VANISHING VISCOSITY WITH SHORT WAVE LONG WAVE INTERACTIONS FOR SYSTEMS OF CONSERVATION LAWS

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Abstract. Motivated by Benney's general theory, we propose new models for short wave long wave interactions when the long waves are described by nonlinear systems of conservation laws. We prove the strong convergence of the solutions of the vanishing viscosity and short wave long wave interactions systems by using compactness results from the compensated compactness theory and new energy estimates obtained for the coupled systems. We analyse several of the representative examples such as scalar conservation laws, general symmetric systems, nonlinear elasticity, nonlinear electromagnetism.

1. Introduction

Motivated by Benney [3], we introduce here several examples of systems of conservation laws coupled with a semilinear Schrödinger equation modelling short wave long wave (SW-LW) interactions for a broad class of physical problems. Several systems motivated by Benney's general theory for SW-LW interactions have appeared in the literature. The case where long waves are described by linear equations was considered in, e.g., [24, 25]. The case where long waves are described by the KdV and other dispersive equations was considered in, e.g., [1, 2].

Here we formulate a general framework for SW-LW interactions when long waves are described by a nonlinear system of conservation laws. In order to upgrade Benney's theory to allow interactions involving nonlinear systems we generalize the scope of that theory and for each particular class of systems we propose the corresponding nonlinear SW-LW interaction coupling based on its own characteristic structure.

We now briefly describe the models of SW-LW interactions considered in this paper, which cover most of the well known examples of systems of conservation laws. For all these models we study the convergence of the vanishing viscosity method. In general, the models present a viscosity parameter ε and an *interaction parameter* α. In the first three applications we will be concerned with the convergence of the sequence of solutions when the viscosity ε goes to 0, while the interaction parameter α is kept fixed. The limit of the vanishing viscosity solutions is a weak solution of the Cauchy problem for a system of Benney's type presenting a nonlinear coupling between the Schrödinger equation and the respective nonlinear hyperbolic system of conservation laws.

In the last three examples we analise the convergence of the solutions when both viscosity and interaction parameters, ε, α, go to zero with $\alpha = O(\varepsilon^{1/2})$. In particular, for these examples, in the limit $\varepsilon = 0$ the Schrödinger equation decouples from the respective nonlinear system of conservation laws and the limit of the vanishing viscosity solutions is a weak solution of the decoupled system. This decoupling is in fact a more realistic situation which is in accordance with the fact that shock wave is a macroscopic phenomenon while Schrödinger equation describes a microcopic one.

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1.1. **Scalar conservation laws.** To warm up, we begin with the simplest case, that is, scalar conservation laws. In this case our general approach proposes the following system:

$$iu_t + u_{xx} = u|u|^2 + \alpha g(v)u, \tag{1.1}$$

$$v_t + f(v)_x = \alpha(g'(v)|u|^2)_x + \varepsilon v_{xx}. \tag{1.2}$$

The new ingredient here, comparing to the original Benney's formulation, is the appearance of a general function $g(v)$ (*interaction function*) in the Schrödinger equation and the corresponding derivative $g'(v)$ in the scalar conservation law, instead of v and 1 prescribed by that formulation. If one chooses $g(v)$ such that $g'(v)$ has a suitable support, the new system preserve physically significant domains. For instance, if v describes a density or a concentration, setting $g(v) = v$ may lead to solutions with no physical meaning taking negative values. In such cases, it is natural to replace v by a suitable function, say, a smooth approximation of $(v)_+$, such that $\operatorname{supp} g' \subseteq [0,\infty)$, which we easily prove to be sufficient to guarantee the nonnegativity of the solution. Thus, choosing $g(v)$ that preserves physically relevant domains we are also able to easily prove the strong convergence of the vanishing viscosity solutions as $\varepsilon \to 0$, for any fixed $\alpha > 0$, independent of ε.

We remark that the convergence of the vanishing viscosity solutions of the Cauchy problem for (1.1),(1.2) for $g(v) = v$, in the particular case $f(v) = a\,v^2 - b\,v^3$, with $b > 0$, was proved in [8], while the existence of local in time smooth solution of the Cauchy problem for (1.1),(1.2), with $\varepsilon = 0$ (hyperbolic case), $g(v) = v$, and a general flux function f was obtained in [9]. In both of these papers α is taken $= 1$ for simplicity.

1.2. **A degenerate symmetric system.** As a bridge from the scalar case and more difficult nonlinear systems of conservation laws, we consider the degenerate symmetric system introduced by Keyfitz and Kranzer [14]. As is well known, this system, in a vector dependent variable v, is basically the combination of a scalar conservation law for $r = |v|$ with transport equations for the components of v/r. For this system we propose the following model of SW-LW interactions

$$iu_t + u_{xx} = u|u|^2 + \alpha g(r)u, \tag{1.3}$$

$$v_t + (\varphi(r)v)_x = \alpha(|u|^2 g'(r)\frac{v}{r})_x + \varepsilon(r_x\frac{v}{r})_x, \qquad r = |v|. \tag{1.4}$$

Here also, since the interaction function depends only on r, our approach allows the invariance of the physically meaningful region $r \geq 0$. Based on the just mentioned result for scalar conservation laws we easily obtain the convergence of the vanishing viscosity solutions as $\varepsilon \to 0$ for any fixed $\alpha > 0$ independent of ε.

1.3. **The p-system of the nonlinear elasticity (I).** As an example involving an authentic nonlinear system of conservation laws, we consider one of the the most representative of all 2×2 systems, namely, that modeling nonlinear elasticity. For this system we propose the following for SW-LW interactions

$$iu_t + u_{xx} = u|u|^2 + \alpha g(v)u, \tag{1.5}$$

$$w_t - \sigma(v)_x = \alpha(g'(v)|u|^2)_x + \varepsilon w_{xx}, \tag{1.6}$$

$$v_t - w_x = 0. \tag{1.7}$$

As is well known, in this case no positively invariant region in the plane is available for the corresponding (Navier-Stokes with nonsingular pressure) system (1.6)-(1.7), when the SW-LW interaction term $\alpha(g'(v)|u|^2)_x$ is omited. In this way, there is a larger flexibility in the choice of the function $g(v)$. In particular, we may choose $g(v) = v$. The lack of bounded invariant domains was surmounted by Serre and Shearer in [22] through an analysis based on L^p estimates with $p < \infty$ and compensated compactness (see also [10]). In this paper, we show how to get the necessary L^p bounds so as to allow the application of the compactness result in [22]. In this way, for the special choice $g(v) = v$, we obtain the strong convergence of the solutions with vanishing viscosity as $\varepsilon \to 0$, for any fixed $\alpha > 0$ independent of ε.

1.4. **The p-system of the nonlinear elasticity (II).** We also consider the case where $g(v)$ is a general nonlinear interaction function subjected to suitable nonlinearity restrictions. For such general interaction functions we address the question of the convergence of the solutions of the Cauchy problem for (1.5),(1.6),(1.7) when ε and α tend to 0. Under suitable nonlinearity assumptions for σ and g, we prove the strong convergence of these solutions when $\varepsilon \to 0$ and $\alpha = O(\varepsilon^{1/2})$.

1.5. **Nonlinear electromagnetism.** As a second example of vanishing viscosity and interaction parameters involving an authentic nonlinear 2×2 system, we consider the equations from nonlinear eletromagnetism first thoroughly mathematically analyzed in [21]. In this case, the original 6×6 system reduces to a 4×4 system in the vector dependent variable $v = (v_1, v_2, v_3, v_4)$, combining a nonlinear 2×2 system of conservation laws for $p = \sqrt{v_1^2 + v_2^2}$ and $q = \sqrt{v_3^2 + v_4^2}$ and transport equations for $a = v_1/p$ and $b = v_3/q$. We propose the following model for SW-LW interactions in this case

$$iu_t + u_{xx} = u|u|^2 + \alpha g(r)u, \qquad r = |v| = \sqrt{v_1^2 + v_2^2 + v_3^2 + v_4^2} \tag{1.8}$$

$$p_t + (\varphi(r)p)_x = -\alpha(g'(r)\frac{p}{r}|u|^2)_x + \varepsilon p_{xx}, \qquad a_t + (\varphi(r) + \alpha\frac{g'(r)}{r}|u|^2 - \varepsilon p_x)a_x = 0, \quad p = \sqrt{v_1^2 + v_2^2} \tag{1.9}$$

$$q_t - (\varphi(r)q)_x = \alpha(g'(r)\frac{q}{r}|u|^2)_x + \varepsilon q_{xx}, \qquad b_t - (\varphi(r) + \alpha\frac{g'(r)}{r}|u|^2 + \varepsilon q_x)b_x = 0, \quad q = \sqrt{v_3^2 + v_4^2} \tag{1.10}$$

We choose $g(r)$ with compact support in the interior of the physically relevant region $r \geq 0$ so as to preserve invariant domains in this region which contain the support of $g(r)$. In this way, our a priori estimates plus the compactness result in [21] allow us to obtain the convergence of the solutions of the Cauchy problem for (1.8),(1.9),(1.10) as $\varepsilon \to 0$ and $\alpha = O(\varepsilon^{1/2})$.

1.6. **General symmetric systems.** For general symmetric systems of conservation laws, that is, when the vector flux-function is the gradient of a real-valued function $\varphi(v)$, of a vector variable $v \in \mathbb{R}^n$, we propose the following model for SW-LW interactions

$$iu_t + u_{xx} = u|u|^2 + \alpha g(v)u, \tag{1.11}$$

$$v_t + (\nabla_v\varphi(v))_x = \alpha(\nabla_v g(v)|u|^2)_x + \varepsilon v_{xx}. \tag{1.12}$$

As concrete examples, we consider the quadratic systems thoroughly analyzed in [5, 6] (see also, e.g., [11, 12] for other examples). For systems possessing bounded invariant domains, as the quadratic systems analized in [5, 6], we may suitably choose $g(v)$ with support contained in the interior of the invariant domain, so that the compactness result obtained in those papers together with our a priori estimates lead to the strong convergence of the solutions of the Cauchy problem for (1.11),(1.12) as $\varepsilon \to 0$ and $\alpha = O(\varepsilon^{1/2})$.

We remark that the case of the p-system of isentropic gas dynamics will be addressed elsewhere since it involves a longer discussion concerning the local uniform boundedness of the solutions of the vanishing viscosity and interaction coefficient systems. For a basic account on the theory of conservation laws we refer to the recent books [4, 7, 13, 15, 20]. The facts about the Schrödinger equation which will be used here belong to the very basic theory that can be found in general books of PDE such as, e.g., [19].

1.7. **Containts of the remaining sections.** The remaining of this paper is organized according to the just made exposition. So, in Section 2 we briefly consider the simple case of scalar conservation laws. We prove the convergence of the vanishing viscosity solutions with a fixed interaction parameter α, sufficiently small. In Section 3 we complement the study of the scalar case with the consideration of the degenerate symmetric system of 1.2. In Section 4 we consider SW-LW interactions for the p-system of the nonlinear elasticity with the special interaction function $g(v) = v$. Also here we prove the convergence of the vanishing viscosity solutions with a fixed interaction parameter $\alpha = 1$. In the remaining three sections we consider the convergence of the solutions of the Cauchy problem for the SW-LW interactions for nonlinear systems of conservation laws with vanishing viscosity and interaction parameters. So, in Section 5 we consider again the SW-LW interactions for the p-system of nonlinear elasticity, now with a general interaction function $g(v)$. We prove the convergence of the solutions when $\varepsilon \to 0$ and $\alpha = O(\varepsilon^{1/2})$. In Section 6, we address the corresponding discussion for the system from nonlinear electromagnetism of 1.5. The general symmetric system and the particular example of the quadratic systems are considered in Section 7.

2. SW-LW interactions with scalar conservation laws

We consider the system (1.1),(1.2), which we repeat here for convenience

$$iu_t + u_{xx} = |u|^2 u + \alpha g(v)\, u \tag{2.1}$$

$$v_t + f(v)_x = \alpha (g'(v)\,|u|^2)_x + \varepsilon v_{xx}, \tag{2.2}$$

$x \in \mathbb{R}$, $t > 0$. Here and henceforth the dependent variable $u(x,t)$ will always assume complex values, that is $u(x,t) \in \mathbb{C}$, while all other dependent variables will assume real values. In particular, $v(x,t) \in \mathbb{R}$.

Remark 2.1. We will not distinguish the notation between the complex valued and the corresponding real valued function spaces of the dependent variables, with the agreement that except for u all others that appear in this paper (v, r, w, p, q, etc.) will be real valued (either scalar or vector, which will be clear in each context).

On the onset, we assume that $f, g \in C^3(\mathbb{R})$ are real functions such that $f(0) = 0$ and g' has compact support. Besides, to allow the application of the compactness result in [23], based on the compensated compactness theory of Tartar and Murat [23, 18], we impose the following nonlinearity condition:

$$\forall \kappa > 0, \text{ the set } \{s \,:\, f''(s) - \kappa g'''(s) \neq 0\} \text{ is dense in } \mathbb{R}. \tag{2.3}$$

We will study the Cauchy problem formed by (2.1),(2.2) and the initial data

$$(u(x,0), v(x,0)) = (u_0(x), v_0(x)), \qquad \text{with } u_0,\ v_0 \in H^1(\mathbb{R}). \tag{2.4}$$

Remark 2.2. For simplicity of notation, throughout this paper we will always take the initial data independent of ε. However, as it will become clear from the proofs, all the convergence results remain valid with essencially the same proof if the initial data relative to the conservation laws with viscosity and interaction terms (i.e., v_0, in the present case) were taken depending on ε, uniformily bounded in $L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$, and converging in $L^1_{\text{loc}}(\mathbb{R})$ to a function in $L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$. In particular, for the present section, we could assume $v(x,0) = v_0^\varepsilon(x)$, $v_0^\varepsilon \in H^1(\mathbb{R})$, $\|v_0^\varepsilon\|_{L^2(\mathbb{R})} + \|v_0^\varepsilon\|_{L^\infty(\mathbb{R})} \le M$, for some $M > 0$ independent of ε, and $v_0^\varepsilon \to v_0$ in $L^1_{\text{loc}}(\mathbb{R})$, with $v_0 \in L^\infty(\mathbb{R}) \cap L^2(\mathbb{R})$.

Existence of a local in time smooth solution for the Cauchy problem (2.1),(2.2),(2.4) is obtained in a standard way using Duhamel's principle and Banach fixed point theorem (see, e.g., [8]). We briefly sketch the argument as follows. For $R, T > 0$, let

$$B_R^T := \{ w \in X_T := C([0,T]; H^1(\mathbb{R})) \,:\, \|w\|_{L^\infty(0,T;H^1(\mathbb{R}))} \le R \}.$$

We take $R > \max\{\|u_0\|_{H^1(\mathbb{R})},\ \|v_0\|_{H^1(\mathbb{R})}\}$. Define the map $\Phi : B_R^T \times B_R^T \to X_T \times X_T$ which associates to each pair $(\tilde u, \tilde v) \in B_R^T \times B_R^T$ the solution (u, v) of the Cauchy problem

$$\begin{cases} iu_t + u_{xx} = |\tilde u|^2 \tilde u + \alpha g(\tilde v)\, \tilde u, \\ v_t - \varepsilon v_{xx} = -f(\tilde v)_x + \alpha (g'(\tilde v)\,|\tilde u|^2)_x, \\ u(0) = u_0, \qquad v(0) = v_0. \end{cases} \tag{2.5}$$

Hence, by Duhamel's principle, we have

$$u(t) = U(t)u_0 + \int_0^t U(t-s)\left[|\tilde u|^2\tilde u + \alpha g(\tilde v)\,\tilde u\right](s)\,ds,$$
$$v(t) = S(t)v_0 - \int_0^t S(t-s)\left[f(\tilde v)_x + \alpha(g'(\tilde v)\,|\tilde u|^2)_x\right](s)\,ds,$$

where $U(t): L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is the continuous unitary propagator associated with the Schrödinger equation and $S(t): L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is the continuous (contractive) semigroup associated with the heat equation. Now, as it is well known, $U(t)\left(H^1(\mathbb{R})\right) = H^1(\mathbb{R})$, while $S(t)\left(L^2(\mathbb{R})\right) \subseteq H^1(\mathbb{R})$ for all $t>0$. Since, using the well known inclusion $H^1(\mathbb{R}) \subseteq L^\infty(\mathbb{R})$, we have

$$\left[|\tilde u|^2\tilde u + \alpha g(\tilde v)\,\tilde u\right] \in X_T,$$

and

$$\left[f(\tilde v)_x + \alpha(g'(\tilde v)\,|\tilde u|^2)_x\right] \in C([0,T];L^2(\mathbb{R})),$$

if $(\tilde u, \tilde v) \in X_T \times X_T$, we deduce that for $T>0$ sufficiently small Φ is a (strict) contraction over the complete metric space $B_R^T \times B_R^T$. Therefore, we can apply Banach fixed point theorem to obtain a unique local in time solution (u,v) of the Cauchy problem (2.1),(2.2),(2.4).

To be able to extend the local in time solution, whose existence and uniqueness we have just reviewed, we need some a priori estimates which will be also fundamental for the proof of the convergence of the vanishing viscosity sequence, to be established subsequently, since they will not depend on ε.

These a priori estimate will guarantee, in particular, that for any $T>0$, if $(u(t),v(t)) \in C([0,T);H^1(\mathbb{R}))$ is a solution of (2.1),(2.2),(2.4), then $u(x,t), v(x,t)$ satisfy

$$iu_t + u_{xx} = h_1, \tag{2.6}$$
$$v_t - \varepsilon v_{xx} = h_2, \tag{2.7}$$

with $h_1 \in L^2([0,T];H^1(\mathbb{R}))$ and $h_2 \in L^2([0,T];L^2(\mathbb{R}))$. Therefore, applying Duhamel's formula to the equation for u_x obtained from (2.6) by taking the x derivative, we obtain $u \in C([0,T];H^1(\mathbb{R}))$. On the other hand, from a well known result concerning equation (2.7) it follows that $v \in L^2([0,T];H^2(\mathbb{R}))$ and $v_t \in L^2([0,T];L^2(\mathbb{R}))$, which in turn, also by a well known interpolation argument implies $v \in C([0,T];H^1(\mathbb{R}))$ (see, e.g., [17]). We then conclude that $u(t), v(t)$ may be extended to functions in $C([0,\infty);H^1(\mathbb{R}))$ which uniquely solve the Cauchy problem (2.1),(2.2),(2.4) in $C([0,\infty);H^1(\mathbb{R}))^2$.

Let us denote by $\|\cdot\|_p$ the norm in $L^p(\mathbb{R})$, $1 \le p \le \infty$ and agree that $\|\cdot\|_\infty$ will also be used as the sup norm of functions of the dependent variables. In what follows c will denote any constant depending only on the data of the problem; its value may change from one occurrence to the next.

The first a priori estimate is an uniform bound for $\|v\|_\infty$ which is obtained from a sort of maximum principle.

Lemma 2.1. *If $(u,v) \in C([0,T];H^1(\mathbb{R}))$ is a solution of (2.1),(2.2),(2.4), then*

$$\|v(t)\|_\infty \le M, \qquad \text{for all } t \in [0,T] \tag{2.8}$$

for some $M>0$ independent of ε and t.

Proof. 1. Let $M>0$ be such that $[-M,M]$ is a closed interval whose interior $(-M,M)$ contains the support of g' and the image of v_0.

2. First, it suffices to prove (2.8) for the smooth solutions of the Cauchy problem obtained from (2.1),(2.2),(2.4) by replacing (2.2) by

$$v_t + f(v)_x = \alpha(g'(v)\,|u|^2)_x + \nu h(v) + \varepsilon v_{xx}, \tag{2.9}$$

for $\nu \in (0,1)$, for $h \in C^3(\mathbb{R})$ such that $h(-M)>0$, $h(0)=0$, $h(M)<0$ (say, $h(v)=-v$), and (2.4) by

$$(u(x,0),v(x,0)) = (u_0^\nu(x), v_0^\nu(x)), \qquad (u_0^\nu, v_0^\nu) \in C^1([0,T],H^3(\mathbb{R})),\ v_0^\nu(x) \in (-M,M),\ x \in \mathbb{R}. \tag{2.10}$$

3. Indeed, an unique local in time solution for (2.1),(2.9),(2.10), $(u^\nu, v^\nu) \in C([0,T'];H^1(\mathbb{R}))$, for some $T' > 0$ independent of ν, is obtained by using a fixed point argument totally similar to the one described above for obtaining the local in time solution of (2.1),(2.2),(2.4). Now, using the fact that $H^k(\mathbb{R})$ is an algebra, $U(t)\left(H^k(\mathbb{R})\right) = H^k(\mathbb{R})$ and $S(t)\left(H^{k-1}(\mathbb{R})\right) \subseteq H^k(\mathbb{R})$, with $k = 3$, we conclude that $(u^\nu, v^\nu) \in C([0,T'];H^3(\mathbb{R})) \cap C^1([0,T'];H^1(\mathbb{R}))$. This is true because we can approach the fixed point in $C([0,T'];H^1(\mathbb{R}))$ by a sequence uniformly bounded in $C([0,T'];H^3(\mathbb{R})) \cap C^1([0,T'];H^1(\mathbb{R}))$. In particular, $(u^\nu, v^\nu) \in C^1([0,T'];C^2([-R,R]))$ for all $R > 0$.

4. We may easily obtain bounds independent of ν for v^ν in $L^2([0,T'];H^2(\mathbb{R}))$, for v^ν_t in $L^2([0,T'];L^2(\mathbb{R}))$, for u^ν in $L^2([0,T'];H^1(\mathbb{R}))$ and for u^ν_t in $L^2([0,T];H^{-1}(\mathbb{R}))$. Hence, when $\nu \to 0$, if $(u^\nu_0, v^\nu_0) \to (u_0, v_0)$ in $H^1(\mathbb{R})$, (u^ν, v^ν) converges in $L^2([0,T];L^2(\mathbb{R}))$ to a pair $(u_*, v_*) \in L^2([0,T];H^1(\mathbb{R}))$ which satisfies the Duhamel's formulas

$$u(t) = U(t)u_0 + \int_0^t U(t-s)\left[|u|^2 u + \alpha g(v)\, u\right](s)\, ds, \tag{2.11}$$

$$v(t) = S(t)v_0 - \int_0^t S(t-s)\left[f(v)_x + \alpha(g'(v)\,|u|^2)_x\right](s)\, ds. \tag{2.12}$$

We then get that $(u_*, v_*) \in C([0,T'];H^1(\mathbb{R}))$ and, thus, it coincides with $(u(t), v(t))$ by the uniqueness of the local in time solution of (2.1),(2.2),(2.4) in $C([0,T'];H^1(\mathbb{R}))$. If $\|v^\nu\|_\infty \le M$ for $0 \le t \le T'$, for all $\nu \in (0,1)$, it follows that $\|v(t)\|_\infty \le M$, $0 \le t \le T'$.

5. We prove that $v^\nu(x,t) \in [-M, M]$, for all $(x,t) \in \mathbb{R} \times [0,T']$ and $\nu \in (0,1)$. Indeed, since the support of g' is contained in $(-M, M)$ and $v^\nu_0(x) \in (-M, M)$, $x \in \mathbb{R}$, the assumption that v^ν assumes values not belonging to $[-M, M]$ implies that there is a smallest time t_*, with $0 < t_* < T'$, such that $v(x_*, t_*) \in \{-M, M\}$, for some $x_* \in \mathbb{R}$. This leads easily to a contradiction by standard maximum principle arguments.

6. By choosing T' depending only on $\sup_{t\in[0,T]} \|(u(t), v(t)\|_{H^1(\mathbb{R})}$ and since $NT' \le T < (N+1)T'$, for some $N \in \mathbb{N} \cup \{0\}$, we may repeat the above argument for the time intervals $[kT', (k+1)T']$, $k = 0, \dots, N$, to obtain $\|v(t)\|_\infty \le M$ for all $t \in [0,T]$ which is (2.8).

□

We now establish three relations which will be decisive for the achievement of our main energy estimates.

Lemma 2.2. *Let* $(u, v) \in \left(C([0,T);H^1(\mathbb{R}))\right)^2$, $T > 0$, *be a solution of the Cauchy problem* (2.1),(2.2),(2.4). *Then, we have*

$$\frac{d}{dt}\int_{\mathbb{R}} |u|^2 dx = 0. \tag{2.13}$$

$$\begin{aligned}
&\frac{d}{dt}\left[\int_{\mathbb{R}} |u_x|^2 dx + \frac{1}{2}\int_{\mathbb{R}} |u|^4 dx + \alpha\int_{\mathbb{R}} g(v)\,|u|^2 dx - \int_{\mathbb{R}} F(v) dx\right] \\
&\qquad + \varepsilon\alpha\int_{\mathbb{R}} (|u|^2)_x g'(v) v_x dx + \varepsilon\alpha\int_{\mathbb{R}} |u|^2 g''(v)\,(v_x)^2 dx - \varepsilon\int_{\mathbb{R}} f'(v)(v_x)^2 dx = 0 \\
&\qquad\qquad \text{where} \quad F(v) = \int_0^v f(s)\, ds.
\end{aligned} \tag{2.14}$$

$$\frac{d}{dt}\left[\frac{1}{2}\int_{\mathbb{R}} v^2 dx + \operatorname{Im}\int_{\mathbb{R}} u\bar{u}_x dx\right] + \varepsilon\int_{\mathbb{R}} (v_x)^2 dx = 0. \tag{2.15}$$

Proof. 1. By approximating the initial data in $H^1(\mathbb{R})$ by functions in $C_0^\infty(\mathbb{R})$ and using a limit argument as in Lemma 2.1, we may assume that $(u(x,t), v(x,t))$ is as smooth as needed. For simplicity we will make

$\alpha = 1$. Now, equation (2.13) is obtained in a standard way: we multiply (2.1) by $\bar{u}$ to obtain an equation $(E1)$; take the conjugate of $(E1)$ to obtain an equation $(E2)$; make $(E1) - (E2)$ to obtain

$$i(|u|^2)_t + (\bar{u}u_x - u\bar{u}_x)_x = 0, \tag{2.16}$$

which gives (2.13) upon integrating in $\mathbb{R}$.

2. Equation (2.15) is obtained by first multiplying (2.2) by v, integrating in $\mathbb{R}$ and using integration by parts to obtain

$$\frac{d}{dt}\frac{1}{2}\int_{\mathbb{R}} v^2\,dx + \varepsilon \int_{\mathbb{R}} (v_x)^2 = -\int_{\mathbb{R}} g(v)_x |u|^2\,dx. \tag{2.17}$$

On the other hand, multiplying (2.1) by $\bar{u}_x$ to obtain an equation $(E3)$, taking the conjugate of $(E3)$ to obtain an equation $(E4)$ and making $(E3) + (E4)$, after a trivial manipulation, we obtain

$$-(\mathrm{Im}\,[u\bar{u}_x])_t + (|u_x|^2)_x = (\frac{1}{2}|u|^4)_x + (g(v)|u|^2)_x - g(v)_x|u|^2,$$

which upon integration in $\mathbb{R}$ gives

$$\frac{d}{dt}\mathrm{Im}\int_{\mathbb{R}} u\bar{u}_x\,dx = \int_{\mathbb{R}} g(v)_x|u|^2\,dx.$$

We then apply the last equation in (2.17) to obtain (2.15) (*cf.* [8]).

3. As to (2.14), we begin by taking the x derivative of (2.1), obtaining an equation $(E5)$, then multiplying $(E5)$ by $\bar{u}_x$ to obtain an equation $(E6)$, then taking the conjugate of $(E6)$ to obtain an equation $(E7)$ and next making $(E6) - (E7)$ obtaining, after some trivial manipulation,

$$i(|u_x|^2)_t + (\bar{u}_x u_{xx} - u_x\bar{u}_{xx})_x = [(\bar{u}_x u - u_x\bar{u})(|u|^2 + g(v))]_x - (|u|^2 + g(v))(\bar{u}_x u - u_x\bar{u})_x,$$

which, by using (2.16), gives

$$i(|u_x|^2)_t + (\bar{u}_x u_{xx} - u_x\bar{u}_{xx})_x = [(\bar{u}_x u - u_x\bar{u})(|u|^2 + g(v))]_x - i(\frac{1}{2}|u|^4)_t - i(g(v)|u|^2)_t + ig(v)_t|u|^2. \tag{2.18}$$

Integrating (2.18) in $\mathbb{R}$ we obtain

$$\frac{d}{dt}\int_{\mathbb{R}} [|u_x|^2 + \frac{1}{2}|u|^4 + g(v)|u|^2]\,dx = \int_{\mathbb{R}} g(v)_t|u|^2\,dx. \tag{2.19}$$

To compute $\int_{\mathbb{R}} g(v)_t|u|^2\,dx$, we multiply (2.2) by $g'(v)|u|^2$, obtaining, after some trivial manipulation,

$$g(v)_t|u|^2 + (|u|^2 g'(v)f(v))_x - (|u|^2 g'(v))_x f(v) + \varepsilon v_x) = (\frac{1}{2}(|u|^2 g'(v))^2)_x + \varepsilon g'(v)|u|^2 v_{xx},$$

which, by using $(g'(v)|u|^2)_x = v_t + f(v)_x - \varepsilon v_{xx}$, gives

$$\begin{aligned} g(v)_t|u|^2 &+ (|u|^2 g'(v)f(v))_x - (v_t + f(v)_x - \varepsilon v_{xx})f(v) \\ &= (\frac{1}{2}(|u|^2 g'(v))^2)_x + \varepsilon(g'(v)|u|^2 v_x)_x - \varepsilon g''(v)(v_x)^2|u|^2 - \varepsilon g'(v)v_x(|u|^2)_x. \end{aligned}$$

Integrating in $\mathbb{R}$, it follows

$$\begin{aligned} \int_{\mathbb{R}} |u|^2 g(v)_t\,dx &= \int_{\mathbb{R}} (v_t + f(v)_x - \varepsilon v_{xx})f(v)\,dx \\ &\quad - \varepsilon\int_{\mathbb{R}} (|u|^2)_x g'(v)v_x\,dx - \varepsilon\int_{\mathbb{R}} |u|^2 g''(v)(v_x)^2\,dx \\ &= \frac{d}{dt}\int_{\mathbb{R}} F(v)\,dx + \varepsilon\int_{\mathbb{R}} (v_x)^2 f'(v)dx \\ &\quad - \varepsilon\int_{\mathbb{R}} (|u|^2)_x g'(v)v_x\,dx - \varepsilon\int_{\mathbb{R}} |u|^2 g''(v)(v_x)^2\,dx, \end{aligned}$$

thus establishing (2.14). $\square$

From (2.13) and (2.15) we deduce

$$\frac{1}{2}\|v(t)\|_2^2 + \varepsilon \int_0^t \|v_x(\tau)\|_2^2\, d\tau \le c\,(1 + \|u_x(t)\|_2)\,, \tag{2.20}$$

with c independent of ε and T. Thus, by (2.8), (2.14) and (2.20) it follows

$$\|u_x(t)\|_2^2 \le c + c\,\varepsilon\alpha \int_0^t \int_{\mathbb{R}} |u|\,|u_x|\,|g'(v)|\,|v_x|\,dx\,d\tau + c\,\varepsilon\alpha \int_0^t \int_{\mathbb{R}} |u|^2 |g''(v)|(v_x)^2\,dx\,d\tau. \tag{2.21}$$

Lemma 2.3. *Let $(u,v) \in \big(C([0,T); H^1(\mathbb{R}))\big)^2$, $T > 0$, be a solution of the Cauchy problem* (2.1),(2.2),(2.4). *Then there exists $\alpha_0 \in (0,1]$ such that , for $0 < \alpha \le \alpha_0$, $0 < \varepsilon \le 1$,*

$$\|u_x(t)\|_2 \le h(t), \tag{2.22}$$

where $h \in C^\infty([0,+\infty))$ is a positive function independent of T and ε.

Proof. Let $q(t) = 1 + \|u_x(t)\|_2^2$. From (2.8), (2.13) and (2.21), using $\|u\|_\infty^2 \le c\,\|u\|_2\|u_x\|_2$, we deduce

$$q(t) \le c + c\,\varepsilon\alpha \int_0^t \|u_x\|_2^{3/2}\|v_x\|_2\,d\tau + c\,\varepsilon\alpha \int_0^t \|u_x\|_2\|v_x\|_2^2\,d\tau.$$

Therefore,

$$q(t) \le \theta(t) := c + c\,\varepsilon\alpha \int_0^t q(\tau)^{3/4}\|v_x(\tau)\|_2\,d\tau + c\,\varepsilon\alpha \int_0^t q(\tau)^{1/2}\|v_x(\tau)\|_2^2\,d\tau. \tag{2.23}$$

We will prove that if α if sufficiently small, then

$$\theta(t) \le c2^{t+1}, \qquad \text{for } t \ge 0, \tag{2.24}$$

where c is the same constant apearing in the definition of θ.

Indeed, from (2.23) we have

$$\begin{aligned}\theta(t) &\le c + c\varepsilon\alpha \sup_{0\le\tau\le t}\theta(\tau)^{3/4} t^{1/2}\left(\int_0^t \|v_x(\tau)\|_2^2\,d\tau\right)^{1/2} + c\,\varepsilon\alpha \sup_{0\le\tau\le t}\theta(\tau)^{1/2}\int_0^t \|v_x(\tau)\|_2^2\,d\tau \\ &\le c + c\varepsilon\alpha(t^{1/2}+1)\sup_{0\le\tau\le t}\theta(\tau),\end{aligned}$$

where we have used (2.20). We then get

$$\sup_{0\le\tau\le t}\theta(\tau) \le c + c\varepsilon\alpha(t^{1/2}+1)\sup_{0\le\tau\le t}\theta(\tau).$$

Let $\alpha_0 > 0$ be given by $\alpha_0 := 1/(4c+1)$. Thus, for $0 < \alpha \le \alpha_0$, if $0 \le t \le 1$, we obtain

$$\theta(t) \le \sup_{0\le\tau\le t}\theta(\tau) \le 2c.$$

Now, we prove by induction that

$$\theta(t) \le c2^n \qquad \text{if } (n-1) < t \le n. \tag{2.25}$$

The assertion is true for $n = 1$ as we have just seen. Assume it holds for $(n-2) < t \le (n-1)$. Let $(n-1) < t \le n$. By the definition of $\theta(t)$, we have

$$\theta(t) = \theta(n-1) + c\,\varepsilon\alpha \int_{(n-1)}^t q(\tau)^{3/4}\|v_x(\tau)\|_2\,d\tau + c\,\varepsilon\alpha \int_{(n-1)}^t q(\tau)^{1/2}\|v_x(\tau)\|_2^2\,d\tau.$$

By the same estimates made in the case $n = 1$, we get

$$\sup_{n-1\le\tau\le t}\theta(\tau) \le \theta(n-1) + c\varepsilon\alpha((t-(n-1))^{1/2}+1)\sup_{n-1\le\tau\le t}\theta(\tau).$$

As before, for $0 < \alpha \le \alpha_0$ we get

$$\theta(t) \le \sup_{n-1\le\tau\le t} \theta(\tau) \le 2\theta(n-1) \le c2^n,$$

by the induction hypothesis, which concludes the proof by induction of (2.25), which in turn gives (2.24). $\square$

Finally, using the estimates obtained in Lemmas 2.1 and 2.3 we next establish the existence and uniqueness of global solutions $(u^\varepsilon, v^\varepsilon)$ of (2.1),(2.2),(2.4), $0 < \varepsilon \le 1$, and their convergence as $\varepsilon \to 0$ to a weak solution of

$$iu_t + u_{xx} = |u|^2 u + \alpha g(v)\, u, \tag{2.26}$$

$$v_t + f(v)_x = \alpha(g'(v)\,|u|^2)_x, \tag{2.27}$$

$$(u(x,0), v(x,0)) = (u_0(x), v_0(x)). \tag{2.28}$$

Before stating the result we introduce the definition of entropy solution for system (2.26),(2.27),(2.28).

Definition 2.1. We say that the pair $(u(x,t), v(x,t)) \in L^\infty_{\rm loc}(\mathbb{R}\times[0,\infty))^2$ is an entropy solution for (2.26),(2.27),(2.28) if:

(i) $u \in L^\infty_{\rm loc}([0,\infty); H^1(\mathbb{R})) \cap C([0,\infty); L^2(\mathbb{R}))$ with $u(0) = u_0$ in $L^2(\mathbb{R})$ and

$$\int_0^t \int_{\mathbb{R}} i\, u\, \varphi_t + u_x\varphi_x + (|u|^2 u + \alpha g(v) u)\varphi \, dx\, dt = 0, \qquad \forall \varphi \in C_0^\infty(\mathbb{R}\times(0,\infty)); \tag{2.29}$$

(ii) for any $\eta \in C^2(\mathbb{R})$ convex and $q_1, q_2 \in C^2(\mathbb{R})$ satisfying $q_1'(v) = \eta'(v) f'(v)$, $q_2'(v) = \eta'(v) g''(v)$ we have

$$\int_0^\infty \int_{\mathbb{R}} \eta(v)\phi_t + (q_1(v) - \alpha|u|^2 q_2(v))\phi_x + \alpha(\eta'(v)g'(v) - q_2(v))(|u|^2)_x\phi \, dx\, dt + \int_{\mathbb{R}} \eta(v_0(x))\phi(x,0)\, dx \ge 0, \tag{2.30}$$

for all non-negative $\phi \in C_0^\infty(\mathbb{R}^2)$.

Theorem 2.1. *For each $0 < \varepsilon \le 1$ and $0 < \alpha \le \alpha_0$, with α_0 given by Lemma 2.2, there exists a unique global solution $(u^\varepsilon, v^\varepsilon) \in C([0,+\infty); H^1(\mathbb{R}))^2$ of the Cauchy problem* (2.1),(2.2),(2.4). *This solution satisfies* (2.8), (2.13), (2.14), (2.15), (2.20) *and* (2.22), *uniformly in ε. Moreover, there exists a subsequence of $(u^\varepsilon, v^\varepsilon)$ converging in $L^1_{loc}(\mathbb{R}\times[0,\infty))^2$ to a pair of functions (u,v), with $u \in C([0,\infty); H^1(\mathbb{R}))$ and $v \in L^\infty(\mathbb{R}\times[0,\infty))$, which is an entropy solution of the problem* (2.26),(2.27),(2.28).

Proof. 1. As we have already explained, the possibility of extending the local in time solution of (2.1),(2.2),(2.4), $(u^\varepsilon, v^\varepsilon) \in C([0,T); H^1(\mathbb{R}))^2$, follows from the fact that the estimates (2.8), (2.20) and (2.22) imply that u^ε and v^ε satisfy (2.6) and (2.7), respectively, with $h_1 \in L^2([0,T]; H^1(\mathbb{R}))$ and $h_2 \in L^2([0,T]; L^2(\mathbb{R}))$. Therefore, we can extend $(u^\varepsilon, v^\varepsilon)$ to a unique global solution of (2.1),(2.2),(2.4) in $C([0,\infty); H^1(\mathbb{R}))^2$.

2. The estimates (2.8), (2.20) and (2.22) imply that u^ε is bounded in $L^\infty([0,\infty); H^1(\mathbb{R}))$ and v^ε is bounded in $L^\infty(\mathbb{R}\times[0,\infty))$ uniformly with respect to ε. A well known compactness lemma by Aubin (see, e.g., [16]) applied to (2.1) then implies the compactness of u^ε in $L^2_{\rm loc}([0,\infty); L^2(\mathbb{R}))$.

3. As to v^ε, the estimates (2.8), (2.20) and (2.22) also imply that for any $\eta \in C^2(\mathbb{R})$, if $q_1, q_2 \in C^2(\mathbb{R})$ are such that $q_1'(v) = \eta'(v) f'(v)$ and $q_2'(v) = \eta'(v) g''(v)$, then we have

$$\frac{\partial}{\partial t}\eta(v^\varepsilon) + \frac{\partial}{\partial x}(q_1(v^\varepsilon) - |u^\varepsilon|^2 q_2(v^\varepsilon)) \in \{\,\text{compact of } W^{-1,2}_{\rm loc}(\mathbb{R}\times(0,\infty))\,\}. \tag{2.31}$$

Indeed, multiplying (2.2) by $\eta'(v)$ we get, after simplification,

$$\frac{\partial}{\partial t}\eta(v) + \frac{\partial}{\partial x}(q_1(v) - |u|^2 q_2(v)) = (\eta'(v)g'(v) - q_2(v))(|u|^2)_x + \varepsilon\frac{\partial^2}{\partial x^2}\eta(v) - \varepsilon\eta''(v)(v_x)^2. \tag{2.32}$$

Now, in the right-hand side of (2.32) the terms $(\eta'(v)g'(v)-q_2(v))(|u|^2)_x-\varepsilon\eta''(v)(v_x)^2$ are uniformly bounded in $L^1_{\rm loc}(\mathbb{R}\times(0,\infty))$, and hence compact in $W^{-1,q}_{\rm loc}(\mathbb{R}\times(0,\infty))$, for $1<q<2$, while the term $\varepsilon\frac{\partial^2}{\partial x^2}\eta(v)$ is clearly compact in $W^{-1,2}_{\rm loc}(\mathbb{R}\times(0,\infty))$. Since the left-hand side of (2.32) is uniformly bounded in $W^{-1,\infty}_{\rm loc}(\mathbb{R}\times(0,\infty))$, a well known interpolation argument in [23] gives (2.31).

4. Now (2.31) and the assumption (2.3) allow us to apply Tartar's method in [23] to achieve the compactness of $\{v^\varepsilon\}_{0<\varepsilon\le1}$ in $L^1_{\rm loc}(\mathbb{R}\times[0,\infty))$, which together with the compactness of $\{u^\varepsilon\}_{0<\varepsilon\le1}$ in the same space implies the existence of a subsequence of $(u^\varepsilon,v^\varepsilon)$ converging to a pair of functions (u,v), with $u\in L^\infty_{\rm loc}([0,\infty);H^1(\mathbb{R}))$, $v\in L^\infty(\mathbb{R}\times[0,\infty))$, as $\varepsilon\to0$.

5. We easily verify that u satisfies Duhamel's formula (2.11) and so $u\in C([0,\infty);L^2(\mathbb{R}))$ and $u(0)=u_0$ in $L^2(\mathbb{R})$. The verification of (2.29) is immediate while (2.30) follows from (2.32) by multiplying this equation by ϕ, using integration by parts, and passing to the limit as $\varepsilon\to0$. We then conclude that (u,v) is an entropy solution of (2.26),(2.27),(2.28). $\square$

3. SW-LW interactions with a degenerate symmetric system

In this section, we briefly consider the convergence of vanishing viscosity solutions of the system (1.3),(1.4) which couples Schrödinger equation with the degenerate symmetric system introduced by Keyfitz and Kranzer [14] with viscosity. For convenience we repeat the system here:

$$iu_t+u_{xx}=u|u|^2+\alpha g(r)u, \tag{3.1}$$

$$v_t+(\varphi(r)v)_x=\alpha(|u|^2g'(r)\frac{v}{r})_x+\varepsilon(r_x\frac{v}{r})_x,\qquad r=|v|. \tag{3.2}$$

The reason for including this system and considering it just after the SW-LW interactions with scalar conservation laws is first to emphasize our criterion for suggesting a SW-LW coupling motivated by the structure of the system governing the long waves, and second because the analysis pratically reduces to the scalar case. Indeed, the system (3.2), for the vector dependent variable $v\in\mathbb{R}^n$, may be split into a scalar conservation law with viscosity for r

$$r_t+(\varphi(r)r)_x=\alpha(|u|^2g'(r))_x+\varepsilon r_{xx}, \tag{3.3}$$

and transport equations for the components of $\varpi:=v/r$

$$r\varpi_t+(r\varphi(r)-\alpha|u|^2g'(r)-\varepsilon r_x)\varpi_x=0. \tag{3.4}$$

We prescribe initial data for the system (3.1),(3.2)

$$(u(x,0),v(x,0))=(u_0(x),v_0(x)), \tag{3.5}$$

and assume that $v_0(x)=r_0(x)\varpi_0(x)$ with $|\varpi_0(x)|=1$, $r_0(x)>0$ for $x\in\mathbb{R}$, and, for certain $r_*>0$, we have

$$u_0\in H^1(\mathbb{R}),\qquad r_0-r_*\in H^2(\mathbb{R})\quad\text{and}\quad \varpi_0\in L^\infty(\mathbb{R}). \tag{3.6}$$

Remark 3.1. As in Remark 2.2 we could replace v_0 by v_0^ε with $v_0^\varepsilon(x)=r_0^\varepsilon(x)\varpi_0(x)$, ask that $r_0^\varepsilon-r_*^\varepsilon\in H^2(\mathbb{R})$, $r_*^\varepsilon>0$, $r_0^\varepsilon(x)>0$ for $x\in\mathbb{R}$, and impose the convergences $r_0^\varepsilon\to r_0$ in $L^1_{\rm loc}(\mathbb{R})$, with $r_0\in L^\infty(\mathbb{R})$, $r_0(x)>\delta_L>0$ for $x\in[-L,L]$, for all $L>0$, and $r_*^\varepsilon\to0$, without changing the conclusions in this section.

As to φ and g we assume $\varphi,g\in C^3([0,\infty))$ and we also impose the following analogue of (2.3)

$$\forall\kappa>0,\text{ the set }\{s\,:\,(r\phi(r))''-\kappa g'''(r)\ne0\}\text{ is dense in }[0,\infty). \tag{3.7}$$

For the sake of reference let us write down separately the initial data for the system formed by (3.1),(3.3):

$$(u(x,0),r(x,0))=(u_0(x),r_0(x))\quad\text{for }x\in\mathbb{R},\ (u_0,r_0)\in H^1(\mathbb{R})\times H^2(\mathbb{R}). \tag{3.8}$$

A global and unique solution for (3.1),(3.3),(3.8), (u,r) with $(u,r-r_*)\in C([0,\infty);H^1(\mathbb{R}))^2$, is obtained by a trivial adaptation of the analysis developed in the last section concerning the problem (2.1),(2.2),(2.4). Here we have an extra regularity for $r(x,t)$, namely $r-r_*\in C([0,\infty);H^2(\mathbb{R}))$ which easily follows from the assumption that $r_0 - r_* \in H^2(\mathbb{R})$. In particular, Lemmas 2.1, 2.2 and 2.3 can easily be adapted to provide analogous results for the solution of the problem (3.1),(3.3),(3.8). More specifically, the arguments in Lemma 2.1 may easily be adapted to give the proof that for certain $r_{**} > 0$, $r(x,t) \in [r_{**}, M]$, where r_{**}, M are such that the support of $g'(r)$ and the values of r_0 are contained in the open interval (r_{**}, M). Also, in Lemma 2.2 relations (2.13) and (2.14) have trivial analogues with v replaced by r and $f(v)$ replaced by $r\varphi(r)$, while in relation (2.15) we just need to replace v by $r - r_*$ to obtain an analogue for the solution of the problem (3.1),(3.3),(3.8).

Next we introduce the definition of renormalized solution of problem (3.1),(3.2),(3.5).

Definition 3.1. The pair $(u, v) \in C([0,\infty); L^1_{\text{loc}}(\mathbb{R}))$ is a renormalized solution of (3.1),(3.2),(3.5) with (3.6) if $(u(0), v(0)) = (u_0, v_0)$ a.e. in $\mathbb{R}$, $u \in C([0,\infty); H^1(\mathbb{R}))$, $v(x,t) = r(x,t)\varpi(x,t)$ where $r(x,t) > 0$ for $(x,t) \in \mathbb{R}\times[0,\infty)$ and $r - r_* \in C([0,\infty); H^2(\mathbb{R}))$, $\varpi \in C([0,\infty); L^\infty(\mathbb{R}))$, $|\varphi(x,t)| = 1$ a.e. in $\mathbb{R}\times[0,\infty)$, and such that (u, r) is the unique solution of (3.1),(3.3),(3.8) in $C([0,\infty); H^1(\mathbb{R}))^2$ and ϖ is a weak solution of (3.4) in the sense that for all $\phi \in C_0^\infty(\mathbb{R}\times(0,\infty))$ we have

$$\int_0^\infty \int_{\mathbb{R}} r\varpi\phi_t + (r\varphi(r) - \alpha|u|^2 g'(r) - \varepsilon r_x)\varpi\phi_x \, dx\, dt = 0. \tag{3.9}$$

Existence of a renormalized solution of (3.1),(3.2),(3.5) with (3.6) is obtained from the unique solution of (3.1),(3.3),(3.8), (u, r) with $u \in C([0,\infty); H^1(\mathbb{R}))$ and $r - r_* \in C([0,\infty); H^2(\mathbb{R}))$, discussed above, in the following standard way. We defined a bi-Lipschitz change of coordinates $(x,t) \mapsto (y,t)$ where $y(x,t)$ is the solution of

$$\begin{cases} \frac{\partial y}{\partial x} = r, \\ \frac{\partial y}{\partial t} = -r\varphi(r) + \alpha|u|^2 g'(r) + \varepsilon r_x, \\ y(x,0) = \int_0^x r_0(z)\, dz. \end{cases}$$

We then define

$$\varpi(x,t) := \varpi_0(y(x,t)). \tag{3.10}$$

It is then easy to verify that $\varpi(x,t)$ so defined satisfies (3.9) for all $\phi \in C_0^\infty(\mathbb{R}\times(0,\infty))$ and also that $\varpi \in C([0,\infty); L^\infty(\mathbb{R}))$ and $|\varphi(x,t)| = 1$ a.e. in $\mathbb{R}\times[0,\infty)$. Hence, defining $v(x,t) := r(x,t)\varpi(x,t)$, we obtain that $(u(x,t), v(x,t))$ is a renormalized solution of (3.1),(3.2),(3.5). Uniqueness of the renormalized solution is easily achieved by using the just mentioned change of coordinates $(x,t) \mapsto (y,t)$ in (3.9) and taking suitable test functions to conclude that (3.10) must hold a.e. in $\mathbb{R}\times[0,\infty)$.

We are about to establish the convergence of a subsequence of the renormalized solutions of (3.1),(3.2),(3.5), $(u^\varepsilon, v^\varepsilon)$, as $\varepsilon \to 0$, to a weak solution of

$$iu_t + u_{xx} = u|u|^2 + \alpha g(r)u, \tag{3.11}$$

$$v_t + (\varphi(r)v)_x = \alpha(|u|^2 g'(r)\frac{v}{r})_x, \qquad r = |v|. \tag{3.12}$$

This weak solution $(u(x,t), v(x,t))$ will be such that $v(x,t) = r(x,t)\varpi(x,t)$ where $r(x,t)$ is an entropy solution of

$$r_t + (\varphi(r)r)_x = \alpha(|u|^2 g'(r))_x, \tag{3.13}$$

and $\varpi(x,t)$ is a weak solution of

$$r\varpi_t + (r\varphi(r) - \alpha|u|^2 g'(r))\varpi_x = 0. \tag{3.14}$$

We make this assertion more precise with the following definition.

Definition 3.2. We say that the pair $(u(x,t), v(x,t)) \in L^\infty_{\text{loc}}(\mathbb{R}\times[0,\infty))^2$ is a renormalized entropy solution for (3.11),(3.12),(3.5) if:

(i) $v(x,t) = r(x,t)\varpi(x,t)$ with $r, \varpi \in L^\infty(\mathbb{R}\times[0,\infty))$, $r(x,t) > 0$ and $|\varpi(x,t)| = 1$ for almost all $(x,t) \in \mathbb{R}\times[0,\infty)$;

(ii) (u,r) is an entropy solution of (3.11),(3.13),(3.8) in the sense of Definition 2.1;

(iii) ϖ is a renormalized solution of (3.14) in the sense that for all $H \in C(\mathbb{R}^n)$ and $\phi \in C_0^\infty(\mathbb{R}\times(0,\infty))$ we have

$$\int_0^\infty \int_{\mathbb{R}} rH(\varpi)\phi_t + (r\varphi(r) - \alpha|u|^2 g'(r))H(\varpi)\phi_x \, dx\, dt = 0. \tag{3.15}$$

Concerning the convergence of the renormalized solutions $(u^\varepsilon, v^\varepsilon)$ of (3.1),(3.2),(3.5) when $\varepsilon \to 0$ we have the following result.

Theorem 3.1. *There exists $\alpha_0 \in (0,1]$ such that for each $0 < \varepsilon \le 1$ and $0 < \alpha \le \alpha_0$, there exists a unique global renormalized solution $(u^\varepsilon, v^\varepsilon)$ of the Cauchy problem (3.1),(3.2),(3.5). Moreover, there exists a subsequence of $(u^\varepsilon, v^\varepsilon)$ converging in $L^1_{loc}(\mathbb{R}\times[0,\infty))^2$ to a pair of functions (u,v), with $u \in C([0,\infty);H^1(\mathbb{R}))$ and $v \in L^\infty(\mathbb{R}\times[0,\infty))$, which is a renormalized entropy solution of the problem (3.11),(3.12),(3.5).*

Proof. The existence of a unique global renormalized solution $(u^\varepsilon, v^\varepsilon)$ of (3.1),(3.2),(3.5) was explained above. For the compactness of $(u^\varepsilon, r^\varepsilon)$ we apply the Theorem 2.1. The fact that ϖ^ε is also compact in $L^1_{\rm loc}(\mathbb{R}\times(0,\infty))$ follows by a well known argument of D. Serre (see, e.g., [20]) which may be explained as follows. Since ϖ^ε is uniformly bounded in $L^\infty(\mathbb{R}\times(0,\infty))$, by the classical result on the existence of Young measures (see, e.g., [23]), there is a subsequence still denoted by ϖ^ε and a Young measure $\mu_{x,t}$ such for all $H \in C(\mathbb{R}^n)$, the sequence $H(\varpi^\varepsilon)$ weak-* converges to $\langle \mu_{x,t}, H(\,\cdot\,)\rangle$. Now, for any $H \in C^2(\mathbb{R}^n)$ and $\phi \in C_0^\infty(\mathbb{R}^2)$, multiplying by $H'(\varpi^\varepsilon)\phi$ (3.4) (with ϖ replaced by ϖ^ε), integrating in $\mathbb{R}\times(0,\infty)$, using integration by parts, and making $\varepsilon \to 0$ we obtain that

$$\int_0^\infty \int_{\mathbb{R}} r\langle \mu_{x,t}, H(\,\cdot\,)\rangle\phi_t + (r\varphi(r) - \alpha g'(r)|u|^2)\langle \mu_{x,t}, H(\,\cdot\,)\rangle\phi_t \, dx\, dt + \int_{\mathbb{R}} r_0(x)H(\varpi_0(x))\phi(x,0)\, dx = 0. \tag{3.16}$$

A standard approximation argument extends this relation to all $H \in C(\mathbb{R}^n)$. We then apply to (3.16) the change of coordinates $(x,t) \mapsto (y,t)$ with y given by

$$\begin{cases} \frac{\partial y}{\partial x} = r, \\ \frac{\partial y}{\partial t} = -r\varphi(r) + \alpha|u|^2 g'(r), \\ y(x,0) = \int_0^x r_0(z)\, dz, \end{cases}$$

to get that $\langle \mu_{y,t}, H(\,\cdot\,)\rangle = H(\varpi_0(y))$ and so we obtain the strong convergence of ϖ^ε to a renormalized solution of (3.14) as defined in Definition 3.2, which concludes the proof.

□

4. SW-LW interactions with the p-system of the nonlinear elasticity (I)

Here we consider the system (1.5)-(1.7) in the special case where $g(v) = v$. Since in this case it will not be needed any restriction on α we just set $\alpha = 1$. So we are going to study the Cauchy problem for the system

$$iu_t + u_{xx} = u|u|^2 + vu, \tag{4.1}$$

$$w_t - \sigma(v)_x = (|u|^2)_x + \varepsilon w_{xx}, \tag{4.2}$$

$$v_t - w_x = 0, \tag{4.3}$$

with initial data

$$(u(x,0), w(x,0), v(x,0)) = (u_0(x), w_0(x), v_0(x)), \qquad x \in \mathbb{R}, \tag{4.4}$$

satisfying

$$u_0,\, w_0,\, v_0 \in H^1(\mathbb{R}) \tag{4.5}$$

Following [22], we define $\Sigma(v) = \int_0^v \sigma(s)ds$ (we may assume $\sigma(0) = 0$) and we impose the same conditions on the stress function $\sigma \in C^3(\mathbb{R})$:

(H1) $\sigma'(v) \geq \sigma_0 > 0$ with σ_0 constant.
(H2) $\sigma''(\lambda_0) = 0$ and $\sigma''(\lambda) \neq 0$ for $\lambda \neq \lambda_0$.
(H3) $\dfrac{\sigma''}{(\sigma')^{5/4}}$, $\dfrac{\sigma'''}{(\sigma')^{7/4}} \in L^2(\mathbb{R})$, $\dfrac{\sigma''}{(\sigma')^{3/2}}$, $\dfrac{\sigma'''}{(\sigma')^2} \in L^\infty(\mathbb{R})$.
(H4) $\dfrac{\sigma(v)}{\Sigma(v)} \longrightarrow 0$ as $|v| \to \infty$ and there exists constants $c > 0$ and $q > 1/2$ such that $(\sigma'(v))^q \leq c(1+\Sigma(v))$.

In particular we have $\Sigma(v) \geq (\sigma_0/2)v^2$. To simplify we also suppose that $v \in H^1(\mathbb{R})$ implies $\int_{\mathbb{R}} \Sigma(v)dx < +\infty$. A typical example is given by $\sigma(v) = v^3 + v$.

Since the questions of global existence, uniqueness and corresponding a priori estimates for solutions of the problem (4.1)-(4.4) are addressed in the same way for general interaction functions $g(v)$ we will establish these results in the more general case of system (1.5)-(1.7), which we repeat now for convenience

$$iu_t + u_{xx} = u|u|^2 + \alpha g(v)u, \tag{4.6}$$
$$w_t - \sigma(v)_x = \alpha(g'(v)|u|^2)_x + \varepsilon w_{xx}, \tag{4.7}$$
$$v_t - w_x = 0. \tag{4.8}$$

and assume $g \in C^3(\mathbb{R})$ with g', g'' uniformly bounded in $\mathbb{R}$.

We start by proving the following lemma, analogue of Lemma 2.2, for local in time smooth solutions of (4.6),(4.7),(4.8),(4.4).

Lemma 4.1. *Let $(u, v, w) \in C^1([0, T); H^1(\mathbb{R}))^3$, $T > 0$, be a solution to the Cauchy problem* (4.6),(4.7),(4.8),(4.4) *in the interval $[0, T)$. We have*

$$\frac{d}{dt}\int_{\mathbb{R}} |u|^2 dx = 0 \tag{4.9}$$

$$\begin{aligned}\frac{d}{dt}\Big[\int_{\mathbb{R}} |u_x|^2 dx + \frac{1}{2}\int_{\mathbb{R}} |u|^4 dx + \alpha\int_{\mathbb{R}} g(v)\,|u|^2 dx \\ + \frac{1}{2}\int_{\mathbb{R}} w^2 dx + \int_{\mathbb{R}} \Sigma(v)dx\Big] + \varepsilon\int_{\mathbb{R}} (w_x)^2 dx = 0\end{aligned} \tag{4.10}$$

Besides, if $\alpha\|g''\|_\infty$ is smaller than certain constant $\alpha_0 > 0$ depending only on the data (which is always true when $g(v) = v$) we have

$$\frac{\varepsilon\sigma_0}{2}\int_0^t \int_{\mathbb{R}} (v_x)^2 dx\, d\tau + \varepsilon^2 \int_{\mathbb{R}} (v_x)^2 dx \leq c_0(1 + t), \tag{4.11}$$

for $\varepsilon \leq 1$, where $c_0 > 0$ is a constant independent of ε and T.

Proof. 1. The proof of (4.9) is identical as the corresponding relation in Lemma 2.2.

2. Concerning (4.7), as for the corresponding relation in Lemma 2.2, it is enough to compute $\int_{\mathbb{R}} |u|^2 g(v)_t\, dx$, assuming without loss of generality $\alpha = 1$ and enough regularity for the solution. Now, to do that we multiply (4.8) by $|u|^2 g'(v)$ and use (4.7) to obtain

$$|u|^2 g(v)_t = (|u|^2 g'(v)w)_x - (|u|^2 g'(v))_x w = (|u|^2 g'(v)w)_x - (w_t - \sigma(v)_x - \varepsilon w_{xx})w.$$

Integrating in $\mathbb{R}$ and using integration by parts we then get

$$\begin{aligned}\int_{\mathbb{R}} |u|^2 g(v)_t\,dx &= -\int_{\mathbb{R}}(w_t-\sigma(v)_x)w\,dx+\varepsilon\int_{\mathbb{R}} ww_{xx}\,dx\\ &=-\frac12\frac{d}{dt}\int_{\mathbb{R}} w^2\,dx-\int_{\mathbb{R}}\sigma(v)v_t\,dx-\varepsilon\int_{\mathbb{R}}(w_x)^2\,dx\\ &=-\frac12\frac{d}{dt}\int_{\mathbb{R}} w^2\,dx-\frac{d}{dt}\int_{\mathbb{R}}\Sigma(v)\,dx-\varepsilon\int_{\mathbb{R}}(w_x)^2\,dx.\end{aligned}$$

3. To prove (4.8) we follow closely the proof of Lemma 1, (ii) in [22]. We multiply (4.7) by v_x, integrate the resulting equation in $\mathbb{R}$, use (4.8) and integration by parts, to obtain

$$\int_{\mathbb{R}}(w_tv_x-\sigma'(v)(v_x)^2)\,dx=\alpha\int_{\mathbb{R}}(g'(v)|u|^2)_xv_x\,dx+\frac{\varepsilon}{2}\int_{\mathbb{R}}\frac{\partial}{\partial t}(v_x)^2dx.$$

Now, observing that $w_tv_x=(wv_x)_t-ww_{xx}$ we obtain, after integration by parts,

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}}(w_xv)\,dx+\int_{\mathbb{R}}(w_x)^2dx-\int_{\mathbb{R}}\sigma'(v)(v_x)^2dx\\ &\qquad=\alpha\int_{\mathbb{R}}(g'(v)|u|^2)_xv_x\,dx+\frac{\varepsilon}{2}\frac{d}{dt}\int_{\mathbb{R}}(v_x)^2dx.\end{aligned}$$

Integrating in $[0,t]$ we get

$$\begin{aligned}&\int_{\mathbb{R}}(w_xv)\,dx-\int_{\mathbb{R}}w_{0x}v_0dx+\int_0^t\int_{\mathbb{R}}(w_x)^2dx\,d\tau-\int_0^t\int_{\mathbb{R}}\sigma'(v)(v_x)^2dx\,d\tau\\ &\qquad=\alpha\int_0^t\int_{\mathbb{R}}(g'(v)|u|^2)_xv_x\,dx\,d\tau+\frac{\varepsilon}{2}\int_{\mathbb{R}}(v_x)^2dx-\frac{\varepsilon}{2}\int_{\mathbb{R}}(v_{0x})^2dx,\end{aligned}$$

and then

(4.12)

$$\begin{aligned}&\int_0^t\int_{\mathbb{R}}\sigma'(v)(v_x)^2dx\,d\tau+\frac{\varepsilon}{2}\int_{\mathbb{R}}(v_x)^2dx\ \le\ \frac{\varepsilon}{4}\int_{\mathbb{R}}(v_x)^2dx+\frac{1}{\varepsilon}\int_{\mathbb{R}}w^2dx\\ &+\int_{\mathbb{R}}|w_{0x}|\,|v_0|\,dx+\frac{\varepsilon}{2}\int_{\mathbb{R}}(v_{0x})^2dx+\int_0^t\int_{\mathbb{R}}(w_x)^2dx\,d\tau+\alpha\int_0^t\int_{\mathbb{R}}|g''(v)||u|^2|v_x|^2+2|u|\,|u_x|\,|v_x|\,dx\,d\tau.\end{aligned}$$

Moreover, by (4.6) and (4.7), we have

$$\|u\|_\infty\le c, \tag{4.13}$$

$$2\int_0^t\int_{\mathbb{R}}|u|\,|u_x|\,|v_x|\,dx\,d\tau\le c\,t+\frac{\sigma_0}{4}\int_0^t\int_{\mathbb{R}}(v_x)^2dx\,d\tau, \tag{4.14}$$

with c independent of ε. By the inequalities (4.12), (4.13), (4.14) and (4.6), (4.7) we deduce (4.8) for $\alpha\|g''\|_\infty$ sufficiently small, depending only on the data, and $\varepsilon\le 1$. □

The well posedness of the Cauchy problem (4.6),(4.7),(4.8),(4.4) is established in the following result.

Lemma 4.2. *For each $\varepsilon\le 1$ and $\alpha\|g''\|_\infty$ sufficiently small, depending only on the data, the Cauchy problem (4.6),(4.7),(4.8),(4.4) has a unique global solution $(u^\varepsilon,w^\varepsilon,v^\varepsilon)\in C([0,+\infty);H^1(\mathbb{R}))^3$.*

Proof. The proof is very similar to that of the existence part of Theorem 2.1. Preserving the same notation, we consider the map

$$(\tilde u,\tilde v)\longrightarrow\Phi(\tilde u,\tilde v)=(u,v)\in X_T\times X_T,$$

where $(u(x,t), v(x,t))$ is the solution of the linear problem (we put $\varepsilon = 1$ to simplify)

$$iu_t + u_{xx} = |\tilde{u}|^2\tilde{u} + \alpha g(\tilde{v})\tilde{u}$$
$$v(t) = v_0 + \int_0^t w_x(\tau)\, d\tau$$

where $w \in X_T$, satisfying $w \in L^2((0,T); H^2)$, $w_t \in L^2((0,T); L^2)$, is the unique solution of the linear initial value problem

$$w_t - w_{xx} = \sigma(\tilde{v})_x + \alpha(g'(\tilde{v})|\tilde{u}|^2)_x,$$
$$w(0) = w_0.$$

Recall that,

$$u(t) = U(t)u_0 + \int_0^t U(t-s)[\,|\tilde{u}|^2\tilde{u} + \alpha g(\tilde{v})\tilde{u}\,](s)\, ds$$

and $(\sigma(\tilde{v}) + \alpha\,|\tilde{u}|^2)_x \in C([0,T]; L^2)$. The existence of a local solution is again an easy consequence of the Banach fixed point theorem for suitables R and T.

Moreover, since we have

$$w_t - w_{xx} = \sigma'(v)v_x + \alpha g''(v)v_x|u|^2 + 2\alpha g'(v)\Re(u\bar{u}_x),$$

we deduce, from (4.6), (4.7) and (4.9), an apriori estimate

$$\|w_t - w_{xx}\|_{L^2((0,T);L^2(\mathbb{R}))} \leq c(T), \quad c \in C([0,+\infty))$$

and so $w \in L^2((0,T); H^2)$ with a similar estimate for $\|w\|_{L^2((0,T);H^2(\mathbb{R}))}$. It is now easy to obtain suitable apriori estimates for u, w and v in $C([0,T]; H^1)$ and the proof is completed.

□

We now establish the convergence of the vanishing viscosity solutions of (4.1)-(4.4). The main point in this special case where $g(v) = v$ is that the term $(|u|^2)_x$ is independent of v and by the Lemma 4.1 it is bounded in $L^1_{\text{loc}}(\mathbb{R} \times [0,\infty))$ uniformly with respect to ε. Below, by $v \in L^{\Sigma}_{\text{loc}}(\mathbb{R} \times [0,\infty))$ we mean that $\int_K \Sigma(v)\, dx\, dt < +\infty$ for any compact set $K \subseteq \mathbb{R} \times [0,\infty)$.

Theorem 4.1. *Let $(u^\varepsilon, w^\varepsilon, v^\varepsilon)$ be the global solution of* (4.1)-(4.4) *given by Lemma 4.2. There exists a subsequence of $(u^\varepsilon, w^\varepsilon, v^\varepsilon)$ converging in $L^1_{loc}(\mathbb{R} \times [0,\infty))^3$ to a function (u, w, v), with $u \in C([0,\infty); H^1(\mathbb{R}))$, $w \in L^2_{loc}(\mathbb{R} \times [0,\infty))$ and $v \in L^{\Sigma}_{loc}(\mathbb{R} \times [0,\infty))$, which is a weak solution of the problem*

$$iu_t + u_{xx} = u|u|^2 + vu, \tag{4.15}$$
$$w_t - \sigma(v)_x = (|u|^2)_x, \tag{4.16}$$
$$v_t - w_x = 0, \tag{4.17}$$
$$(u(x,0), w(x,0), v(x,0)) = (u_0(x), w_0(x), v_0(x)), \qquad x \in \mathbb{R}. \tag{4.18}$$

Proof. 1. As in the proof of Theorem 2.1, the compactness of u^ε is a consequence of the classical Aubin's lemma [16], since by Lemma 4.1, and for each $T > 0$, we have

$$\{u^\varepsilon\}_\varepsilon \quad \text{bounded in} \quad L^\infty((0,T); H^1),$$
$$\{u^\varepsilon_t\}_\varepsilon \quad \text{bounded in} \quad L^\infty((0,T); H^{-1}),$$

and so $\{u^\varepsilon\}_\varepsilon$ lies in a compact set of $L^2((0,T); L^2(I_R))$ for each interval $I_R = \{x \in \mathbb{R} : |x| \leq R\}$, $R > 0$. Hence, by applying a standard diagonalization procedure, we conclude that there exists $u \in L^\infty(\mathbb{R}_+; H^1(\mathbb{R}))$ and a subsequence of $\{u^\varepsilon\}_\varepsilon$, still denoted by $\{u^\varepsilon\}_\varepsilon$, such that $u^\varepsilon \to u$ in $L^1_{\text{loc}}(\mathbb{R} \times (0,\infty))$.

2. Also by Lemma 4.1, we have that w^ε is uniformly bounded in $L^2_{\rm loc}(\mathbb{R}\times(0,\infty))$, v^ε is uniformly bounded in $L^\Sigma_{\rm loc}(\mathbb{R}\times(0,\infty))$ and for any entropy-entropy flux pair (η,q) for (4.16),(4.17) with $\eta_w(w,v),\eta_v(w,v)$ uniformly bounded in $\mathbb{R}^2$ we have

$$\eta(w^\varepsilon,v^\varepsilon)_t+q(w^\varepsilon,v^\varepsilon)_x\in\{\,\text{compact of }\ W^{-1,2}_{\rm loc}(\mathbb{R}\times(0,\infty))\,\}. \tag{4.19}$$

Hence, we can use the compactness result in [22] (see also [10]) to conclude that $(w^\varepsilon,v^\varepsilon)$ is precompact in $L^1_{\rm loc}(\mathbb{R}\times(0,\infty))$ and also that any limit of a subsequence of $(u^\varepsilon,w^\varepsilon,v^\varepsilon)$ is a weak solution of (4.15)-(4.18). □

5. SW-LW interactions with the p-system of the nonlinear elasticity (II)

It is somehow surprising that one is able to carry out the vanishing viscosity method for the solutions of (4.1)-(4.4) obtaining thus a weak solution of (4.15)-(4.18), since SW-LW interactions constitute a microscopic phenomenon while the limit as $\varepsilon\to0$ is only physically meaningful at a macroscopic level. This comment applies as well to the two cases considered before, namely, the case of scalar conservation laws and that of the symmetric degenerate systems, but one could argue that these are less realistic models.

In the specific case of the nonlinear elasticity with physical viscosity, the method that we used to justify the vanishing viscosity limit is only able to handle the particular case $g(v)=v$, because, as we already mentioned, in this case the term $(g'(v)|u|^2)_x$ in (4.7) is independent of v. In the more general case of the problem (4.6)-(4.9), assuming only that $g'(v)$ and $g''(v)$ are bounded in $\mathbb{R}$, a similar procedure can justify the limit when both viscosity ε and interaction coefficient α go to zero, with $\alpha=O(\sqrt{\varepsilon})$. This will be also the limit process to be studied in the remaining sections, which we believe to be a more realistic one.

Thus we next establish the convergence of the vanishing viscosity and interaction coefficient for the solutions of (4.6)-(4.9).

Theorem 5.1. *Let $(u^\varepsilon,w^\varepsilon,v^\varepsilon)$ be the global solution of (4.6)-(4.9) given by Lemma 4.2 and assume $\alpha=O(\sqrt{\varepsilon})$. There exists a subsequence of $(u^\varepsilon,w^\varepsilon,v^\varepsilon)$ converging in $L^1_{loc}(\mathbb{R}\times[0,\infty))^3$ to a function (u,w,v), with $u\in C([0,\infty);H^1(\mathbb{R}))$, $w\in L^2_{loc}(\mathbb{R}\times[0,\infty))$ and $v\in L^\Sigma_{loc}(\mathbb{R}\times[0,\infty))$, which is a weak solution of the problem*

$$iu_t+u_{xx}=u|u|^2, \tag{5.1}$$

$$w_t-\sigma(v)_x=0, \tag{5.2}$$

$$v_t-w_x=0, \tag{5.3}$$

$$(u(x,0),w(x,0),v(x,0))=(u_0(x),w_0(x),v_0(x)),\qquad x\in\mathbb{R}. \tag{5.4}$$

Proof. This follows as in the proof of Theorem 4.1, since the fact that $\alpha=O(\sqrt{\varepsilon})$ gives us that the term $\alpha(g'(v)|u|^2)_x$ is again uniformly bounded in $L^1_{\rm loc}(\mathbb{R}\times(0,T))$. Thus, for any entropy-entropy flux pair (η,q) for (5.2),(5.3) with $\eta_w(w,v),\eta_v(w,v)$ uniformly bounded in $\mathbb{R}^2$ we again have

$$\eta(w^\varepsilon,v^\varepsilon)_t+q(w^\varepsilon,v^\varepsilon)_x\in\{\,\text{compact of }\ W^{-1,2}_{\rm loc}(\mathbb{R}\times(0,\infty))\,\}, \tag{5.5}$$

which allows us to apply the compactness result in [22]. □

6. SW-LW interactions with the equations of nonlinear electromagnetism

In this section we consider the convergence of the vanishing viscosity and interaction coefficient solutions of the Cauchy problem for the system (1.8)-(1.10) representing the interaction of short waves described by the nonlinear Schrödinger equation with long waves modelling nonlinear electromagnetism. Convergence of the vanishing viscosity method for the system of nonlinear eletromagnetism was proved in [21] and we are going to apply the compactness result in [21] to study the convergence of solutions of the Cauchy problem for (1.8)-(1.10) when $\varepsilon,\alpha\to0$ with $\alpha=O(\varepsilon^{1/2})$.

Since the transport equations in (1.9) and (1.10) are treated exactly as in the case of the degenerate symmetric systems in Section 3, here we just analyse the equations for u, p, q, that is,

$$iu_t + u_{xx} = u|u|^2 + \alpha g(r)u, \qquad r = \sqrt{p^2+q^2} \tag{6.1}$$

$$p_t + (\varphi(r)p)_x = -\alpha(g'(r)\frac{p}{r}|u|^2)_x + \varepsilon p_{xx}, \tag{6.2}$$

$$q_t - (\varphi(r)q)_x = \alpha(g'(r)\frac{q}{r}|u|^2)_x + \varepsilon q_{xx}. \tag{6.3}$$

We suppose that we are given initial data

$$(u(x,0), p(x,0), q(x,0)) = (u_0(x), p_0(x), q_0(x)). \tag{6.4}$$

As in the case of the degenerate symmetric systems, we assume $p_0(x) > 0$, $q_0(x) > 0$, for $x \in \mathbb{R}$, and for certain $p_* > 0$ and $q_* > 0$ we have

$$u_0 \in H^1(\mathbb{R}) \text{ and } p_0 - p_*, q_0 - q_* \in H^2(\mathbb{R}). \tag{6.5}$$

A remark similar to Remark 3.1 also holds here for both p_0 and q_0.

We also assume that $g \in C^3([0,\infty))$ and g' has compact support contained in $(0,\infty)$, and $\varphi \in C^2([0,+\infty))$ is a real function such that

$$\varphi(r) \geq a > 0 \text{ and } \varphi'(r) \geq 0 \text{ for } r \in [0,+\infty). \tag{6.6}$$

Positively invariant regions in the quadrant $p \geq 0$, $q \geq 0$ for the system (6.2),(6.3) without the interaction terms, i.e. $\alpha = 0$, were obtained in [21]. When these terms are present as in the coupled system (6.1)-(6.3), positively invariant regions are obtained as in the case of the scalar conservation laws, due to our assumption that the support of g' is compact in $(0,\infty)$. More specifically, we have the following.

Lemma 6.1. *Let $(u,p,q)(x,t)$ be a smooth solution of* (6.1)-(6.4) *defined in $\mathbb{R} \times [0,T)$ with $u_0, p_0, q_0, g, \varphi$ satisfying the hypotheses above. Then there exists a constant $M_0 > 0$ such that*

$$0 \leq p(x,t) \leq M_0, \quad 0 \leq q(x,t) \leq M_0, \quad x \in \mathbb{R}, \quad t \in [0,T), \tag{6.7}$$

with c independent of ε and T.

Proof. We recall that the positively invariant regions for the system (6.2),(6.3) with $\alpha = 0$ obtained in [21] are convex regions of the form $D = \{(p,q) \in [0,\infty)^2 : 0 \leq p \leq h(p),\ 0 \leq q \leq h(q)\}$ where $h \in C^3([0,\infty))$ is decreasing with $h(0) > 0$, $h(R) = 0$ for some $R > 0$, $h''(s) < 0$ for $0 \leq s < s_*$, $h''(s) > 0$ for $s_* < s \leq R$, for certain $s_* \in [0,R]$ satisfying $h(s_*) = s_*$ and $h'(s_*) < -1$. Let the support of $g'(r)$ be contained in an interval $[m,M]$ with $0 < m < M$, let D be such an invariant domain obtained in [21], and assume that:

(i) the image of (p_0, q_0) is contained in the interior of D;
(ii) $\{(p,q) \in [0,\infty)^2 : m \leq r = \sqrt{p^2+q^2} \leq M\} \subseteq D$.

As in the scalar case in Section 2, we first consider the perturbed system

$$iu_t + u_{xx} = u|u|^2 + \alpha g(r)u, \qquad r = \sqrt{p^2+q^2} \tag{6.8}$$

$$p_t + (\varphi(r)p)_x = -\alpha(g'(r)\frac{p}{r}|u|^2)_x + \nu(p_\bullet - p) + \varepsilon p_{xx}, \tag{6.9}$$

$$q_t - (\varphi(r)q)_x = \alpha(g'(r)\frac{q}{r}|u|^2)_x + \nu(q_\bullet - q) + \varepsilon q_{xx}, \tag{6.10}$$

where $(p_\bullet, q_\bullet)$ is any point in the interior of D. Because D satisfies (i) and (ii) above, and the terms $-\alpha(g'(r)\frac{p}{r}|u|^2)_x$ and $\alpha(g'(r)\frac{q}{r}|u|^2)_x$ in (6.9) and (6.10), respectively, vanish at ∂D, assuming that there is a first time at which the solution leaves the region D leads us to contradiction exactly as in the case where $\alpha = 0$ addressed in [21]. A standard limit argument as $\nu \to 0$ then gives the desired boundedness for the solution of (6.1)-(6.4).

□

We now establish two identities similar to (2.13) and (2.14) in Lemma 2.2.

Lemma 6.2. *Let (u, p, q) with $(u, p - p_*, q - p_*) \in C([0, T); H^1(\mathbb{R}))^3$, $T > 0$, be a solution of the Cauchy problem* (6.1)-(6.4). *We have*

$$\frac{\partial}{\partial t} \int_{\mathbb{R}} |u|^2 dx = 0 \tag{6.11}$$

$$\begin{aligned} &\frac{\partial}{\partial t} \left[\frac{1}{2} \int_{\mathbb{R}} |u_x|^2 dx + \frac{1}{4} \int_{\mathbb{R}} |u|^4 dx + \frac{\alpha}{2} \int_{\mathbb{R}} g(r)\, |u|^2 dx + \frac{1}{4} \int_{\mathbb{R}} H(r) dx \right] \\ &\quad + \frac{\varepsilon}{2} \int_{\mathbb{R}} \varphi(r)\, (|p_x|^2 + |q_x|^2) dx + \frac{\varepsilon}{2\alpha} \int_{\mathbb{R}} \varphi'(r) r\, (r_x)^2 dx \\ &\quad + \frac{\varepsilon\alpha}{2} \int_{\mathbb{R}} \left(|u|^2\right)_x g'(r) r_x dx + \frac{\varepsilon\alpha}{2} \int_{\mathbb{R}} |u|^2 \left(g''(r) - \frac{g'(r)}{r} \right) (r_x)^2 dx = 0, \end{aligned} \tag{6.12}$$

where $H(r) = \displaystyle\int_0^r s\varphi(s)\, ds$.

Proof. 1. The proof of (6.11) is identical to that of (2.13) so we omit it.

2. As to the proof of (6.12), it is very similar to that of (2.14) and after proceeding as in the beginning of that proof we arrive at the point where we need to compute $\int_{\mathbb{R}} g(r)_t |u|^2\, dx$. To do that, now we have to multiply equation (6.2) by $g'(r)\frac{p}{r}|u|^2$ obtaining an equation $(E1)$, multiply equation (6.3) by $g'(r)\frac{q}{r}|u|^2$ obtaining an equation $(E2)$, add up $(E1) + (E2)$ obtaining an equation $(E3)$ and integrate $(E3)$. The remaining of the proof follows as in the proof of (2.14). □

We need one more relation which together with (6.11) and (6.12) will allow us to obtain the necessary energy estimates.

Lemma 6.3. *Under the hypothesis of the Lemma 6.2 we have*

$$\begin{aligned} &\int_{\mathbb{R}} H(r) dx + a\, \varepsilon \int_0^t \int_{\mathbb{R}} (|p_x|^2 + |q_x|^2)\, dx\, d\tau \\ &\qquad \leq c \left(1 + \alpha \int_0^t \int_{\mathbb{R}} |u|^2 (|p_x| + |q_x|) dx\, d\tau \right), \end{aligned} \tag{6.13}$$

with c independent of ε, T and α.

Proof. We deduce from (6.2), (6.3)

$$\int_{\mathbb{R}} p_t \varphi(r)\, p\, dx = -\alpha \int_{\mathbb{R}} \left(g'(r)\, \frac{p}{r}\, |u|^2 \right)_x \varphi(r)\, p\, dx + \varepsilon \int_{\mathbb{R}} \varphi(r) p\, p_{xx}\, dx,$$

$$\int_{\mathbb{R}} q_t \varphi(r)\, q\, dx = \alpha \int_{\mathbb{R}} \left(g'(r)\, \frac{q}{r}\, |u|^2 \right)_x \varphi(r)\, q\, dx + \varepsilon \int_{\mathbb{R}} \varphi(r) q\, q_{xx}\, dx,$$

and so,

$$\frac{\partial}{\partial t}\int_{\mathbb{R}} H(r)dx = \int_{\mathbb{R}} \varphi(r)\, r\left(\frac{p}{r}\, p_t + \frac{q}{r}\, q_t\right) dx = \alpha \int_{\mathbb{R}} \left(g'(r)\,\frac{p}{r}\,|u|^2\right)(\varphi(r)\,p)_x dx -$$
$$- \varepsilon \int_{\mathbb{R}} \varphi'(r)\, p\left(\frac{p}{r}\, p_x + \frac{q}{r}\, q_x\right) p_x dx - \varepsilon \int_{\mathbb{R}} \varphi(r)(p_x)^2 dx -$$
$$- \alpha \int_{\mathbb{R}} \left(g'(r)\,\frac{p}{r}\,|u|^2\right)(\varphi(r)\,q)_x dx - \varepsilon \int_{\mathbb{R}} \varphi'(r)\, p\left(\frac{p}{r}\, p_x + \frac{q}{r}\, q_x\right) q_x dx$$
$$- \varepsilon \int_{\mathbb{R}} \varphi(r)\,(q_x)^2 dx.$$

We then have

$$\frac{\partial}{\partial t}\int_{\mathbb{R}} H(r)dx = \alpha \int_{\mathbb{R}} \left(g'(r)\,|u|^2\right)\varphi'(r)\,(p\,p_x + q\,q_x)\,\frac{p^2-q^2}{r^2}\, dx$$
$$+ \alpha \int_{\mathbb{R}} \left(g'(r)\,|u|^2\right)\varphi(r)\,\frac{pp_x - qq_x}{r} dx$$
$$- \varepsilon \int_{\mathbb{R}} \varphi'(r)\, r\,(r_x)^2 dx - \varepsilon \int_{\mathbb{R}} \varphi(r)\left(|p_x|^2 + |q_x|^2\right) dx.$$

Hence, by (6.7),

$$\frac{\partial}{\partial t}\int_{\mathbb{R}} H(r)dx + a\,\varepsilon \int_{\mathbb{R}} \left(|p_x|^2 + |q_x|^2\right) dx \le c\,\alpha \int_{\mathbb{R}} |u|^2\left(|p_x| + |q_x|\right) dx,$$

with c independent of ε, T and α.

□

Now, assume $\alpha = O(\varepsilon^{1/2}) > 0$ and let $\beta > 0$ be such that

$$\frac{\alpha}{\varepsilon^{1/2}} \le \beta. \tag{6.14}$$

From (6.13) we obtain the following.

Corollary 6.1. *Under the hypothesis of the Lemma 6.2, we have*

$$\int_{\mathbb{R}} r^2 dx + \varepsilon \int_0^t \int_{\mathbb{R}} (|p_x|^2 + |q_x|^2)\, dx\, d\tau \le c\,(1 + \beta^2 \int_0^t \|u_x\|_2 d\tau) \tag{6.15}$$

with c independent of ε and T, where β is defined by (6.14),

Proof. Since $\varphi(r) \ge a > 0$, we easily deduce from (6.13), (6.14),

$$\int_{\mathbb{R}} r^2 dx + \varepsilon \int_0^t \int_{\mathbb{R}} (|p_x|^2 + |q_x|^2)\, dx\, d\tau \le c\,(1 + \beta^2 \int_0^t \int_{\mathbb{R}} \|u\|_4^4 dx\, d\tau).$$

Now we use that $\|u\|_4 \le c\,\|u_x\|_2^{1/4}\|u\|_2^{3/4}$.

□

We next establish the following analogue of Lemma 2.3.

Lemma 6.4. *Let (u,p,q) with $(u, p-p_*, q-q_*) \in C([0,T); H^1(\mathbb{R}))^3$, $T>0$, be a solution of the Cauchy problem* (6.1)-(6.4)*. Then, there exists $\varepsilon_0 > 0$ such that if $0 < \varepsilon < \varepsilon_0$, we have*

$$\|u_x(t)\|_2 \le h(t), \tag{6.16}$$

where $h \in C^\infty([0,+\infty))$ is a positive function independent of T and ε.

Proof. Define $\theta(t) := \big(1 + \max_{\tau\in[0,t]} \|u_x(\tau)\|_2^2\big)^{1/2}$. From (6.12) and (6.15), we have

$$\begin{aligned}\|u_x(t)\|_2^2 &\le c + c\,\varepsilon\alpha \int_0^t \int_{\mathbb{R}} |u|\,|u_x|\,(|p_x| + |q_x|)\,dx\,d\tau + c\,\varepsilon\alpha \int_0^t \int_{\mathbb{R}} |u|^2\,(|p_x|^2 + |q_x|^2)\,dx\,d\tau \\ &\le c + c\,\varepsilon\alpha \int_0^t \|u_x\|_2^{3/2} \||p_x| + |q_x|\|_2\,d\tau + c\,\varepsilon\alpha \int_0^t \|u_x\|_2\,\||p_x| + |q_x|\|_2^2\,d\tau.\end{aligned}$$

It follows then that

$$\begin{aligned}\theta^2(t) &\le c\, + c\,\varepsilon\beta\theta(t)^{3/2}t^{1/2}\left(1 + \beta t^{1/2}\theta^{1/2}(t)\right) + c\,\varepsilon^{1/2}\beta\theta(t)(1 + \beta^2\,t\,\theta(t)) \\ &\le c + c\varepsilon t^{1/2}(1 + t^{1/2})\theta(t)^2 + c\varepsilon^{1/2}(1 + t)\theta(t)^2. \end{aligned} \tag{6.17}$$

Let $\varepsilon_0 > 0$ be such that

$$2c(\varepsilon_0 + \varepsilon_0^{1/2}) < \frac{1}{2}.$$

Then, for $0 \le t < 1$, from (6.17) we deduce

$$\theta(t)^2 \le 2c$$

By induction, as in the proof of Lemma 2.3, we prove that for $n \le t < n + 1$, we have

$$\theta(t)^2 \le 2^{n+1}c.$$

Hence, we get

$$\theta(t)^2 \le 2^{t+1}c, \qquad \text{for } t \ge 0,$$

which concludes the proof. □

We finally establish the following result on the global well posedness and convergence of the vanishing viscosity and interaction solutions of (6.1)-(6.4).

Theorem 6.1. *There exists $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$ the problem (6.1)-(6.4) admits a unique global solution $(u^\varepsilon, p^\varepsilon, q^\varepsilon)$ with $(u^\varepsilon, p^\varepsilon - p_*, q^\varepsilon - q_*) \in C([0,\infty); H^1(\mathbb{R}))^3$. Moreover, as $\varepsilon \to 0$ and $\alpha \to 0$ with $\alpha = O(\varepsilon^{1/2})$, we may extract a subsequence of solutions $(u^\varepsilon, p^\varepsilon, q^\varepsilon)$ converging in $L^1_{loc}(\mathbb{R} \times [0,\infty))$ to some $(u, p, q) \in C([0,\infty); H^1(\mathbb{R})) \times L^\infty(\mathbb{R} \times [0,\infty))^2$ which is a weak (entropy) solution of*

$$iu_t + u_{xx} = u|u|^2, \tag{6.18}$$

$$p_t + (\varphi(r)p)_x = 0, \qquad r = \sqrt{p^2 + q^2}, \tag{6.19}$$

$$q_t - (\varphi(r)q)_x = 0. \tag{6.20}$$

$$(u(x,0), p(x,0), q(x,0)) = (u_0(x), p_0(x), q_0(x)). \tag{6.21}$$

Proof. 1. The existence and uniqueness of a global solution of (6.1)-(6.4), $(u^\varepsilon, p^\varepsilon, q^\varepsilon)$, with $(u^\varepsilon, p^\varepsilon - p_*, q^\varepsilon - q_*) \in C([0,\infty); H^1(\mathbb{R}))^3$ follows from the estimates provided by Lemmas 6.1–6.4, which allow the extension of the unique local in time solution, as in the earlier sections.

2. The compactness of $\{u^\varepsilon\}$ follows as before from Aubin's lemma, in view of the referred estimates. The fact that the limits of converging subsequences must belong to $C([0,\infty); H^1(\mathbb{R}))$ is a consequence of the uniform boundedness of u^ε in $L^\infty_{\text{loc}}([0,\infty); H^1(\mathbb{R}))$ and Duhamel's formula.

3. As to the compactness of $(p^\varepsilon, q^\varepsilon)$ in $L^1_{\text{loc}}(\mathbb{R} \times (0,\infty))$, since $\alpha = O(\sqrt{\varepsilon})$ the estimates given by Lemmas 6.1–6.4 give that the term $\alpha(g'(r)|u|^2)_x$ is uniformly bounded in $L^1_{\text{loc}}(\mathbb{R} \times (0,T))$. Thus, using those estimates, we deduce that for any entropy-entropy flux pair for (6.19),(6.20), $(E(p,q), F(p,q)) \in C^2((0,\infty) \times (0,\infty))^2$, we have

$$E(p^\varepsilon, q^\varepsilon)_t + F(p^\varepsilon, q^\varepsilon)_x \in \{\,\text{compact of } W^{-1,2}_{\text{loc}}(\mathbb{R} \times (0,\infty))\,\}, \tag{6.22}$$

which allows us to apply the compactness result in [21].

□

7. SW-LW interactions with general symmetric systems

In this section we consider a general symmetric system of conservation laws, that is, the vector flux-function is the gradient of a real-valued C^3 function $\varphi(v)$, $v=(v_1,\dots,v_n)\in\mathbb{R}^n$. We propose the following model for SW-LW interactions

$$iu_t + u_{xx} = |u|^2u + \alpha\, g(v)\, u \tag{7.1}$$

$$v_t + (\nabla_v\varphi(v))_x = \alpha\,\left(\nabla_v g(v)|u|^2\right)_x + \varepsilon\, v_{xx}, \quad \alpha>0,\, \varepsilon>0. \tag{7.2}$$

For simplicity we will assume that the corresponding nonlinear symmetric system of conservation laws with viscosity

$$v_t + (\nabla_v\varphi(v))_x = \varepsilon v_{xx}, \tag{7.3}$$

admits a bounded invariant domain D, which is the case of the systems considered in [5, 6, 11], for instance.

Concerning g, we assume that it is a real-valued C^2 function such that g' has support contained in the interior of the invariant domain D.

We assume $\nabla_v\varphi(0)=0$ and that $\nabla_v^2\varphi(0)$ (Hessian matrix of φ at $v=0$) is equal to the $n\times n$ zero matrix. We consider the Cauchy problem for (7.1),(7.2) with initial data given by

$$(u(x,0),v(x,0)) = (u_0(x),v_0(x)), \tag{7.4}$$

with

$$(u_0, v_0 - v_*) \in H^1(\mathbb{R})^{n+1}, \tag{7.5}$$

where v_* is a point in the interior of the invariant domain D and $v_0(x)\in D$ for all $x\in\mathbb{R}$.

Exactly as in Lemma 6.1 we obtain the following estimate.

Lemma 7.1. *Let (u,v) be a local in time solution of (7.1)-(7.4) with $(u, v-v_*)\in C([0,T);H^1(\mathbb{R}))$, for some $T>0$. Then*

$$|v(t)|_{L^\infty(\mathbb{R})} \le c \tag{7.6}$$

where c is independent of ε, t, T.

Using (7.6) and the symmetry of (7.4) we obtain, with essentially the same proofs, the analogues of lemmas are similar to Lemmas 2.2 and 2.3, which we only state below for the sake of reference.

Lemma 7.2. *Let $(u,v)\in C([0,T);H^1(\mathbb{R}))^{n+1}$, $T>0$, be a solution of the Cauchy problem (7.1),(7.2),(7.4). We have:*

$$\frac{\partial}{\partial t}\int_{\mathbb{R}} |u|^2 dx = 0. \tag{7.7}$$

$$\begin{aligned}
&\frac{\partial}{\partial t}\left[\frac12\int_{\mathbb{R}}|u_x|^2dx + \frac14\int_{\mathbb{R}}|u|^4dx + \frac{\alpha}{2}\int_{\mathbb{R}} g(v)\,|u|^2dx - \frac12\int_{\mathbb{R}}\varphi(v)dx\right]\\
&\quad + \frac{\varepsilon\alpha}{2}\int_{\mathbb{R}}(|u|^2)_x\nabla_v g(v)\cdot v_x dx + \frac{\varepsilon\alpha}{2}\int_{\mathbb{R}}|u|^2(v_x^\top\,\nabla_v^2 g(v)\, v_x)\, dx\\
&\quad - \frac{\varepsilon}{2}\int_{\mathbb{R}} v_x^\top\,\nabla_v^2\varphi(v)\, v_x\, dx = 0.
\end{aligned} \tag{7.8}$$

$$\frac{\partial}{\partial t}\left[\frac12\int_{\mathbb{R}}|v|^2dx + \mathrm{Im}\int_{\mathbb{R}} u\bar{u}_x dx\right] + \varepsilon\int_{\mathbb{R}}|v_x|^2dx = 0. \tag{7.9}$$

Lemma 7.3. *Assume* (7.6) *and suppose that* $\alpha = O(\varepsilon^{1/2})$. *Under the hypothesis of the Lemma 7.1 there exists* $\varepsilon_0 > 0$, *independent of* T, *such that for* $\varepsilon \leq \varepsilon_0$

$$|u_x|_{L^2(\mathbb{R})}(t) \leq h(t), \tag{7.10}$$

where $h \in C^\infty([0, +\infty))$ *is a positive function independent of* T *and* ε.

As in the earlier section, using the estimates given by Lemmas 7.1–7.3 we arrive at the following result.

Theorem 7.1. *Let* $\alpha = O(\varepsilon^{1/2})$. *There exists* $\varepsilon_0 > 0$ *such that for* $\varepsilon \leq \varepsilon_0$ *there is a unique global solution* $(u^\varepsilon, v^\varepsilon) \in \left(C([0, +\infty[; H^1(\mathbb{R}))\right)^{n+1}$ *of the Cauchy problem* (7.1),(7.2),(7.4). *Moreover, there exists* $u \in L^\infty_{loc}([0, +\infty[; H^1(\mathbb{R}))$ *and a subsequence of* $\{u^\varepsilon\}_\varepsilon$ *converging in* $L^1_{loc}(\mathbb{R} \times [0, \infty))$ *to* u. *Further, if uniformly bounded solutions of* (7.5) *as* $\varepsilon \to 0$ *form a precompact sequence in* $L^1_{loc}(\mathbb{R} \times (0, \infty))$ *(as in* [5, 6, 11], *for instance),then* $\{v^\varepsilon\}$ *is precompact in* $L^1_{loc}(\mathbb{R} \times (0, \infty))$ *and any limit of a convergent subsequence of* v^ε *is an entropy solution of the symmetric nonlinear system of conservation laws obtained from* (7.2) *when* $\alpha = \varepsilon = 0$.

Remark 7.1. We observe in conclusion that some symmetric systems with unbounded invariant domains, such as the one in [12], admit a completely similar result.

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