# ACOUSTIC LIMIT OF THE BOLTZMANN EQUATION: CLASSICAL SOLUTIONS

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ABSTRACT. We study the acoustic limit from the Boltzmann equation in the framework of classical solutions. For a solution $F_\varepsilon = \mu + \varepsilon\sqrt{\mu} f_\varepsilon$ to the rescaled Boltzmann equation in the acoustic time scaling

$$\partial_t F_\varepsilon + v\cdot\nabla_x F_\varepsilon = \frac{1}{\varepsilon}\mathcal{Q}(F_\varepsilon, F_\varepsilon)\,,$$

inside a periodic box $\mathbb{T}^3$, we establish the global-in-time uniform energy estimates of $f_\varepsilon$ in $\varepsilon$ and prove that $f_\varepsilon$ converges strongly to $f$ whose dynamics is governed by the acoustic system. The collision kernel $\mathcal{Q}$ includes hard-sphere interaction and inverse-power law with an angular cutoff.

## 1. Introduction

The acoustic system is the linearization about the homogeneous state of the compressible Euler system. After a suitable choice of units, in this model the fluid fluctuations $(\rho, u, \theta)$ satisfy

$$\begin{aligned} &\partial_t\rho + \nabla_x\cdot u = 0\,, && \rho(x,0) = \rho^0(x)\,,\\ &\partial_t u + \nabla_x(\rho+\theta) = 0\,, && u(x,0) = u^0(x)\,,\\ &\partial_t\theta + \tfrac{2}{3}\nabla_x\cdot u = 0\,, && \theta(x,0) = \theta^0(x)\,. \end{aligned} \tag{1.1}$$

In this paper, we consider the periodic boundary condition, i.e $x \in \mathbb{T}^3$.

This is one of the simplest system of fluid dynamical equations imaginable, being essentially the wave equation. It may be derived directly from the Boltzman equation as the formal limit of moment equations for an appropriately scaled family of Boltzmann solutions as the Knudsen number tends to zero.

The program initiated by Bardos, Golse, and Levermore [1] was to derive the fluid limits which include incompressible Stokes, Navier-Stokes, Euler equations, and acoustic system from the *DiPerna-Lions renormalized* solutions. This program has been developed with great success during the last decade, here we only mention [1, 2, 3, 4, 10, 11, 12] among others. In particular, Golse and Saint-Raymond [4] justified the first complete incompressible Navier-Stokes limit from the Boltzmann equation without any compactness assumption. On the other hand, higher order approximations with the unified energy method have been shown by Guo [7] to give rise to a rigorous passage from the Boltzmann equations to the Navier-Stokes-Fourier systems beyond the Navier-Stokes approximations in the framework of classical solutions.

Surprisingly, the status for rigorously deriving the acoustic system from DiPerna-Lions solutions of Boltzmann equation is still incomplete. This is mainly because DiPerna-Lions solutions do not have some properties which are formally satisfied such as local conservation laws. In [2], the acoustic limit was justified for Maxwell molecular collisions under some assumption on the amplitude of fluctuations. The result was significantly improved in [3] to a large class of hard potentials and the assumption of the amplitude of fluctuations was relaxed to the order $\varepsilon^m$ with $m > \frac{1}{2}$. Recently, the borderline case $m = \frac{1}{2}$ was covered in [9] for soft potentials.

In this paper, we take the first step to establish the acoustic limit from the Boltzmann equation in the framework of classical solutions. Working with classical solutions has several advantages than working with the DiPerna-Lions solutions. For example, the classical solutions automatically satisfy local conservation laws and have good regularities; the nonlinear interaction can be controlled by linear dissipation for small solutions.

We employ the nonlinear energy method developed by Guo [5, 6, 7] in recent years which has been turned out to be applicable to other problems, for instance see [8]. We justify the limit for the case that the amplitude of fluctuation is $\varepsilon$, which is not being optimal. However, our work has advantages in that we can treat for a large class of collision kernels in a rather uniform way, including hard potentials, soft potentials and especially Landau kernels which were not covered in the framework of the renormalized solutions. Furthermore, different dissipation mechanisms for macroscopic parts and microscopic parts in the limit process are clearly presented by the energy dissipation rate. To our best knowledge, this is the first global-in-time acoustic limit result in the class of classical solutions.

The paper is organized as follows: the next section contains the formulation of the Boltzmann equation for different collision kernels. Some preliminary lemmas regarding the estimates on the collision operators are listed in Section 3. Then we give a very brief formal derivation. Section 5 and 6 are devoted to the energy estimates.

## 2. Formulation and Notations

Consider the following rescaled Boltzmann equation:

$$\partial_t F_\varepsilon + v\cdot\nabla_x F_\varepsilon = \frac{1}{\varepsilon}\mathcal{Q}(F_\varepsilon, F_\varepsilon) \tag{2.1}$$

In this paper, as in [7], we consider two classes of collision kernels, the first is given by the standard Boltzmann collision operator $\mathcal{Q}(G_1, G_2)$:

$$\mathcal{Q}(G_1, G_2) = \int_{\mathbb{R}^3\times S^2} |u-v|^\gamma B(\theta)|\{G_1(v')G_2(u') - G_1(v)G_2(u)\}du d\omega, \tag{2.2}$$

where $-3 < \gamma \le 1$, $B(\theta) \le C|\cos\theta|$, $v' = v - [(v-u)\cdot\omega]\omega$ and $u' = u + [(v-u)\cdot\omega]\omega$. These collision operators cover hard-sphere interactions and inverse-power law with an angular cutoff. The hard potential means $0 \le \gamma \le 1$, and the soft potential means $-3 < \gamma < 0$.

The second class is the Landau collision operator

$$\mathcal{Q}(G_1, G_2) = \sum_{1\le i,j\le 3} \partial^i \int \phi_{ij}(v-u)\{G_1(u)\partial^j G_2(v) - G_2(v)\partial^j G_1(u)\}\mathrm{d}u\,, \tag{2.3}$$

where $\partial^i = \partial_{v_i}$ and

$$\phi_{ij} \equiv \frac{1}{|v|}\left\{\delta_{ij} - \frac{v_i v_j}{|v|^2}\right\}\,. \tag{2.4}$$

Let

$$F_\varepsilon = \mu + \varepsilon\sqrt{\mu} f_\varepsilon$$

be the perturbation around the global Maxwellian

$$\mu = \frac{1}{(2\pi)^{3/2}} e^{-\frac{|v|^2}{2}}.$$

Define $\mathcal{L}$, the linearized collision operator, as follows

$$\mathcal{L}g \equiv -\frac{1}{\sqrt{\mu}}\{\mathcal{Q}(\mu, \sqrt{\mu}g) + \mathcal{Q}(\sqrt{\mu}g, \mu)\}, \tag{2.5}$$

and the nonlinear collision operator $\Gamma$ as

$$\Gamma(g, h) = \frac{1}{\sqrt{\mu}} \mathcal{Q}(\sqrt{\mu}g, \sqrt{\mu}h). \tag{2.6}$$

The rescaled Boltzmann equation (2.1) is written in terms of the perturbation $f_\varepsilon$ as follows:

$$\partial_t f_\varepsilon + v \cdot \nabla_x f_\varepsilon + \frac{1}{\varepsilon}\mathcal{L} f_\varepsilon = \Gamma(f_\varepsilon, f_\varepsilon). \tag{2.7}$$

We first recall that the operator $\mathcal{L} \geq 0$, and for any fixed $(t, x)$, the null space of $\mathcal{L}$ is generated by $[\sqrt{\mu}, v\sqrt{\mu}, |v|^2\sqrt{\mu}]$. For any function $f(t, x, v)$ we thus can decompose

$$f = \mathbf{P}f + (\mathbf{I} - \mathbf{P})f$$

where $\mathbf{P}f$ (the hydrodynamic part) is the $L^2_v$ projection on the null space for $\mathcal{L}$ for given $(t, x)$. We can further denote

$$\mathbf{P}f = \{\rho_f(t, x) + v \cdot u_f(t, x) + (\tfrac{|v|^2}{2} - \tfrac{3}{2})\theta_f(t, x)\}\sqrt{\mu}. \tag{2.8}$$

Here we define the *hydrodynamic field* of $f$ as

$$[\rho_f(t, x), u_f(t, x), \theta_f(t, x)]$$

which represents the density, velocity and temperature fluctuations physically.

In order to state our results precisely, we introduce the following norms and notations. We use $\langle \cdot\,, \cdot\rangle$ to denote the standard $L^2$ inner product in $\mathbb{R}^3_v$, while we use $(\cdot\,, \cdot)$ to denote the $L^2$ inner product either in $\mathbb{T}^3 \times \mathbb{R}^3$ or in $\mathbb{T}^3$ with corresponding the $L^2$ norm $\|\cdot\|$. We use the standard notation $H^s$ to denote the Sobolev space $W^{s,2}$. For the Boltzmann collision operator (2.2), we define the collision frequency as

$$\nu(v) \equiv \int_{\mathbb{R}^3} |v - v'|^\gamma \mu(v') dv', \tag{2.9}$$

which behaves like $|v|^\gamma$ as $|v| \to \infty$. It is natural to define the following weighted $L^2$ norm to characterize the dissipation rate.

$$|g|^2_\nu \equiv \int_{\mathbb{R}^3} g^2(v)\nu(v) dv, \quad \|g\|^2_\nu \equiv \int_{\mathbb{T}^3\times\mathbb{R}^3} g^2(x, v)\nu(v) \mathrm{d}v\mathrm{d}x.$$

For the Landau operator (2.3). let

$$\sigma_{ij}(v) = \int_{\mathbb{R}^3} \frac{1}{|v - u|}\left\{\delta_{ij} - \frac{(v - u)_i(v - u)_j}{|v - u|^2}\right\}\mu(u)\,\mathrm{d}u\,. \tag{2.10}$$

The natural norms are given by the $\sigma$-norm

$$|g|_\sigma^2 \equiv \sum_{1\le i,j\le 3}\int_{\mathbb{R}}^3 \{\sigma_{ij}\partial^i g\partial^j g+\sigma_{ij}v^iv^jg^2\}\,\mathrm{d}v\,,$$

$$\|g\|_\sigma^2 \equiv \sum_{1\le i,j\le 3}\int_{\mathbb{R}^3\times\mathbb{T}^3} \{\sigma_{ij}\partial^i g\partial^j g+\sigma_{ij}v^iv^jg^2\}\,\mathrm{d}v\mathrm{d}x\,.$$

We also use a unified notation for the dissipation as $|g|_D$ and $\|g\|_D$ to denote either $|g|_\nu$ or $|g|_\sigma$, $\|g\|_\nu$ or $\|g\|_\sigma$ respectively. Let the weight function $w(v)$ be

$$w(v)\equiv |(1+|v|^2)^{\frac{1}{2}}.$$

For both Boltzmann and Landau kernels we have

$$\|w^{-3/2}g\|\le C\|g\|_D\,. \tag{2.11}$$

See [7] for the details.

In order to be consistent with the hydrodynamic equations, we define

$$\partial_\alpha^\beta=\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_{x_3}^{\alpha_3}\partial_{v_1}^{\beta_1}\partial_{v_2}^{\beta_2}\partial_{v_3}^{\beta_3} \tag{2.12}$$

where $\alpha=[\alpha_1,\alpha_2,\alpha_3]$ is related to the space derivatives, while $\beta=[\beta_1,\beta_2,\beta_3]$ is related to the velocity derivatives.

We now define instant energy functionals and the dissipation rate.

**Definition 1 (Instant Energy)** *For $N\ge 8$, for some constant $C>0$, an instant energy functional $\mathcal{E}_{N,l}(f)(t)\equiv\mathcal{E}_{N,l}(t)$ satisfies:*

(i) for hard potentials with $0\le\gamma\le 1$ in (2.2)

$$\frac{1}{C}\mathcal{E}_{N,l}(t)\le\sum_{|\alpha|\le N+1}\|\partial_\alpha f\|^2+\sum_{|\alpha|+|\beta|\le N}\|w^l\partial_\alpha^\beta f\|^2\le C\mathcal{E}_{N,l}(t)\,; \tag{2.13}$$

(ii) for soft potentials with $-3<\gamma<0$ in (2.2)

$$\frac{1}{C}\mathcal{E}_{N,l}(t)\le\sum_{|\alpha|\le N+1}\|\partial_\alpha f\|^2+\sum_{|\alpha|+|\beta|\le N}\|w^{\{l-|\beta|\}|\gamma|}\partial_\alpha^\beta f\|^2\le C\mathcal{E}_{N,l}(t)\,; \tag{2.14}$$

(iii) for the Landau kernel (2.3),

$$\frac{1}{C}\mathcal{E}_{N,l}(t)\le\sum_{|\alpha|\le N+1}\|\partial_\alpha f\|^2+\sum_{|\alpha|+|\beta|\le N}\|w^{l-|\beta|}\partial_\alpha^\beta f\|^2\le C\mathcal{E}_{N,l}(t).\,, \tag{2.15}$$

for all functions $f(t,x,v)$.

**Definition 2 (Dissipation Rate)** *For $N\ge 8$, the dissipation rate $\mathcal{D}_N(t)$ is defined as*

(i) for hard potentials with $0\le\gamma\le 1$ in (2.2)

$$\begin{aligned}\mathcal{D}_{N,l}(t)=&\sum_{|\alpha|\le N+1}\left(\varepsilon\|\partial_\alpha\mathbf{P}f\|^2(t)+\frac{1}{\varepsilon}\|\partial_\alpha(\mathbf{I}-\mathbf{P})f\|_\nu^2\right)\\&+\frac{1}{\varepsilon}\sum_{|\alpha|+|\beta|\le N}\|w^l\partial_\alpha^\beta(\mathbf{I}-\mathbf{P})f\|_\nu^2;\end{aligned} \tag{2.16}$$

(ii) for soft potentials with $-3<\gamma<0$ in (2.2)

$$\begin{aligned}\mathcal{D}_{N,l}(t) = \sum_{|\alpha|\le N+1}\left(\varepsilon\|\partial_\alpha \mathbf{P}f\|^2(t)+\frac{1}{\varepsilon}\|\partial_\alpha(\mathbf{I}-\mathbf{P})f\|_\nu^2\right)\\ +\frac{1}{\varepsilon}\sum_{|\alpha|+|\beta|\le N}\|w^{\{l-|\beta|\}|\gamma|}\partial_\alpha^\beta(\mathbf{I}-\mathbf{P})f\|_\nu^2.\end{aligned} \tag{2.17}$$

(iii) for the Landau kernel (2.3),

$$\begin{aligned}\mathcal{D}_{N,l}(t) = \sum_{|\alpha|\le N+1}\left(\varepsilon\|\partial_\alpha \mathbf{P}f\|^2(t)+\frac{1}{\varepsilon}\|\partial_\alpha(\mathbf{I}-\mathbf{P})f\|_\nu^2\right)\\ +\frac{1}{\varepsilon}\sum_{|\alpha|+|\beta|\le N}\|w^{l-|\beta|}\partial_\alpha^\beta(\mathbf{I}-\mathbf{P})f\|_\nu^2.\end{aligned} \tag{2.18}$$

Both the instant energy and the dissipation rate are carefully designed to capture the structure of the rescaled Boltzmann equation (2.1) in the acoustic regime. For soft potentials, $\mathcal{E}_{N,l}$ and $\mathcal{D}_{N,l}$ involve a weight function in $v$ which depends on the number of velocity derivatives $\partial^\beta$. This is designed to control the velocity derivatives for the streaming terms $v\cdot\nabla_x$ by a weak dissipation rate as proposed in [7]. In particular, the dissipation rates in (2.16), (2.17), (2.18) in which the hydrodynamic part has $\varepsilon$ scale reflect that we do not observe the dissipation in the limit, which is exactly the case of the acoustic system.

We state the main result of this article.

**Theorem 2.1.** *Let $N\ge 8$. Let $0<\varepsilon\le\frac{1}{4}$ be given. Suppose $f_\varepsilon(0,x,v)=f_0^\varepsilon(x,v)$ satisfies the mass, momentum, and energy conservation laws*

$$(f_0^\varepsilon,[1,v,|v|^2]\sqrt{\mu})=0, \tag{2.19}$$

*and $F_\varepsilon(0,x,v)=\mu+\varepsilon f_0^\varepsilon(x,v)\ge 0$. If $\mathcal{E}_{N,l}(f^\varepsilon)(0)$ is sufficiently small, then there exists a unique global-in-time solution $f_\varepsilon(t,x,v)$ to (2.7), and moreover there exists an instant energy functional $\mathcal{E}_{N,l}(f^\varepsilon)(t)$ such that*

$$\frac{d}{dt}\mathcal{E}_{N,l}(f^\varepsilon)(t)+\mathcal{D}_{N,l}(f^\varepsilon)(t)\le 0. \tag{2.20}$$

*In particular, we have the following global energy bound:*

$$\sup_{0\le t\le\infty}\mathcal{E}_{N,l}(f^\varepsilon)(t)\le\mathcal{E}_{N,l}(f^\varepsilon)(0). \tag{2.21}$$

*Remark* 2.2. The global existence of solutions $f_\varepsilon$ to (2.7) follows from the a priori global energy bound (2.21) by rather standard method. In this article, we focus on proving the uniform bound.

*Remark* 2.3. Note that due to the weak dissipation (2.16), we cannot deduce the time decay estimate from the energy inequality (2.20) unlike the incompressible Navier-Stokes-Fourier case in [7, 8]. Indeed, physically, we do not expect any time decay of our instant energy $\mathcal{E}_{N,l}(f^\varepsilon)(t)$, since the acoustic system preserves the initial energy for all time. See Lemma 4.2.

## 3. Basic estimates of collision operators

In this section, we sum up some basic estimates of collision operators for various kernels considered in this paper. The proofs can be found in [7]. The following is the coercivity of $\mathcal{L}$.

**Lemma 3.1.** *There exists $\delta > 0$ such that for any $f \in L^2(\mathbb{R}^3_v)$*

$$\langle \mathcal{L}f, f\rangle \geq \delta |(\mathbf{I}-\mathbf{P})f|^2_\nu. \tag{3.1}$$

**Lemma 3.2.** *For hard potential with $\gamma \geq 0$, there exits $C_{|\beta|}$, $C > 0$ such that*

$$(w^{2l}\partial^\beta_\alpha \mathcal{L}f\,,\partial^\beta_\alpha f) \geq \frac{1}{2}\|w^l\partial^\beta_\alpha f\|^2_\nu - C_{|\beta|}\|f\|^2_\nu\,, \tag{3.2}$$

$$(\partial^\beta_\alpha \Gamma(f,g),\partial^\beta_\alpha h) \leq C\{\|w^l\partial^{\beta_1}_{\alpha_1} f\|\cdot\|w^l\partial^{\beta_2}_{\alpha_2} g\|_\nu + \|w^l\partial^{\beta_1}_{\alpha_1} g\|\cdot\|w^l\partial^{\beta_2}_{\alpha_2} f\|_\nu\}\|w^l\partial^\beta_\alpha h\|_\nu\,. \tag{3.3}$$

*where $l \geq 0$, and summation is for $|\alpha|+|\beta| \leq N$ with $\beta_1+\beta_2 \leq \beta$ and $\alpha_2 \leq \alpha$ componentwise.*

**Lemma 3.3.** *For the inverse power law with $-3 < \gamma < 0$, for any $l \geq 0$, there exist $C_{|\beta|}, C > 0$ such that*

$$(w^{\{2l-2|\beta|\}|\gamma|}\partial^\beta_\alpha \mathcal{L}f\,,\partial^\beta_\alpha f) \geq \frac{1}{2}\|w^{\{l-|\beta|\}|\gamma|}\partial^\beta_\alpha f\|^2_\nu - C_{|\beta|}\|f\|^2_\nu\,, \tag{3.4}$$

$$\begin{aligned}(w^{\{2l-2|\beta|\}|\gamma|}\partial^\beta_\alpha \Gamma(f,g)\,,\partial^\beta_\alpha h) \leq C\{&\|w^{\{l-|\beta_1|\}|\gamma|}\partial^{\beta_1}_{\alpha_1} f\|\cdot\|w^{\{l-|\beta_2|\}|\gamma|}\partial^{\beta_2}_{\alpha_2} g\|_\nu \\ &+ \|w^{\{l-|\beta_1|\}|\gamma|}\partial^{\beta_1}_{\alpha_1} g\|\cdot\|w^{\{l-|\beta_2|\}|\gamma|}\partial^{\beta_2}_{\alpha_2} f\|_\nu\} \\ &\times \|w^{\{l-|\beta|\}|\gamma|}\partial^\beta_\alpha h\|_\nu\,,\end{aligned} \tag{3.5}$$

*where the summation is taken over $|\alpha_1|+|\beta_1| \leq |\alpha|+|\beta| \leq [\frac{N}{2}]+4$, and $\alpha_2 \leq \alpha$ and $\beta_2 \leq \beta$ componentwise.*

**Lemma 3.4.** *For the Landau kernel, for any $l \geq 0$, there exist $C_{|\beta|}, C > 0$, such that*

$$(w^{2l-2|\beta|}\partial^\beta_\alpha \mathcal{L}f\,,\partial^\beta_\alpha f) \geq \frac{1}{2}\|w^{l-|\beta|}\partial^\beta_\alpha f\|^2_\sigma - C_{|\beta|}\|f\|^2_\sigma\,, \tag{3.6}$$

$$\begin{aligned}(w^{2l-2|\beta|}\partial^\beta_\alpha \Gamma(f,g)\,,\partial^\beta_\alpha h) \leq C\{&\|w^{l-|\beta_1|}\partial^{\beta_1}_{\alpha_1} f\|\cdot\|w^{l-|\beta_2|}\partial^{\beta_2}_{\alpha_2} g\|_\sigma \\ &+ \|w^{l-|\beta_1|}\partial^{\beta_1}_{\alpha_1} g\|\cdot\|w^{l-|\beta_2|}\partial^{\beta_2}_{\alpha_2} f\|_\sigma\} \\ &\times \|w^{l-|\beta|}\partial^\beta_\alpha h\|_\sigma\,,\end{aligned} \tag{3.7}$$

*where the summation is taken over $|\alpha|+|\beta| \leq N$, and $\beta_1+\beta_2 \leq \beta$ and $\alpha_2 \leq \alpha$ componentwise.*

As a direct consequence of (3.3) in the above lemmas, we can estimate the pure spatial derivatives for the nonlinear collision operator $\Gamma$.

**Lemma 3.5.** *Let $\zeta(v)$ be a smooth function that decays exponentially, then there is a given instant energy functional $\mathcal{E}_{N,0}(f)$ and $C_\zeta > 0$, such that for summation over $\alpha_1+\alpha_2=\alpha$, $|\alpha| \leq N$,*

$$\begin{aligned}(\partial_\alpha \Gamma(f\,,g)\,,\partial_\alpha h) &\leq \{\mathcal{E}^{1/2}_{N,0}(f)\|\partial_{\alpha_2} g\|_\nu + \mathcal{E}^{1/2}_{N,0}(g)\|\partial_{\alpha_2} f\|_\nu\}\|\partial_{\alpha_3} h\|_\nu\,, \\ \left\|\int \partial_\alpha \Gamma(f\,,g)\zeta\,dv\right\| &\leq C_\zeta\{\mathcal{E}^{1/2}_{N,0}(f)\cdot\|\partial_{\alpha_2} g\|_\nu + \mathcal{E}^{1/2}_{N,0}(g)\cdot\|\partial_{\alpha_2} f\|_\nu\}\,.\end{aligned} \tag{3.8}$$

## 4. Derivation of Acoustic System

In this section, we derive the acoustic system as the hydrodynamic limit of solutions $f_\varepsilon$ to the rescaled Boltzmann equation (2.7). Since we have the uniform energy bound in $\varepsilon$ by Theorem 2.1, there exists the unique limit $f$ of $f_\varepsilon$ in $\varepsilon$ and we remark that due to higher order energy bound, all the limits in the below are strongly convergent. First, by letting $\varepsilon \to 0$ in (2.7), one finds that $\mathcal{L}f = 0$. Thus $f$ can be written as follows:

$$f = \{\rho + v \cdot u + \left(\tfrac{|v|^2}{2} - \tfrac{3}{2}\right)\theta\}\sqrt{\mu},$$

for $\rho$, $u$, $\theta$ are functions of $t, x$. In order to determine the dynamics of $\rho$, $u$, $\theta$, project (2.7) onto $\{\sqrt{\mu}, v\sqrt{\mu}, (\frac{|v|^2}{2} - \frac{3}{2})\sqrt{\mu}\}$: by collision invariants, first we get

$$\langle \partial_t f_\varepsilon + v \cdot \nabla_x f_\varepsilon,\ \{1, v, (\tfrac{|v|^2}{3} - 1)\}\sqrt{\mu}\rangle = 0$$

and take the limit $\varepsilon \to 0$ to get

$$\langle \partial_t f + v \cdot \nabla_x f,\ \{1, v, (\tfrac{|v|^2}{3} - 1)\}\sqrt{\mu}\rangle = 0$$

Since $f = \mathbf{P}f$, this is equivalent to

$$\begin{aligned} &\partial_t \rho + \nabla_x \cdot u = 0 \\ &\partial_t u + \nabla_x(\rho + \theta) = 0 \\ &\partial_t \theta + \tfrac{2}{3}\nabla_x \cdot u = 0 \end{aligned} \tag{4.1}$$

Thus we have shown the following proposition on the mathematical derivation of the acoustic system from the Boltzmann equation.

**Proposition 4.1.** *Assume that $F_\varepsilon = \mu + \varepsilon\sqrt{\mu} f_\varepsilon$ solves the rescaled Boltzmann equation* (2.1) *where $f_\varepsilon$ is obtained from Theorem 2.1. Then there exists the hydrodynamic limit $f$ of $f_\varepsilon$ such that $f = \mathbf{P}f$, and furthermore its macroscopic variables $\rho$, $u$, $\theta$ solve the acoustic system* (4.1)*.*

The acoustic system is a linear system and it is globally well-posed in the Sobolev space.

**Lemma 4.2.** *The acoustic system* (4.1) *is globally well-posed in $H^s(\mathbb{T}^3)$ space, for any $s \geq 0$. Moreover, we obtain the following estimates:*

$$\frac{d}{dt}\{||\rho_1||^2_{H^s} + ||u_1||^2_{H^s} + \tfrac{3}{2}||\theta_1||^2_{H^s}\} = 0 \tag{4.2}$$

*Proof.* The existence of solutions can be verified, for instance by solving the ordinary differential equation after taking Fourier transform in $x \in \mathbb{T}^3$. The energy estimates give rise to the conservation of energy (4.2). The uniqueness is easily deduced. □

## 5. uniform spatial energy estimates

In this section, we shall establish a uniform spatial energy estimate for $f_\varepsilon$, a solution to (2.7):

$$\partial_t f_\varepsilon + v \cdot \nabla_x f_\varepsilon + \frac{1}{\varepsilon}\mathcal{L} f_\varepsilon = \Gamma(f_\varepsilon, f_\varepsilon)$$

For the convenience, we rewrite the fluid part $\mathbf{P}f_\varepsilon$ as follows:

$$\mathbf{P}f_\varepsilon = \{a^\varepsilon(t,x) + b^\varepsilon(t,x) \cdot v + c^\varepsilon(t,x)|v|^2\}\sqrt{\mu}$$

Our goal is to estimate $a^\varepsilon(t,x)$, $b^\varepsilon(t,x)$, and $c^\varepsilon(t,x)$ in terms of $(\mathbf{I} - \mathbf{P})f_\varepsilon$.

**Lemma 5.1.** *Assume $f_\varepsilon$ is a solution to (2.7) satisfying conservation of mass, momentum and energy:*

$$(f^\varepsilon(t),[1,v,|v|^2]\sqrt{\mu})=0. \tag{5.1}$$

*Then there exists $C_1>0$ such that*

(5.2)

$$\varepsilon\sum_{|\alpha|\le N+1}\|\partial_\alpha \mathbf{P}f_\varepsilon\|^2\le \varepsilon\frac{dG(t)}{dt}+\frac{C_1}{\varepsilon}\sum_{|\alpha|\le N+1}\|\partial_\alpha(\mathbf{I}-\mathbf{P})f_\varepsilon\|_\nu^2+C_1\varepsilon\sum_{|\alpha|\le N}\|\partial_\alpha\Gamma(f_\varepsilon,f_\varepsilon)_{\|}\|^2$$

*where $G(t)$ is defined as*

$$\begin{aligned}&-\sum_{|\alpha|\le N}\int_{\mathbb{T}^3}\left(\langle(\mathbf{I}-\mathbf{P})\partial_\alpha f_\varepsilon\,,\zeta_{ij}\rangle\cdot\partial_j\partial_\alpha b^\varepsilon-\langle(\mathbf{I}-\mathbf{P})\partial_\alpha f_\varepsilon\,,\zeta_c\rangle\cdot\nabla_x\partial_\alpha c^\varepsilon\right)\,dx\\ &-\sum_{|\alpha|\le N}\int_{\mathbb{T}^3}\left(\langle(\mathbf{I}-\mathbf{P})\partial_\alpha f_\varepsilon\,,\zeta\rangle\cdot\nabla_x\partial_\alpha a^\varepsilon\,-\partial_\alpha b^\varepsilon\cdot\nabla_x\partial_\alpha a^\varepsilon\right)\,dx\,.\end{aligned}\tag{5.3}$$

*Here $\zeta_{ij}(v)\,,\zeta_c(v)\,,\zeta_a(v)$ are some fixed linear combinations of the basis*

$$[\sqrt{\mu},v_i\sqrt{\mu},v_iv_j\sqrt{\mu},v_i|v|^2\sqrt{\mu}]$$

*for $1\le i\,,j\le 3$, and $f_{\|}$ is the $L^2_v$ projection of $f$ onto the subspace generated by the same basis. It is obvious that $|G(t)|\le C\mathcal{E}_{N,0}(t)$.*

*Proof.* The proof is similar to the one of Lemma 6.1 in [7]. For the clear presentation of this article, we provide the key ingredients and estimates and point out the difference. From the conservation of mass, momentum, and energy (5.1), it follows that

$$\int_{\mathbb{T}^3}a^\varepsilon(t,x)dx=\int_{\mathbb{T}^3}b^\varepsilon(t,x)dx=\int_{\mathbb{T}^3}c^\varepsilon(t,x)dx=0$$

By Poincaré inequality, it suffices to estimate

$$\nabla_x\partial_\alpha a^\varepsilon,\ \nabla_x\partial_\alpha b^\varepsilon,\ \nabla_x\partial_\alpha c^\varepsilon,$$

for $|\alpha|\le N$. First, we use the local conservation laws: Multiply $\sqrt{\mu},v\sqrt{\mu},|v|^2\sqrt{\mu}$ with (2.7) and integrate in $v\in\mathbb{R}^3$. By the collision invariants, we obtain

$$\begin{aligned}&\partial_t a^\varepsilon=\frac12\langle v\cdot\nabla_x(\mathbf{I}-\mathbf{P})f_\varepsilon,|v|^2\sqrt{\mu}\rangle\\ &\partial_t c^\varepsilon+\frac13\nabla_x\cdot b^\varepsilon=-\frac16\langle v\cdot\nabla_x(\mathbf{I}-\mathbf{P})f_\varepsilon,|v|^2\sqrt{\mu}\rangle\\ &\partial_t b^\varepsilon+\nabla_x a^\varepsilon+5\nabla_x c^\varepsilon=-\langle v\cdot\nabla_x(\mathbf{I}-\mathbf{P})f_\varepsilon,v\sqrt{\mu}\rangle\end{aligned}\tag{5.4}$$

The second ingredient of the proof is the macroscopic equations. By plugging $f_\varepsilon=\mathbf{P}f_\varepsilon+(\mathbf{I}-\mathbf{P})f_\varepsilon$ into (2.7), we get

$$\begin{aligned}\{\partial_t a^\varepsilon+\partial_t b^\varepsilon\cdot v+\partial_t c^\varepsilon|v|^2\}\sqrt{\mu}&+v\cdot\{\nabla_x a^\varepsilon+\nabla_x b^\varepsilon\cdot v+\nabla_x c^\varepsilon|v|^2\}\sqrt{\mu}\\ &=-\{\partial_t+v\cdot\nabla_x\}(\mathbf{I}-\mathbf{P})f_\varepsilon-\frac{1}{\varepsilon}L(\mathbf{I}-\mathbf{P})f_\varepsilon+\Gamma(f_\varepsilon,f_\varepsilon)\end{aligned}$$

Fix $t,x$, and compare the coefficients on both sides in front of

$$[\sqrt{\mu},v_i\sqrt{\mu},v_iv_j\sqrt{\mu},v_i|v|^2\sqrt{\mu}].$$

Then we get the following macroscopic equations as

$$\partial_i c^\varepsilon = l_c^\varepsilon + h_c^\varepsilon \tag{5.5}$$

$$\partial_t c^\varepsilon + \partial_i b_i^\varepsilon = l_i^\varepsilon + h_i^\varepsilon \tag{5.6}$$

$$\partial_i b_j^\varepsilon + \partial_j b_i^\varepsilon = l_{ij}^\varepsilon + h_{ij}^\varepsilon,\ i \neq j \tag{5.7}$$

$$\partial_t b_i^\varepsilon + \partial_i a^\varepsilon = l_{bi}^\varepsilon + h_{bi}^\varepsilon \tag{5.8}$$

$$\partial_t a^\varepsilon = l_a^\varepsilon + h_a^\varepsilon \tag{5.9}$$

Here the linear parts $l_c^\varepsilon, l_i^\varepsilon, l_{ij}^\varepsilon, l_{bi}^\varepsilon, l_a^\varepsilon$ are of the form

$$\langle -\{\partial_t + v\cdot\nabla_x\}(\mathbf{I}-\mathbf{P})f_\varepsilon - \frac{1}{\varepsilon}\mathcal{L}(\mathbf{I}-\mathbf{P})f_\varepsilon, \zeta\rangle \tag{5.10}$$

where $\zeta$ is a linear combination of the basis

$$[\sqrt{\mu}, v_i\sqrt{\mu}, v_iv_j\sqrt{\mu}, v_i|v|^2\sqrt{\mu}],$$

and accordingly, $h_c^\varepsilon, h_i^\varepsilon, h_{ij}^\varepsilon h_{bi}^\varepsilon, h_a^\varepsilon$ are defined as $\langle\Gamma(f_\varepsilon, f_\varepsilon),\zeta\rangle$ with the same choices of $\zeta$.

Following the proof of Lemma 6.1 in [7], we first deduce

$$\begin{aligned}
\|\nabla\partial_\alpha b^\varepsilon\|^2 \le &-\frac{d}{dt}\int_{\mathbb{T}^3}\langle(\mathbf{I}-\mathbf{P})\partial_\alpha f^\varepsilon, \zeta_{ij}\rangle\cdot\partial_j\partial_\alpha b^\varepsilon dx\\
&+C\|\nabla_x\partial_\alpha(\mathbf{I}-\mathbf{P})f^\varepsilon\|_\nu\{\|\nabla\partial_\alpha a^\varepsilon\| + \|\nabla\partial_\alpha c^\varepsilon\|\}\\
&+C\{\|\partial_\alpha(\mathbf{I}-\mathbf{P})f^\varepsilon\|_\nu^2 + \|\nabla_x\partial_\alpha(\mathbf{I}-\mathbf{P})f^\varepsilon\|_\nu^2\}\\
&+\frac{C}{\varepsilon}\{\|\nabla_x\partial_\alpha(\mathbf{I}-\mathbf{P})f^\varepsilon\|_\nu + \|\partial_\alpha(\mathbf{I}-\mathbf{P})f^\varepsilon\|_\nu\}\|\nabla\partial_\alpha b^\varepsilon\| + C\|\partial_\alpha h_\parallel^\varepsilon\|\cdot\|\nabla\partial_\alpha b^\varepsilon\|.
\end{aligned}$$

Note that the difference from the estimate in [7] so far is that the scaling parameter $\varepsilon$ is absent in the $t$-derivative term due to the acoustic scaling. Now multiply it by $\varepsilon$ and apply the Cauchy-Schwarz inequality to get

$$\begin{aligned}
\varepsilon\|\nabla\partial_\alpha b^\varepsilon\|^2 \le &-\varepsilon\frac{d}{dt}\int_{\mathbb{T}^3}\langle(\mathbf{I}-\mathbf{P})\partial_\alpha f^\varepsilon, \zeta_{ij}\rangle\cdot\partial_j\partial_\alpha b^\varepsilon dx + \frac{\varepsilon^2}{2}\{\|\nabla\partial_\alpha a^\varepsilon\|^2 + \|\nabla\partial_\alpha c^\varepsilon\|^2\}\\
&+\frac{C}{\varepsilon}\{\|\nabla_x\partial_\alpha(\mathbf{I}-\mathbf{P})f^\varepsilon\|_\nu^2 + \|\partial_\alpha(\mathbf{I}-\mathbf{P})f^\varepsilon\|_\nu^2\} + C\varepsilon\|\partial_\alpha h_\parallel^\varepsilon\|^2 + \frac{\varepsilon}{2}\|\nabla\partial_\alpha b^\varepsilon\|^2.
\end{aligned}$$

By the same token, we obtain the similar estimates on $\nabla\partial_\alpha c^\varepsilon$ and $\nabla\partial_\alpha a^\varepsilon$ as follows:

$$\begin{aligned}
\varepsilon\|\nabla\partial_\alpha c^\varepsilon\|^2 \le &-\varepsilon\frac{d}{dt}\int_{\mathbb{T}^3}\langle(\mathbf{I}-\mathbf{P})\partial_\alpha f^\varepsilon, \zeta_c\rangle\cdot\nabla_x\partial_\alpha c^\varepsilon dx + \frac{\varepsilon^2}{2}\|\nabla\partial_\alpha b^\varepsilon\|^2\\
&+\frac{C}{\varepsilon}\{\|\nabla_x\partial_\alpha(\mathbf{I}-\mathbf{P})f^\varepsilon\|_\nu^2 + \|\partial_\alpha(\mathbf{I}-\mathbf{P})f^\varepsilon\|_\nu^2\} + C\varepsilon\|\partial_\alpha h_\parallel^\varepsilon\|^2 + \frac{\varepsilon}{2}\|\nabla\partial_\alpha c^\varepsilon\|^2,\\
\varepsilon\|\nabla\partial_\alpha a^\varepsilon\|^2 \le &-\varepsilon\frac{d}{dt}\{\int_{\mathbb{T}^3}\langle(\mathbf{I}-\mathbf{P})\partial_\alpha f^\varepsilon, \zeta\rangle\cdot\nabla_x\partial_\alpha a^\varepsilon dx + \int_{\mathbb{T}^3}\partial_\alpha b^\varepsilon\cdot\nabla_x\partial_\alpha a^\varepsilon dx\} + \frac{\varepsilon^2}{2}\|\nabla\partial_\alpha b^\varepsilon\|^2\\
&+\frac{C}{\varepsilon}\{\|\nabla_x\partial_\alpha(\mathbf{I}-\mathbf{P})f^\varepsilon\|_\nu^2 + \|\partial_\alpha(\mathbf{I}-\mathbf{P})f^\varepsilon\|_\nu^2\} + C\varepsilon\|\partial_\alpha h_\parallel^\varepsilon\|^2 + \frac{\varepsilon}{2}\|\nabla\partial_\alpha a^\varepsilon\|^2.
\end{aligned}$$

By absorbing the hydrodynamic terms in the right hand sides into the left hand sides, we obtain the desired estimates (5.2). □

Next we perform the energy estimates of spatial derivatives.

**Lemma 5.2.** *Assume that $f_\varepsilon$ is a solution to equation* (2.7) *and satisfies* (5.1)*; then there exists a constant $C_1 \geq 1$ such that the following energy estimate is valid:*

$$
\begin{aligned}
&\frac{d}{dt}\{C_1 \sum_{|\alpha|\leq N+1} \|\partial_\alpha f_\varepsilon\|^2 - \varepsilon\delta G(t)\} + \delta \sum_{|\alpha|\leq N+1} \{\varepsilon \|\partial_\alpha \mathbf{P} f_\varepsilon\|^2 + \frac{1}{\varepsilon}\|\partial_\alpha(\mathbf{I}-\mathbf{P})f_\varepsilon\|_\nu^2\} \\
&\qquad \leq 2C_1 \sum_{|\alpha|\leq N+1} (\partial_\alpha \Gamma(f_\varepsilon, f_\varepsilon), \partial_\alpha f_\varepsilon) + \varepsilon\delta \sum_{|\alpha|\leq N} \|\partial_\alpha \Gamma(f_\varepsilon, f_\varepsilon)_\|\|^2 \\
&\qquad \leq C\{\mathcal{E}_{N,0}^{1/2}(f_\varepsilon) + \mathcal{E}_{N,0}(f_\varepsilon)\}\mathcal{D}_{N,0}(f_\varepsilon)
\end{aligned} \tag{5.11}
$$

*Proof.* We take $\partial_\alpha$ of (2.7) and sum over $\alpha$ to get

$$
\frac{1}{2}\frac{d}{dt}\|\partial_\alpha f_\varepsilon\|^2 + \frac{\delta}{\varepsilon}\|(\mathbf{I}-\mathbf{P})\partial_\alpha f_\varepsilon\|_\nu^2 \leq (\partial_\alpha \Gamma(f_\varepsilon, f_\varepsilon), \partial_\alpha f_\varepsilon)\,.
$$

By Lemma 5.1, there is a constant $C_1 \geq 1$ such that

$$
\begin{aligned}
&\frac{\delta}{2\varepsilon} \sum_{|\alpha|\leq N+1} \|\partial_\alpha(\mathbf{I}-\mathbf{P})f_\varepsilon\|_\nu^2 \\
&\qquad \geq \frac{\delta\varepsilon}{2C_1} \sum_{|\alpha|\leq N+1} \|\partial_\alpha \mathbf{P} f_\varepsilon\|^2 - \frac{\delta\varepsilon}{2C_1}\frac{dG}{dt} - \frac{\delta\varepsilon}{2}\sum_{|\alpha|\leq N} \|\partial_\alpha \Gamma(f_\varepsilon, f_\varepsilon)_\|\|^2\,.
\end{aligned} \tag{5.12}
$$

Multiply by $C_1$ and collecting terms, we deduce the first inequality in (5.11). By the nonlinear estimate in (3.8), it is easy to derive that for $|\alpha| \leq N$

$$
\|\partial_\alpha \Gamma(f_\varepsilon\,, f_\varepsilon)_\|\|^2 \leq C\mathcal{E}_{N,0}(f_\varepsilon)\mathcal{D}_{N,0}(f_\varepsilon)\,, \tag{5.13}
$$

and

$$
\begin{aligned}
(\partial_\alpha \Gamma(f_\varepsilon\,, f_\varepsilon)\,, \partial_\alpha f_\varepsilon) &\leq C\mathcal{E}_{N,0}^{1/2}(f_\varepsilon)\|\varepsilon^{1/2}\partial_\alpha f_\varepsilon\|_\nu \|\varepsilon^{-1/2}\partial_\alpha(\mathbf{I}-\mathbf{P})f_\varepsilon\|_\nu \\
&\leq C\mathcal{E}_{N,0}^{1/2}(f_\varepsilon)\mathcal{D}_{N,0}(f_\varepsilon)\,.
\end{aligned} \tag{5.14}
$$

Thus, the second inequality in (3.8) follows and this finishes the proof of the lemma. □

## 6. The first order remainder

In this section we finish the proof of Theorem 2.1. We already established a pure spatial energy estimate for all collision kernels in Lemma 5.2. For general derivatives $\partial_\alpha^\beta$, different collision kernels require different weight functions, we treat separately in two cases: hard potentials then soft potentials and Landau kernel.

### 6.1. **Proof of hard potential case of Theorem 2.1.**

*Proof.* First note that for the hydrodynamic part $\mathbf{P} f_\varepsilon$,

$$
\|\partial_\alpha^\beta \mathbf{P} f_\varepsilon\| \leq C\|\partial_\alpha \mathbf{P} f_\varepsilon\|
$$

which has been estimated in Lemma 5.2. In order to prove Theorem 2.1, it remains to estimate the remaining microscopic part $\partial_\alpha^\beta(\mathbf{I}-\mathbf{P})f_\varepsilon$ for $|\alpha| + |\beta| \leq N$. We take $\partial_\alpha^\beta$ of equation (2.7) and sum over $|\alpha|+|\beta|\le N$ to get

$$\begin{aligned}
&\partial_t\partial_\alpha^\beta(\mathbf{I}-\mathbf{P})f_\varepsilon+v\cdot\nabla_x\partial_\alpha^\beta(\mathbf{I}-\mathbf{P})f_\varepsilon+\frac{1}{\varepsilon}\partial_\alpha^\beta\mathcal{L}(\mathbf{I}-\mathbf{P})f_\varepsilon\\
&\qquad+\left(\partial_t\partial_\alpha^\beta\mathbf{P}f_\varepsilon+v\cdot\nabla_x\partial_\alpha^\beta\mathbf{P}f_\varepsilon+\binom{\beta_1}{\beta}\partial_{\beta_1}v\cdot\nabla_x\partial_\alpha^{\beta-\beta_1}f_\varepsilon\right)\\
&\qquad=\partial_\alpha^\beta\Gamma(f_\varepsilon\,,f_\varepsilon)\,,
\end{aligned}\tag{6.1}$$

where $|\beta_1|=1$. Taking the inner product with $w^{2l}\partial_\alpha^\beta(\mathbf{I}-\mathbf{P})f_\varepsilon$, we get

$$\begin{aligned}
&\frac{d}{dt}\left\{\frac{1}{2}\|w^l\partial_\alpha^\beta(\mathbf{I}-\mathbf{P})f_\varepsilon\|^2\right\}+\frac{1}{\varepsilon}(w^{2l}\partial_\alpha^\beta\mathcal{L}(\mathbf{I}-\mathbf{P})f_\varepsilon\,,\partial_\alpha^\beta(\mathbf{I}-\mathbf{P})f_\varepsilon)\\
&\quad+\left(\partial_t\partial_\alpha^\beta\mathbf{P}f_\varepsilon+v\cdot\nabla_x\partial_\alpha^\beta\mathbf{P}f_\varepsilon+\binom{\beta_1}{\beta}\partial_\alpha^{\beta_1}v\cdot\nabla_x\partial_\alpha^{\beta-\beta_1}f_\varepsilon\,,w^{2l}\partial_\alpha^\beta(\mathbf{I}-\mathbf{P})f_\varepsilon\right)\\
&\quad\le\left(w^{2l}\partial_\alpha^\beta\Gamma(f_\varepsilon\,,f_\varepsilon)\,,\partial_\alpha^\beta(\mathbf{I}-\mathbf{P})f_\varepsilon\right)\,.
\end{aligned}\tag{6.2}$$

By the linear estimate (3.2), we have

$$\frac{1}{\varepsilon}(w^{2l}\partial_\alpha^\beta\mathcal{L}(\mathbf{I}-\mathbf{P})f_\varepsilon\,,\partial_\alpha^\beta(\mathbf{I}-\mathbf{P})f_\varepsilon)\ge\frac{1}{2\varepsilon}\|w^l\partial_\alpha^\beta(\mathbf{I}-\mathbf{P})f_\varepsilon\|_\nu^2-\frac{C}{\varepsilon}\|\partial_\alpha(\mathbf{I}-\mathbf{P})f_\varepsilon\|_\nu^2\,.\tag{6.3}$$

From the local conservation laws (5.4) and the estimate (5.13),

$$\begin{aligned}
\|w^{2l}\partial_t\partial_\alpha^\beta\mathbf{P}f_\varepsilon\|&\le C\sum_{|\alpha|\le N}(\|\partial_t\partial_\alpha a^\varepsilon\|+\|\partial_t\partial_\alpha b^\varepsilon\|+\|\partial_t\partial_\alpha c^\varepsilon\|)\\
&\le C\left(\sum_{|\alpha|\le N+1}\|\partial_\alpha f_\varepsilon\|_\nu+\sum_{|\alpha|\le N}\|\partial_\alpha h_\parallel^\varepsilon\|\right)\\
&\le C\left(\sum_{|\alpha|\le N+1}\|\partial_\alpha f_\varepsilon\|_\nu+\mathcal{E}_{N,0}^{1/2}(f_\varepsilon)\mathcal{D}_{N,0}^{1/2}(f_\varepsilon)\right)\,.
\end{aligned}\tag{6.4}$$

We also have

$$\|w^{2l}v\cdot\nabla_x\partial_\alpha^\beta\mathbf{P}f_\varepsilon\|\le C\sum_{|\alpha|\le N}\|\nabla_x\partial_\alpha\mathbf{P}f_\varepsilon\|\le C\sum_{|\alpha|\le N+1}\|\partial_\alpha\mathbf{P}f_\varepsilon\|\,.\tag{6.5}$$

Thus the first two inner products in the second line of (6.2) is bounded by

$$\frac{1}{8\varepsilon}\sum_{|\alpha|+|\beta|\le N}\|\partial_\alpha^\beta(\mathbf{I}-\mathbf{P})f_\varepsilon\|_\nu^2+C\left(\sum_{|\alpha|\le N+1}\|\partial_\alpha f_\varepsilon\|_\nu^2+\mathcal{E}_{N,0}^{1/2}(f_\varepsilon)\mathcal{D}_{N,0}(f_\varepsilon)\right)\,.\tag{6.6}$$

The last term in the second line of (6.2) is bounded by

$$\begin{aligned}
&C|(\partial^{\beta_1}v\cdot\nabla_x\partial_\alpha^{\beta-\beta_1}(\mathbf{I}-\mathbf{P})f_\varepsilon\,,w^{2l}\partial_\alpha^\beta(\mathbf{I}-\mathbf{P})f_\varepsilon)|\\
&\quad+C|(\partial^{\beta_1}v\cdot\nabla_x\partial_\alpha^{\beta-\beta_1}\mathbf{P}f_\varepsilon\,,w^{2l}\partial_\alpha^\beta(\mathbf{I}-\mathbf{P})f_\varepsilon)|\\
&\quad\le C\|\nabla_x\partial_\alpha^{\beta-\beta_1}(\mathbf{I}-\mathbf{P})f_\varepsilon\|_\nu^2+\frac{1}{8\varepsilon}\|w^l\partial_\alpha^\beta(\mathbf{I}-\mathbf{P})f_\varepsilon\|_\nu^2+C\varepsilon\|\partial_\alpha\mathbf{P}f_\varepsilon\|^2\\
&\quad\le C\varepsilon\mathcal{D}_{N,l}(f_\varepsilon)+\frac{1}{8\varepsilon}\|w^l\partial_\alpha^\beta(\mathbf{I}-\mathbf{P})f_\varepsilon\|_\nu^2+C\varepsilon\|\partial_\alpha\mathbf{P}f_\varepsilon\|^2\,,
\end{aligned}\tag{6.7}$$

since $\nu(v)$ is bounded from below for hard potential.

Now we estimate the nonlinear term in (6.2). By the nonlinear estimate in (3.3),

$$\begin{aligned}(w^{2l}\partial_\alpha^\beta\Gamma(f_\varepsilon, f_\varepsilon)\,,\partial_\alpha^\beta(\mathbf{I}-\mathbf{P})f_\varepsilon) &\le C\mathcal{E}_{N,l}^{1/2}(f_\varepsilon)\|\varepsilon^{1/2}w^l\partial_\alpha f_\varepsilon\|_\nu\|\varepsilon^{-1/2}w^l\partial_\alpha(\mathbf{I}-\mathbf{P})f_\varepsilon\|_\nu \\ &\le C\mathcal{E}_{N,l}^{1/2}(f_\varepsilon)\mathcal{D}_{N,l}(f_\varepsilon)\,.\end{aligned} \tag{6.8}$$

Using the coercivity of $\mathcal{L}$ (3.1) and absorbing a total of $\frac{1}{\varepsilon}\|w^l\partial_\alpha^\beta(\mathbf{I}-\mathbf{P})f_\varepsilon\|_\nu^2$ from the right-hand side, we have

$$\begin{aligned}\sum_{|\alpha|+|\beta|\le N}&\left(\frac{d}{dt}\left\{\frac{1}{2}\|w^l\partial_\alpha^\beta(\mathbf{I}-\mathbf{P})f_\varepsilon\|^2\right\}+\frac{1}{4\varepsilon}\|w^l\partial_\alpha^\beta(\mathbf{I}-\mathbf{P})f_\varepsilon\|_\nu^2\right)\\ &\le C\sum_{aaN}\|\partial_\alpha f_\varepsilon\|_\nu^2+C\left(\mathcal{E}_{N,l}^{1/2}(f_\varepsilon)+\varepsilon\right)\mathcal{D}_{N,l}(f_\varepsilon)\,.\end{aligned} \tag{6.9}$$

Multiplying (6.9) by a factor and adding a large multiple $K$ of (5.11), we have

$$\begin{aligned}&\frac{d}{dt}(K\{C_1\sum_{|\alpha|\le N+1}\|\partial_\alpha f_\varepsilon\|^2-\varepsilon\delta G(t)\}+2\sum_{|\alpha|+|\beta|\le N}\|w^l\partial_\alpha^\beta(\mathbf{I}-\mathbf{P})f_\varepsilon\|^2)+\mathcal{D}_{N,l}(f_\varepsilon)\\ &\le C_K\left(\mathcal{E}_{N,l}^{1/2}(f_\varepsilon)+\mathcal{E}_{N,l}(f_\varepsilon)+\varepsilon\right)\mathcal{D}_{N,l}(f_\varepsilon)\,.\end{aligned} \tag{6.10}$$

Notice that

$$\|w^l\partial_\alpha^\beta\mathbf{P}f_\varepsilon\|^2\le C\|\partial_\alpha\mathbf{P}f_\varepsilon\|^2\le C\|\partial_\alpha f_\varepsilon\|^2\,, \tag{6.11}$$

and

$$G(t)\le C\sum_{aaN}\|\partial_\alpha\mathbf{P}f_\varepsilon\|(\|\mathbf{I}-\mathbf{P}\partial_\alpha f_\varepsilon\|+\|\partial_\alpha\mathbf{P}f_\varepsilon\|)\,. \tag{6.12}$$

Thus we can redefine the instant energy by

$$\mathcal{E}_{N,l}(f_\varepsilon)=K\{C_1\sum_{|\alpha|\le N+1}\|\partial_\alpha f_\varepsilon\|^2-\varepsilon\delta G(t)\}+2\sum_{|\alpha|+|\beta|\le N}\|w^l\partial_\alpha^\beta(\mathbf{I}-\mathbf{P})f_\varepsilon\|^2 \tag{6.13}$$

for $\varepsilon$ sufficiently small. By a standard continuity argument, we deduce our main estimate (2.20) by letting $\mathcal{E}_{N,l}(f_\varepsilon)$ be sufficiently small initially. $\square$

6.2. **Proof of soft potential and Landau cases for Theorem 2.1.** We follow the same idea as in the hard potential case to establish (2.20) for both soft potential and Landau kernels. First, for soft potential cases, we take inner product of $w^{2(l-|\beta|)|\gamma|}\partial_\alpha^\beta(\mathbf{I}-\mathbf{P})f_\varepsilon$ with the equation (6.1) and sum over $|\alpha|+|\beta|\le N$ to get

$$\begin{aligned}&\frac{d}{dt}\left\{\frac{1}{2}\|w^{\{l-|\beta|\}|\gamma|}\partial_\alpha^\beta(\mathbf{I}-\mathbf{P})f_\varepsilon\|^2\right\}+\frac{1}{\varepsilon}(w^{2\{l-|\beta|\}|\gamma|}\partial_\alpha^\beta\mathcal{L}(\mathbf{I}-\mathbf{P})f_\varepsilon\,,\partial_\alpha^\beta(\mathbf{I}-\mathbf{P})f_\varepsilon)\\ &+\left(\partial_t\partial_\alpha^\beta\mathbf{P}f_\varepsilon+v\cdot\nabla_x\partial_\alpha^\beta\mathbf{P}f_\varepsilon+\binom{\beta_1}{\beta}\partial_\alpha^{\beta_1}v\cdot\nabla_x\partial_\alpha^{\beta-\beta_1}f_\varepsilon\,,w^{2\{l-|\beta|\}|\gamma|}\partial_\alpha^\beta(\mathbf{I}-\mathbf{P})f_\varepsilon\right)\\ &\le\left(w^{2\{l-|\beta|\}|\gamma|}\partial_\alpha^\beta\Gamma(f_\varepsilon\,,f_\varepsilon)\,,\partial_\alpha^\beta(\mathbf{I}-\mathbf{P})f_\varepsilon\right)\,,\end{aligned} \tag{6.14}$$

for $|\beta_1|=1$. By the linear estimate (3.4), we have

$$\begin{aligned}&\frac{1}{\varepsilon}(w^{2\{l-|\beta|\}|\gamma|}\partial_\alpha^\beta\mathcal{L}(\mathbf{I}-\mathbf{P})f_\varepsilon\,,\partial_\alpha^\beta(\mathbf{I}-\mathbf{P})f_\varepsilon)\\ &\ge\frac{1}{2\varepsilon}\|w^{\{l-|\beta|\}|\gamma|}\partial_\alpha^\beta(\mathbf{I}-\mathbf{P})f_\varepsilon\|_\nu^2-\frac{C}{\varepsilon}\|\partial_\alpha(\mathbf{I}-\mathbf{P})f_\varepsilon\|_\nu^2\,.\end{aligned} \tag{6.15}$$

From the local conservation laws (5.4), we have

$$\|w^{2\{l-|\beta|\}|\gamma|}\partial_t\partial_\alpha^\beta \mathbf{P} f_\varepsilon\| \le C\left(\sum_{|\alpha|\le N+1}\|\partial_\alpha f_\varepsilon\|_\nu + \mathcal{E}_{N,0}^{1/2}(f_\varepsilon)\mathcal{D}_{N,0}^{1/2}(f_\varepsilon)\right). \tag{6.16}$$

We also have

$$\|w^{2\{l-|\beta|\}|\gamma|}v\cdot\nabla_x\partial_\alpha^\beta \mathbf{P} f_\varepsilon\| \le C\sum_{|\alpha|\le N}\|\nabla_x\partial_\alpha \mathbf{P} f_\varepsilon\| \le C\sum_{|\alpha|\le N+1}\|\partial_\alpha \mathbf{P} f_\varepsilon\|. \tag{6.17}$$

Note that $\|\cdot\|_\nu$ is equivalent to $\|w^{\gamma/2}\cdot\|$, the first two inner products in the second line of (6.14) is bounded by

$$\frac{1}{8\varepsilon}\sum_{|\alpha|+|\beta|\le N}\|w^{\{l-|\beta|\}|\gamma|}\partial_\alpha^\beta(\mathbf{I}-\mathbf{P})f_\varepsilon\|_\nu^2 + C\left(\sum_{|\alpha|\le N+1}\|\partial_\alpha f_\varepsilon\|_\nu^2 + \mathcal{E}_{N,0}^{1/2}(f_\varepsilon)\mathcal{D}_{N,0}(f_\varepsilon)\right). \tag{6.18}$$

The weight function $w^{|\beta|\gamma}$ is so designed to treat the last term in the second line of (6.14)

$$\begin{aligned}
&C|(\partial^{\beta_1}v\cdot\nabla_x\partial_\alpha^{\beta-\beta_1}(\mathbf{I}-\mathbf{P})f_\varepsilon\,, w^{2\{l-|\beta|\}|\gamma|}\partial_\alpha^\beta(\mathbf{I}-\mathbf{P})f_\varepsilon)|\\
&\quad + C|(\partial^{\beta_1}v\cdot\nabla_x\partial_\alpha^{\beta-\beta_1}\mathbf{P}f_\varepsilon\,, w^{2\{l-|\beta|\}|\gamma|}\partial_\alpha^\beta(\mathbf{I}-\mathbf{P})f_\varepsilon)|\\
&\quad \le C\|w^{l+|\beta-\beta_1|\gamma}\nabla_x\partial_\alpha^{\beta-\beta_1}(\mathbf{I}-\mathbf{P})f_\varepsilon\|_\nu^2 + \frac{1}{8\varepsilon}\|w^{\{l-|\beta|\}|\gamma|}\partial_\alpha^\beta(\mathbf{I}-\mathbf{P})f_\varepsilon\|_\nu^2 + C\varepsilon\|\partial_\alpha\mathbf{P}f_\varepsilon\|^2\\
&\quad \le C\varepsilon\mathcal{D}_{N,l}(f_\varepsilon) + \frac{1}{8\varepsilon}\|w^{\{l-|\beta|\}|\gamma|}\partial_\alpha^\beta(\mathbf{I}-\mathbf{P})f_\varepsilon\|_\nu^2 + C\varepsilon\|\partial_\alpha\mathbf{P}f_\varepsilon\|^2.
\end{aligned} \tag{6.19}$$

The nonlinear term in (6.14) is estimated by (3.5),

$$\begin{aligned}
&(w^{2\{l-|\beta|\}|\gamma|}\partial_\alpha^\beta\Gamma(f_\varepsilon, f_\varepsilon)\,, \partial_\alpha^\beta(\mathbf{I}-\mathbf{P})f_\varepsilon)\\
&\le C\mathcal{E}_{N,l}^{1/2}(f_\varepsilon)\|\varepsilon^{1/2}w^{\{l-|\beta|\}|\gamma|}\partial_\alpha f_\varepsilon\|_\nu\|\varepsilon^{-1/2}w^{\{l-|\beta|\}|\gamma|}\partial_\alpha(\mathbf{I}-\mathbf{P})f_\varepsilon\|_\nu\\
&\le C\mathcal{E}_{N,l}^{1/2}(f_\varepsilon)\mathcal{D}_{N,l}(f_\varepsilon).
\end{aligned} \tag{6.20}$$

The rest of the proof is similar to the hard potential case, the nonlinear estimate (2.20) can be deduced by letting

$$\begin{aligned}
\mathcal{E}_{N,l}(f_\varepsilon) = K\{C_1&\sum_{|\alpha|\le N+1}\|\partial_\alpha f_\varepsilon\|^2 - \varepsilon\delta G(t)\}\\
&+ 2\sum_{|\alpha|+|\beta|\le N}\|w^{\{l-|\beta|\}|\gamma|}\partial_\alpha^\beta(\mathbf{I}-\mathbf{P})f_\varepsilon\|^2.
\end{aligned} \tag{6.21}$$

To establish the estimate (2.20) for the Landau case for which the power of weight is $\gamma = -1$. We follow the same procedure as in the soft potential case. Take the inner product with $w^{2l-2|\beta|}\partial_\alpha^\beta(\mathbf{I}-\mathbf{P})f_\varepsilon$ for equation (6.1) to get (6.14) with $\gamma = -1$. All the estimates for the soft potential case can applied for the case $\gamma = -1$. So we omit the details here.

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