LIPSCHITZ METRIC FOR THE HUNTER–SAXTON EQUATION

ALBERTO BRESSAN, HELGE HOLDEN, AND XAVIER RAYNAUD

Abstract. We study stability of solutions of the Cauchy problem for the Hunter–Saxton equation $u_t + uu_x = \frac{1}{4}(\int_{-\infty}^{x} u_x^2\,dx - \int_x^{\infty} u_x^2\,dx)$ with initial data u_0. In particular, we derive a new Lipschitz metric $d_{\mathcal{D}}$ with the property that for two solutions u and v of the equation we have $d_{\mathcal{D}}(u(t), v(t)) \le e^{Ct} d_{\mathcal{D}}(u_0, v_0)$.

Contents

1. Introduction

The initial value problem for the Hunter–Saxton equation

$$u_t + uu_x = \frac{1}{4}\Big(\int_{-\infty}^{x} u_x^2\,dx - \int_x^{\infty} u_x^2\,dx\Big), \quad u|_{t=0} = u_0, \tag{1}$$

or alternatively

$$(u_t + uu_x)_x = \frac{1}{2}u_x^2, \quad u|_{t=0} = u_0, \tag{2}$$

has been widely studied since it was introduced [10] as a model for liquid crystals. It possesses a number of startling properties, being completely integrable, having infinitely many conserved quantities and a Lax pair. Furthermore, it is bi-variational and bi-Hamiltonian [11]. The initial value problem

Date: April 23, 2009.
2000 *Mathematics Subject Classification.* Primary: 35Q53, 35B35; Secondary: 35Q20.
Key words and phrases. Hunter–Saxton equation, Lipschitz metric, conservative solutions.
This paper was written as part of the international research program on Nonlinear Partial Differential Equations at the Centre for Advanced Study at the Norwegian Academy of Science and Letters in Oslo during the year 2008–09. Research supported in part by the Research Council of Norway.

has been extensively studied [12, 13, 15, 16, 17, 3]. A convergent finite difference scheme exists for the equation [7]. The simplest conservation law reads

$$(u_x^2)_t + (uu_x^2)_x = 0. \tag{3}$$

Furthermore, the equation enjoys wave breaking in finite time. More precisely, if the initial data is *not* monotone increasing, then

$$\inf(u_x) \to -\infty \text{ as } t \uparrow t^* = 2/\sup(-u_0'). \tag{4}$$

Past wave breaking there are at least two different classes of solutions. Consider the example [13] with initial data $u_0(x) = -x\chi_{[0,1]}(x) - \chi_{[1,\infty)}(x)$. For $t \in [0,2)$ the solution reads

$$u(t,x) = \frac{2x}{t-2}\chi_{(0,(2-t)^2/4)}(x) + \frac{1}{2}(t-2)\chi_{((2-t)^2/4,\infty)}(x), \quad t < 2. \tag{5}$$

Observe that $u(t,x) \to 0$ pointwise almost everywhere as $t \to 2-$. A careful analysis of the solution reveals that the energy density $u_x^2 dx$ approaches a Dirac delta mass at the origin as $t \to 2$. Two continuations past $t = 2$ are possible: The dissipative solution

$$u(t,x) = 0, \quad x \in \mathbb{R},\, t > 2, \tag{6}$$

and the conservative solution

$$u(t,x) = \frac{2x}{t-2}\chi_{(0,(2-t)^2/4)}(x) + \frac{1}{2}(t-2)\chi_{((2-t)^2/4,\infty)}(x), \quad t > 2. \tag{7}$$

Another example [10] is the following with initial data $u_0(x) = 0$. One solution (the dissipative) is clearly $u(t,x) = 0$ everywhere. Another solution (the conservative) solution reads

$$u(t,x) = -2t\chi_{(-\infty,-t^2)}(x) + \frac{2x}{t}\chi_{(-t^2,t^2)}(x) + 2t\chi_{(t^2,\infty)}(x). \tag{8}$$

As a consequence of this the existence theory for the Hunter–Saxton equation is complicated, and there is a dichotomy between the dissipative and the conservative solutions.

Zhang and Zheng [17] have proved global existence and uniqueness of both conservative and dissipative solutions (on the half-line $x > 0$) using Young measures and mollification techniques for compactly supported square integrable initial data. An alternative approach was developed in [3] for the Hunter–Saxton equation and in [6] for a somewhat more general class of nonlocal wave equations, by rewriting the equation in terms of an "energy variable", and showing the existence of a continuous semigroup of solutions. Furthermore, the papers [3] and [5] introduce a new distance function which renders Lipschitz continuous this semigroup of solutions. This is important because it establishes the uniqueness and continuous dependence for the Cauchy problem.

We remark that this is a nontrivial issue for nonlinear partial differential equations. For scalar conservation laws, where $u = u(t,x) \in \mathbb{R}$ satisfies $u_t + \nabla_x \cdot f(u) = 0$, as proved in [14] every couple of entropy weak solutions satisfies $\|u(t) - v(t)\|_{\mathbf{L}^1} \leq \|u(0) - v(0)\|_{\mathbf{L}^1}$ for all $t \geq 0$. For a hyperbolic system of conservation laws in one space dimension $u_t + f(u)_x = 0$, it is well known that, for initial data with sufficiently small total variation, one has $\|u(t)-v(t)\|_{\mathbf{L}^1} \le C\|u(0)-v(0)\|_{\mathbf{L}^1}$ for a suitable constant C and all t positive [1, 9].

The problem at hand can nicely be illustrated in the simpler context of an ordinary differential equation. Consider three differential equations:

$$\dot{x} = a(x), \qquad x(0)=x_0, \quad a \text{ Lipschitz}, \tag{9a}$$

$$\dot{x} = 1+\alpha H(x), \qquad x(0)=x_0, \quad H \text{ the Heaviside function}, \ \alpha>0, \tag{9b}$$

$$\dot{x} = |x|^{1/2}, \qquad x(0)=x_0, \qquad t\mapsto x(t) \text{ strictly increasing}. \tag{9c}$$

Straightforward computations give as solutions

$$x(t) = x_0 + \int_0^t a(x(s))\,ds, \tag{10a}$$

$$x(t) = (1+\alpha H(t-t_0))(t-t_0), \quad t_0 = -x_0/(1+\alpha H(x_0)), \tag{10b}$$

$$x(t) = \operatorname{sgn}\big(\frac{t}{2}+v_0\big)\big(\frac{t}{2}+v_0\big)^2 \text{ where } v_0 = \operatorname{sgn}(x_0)\,|x_0|^{1/2}. \tag{10c}$$

We find that

$$|x(t)-\bar{x}(t)| \le e^{Lt}\,|x_0-\bar{x}_0|\,, \quad L = \|a\|_{\mathrm{Lip}}\,, \tag{11a}$$

$$|x(t)-\bar{x}(t)| \le (1+\alpha)\,|x_0-\bar{x}_0|\,, \tag{11b}$$

$$x(t)-\bar{x}(t) = tv_0 + |x_0|\,, \quad \text{when } \bar{x}_0 = 0. \tag{11c}$$

Thus we see that in the regular case (9a) we get a Lipschitz estimate with constant e^{Lt} uniformly bounded as t ranges on a bounded interval. In the second case (9b) we get a Lipschitz estimate uniformly valid for all $t\in\mathbb{R}$. In the final example (9c), by restricting attention to strictly increasing solutions of the ordinary differential equations, we achieve uniqueness and continuous dependence on the initial data, but without any Lipschitz estimate at all. We observe that, by introducing the Riemannian metric

$$d(x,\bar{x}) = \left|\int_x^{\bar{x}} \frac{dz}{|z|^{1/2}}\right|, \tag{12}$$

an easy computation reveals that

$$d(x(t),\bar{x}(t)) = d(x_0,\bar{x}_0). \tag{13}$$

Let us explain why this metric can be considered as a Riemannian metric. The Euclidean metric between the two points is then given

$$|x_0-\bar{x}_0| = \inf_x \int_0^1 |x_s(s)|\;ds \tag{14}$$

where the infimum is taken over all paths $x\colon[0,1]\to\mathbb{R}$ that join the two points x_0 and $\bar{x}_0$, that is, $x(0)=x_0$ and $x(1)=x_1$. However, as we have seen, the solutions are not Lipschitz for the Euclidean metric. Thus we want to measure the infinitesimal variation x_s in an alternative way, which makes solutions of equation (9c) Lipschitz continuous. We look at the evolution equation that governs x_s and, by differentiating (9c) with respect to s, we get

$$\dot{x}_s = \frac{\operatorname{sgn}(x)x_s}{2\sqrt{|x|}},$$

and we can check that

$$\frac{d}{dt}\left(\frac{|x_s|}{\sqrt{|x|}}\right) = 0. \tag{15}$$

Let us consider the real line as a Riemannian manifold where, at any point $x \in \mathbb{R}$, the Riemannian norm, for any tangent vector $v \in \mathbb{R}$ in the tangent space of x, is given by $|v| / \sqrt{|x|}$. From (15), one can see that at the infinitesimal level, this Riemannian norm is exactly preserved by the evolution equation. The distance on the real line which is naturally inherited by this Riemannian is given by

$$d(x_0, \bar{x}_0) = \inf_x \int_0^1 \frac{|x_s|}{\sqrt{|x|}}\, ds$$

where the infimum is taken over all paths $x\colon [0,1] \to \mathbb{R}$ joining x_0 and $\bar{x}_0$. It is quite reasonable to restrict ourselves to paths that satisfy $x_s \geq 0$ and then, by a change of variables, we recover the definition (12).

We remark that, for a wide class of ordinary differential equations of the form $\dot{x} = f(t, x)$, $x \in \mathbb{R}^n$, a Riemannian metric that is contractive with respect to the corresponding flow has been constructed in [2]. Here the coefficient of the metric at a point $P = (t, x)$ depends on the total directional variation of the (possibly discontinuous) vector field f up to the point P. The equations (9a) and (9b) provide two examples covered by this approach.

The aim of this paper is to construct a Riemannian metric on a functional space, which renders Lipschitz continuous the flow generated by the Hunter–Saxton equation in the conservative case. Let us describe the result of the paper in a non-technical manner. From the examples above, it is clear that the solution itself is insufficient to describe a unique solution. Similar to the treatment of the Camassa–Holm equation [8, 4], it turns out that the appropriate way to resolve this issue to consider the pair (u, μ) where we have added the energy measure μ with absolute continuous part satisfying $\mu_{\mathrm{ac}} = u_x^2 dx$. To obtain a Lipschitz metric we introduce new variables. To that end assume first that one has a solution $u = u(t, x)$, and consider the characteristics $y_t(t, \xi) = u(t, y(t, \xi))$, the Lagrangian velocity $U(t, \xi) = u(t, y(t, \xi))$, and the Lagrangian cumulative energy $H(t, \xi) = \int_{-\infty}^{y(t,\xi)} u_x^2(t, x)\, dx$. Formally, the Hunter–Saxton equation is equivalent to the linear system of ordinary differential equations

$$\begin{aligned} y_t &= U, \\ U_t &= \frac{1}{2}H - \frac{1}{4}H(\infty), \\ H_t &= 0 \end{aligned} \tag{16}$$

in an Hilbert space. The quantity $H(\infty) = \int_{\mathbb{R}} u_x^2(t, x)\, dx$ is a constant. We first prove the existence of a global solution, see Theorem 2.3, and the existence of a continuous semigroup. However, in order to return to Eulerian variables it is necessary to resolve the redundancy, denoted relabeling, in Lagrangian coordinates, see Section 2.3. We introduce an equivalence relation for the Lagrangian variables corresponding to one and the same Eulerian solution. Next, we introduce a Riemannian metric d in Lagrangian variables. Denote by $X = (y, U, H)$. The natural choice of letting the distance between two elements X_0 and X_1 as the infimum of $\|X_0 \circ f - X_1 \circ f\|$ over all relabelings f, fails as it does not satisfy the triangle inequality. At each point X, we consider the elements that coincide to X under relabelings. Formally it corresponds to a Riemaniann submanifold whose structure is inherited from the ambiant Hilbert space. At each point X, we show that the tangent space to the relabeling submanifold corresponds to the set of all elements V such that $V = gX_\xi$ for some scalar function g. Given X and a tangent vector V to X, we can consider the scalar function g which minimizes the norm $\|V - gX_\xi\|$. This function g exists, is unique and is computed by solving of an elliptic equation, see Definition 3.1. We then define the seminorm $|||V||| = \|V - gX_\xi\|$ and consider the distance given by the infimum of $\int_0^1 |||\dot{X}(s)|||_{X(s)}\, ds$ over all paths $X(s)$ joining X_0 and X_1, that is, $X(0) = X_0$ and $X(1) = X_1$. The seminorm $|||\cdot|||$ has the property that it vanishes on the tangent space of all elements that coincide under relabelings, and, in particular, it implies that if X_1 is a relabeling of X_0 then $d(X_0, X_1) = 0$, see Section 3. With the proper definitions we find, see Theorem 3.14, that $d(\tilde{S}_t(X_0), \tilde{S}_t(X_1)) \leq e^{Ct} d(X_0, X_1)$ for some positive constant C, where $\tilde{S}_t$ denotes the semigroup that advances the system (16) by a time t. By transfering this metric to Eulerian variables we finally get a metric $d_{\mathcal{D}}$ such that $d_{\mathcal{D}}(T_t(u,\mu), T_t(\bar{u},\bar{\mu})) \leq e^{Ct} d_{\mathcal{D}}((u,\mu),(\bar{u},\bar{\mu}))$, where T_t is the semigroup in Eulerian variables.

In Section 5, we compare the metric $d_{\mathcal{D}}$ with other natural topologies. In particular, in Proposition 5.2 we show that if (u_n, μ_n) converges in the topology induced by $d_{\mathcal{D}}$, then u_n converges in $L^\infty(\mathbb{R})$. Furthermore, if u_n converges in $L^\infty(\mathbb{R})$ and $u_{x,n}$ converges in $L^2(\mathbb{R})$, then the mapping $u \mapsto (u, u_x^2 dx)$ is continuous on $\mathcal{D}$.

2. Semi-group of solutions in Lagrangian coordinates

2.1. **Equivalent system.**

The Hunter–Saxton equation equals

$$u_t + uu_x = \frac{1}{4}\Big(\int_{-\infty}^{x} u_x^2\, dx - \int_{x}^{\infty} u_x^2\, dx\Big). \tag{17}$$

Formally, the solution satisfies the following transport equation for the energy density $u_x^2\, dx$,

$$(u_x^2)_t + (uu_x^2)_x = 0 \tag{18}$$

so that $\int_{\mathbb{R}} u_x^2\, dx$ is a preserved quantity. Next, we rewrite the equation in Lagrangian coordinates. We introduce the characteristics

$$y_t(t,\xi) = u(t, y(t,\xi)).$$

The Lagrangian velocity U reads

$$U(t,\xi) = u(t, y(t,\xi)).$$

Furthermore, we define the Lagrangian cumulative energy by

$$H(t,\xi) = \int_{-\infty}^{y(t,\xi)} u_x^2(t,x)\, dx.$$

From (17), we get that

$$U_t = u_t \circ y + y_t u_x \circ y = \frac{1}{4}\Big(\int_{-\infty}^{y} u_x^2\,dx - \int_{y}^{\infty} u_x^2\,dx\Big) = \frac{1}{2}H(t,\xi) - \frac{1}{4}H(t,\infty)$$

and

$$\begin{aligned} H_t &= \int_{-\infty}^{y(t,\xi)} (u_x^2(t,x))_t\,dx + y_t u_x^2(t,y) \\ &= \int_{-\infty}^{y(t,\xi)} ((u_x^2)_t + (uu_x^2)_x)(t,x)\,dx \\ &= 0 \end{aligned}$$

by (18). Hence, the Hunter–Saxton equation formally is equivalent to the following system of ordinary differential equations:

$$y_t = U, \tag{19a}$$

$$U_t = \frac{1}{2}H - \frac{1}{4}H(\infty), \tag{19b}$$

$$H_t = 0. \tag{19c}$$

We have that $H(\infty) = H_0$ is a constant which does not depend on time, and global existence of solutions to (19) follows from the linear nature of the system. There is no exchange of energy across the characteristics and the system (19) can be solved explicitly, in contrast with the Camassa–Holm equation where energy is exchanged across characteristics. We have

$$y(t,\xi) = (\frac{1}{4}H(0,\xi) - \frac{1}{8}H(0,\infty))t^2 + U(0,\xi)t + y(0,\xi), \tag{20a}$$

$$U(t,\xi) = (\frac{1}{2}H(0,\xi) - \frac{1}{4}H(0,\infty))t + U(0,\xi), \tag{20b}$$

$$H(t,\xi) = H(0,\xi). \tag{20c}$$

Our goal is now to construct a continuous semigroup of solutions in Eulerian coordinates, i.e., for the original variable, u. The idea is to establish a mapping between the variables in Eulerian and Lagrangian coordinates, and we have to decide which function space we are going to use for the solutions of (19). Later, we will introduce a projection and therefore we need the framework of Hilbert spaces. A Riemannian metric also comes from an underlying Hilbert space structure. Given a natural number p, let us introduce the Banach space (if $p > 1$, then $E^p = H^p(\mathbb{R})$)

$$E^p = \{f \in L^\infty(\mathbb{R}) \mid f^{(i)} \in L^2(\mathbb{R}) \text{ for } i = 1,\dots,p\}$$

and the Hilbert spaces

$$H_1^p = H^p(\mathbb{R}) \times \mathbb{R}, \quad H_2^p = H^p(\mathbb{R}) \times \mathbb{R}^2.$$

We write $\mathbb{R}$ as $\mathbb{R} = (-\infty, 1) \cup (-1, \infty)$ and consider the corresponding partition of unity χ^+ and χ^- (so that χ^+ and $\chi^- \in C^\infty(\mathbb{R})$, $\chi^+ + \chi^- = 1$, $0 \le \chi^+ \le 1$, $\text{supp}(\chi^+) \subset (-1,\infty)$ and $\text{supp}(\chi^-) \subset (-\infty, 1)$). Introduce the linear mapping $\mathcal{R}_1$ from H_1^p to E^p defined as

$$(\bar f, a) \xrightarrow{\mathcal{R}_1} f(\xi) = \bar f(\xi) + a\chi^+(\xi)\ ,$$

and the linear mapping $\mathcal{R}_2$ from H_2^p to E^p defined as

$$(\bar{f}, a, b) \xmapsto{\mathcal{R}_2} f(\xi) = \bar{f}(\xi) + a\chi^+(\xi) + b\chi^-(\xi).$$

The mappings $\mathcal{R}_1$ and $\mathcal{R}_2$ are linear, continuous and injective. Let us introduce E_1^p and E_2^p, the images of H_1^p and H_2^p by $\mathcal{R}_1$ and $\mathcal{R}_2$, respectively, that is,

$$E_1^p = \mathcal{R}_1(H_1^p) \text{ and } E_2^p = \mathcal{R}_2(H_2^p).$$

One can check that the mappings $R_1\colon H_1^p \to E_1^p$ and $R_2\colon H_2^p \to E_2^p$ are homeomorphisms. It follows that E_1^p can be equipped with two equivalent norms $\|\cdot\|_E$ and $\left\|\mathcal{R}_1^{-1}(\,\cdot\,)\right\|_{H_1^p}$ (and similarly for E_2^p) and, through the mappings $\mathcal{R}_1$ and $\mathcal{R}_2$, E_1^p and E_2^p can be seen as Hilbert spaces. We denote

$$B^p = E_2^p \times E_2^p \times E_1^p.$$

We will mostly be concerned with the case $p = 1$ and to ease the notation, we will not write the superscript p for $p = 1$, that is, $B = B^1$, $E_j = E_j^1$, etc. In the same way that one proves that $H^1(\mathbb{R})$ is a continous algebra, one proves the following lemma, which we use later,

Lemma 2.1. *The space E is a continuous algebra, that is, for any $f, g \in E$, then the product fg belongs to E and there exists constant C such that*

$$\|fg\|_E \leq C \|f\|_E \|g\|_E$$

for any $f, g \in E$.

Definition 2.2. *The set $\mathcal{F}$ consists of the elements $(\zeta, U, H) \in B = E_2 \times E_2 \times E_1$ such that*

(i) $(\zeta, U, H) \in (W^{1,\infty})^3$, where $\zeta(\xi) = y(\xi) - \xi$;

(ii) $y_\xi \geq 0$, $H_\xi \geq 0$ and $y_\xi + H_\xi \geq c$, almost everywhere, where c is a strictly positive constant;

(iii) $y_\xi H_\xi = U_\xi^2$ almost everywhere.

Theorem 2.3. *The solution of the equivalent system given by* (19) *constitutes a semigroup S_t in $\mathcal{F}$ which is continuous with respect to the B-norm. Thus $X(t) = (y(t), U(t), H(t)) = S_t(X_0)$ denotes the solution of* (19) *at time t with initial data X_0. Moreover, the function $\xi \to y(t, \xi)$ is invertible for almost every t and we have, for almost every t, that*

$$y_\xi(t, \xi) > 0 \text{ for almost every } \xi \in \mathbb{R}. \tag{21}$$

Proof. Let $(\bar{\zeta}, \zeta_\infty, \zeta_{-\infty})$, $(\bar{U}, U_\infty, U_{-\infty})$ be the preimage of ζ and U by $\mathcal{R}_2$, respectively, and $(\bar{H}, H_\infty)$ the preimage of H by $\mathcal{R}_1$. Inserting these variables into (19), we obtain the following linear system of equations

$$\begin{aligned}
\bar{y}_t &= \bar{U},\\
\bar{U}_t &= \frac{1}{2}\bar{H},\\
\bar{H}_t &= 0,
\end{aligned}$$

and

$$(y_{\pm\infty})_t = U_{\pm\infty},$$

$$(U_{\pm\infty})_t = \pm\frac{1}{4}H_\infty,$$
$$(H_{\pm\infty})_t = 0.$$

Since it is linear, the system has a global solution in B, and we have Lipshitz stability with respect to the B-norm. Again due to the linearity, it is clear that the space $(W^{1,\infty}(\mathbb{R}))^3$ is invariant. After differentiating (19) with respect to ξ, we obtain

$$y_{\xi t} = U_\xi, \tag{22a}$$

$$U_{\xi t} = \frac{1}{2}H_\xi, \tag{22b}$$

$$H_{\xi t} = 0. \tag{22c}$$

Hence,

$$\frac{d}{dt}(y_\xi H_\xi - U_\xi^2) = 0$$

so that if the relation

$$y_\xi(t,\xi)H_\xi(t,\xi) = U_\xi^2(t,\xi) \tag{23}$$

holds for $t = 0$, then it holds for all t. By assumption, since $(y, U, H)_{t=0} \in \mathcal{F}$, we have

$$(y_\xi + H_\xi)(t,\xi) > 0 \tag{24}$$

for $t = 0$. By continuity, (24) is true in a vicinity of $t = 0$, and we denote by $[0, T)$ the largest interval where it holds. For $t \in [0, T)$, it follows from (23) that

$$y_\xi(t,\xi) \geq 0, \quad H_\xi(t,\xi) \geq 0, \tag{25}$$

and

$$|U_\xi| \leq \frac{1}{2}(y_\xi + H_\xi). \tag{26}$$

Hence,

$$\frac{d}{dt}\left(\frac{1}{y_\xi + H_\xi}\right) = \frac{U_\xi}{(y_\xi + H_\xi)^2} \leq \frac{1}{2(y_\xi + H_\xi)},$$

and, by the Gronwall lemma,

$$\frac{1}{y_\xi + H_\xi}(t,\xi) \leq \frac{1}{y_\xi + H_\xi}(0,\xi)e^{t/2} \tag{27}$$

for $t \in [0, T)$. It implies that $T = \infty$ and we have proved that $(y(t), U(t), H(t))$ remains in $\mathcal{F}$ for all t. The proof of statement (21) goes as in [8, Lemma 2.7] and we only give here a sketch of the argument. Given a fixed $\xi \in \mathbb{R}$, let

$$\mathcal{N}_\xi = \{t \in [0, T] \mid y_\xi(t,\xi) = 0\}.$$

For any $t^* \in \mathcal{N}_\xi$, we have

$$y_\xi(t^*,\xi) = 0, \qquad \text{from the definition of } t^*, \tag{28}$$

$$y_{\xi,t}(t^*,\xi) = 0, \qquad \text{by (28) and (23)}, \tag{29}$$

$$y_{\xi,tt}(t^*,\xi) = \frac{1}{2}H_\xi(t^*,\xi) > 0, \qquad \text{by (28) and (27)}. \tag{30}$$

Since the second derivative in time is strictly positive, the function $t \to y_\xi(t,\xi)$ is strictly positive at least on a small neighborhood of t^* excluding

t^* where it is equal to zero. Note that we can also use the explicit formulation given by (20) to get the same conclusion. We use Fubini's theorem to conclude this argument, see [8] for the details. □

2.2. **Functional setting in Eulerian variables.** Let us define $m = u_{xx}$. After differentiating (17) twice, we obtain

$$m_t + um_x + 2u_x m = 0. \tag{31}$$

Note that if we replace m by $u - u_{xx}$, then (31) will give the Camassa–Holm equation. For the Camassa–Holm equation there exists a particular class of solutions that takes the form

$$m = \sum_{i=1}^{N} p_i(t)\delta_{q_i(t)}.$$

Such particular solutions also exist for the Hunter–Saxton equation, and they correspond to piecewise linear functions (indeed, $u_{xx} = 0$ if u is linear). Let

$$y_1(t) = -\frac{t^2}{8}, \quad U_1(t) = -\frac{t}{4}, \quad H_1(t) = 0,$$

and

$$y_2(t) = \frac{t^2}{8}, \quad U_2(t) = \frac{t}{4}, \quad H_2(t) = 1.$$

Then (y_1, U_1, H_1) and (y_2, U_2, H_2) are solutions of (19) for the total energy $H(\infty) = 1$. One can check that the function u defined as

$$u(t,x) = \begin{cases} U_1(t) & \text{if } x \le y_1(t), \\ \frac{y_1(t)-x}{y_2(t)-y_1(t)} U_1(t) + \frac{x-y_2(t)}{y_2(t)-y_1(t)} U_2(t) & \text{if } y_1(t) < x \le y_2(t), \\ U_2(t) & \text{if } x > y_2(t), \end{cases}$$

is a weak solution of (17). At $t = 0$, we have $u(0,x) = 0$. However zero is also solution to (17) and therefore, if we want to construct a semigroup of solution, the function u at $t = 0$ does not provide us with all the necessary information. We need to know the location and the amount of energy that has concentrated on singular set. In the above example, the whole energy is concentrated at the origin when $t = 0$. The correct space where to construct global solution of the Hunter–Saxton equation is given by $\mathcal{D}$ defined as follows.

Definition 2.4. *The set $\mathcal{D}$ consists of all pairs (u, μ) such that*

(i) $u \in E$, μ is a finite Radon measure;

(ii) we have

$$\mu_{\text{ac}} = u_x^2 dx \tag{32}$$

where μ_{ac} denotes the absolute continuous part of μ with respect to the Lebesgue measure.

We introduce the subset $\mathcal{F}_0$ of $\mathcal{F}$ defined as follows

$$\mathcal{F}_0 = \{X = (y, U, H) \in \mathcal{F} \mid y + H = \text{Id}\}. \tag{33}$$

We can define a mapping, denoted L, from $\mathcal{D}$ to $\mathcal{F}_0 \subset \mathcal{F}$ as follows.

Definition 2.5. *For any* (u,μ) *in* $\mathcal{D}$*, let*

$$y(\xi) = \sup\{y \mid \mu((-\infty,y)) + y < \xi\}, \tag{34a}$$

$$H(\xi) = \xi - y(\xi), \tag{34b}$$

$$U(\xi) = u \circ y(\xi). \tag{34c}$$

Then $X = (\zeta, U, H) \in \mathcal{F}_0$ *and we denote by* $L\colon \mathcal{D} \to \mathcal{F}_0$ *the mapping which to any* $(u,\mu) \in \mathcal{D}$ *associates* $(\zeta, U, H) \in \mathcal{F}_0$ *as given by* (34)*.*

Thus, from any initial data $(u_0, \mu_0) \in \mathcal{D}$, we can construct a solution of (19) in $\mathcal{F}$ with initial data $X_0 = L(u_0, \mu_0) \in \mathcal{F}$. It remains to go back to the original variables, which is the purpose of the mapping M defined as follows.

Definition 2.6. *Given any element* X *in* $\mathcal{F}$*. Then, the pair* (u,μ) *defined as follows*[1]

$$u(x) = U(\xi) \text{ for any } \xi \text{ such that } x = y(\xi), \tag{35a}$$

$$\mu = y_{\#}(H_\xi \, d\xi) \tag{35b}$$

belongs to $\mathcal{D}$*. We denote by* $M\colon \mathcal{F} \to \mathcal{D}$ *the mapping which to any* X *in* $\mathcal{F}$ *associates* (u,μ) *as given by* (35)*.*

The proofs of the well-posedness of the Definitions 2.5 and 2.6 are the same as in [8, Theorems 3.8 and 3.11].

2.3. **Relabeling symmetry.** When going from Eulerian to Lagrangian coordinates, there exists an additional degree of freedom which corresponds to relabeling. Let us explain this schematically. We consider two elements X and $\bar{X}$ in $\mathcal{F}$ such that $\bar{X} = X \circ f$, for some function f, where $X \circ f$ denotes $(y \circ f, U \circ f, H \circ f)$. The two element X and $\bar{X}$ correspond to functions in Eulerian coordinates denoted u and $\bar{u}$, respectively. We have

$$U(\xi) = u \circ y(\xi), \text{ and } \bar{U}(\xi) = \bar{u} \circ \bar{y}(\xi).$$

Then, if y and $\bar{y}$ are invertible, we get

$$\bar{u} = \bar{U} \circ \bar{y}^{-1} = U \circ f \circ (y \circ f)^{-1} = U \circ y = u$$

so that X and $\bar{X}$, which may be distinct, correspond to the same Eulerian configuration. We can put this statement in a more rigorous framework by introducing the subgroup G of the group of homeomorphisms from $\mathbb{R}$ to $\mathbb{R}$ defined as

$$f - \mathrm{Id} \text{ and } f^{-1} - \mathrm{Id} \text{ both belong to } W^{1,\infty}(\mathbb{R}). \tag{36}$$

For any $\alpha > 1$, we introduce the subsets G_α of G defined by

$$G_\alpha = \{f \in G \mid \|f - \mathrm{Id}\|_{W^{1,\infty}(\mathbb{R})} + \left\|f^{-1} - \mathrm{Id}\right\|_{W^{1,\infty}(\mathbb{R})} \le \alpha\}.$$

The subsets G_α do not possess the group structure of G but they are closed sets.

[1]The push-forward of a measure ν by a measurable function f is the measure $f_{\#}\nu$ defined by $f_{\#}\nu(B) = \nu(f^{-1}(B))$ for all Borel sets B.

Definition 2.7. *Given $\alpha \geq 0$, the set $\mathcal{F}_\alpha$ (respectively $\mathcal{G}_\alpha$) consists of the elements $(\zeta, U, H) \in B = E_2 \times E_2 \times E_1$ such that*

$$(\zeta, U, H) \in (W^{1,\infty})^3, \tag{37a}$$

$$y + H \in G_\alpha, \tag{37b}$$

$$y_\xi H_\xi = U_\xi^2 \text{ (respectively } y_\xi H_\xi \geq U_\xi^2\text{)}, \tag{37c}$$

where $\zeta(\xi) = y(\xi) - \xi$.

We have $\mathcal{F}_\alpha \subset \mathcal{G}_\alpha$. One can check, using [8, Lemma 3.2], that $\mathcal{F} = \bigcup_{\alpha \geq 0} \mathcal{F}_\alpha$, and we denote $\mathcal{G} = \bigcup_{\alpha \geq 0} \mathcal{G}_\alpha$. The following proposition holds.

Proposition 2.8. *(i) The mapping $(X, f) \mapsto \bar{X}$ from $\mathcal{F} \times G$ to $\mathcal{F}$ given by $\bar{X} = X \circ f$ defines an action of the group G on $\mathcal{F}$. Hence, we can define the equivalence relation on $\mathcal{F}$ by*

$$X \sim \bar{X} \text{ if and only if there exists } f \in G \text{ such that } \bar{X} = X \circ f,$$

and the corresponding quotient is denoted $\mathcal{F}/G$.
(ii) If $X \sim \bar{X}$, then $M(X) = M(\bar{X})$, i.e., the relabeling of an element in $\mathcal{F}$ corresponds to the same element in $\mathcal{D}$.

The proof of this proposition and of the remaining propositions in this section can be found in [8] with only minor adaptions. Given $X \in \mathcal{F}$, we denote by $[X]$ the element of $\mathcal{F}/G$ which corresponds to the equivalence class of X. We shall see that we can identify $\mathcal{F}/G$ with the subset $\mathcal{F}_0$ of $\mathcal{F}$.

Definition 2.9. *We define the projection $\Pi\colon \mathcal{G} \to \mathcal{G}_0$ as follows*

$$\Pi(X) = X \circ (y + H)^{-1}.$$

We have $\Pi(\mathcal{F}) = \mathcal{F}_0$.

We have the following proposition.

Proposition 2.10. *(i) For X and $\bar{X}$ in $\mathcal{F}$,*

$$X \sim \bar{X} \text{ if and only if } \Pi(X) = \Pi(\bar{X}).$$

(ii) The injection $X \mapsto [X]$ is a bijection from $\mathcal{F}_0$ to $\mathcal{F}/G$.

Proposition 2.11. *(i) The sets $\mathcal{D}$ and $\mathcal{F}_0$ are in bijection. We have*

$$M \circ L = \mathrm{Id}_{\mathcal{D}} \text{ and } L \circ M|_{\mathcal{F}_0} = \mathrm{Id}_{\mathcal{F}_0}.$$

(ii) The sets $\mathcal{D}$ and $\mathcal{F}/G$ are in bijection.

The following proposition says that the solutions to the system (19) are invariant under relabeling.

Proposition 2.12. *The mapping $S_t\colon \mathcal{F} \to \mathcal{F}$ is G-equivariant, that is,*

$$S_t(X \circ f) = S_t(X) \circ f \tag{38}$$

for any $X \in \mathcal{F}$ and $f \in G$. This implies that

$$\Pi \circ S_t \circ \Pi = \Pi \circ S_t.$$

Hence, we can define a semigroup of solutions on $\mathcal{F}/G$. It corresponds to the mapping $\tilde{S}_t$ from $\mathcal{F}_0$ to $\mathcal{F}_0$ given by

$$\tilde{S}_t = \Pi \circ S_t \tag{39}$$

which defines a semigroup on $\mathcal{F}_0$.

We can rewrite system (19) as

$$X_t = F(X) \tag{40}$$

where $F\colon B \to B$ is given by

$$F(y, U, H) = (U, \frac{1}{2}H - \frac{1}{4}H(\infty), 0). \tag{41}$$

Proposition 2.12 follows from the fact, which can be verified directly by looking at (41), that

$$F(X \circ f) = F(X) \circ f. \tag{42}$$

We want to define a distance in $\mathcal{F}_0$ which makes the semigroup $\tilde{S}_t$ Lipschitz continuous.

3. A Riemannian metric

We want to define a mapping d from $\mathcal{F} \times \mathcal{F}$ to $\mathbb{R}$, which is symmetric and satisfies the triangle inequality, and such that

$$d(X, \bar{X}) = 0 \text{ if and only if } X \sim \bar{X}, \tag{43}$$

and

$$d(S_t X, S_t \bar{X}) \leq C\, d(X, \bar{X}), \tag{44}$$

because such mapping can in a natural way be used to define a distance on $\mathcal{F}/G$ which also makes the semigroup of solutions continuous. Since the stability of the semigroup S_t holds for the B-norm, it is natural to use this norm to construct the mapping d. A natural candidate would be

$$d(X, \bar{X}) = \inf_{f, \bar{f} \in G} \left\| X \circ f - \bar{X} \circ \bar{f} \right\|_B,$$

which is likely to fulfill (43) and (44). However it does not satisfy the triangle inequality. Formally, let us explain our construction, which is inspired by ideas originating in Riemannian geometry. Let us think of $\mathcal{F}$ as a Riemannian manifold embedded in the Hilbert space B. There is a natural scalar product in the tangent bundle of $T\mathcal{F}$ of $\mathcal{F}$ which is inherited from B. We can then define a distance in $\mathcal{F}$ by considering geodesics, namely,

$$d(X_0, X_1) = \inf_X \int_0^1 \left\| \dot{X}(s) \right\|_B ds \tag{45}$$

for any $X_0, X_1 \in \mathcal{F}$ and where the infimum is taken over all smooth paths $X(s)$ in $\mathcal{F}$ joining X_0 and X_1. The distance equals to the B-norm. It makes the semigroup stable but it clearly separates points which belong to the same equivalence class and so does not fulfill (43). For a given element $X \in \mathcal{F}$, we consider the subset $\Gamma \subset \mathcal{F}$ which corresponds to all relabelings of X, that is, $\Gamma = [X] = \{X \circ f \mid f \in G\}$. If we substitute in (45) the following definition

$$d(X_0, X_1) = \inf_X \int_0^1 \left|\left|\left| \dot{X}(s) \right|\right|\right|_{X(s)} ds \tag{46}$$

where $|||\cdot|||$ is a seminorm in $T\mathcal{F}$ with the extra property that it vanishes on $T\Gamma_{X(s)}$, then the property (43) will follow in a natural way, and we expect the stability property (44) to be a consequence of the equivariance of S_t, as stated in Proposition 2.12. We will carry out the plan next.

Let us first investigate the local structure of Γ around X. Given a smooth function $g(\xi)$ (one should actually think of g as an element of $TG|_{\mathrm{Id}}$), we consider the curve f in G given by

$$f(\theta,\xi)=\xi+\theta g(\xi).$$

It leads to the curve in $X\circ f(\theta)$ in Γ that we differentiate, and we obtain

$$\frac{d}{d\theta}(X\circ f(\theta))=gX_\xi.$$

We now define the subspace $E(X)$ which formally corresponds to the subspace $T\Gamma_X$ of $T\mathcal{G}$.

Definition 3.1. *Given a fixed element $X\in\mathcal{G}\cap B^2$, we consider the subspace $E(X)$ defined as*

$$E(X)=\{g(\xi)X_\xi(\xi)\mid g\in E_2\},$$

where $X_\xi(\xi)=(y_\xi(\xi),U_\xi(\xi),H_\xi(\xi))^T$.

Lemma 3.2. *Given any $X\in B^2$, the bilinear form a_X defined as*

$$a_X(g,h)=\langle gX_\xi,hX_\xi\rangle$$

is coercive, that is, there exists a constant $C>0$ such that

$$\frac{1}{C}\|g\|_{E_2}^2\le a_X(g,g)=\|gX_\xi\|_B^2 \tag{47}$$

for all $g\in E_2$. Moreover, the constant C depends only on $\|X\|_{B^2}$ and $\left\|\frac{1}{y_\xi+H_\xi}\right\|_{L^\infty}$.

Proof. Given $g\in E_2$, let $(\bar g,g_{-\infty},g_\infty)=\mathcal{R}_2^{-1}(g)$, we have the following decomposition, $g=\bar g+g_{-\infty}\chi^-+g_\infty\chi^+$ and, by definition,

$$\|g\|_{E_2}^2=\|\bar g\|_{H^1}^2+|g_{-\infty}|^2+|g_\infty|^2.$$

Let us denote $\tilde g=g_{-\infty}\chi^-+g_\infty\chi^+$. Given $X\in B^2$, we have $\lim_{\xi\to\pm\infty}y_\xi(\xi)=1$ and $\lim_{\xi\to\pm\infty}(|\zeta_\xi|+|U_\xi|+|H_\xi|)(\xi)=0$. The following decomposition hold

$$gy_\xi=\bar gy_\xi+\tilde g\zeta_\xi+g_{-\infty}\chi^-+g_{+\infty}\chi^+$$

so that $\mathcal{R}_2^{-1}(gy_\xi)=(\bar gy_\xi+\tilde g\zeta_\xi,g_{-\infty},g_\infty)$. We have also that $\mathcal{R}_2^{-1}(gU_\xi)=(gU_\xi,0,0)$ and $\mathcal{R}_1^{-1}(gH_\xi)=(gH_\xi,0)$. Hence,

$$\|gX_\xi\|_B^2=\|\bar gy_\xi+\tilde g\zeta_\xi\|_{H^1}^2+\|gU_\xi\|_{H^1}^2+\|gH_\xi\|_{H^1}^2+|g_{-\infty}|^2+|g_\infty|^2.$$

Let us prove that

$$\|\bar g\|_{L^2}\le C\|gX_\xi\|_B. \tag{48}$$

We have

$$\|\bar gy_\xi+\tilde g\zeta_\xi\|_{L^2}^2+\|gU_\xi\|_{L^2}^2+\|gH_\xi\|_{L^2}^2=\int_{\mathbb{R}}\bar g^2(y_\xi^2+U_\xi^2+H_\xi^2)\,d\xi \tag{49}$$
$$+\int_{\mathbb{R}}\tilde g^2(\zeta_\xi^2+U_\xi^2+H_\xi^2)\,d\xi+2\int_{\mathbb{R}}(\bar gy_\xi\tilde g\zeta_\xi+\bar gU_\xi\tilde gU_\xi+\bar gH_\xi\tilde gH_\xi)\,d\xi$$

For all $\varepsilon > 0$, we have

$$2\int_{\mathbb{R}} (\bar{g}y_\xi \tilde{g}\zeta_\xi + \bar{g}U_\xi \tilde{g}U_\xi + \bar{g}H_\xi \tilde{g}H_\xi)\,d\xi \geq -\varepsilon \int_{\mathbb{R}} \bar{g}^2 (y_\xi^2 + U_\xi^2 + H_\xi^2)\,d\xi - \frac{1}{\varepsilon}\int_{\mathbb{R}} \tilde{g}^2(\zeta_\xi^2 + U_\xi^2 + H_\xi^2)\,d\xi,$$

and, by taking ε sufficiently small and inserting this inequality into (49), it yields

$$\begin{aligned}\int_{\mathbb{R}} \bar{g}^2 (y_\xi^2 + U_\xi^2 + H_\xi^2)\,d\xi &\leq C\big(\|\bar{g}y_\xi + \tilde{g}\zeta_\xi\|_{L^2}^2 + \|gU_\xi\|_{L^2}^2 \\ &\qquad + \|gH_\xi\|_{L^2}^2 + \int_{\mathbb{R}} \tilde{g}^2(\zeta_\xi^2 + U_\xi^2 + H_\xi^2)\,d\xi \big) \\ &\leq C(\|gX_\xi\|_B^2 + |g_{-\infty}|^2 + |g_{+\infty}|^2) \\ &\leq C\,\|gX_\xi\|_B^2 .\end{aligned}$$

Since $y_\xi^2 + U_\xi^2 + H_\xi^2(\xi) \geq \frac{1}{2}(y_\xi + H_\xi)^2$, (48) follows. Similarly, by using (48) and a decomposition using ε and $\frac{1}{\varepsilon}$ as above, one proves that

$$\|\bar{g}_\xi\|_{L^2} \leq C\,\|gX_\xi\|_B ,$$

which concludes the proof of the lemma. □

From Lemma 3.2 and Lax–Milgram theorem, we obtain the following definition.

Definition 3.3. *Given any $X \in B^2$ and $V \in B$, there exists a unique $g \in E_2$, that we denote $g(X, V)$, such that*

$$\langle gX_\xi, hX_\xi\rangle = \langle V, hX_\xi\rangle \quad \text{for all } h \in E_2, \tag{50}$$

and we have

$$\|V - gX_\xi\| \leq \|X - hX_\xi\| \quad \text{for all } h \in E_2.$$

Given $X \in B^2$ and $V \in B$ and $g = g(X, V)$, let $(\bar{g}, g_{-\infty}, g_\infty) = R_2^{-1}(g)$. When V is smooth (say $V \in B^2$), one can show that the following system of equations for $\bar{g}$, $g_{-\infty}$ and g_∞ is equivalent to (50),

$$\begin{aligned} &-|X_\xi|^2\,\bar{g}_{\xi\xi} + 2(X_{\xi\xi}\cdot X_\xi)\bar{g}_\xi + (\|X_\xi\|^2 + X_\xi\cdot X_{\xi\xi\xi})\bar{g} \\ &\qquad = \bar{V} - \bar{V}_{\xi\xi} + X_\xi\cdot\big((\mathrm{Id} - \partial_\xi^2)\big((g_{-\infty}\chi^- + g_\infty\chi^+)[\zeta_\xi, U_\xi, H_\xi]^T\big)\big),\end{aligned} \tag{51}$$

and

$$(1 + \|\alpha\|_{H^1}^2)g_\infty + \langle\alpha,\beta\rangle_{H^1}\,g_{-\infty} = V_\infty - \langle\bar{g}X_\xi,\alpha\rangle\,, \tag{52a}$$

$$\langle\alpha,\beta\rangle_{H^1}\,g_\infty + (1 + \|\beta\|_{H^1}^2)g_{-\infty} = V_{-\infty} - \langle\bar{g}X_\xi,\beta\rangle\,, \tag{52b}$$

where $\alpha(\xi) = \chi^+(\xi)[\zeta_\xi, U_\xi, H_\xi]^T$ and $\beta(\xi) = \chi^-(\xi)[\zeta_\xi, U_\xi, H_\xi]^T$ are known functions as they depend only on X, which is given. By Cauchy–Schwarz, the determinant of system (52) for the unknowns $g_{-\infty}$ and g_∞ is strictly bigger than 1, and therefore we can write $g_{-\infty}$ and g_∞ as functions of V, X_ξ and integral terms which contain $\bar{g}$. Since $|X_\xi|^2$ is strictly bounded away from zero, equation (51) for $\bar{g}$ is elliptic.

Lemma 3.4. *The mapping* $g : B^2 \times B \to E$ *is continuous and*

$$\|g(X_1, V_1) - g(X_0, V_0)\| \leq C(\|X - \bar{X}\|_{B^2} + \|V - \bar{V}\|_B)$$

for some constant C *which depends only on* $\|V_1\|$, $\|V_0\|$, $\|X_1\|_{B^2}$, $\|X_0\|_{B^2}$, $\|(y_{0\xi} + H_{0\xi})^{-1}\|_{L^\infty}$, $\|(y_{1\xi} + H_{1\xi})^{-1}\|_{L^\infty}$.

Proof. From Lemma 2.1, it follows that $\|gX_\xi\|_B \leq \|g\|_{E_2} \|X_\xi\|_B$ for any $X \in B^2$ and $g \in E_2$. By (50) and (47), we get $\|g\|_{E_2}^2 C \leq \|V\|_B \|g\|_{E_2} \|X_\xi\|$ which implies $\|g\|_{E_2} \leq C \|V\|_B$, for a constant C which depends only on $\|X\|_{B^2}$. We have, for all $h \in E_2$,

$$\begin{aligned} (53) \quad \langle (g_1 - g_0)X_{1\xi}, hX_{1\xi}\rangle &= -\langle g_0(X_{1\xi} - X_{0\xi}), hX_{1\xi}\rangle \\ &- \langle g_0X_{0\xi}, h(X_{1\xi} - X_{0\xi})\rangle + \langle V_1 - V_0, hX_{1\xi}\rangle + \langle V_1, h(X_{1\xi} - X_{0\xi})\rangle, \end{aligned}$$

which gives

$$|\langle (g_1 - g_0)X_{1\xi}, hX_{1\xi}\rangle| \leq C \|h\|_{E_2} (\|V_1 - V_0\|_B + \|X_1 - X_0\|_{B^2}).$$

The results follows by taking $h = \frac{g_1 - g_0}{\|g_1 - g_0\|_{E_2}}$ and using (47). □

We can now define a seminorm on $T\mathcal{F}|_X \subset B$.

Definition 3.5. *Given* $X \in B^2$, *we define the seminorm* $|||\cdot|||$ *on* B *as follows: For any element* $V \in B$, *we set*

$$|||V|||_X = \|V - g(X, V)X_\xi\|_B.$$

Using the definition (46) we then get that

$$\text{(54)} \qquad \text{if } X_0 \sim X_1, \text{ then } d(X_0, X_1) = 0.$$

Indeed, If $X_0 \sim X_1$, there exists a function $f \in G$ such that $X_1 = X_0 \circ f$. We consider the path $X(s, \xi) = X_0((1-s)\xi + sf(\xi))$ which joins X_0 and X_1. We have

$$X_s = (f - 1)X_{0,\xi}((1 - s)\xi + sf(\xi)).$$

Furthermore

$$X_\xi = ((1 - s)\,\mathrm{Id} + sf'(\xi))X_{0,\xi}((1 - s)\xi + sf(\xi)).$$

We see that $(1 - s)\,\mathrm{Id} + sf'(\xi) \geq \min(\mathrm{Id}, f') > 0$. Thus

$$X_s = \frac{(f - 1)}{(1 - s)\,\mathrm{Id} + sf'(\xi)} X_\xi,$$

and $X_s \in B$, which implies that $P(X_s) = 0$ and therefore $|||X_s|||_{X(s)} = 0$, for all $s \in [0, 1]$. Then, (54) follows from (46). In (46), we consider the infimum over curves in $\mathcal{F}$. However, for any $\alpha \geq 0$, the set $\mathcal{F}_\alpha$ is not convex due to the condition (37c) in Definition 2.7. We relax this condition and consider instead the set $\mathcal{G}_\alpha$ which is preserved by the semigroup and which is convex for $\alpha = 0$.

Lemma 3.6. *The set* $\mathcal{G}_0$ *is convex.*

Proof. The set G_0 is convex. The condition (37b) implies, for $\alpha = 0$, that

$$y_\xi + H_\xi = 1 \tag{55}$$

which gives $y_\xi^2 + 2y_\xi H_\xi + H_\xi^2 = 1$. Then the condition (37c) is equivalent to

$$y_\xi^2 + H_\xi^2 + 2U_\xi^2 \leq 1 \tag{56}$$

which defines a convex set. □

The solution semigroup can be extended to curves in $\mathcal{G}$. First we define the class of curves we will be considering.

Definition 3.7. *Given $\alpha \geq 0$, we denote by $\mathcal{C}_\alpha$ the set of curves $X(s) = (\zeta(s), U(s), H(s))$ where*

$$X\colon [0,1] \to \mathcal{G}_\alpha \cap B^2,$$

and such that

$$X \in C([0,1], B^2) \quad \textit{and} \quad X_s \in C_{\mathrm{pc}}([0,1], B)$$

where $C_{\mathrm{pc}}([0,1], B)$ denotes the set of functions from $[0,T]$ to B which are piecewise continuous.

We denote $\mathcal{C} = \bigcup_\alpha \mathcal{C}_\alpha$. The solution operator S_t naturally extends to curves in $\mathcal{C}$.

Lemma 3.8. *For any initial curve $X_0 \in \mathcal{C}$, there exists a solution curve $X\colon [0,1] \times \mathbb{R}_+ \to B^2$ such that*
(i) $X(s, 0) = X_0(s)$;
(ii) for each fixed $t \in \mathbb{R}_+$, $X(\,\cdot\,, t)\colon [0,1] \to B^2$ belongs to $\mathcal{C}$;
(iii) for each fixed $s \in [0,1]$, $X(s, \cdot\,)\colon \mathbb{R}_+ \to B^2$ is a solution of (19) with initial data $X_0(s)$.
Moreover, we have

$$(y + H)(t, \,\cdot\,) \in G_{\alpha(t)} \textit{ with } \alpha(t) \leq e^{Ct} \tag{57}$$

for some constant C.

Proof. The proof follows as the proof of Theorem 2.3. We use a fixed point argument, for T small enough, on the set $C([0,T], \bar{\mathcal{C}})$ where $\bar{\mathcal{C}}$ is the Banach space of curves with piecewise constant derivatives, i.e.,

$$\bar{\mathcal{C}} = \{X \in C([0,1], B^2) \mid X_s \in C([s_i, s_{i+1}], B),\ i = 1, \dots, n\}$$

where the sequence $0 = s_1 \leq \cdots \leq s_n = 1$ is chosen such that $X_0 \in \bar{C}$. We then extend the solution globally in time and obtain (57) as in the proof of Theorem 2.3. □

Define a metric on $\mathcal{G}_0$ as follows.

Definition 3.9. *For two elements $X_0, X_1 \in \mathcal{G}_0 \cap B^2$, we define*

$$d(X_0, X_1) = \inf_{X \in \mathcal{C}_0} \int_0^1 |||X_s(s)|||_{X(s)}\, ds. \tag{58}$$

Note that the definition is well-posed because $\mathcal{C}_0$ is nonempty since, as $\mathcal{G}_0 \cap B^2$ is convex, we can always join two elements in C_0 by a straight line.

Lemma 3.10. *The mapping $d\colon \mathcal{G}_0 \times \mathcal{G}_0 \to \mathbb{R}_+$ is a distance on $\mathcal{G}_0 \cap B^2$.*

Proof. Let us first prove that $d(X_0, X_1) = 0$ implies $X_0 = X_1$. For any $\varepsilon \geq 0$, we consider $X \in \mathcal{C}_0$ such that

$$\int_0^1 |||X_s|||_{X(s)}\, ds \leq \varepsilon. \tag{59}$$

Since $y(s,\xi) + H(s,\xi) = \xi$ for all ξ, we get

$$y_s + H_s = 0, \quad \text{and} \quad y_\xi + H_\xi = 1.$$

We consider the orthogonal decomposition of X_s, i.e.,

$$X_s(s,\xi) = g(s,\xi)X_\xi(s,\xi) + R(s,\xi). \tag{60}$$

It follows, by adding the first and third components in (60), that

$$0 = y_s + H_s = g(s,\xi)(y_\xi + H_\xi) + R_1 + R_3 = g(s,\xi) + R_1 + R_3$$

(where R_1 and R_3 denotes the first and third components of R) and therefore

$$g(s,\xi) = R_1(s,\xi) + R_3(s,\xi). \tag{61}$$

Since, in a Euclidean space the shortest path between two points is a straight line, we have

$$\|X_1 - X_0\|_{L^\infty(\mathbb{R})} \leq \int_0^1 \|X_s(s,\,\cdot\,)\|_{L^\infty}\, ds. \tag{62}$$

From the definition of $\mathcal{G}_0$, it follows that y_ξ, H_ξ and U_ξ are bounded by one in $L^\infty(\mathbb{R})$, see (56). Therefore, (62) and (60) imply

$$\begin{aligned}
\|X_1 - X_0\|_{L^\infty(\mathbb{R})} &\leq \int_0^1 \left(\|g(s,\,\cdot\,)\|_{L^\infty} + \|R(s,\,\cdot\,)\|_{L^\infty}\right) ds \\
&\leq 2\int_0^1 \|R(s,\,\cdot\,)\|_{L^\infty}\, ds \quad \text{(by (61))} \\
&\leq 2\int_0^1 \|R(s,\,\cdot\,)\|_B\, ds = 2\int_0^1 |||X(s,\,\cdot\,)|||_{X(s)}\, ds \leq \varepsilon.
\end{aligned}$$

Since ε is arbitrary, it follows that $X_1 = X_0$. The triangle inequality is obtained by patching two curves together and reparametrizing them while the symmetry of d is also obtained by reparametrization. Both proofs are somehow standard. □

On $\mathcal{G}_0$, the distance d is weaker than the B-norm as the next lemma shows.

Lemma 3.11. *For any $X_0, X_1 \in \mathcal{G}_0 \cap B^2$, we have*

$$d(X_0, X_1) \leq \|X_1 - X_0\|. \tag{63}$$

Proof. Consider $X \in \mathcal{C}_0$ defined as follows

$$X(s) = (1-s)X_0 + sX_1.$$

We have

$$d(X_0, X_0) \leq \int_0^1 |||X_s(s)|||_{X(s)}\, ds \leq \int_0^1 \|X_s(s)\|_{X(s)}\, ds = \|X_1 - X_0\|$$

because $|||X_s||| = \|P(X_s)\| \leq \|X_s\|$ as P is an orthogonal projection. □

Definition 3.12. *For two elements $X_0, X_1 \in \mathcal{G}_0$, we define*

$$d(X_0, X_1) = \lim_{n\to\infty} d(X_0^n, X_1^n) \tag{64}$$

for any sequences X_0^n and X_1^n in $\mathcal{G}_0 \cap B^2$ which converge in B to X_0 and X_1, respectively.

Definition 3.12 is well-posed thanks to (63). The mapping S_t maps $\mathcal{F}$ to $\mathcal{F}$, and we can formally define what is called in differential geometry the tangent map of S_t, TS_t, which is a mapping $T\mathcal{F}_X$ to $T\mathcal{F}_{S_tX}$. The following theorem expresses the fact that TS_t is uniformly continuous (in time) with respect to the seminorm $|||\cdot|||$.

Theorem 3.13. *There exists a constant C such that, for any initial curve $X_0(s,\xi) \in \mathcal{C}_0$, if we consider the curve solution $X(t,s,\xi)$ with initial data $X_0(s,\xi)$ given by Lemma 3.8, we have*

$$|||X_s(s,t)|||_{X(s,t)} \leq e^{Ct} |||X_s(s,0)|||_{X(s,0)}. \tag{65}$$

Proof. We rewrite the system

$$X_t = F(X) \tag{66}$$

where F is given by (41). The mapping F is linear and therefore differentiable and we have, for any $X, \bar{X} \in B$,

$$DF[X](\bar{X}) = F(\bar{X}) \tag{67}$$

where $DF[X]$ denotes the diffential of F at X. For $X \in B^2$, since $X_\xi \in H^1$, we have $\lim_{\xi\to\infty} H_\xi(\xi) = 0$ and one can then check directly that, for any $g \in E_2$,

$$DF[X](g(\xi)X_\xi(\xi)) = g(\xi)(DF[X](X_\xi(\xi))). \tag{68}$$

However, the simplicity of system (19) may hide the more fundamental nature of relation (68), which in fact corresponds to the infinitesimal version of the equivariance property of F stated in (42). Indeed, given a smooth function g, we consider the family of diffeomorphisms parametrized by θ given by $f^\theta(\xi) = \xi + \theta g(\xi)$. The equivariance property (42) of F gives

$$F(X \circ f^\theta) = F(X) \circ f^\theta,$$

which after differentiation by θ and taking the value at $\theta = 0$ yields (68). After differentiating (66) with respect to s, we get

$$X_{st} = DF[X](X_s) \tag{69}$$

while differentiating it with respect to ξ yields

$$X_{\xi t} = DF[X](X_\xi). \tag{70}$$

We consider the decomposition of X_s given by

$$X_s = g(X,V)X_\xi + R. \tag{71}$$

Since, for every $s \in [0,1]$, $X_s \in C^1([0,T],B)$, $X_\xi \in C^1([0,T],B^2)$ and (57) holds, we can use Lemma 3.4 to prove that $g \in C^1([0,T],E_2)$, for any $s \in [0,1]$. By differentiating

$$\langle gX_\xi, hX_\xi\rangle = \langle X_s, hX_\xi\rangle$$

we obtain that g_t is defined as the unique element in E_2 such that

$$\langle g_t X_\xi, hX_\xi\rangle = \langle X_{st}, hX_\xi\rangle + \langle X_s, hX_{\xi t}\rangle - \langle gX_{\xi t}, hX_\xi\rangle - \langle gX_\xi, hX_{\xi t}\rangle$$

for all $h \in E_2$. We differentiate (71) and get

$$X_{st} = g_t X_\xi + gX_{\xi t} + R_t.$$

After using (69) and (70), it yields

$$DF[X](X_s) = g_t X_\xi + g(DF[X](X_\xi)) + R_t.$$

Using (68), this identity rewrites

$$R_t = DF[X](X_s - gX_\xi) - g_t X_\xi$$

or

$$R_t = DF[X]R - g_t X_\xi. \tag{72}$$

We take the scalar product of R_t and, since $g_t X_\xi$ and R are orthogonal, we obtain

$$\begin{aligned} \langle R_t, R\rangle &= \langle DF[X](R), R\rangle \\ &\leq \|DF[X](R)\| \, \|R\| \\ &\leq C \, \|R\|^2 \end{aligned} \tag{73}$$

because the mapping $DF[X] \colon B \to B$ is uniformly bounded, see (67). Thus, (73) yieds

$$\frac{d}{dt} \|R\|^2 \leq C \, \|R\|^2 .$$

By Gronwall's inequality, it implies

$$|||X_s(t)||| = \|R(t)\| \leq \|R(0)\| \, e^{Ct} = |||X_s(0)|||e^{Ct}.$$

□

Theorem 3.14. *The semigroup $\tilde{S}_t \colon \mathcal{G}_0 \to \mathcal{G}_0$ is Lipschitz continuous with respect to the metric d. We have, for some constant C,*

$$d(\tilde{S}_t(X_0), \tilde{S}_t(X_1)) \leq e^{Ct} d(X_0, X_1) \tag{74}$$

for all $X_0, X_1 \in \mathcal{G}_0$.

Proof. We consider first initial conditions $X_0, X_1 \in \mathcal{F}_0$. There exists a curve $X(s)$ in $\mathcal{C}_0$ such that

$$\int_0^1 |||X_s(s)|||_{X(s)} \, ds \leq d(X_0, X_1) + \varepsilon.$$

We consider the corresponding solution given by Lemma 3.8, that we simply denote $X(s,t)$. By Theorem 3.13, we have

$$|||X_s(s,t)|||_{X(s,t)} \leq e^{Ct} |||X_s(s,0)|||_{X(s,0)}. \tag{75}$$

Given a time T, we consider the projection of the curve $X(s, T, \,\cdot\,)$ on $\mathcal{G}_0$, that we denote $\bar{X}(s,\xi)$, which is given by

$$\bar{X}(s, \,\cdot\,) = \Pi(X(s, T, \,\cdot\,)).$$

We denote by $f(s,t,\xi)$, the inverse of $(y+H)(s,t,\xi)$ with respect to ξ, which is allways well-defined and bounded as $(y+H)(s,t,\,\cdot\,)\in G_\alpha$ for some $\alpha\leq e^{Ct}$, see Lemma 3.8. The definition of Π gives

$$\bar{X}(s,\xi)=X(s,T,f(s,T,\xi)).$$

We have $X(0,\,\cdot\,)=\tilde{S}_TX_0$ and $X(1,\,\cdot\,)=\tilde{S}_TX_1$ and the curve $\bar{X}$ belongs to $\mathcal{C}_0$. We have

$$\bar{X}_s(s,\xi)=X_s(s,T,f)+f_sX_\xi(s,T,f) \tag{76}$$

and

$$\bar{X}_\xi(s,\xi)=f_\xi X_s(s,T,f). \tag{77}$$

We consider decomposition of X_s given by

$$X_s(s,T,\xi)=g(s,T,\xi)X_\xi(s,T,\xi)+R(s,T,\xi). \tag{78}$$

where $g(s,T,\cdot)=g(X(s,T,\cdot),X_s(s,T,\cdot))$. Combining (76), (77) and (78), we end up with

$$\bar{X}_s(s,\xi)=\left(\frac{g(s,T,f(\xi))}{f_\xi(s,T,\xi)}+f_s(s,T,\xi)\right)\bar{X}_\xi(s,\xi)+R(s,t,f(s,T,\xi)). \tag{79}$$

Hence,

$$|||\bar{X}_s(s,\xi)|||\leq\left\|R(s,t,f(s,T,\xi))\right\|. \tag{80}$$

Let us prove that

$$\left\|R(s,T,f(s,T,\xi))\right\|\leq e^{Ct}\left\|R(s,T,\xi)\right\| \tag{81}$$

for some constant C. We have to prove that for any $g\in E_2$, we have

$$\left\|g\circ f\right\|_{E_2}\leq e^{Ct}\left\|g\right\|_{E_2}. \tag{82}$$

We have

$$\left\|g\circ f\right\|_{L^\infty(\mathbb{R})}\leq\left\|g\right\|_{L^\infty(\mathbb{R})} \tag{83}$$

and

$$\begin{aligned}\left\|(g\circ f)_\xi\right\|^2_{L^2(\mathbb{R})}&=\int_{\mathbb{R}}(g_\xi\circ f)^2f_\xi^2\,d\xi\\&\leq\left\|f_\xi\right\|_{L^\infty(\mathbb{R})}\int_{\mathbb{R}}(g_\xi\circ f)^2f_\xi\,d\xi\\&=\left\|f_\xi\right\|_{L^\infty(\mathbb{R})}\left\|g\right\|^2_{L^2(\mathbb{R})}.\end{aligned} \tag{84}$$

Since $f_\xi=\frac{1}{y_\xi+H_\xi}\circ f$, we have $\left\|f_\xi\right\|_{L^\infty(\mathbb{R})}\leq e^{CT}$ by (57), as $(y_\xi+H_\xi)(s,0,\xi)=1$ for all ξ, and (83), (84) imply (82). Using (81), it follows from (80) that

$$|||\bar{X}_s(s,\xi)|||\leq C|||X_s(s,T,\xi)||| \tag{85}$$

because $\left\|R(s,T,\xi)\right\|=|||X_s(s,T,\xi)|||$. Hence, we finally get

$$\begin{aligned}d(\tilde{S}_tX_0,\tilde{S}_tX_1)&\leq\int_0^1|||\bar{X}_s|||_{\bar{X}(s)}\,ds\\&\leq e^{CT}\int_0^1|||X_s(s,T)|||_{X(s,T)}\,ds\quad\text{(by (85))}\\&\leq e^{2CT}\int_0^1|||X_s(s,0)|||_{X(s,0)}\,ds\quad\text{(by (75))}\end{aligned}$$

$$\leq e^{2CT}(d(X_0, X_1) + \varepsilon)$$

which implies (74) as ε is arbitrary. To extend this result to any $X_0, X_1 \in \mathcal{G}_0$, we use the fact that the mapping $\tilde{S}_t$ is continuous with respect to the B-norm (Lemma 3.15) and $\mathcal{G}_0 \cap B^2$ is dense in $\mathcal{G}_0$ (Lemma 3.16). □

Lemma 3.15. *The mapping* $\Pi\colon \mathcal{F}_\alpha \to \mathcal{F}_0$ *is continuous with respect to the* B*-norm. It follows that* $\tilde{S}_t$ *is a continuous semigroup with respect to the* B*-norm.*

Proof. The proof of the continuity of Π is the same as in in [8, Lemma 3.5]. The continuity of $\tilde{S}_t$ then follows from (39) and the fact that $S_t\colon \mathcal{F}_0 \to \mathcal{F}_{\alpha(t)}$ for $\alpha \leq e^{Ct}$. □

Lemma 3.16. *The set* $\mathcal{G}_0 \cap B^2$ *is dense in* $\mathcal{G}_0$.

Proof. Given $X_0 \in \mathcal{G}_0$, we first assume that $X_{0,\xi}$ has compact support. We consider a mollifier ρ^ε. Given $X \in \mathcal{G}_0$, we consider the approximation $X^\varepsilon = X \star \rho^\varepsilon = (\zeta \star \rho^\varepsilon, U \star \rho^\varepsilon, H \star \rho^\varepsilon)$. By the Jensen inequality, since $\rho^\varepsilon \geq 0$ and $\int_{\mathbb{R}} \rho^\varepsilon(\eta)\, d\eta = 1$, we have

$$\left(\int_{\mathbb{R}} \zeta_\xi(\xi - \eta)\rho^\varepsilon(\eta)\, d\eta\right)^2 \leq \int_{\mathbb{R}} \zeta_\xi(\xi - \eta)^2 \rho^\varepsilon(\eta)\, d\eta$$

and similar inequalities for U_ξ and H_ξ. Hence, since X satisfies (56),

$$\begin{aligned}((y^\varepsilon_\xi)^2 + (H^\varepsilon_\xi)^2 + 2(U^\varepsilon_\xi)^2)(\xi) &\leq \int_{\mathbb{R}} ((y_\xi)^2 + (H_\xi)^2 + 2(U_\xi)^2)(\xi - \eta)\rho^\varepsilon(\eta)\, d\eta \\ &\leq \int_{\mathbb{R}} \rho^\varepsilon(\eta)\, d\eta = 1,\end{aligned}$$

and X^ε also satisfies (56). Since $y + H = \mathrm{Id}$, we have

$$y^\varepsilon + H^\varepsilon = \int_{\mathbb{R}} (\xi - \eta)\rho^\varepsilon(\eta)\, d\eta = \xi$$

(we consider an even mollifier) and X^ε satisfies (37b) for $\alpha = 0$. Since X_ξ has a compact support, which we denote K, $X(\xi)$ is constant for $\xi \in K^c$ and $X^\varepsilon = X$ on a the complement of a compact neighborhood of K, for ε small enough. Since $X^\varepsilon \to X$ on any compact set, it follows that $X^\varepsilon \to X$ in $L^\infty(\mathbb{R})$. By the standard convergence properties of approximating sequences, we have $X^\varepsilon_\xi \to X_\xi$ in $L^2(\mathbb{R})$ so that, finally, $X^\varepsilon \to X$ in B. Let us now consider the case where $X \in \mathcal{G}_0$ does not have a compact support. For any integer n, we define $X^n \in \mathcal{G}_0$ as follows

$$X^n(\xi) = \begin{cases} X^n(-n) & \text{if } \xi \leq -n, \\ X^n(\xi) & \text{if } -n < \xi < n, \\ X^n(n) & \text{if } \xi \geq n. \end{cases}$$

We have

$$X^n_\xi = \begin{cases} X_\xi & \text{if } \xi \in (-n, n), \\ 0 & \text{otherwise,} \end{cases}$$

so that X^n_ε has a compact support and the condition (37c) is satisfied. Since $X \in B$, we have $\lim_{\eta \to \pm\infty} X(\xi) = X(\pm\infty)$ and X^n tends to X in $L^\infty(\mathbb{R})$. Since X^n_ξ is a cut-off of X_ξ with a growing support, X^n_ξ tends to X_ξ in $L^2(\mathbb{R})$.

Therefore X^n tends to X in B and we have proved that the functions $X \in \mathcal{G}_0$ such that X_ξ has compact support are dense in $\mathcal{G}_0$. $\square$

4. Semi-group of solutions in Eulerian coordinates

We now return to the Eulerian variables.

Definition 4.1. *Let*

$$T_t = MS_tL \colon \mathcal{D} \to \mathcal{D}. \tag{86}$$

Next we show that T_t is a Lipschitz continuous semigroup by introducing a metric on $\mathcal{D}$.

Using the bijection L we can transport the topology from $\mathcal{F}_0$ to $\mathcal{D}$.

Definition 4.2. *Define the metric $d_{\mathcal{D}} \colon \mathcal{D} \times \mathcal{D} \to [0,\infty)$ by*

$$d_{\mathcal{D}}((u,\mu),(\bar u,\bar\mu)) = d(L(u,\mu), L(\bar u,\bar\mu)). \tag{87}$$

The final result in Eulerian variables reads as follows.

Theorem 4.3. *We have that $(T_t, d_{\mathcal{D}})$ is a continuous semigroup on $\mathcal{D}$.*

Proof. We have the following calculation

$$\begin{aligned}
d_{\mathcal{D}}(T_t(u,\mu), T_t(\bar u,\bar\mu)) &= d(L(T_t(u,\mu)), L(T_t(\bar u,\bar\mu))) \\
&= d(LT_tML(u,\mu), LT_tML(\bar u,\bar\mu)) \\
&= d(S_tL(u,\mu), S_tL(\bar u,\bar\mu)) \\
&\le e^{Ct} d(L(u,\mu), L(\bar u,\bar\mu)) \\
&= e^{Ct} d_{\mathcal{D}}((u,\mu),(\bar u,\bar\mu)).
\end{aligned}$$

$\square$

By a weak solution of (1) we mean the following.

Definition 4.4. *Let $u \colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ that satisfies:*
(i) $u \in C([0,\infty); L^\infty(\mathbb{R}))$ and $u_x \in L^\infty([0,\infty); L^2(\mathbb{R}))$;
(ii) the equation

$$\iint_{[0,\infty)\times\mathbb{R}} \big(u\phi_t - (uu_x - V)\phi\big)\,dxdt = \int_{\mathbb{R}} u_0\phi|_{t=0}\,dx \tag{88}$$

holds for all $\phi \in C_0^\infty([0,\infty)\times\mathbb{R})$. Here $V(t,x) = \frac14\left(\int_{-\infty}^x u_x^2\,dx - \int_x^\infty u_x^2\,dx\right)$ is in $L^\infty([0,\infty); L^\infty(\mathbb{R}))$. Then we say that u is a weak global conservative solution of the Hunter–Saxton equation (1)*.*

Theorem 4.5. *Given any initial condition $(u_0,\mu_0) \in \mathcal{D}$, we denote $(u,\mu)(t) = T_t(u_0,\mu_0)$. Then, $u(t,x)$ is a global solution of the Hunter–Saxton equation.*

Proof. After making the change of variables $x = y(t,\xi)$, we get, on the one hand,

$$\begin{aligned}
\iint_{[0,\infty)\times\mathbb{R}} u\phi_t\,dxdt &= \iint_{[0,\infty)\times\mathbb{R}} u(t,y(t,\xi))\phi_t(t,y(t,\xi))y_\xi(t,\xi)\,d\xi dt \\
&= \iint_{[0,\infty)\times\mathbb{R}} U(\phi(t,y)_t - y_t\phi_x(t,y))y_\xi\,d\xi dt
\end{aligned}$$

$$
\begin{aligned}
&= -\iint_{[0,\infty)\times\mathbb{R}} (U_t y_\xi + y_{\xi,t} U)\phi(t,y)\,d\xi dt \\
&\quad - \iint_{[0,\infty)\times\mathbb{R}} U^2 \phi(t,y)_\xi \,d\xi dt \\
&\quad + \int_{\mathbb{R}} (U y_\xi)(0,\xi)\phi(0,y(0,\xi)) y_\xi(0,\xi)\,d\xi \\
&= -\iint_{[0,\infty)\times\mathbb{R}} \big((\frac{1}{2}H - \frac{1}{4}H(\infty))y_\xi\big)\phi(t,y)\,d\xi dt \\
&\quad + \iint_{[0,\infty)\times\mathbb{R}} (U_\xi U)\phi(t,y)\,d\xi dt \\
&\quad + \int_{\mathbb{R}} u(0,x)\phi(0,x)\,dx,
\end{aligned} \tag{89}
$$

and, on the other hand,

$$
\begin{aligned}
&\iint_{[0,\infty)\times\mathbb{R}} \big(uu_x - V\big)\phi\,dxdt \\
&\qquad = \iint_{[0,\infty)\times\mathbb{R}} \big(uu_x - V\big)(t,y)\phi(t,y)y_\xi\,d\xi dt \\
&\qquad = \iint_{[0,\infty)\times\mathbb{R}} \big(UU_\xi\big)\phi(t,y)\,d\xi dt \\
&\qquad\quad - \iint_{[0,\infty)\times\mathbb{R}} V(t,y)\phi(t,y)y_\xi\,d\xi dt.
\end{aligned} \tag{90}
$$

By using (37c) and the fact that $U_\xi = u_x \circ y y_\xi$, we get

$$
\int_{\mathbb{R}} u_x^2\,dx = \int_{\mathbb{R}} u_x^2 \circ y y_\xi\,d\xi = \int_{\{\xi\in\mathbb{R} \mid y_\xi(t,\xi)>0\}} \frac{U_\xi^2}{y_\xi}\,d\xi = \int_{\{\xi\in\mathbb{R} \mid y_\xi(t,\xi)>0\}} H_\xi\,d\xi. \tag{91}
$$

The statement (21) implies that, for almost every $t \in \mathbb{R}$, the set $\{\xi \in \mathbb{R} \mid y_\xi(t,\xi) > 0\}$ is of full measure and therefore (91) yields

$$
\int_{\mathbb{R}} u_x^2\,dx = \int_{\mathbb{R}} H_\xi\,d\xi = H(\infty), \tag{92}
$$

for almost every $t \in \mathbb{R}$. Similarly, for almost every $t \in \mathbb{R}$, we get

$$
\begin{aligned}
V(t,y(t,\xi)) &= \frac{1}{2}\int_{\infty}^{y(t,\xi)} u_x^2\,dx - \frac{1}{4}\int_{\mathbb{R}} u_x^2\,dx \\
&= \frac{1}{2}\int_{\infty}^{\xi} u_x^2(t,y(t,\xi))y_\xi(t,\xi)\,dx - \frac{1}{4}H(\infty) \\
&= \frac{1}{2}\int_{\infty}^{\xi} H_\xi(t,\xi)\,d\xi - \frac{1}{4}H(\infty) \\
&= \frac{1}{2}H(t,\xi) - \frac{1}{4}H(\infty).
\end{aligned} \tag{93}
$$

After gathering (89), (90) and (93), we obtain that u is a weak solution of the Hunter–Saxton equation. It follows from (91) that

$$\int_{\mathbb{R}} u_x^2(t,x)\,dx \le H(t,\infty) = H(0,\infty) = \mu_0(\mathbb{R})$$

so that $u_x \in L^\infty(\mathbb{R}, L^2(\mathbb{R}))$. By construction of the semigroup T_t, we know that $(u,\mu)(t) \in C(\mathbb{R},\mathcal{D})$ where $\mathcal{D}$ is equipped by the metric $d_{\mathcal{D}}$. Proposition 5.2 below then implies that $u \in C(\mathbb{R}, L^\infty(\mathbb{R}))$. □

5. The topology induced by the metric d_D

Proposition 5.1. *The mapping*

$$u \mapsto (u, u_x^2\,dx)$$

is continuous from E_2 into $\mathcal{D}$. In other words, given a sequence $u_n \in E_2$ converging to u in E_2, that is,

$$u_n \to u \text{ in } L^\infty(\mathbb{R}) \text{ and } u_{n,x} \to u_x \text{ in } L^2(\mathbb{R}),$$

then $(u_n, u_{nx}^2\,dx)$ converges to $(u, u_x^2\,dx)$ in $\mathcal{D}$.

Proof. Let $X_n = (y_n, U_n, H_n) = L(u_n, u_{nx}^2\,dx)$ and $X = (y,U,H) = L(u, u_x^2\,dx)$, see (34). Following the proof of [8, Proposition 5.1], one can prove that

$$X_n \to X \text{ in } B.$$

Hence, by (63) in Lemma 3.11, we get that $\lim_{n\to\infty} d(X_n, X) = 0$ and therefore

$$(u_n, u_{nx}^2\,dx) \to (u, u_x^2\,dx) \text{ in } \mathcal{D}.$$

□

Proposition 5.2. *Let (u_n,μ_n) be a sequence in $\mathcal{D}$ that converges to (u,μ) in $\mathcal{D}$. Then*

$$u_n \to u \text{ in } L^\infty(\mathbb{R}).$$

Proof. Let $X_n = (y_n, U_n, H_n) = L(u_n,\mu_n)$ and $X = (y,U,H) = L(u,\mu)$, see (34). By the definition of the metric $d_{\mathcal{D}}$, we have $\lim_{n\to\infty} d(X_n, X) = 0$. We claim that

$$X_n \to X \text{ in } L^\infty(\mathbb{R}). \tag{94}$$

The proof of this claim follows the same lines as the proof of Lemma 3.10. For any $\varepsilon > 0$, there exists N such that for any $n \ge N$ there exist a path $X^n \in \mathcal{C}_0$ joining X_n and X such that

$$\int_0^1 |||X_s^n|||_{X^n(s)}\,ds \le \frac{\varepsilon}{2}. \tag{95}$$

We have the decomposition

$$X_s^n(s,\xi) = g^n(s,\xi)X_\xi^n(s,\xi) + R^n(s,\xi). \tag{96}$$

In the same way that we obtained (61), we now obtain

$$g^n(s,\xi) = R_1^n(s,\xi) + R_3^n(s,\xi). \tag{97}$$

and it follows that

$$\|X_n - X\|_{L^\infty(\mathbb{R})} \le \int_0^1 \|X_s^n(s,\cdot)\|_{L^\infty}\,ds$$

$$\leq \int_0^1 (\|g^n(s,\,\cdot\,)\|_{L^\infty} + \|R^n(s,\,\cdot\,)\|_{L^\infty})\,ds$$
$$\leq 2\int_0^1 \|R^n(s,\,\cdot\,)\|_{L^\infty}\,ds \quad \text{(by (97))}$$
$$\leq 2\int_0^1 \|R^n(s,\,\cdot\,)\|_B\,ds = 2\int_0^1 |||X^n(s,\,\cdot\,)|||_{X(s)}\,ds \leq \varepsilon.$$

and this concludes the proof of the claim (94). The rest of the proof is similar to the proof in [8, Proposition 5.2]. We reproduce it here for the sake of completeness. For any $x \in \mathbb{R}$, there exists ξ_n and ξ, which may not be unique, such that $x = y_n(\xi_n)$ and $x = y(\xi)$. We set $x_n = y_n(\xi)$. We have

$$u_n(x) - u(x) = u_n(x) - u_n(x_n) + U_n(\xi) - U(\xi) \tag{98}$$

and

$$\begin{aligned}
|u_n(x) - u_n(x_n)| &= \left| \int_\xi^{\xi_n} U_{n,\xi}(\eta)\,d\eta \right| \\
&\leq \sqrt{\xi_n - \xi}\left(\int_\xi^{\xi_n} U_{n,\xi}^2\,d\eta\right)^{1/2} && \text{(Cauchy–Schwarz)} \\
&= \sqrt{\xi_n - \xi}\left(\int_\xi^{\xi_n} y_{n,\xi}H_{n,\xi}\,d\eta\right)^{1/2} && \text{(from (37c))} \\
&\leq \sqrt{\xi_n - \xi}\sqrt{|y_n(\xi_n) - y_n(\xi)|} && \text{(since } H_{n,\xi} \leq 1) \\
&= \sqrt{\xi_n - \xi}\sqrt{y(\xi) - y_n(\xi)} \\
&\leq \sqrt{\xi_n - \xi}\,\|y - y_n\|_{L^\infty(\mathbb{R})}^{1/2}. && (99)
\end{aligned}$$

From (34a), one can prove that

$$|y(\xi) - \xi| \leq \mu(\mathbb{R})$$

and it follows that

$$|\xi_n - \xi| \leq 2\mu_n(\mathbb{R}) + |y_n(\xi_n) - y_n(\xi)| = 2H_n(\infty) + |y(\xi) - y_n(\xi)|$$

and, therefore, since $H_n \to H$ and $y_n \to y$ in $L^\infty(\mathbb{R})$, $|\xi_n - \xi|$ is bounded by a constant C independent of n. Then, (99) implies

$$|u_n(x) - u_n(x_n)| \leq C\,\|y - y_n\|_{L^\infty(\mathbb{R})}^{1/2}. \tag{100}$$

Since $y_n \to y$ and $U_n \to U$ in $L^\infty(\mathbb{R})$, it follows from (98) and (100) that $u_n \to u$ in $L^\infty(\mathbb{R})$.

□

References

[1] A. Bressan. *Hyperbolic Systems of Conservation Laws. The One Dimensional Cauchy Problem.* Oxford University Press, Oxford 2000.

[2] A. Bressan and G. Colombo. Existence and continuous dependence for discontinuous O.D.E.'s. *Boll. Un. Mat. Ital.*, 4-B:295–311, 1990.

[3] A. Bressan and A. Constantin. Global solutions of the Hunter–Saxton equation. *SIAM J. Math. Anal.*, 37:996–1026, 2005.

[4] A. Bressan and A. Constantin. Global conservative solutions to the Camassa–Holm equation. *Arch. Rat. Mech. Anal.* 183:215–239, 2007.

[5] A. Bressan and M. Fonte. An optimal transportation metric for solutions of the Camassa–Holm equation. *Methods and Applications of Analysis*, 12:191–220, 2005.
[6] A. Bressan, P. Zhang, and Y. Zheng. Asymptotic variational wave equations. *Arch. Rat. Mech. Anal.*, 183:163–185, 2007.
[7] H. Holden, K. H. Karlsen, and N. H. Risebro. Convergent difference schemes for the Hunter–Saxton equation. *Math. Comp.*, 76(258):699–744, 2005.
[8] H. Holden and X. Raynaud. Global conservative solutions of the Camassa–Holm equation—a Lagrangian point of view. *Comm. Partial Differential Equations*, 32(10-12):1511–1549, 2007.
[9] H. Holden and N. H. Risebro. *Front Tracking for Hyperbolic Conservation Laws.* Springer, Berlin, 2nd printing, 2007.
[10] J. K. Hunter and R. Saxton. Dynamics of director fields. *SIAM J. Appl. Math.*, 51(6):1498–1521, 1991.
[11] J. K. Hunter and Y. Zheng. On a completely integrable nonlinear hyperbolic variational equation. *Phys. D*, 79(2-4):361–386, 1994.
[12] J. K. Hunter and Y. Zheng. On a nonlinear hyperbolic variational equation. I. Global existence of weak solutions. *Arch. Rational Mech. Anal.*, 129(4):305–353, 1995.
[13] J. K. Hunter and Y. Zheng. On a nonlinear hyperbolic variational equation. II. The zero-viscosity and dispersion limits. *Arch. Rational Mech. Anal.*, 129(4):355–383, 1995.
[14] S. Kružkov. First order quasilinear equations with several space variables. *Math. USSR Sbornik* 10:217–243, 1970.
[15] P. Zhang and Y. Zheng. On oscillations of an asymptotic equation of a nonlinear variational wave equation. *Asymptot. Anal.*, 18(3-4):307–327, 1998.
[16] P. Zhang and Y. Zheng. On the existence and uniqueness of solutions to an asymptotic equation of a variational wave equation. *Acta Math. Sin. (Engl. Ser.)*, 15(1):115–130, 1999.
[17] P. Zhang and Y. Zheng. Existence and uniqueness of solutions of an asymptotic equation arising from a variational wave equation with general data. *Arch. Ration. Mech. Anal.*, 155(1):49–83, 2000.

(Bressan)
Department of Mathematics, Pennsylvania State University, University Park, PA 16802, USA
E-mail address: `bressan@math.psu.edu`
URL: `www.math.psu.edu/bressan/`

(Holden)
Department of Mathematical Sciences, Norwegian University of Science and Technology, NO–7491 Trondheim, Norway, and
Centre of Mathematics for Applications, University of Oslo, P.O. Box 1053, Blindern, NO–0316 Oslo, Norway
E-mail address: `holden@math.ntnu.no`
URL: `www.math.ntnu.no/~holden/`

(Raynaud)
Department of Mathematical Sciences, Norwegian University of Science and Technology, NO–7491 Trondheim, Norway, and
Centre of Mathematics for Applications, University of Oslo, P.O. Box 1053, Blindern, NO–0316 Oslo, Norway
E-mail address: `raynaud@math.ntnu.no`
URL: `www.math.ntnu.no/~raynaud/`