EXISTENCE AND UNIQUENESS FOR MULTIDIMENSIONAL SCALAR CONSERVATION LAW WITH DISCONTINUOUS FLUX

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Abstract. We prove existence and uniqueness of an entropy solution to a multidimensional scalar conservation law with discontinuous flux. The proof is based on the corresponding kinetic formulation of the considered equation and a "smart" change of an unknown function.

1. Introduction

In the current contribution, we consider the following Cauchy problem:

$$\partial_t u + \mathrm{div}_x f(x,u) = 0, \quad u = u(t,x), \quad t \geq 0, \ x \in I\!R^d. \tag{1}$$

$$u|_{t=0} = u_0(x) \in L^1(I\!R^d), \quad a \leq u_0 \leq b. \tag{2}$$

Here, the flux vector $f(x,\lambda) = (f_1(x,\lambda), \dots, f_d(x,\lambda))$, $\lambda \in I\!R$, is assumed to be continuously differentiable with respect to $u \in I\!R$ and discontinuous with respect to $x \in I\!R^d$ so that for every $\lambda \in I\!R$ the discontinuity is placed on the manifold $\Gamma \subset I\!R^d$ of codimension one which divides the space $I\!R^d$ on two domains.

More precisely, we assume that there exist two domains Ω_L and Ω_R such that

$$I\!R^d = \Omega_L \cup \Gamma \cup \Omega_R, \quad \overline{\Omega_L} \cap \overline{\Omega_R} = \Gamma, \tag{3}$$

and that, denoting

$$\kappa_L(x) = \begin{cases} 1, & x \in \Omega_L \\ 0, & x \notin \Omega_L \end{cases}, \quad \kappa_R(x) = \begin{cases} 1, & x \in \Omega_R \\ 0, & x \notin \Omega_R \end{cases},$$

we can rewrite (1) in the form

$$\partial_t u + \mathrm{div}_x \left(g_L(x,u)\kappa_L(x) + g_R(x,u)\kappa_R(x) \right) = 0. \tag{4}$$

Furthermore, we assume that the functions $g_L, g_R \in C^1(I\!R^{d+1}; I\!R^d)$ have the form:

$$g_L(x,u) = (g_{1L}(\hat{x_1},u), \dots, g_{dL}(\hat{x_d},u)),$$
$$g_R(x,u) = (g_{1R}(\hat{x_1},u), \dots, g_{dR}(\hat{x_d},u)).$$

Remark 1. Here and in the sequel, by $A(\hat{x_i})$ we imply that the quantity A does not depend on x_i but only on $x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_d$.

Scalar conservation law with discontinuous flux attracted great deal of attention in recent years. It models different physical phenomena, for instance flow in porous media, sedimentation processes, traffic flow, radar shape-from-shading problems, blood flow. Still, almost all results were restricted to one dimensional case. The

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following incomplete list ranges over different admissibility concept and methods for proving existence and/or uniqueness of a weak solution to the one dimensional scalar conservation law [5, 14, 7, 8, 16, 1, 15, 4, 6, 3, 9, 12, 25, 22]. Beside confinement on the dimension, in all listed papers some structural demands were imposed on the flux (such as genuine nonlinearity, convexity, crossing condition) or on the form of a solution (such as piecewise smoothness). In [21], we succeeded to replace all the latter conditions by the assumptions that the initial data belong to the BV-class, and to obtain both, existence and uniqueness.

On the other hand, there are incomparably less results concerning questions of existence and uniqueness for a multidimensional scalar conservation law with a discontinuous flux. In the two-dimensional case, existence of a weak solution to the corresponding Cauchy problem is obtained in [18] by using the compensated compactness [28] under the genuine nonlinearity assumptions on the flux. Under the same assumptions, in [25] the existence is proved in the d-dimensional case, $d \in \mathbb{N}$ arbitrary. The basic tool was a modification of the H-measures [29, 13, 24]. Probably only result on the uniqueness of certain class of solutions to Cauchy problem (1), (2) can be found in [26]. There, the flux vector $f = (f_1, \dots, f_d)$ has rather special form. Namely, it is assumed that $f = f(\beta(x,u)) = (f_1(\beta(x,u), \dots, f_d(\beta(x,u))$, where the function $\beta \in C^1(\mathbb{R}_u; L^1(\mathbb{R}^d))$ is increasing with respect to $u \in \mathbb{R}$ and discontinuous with respect to $x \in \mathbb{R}^d$. Since the function β is increasing with respect to u, there exists a function $\alpha(x,v)$ such that

$$\beta(x,u) = v \Rightarrow v = \alpha(x,u).$$

Thus, equation (1) can be rewritten as

$$\begin{aligned} \partial_t \alpha(x,v) + \operatorname{div} f(v) &= 0, \\ v|_{t=0} &= \alpha(x, u_0(x)). \end{aligned}$$

Since the discontinuity in $x \in \mathbb{R}^d$ is removed out of the derivative in x, we can apply standard Kruzhkov theory to prove the uniqueness.

In this paper, we shall prove existence and uniqueness assuming that:

a) the flux $f = f(x,u)$, $x \in \mathbb{R}^d$, $u \in \mathbb{R}$, from (1) has compact support with respect to $u \in \mathbb{R}$ which is usual and rather natural assumption which provides the maximum principle for the considered problem.

b) that the discontinuity manifold is such that it holds for every $i = 1, \dots, d$:

$$\begin{aligned} &\Gamma = \{x \in \mathbb{R}^d : \ x_i = \alpha_i(\hat{x_i})\}, \quad \text{and} \\ &x \in \Omega_L \text{ if } x_i \leq \alpha_i(\hat{x_i}) \ \text{ and } \ x \in \Omega_R \text{ if } x_i > \alpha_i(\hat{x_i}). \end{aligned} \tag{5}$$

The proof is based on a combination of approaches used in [5] and [21]. We remark that a class of admissible solutions is wider in [21] than here (compare (16) here and [21, Figure 2]).

In the first part of the paper, we shall consider the case which we call "the special case". More precisely, we shall assume that for every $i = 1, \dots, d$, there exist the functions $\alpha_i = \alpha(\hat{x_i}) \in C^1(\mathbb{R}^{d-1})$, $i = 1, \dots, d$, such that:

According to the latter assumptions, we can rewrite equation (4) in the form:

$$\partial_t u + \sum_{i=1}^d \partial_{x_i} \left(g_{iL}(\hat{x_i}, u) H(\alpha_i(\hat{x_i}) - x_i) + g_{iR}(\hat{x_i}, u) H(x_i - \alpha_i(\hat{x_i})) \right) = 0, \tag{6}$$

where H is the Heaviside function and the functions $g_{iL}(\hat{x_i},\lambda)$ and $g_{iR}(\hat{x_i},\lambda)$ depend on $\lambda\in I\!R$ and the coordinates $x_1,\dots,x_{i-1},x_{i+1},\dots,x_d$. Furthermore, we assume that they are nonnegative, continuously differentiable with respect to all variables and that for every $i=1,\dots,d$

$$\mathrm{supp} g_{iL}(\hat{x_i},\cdot), \mathrm{supp} g_{iR}(\hat{x_i},\cdot)\subset(a,b)\subset I\!R, \tag{7}$$

independently on $x\in I\!R^d$.

Example 2. *We give two examples of equation* (6).

a) *Assume that we have two-dimensional scalar conservation law of form* (6) *where the corresponding discontinuity manifold is hyperplane* $x_2=0$. *More precisely, we assume that* $\Gamma=\{(x_1,x_2)\in I\!R^2:\ x_2=0\}$.

Conservation law (6) *can be rewritten as*

$$\partial_t u+\partial_{x_1} g_1(x_2,u)+\partial_{x_2}\left(g_{2L}(x_1,u)H(-x_2)+g_{1L}(x_1,u)H(x_2)\right)=0,$$

i.e. here $g_{1L}(x_2,u)=g_{1R}(x_2,u)=g_1(x_2,u)$ *since there is no discontinuity with respect to* x_1 *and we do not need the function* α_1, *while* $\alpha_2(\hat{x_2})=\alpha_2(x_1)=0$.

b) *We consider again two dimensional scalar conservation law of form* (6), *this time assuming that* $\Gamma=\{(x_1,x_2)\in I\!R^2:\ x_2+x_1=1\}$.

Equation (6) *can be rewritten as*

$$\begin{aligned}\partial_t u&+\partial_{x_1}\left(g_{1L}(x_2,u)H(x_2+x_1)+g_{1R}(x_2,u)H(x_1+x_2)\right)\\&+\partial_{x_2}\left(g_{2L}(x_1,u)H(x_1+x_2)+g_{1L}(x_1,u)H(x_2+x_1)\right)=0,\end{aligned}$$

i.e. $\alpha_1(\hat{x_1})=\alpha_1(x_2)=-x_2$ *and* $\alpha_2(\hat{x_2})=\alpha_2(x_1)=-x_1$.

In the last part of the paper, we consider equation (4) in "the general case". It represents a slight generalization of "the special case" (see Example 2).

More precisely, we assume that there exists, without loosing on generality, finite partition of the set $I\!R^d$ so that

$$Cl(\dot{\bigcup}_{j=1}^{n}\Omega_j)=I\!R^d,$$

where $\Omega_j\subset I\!R^d$ are domains in $I\!R^d$, Cl is closure of a set, and $\dot{\bigcup}$ denotes disjoint union. Furthermore, we assume that for every $j=1,\dots,n$, there exist functions $\alpha_i^j=\alpha_i^j(\hat{x_i})\in C^1(I\!R^{d-1})$, $i=1,\dots,d$, so that:

$$\Gamma\cap\Omega_j=\{x\in\Omega_j:\ x_i=\alpha_i^j(\hat{x_i})\}.$$

Also, we assume that

$$codim(\overline{\Omega_p}\cap\overline{\Omega_q}\cap\Gamma)\geq 2,\quad p,q=1,\dots,n.$$

Denote

$$\kappa_j(x)=\begin{cases}1, & x\in\Omega_j\\ 0, & x\notin\Omega_j\end{cases},$$

i.e. κ_j are the characteristic function of the set Ω_j, $j=1,\dots,n$.

According to the latter assumptions, we can rewrite equation (4) in the form:

$$\partial_t u+\sum_{i=1}^d\partial_{x_i}\Big(\sum_{j=1}^n\kappa_j(x)\left(g_{iL}^j(\hat{x_i},u)H(\alpha_i^j(\hat{x_i})-x_i)+g_{iR}^j(\hat{x_i},u)H(x_i-\alpha_i^j(\hat{x_i}))\right)\Big)=0, \tag{8}$$

where g_{iL}^j and g_{iR}^j are nonnegative functions such that for every $j=1,\dots,n$

$$\begin{aligned}&g_{iL}^j(\hat{x}_i,\cdot)=g_{iL}(\hat{x}_i,\cdot) \text{ and } g_{iR}^j(\hat{x}_i,\cdot)=g_{iR}(\hat{x}_i,\cdot), \quad i=1,\dots,d, \ \textit{or}\\ &g_{iL}^j(\hat{x}_i,\cdot)=g_{iR}(\hat{x}_i,\cdot) \text{ and } g_{iR}^j(\hat{x}_i,\cdot)=g_{iL}(\hat{x}_i,\cdot), \quad i=1,\dots,d.\end{aligned} \tag{9}$$

The discontinuity manifold in the following example does not satisfy conditions (9). We hope to remove the conditions in future work. Still, in one dimensional case, conditions (9) is always fulfilled making the current work step forward with respect to the previous contributions (since we do not have any confinements on the flux or the initial data).

Example 3. *We shall give an example of two dimensional variant of equation* (8) *when the discontinuity manifold* Γ *is a unit circle.*

More precisely, we assume that we deal with the equation:

$$\begin{aligned}&\partial_t u+\partial_{x_1}\left(g_{1L}(x_2,u)\kappa_{D(0,1)}+g_{1R}(x_2,u)\kappa_{D^C(0,1)}\right)\\ &\quad+\partial_{x_2}\left(g_{2L}(x_1,u)\kappa_{D(0,1)}+g_{2R}(x_1,u)\kappa_{D^C(0,1)}\right)=0,\end{aligned}$$

where $D(0,1)\subset \mathbb{R}^2$ *is the unit disc centered in* $0\in\mathbb{R}^2$ *and* $D^C(0,1)$ *is its complement.*

With the previous notations, we partition the space $\mathbb{R}^2$ *on four domains* Ω_i, $i=1,2,3,4$*:*

$$\begin{aligned}\Omega_1=&\{(x_1,x_2)\in\mathbb{R}^2:\ x_1>0,\ x_2>0\},\\ \Omega_2=&\{(x_1,x_2)\in\mathbb{R}^2:\ x_1<0,\ x_2>0\},\\ \Omega_3=&\{(x_1,x_2)\in\mathbb{R}^2:\ x_1<0,\ x_2<0\},\\ \Omega_4=&\{(x_1,x_2)\in\mathbb{R}^2:\ x_1>0,\ x_2<0\}.\end{aligned}$$

The functions g_{iL}^j *and* g_{iR}^j *as well as* α_i^j, $i=1,2$, $j=1,2,3,4$, *are given by:*

$$j=1:\ \begin{cases} g_{1L}^1(x_2,u)=g_{1L}(x_2,u),\ g_{1R}^1(x_2,u)=g_{1R}(x_2,u),\\ g_{2L}^1(x_1,u)=g_{2L}(x_1,u),\ g_{2R}^1(x_1,u)=g_{2R}(x_1,u),\\ \alpha_1^1(x_2)=\sqrt{1-x_1^2},\quad \alpha_2^1(x_1)=\sqrt{1-x_2^2},\end{cases}$$

$$j=2:\ \begin{cases} g_{1L}^2(x_2,u)=g_{1R}(x_2,u),\ g_{1R}^2(x_2,u)=g_{1L}(x_2,u),\\ g_{2L}^2(x_1,u)=g_{2L}(x_1,u),\ g_{2R}^2(x_1,u)=g_{2R}(x_1,u),\\ \alpha_1^2(x_2)=\sqrt{1-x_1^2},\quad \alpha_2^2(x_1)=-\sqrt{1-x_2^2},\end{cases}$$

$$j=3:\ \begin{cases} g_{1L}^3(x_2,u)=g_{1R}(x_2,u),\ g_{1R}^3(x_2,u)=g_{1L}(x_2,u),\\ g_{2L}^3(x_1,u)=g_{2R}(x_1,u),\ g_{2R}^3(x_1,u)=g_{2L}(x_1,u),\\ \alpha_1^1(x_2)=-\sqrt{1-x_1^2},\quad \alpha_2^1(x_1)=-\sqrt{1-x_2^2},\end{cases}$$

$$j=4:\ \begin{cases} g_{1L}^4(x_2,u)=g_{1L}(x_2,u),\ g_{1R}^4(x_2,u)=g_{1R}(x_2,u),\\ g_{2L}^4(x_1,u)=g_{2R}(x_1,u),\ g_{2R}^4(x_1,u)=g_{2L}(x_1,u),\\ \alpha_1^4(x_2)=-\sqrt{1-x_1^2},\quad \alpha_2^4(x_1)=\sqrt{1-x_2^2}.\end{cases}$$

It is clear that in the domains Ω_2 *and* Ω_4 *we do not have conditions* (9) *fulfilled.*

The paper contains the following sections.

Section 1 is the Introduction where we formulate and explain problems that we will consider.

In Section 2, we introduce several admissibility concepts. They are based on a combination of concepts used in [21] and [5]. We formulate the main theorem about existence and uniqueness of certain classes of our entropy solutions to Cauchy problem (6), (2).

Section 3 is a collection of notions and auxiliary results that we shall use in the rest of the paper.

Section 4 is the proof of the main theorem stating the existence and uniqueness for certain classes of entropy solutions to (6), (2).

Section 5 deals with Cauchy problem (8), (2). We prove existence and uniqueness of appropriate entropy admissible solution to the latter Cauchy problem.

2. Admissibility conditions

First, we shall introduce admissibility conditions similar to the ones that we used in [21].

We need the following step function:

$$k(x) = \begin{cases} k_L, & x_i \leq \alpha_i(\hat{x}_i) \\ k_R, & x_i > \alpha_i(\hat{x}_i) \end{cases}, \quad k_L, k_R \in I\!R, \quad i = 1, \dots, d. \tag{10}$$

Notice that, according to the assumptions on the discontinuity manifold Γ, the function k is well defined.

In the sequel, we denote as usual $I\!R^+ = (0, \infty)$ and:

$$|z|^+ = \begin{cases} z, & z > 0 \\ 0, & z \leq 0 \end{cases}, \quad |z|^- = \begin{cases} 0, & z > 0 \\ -z, & z \leq 0 \end{cases}$$

$$\mathrm{sgn}_\pm(z) = (|z|^\pm)'.$$

Definition 4. We say that the weak solution $u \in L^\infty([0,\infty) \times I\!R^d)$ to Cauchy problem (6), (2), is the k-entropy weak super(sub) solution if the function $v(t,x) = u(t,x) - k(x)$ satisfies for every $\xi \in I\!R$

$$\begin{aligned}\partial_t |v - \xi|^\pm + \sum_{i=1}^d \partial_{x_i} \mathrm{sgn}_\pm(v-\xi)\Big(&\big(g_{iL}(\hat{x}_i, v + k_L) - g_{iL}(\hat{x}_i, \xi + k_L)\big) H(\alpha_i(\hat{x}_i) - x_i) \\ &+ \left(g_{iR}(\hat{x}_i, v + k_R) - g_{iR}(\hat{x}_i, \xi + k_R)\right) H(x_i - \alpha_i(\hat{x}_i))\Big) \\ - \sum_{i=1}^d |g_{iL}(\hat{x}_i, \xi + k_R) &- g_{iR}(\hat{x}_i, \xi + k_L)|^\pm \delta(x_i - \alpha_i(\hat{x}_i)) \leq 0,\end{aligned}$$

where $\delta(x_i - \alpha_i(\hat{x}_i))$ is the Dirac δ distribution supported at $x_i = \alpha_i(\hat{x}_i)$.

If the function u is the k-entropy weak super and sub solution at the same time then we call it k-entropy weak solution.

In order to prove that the latter admissible solution exists, in one dimensional case [21], we used a special vanishing viscosity approximation and a result from [25] stating about strong L^1_{loc} precompactness of the family of solutions to the conservation law with such vanishing viscosity. The crucial [25, Theorem 2] that we used in [21] demands a kind of the genuine non linearity condition (see [25, Definition 2]). In [21], by using the Lipschitz regularity in time of a sequence of approximate solutions [18, Lemma 4.2], we succeeded to replace the genuine nonlinearity condition by the BV assumptions on the initial data.

Here, we are dealing with the multidimensional case and we can not use tricks from [21]. We shall need more subtle arguments similar to those given in [5]. We shall need kinetic formulation for conservation laws [23, 27] as well as notions of the nonlinear weak-$\star$ convergence and entropy process sub and super solution.

Definition 5. Let Ω be an open subset of $I\!R^d$ and $(u_n) \subset L^\infty(\Omega)$ and $u \in L^\infty(\Omega \times (0,1))$. The sequence (u_n) converges toward u in the nonlinear weak-$\star$ sense if

$$\int_\Omega g(u_n(x))\psi(x)dx \to \int_0^1 \int_\Omega g(u(x,\lambda)\psi(x)dxd\lambda \ \ as \ \ n \to \infty,$$
$$\forall \psi \in L^1(\Omega), \ \ \forall g \in C(I\!R).$$

Any bounded sequence of $L^\infty(\Omega)$ has a subsequence converging in the nonlinear weak-$\star$ sense.

Theorem 6. *Let Ω be an open subset of $I\!R^d$ and (u_n) be a bounded sequence of $L^\infty(\Omega)$. Then (u_n) admits a subsequence converging in the nonlinear weak-$\star$ sense.*

This result is established in [11]. It is a modification of the Young measures concept [10] which is more convenient to work with since instead of measures, we are dealing with L^∞ functions.

Now, refereing on [5], we can introduce the notion of the weak entropy process sub and super solutions.

Definition 7. Let $u_0 \in L^\infty(I\!R^d)$, $a \leq u_0 \leq b$ a.e. on $I\!R^d$. Let $u \in L^\infty([0,\infty) \times I\!R^d \times (0,1))$.

1. The function u is a k-weak entropy process subsolution (respectively k-weak entropy process supersolution) of problem (6), (2) if the function $v = v(t,x,\lambda) = u(t,x,\lambda) - k(x)$ satisfies for any $\xi \in I\!R$ and any $\varphi \in C^1_0(I\!R \times I\!R^d)$:

$$\begin{aligned}
&\int_0^1 d\lambda \int_{I\!R^+ \times I\!R^d} (v-\xi)^\pm \partial_t \varphi dtdx \qquad (11)\\
&+ \int_0^1 d\lambda \int_{I\!R^+ \times I\!R^d} \sum_{i=1}^d \mathrm{sgn}_\pm (v-\xi) \times \\
&\quad \times \Big(\left(g_{iL}(\hat{x}_i, v+k_L) - g_{iL}(\hat{x}_i, \xi+k_L)\right) H(\alpha_i(\hat{x}_i) - x_i) \\
&\qquad + \left(g_{iR}(\hat{x}_i, v+k_R) - g_{iR}(\hat{x}_i, \xi+k_R)\right) H(x_i - \alpha_i(\hat{x}_i)) \Big) \partial_{x_i}\varphi \\
&+ \int_{I\!R^d} (u_0 + k(x) - \xi)^\pm \varphi(0,x)dx \\
&- \sum_{i=1}^d \int_{I\!R^+ \times I\!R^{d-1}} \left(g_{iL}(\hat{x}_i, \xi+k_L) - g_{iR}(\hat{x}_i, \xi+k_R)\right)^\pm \varphi|_{x_i = \alpha_i(\hat{x}_i)} d\hat{x}_i \geq 0.
\end{aligned}$$

2. The function u is k-weak entropy process solution if it is weak k-entropy process sub and super solution ad the same time.

It is not difficult to prove existence of a k-weak entropy process solution to (6), (2) for any step function k from (10).

Theorem 8. *There exists k-weak entropy process solution to* (6), (2) *for every step function k from* (10).

Proof: In order to construct wanted solution, we use a procedure similar to the one from [21].

First, we introduce the following change of the unknown function

$$u(t,x)=v(t,x)+k(x), \quad t\geq 0, \quad x\in I\!R^d,$$

for the function k from (10).

Equation (6) becomes:

$$\partial_t v+\sum_{i=1}^d \partial_{x_i}\left(g_{iL}(\hat{x}_i,v+k_L)H(\alpha_i(\hat{x}_i)-x_i)+g_{iR}(\hat{x}_i,v+k_R)H(x_i-\alpha_i(\hat{x}_i))\right)=0. \tag{12}$$

Then, we proceed as in [5]. Consider the sequence (v_ε) of Kruzhkov entropy admissible solutions to the following smoothed flux regularization to (12):

$$\partial_t v_\varepsilon+\sum_{i=1}^d \partial_{x_i}\left(g_{iL}(\hat{x}_i,v_\varepsilon+k_L)H_\varepsilon(\alpha_i(\hat{x}_i)-x_i)+g_{iR}(\hat{x}_i,v_\varepsilon+k_R)H_\varepsilon(x_i-\alpha_i(\hat{x}_i))\right)=0, \tag{13}$$

augmented with initial data (2). Above, $H_\varepsilon(z)=\int_{-\infty}^{z/\varepsilon}\omega(p)dp$ for a smooth even compactly supported function ω with total mass one, represents a regularization of the Heaviside function.

Next, notice that for an A such that $A<a-\max\{a,b\}\leq u_0(x)-k(x)$, it holds $g_{iL}(\hat{x}_i,A+k_L)=g_{iR}(\hat{x}_i,A+k_R)=0$, for all $i=1,\dots,d$ and every $x\in I\!R^d$. Similarly, for a B such that $B>b+\max\{a,b\}\geq u_0(x)-k(x)$, it holds $g_{iL}(\hat{x}_i,B+k_L)=g_{iR}(\hat{x}_i,B+k_R)=0$, for all $i=1,\dots,d$ and every $x\in I\!R^d$. Therefore, the constants $A\leq u_0(x)-k(x)$ and $B\geq u_0(x)-k(x)$ represent Kružkov entropy solutions to equation (13). According to the maximum principle, we conclude that the Kružkov entropy solution v_ε to equation (13) with the initial condition $v_\varepsilon|_{t=0}=u_0(x)-k(x)$ satisfies

$$A\leq v_\varepsilon\leq B, \quad \varepsilon>0.$$

Furthermore, since it is the Kružkov entropy solution, the function v_ε satisfies for any $\varphi\in C^1_0(I\!R\times I\!R^d)$:

$$\begin{aligned}
&\int_{I\!R^+\times I\!R^d}(v_\varepsilon(t,x)-\xi)^\pm\partial_t\varphi(t,x)dtdx+\int_{I\!R^+\times I\!R^d}\sum_{i=1}^d \operatorname{sgn}_\pm(v_\varepsilon(t,x)-\xi)\times \qquad (14)\\
&\quad\times\Big(\left(g_{iL}(\hat{x}_i,v_\varepsilon(t,x)+k_L)-g_{iL}(\hat{x}_i,\xi+k_L)\right)H(\alpha_i(\hat{x}_i)-x_i)\\
&\qquad+\left(g_{iR}(\hat{x}_i,v_\varepsilon(t,x)+k_R)-g_{iR}(\hat{x}_i,\xi+k_R)\right)H(x_i-\alpha_i(\hat{x}_i))\Big)\partial_{x_i}\varphi(t,x)dtdxd\lambda\\
&\qquad-\int_{I\!R^d}(u_0(x)+k(x)-\xi)^\pm\varphi(0,x)dx\\
&\quad+\sum_{i=1}^d\int_{I\!R^+\times I\!R^d}\operatorname{sgn}_\pm(v_\varepsilon(t,x)-\xi)\frac{1}{\varepsilon}\omega(\frac{x_i-\alpha(\hat{x}_i)}{\varepsilon})\times\\
&\qquad\qquad\times\left(g_{iL}(\hat{x}_i,\xi+k_L)-g_{iR}(\hat{x}_i,\xi+k_R)\right)\varphi(t,x)dxdt\geq 0.
\end{aligned}$$

Noticing that

$$-\operatorname{sgn}_\pm(v_\varepsilon-\xi)\left(g_{iL}(\hat{x}_i,\xi+k_L)-g_{iR}(\hat{x}_i,\xi+k_R)\right)\leq\left(g_{iL}(\hat{x}_i,\xi+k_L)-g_{iR}(\hat{x}_i,\xi+k_R)\right)^\mp$$

from (14) after letting $\varepsilon \to 0$ along a subsequence and taking Theorem 6 into account, we arrive at (11) for a nonlinear weak-$\star$ limit of a subsequence $(v_{\varepsilon_n}) \subset (v_\varepsilon)$.

❏

We shall prove the following comparison principle which gives uniqueness and existence of certain classes of the k-admissible weak solution to (6), (2).

Theorem 9. *Assume that the step function k from* (10) *is such that there exists an interval $(c,d) \subset I\!R$ such that for every $x \in I\!R^d$:*

$$g_{iL}(\hat{x_i}, \xi + k_L) \equiv 0,\ \xi \geq c \ \text{ and } \ g_{iR}(\hat{x_i}, \xi + k_R) \equiv 0,\ \xi \leq d, \quad \forall i = 1, \dots, d,$$

or

$$g_{iR}(\hat{x_i}, \xi + k_L) \equiv 0,\ \xi \geq c \ \text{ and } \ g_{iL}(\hat{x_i}, \xi + k_R) \equiv 0,\ \xi \leq d, \quad \forall i = 1, \dots, d.$$

Then, for any two k-weak entropy process solutions u and v to (6) *with the initial conditions u_0 and v_0, respectively, it holds for any $T > 0$ and any ball $B(0,R) \subset I\!R^d$:*

$$\int_0^1 d\lambda \int_0^1 d\eta \int_0^T \int_{B(0,R)} (u(t,x,\lambda) - v(t,x,\eta))^{\pm} dx dt \leq T \int_{B(0,R+CT)} (u_0(x) - v_0(x))^{\pm} dx, \tag{15}$$

for a constant $C > 0$ independent on $T, R > 0$.

Remark 10. In the sequel, we shall assume that

$$\begin{aligned} g_{iL}(\hat{x_i}, \xi + k_R) &\equiv 0, \quad \xi \geq c \\ g_{iR}(\hat{x_i}, \xi + k_L) &\equiv 0, \quad \xi \leq d. \end{aligned} \tag{16}$$

The proof of Theorem 9 is based on a kinetic formulation of (6) to be introduced in the next subsection.

2.1. **Kinetic formulation.**

We denote for functions $u \in L^\infty(I\!R^+ \times I\!R^d \times (0,1))$, $u_0 \in L^\infty(I\!R^d; [a,b])$ and the function k from (10):

$$\begin{aligned} h_\pm^k(t,x,\lambda,\xi) &= \operatorname{sgn}_\pm(u(t,x,\lambda) + k(x) - \xi), \\ h_{\pm,k}^0(x,\xi) &= \operatorname{sgn}_\pm(u_0(x) + k(x) - \xi). \end{aligned}$$

The functions $h_\pm$ we call equilibrium functions.

Definition 11. Denote

$$G_{iL}(x,\xi) = \partial_\xi g_{iL}(x,\xi), \quad G_{iR}(x,\xi) = \partial_\xi g_{iR}(x,\xi).$$

Let $u_0 \in L^\infty(I\!R^d; [a,b])$ and $u \in L^\infty(I\!R^+ \times I\!R^d \times (0,1))$.

The function u is k-kinetic process supersolution (respectively k-kinetic process subsolution) to (6), (2) if for the function $v = u - k \in L^\infty(I\!R^+ \times I\!R^d \times (0,1))$, k given by (10), there exists $m_\pm \in C(I\!R_\xi; w - \star\mathcal{M}_+(I\!R^+ \times I\!R^d))$ such that $m_+(\cdot,\xi)$ vanishes for large ξ (respectively $m_-(\cdot,\xi)$ vanishes for large $-\xi$), and such that for any $\varphi \in C^1(I\!R^+ \times I\!R^d \times (0,1))$,

$$\int_0^1 d\lambda \int_{t,x,\xi} h^k_\pm \times \tag{17}$$

$$\times\Big(\partial_t + \sum_{i=1}^d \left(G_{iL}(\hat{x}_i, \xi+k_L) H(\alpha_i(\hat{x}_i) - x_i) + G_{iR}(\hat{x}_i, \xi+k_R) H(x_i - \alpha_i(\hat{x}_i))\right)\partial_{x_i}\Big)\varphi$$

$$+ \int_{x,\xi} h^0_{\pm,k}\varphi|_{t=0} dx d\xi - \int_{t,\hat{x}_i,\xi} \left(g_{iL}(\hat{x}_i, \xi+k_L) - g_{iR}(\hat{x}_i, \xi+k_R)\right)^\pm \partial_\xi\varphi|_{x_i=\alpha(\hat{x}_i)} d\hat{x}_i dt d\xi$$

$$= \int_{t,x,\xi} \partial_\xi \varphi dm_\pm d\xi.$$

It can be proved by a simple modification of the procedure from [5] that the notions of k-weak entropy process solution and k-kinetic entropy process solution are equivalent. Still, for our purposes, it will be enough to prove that the k-weak entropy process solution is, at the same time, the k-kinetic process solution.

Proposition 12. *The k-weak entropy process admissible solution is at the same time the k-kinetic process solution.*

Proof: Take an arbitrary k-weak entropy process solution to (6), (2).

According to the Schwartz lemma for non-negative distributions, for every fixed $\xi \in I\!R$ there exist non-negative Radon measures $m_\pm(\cdot,\xi) \in \mathcal{M}_+(I\!R^+ \times I\!R^d)$ satisfying for every $\varphi \in C^1_0(I\!R^{d+2})$

$$\int_{t,x,\xi} \partial_\xi \varphi m_\pm d\xi = \int_0^1 d\lambda \int_{t,x,\xi} (v-\xi)^\pm \partial_t \partial_\xi \varphi$$

$$+ \int_0^1 \int_{t,x,\xi} \sum_{i=1}^d \mathrm{sgn}_\pm(v-\xi) \times$$

$$\times \Big(\left(g_{iL}(\hat{x}_i, v+k_L) - g_{iL}(\hat{x}_i, \xi+k_L)\right) H(\alpha_i(\hat{x}_i) - x_i)$$

$$+ \left(g_{iR}(\hat{x}_i, v+k_R) - g_{iR}(\hat{x}_i, \xi+k_R)\right) H(x_i - \alpha_i(\hat{x}_i))\Big) \partial_{x_i}\partial_\xi\varphi$$

$$+ \int_x (u_0(x) + k(x) - \xi)^\pm \partial_\xi \varphi(0,x,\xi) dx d\xi$$

$$+ \sum_{i=1}^d \int_{t,\hat{x}_i,\xi} \left(g_{iL}(\hat{x}_i, \xi+k_L) - g_{iR}(\hat{x}_i, \xi+k_R)\right)^\pm \partial_\xi \varphi|_{x_i=\alpha_i(\hat{x}_i)} d\hat{x}_i dt d\xi$$

Integrating by part in $\xi \in I\!R$ the right-hand side of the previous expression we arrive at (17).

It is clear that the measures $m_\pm \in C(I\!R_\xi; w - \star\mathcal{M}_+(I\!R^+ \times I\!R^d))$ satisfy the conditions of the theorem. ❑

3. Auxiliary results

We will prove in this section some of the results and introduce some of the notions that we shall use in the proof of Theorem 9.

In the sequel, we shall denote by $h^k_\pm$ and $j^k_\pm$ equilibrium functions corresponding to k-weak entropy process solutions u and v to (6) with the initial conditions $u_0 \in L^\infty(I\!R^d; (a,b))$ and $v_0 \in L^\infty(I\!R^d; (a,b))$, respectively.

Introduce the cut-off function

$$\omega_\varepsilon(s)=\int_0^{|s|}\rho_\varepsilon(r)dr,\quad \rho_\varepsilon(r)=\varepsilon^{-1}\rho(\varepsilon^{-1}r),\quad s\in I\!R^d,\ r\in I\!R, \tag{18}$$

where ρ is compactly supported non-negative function with total mass one.

Let $\psi_L,\psi_R\in C^\infty(I\!R)$ be nonnegative monotonic functions such that

$$\begin{cases}\psi_L(\xi)+\psi_R(\xi)\equiv 1, & \xi\in I\!R,\\ \psi_L(\xi)\equiv 0, & \xi\geq d,\\ \psi_R(\xi)\equiv 0, & \xi\leq c.\end{cases} \tag{19}$$

Next, take the functions

$$I\!R^+\times I\!R^d\times I\!R\ni(t,x,\xi)\mapsto\rho_{\varepsilon,\sigma,\zeta}(t,x,\xi)=\sum_{i=1}^d\rho^i_{\varepsilon,\sigma,\zeta}(t,x,\xi)=\sum_{i=1}^d\rho_\varepsilon(t)\rho_\zeta(\xi)\rho_\sigma(x_i),$$

$$I\!R^+\times I\!R^d\ni(t,x)\mapsto\rho_{\varepsilon,\sigma}(t,x)=\sum_{i=1}^d\rho^i_{\varepsilon,\sigma}(t,x)=\sum_{i=1}^d\rho_\varepsilon(t)\rho_\sigma(x_i),$$

where ρ_ε is defined in (18), and let

$$\begin{aligned}&j^k_{-,\varepsilon_j,\sigma_j,\zeta_j}(t,x,\xi,\eta)=j^k_-\star\rho_{\varepsilon_j,\sigma_j,\zeta_j}(t,x,\xi,\eta)\\ &h^k_{+,\varepsilon_h,\sigma_h,\zeta_h}(t,x,\xi,\lambda)=h^k_+\star\rho_{\varepsilon_h,\sigma_h,\zeta_h}(t,x,\xi,\lambda),\end{aligned}$$

and

$$\begin{aligned}j^k_{-,\varepsilon_j,\sigma_j}&=\lim_{\zeta_j\to 0}j^k_{-,\varepsilon_j,\sigma_j,\zeta_j}=j^k_-\star\rho_{\varepsilon_j,\sigma_j}\\ h^k_{+,\varepsilon_h,\sigma_h}&=\lim_{\zeta_h\to 0}h^k_{+,\varepsilon_h,\sigma_h,\zeta_h}=h^k_+\star\rho_{\varepsilon_h,\sigma_h},\end{aligned}$$

where the limit is understood in the strong L^1_{loc} sense.

We shall need the following lemma:

Lemma 13. *Assume that for every $i=1,\dots,d$*

$$g_{iL}(\hat{x_i},\xi+k_L)\geq g_{iR}(\hat{x_i},\xi+k_R),\quad x\in I\!R^d,\ \xi\leq p\in I\!R. \tag{20}$$

Then, for $\psi\in C^1((-\infty,p))$, $\theta\in C^1_0(I\!R^+\times I\!R^d\times I\!R)$:

$$\begin{aligned}\int_0^1 d\lambda\int_{t,x,\xi}h^k_-\psi\theta\sum_{i=1}^d\Big(&G_{iL}(\hat{x_i},\xi+k_L)H(\alpha_i(\hat{x_i})-x_i)\\ &+G_{iR}(\hat{x_i},\xi+k_R)H(x_i-\alpha_i(\hat{x_i}))\Big)\partial_{x_i}\omega_{1/n}(x_1-\alpha_1(\hat{x_1}))\\ =-\int_{t,x,\xi}&(1-\omega_{1/n}(x_1-\alpha_1(\hat{x_1})))\partial_\xi(\psi\theta)dm_-d\xi+o_n(1),\end{aligned} \tag{21}$$

where $o_n(1)\to 0$ as $n\to\infty$ and it depends only on θ, $\partial_x\theta$ and $\partial_t\theta$.

Similarly, if for every $i=1,\dots,d$

$$g_{iL}(\hat{x_i},\xi+k_L)\leq g_{iR}(\hat{x_i},\xi+k_R),\quad x\in I\!R^d,\ \xi\geq p\in I\!R. \tag{22}$$

Then, for $\psi \in C^1((p,\infty))$, $\theta \in C^1_0(I\!R^+ \times I\!R^d \times I\!R)$:

$$\int_0^1 d\lambda \int_{t,x,\xi} h_+^k \psi\theta \sum_{i=1}^d \Big(G_{iL}(\hat{x_i}, \xi + k_L) H(\alpha_i(\hat{x_i}) - x_i) \tag{23}$$
$$+ G_{iR}(\hat{x_i}, \xi + k_R) H(x_i - \alpha_i(\hat{x_i})) \Big) \partial_{x_i} \omega_{1/n}(x_1 - \alpha_1(\hat{x_1}))$$
$$= - \int_{t,x,\xi} (1 - \omega_{1/n}(x_1 - \alpha_1(\hat{x_1}))) \partial_\xi(\psi\theta) dm_+ d\xi + o_n(1),$$

where $o_n(1) \to 0$ as $n \to \infty$ and it depends only on θ, $\partial_x\theta$ and $\partial_t\theta$.

Proof: We will prove (21). Relation (23) is proved analogically.

It is enough to choose in (17):

$$\varphi(t,x,\xi) = \theta(t,x,\xi)\psi(\xi)(1 - \omega_{1/n}(x_1 - \alpha_1(\hat{x_1})))$$

and to notice that $1 - \omega_{1/n}(x_1 - \alpha(\hat{x_1})) \to 0$ almost everywhere. We get:

$$\int_0^1 d\lambda \int_{t,x,\xi} h_-^k \psi\theta \sum_{i=1}^d (G_{iL}(\hat{x_i}, \xi + k_L) H(\alpha_i(\hat{x_i}) - x_i) + G_{iR}(\hat{x_i}, \xi + k_R) H(x_i - \alpha_i(\hat{x_i}))) \times$$
$$\times \partial_{x_i}(1 - \omega_{1/n}(x_1 - \alpha_1(\hat{x_1})))$$
$$- \int_{t,\hat{x_i},\xi} (g_{iL}(\hat{x_i}, \xi + k_L) - g_{iR}(\hat{x_i}, \xi + k_R))^- \, \partial_\xi(\psi\theta|_{x_i = \alpha(\hat{x_i})}) d\hat{x_i} dt d\xi$$
$$= \int_{t,x,\xi} (1 - \omega_{1/n}(x_1 - \alpha_1(\hat{x_1}))) \partial_\xi(\psi\theta) dm_- d\xi + o_n(1).$$

Due to assumption (20), we conclude from here

$$\int_0^1 d\lambda \int_{t,x,\xi} h_-^k \psi\theta \sum_{i=1}^d (G_{iL}(\hat{x_i}, \xi + k_L) H(\alpha_i(\hat{x_i}) - x_i) + G_{iR}(\hat{x_i}, \xi + k_R) H(x_i - \alpha_i(\hat{x_i}))) \times$$
$$\times \partial_{x_i} \omega_{1/n}(x_1 - \alpha_1(\hat{x_1}))$$
$$= - \int_{t,x,\xi} (1 - \omega_{1/n}(x_1 - \alpha_1(\hat{x_1}))) \partial_\xi(\psi\theta) dm_- d\xi + o_n(1),$$

which we wanted to obtain.

❑

Remark 14. Notice that if we assume that $\psi \geq 0$ and $\theta \in C^1_0(I\!R^+ \times I\!R^d; L^\infty(I\!R))$ such that, in the sense of distributions, $\partial_\xi \theta \geq 0$, we can write instead (21):

$$\int_0^1 d\lambda \int_{t,x,\xi} h_-^k \psi\theta \sum_{i=1}^d \Big(G_{iL}(\hat{x_i}, \xi + k_L) H(\alpha_i(\hat{x_i}) - x_i) \tag{24}$$
$$+ G_{iR}(\hat{x_i}, \xi + k_R) H(x_i - \alpha_i(\hat{x_i})) \Big) \partial_{x_i} \omega_{1/n}(x_1 - \alpha_1(\hat{x_1}))$$
$$\leq - \int_{t,x,\xi} (1 - \omega_{1/n}(x_1 - \alpha_1(\hat{x_1}))) \theta \partial_\xi \psi dm_- d\xi + o_n(1), \quad n \to \infty,$$

Similarly, instead of (23)

$$
\begin{aligned}
\int_0^1 d\lambda \int_{t,x,\xi} h_+^k \psi\theta \sum_{i=1}^d \Big(& G_{iL}(\hat{x_i},\xi+k_L)H(\alpha_i(\hat{x_i})-x_i) \\
& +G_{iR}(\hat{x_i},\xi+k_R)H(x_i-\alpha_i(\hat{x_i}))\Big)\partial_{x_i}\omega_{1/n}(x_1-\alpha_1(\hat{x_1})) \\
\leq -\int_{t,x,\xi} & (1-\omega_{1/n}(x_1-\alpha_1(\hat{x_1})))\theta\partial_\xi\psi dm_+ d\xi + o_n(1), \quad n\to\infty,
\end{aligned} \tag{25}
$$

We shall also need the following known formula. It holds for a $\theta \in C_0^1(I\!R^+\times I\!R^d)$:

$$
\begin{aligned}
& \int_0^1 d\lambda \int_0^1 d\eta \int_{t,x,\xi} (-h_+^k j_-^k)\Big(\partial_t\theta \\
& \qquad + \sum_{i=1}^d \left(G_{iL}(\hat{x_i},\xi)H(\alpha_i(\hat{x_i})-x_i)+G_{iR}(\hat{x_i},\xi)H(x_i-\alpha_i(\hat{x_i}))\right)\partial_{x_i}\theta\Big) \\
& = \int_0^1 d\lambda \int_0^1 d\eta \int_{t,x} \Big(|u(t,x,\lambda)-v(t,x,\eta)|^+\partial_t\theta \\
& \qquad + \sum_{i=1}^d \mathrm{sgn}_+(u(t,x,\lambda)-v(t,x,\eta))\times \\
& \qquad \times \Big(\big(g_{iL}(\hat{x_i},u(t,x,\lambda))-g_{iL}(\hat{x_i},v(t,x,\eta))\big)H(\alpha_i(\hat{x_i})-x_i) \\
& \qquad\quad + \big(g_{iR}(\hat{x_i},u(t,x,\lambda))-g_{iR}(\hat{x_i},v(t,x,\eta))\big)H(x_i-\alpha_i(\hat{x_i}))\Big)\partial_{x_i}\theta
\end{aligned} \tag{26}
$$

We finish this section by a lemma stating about traces of the k-entropy process solutions at the line $t=0$.

Lemma 15. *Assume that the functions $u=u(t,x,\lambda)$ and $v=v(t,x,\eta)\in L^\infty$ are two k-entropy process solutions to* (6) *corresponding to the initial condition $u_0\in L^\infty(I\!R^d;[a,b])$ and $v_0\in L^\infty(I\!R^d;[a,b])$, respectively.*

It holds for every $\varphi\in C_0^1(I\!R\times I\!R^d)$:

$$
\begin{aligned}
\lim_{n\to\infty}\int_0^1 d\lambda\int_0^1 d\eta\int_{I\!R^+\times I\!R^d} & |u(t,x,\lambda)-v(t,x,\eta)|^{\pm}\omega'_{1/n}(t)\varphi(t,x)dtdx \\
& \leq \int_{I\!R^d}|u_0(x)-v_0(x)|^{\pm}\varphi(0,x)dtdx
\end{aligned} \tag{27}
$$

Proof: By using standard Kruzhkov's doubling of variables method [20], it is not difficult to prove that for every $\varphi\in C^1_0(I\!R\times(I\!R^d\backslash\Gamma))$:

$$\int_0^1 d\lambda\int_0^1 d\eta\int_{I\!R^+\times I\!R^d}(u(t,x,\lambda)-v(t,x,\eta))^{\pm}\partial_t\varphi dtdx$$
$$+\int_0^1 d\lambda\int_0^1 d\eta\int_{I\!R^+\times I\!R^d}\sum_{i=1}^d \mathrm{sgn}_{\pm}(u(t,x,\lambda)-v(t,x,\eta))\times$$
$$\times\Big(\left(g_{iL}(\hat{x_i},u(t,x,\lambda)+k_L)-g_{iL}(\hat{x_i},v(t,x,\eta)+k_L)\right)H(\alpha_i(\hat{x_i})-x_i)$$
$$+\left(g_{iR}(\hat{x_i},u(t,x,\lambda)+k_R)-g_{iR}(\hat{x_i},v(t,x,\eta)+k_R)\right)H(x_i-\alpha_i(\hat{x_i}))\Big)\partial_{x_i}\varphi$$
$$+\int_{I\!R^d}(u_0(x)-v_0(x))^{\pm}\varphi(0,x)dx\geq 0.$$

We put here

$$\varphi(t,x)=(1-\omega_{1/n}(t))\omega_\varepsilon(x_1-\alpha(\hat{x_1}))\theta(t,x),$$

where $\theta\in C^1_0(I\!R\times I\!R^d)$, and let $n\to\infty$. We get:

$$-\lim_{n\to\infty}\int_0^1 d\lambda\int_0^1 d\eta\int_{I\!R^+\times I\!R^d}(u(t,x,\lambda)-v(t,x,\eta))^{\pm}\omega'_{1/n}(t)\theta(t,x)dt\omega_\varepsilon(x_1-\alpha_1(\hat{x_1}))dx$$
$$+\int_{I\!R^d}(u_0(x)-v_0(x))^{\pm}\omega_\varepsilon(x_1-\alpha(\hat{x_1}))\varphi(0,x)dx\geq 0.$$

Finally, letting here $\varepsilon\to 0$ we arrive at (27). ❑

4. Proof of Theorem 9

The proof is based on the procedure from [5].

On the first step, choose the following test function

$$\varphi(t,x,\xi)=\theta\star\rho_{\varepsilon,\sigma,\zeta},$$

where $\mathrm{supp}\theta\subset I\!R^+\times(I\!R^d\backslash\Gamma)\times I\!R$, in the place of the function φ from (17).

For ε,σ,ζ small enough, it holds as well

$$\mathrm{supp}\theta\star\rho_{\varepsilon,\sigma,\zeta}\subset I\!R^+\times(I\!R^d\backslash\Gamma)\times I\!R.$$

Therefore, (17) becomes for the equilibrium functions $h_\pm$:

$$\int_0^1 d\lambda\int_{t,x,\xi}h^k_\pm\star\rho_{\varepsilon_h,\sigma_h,\zeta_h}\partial_t\theta \tag{28}$$
$$+\sum_{i=1}^d\Big(h^k_\pm\big(G_{iL}(\hat{x_i},\xi+k_L)H(\alpha_i(\hat{x_i})-x_i)$$
$$+G_{iR}(\hat{x_i},\xi+k_R)H(x_i-\alpha_i(\hat{x_i}))\big)\Big)\star\rho^i_{\varepsilon_h,\sigma_h,\zeta_h}\partial_{x_i}\theta$$
$$=\int_{t,x,\xi}\partial_\xi\theta m_\pm^{\varepsilon_h,\sigma_h,\zeta_h}dtdx.$$

where $m_\pm^{\varepsilon_h,\sigma_h,\zeta_h}=m_\pm\star\rho_{\varepsilon_h,\sigma_h,\zeta_h}$, while for the equilibrium functions $j_\pm$

$$\begin{aligned}\int_0^1 d\eta\int_{t,x,\xi} & j^k_\pm\star\rho_{\varepsilon_j,\sigma_j,\zeta_j}\partial_t\theta \\ &+\sum_{i=1}^d\Big(j^k_\pm(v,\xi)\big(G_{iL}(\hat{x_i},\xi+k_L)H(\alpha_i(\hat{x_i})-x_i) \\ &\qquad+G_{iR}(\hat{x_i},\xi+k_R)H(x_i-\alpha_i(\hat{x_i}))\big)\Big)\star\rho^i_{\varepsilon_j,\sigma_j,\zeta_j}\partial_{x_i}\theta \\ &=\int_{t,x,\xi}\partial_\xi\theta q_\pm^{\varepsilon_j,\sigma_j,\zeta_j}dtdx,\end{aligned}\tag{29}$$

where $q_\pm^{\varepsilon_j,\sigma_j,\zeta_j}=q_\pm\star\rho_{\varepsilon_j,\sigma_j,\zeta_j}$.

With the notation from Section 3, take in (28) instead of $\pm$ the sign $+$ and $\theta(t,x,\xi)=-\psi_L(\xi)\varphi(t,x)j^k_{-,\varepsilon_j,\sigma_j,\zeta_j}$ where φ disappears in a neighborhood of the discontinuity manifold Γ, and integrate over $\eta\in(0,1)$. Similarly, for the same function φ, take in (29) instead of $\pm$ the sign $-$ and $\theta(t,x,\xi)=-\psi_L(\xi)\varphi(t,x)h^k_{+,\varepsilon_h,\sigma_h,\zeta_h}$, and integrate over $\lambda\in(0,1)$.

By adding the resulting expressions, we obtain:

$$\begin{aligned}\int_0^1 d\lambda\int_0^1 d\eta\int_{t,x,\xi} & (-h^k_{+,\varepsilon_h,\sigma_h,\zeta_h}j^k_{-,\varepsilon_j,\sigma_j,\zeta_j})\psi_L\Big(\partial_t \\ &+\sum_{i=1}^d\left(G_{iL}(\hat{x_i},\xi+k_L)H(\alpha_i(\hat{x_i})-x_i)+G_{iR}(\hat{x_i},\xi+k_R)H(x_i-\alpha_i(\hat{x_i}))\right)\partial_{x_i}\Big)\varphi \\ \geq&\int_0^1 d\eta\int_{t,x,\xi}\varphi\partial_\xi(-\psi_L j^k_{-,\varepsilon_j,\sigma_j,\zeta_j})m_+^{\varepsilon_h,\sigma_h,\zeta_h}dtdxd\xi \\ &+\int_0^1 d\lambda\int_{t,x,\xi}\varphi\partial_\xi(-\psi_L h^k_{+,\varepsilon_h,\sigma_h,\zeta_h})q_-^{\varepsilon_j,\sigma_j,\zeta_j}dtdxd\xi \\ &+R_{\varepsilon_h,\sigma_h,\zeta_h}(\varphi j^k_{-,\varepsilon_j,\sigma_j,\zeta_j})+Q_{\varepsilon_j,\sigma_j,\zeta_j}(\varphi h^k_{+,\varepsilon_h,\sigma_h,\zeta_h}),\end{aligned}\tag{30}$$

where,

$$\begin{aligned}R_{\varepsilon_h,\sigma_h,\zeta_h}(\varphi)=&h^k_{+,\varepsilon_h,\sigma_h,\zeta_h}\sum_{i=1}^d\Big(G_{iL}(\hat{x_i},\xi+k_L)H(\alpha_i(\hat{x_i})-x_i) \\ &\qquad+G_{iR}(\hat{x_i},\xi+k_R)H(x_i-\alpha_i(\hat{x_i}))\Big)\partial_{x_i}\varphi \\ &-\sum_{i=1}^d\Big(h^k_+\left(G_{iL}(\hat{x_i},\xi+k_L)H(\alpha_i(\hat{x_i})-x_i)+G_{iR}(\hat{x_i},\xi+k_R)H(x_i-\alpha_i(\hat{x_i}))\right)\Big)\star\rho^i_{\varepsilon_j,\sigma_j,\zeta_j}\partial_{x_i}\varphi,\end{aligned}$$

$$\begin{aligned}Q_{\varepsilon_j,\sigma_j,\zeta_j}(\varphi)=&j^k_{+,\varepsilon_h,\sigma_h,\zeta_h}\sum_{i=1}^d\Big(G_{iL}(\hat{x_i},\xi+k_L)H(\alpha_i(\hat{x_i})-x_i) \\ &\qquad+G_{iR}(\hat{x_i},\xi+k_R)H(x_i-\alpha_i(\hat{x_i}))\Big)\partial_{x_i}\varphi \\ &-\Big(j^k_-\sum_{i=1}^d\left(G_{iL}(\hat{x_i},\xi+k_L)H(\alpha_i(\hat{x_i})-x_i)+G_{iR}(\hat{x_i},\xi+k_R)H(x_i-\alpha_i(\hat{x_i}))\right)\Big)\star\rho_{\varepsilon_j,\sigma_j,\zeta_j}\partial_{x_i}\varphi,\end{aligned}$$

and, according to the Friedrichs lemma:

$$R_{\varepsilon_h,\sigma_h,\zeta_h}(\varphi j^k_{-,\varepsilon_j,\sigma_j,\zeta_j})=\mathcal{O}(\frac{\zeta_j}{\varepsilon_h\sigma_h}),\quad Q_{\varepsilon_j,\sigma_j,\zeta_j}(\varphi h^k_{+,\varepsilon_h,\sigma_h,\zeta_h})=\mathcal{O}(\frac{\zeta_h}{\varepsilon_j\sigma_j}).$$

Finding the derivative in ξ on the right-hand of (30), and having in mind that $\partial_\xi(-j^k_{-,\varepsilon_j,\sigma_j,\zeta_j})>0$ and $\partial_\xi(-h^k_{+,\varepsilon_h,\sigma_h,\zeta_h})>0$, we conclude from (30) after letting $\zeta_h,\zeta_j\to 0$:

$$\begin{aligned}&\int_0^1 d\lambda\int_0^1 d\eta\int_{t,x,\xi}(-h^k_{+,\varepsilon_h,\sigma_h}j^k_{-,\varepsilon_j,\sigma_j})\psi_L\Big(\partial_t\\&\qquad+\sum_{i=1}^d\left(G_{iL}(\hat{x}_i,\xi+k_L)H(\alpha_i(\hat{x}_i)-x_i)+G_{iR}(\hat{x}_i,\xi+k_R)H(x_i-\alpha_i(\hat{x}_i))\right)\partial_{x_i}\Big)\varphi\\&\geq-\int_0^1 d\eta\int_{t,x,\xi}\varphi j^k_{-,\varepsilon_j,\sigma_j}\partial_\xi\psi_L m_+^{\varepsilon_h,\sigma_h}\,dtdxd\xi\\&-\int_0^1 d\lambda\int_{t,x,\xi}\varphi h^k_{+,\varepsilon_h,\sigma_h}\partial_\xi\psi_L q_-^{\varepsilon_j,\sigma_j}\,dtdxd\xi\end{aligned}\tag{31}$$

Next, notice that, it holds according to (16):

$$g_{iL}(\hat{x}_i,\xi+k_L)\geq g_{iR}(\hat{x}_i,\xi+k_R)=0,\quad x\in I\!R^d,\quad \xi\in\mathrm{supp}\psi_L.\tag{32}$$

Now, we let in (31) $\varepsilon_j,\sigma_j\to 0$:

$$\begin{aligned}&\int_0^1 d\lambda\int_0^1 d\eta\int_{t,x,\xi}(-h^k_{+,\varepsilon_h,\sigma_h}j^k_{-})\psi_L\Big(\partial_t\\&\qquad+\sum_{i=1}^d\left(G_{iL}(\hat{x}_i,\xi+k_L)H(\alpha_i(\hat{x}_i)-x_i)+G_{iR}(\hat{x}_i,\xi+k_R)H(x_i-\alpha_i(\hat{x}_i))\right)\partial_{x_i}\Big)\varphi\\&\geq-\int_0^1 d\eta\int_{t,x,\xi}\varphi j^k_{-}\partial_\xi\psi_L m_+^{\varepsilon_h,\sigma_h}\,dtdxd\xi-\int_0^1 d\lambda\int_{t,x,\xi}\varphi h^k_{+,\varepsilon_h,\sigma_h}\partial_\xi\psi_L dq_-d\xi\end{aligned}\tag{33}$$

Let us now remove the conditions imposed on the support of the function φ. Put in (33):

$$\varphi(t,x)=\theta(t,x)\omega_{1/n}(x_1-\alpha_1(\hat{x}_1)),$$

for an arbitrary function $\theta \in C_0^1(I\!R^+ \times I\!R^d)$. We get:

$$\int_0^1 d\lambda \int_0^1 d\eta \int_{t,x,\xi} (-h^k_{+,\varepsilon_h,\sigma_h} j^k_-)\psi_L \omega_{1/n}(x_1 - \alpha_1(\hat{x_1}))\times \tag{34}$$
$$\times\Big(\partial_t + \sum_{i=1}^d \left(G_{iL}(\hat{x_i},\xi+k_L)H(\alpha_i(\hat{x_i})-x_i) + G_{iR}(\hat{x_i},\xi+k_R)H(x_i-\alpha_i(\hat{x_i}))\right)\partial_{x_i}\Big)\theta$$
$$+\int_0^1 d\lambda \int_0^1 d\eta \int_{t,x,\xi} (-h^k_{+,\varepsilon_h,\sigma_h} j^k_-)\psi_L \theta\Big(\partial_t + \sum_{i=1}^d \big(G_{iL}(\hat{x_i},\xi+k_L)H(\alpha_i(\hat{x_i})-x_i)$$
$$+G_{iR}(\hat{x_i},\xi+k_R)H(x_i-\alpha_i(\hat{x_i}))\big)\partial_{x_i}\Big)\omega_{1/n}(x_1 - \alpha_1(\hat{x_1}))$$
$$\geq -\int_0^1 d\eta \int_{t,x,\xi} \theta j^k_- \partial_\xi \psi_L \omega_{1/n}(x_1 - \alpha_1(\hat{x_1})) m_+^{\varepsilon_h,\sigma_h} dtdxd\xi$$
$$-\int_0^1 d\lambda \int_{t,x,\xi} \theta h^k_{+,\varepsilon_h,\sigma_h} \omega_{1/n}(x_1 - \alpha_1(\hat{x_1}))\partial_\xi \psi_L dq_- d\xi.$$

According to Remark 14, more precisely (24), we conclude from (34) since $\partial_\xi(-h^k_{+,\varepsilon_h,\sigma_h}) \geq 0$ (in $\mathcal{D}'$) and $g_{iL}(\hat{x_i},\xi+k_L) \geq g_{iR}(\hat{x_i},\xi+k_R)$ for every $i = 1,\dots,d$, $x \in I\!R^d$, $\xi \in \mathrm{supp}\psi_L$:

$$\int_0^1 d\lambda \int_0^1 d\eta \int_{t,x,\xi} (-h^k_{+,\varepsilon_h,\sigma_h} j^k_-)\psi_L \omega_{1/n}(x_1 - \alpha_1(\hat{x_1}))\times \tag{35}$$
$$\times\Big(\partial_t\theta + \sum_{i=1}^d \left(G_{iL}(\hat{x_i},\xi+k_L)H(\alpha_i(\hat{x_i})-x_i) + G_{iR}(\hat{x_i},\xi+k_R)H(x_i-\alpha_i(\hat{x_i}))\right)\partial_{x_i}\theta\Big)$$
$$-\int_{t,x,\xi} (1-\omega_{1/n}(x_1 - \alpha_1(\hat{x_1})))\theta(-h^k_{+,\varepsilon_h,\sigma_h})\partial_\xi\psi_L dq_- d\xi + o_n(1)$$
$$\geq -\int_0^1 d\eta \int_{t,x,\xi} \theta j^k_- \partial_\xi \psi_L \omega_{1/n}(x_1 - \alpha_1(\hat{x_1})) m_+^{\varepsilon_h,\sigma_h} dtdxd\xi$$
$$-\int_0^1 d\lambda \int_{t,x,\xi} \theta h^k_{+,\varepsilon_h,\sigma_h} \omega_{1/n}(x_1 - \alpha_1(\hat{x_1}))\partial_\xi \psi_L dq_- d\xi.$$

From here, letting $n \to \infty$, we obtain:

$$\int_0^1 d\lambda \int_0^1 d\eta \int_{t,x,\xi} (-h^k_{+,\varepsilon_h,\sigma_h} j^k_-)\psi_L \Big(\partial_t\theta \tag{36}$$
$$+ \sum_{i=1}^d \left(G_{iL}(\hat{x_i},\xi+k_L)H(\alpha_i(\hat{x_i})-x_i) + G_{iR}(\hat{x_i},\xi+k_R)H(x_i-\alpha_i(\hat{x_i}))\right)\partial_{x_i}\theta\Big)$$
$$\geq -\int_0^1 d\eta \int_{t,x,\xi} \theta j^k_- \partial_\xi \psi_L m_+^{\varepsilon_h,\sigma_h} dtdxd\xi - \int_0^1 d\lambda \int_{t,x,\xi} \theta h^k_{+,\varepsilon_h,\sigma_h} \partial_\xi \psi_L dq_- d\xi.$$

Now, take in (28) instead of $\pm$ the sign $+$ and $-\psi_R(\xi)\varphi(t,x) j^k_{-,\varepsilon_j,\sigma_j,\zeta_j}$ in the place of the test function, where $\varphi \in C^1(I\!R^+ \times I\!R^d)$ disappears in the neighborhood of the discontinuity manifold Γ, and integrate over $\eta \in (0,1)$. Similarly, for the same function φ, take in (29) instead of $\pm$ the sign $-$ and $-\psi_R(\xi)\varphi(t,x) h^k_{+,\varepsilon_h,\sigma_h,\zeta_h}$ in the place of the test function, and integrate over $\lambda \in (0,1)$.

Applying the same procedure as for the function ψ_L, we get for an arbitrary $\theta \in C^1_0(I\!R^+ \times I\!R^d)$:

$$\int_0^1 d\lambda \int_0^1 d\eta \int_{t,x,\xi} (-h^k_+ j^k_{-,\varepsilon_j,\sigma_j})\psi_R\Big(\partial_t\theta \tag{37}$$
$$+\sum_{i=1}^d \left(G_{iL}(\hat{x_i},\xi+k_L)H(\alpha_i(\hat{x_i})-x_i)+G_{iR}(\hat{x_i},\xi+k_R)H(x_i-\alpha_i(\hat{x_i}))\right)\partial_{x_i}\theta\Big)$$
$$\geq -\int_0^1 d\eta \int_{t,x,\xi} \theta j^k_{-,\varepsilon_j,\sigma_j}\partial_\xi\psi_R dm_+ d\xi d\xi - \int_0^1 d\lambda \int_{t,x,\xi} \theta h^k_+ \partial_\xi \psi_R q_{-,\varepsilon_j,\sigma_j} dtdxd\xi.$$

Now, we put $\varepsilon = \varepsilon_j = \varepsilon_h$ and $\sigma = \sigma_j = \sigma_h$ in (36) and (37), and add the resulting expressions.

$$\int_0^1 d\lambda \int_0^1 d\eta \int_{t,x,\xi} (-h^k_+ j^k_{-,\varepsilon,\sigma})\psi_R\Big(\partial_t\theta \tag{38}$$
$$+\sum_{i=1}^d \left(G_{iL}(\hat{x_i},\xi+k_L)H(\alpha_i(\hat{x_i})-x_i)+G_{iR}(\hat{x_i},\xi+k_R)H(x_i-\alpha_i(\hat{x_i}))\right)\partial_{x_i}\theta\Big)$$
$$+\int_0^1 d\lambda \int_0^1 d\eta \int_{t,x,\xi} (-h^k_{+,\varepsilon,\sigma} j^k_-)\Big(\partial_t\theta$$
$$+\sum_{i=1}^d \left(G_{iL}(\hat{x_i},\xi+k_L)H(\alpha_i(\hat{x_i})-x_i)+G_{iR}(\hat{x_i},\xi+k_R)H(x_i-\alpha_i(\hat{x_i}))\right)\partial_{x_i}\theta\Big)$$
$$\geq -\int_0^1 d\eta \int_{t,x,\xi} \theta j^k_- \partial_\xi\psi_L m_+^{\varepsilon,\sigma} dtdxd\xi - \int_0^1 d\lambda \int_{t,x,\xi} \theta h^k_{+,\varepsilon,\sigma}\partial_\xi\psi_L dq_- d\xi$$
$$-\int_0^1 d\eta \int_{t,x,\xi} \theta j^k_{-,\varepsilon_j,\sigma_j}\partial_\xi\psi_R dm_+ d\xi d\xi - \int_0^1 d\lambda \int_{t,x,\xi} \theta h^k_+ \partial_\xi\psi_R q_{-,\varepsilon,\sigma} dtdxd\xi.$$

Next, notice that

$$\int_0^1 d\eta \int_{t,x,\xi} \partial_\xi\psi_L \theta j^k_- m_+^{\varepsilon,\sigma} dtdxd\xi + \int_0^1 d\eta \int_{t,x,\xi} \partial_\xi\psi_R \theta j^k_{-,\varepsilon,\sigma} dm_+ d\xi \tag{39}$$
$$= \int_0^1 d\eta \int_{t,x,\xi} \partial_\xi\psi_L \left((\theta j^k_-)\star\rho_{\varepsilon,\sigma} - \theta j^k_{-,\varepsilon,\sigma}\right) dm_+ d\xi \to 0 \quad \text{as} \quad \varepsilon,\sigma\to 0,$$

according to (19) and since

$$\|(\theta j^k_-)\star\rho_{\varepsilon,\sigma} - \theta j^k_{-,\varepsilon,\sigma}\|_{L^\infty(I\!R^+\times I\!R^d)} = \mathcal{O}(\varepsilon+\sigma).$$

Similarly,

$$\int_0^1 d\lambda \int_{t,x,\xi} \partial_\xi\psi_L \theta h^k_+ q_-^{\varepsilon,\sigma} dtdxd\xi + \int_0^1 d\lambda \int_{t,x,\xi} \partial_\xi\psi_R \theta h^k_{+,\varepsilon,\sigma}\partial_\xi dq_- d\xi \tag{40}$$
$$= \int_0^1 d\lambda \int_{t,x,\xi} \partial_\xi\psi_L \left((\theta h^k_+)\star\rho_{\varepsilon,\sigma} - \theta h^k_{+,\varepsilon,\sigma}\right) dq_- d\xi \to 0 \quad \text{as} \quad \varepsilon,\sigma\to 0.$$

Finally, we conclude from (39) and (40) after letting $\varepsilon, \sigma \to 0$:

$$\int_0^1 d\eta \int_0^1 d\lambda \int_{t,x,\xi} (-h_+^k j_-^k)\Big(\partial_t \theta + \sum_{i=1}^d \left(G_{iL}(\hat{x_i}, \xi + k_L) H(\alpha_i(\hat{x_i}) - x_i) + G_{iR}(\hat{x_i}, \xi + k_R) H(x_i - \alpha_i(\hat{x_i}))\right) \partial_{x_i}\theta\Big) \geq 0,$$

and from here, appealing on (26), we conclude:

$$\int_0^1 d\lambda \int_0^1 d\eta \int_{t,x} \Big(|u(t,x,\lambda) - v(t,x,\eta)|^+ \partial_t \theta + \sum_{i=1}^d \mathrm{sgn}_+(u(t,x,\lambda) - v(t,x,\eta)) \times \\ \times \Big(g_{iL}(\hat{x_i}, u(t,x,\lambda)) - g_{iL}(\hat{x_i}, v(t,x,\eta))\Big) H(\alpha_i(\hat{x_i}) - x_i) \\ + \Big(g_{iR}(\hat{x_i}, u(t,x,\lambda)) - g_{iR}(\hat{x_i}, v(t,x,\eta))\Big) H(x_i - \alpha_i(\hat{x_i}))\Big) \partial_{x_i}\theta \geq 0.$$

From here, using the standard procedure (e.g. [20]) and (27), we arrive at (15). This completes the proof.

Simple corollary of Theorem 8 and Theorem 9 is (see e.g. [5, Page 377]):

Corollary 16. *Assume that the function k from* (10) *is such that* (16) *is satisfied. Then, there exists a unique k-entropy weak solution to* (6), (2).

5. General case

In this section, we are concerned with Cauchy problem (8), (2). We shall mainly rely on the results of the previous section. Definitions and concepts are little bit more involved, but they are basically the same as in the case of equation (6), (2). Accordingly, introduce the function

$$k(x) = \begin{cases} k_L, & x \in \Omega_L \\ k_R, & x \in \Omega_R \end{cases}, \quad k_L, k_R \in \mathbb{R}. \tag{41}$$

By k_j we denote restriction of the function k on the set Ω_j, $j = 1, \dots, n$. We write $\mathbb{R} \ni k_L^j = k(x)$ for $x \in \Omega_j$ such that $x_1 \leq \alpha_1^j(\hat{x_1})$, and we write $\mathbb{R} \ni k_R^j = k(x)$ for $x \in \Omega_j$ such that $x_1 > \alpha_1^j(\hat{x_1})$. Notice that instead of $i = 1$, here we could put an arbitrary $i = 1, \dots, d$.

Definition 17. Let $u_0 \in L^\infty(\mathbb{R}^d)$, $a \leq u_0 \leq b$ a.e. on $\mathbb{R}^d$. Let $u \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^d \times (0,1))$.

1. The function u is a k-weak entropy process subsolution (respectively k-weak entropy process supersolution) of problem (8), (2) if for the function $v = v(t,x,\lambda) = u(t,x,\lambda) - k(x)$, k given by (41), every $\xi \in \mathbb{R}$ and:

a) For every fixed $j=1,\dots,n$, it holds for every $\varphi\in C^1_0(I\!R\times\Omega_j)$:

$$\begin{aligned}
&\int_0^1 d\lambda\int_{I\!R^+\times I\!R^d}(v-\xi)^{\pm}\partial_t\varphi dtdx \qquad (42)\\
&+\int_0^1 d\lambda\int_{I\!R^+\times I\!R^d}\sum_{i=1}^d \operatorname{sgn}_{\pm}(v-\xi)\times\\
&\quad\times\Big(\Big(g^j_{iL}(\hat{x}_i,v+k^j_L)-g^j_{iL}(\hat{x}_i,\xi+k^j_L)\Big)H(\alpha^j_i(\hat{x}_i)-x_i)\\
&\quad\quad+\Big(g^j_{iR}(\hat{x}_i,v+k^j_R)-g^j_{iR}(\hat{x}_i,\xi+k^j_R)\Big)H(x_i-\alpha^j_i(\hat{x}_i))\Big)\partial_{x_i}\varphi\\
&+\int_{I\!R^d}(u_0+k(x)-\xi)^{\pm}\varphi(0,x)dx\\
&-\sum_{i=1}^d\int_{I\!R^+\times I\!R^{d-1}}\Big(g^j_{iL}(\hat{x}_i,\xi+k^j_L)-g^j_{iR}(\hat{x}_i,\xi+k^j_R)\Big)^{\pm}\varphi|_{x_i=\alpha^j_i(\hat{x}_i)}d\hat{x}_i dt\geq 0.
\end{aligned}$$

b) For any $\varphi\in C^1_0(I\!R\times\Omega_L)$:

$$\begin{aligned}
&\int_0^1 d\lambda\int_{I\!R^+\times I\!R^d}(v-\xi)^{\pm}\partial_t\varphi dtdx \qquad (43)\\
&+\int_0^1 d\lambda\int_{I\!R^+\times I\!R^d}\sum_{i=1}^d \operatorname{sgn}_{\pm}(v-\xi)\left(g_{iL}(\hat{x}_i,v)-g_{iL}(\hat{x}_i,\xi+k_L)\right)\partial_{x_i}\varphi\\
&+\int_{I\!R^d}(u_0+k_L-\xi)^{\pm}\varphi(0,x)dx\geq 0.
\end{aligned}$$

c) For any $\varphi\in C^1_0(I\!R\times\Omega_R)$:

$$\begin{aligned}
&\int_0^1 d\lambda\int_{I\!R^+\times I\!R^d}(v-\xi)^{\pm}\partial_t\varphi dtdx \qquad (44)\\
&+\int_0^1 d\lambda\int_{I\!R^+\times I\!R^d}\sum_{i=1}^d \operatorname{sgn}_{\pm}(v-\xi)\left(g_{iR}(\hat{x}_i,v)-g_{iR}(\hat{x}_i,\xi+k_R)\right)\partial_{x_i}\varphi\\
&+\int_{I\!R^d}(u_0+k_R-\xi)^{\pm}\varphi(0,x)dx\geq 0.
\end{aligned}$$

2. The function u is k-weak entropy process solution if it is weak k-entropy process sub and super solution ad the same time.

Now, we shall introduce appropriate kinetic formulation of the considered problem.

Denote for functions $u\in L^\infty(I\!R^+\times I\!R^d\times(0,1))$, $u_0\in L^\infty(I\!R^d;[a,b])$, and the function k from (41):

$$\begin{aligned}
h^k_{\pm}(t,x,\lambda,\xi)&=\operatorname{sgn}_{\pm}(u(t,x,\lambda)+k(x)-\xi),\\
h^0_{\pm,k}(x,\xi)&=\operatorname{sgn}_{\pm}(u_0(x)+k(x)-\xi).
\end{aligned}$$

As before, the functions $h_{\pm}$ we call the equilibrium functions.

Definition 18. Denote for $j=1,\dots,n$ and $i=1,\dots,d$:

$$\begin{aligned}
G^j_{iL}(x,\xi)=\partial_\xi g^j_{iL}(x,\xi),\quad G^j_{iR}(x,\xi)=\partial_\xi g^j_{iR}(x,\xi),\\
G_{iL}(x,\xi)=\partial_\xi g_{iL}(x,\xi),\quad G_{iR}(x,\xi)=\partial_\xi g_{iR}(x,\xi).
\end{aligned}$$

Let $u_0 \in L^\infty(\mathbb{R}^d;[a,b])$ and $u \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^d \times (0,1))$.

The function u is k-kinetic process supersolution (respectively k-kinetic process subsolution) to (8), (2) if for the function $v = u+k \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^d \times (0,1))$, k given by (41), there exist $m^j_\pm, m^L_\pm, m^R_\pm \in C(\mathbb{R}_\xi; w-\star\mathcal{M}_+(\mathbb{R}^+\times\mathbb{R}^d))$ such that $m^j_+(\cdot,\xi)$, $m^R_+(\cdot,\xi)$, $m^L_+(\cdot,\xi)$ vanish for large ξ (respectively $m^j_-(\cdot,\xi)$, $m^R_-(\cdot,\xi)$, $m^L_-(\cdot,\xi)$ vanish for large $-\xi$), and such that:

a) For every $j=1,\dots,n$ and every $\varphi \in C^1_0(\mathbb{R}^+\times\Omega_j\times\mathbb{R})$,

$$\int_0^1 d\lambda \int_{t,x,\xi} h^k_\pm \times \tag{45}$$
$$\times\Big(\partial_t + \sum_{i=1}^d \Big(G^j_{iL}(\hat{x}_i,\xi+k^j_L)H(\alpha_i(\hat{x}_i)-x_i) + G^j_{iR}(\hat{x}_i,\xi+k^j_R)H(x_i-\alpha_i(\hat{x}_i))\Big)\partial_{x_i}\Big)\varphi$$
$$+\int_{x,\xi} h^0_{\pm,k}\varphi|_{t=0}dxd\xi$$
$$-\int_{t,\hat{x}_i,\xi}\Big(g^j_{iL}(\hat{x}_i,\xi+k^j_L) - g^j_{iR}(\hat{x}_i,\xi+k^j_R)\Big)^\pm \partial_\xi\varphi|_{x_i=\alpha(\hat{x}_i)}d\hat{x}_i dtd\xi$$
$$=\int_{t,x,\xi}\partial_\xi\varphi dm^j_\pm d\xi.$$

b) For every $\varphi \in C^1_0(\mathbb{R}^+\times\Omega_L\times\mathbb{R})$,

$$\int_0^1 d\lambda \int_{t,x,\xi} h^k_\pm\Big(\partial_t + \sum_{i=1}^d G_{iL}(\hat{x}_i,\xi+k_L)\partial_{x_i}\Big)\varphi \tag{46}$$
$$+\int_{x,\xi} h^0_{\pm,k}\varphi|_{t=0}dxd\xi = \int_{t,x,\xi}\partial_\xi\varphi dm^L_\pm d\xi.$$

c) For every $\varphi \in C^1_0(\mathbb{R}^+\times\Omega_R\times\mathbb{R})$,

$$\int_0^1 d\lambda \int_{t,x,\xi} h^k_\pm\Big(\partial_t + G_{iR}(\hat{x}_i,\xi)\partial_{x_i}\Big)\varphi \tag{47}$$
$$+\int_{x,\xi} h^0_{\pm,k}\varphi|_{t=0}dxd\xi = \int_{t,x,\xi}\partial_\xi\varphi dm^L_\pm d\xi.$$

The following proposition can be proved in the completely same manner as Proposition 12. We leave it without prove.

Proposition 19. *The k-weak entropy process admissible solution to* (8), (2) *is at the same time the k-kinetic process solution* (8), (2).

Using the latter proposition and repeating the proof of Theorem 9, it is not difficult to prove the following theorem. We leave it without prove.

Theorem 20. *Assume that the function k from* (41) *is such that there exists an interval $(c,d)\subset\mathbb{R}$ such that for every $j=1,\dots,n$ and every $x\in\mathbb{R}^d$:*

$$\begin{gathered} g^j_{iL}(\hat{x}_i,\xi+k_L)\equiv 0 \text{ if } \xi\geq c \;\text{ and }\; g^j_{iR}(\hat{x}_i,\xi+k_R)\equiv 0 \text{ if } \xi\leq d, \;\; \forall i=1,\dots,d \\ \text{or} \\ g^j_{iR}(\hat{x}_i,\xi+k_L)\equiv 0 \text{ if } \xi\geq c \;\text{ and }\; g^j_{iL}(\hat{x}_i,\xi+k_R)\equiv 0 \text{ if } \xi\leq d, \;\; \forall i=1,\dots,d. \end{gathered} \tag{48}$$

Then, for any two k-weak entropy process solutions u and v to (8) *with the initial conditions* u_0 *and* v_0*, respectively, it holds for every* $j = 1, \dots, n$ *and every* $\varphi \in C_0^1(I\!R \times \Omega_j)$

$$\begin{aligned}
&\int_0^1 d\lambda \int_{I\!R^+ \times I\!R^d} (v-u)^\pm \partial_t \varphi dt dx \qquad (49)\\
&+ \int_0^1 d\lambda \int_{I\!R^+ \times I\!R^d} \sum_{i=1}^d \mathrm{sgn}_\pm(v-u) \times \\
&\quad \times \Big(\Big(g^j_{iL}(\hat{x}_i, v + k^j_L) - g^j_{iL}(\hat{x}_i, u + k^j_L) \Big) H(\alpha^j_i(\hat{x}_i) - x_i) \\
&\quad + \Big(g^j_{iR}(\hat{x}_i, v + k^j_R) - g^j_{iR}(\hat{x}_i, u + k^j_R) \Big) H(x_i - \alpha^j_i(\hat{x}_i)) \Big) \partial_{x_i} \varphi \\
&+ \int_{I\!R^d} (v_0 - u_0)^\pm \varphi(0, x) dx \geq 0.
\end{aligned}$$

Remark 21. Notice that, according to the assumptions on g^j_{iL} and g^j_{iR} as well as k^j_L and k^j_R, $i = 1, \dots, d$, $j = 1, \dots, n$, we can rewrite (42) for every $j = 1, \dots, n$, and every $\varphi \in C_0^1(I\!R \times \Omega_j)$:

$$\begin{aligned}
&\int_0^1 d\lambda \int_{I\!R^+ \times I\!R^d} (v-u)^\pm \partial_t \varphi dt dx \\
&+ \int_0^1 d\lambda \int_{I\!R^+ \times I\!R^d} \sum_{i=1}^d \mathrm{sgn}_\pm(v-u) \times \\
&\quad \times \Big(\left(g_{iL}(\hat{x}_i, v + k_L) - g_{iL}(\hat{x}_i, u + k_L) \right) \kappa_L(x) \\
&\quad + \left(g_{iR}(\hat{x}_i, v + k_R) - g_{iR}(\hat{x}_i, u + k_R) \right) \kappa_R(x) \Big) \partial_{x_i} \varphi \\
&+ \int_{I\!R^d} (v_0 - u_0)^\pm \varphi(0, x) dx \geq 0.
\end{aligned}$$

As noticed in the proof of Lemma 15, using the standard Kruzhkov doubling of variables method, it can be proved:

Theorem 22. *Any two k-weak entropy process solutions u and v to* (8) *with the initial conditions* u_0 *and* v_0*, respectively, satisfy for every* $\varphi \in C_0^1(I\!R \times (\Omega_L \cup \Omega_R))$

$$\begin{aligned}
&\int_0^1 d\lambda \int_{I\!R^+ \times I\!R^d} (v-u)^\pm \partial_t \varphi dt dx \qquad (50)\\
&+ \int_0^1 d\lambda \int_{I\!R^+ \times I\!R^d} \sum_{i=1}^d \mathrm{sgn}_\pm(v-u) \Big(\left(g_{iL}(\hat{x}_i, v + k_L) - g_{iL}(\hat{x}_i, u + k_L) \right) \kappa_L(x) \\
&\qquad + \left(g_{iR}(\hat{x}_i, v + k_R) - g_{iR}(\hat{x}_i, u + k_R) \right) \kappa_R(x) \Big) \partial_{x_i} \varphi \\
&+ \int_{I\!R^d} (v_0 - u_0)^\pm \varphi(0, x) dx \geq 0.
\end{aligned}$$

From Theorem 22 and Remark 21 we will deduce the following theorem:

Theorem 23. *Assume that the step function k from* (10) *is such that there exists an interval* $(c, d) \subset I\!R$ *such that for every* $i = 1, \dots, d$ *and every* $x \in I\!R^d$ (48) *holds.*

Then, for any two k-weak entropy process solutions u and v to (8) *with the initial conditions u_0 and v_0, respectively, it holds for any $T>0$ and any ball $B(0,R)\subset I\!R^d$:*

$$\int_0^1 d\lambda\int_0^1 d\eta\int_0^T\int_{B(0,R)}(u(t,x,\lambda)-v(t,x,\eta))^{\pm}dxdt\leq T\int_{B(0,R+CT)}(u_0(x)-v_0(x))^{\pm}dx, \tag{51}$$

for a constant $C>0$ independent on $T,R>0$.

Proof: On the first step, denote by $\tilde\Gamma=\cup_{p,q=1}^{n}(\bar\Omega_p\cap\Omega_q\cap\Gamma)$ and notice that $codim\tilde\Gamma\geq 2$.

Then, notice that any test function $\varphi\in C^1_0(I\!R\times(I\!R^d\backslash\tilde\Gamma))$ can be written as a sum

$$\varphi=\varphi_L+\varphi_R+\sum_{j=1}^{n}\varphi_j,$$

where supp$\varphi_L\subset I\!R\times\Omega_L$, supp$\varphi_R\subset I\!R\times\Omega_R$ and $\varphi_j\subset I\!R\times\Omega_j$, $j=1,\dots,n$.

Therefore, from Remark 21 and Theorem 22, we conclude that it holds for every $\varphi\in C^1_0(I\!R\times(I\!R^d\backslash\tilde\Gamma))$:

$$\begin{aligned}&\int_0^1 d\lambda\int_{I\!R^+\times I\!R^d}(v-u)^{\pm}\partial_t\varphi dtdx\\&+\int_0^1 d\lambda\int_{I\!R^+\times I\!R^d}\sum_{i=1}^{d}\mathrm{sgn}_{\pm}(v-u)\times\\&\qquad\times\Big((g_{iL}(\hat x_i,v+k_L)-g_{iL}(\hat x_i,u+k_L))\,\kappa_L(x)\\&\qquad\;+(g_{iR}(\hat x_i,v+k_R)-g_{iR}(\hat x_i,u+k_R))\,\kappa_R(x)\Big)\partial_{x_i}\varphi\\&+\int_{I\!R^d}(v_0-u_0)^{\pm}\varphi(0,x)dx\geq 0.\end{aligned} \tag{52}$$

Now, denote by $\tilde\Gamma_\varepsilon$ an ε-neighborhood of the set $\tilde\Gamma$. Let $\omega_\varepsilon\in C^1(I\!R^d)$ be such that

$$\omega_\varepsilon(x)=\begin{cases}1, & x\notin\Gamma_{2\varepsilon}\\ 0, & x\in\Gamma_\varepsilon.\end{cases}$$

Notice that

$$\begin{aligned}&|\partial_{x_i}\omega_\varepsilon|\leq\frac{C}{\varepsilon}\\&\mathrm{meas}(\mathrm{supp}(\partial_{x_i}\omega_\varepsilon))\leq\tilde C\varepsilon^2,\end{aligned} \tag{53}$$

for some constants C and $\tilde C$, since $codim(\mathrm{supp}(\partial_{x_i}\omega_\varepsilon))\geq 2$.

Then, take an arbitrary $\varphi \in C_0^1(\mathbb{R} \times \mathbb{R}^d)$ and put in (52) $\varphi\omega_\varepsilon$. We conclude from (53):

$$
\begin{aligned}
&\int_0^1 d\lambda \int_{\mathbb{R}^+ \times \mathbb{R}^d} (v-u)^{\pm} \partial_t \varphi dt dx \\
&+ \int_0^1 d\lambda \int_{\mathbb{R}^+ \times \mathbb{R}^d} \sum_{i=1}^{d} \operatorname{sgn}_{\pm}(v-u) \times \\
&\qquad \times \Big(\left(g_{iL}(\hat{x_i}, v+k_L) - g_{iL}(\hat{x_i}, u+k_L) \right) \kappa_L(x) \\
&\qquad\quad + \left(g_{iR}(\hat{x_i}, v+k_R) - g_{iR}(\hat{x_i}, u+k_R) \right) \kappa_R(x) \Big) \partial_{x_i} \varphi \\
&+ \int_{\mathbb{R}^d} (v_0 - u_0)^{\pm} \varphi(0,x) dx \geq \mathcal{O}(\varepsilon).
\end{aligned}
$$

Letting $\varepsilon \to 0$ here, and then using standard procedure [20], we arrive at (51).

❑

As in the special case, simple corollary of the last theorem is existence and uniqueness to the k-admissible solution to (8), (2).

Corollary 24. *Assume that the function k from* (41) *is such that* (48) *is satisfied. Then, there exists a unique k-entropy weak solution to* (8), (2).

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