows that  $\Gamma_{31}^3 = 0$  and  $\Gamma_{21}^2 = 1$  are not equal and thus according to Proposition 3.4 we can expect only trivial general solution. Indeed, taking into account the algebraic relation  $\lambda^2 = \lambda^3$  the differential equations  $r_1(\lambda^2) = \Gamma_{21}^2(\lambda^1 - \lambda^2)$  and  $r_1(\lambda^3) = \Gamma_{31}^3(\lambda^1 - \lambda^3)$  imply that

$$(5.19) \quad r_1(\lambda^2) = (\lambda^2 - \lambda^1) \quad \text{and} \quad r_1(\lambda^3) = 0,$$

which implies  $\lambda^1 = \lambda^2$ . The only non-trivial solution of the  $\lambda$  system is therefore  $\lambda^1 = \lambda^2 = \lambda^3 = \bar{\lambda}$ , where  $\bar{\lambda}$  is an arbitrary constant.

We include an example which illustrates the fact that maximal rank of  $N$  ( $\text{rank}(N) = n - 1$ ) implies that all solutions of the  $\lambda$ -system are trivial (see section 2.3).

**Example 5.5.**  $n = 4$ , SYSTEM WITH MAXIMAL RANK OF ALGEBRAIC CONSTRAINTS ( $\text{rank } N = 3$ ). The  $\lambda$ -system for the following non-rich frame in  $\mathbb{R}^4$

$$R_1 = (1, 0, u^2, u^4), \quad R_2 = (0, 1, u^1, 0), \quad R_3 = (u^3, 0, 1, 0), \quad R_4 = (1, 0, 0, 0)$$

includes three independent algebraic relations:

$$\lambda^2 = \lambda^1, \quad \lambda^3 = \lambda^4, \quad \lambda^3 = \lambda^2,$$

which immediately shows that all solutions must be trivial:  $\lambda^1 = \lambda^2 = \lambda^3 = \lambda^4 \equiv \text{const}$ .

### 5.3. Additional examples with rich frames.

**Example 5.6.**  $n = 2$  SYSTEM. Suppose we want to find the most general system (1.1) of two conservation laws whose eigencurves are hyperbolas and radial straight lines. As a  $2 \times 2$ -system it is necessarily rich and as Riemann invariants we use  $w^1 := u^1 u^2$  and  $w^2 := \frac{u^2}{u^1}$ , where  $(u^1, u^2)$  are standard coordinates on  $\mathbb{R}^2$ . Then the matrix of left eigenvectors is

$$L(u^1, u^2) = \begin{bmatrix} \nabla w^1 \\ \nabla w^2 \end{bmatrix} = \begin{bmatrix} u^2 & u^1 \\ -\frac{u^2}{(u^1)^2} & \frac{1}{u^1} \end{bmatrix}$$

and the matrix of right eigenvectors is  $R = L^{-1}$ . A direct computation shows that  $Z_{21}^2 = \frac{1}{2w^1}$  and  $Z_{12}^1 = 0$ , such that the  $\lambda$ -system (4.2) becomes

$$(5.20) \quad \partial_2 \kappa^1(w^1, w^2) = 0$$

$$(5.21) \quad \partial_1 \kappa^2(w^1, w^2) = \frac{1}{2w^1}(\kappa^1 - \kappa^2).$$

By (5.20) the first eigenvalue is an arbitrary function of  $w^1$  alone,  $\kappa^1 = \kappa^1(w^1)$ , and (5.21) then yields

$$\kappa^2(w^1, w^2) = \frac{1}{\sqrt{w^1}} [h(w^2) + g(w^1)], \quad \text{where } g(w^1) = \int_1^{w^1} \frac{\kappa^1(\xi)}{\sqrt{\xi}} d\xi,$$

and  $h$  is an arbitrary function of  $w^2$  alone. In accordance with Theorem 4.2 we see that the general solution depends on two functions of one variable.

As we proved in Section 4.2.1, for a rich and orthogonal frame the algebraic part of the  $\lambda$ -system becomes trivial. We consider a concrete case in this class.

**Example 5.7.** RICH ORTHOGONAL FRAME. Suppose we want to find the most general system (1.1) of two conservation laws whose eigencurves are the coordinates curves of spherical coordinates (radial lines, latitudes, and longitudes). The corresponding Riemann invariants are polar coordinates  $(r, \theta, \phi)$ . More precisely we let  $\rho : \mathbb{R}^3 \setminus \{(u^1, 0, u^3) \mid u^1 \leq 0, z \leq 0\} \rightarrow \mathbb{R}_+ \times (-\pi, \pi) \times (0, \pi)$  be the change from rectangular Cartesian coordinates  $(u^1, u^2, u^3)$  to spherical polar coordinates  $(r, \theta, \phi)$ ,

$$(5.22) \quad \rho(u, v, w) = (r, \theta, \phi),$$

where

$$r = \sqrt{(u^1)^2 + (u^2)^2 + (u^3)^2}, \quad \theta = \arctan \left( \frac{\sqrt{(u^1)^2 + (u^2)^2}}{u^3} \right), \quad \text{and} \quad \phi = \arctan \left( \frac{u^2}{u^1} \right).$$

Defining  $R$  by  $R^{-1} = D\rho$  we have

$$R = \begin{bmatrix} \sin \theta \cos \phi & r \cos \theta \cos \phi & -r \sin \theta \sin \phi \\ \sin \theta \sin \phi & r \cos \theta \sin \phi & r \sin \theta \cos \phi \\ \cos \theta & -r \sin \theta & 0 \end{bmatrix}.$$

A calculation now yields that  $Z_{12}^1 = Z_{13}^1 = Z_{23}^2 = 0$ ,  $Z_{21}^2 = Z_{31}^3 = \frac{1}{r}$ ,  $Z_{32}^3 = \arctan(\theta)$  and, as expected, all  $Z$ 's with three distinct indices are zero. Thus according to (4.2)-(4.3), the  $\lambda$ -system contains only the following set of differential equations:

$$\begin{aligned} \partial_\theta \kappa^1 &= 0 & \partial_\phi \kappa^1 &= 0, \\ \partial_r \kappa^2 &= \frac{1}{r}(\kappa^1 - \kappa^2), & \partial_\phi \kappa^2 &= 0, \\ \partial_r \kappa^3 &= \frac{1}{r}(\kappa^1 - \kappa^3), & \partial_\theta \kappa^3 &= \arctan \theta (\kappa^2 - \kappa^3). \end{aligned}$$

The general solution of this system depends on three arbitrary functions  $F, G, H$  each depending on a single variable:

$$\begin{aligned} \kappa^1 &= F(r), \\ \kappa^2 &= \frac{H(\theta) + g(r)}{r}, \quad \text{where } g(r) := \int_1^r F(\xi) d\xi, \\ \kappa^3 &= \frac{K(\phi) + \int_{\theta_0}^\theta H(\xi) \cos \xi d\xi}{r \sin \theta} + \frac{g(r)}{r}. \end{aligned}$$

**Example 5.8.** CONSTANT EIGENFIELDS. *If the frame  $\{R_1(u), \dots, R_n(u)\}$  consists of constant vector fields  $R_1(u) \equiv R_i$  then it is rich according to Definition 1. Furthermore, all connection coefficients  $Z_{ij}^k$  vanish and Theorem 4.2 applies. As associated Riemann invariants we may choose  $w^i(u) = L^i \cdot u$ , where each  $L_i$  is constant. A calculation shows that the general solution of  $\lambda$ -system (4.2) is given by  $\kappa^1 = \phi^1(w^1), \dots, \phi^n(w^n)$ , where  $\phi^1, \dots, \phi^n$  are arbitrary functions of one variable, in accordance with Theorem 4.2. That is, the eigenvalues solving Problem 1 in this case are given by  $\lambda^i(u) = \phi^i(L^i \cdot u)$ ,  $i = 1, \dots, n$ .*

We have seen an example of rich  $3 \times 3$ -system with no algebraic constraints ( $\text{rank } N = 0$ ) when we analyzed eigenvector-fields of the Euler system in Example 5.1. The following two examples illustrate our observation that Richness  $\not\Rightarrow \text{rank}(N) = 0$ . In Example 5.9 we have a rich system with  $\text{rank}(N) = 1$  with non trivial general solution depending on 1 arbitrary constant and 1 arbitrary function of 1 variable. In Example 5.10 we have a rich system with  $\text{rank}(N) = 2$  and therefore only trivial solutions.

**Example 5.9.**  $n = 3$ , RICH,  $\text{rank}(N) = 1$ , NON-TRIVIAL. *Consider a system with eigenvectors:*

$$R_1 = (1, 0, u^2)^T, \quad R_2 = (0, 1, u^1)^T, \quad R_3 = (0, 0, -1)^T.$$

*These vectors commute and therefore any system (1.1) with these as eigenvectors is rich. The only non-zero connection components are  $\Gamma_{12}^3 = \Gamma_{21}^3 = -1$ , such that the  $\lambda$ -system implies a single algebraic relation:  $\lambda^1 = \lambda^2$ . Thus this example illustrates the results in Section 4.3. Taking into account this relation the differential equations become:*

$$\begin{aligned} r_1(\lambda^1) &= \partial_1 \lambda^1 + u^2 \partial_3 \lambda^1 = 0, \\ r_1(\lambda^3) &= \partial_1 \lambda^3 + u^2 \partial_3 \lambda^3 = 0, \\ r_2(\lambda^1) &= \partial_2 \lambda^1 + u^1 \partial_3 \lambda^1 = 0, \\ r_2(\lambda^3) &= \partial_2 \lambda^3 + u^1 \partial_3 \lambda^3 = 0, \\ r_3(\lambda^1) &= -\partial_3 \lambda^1 = 0. \end{aligned}$$

*The general solution of this system is  $\lambda^1 = \lambda^2 \equiv \bar{\lambda}$ , and  $\lambda^3 = \varphi(u^3 - u^1 u^2)$ , where  $\bar{\lambda}$  is an arbitrary constant and  $\varphi$  is an arbitrary function of one variable.*

**Example 5.10.**  $n = 3$ , RICH,  $\text{rank } N = 2$ , TRIVIAL. *Consider the vector fields*

$$R_1 = [u^1, u^2, 0]^T, \quad R_2 = [-u^2, u^1, 0]^T, \quad R_3 = [-u^2, u^1, 1]^T.$$

*We are then searching for systems (1.1) whose eigencurves are horizontal lines through the  $u^3$ -axis, horizontal circles, and vertical helices. A calculation shows that these determine a rich frame. We can rewrite this problem in terms of Riemann invariants using the change of coordinates:  $\rho : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  given by*

$$(5.23) \quad \rho^{-1}(r, \theta, h) = (u^1, u^2, u^3) = (r \cos(\theta + h), r \sin(\theta + h), h).$$

*A calculation show that  $Z_{23}^1 = -r$  and  $Z_{13}^2 = \frac{1}{r}$ . It follows from (4.3) and Proposition 2.1 that the only solutions are the trivial solutions  $\lambda^1 = \lambda^2 = \lambda^3 \equiv \text{const}$ .*

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