THE DISCONTINUOUS GALERKIN METHOD FOR FRACTAL CONSERVATION LAWS

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Abstract. We propose, analyze, and demonstrate a discontinuous Galerkin method for fractional conservation laws. Various stability estimates are established along with error estimates for regular solutions of linear equations. Moreover, in the nonlinear case and when piecewise constant elements are utilized, we prove a rate of convergence toward the unique entropy solution. We present numerical results for different types of solutions of linear and nonlinear fractional conservation laws.

1. Introduction

We consider the fractional conservation law

$$\begin{cases} \partial_t u(x,t) + \partial_x f(u(x,t)) = g_\lambda[u(x,t)] & (x,t) \in \mathbb{R} \times (0,T) \\ u(x,0) = u_0(x) & x \in \mathbb{R} \end{cases} \tag{1.1}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function and g_λ is the nonlocal fractional Laplace operator $-(-\partial_x^2)^{\lambda/2}$ for $\lambda \in (0,2)$. This operator can be formally defined by Fourier transform as

$$\widehat{g_\lambda[\varphi(x)]}(\xi) = -|\xi|^\lambda \hat{\varphi}(\xi), \tag{1.2}$$

or, equivalently, by a singular integral (see [27, 20]) as

$$g_\lambda[\varphi(x)] = c_\lambda \int_{|z|>0} \frac{\varphi(x+z) - \varphi(x)}{|z|^{1+\lambda}} dz \qquad \text{for some} \quad c_\lambda > 0.$$

For the sake of brevity, we will often write g instead of g_λ in what follows.

Nonlocal partial differential equations appear in different areas of engineering and sciences. For example, the linear nonlocal partial differential equation

$$\partial_t u - \partial_x^2 u - \partial_x u + u = g_\lambda[u], \tag{1.3}$$

is a nonlocal generalizations of the famous Black-Scholes' equation in finance (see [15]), and have received a lot of attention in the last decade. In recent years, attention has also been given to nonlinear nonlocal equations like

$$\partial_t u + u \partial_x u = g_\lambda[u], \tag{1.4}$$

known as the fractional Burgers' equation. Equation (1.4) finds application in certain models of detonation of gases (see [30]) characterized by an anomalous diffusive behavior which can be described by means of the fractional Laplacian. We refer the reader to [2, 3, 18], and the references therein, for further applications in hydrodynamics, molecular biology, semiconductor growth and dislocation dynamics.

Key words and phrases. Fractal/Fractional conservation law, fractional Laplacian, entropy solution, discontinuous Galerkin method, stability, high-order accuracy, convergence rate.

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Many authors, see [2, 3, 5, 6, 7, 8, 20, 23], have contributed to settle issues like well-posedness and regularity of solutions for the fractional conservation law (1.1). In the case $\lambda \in (1, 2)$, (1.1) is the natural nonlocal generalization of the viscous conservation law $\partial_t u + \partial_x f(u) = \partial_x^2 u$. Such equations turn a merely bounded initial datum into a unique stable smooth solution (see [19]). The case $\lambda \in (0, 1)$ is more delicate. Alibaud's entropy formulation is needed to guarantee well-posedness [2], and the solutions may develop shocks in finite time [3]. The diffusion is no longer strong enough to counterbalance the convection, and equation (1.1) fails to regularize the initial datum. In the critical case $\lambda = 1$, Alibaud's entropy formulation is still needed to ensure well-posedness, however, solutions should be smooth as in the case $\lambda \in (1, 2)$ – see Kiselev *et al.* [25] for the case of the fractional Burgers' equation.

A vast literature is available on numerical methods for nonlocal linear equations like (1.3). The interested reader could see, for example, [12, 11, 10, 17, 29, 4, 14]. However, numerical methods for nonlocal nonlinear equations like (1.1) are far from being abundant. Dedner *et al.* introduced in [16] a general class of differences methods for a nonlinear nonlocal equation similar to (1.1) coming from a specific problem in radiative hydrodynamics. Droniou [18] was the first to analyze a general class of difference methods for (1.1). He proved convergence to Alibaud's entropy solution, but produced no results regarding the rate of convergence of his methods.

In this paper we will study a discontinuous Galerkin (DG) approximation of (1.1). The DG method is a well established numerical method for the pure conservation law $\partial_t u + \partial_x f(u) = 0$. Some of the important features of this method are stability and high-order accuracy. Moreover, when piecewise constant elements are used, the DG method reduces to a conservative monotone difference method (see [22]) which converges to the entropy solution with rate 1/2 (see Kuznetsov [26]). For a detailed presentation of the DG method for pure conservation laws, we refer Cockburn [13].

In this paper we propose a DG approximation of (1.1) in the case $\lambda \in (0, 1)$, and prove that we retain the main features of the DG method in our nonlocal setting. We show L^2-stability, we prove high-order accuracy for linear equations, and we prove that when piecewise constant elements are used, the method is equivalent to a conservative monotone difference method which converges towards Alibaud's entropy solution with rate 1/2. To prove the rate 1/2, we generalize the Kuznetsov argument (see [26]) to our nonlocal setting. As a byproduct, we obtain the following theoretical result: Alibaud's entropy formulation and what Alibaud himself in [2] calls the intermediate formulation are equivalent for all integrable initial data of bounded variation. This equivalence has been recently remarked by Karlsen *et al.* [24] in a more general context using different arguments.

Finally, several numerical experiments have been performed to illustrate the developed theory. Among other things, we are able to reproduce the theoretical results (absence of smoothing effect due to persistence of discontinuities and formations of shocks) obtained by [3, 25] for the fractional Burgers' equation.

2. A semidiscrete method

Let us introduce the space grid $x_i = i\Delta x$, $i \in \mathbb{Z}$, and let us label $I_i = (x_i, x_{i+1})$. We call $P^k(I_i)$ the set of polynomials of degree $k \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$ with support on the interval I_i, and consider the Legendre polynomials (see [13] for details)

$$\{\varphi_{0,i}, \varphi_{1,i}, \ldots, \varphi_{k,i}\},\ \varphi_{j,i} \in P^j(I_i) \text{ for all } j = 0, 1, \ldots, k.$$

Each $\varphi \in P^k(I_i)$ is a linear combination of the functions $\{\varphi_{0,i}, \varphi_{1,i}, \ldots, \varphi_{k,i}\}$.

If we multiply (1.1) by an arbitrary $\varphi \in P^k(I_i)$, integrate over the interval I_i, integrate by parts, and replace the flux f by a numerical flux F, we get

$$\int_{I_i} u_t\varphi - \int_{I_i} f(u)\varphi_x + F(u(x_{i+1}))\varphi(x_{i+1}^-) - F(u(x_i))\varphi(x_i^+) = \int_{I_i} g[u]\varphi. \tag{2.1}$$

As usual for DG methods, the numerical flux $F(u_i) = F(u(x_i^-), u(x_i^+))$ satisfies the following assumptions:

A1: F is Lipschitz continuous on $\mathbb{R}\times\mathbb{R}$,
A2: $F(a,a) = f(a)$ for all $a \in \mathbb{R}$,
A3: F is non-decreasing with respect to its first variable,
A4: F is non-increasing with respect to its second variable.

The goal is to find a function $\tilde{u} : \mathbb{R}\times[0,T] \to \mathbb{R}$,

$$\tilde{u}(x,t) = \sum_{i\in\mathbb{Z}}\sum_{p=0}^{k} U_{p,i}(t)\varphi_{p,i}(x), \tag{2.2}$$

which satisfies (2.1) for all $\varphi \in P^k(I_i)$, $i \in \mathbb{Z}$. Let us fix $\varphi(x) = \sum_{q=0}^{k} \alpha_{q,i}\varphi_{q,i}(x)$ and plug (2.2) into (2.1) to get

$$\sum_{q=0}^{k}\alpha_{q,i}\left\{w_q U_{q,i}'\right\} = \sum_{q=0}^{k}\alpha_{q,i}\left\{\int_{I_i} f(\tilde{u})\varphi_{q,i}' + (-1)^q F(\tilde{u}_i) - F(\tilde{u}_{i+1}) + \int_{I_i} g[\tilde{u}]\varphi_{q,i}\right\}$$

where $F(\tilde{u}_i) = F(\sum_{p=0}^{k} U_{p,i-1}, \sum_{p=0}^{k} U_{p,i}(-1)^p)$. To derive the above expression, we have used some well know properties of the Legendre polynomials: for all $i \in \mathbb{Z}$,

$$\int_{I_i}\varphi_{q,i}^2 dx = \omega_q,\ \int_{I_i}\varphi_{p,i}\varphi_{q,i}dx = 0\ (p\neq q),\ \varphi_{p,i}(x_{i+1}^-) = 1 \text{ and } \varphi_{p,i}(x_i^+) = (-1)^p.$$

The semidiscrete method we study is the following: for all $q = 0, 1, \ldots, k$ and $i \in \mathbb{Z}$,

$$w_q U_{q,i}' = \int_{I_i} f(\tilde{u})\varphi_{q,i}' + (-1)^q F(\tilde{u}_i) - F(\tilde{u}_{i+1}) + \int_{I_i} g[\tilde{u}]\varphi_{q,i} \tag{2.3}$$

$$U_{q,i}(0) = \frac{2q+1}{\Delta x}\int_{I_i} u_0(x)\varphi_{q,i}(x)dx. \tag{2.4}$$

3. Non-linear L^2 stability and convergence in the linear case

As usual for the DG method, let us introduce some integrability into our formulation. Let us require the function (2.2) to be such that

$$\tilde{u}(\cdot,t) \in V^k = \{u : \|u\|_{H^{\lambda/2}(\mathbb{R})} < \infty \text{ and } u|_{I_i} \in P^k(I_i) \text{ for all } i \in \mathbb{Z}\} \tag{3.1}$$

where we have introduced the fractional Sobolev norm (see [1])

$$\|u\|_{H^{\lambda/2}(\mathbb{R})}^2 = \|u\|_{L^2(\mathbb{R})}^2 + \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{(u(z)-u(x))^2}{|z-x|^{1+\lambda}}dzdx. \tag{3.2}$$

Note that the space V^k also contains *discontinuous* piecewise polynomials, see e.g. Lemma 6.5 in [21].

Theorem 3.1. (Stability) *Let $u_0 \in L^2(\mathbb{R})$. Then,*

$$\|\tilde{u}(\cdot,T)\|_{L^2(\mathbb{R})} \leq \|u_0\|_{L^2(\mathbb{R})}.$$

The above result generalizes a well known feature of the DG method for pure conservation laws (see [13], Proposition 2.1 and Theorem 4.2, for details).

Proof. For brevity, write u instead of $\tilde{u}$. By construction, $u(\cdot,t)$ satisfies (2.1) for all $\varphi \in P^k(I_i)$. Sum over all $i \in \mathbb{Z}$, and rearrange the terms in the sum to get

$$\int_{\mathbb{R}} u_t u = \sum_{i\in\mathbb{Z}} \Big[F(u_i)\big(u(x_i^+) - u(x_i^-)\big) + \int_{I_i} f(u)u_x\Big] + \int_{\mathbb{R}} g[u]u.$$

Note that $\int_{I_i} f(u)u_x = \int_{I_i}(\int^{u(x)} f)_x = \int^{u(x_{i+1}^-)} f - \int^{u(x_i^+)} f$ where $\int^u f$ is a primitive of $f(u)u_x$. Thus, after having rearranged the terms in the sum,

$$\int_{\mathbb{R}} u_t u = \sum_{i\in\mathbb{Z}} \Big[F(u_i)\big(u(x_i^+) - u(x_i^-)\big) - \int_{u(x_i^-)}^{u(x_i^+)} f(x)dx\Big] + \int_{\mathbb{R}} g[u]u.$$

It is well known that a flux satisfying *A2*-*A4* is an E-flux (see [13]), i.e.

$$F(u_i)\big(u(x_i^+) - u(x_i^-)\big) - \int_{u(x_i^-)}^{u(x_i^+)} f(x)dx \leq 0 \qquad \text{for all } i \in \mathbb{Z}.$$

Thus, by Corollary A.4,

$$\|u(\cdot,T)\|^2_{L^2(\mathbb{R})} + \int_0^T\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{(u(z,t)-u(x,t))^2}{|z-x|^{1+\lambda}} dzdxdt \leq \|u_0\|_{L^2(\mathbb{R})}.$$

□

In the linear case, problem (1.1) reduces to

$$\partial_t u + c\partial_x u = g[u] \tag{3.3}$$

where $c \in \mathbb{R}$. Let us recall the following result.

Proposition 3.2. *Let $u_0 \in H^{k+1}(\mathbb{R})$ for $k \geq 0$. Then, there exists a unique function $u \in H^{k+1}(\mathbb{R}\times[0,T])$ which solves (3.3). Moreover,*

$$\|u(\cdot,t)\|_{H^{k+1}(\mathbb{R})} \leq \|u_0\|_{H^{k+1}(\mathbb{R})}. \tag{3.4}$$

Proof. Since (3.3) is linear, its Fourier transform, $\partial_t\hat{u} + i\xi c\hat{u} = -|\xi|^\lambda\hat{u}$, has solution

$$\hat{u}(\xi,t) = \hat{u}_0(\xi)e^{-(i\xi c+|\xi|^\lambda)t}.$$

This implies existence plus, using Plancherel theorem, L^2-stability and uniqueness. L^2-stability for (weak) higher derivatives can be obtained as follows: take the derivative of (3.3), and repeat the above procedure; iterate until the kth derivative. Regularity in time can be shown by using equation (3.3) and regularity in space. □

As pointed out by Cockburn in [13], in the linear case all relevant numerical fluxes (Godunov, Engquist-Osher, Lax-Friedrichs, *etc.*) reduce to

$$F(a,b) = \frac{c}{2}(a+b) - \frac{|c|}{2}(b-a). \tag{3.5}$$

We use this flux to prove the following result: the order of semidiscrete method (2.3)-(2.4) increases along with the degree k of the polynomial basis used.

Theorem 3.3. (Convergence) *Let $u_0 \in H^{k+1}(\mathbb{R})$, $k \geq 0$, and $u \in H^{k+1}(\mathbb{R}\times[0,T])$ be the unique solution of (3.3). Then, there exists $c_{k,T} > 0$ such that*

$$\|u(\cdot,T) - \tilde{u}(\cdot,T)\|_{L^2(\mathbb{R})} \leq c_{k,T}(\Delta x)^{k+\frac{1}{2}}.$$

The above result, called high-order accuracy, generalizes a well known feature of the DG method for pure conservation laws (see [13], Theorem 2.1, for details).

Proof. By construction, for all φ such that $\varphi|_{I_i} \in P^k(I_i)$ where $i \in \mathbb{Z}$,

$$\int_{\mathbb{R}} \tilde{u}_t\varphi + \sum_{i\in\mathbb{Z}} \Big(F(\tilde{u}(x_i))\big(\varphi(x_i^-) - \varphi(x_i^+)\big) - \int_{I_i} c\tilde{u}\varphi_x \Big) = \int_{\mathbb{R}} g[\tilde{u}]\varphi.$$

Note that u satisfies an analogous expression. That is,

$$\int_{\mathbb{R}} u_t\varphi + \sum_{i\in\mathbb{Z}} \Big(F(u(x_i))\big(\varphi(x_i^-) - \varphi(x_i^+)\big) - \int_{I_i} cu\varphi_x \Big) = \int_{\mathbb{R}} g[u]\varphi. \tag{3.6}$$

To prove the above relation, multiply (3.3) by a function $\varphi \in P^k(I_i)$ and integrate over the interval I_i. Since u is continuous (by Sobolev imbedding since $u(\cdot,t) \in H^{k+1}(\mathbb{R})$ for $k \geq 0$) and the numerical flux is chosen such that $F(a,a) = f(a)$ for all $a \in \mathbb{R}$, we get that

$$\begin{aligned}
&\int_{I_i} (u_t + cu_x - g[u])\varphi \\
&= \int_{I_i} u_t\varphi - \int_{I_i} cu\varphi_x + F(u(x_{i+1}))\varphi(x_{i+1}^-) - F(u(x_i))\varphi(x_i^+) - \int_{I_i} g[u]\varphi.
\end{aligned}$$

We obtain (3.6) by summing over all $i \in \mathbb{Z}$ and rearranging the terms in the sum. Introduce the bilinear form

$$B(e,\varphi) = \int_{\mathbb{R}} e_t\varphi + \sum_{i\in\mathbb{Z}} \Big(F(e(x_i))\big(\varphi(x_i^-) - \varphi(x_i^+)\big) - \int_{I_i} ce\varphi_x \Big) - \int_{\mathbb{R}} g[e]\varphi$$

where $e = u - \tilde{u}$. Call $\mathbf{e}$ the L^2-projection of e into the set of piecewise polynomials. That is: for all $i \in \mathbb{Z}$, $\mathbf{e}|_{I_i} \in P^k(I_i)$ and

$$\int_{I_i} \big(\mathbf{e}(x) - e(x)\big)\varphi_{ji}(x)dx = 0 \quad \text{for all } j = 0, 1, \ldots, k.$$

Since $B(e,\mathbf{e}) = 0$, then $B(\mathbf{e},\mathbf{e}) = B(\mathbf{e} - e, \mathbf{e}) = B(\mathbf{u} - u, \mathbf{e})$ or

$$\begin{aligned}
\int_0^T \int_{\mathbb{R}} \mathbf{e}_t\mathbf{e} &= \int_0^T \int_{\mathbb{R}} (\mathbf{u} - u)_t\mathbf{e} - \int_0^T \sum_{i\in\mathbb{Z}} \Big(F(\mathbf{e}(x_i))\big(\mathbf{e}(x_i^-) - \mathbf{e}(x_i^+)\big) - \int_{I_i} c\mathbf{e}\mathbf{e}_x \Big) \\
&\quad + \int_0^T \sum_{i\in\mathbb{Z}} \Big(F((\mathbf{u} - u)(x_i)\big(\mathbf{e}(x_i^-) - \mathbf{e}(x_i^+)\big) - \int_{I_i} c(\mathbf{u} - u)\mathbf{e}_x \Big) \\
&\quad + \int_0^T \int_{\mathbb{R}} g[\mathbf{e}]\mathbf{e} - \int_0^T \int_{\mathbb{R}} g[\mathbf{e} - e]\mathbf{e}.
\end{aligned}$$

As shown in [13], Theorem 2.1, the local terms are less than $c_{k,T}(\Delta x)^{2k+1}$. Thus,

$$\int_0^T \int_{\mathbb{R}} \mathbf{e}_t\mathbf{e} \leq c_{k,T}(\Delta x)^{2k+1} + \int_0^T \int_{\mathbb{R}} g[\mathbf{e}]\mathbf{e} - \int_0^T \int_{\mathbb{R}} g[\mathbf{e} - e]\mathbf{e}$$

Denote by $\mathcal{I}$ what it is left to estimate on the right-hand side. By Corollary A.4,

$$\begin{aligned}
\mathcal{I} &= \frac{1}{2}\int_0^T \int_{\mathbb{R}} g[\mathbf{e}]\mathbf{e} + \frac{1}{2}\int_0^T \int_{\mathbb{R}} g[e]e - \frac{1}{2}\int_0^T \int_{\mathbb{R}} g[\mathbf{e} - e](\mathbf{e} - e) \\
&\leq \int_0^T \|(u - \mathbf{u})(\cdot,t)\|^2_{H^{\lambda/2}(\mathbb{R})}dt,
\end{aligned}$$

by Lemma A.2,

$$\|(u - \mathbf{u})(\cdot,t)\|^2_{H^{\lambda/2}(\mathbb{R})} \leq \|(u - \mathbf{u})(\cdot,t)\|^{2-\lambda}_{L^2(\mathbb{R})}\|(u - \mathbf{u})(\cdot,t)\|^{\lambda}_{H^1(\mathbb{R})},$$

and, see [9] (Section 4.4),

$$\begin{aligned}
\|(u - \mathbf{u})(\cdot,t)\|_{L^2(\mathbb{R})} &\leq c_k\|u(\cdot,t)\|_{H^{k+1}(\mathbb{R})}(\Delta x)^{k+1}, \\
\|(u - \mathbf{u})(\cdot,t)\|_{H^1(\mathbb{R})} &\leq c_k\|u(\cdot,t)\|_{H^{k+1}(\mathbb{R})}(\Delta x)^k.
\end{aligned}$$

Thus, by (3.4),

$$\int_0^T \int_{\mathbb{R}} \mathbf{e}_t \mathbf{e} \le c_{k,T} \Big[(\Delta x)^{2k+1} + (\Delta x)^{2k+2-\lambda} \Big]$$

Since $\mathbf{e}(x,0) = 0$ and $\|\mathbf{e}\| = \|(u - \tilde{u}) - (u - \mathbf{u})\| \ge \|e\| - \|u - \mathbf{u}\|$,

$$\|e(\cdot, T)\|^2_{L^2(\mathbb{R})} \le c_{k,T} \Big[(\Delta x)^{2k+1} + (\Delta x)^{2k+2-\lambda} + (\Delta x)^{2k+2} \Big] \le c_{k,T} (\Delta x)^{2k+1}.$$

□

4. Convergence in the nonlinear case

We study the nonlinear case by using only piecewise constant elements ($k = 0$):

$$\{\varphi_{0,i}, \varphi_{1,i}, \ldots, \varphi_{k,i}\} \equiv \{\varphi_{0,i}\}, \ \varphi_{0,i}(x) = \mathbf{1}_{I_i}(x)$$

where $\mathbf{1}_{I_i} : \mathbb{R} \to \mathbb{R}$ is the indicator function of the interval $I_i = (x_i, x_{i+1})$. Starting from the semidiscrete method (2.3)-(2.4), we derive a conservative monotone method which, by adapting Kuznetsov's techniques (see [26]) to our nonlocal setting, we prove converges to Alibaud's entropy solution with rate 1/2. Note that in the nonlinear case, even when pure conservation laws are considered, no results concerning the convergence rate are available for high-order polynomials ($k > 0$).

Let us introduce the time grid $t_n = n\Delta t$, $n = 0, 1, \ldots, N$ where $N\Delta t = T$ and discretize the semidiscrete method (2.3)-(2.4) in time to obtain the explicit method

$$U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x} \Big[F(U_i^n, U_{i+1}^n) - F(U_{i-1}^n, U_i^n) \Big] + \Delta t g\langle U^n \rangle_i \tag{4.1}$$

$$U_i^0 = \frac{1}{\Delta x} \int_{I_i} u_0(x) dx \tag{4.2}$$

where the nonlocal operator

$$g\langle U^n \rangle_i = \frac{1}{\Delta x} \int_{I_i} g[\bar{U}^n(x)] dx$$

and $\bar{U}^n : \mathbb{R} \to \mathbb{R}$ is the step function associated with $U^n : \mathbb{Z} \to \mathbb{R}$.

Proposition 4.1. *For all $i \in \mathbb{Z}$, $g\langle U \rangle_i = \frac{1}{\Delta x} \sum_{j \in \mathbb{Z}} m_j U_{i+j}$ where the weights*

$$m_j(\lambda, \Delta x) = \int_{I_0} g_\lambda[\mathbf{1}_{I_j}(x)] dx,$$

$\sum_{j \in \mathbb{Z}} |m_j| < \infty$, $\sum_{j \in \mathbb{Z}} m_j = 0$, $m_{-j} = m_j > 0$ *for all* $j \neq 0$, $m_0 = -d_\lambda \Delta x^{1-\lambda}$ *and*

$$d_\lambda = c_\lambda \left(\int_{|z|<1} \frac{1}{|z|^\lambda} dz + \int_{|z|>1} \frac{1}{|z|^{1+\lambda}} dz \right) > 0.$$

Proof. See appendix. □

Let us point out two consequences of Proposition 4.1. In the first place, the numerical method (4.1)-(4.2) is conservative. Indeed, since $\sum_{j \in \mathbb{Z}} |m_j| < \infty$,

$$\sum_{i \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} |m_j U_{i+j}| = \sum_{j \in \mathbb{Z}} |m_j| \sum_{i \in \mathbb{Z}} |U_i| < \infty \tag{4.3}$$

whenever $\sum_{i \in \mathbb{Z}} |U_i| < \infty$, and, since $\sum_{j \in \mathbb{Z}} m_j = 0$,

$$\sum_{i \in \mathbb{Z}} g\langle U \rangle_i = \frac{1}{\Delta x} \sum_{i \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} m_j U_{i+j} = \frac{1}{\Delta x} \sum_{j \in \mathbb{Z}} m_j \sum_{i \in \mathbb{Z}} U_i = 0.$$

Thus, for all $n \geq 0$, we have the conservative behavior $\sum_{i\in\mathbb{Z}} U_i^{n+1} = \sum_{i\in\mathbb{Z}} U_i^n$. In the second place, we can make the numerical method (4.1)-(4.2) monotone by using the following CFL condition:

$$(f_1 + f_2)\frac{\Delta t}{\Delta x} + d_\lambda \frac{\Delta t}{\Delta x^\lambda} \leq 1$$

where f_1, f_2 are the Lipschitz constants of F with respect to its first and second variable. Monotonicity means that, for all $i \in \mathbb{Z}$ and $n \geq 0$, (4.1) is increasing in all U^n. In what follows, the above CFL condition is assumed to be satisfied.

Let us introduce the time discretization into (2.2) as follows:

$$\tilde{u}(x,t) = U_i^n \text{ for all } (x,t) \in \big[i\Delta x, (i+1)\Delta x\big) \times \big[n\Delta t, (n+1)\Delta t\big).$$

Theorem 4.2. (Stability) *Let $u_0 \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R}) \cap BV(\mathbb{R})$. Then,*

i) $\|\tilde{u}(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq \|\tilde{u}_0\|_{L^\infty(\mathbb{R})}$,
ii) $\|\tilde{u}(\cdot,t)\|_{L^1(\mathbb{R})} \leq \|\tilde{u}_0\|_{L^1(\mathbb{R})}$,
iii) $|\tilde{u}(\cdot,t)|_{BV(\mathbb{R})} \leq |\tilde{u}_0|_{BV(\mathbb{R})}$.

Proof. Monotonicity plus $\sum_{j\in\mathbb{Z}} m_j = 0$ implies *i)*. The proof of *ii)* and *iii)* follows word by word the one of Theorem 3.6 in Holden *et al.* [22]. □

Let us recall the entropy formulation for problem (1.1).

Definition 4.1. *A function $u : \mathbb{R} \times [0,T] \to \mathbb{R}$, $u \in L^\infty(\mathbb{R} \times (0,T])$, is an entropy solution of (1.1) provided that, for all $k \in \mathbb{R}$ and all nonnegative $\varphi \in C_c^\infty(\mathbb{R}\times[0,T])$,*

$$\begin{aligned}\Lambda[u,\varphi,k] = &\int_0^T \int_{\mathbb{R}} \eta_k(u)\varphi_t + q_k(u)\varphi_x + \eta_k(u(x,t))g[\varphi(x,t)]dxdt \\ &+ \int_{\mathbb{R}} \eta_k(u_0(x))\varphi(x,0)dx - \int_{\mathbb{R}} \eta_k(u(x,T))\varphi(x,T)dx \geq 0 \end{aligned} \tag{4.4}$$

where $\eta_k(u) = |u-k|$ and $q_k(u) = sgn(u-k)(f(u) - f(k))$.

The above definition is different from the one introduced by Alibaud in [2]. Alibaud's definition implies Definition 4.1, but the opposite could be false for general initial data $u_0 \in L^\infty(\mathbb{R})$. However, as shown by Karlsen *et al.* in [24], Definition 4.1 is equivalent to Alibaud's definition for all initial data $u_0 \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R}) \cap BV(\mathbb{R})$.

Theorem 4.3. *Let $u_0 \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R}) \cap BV(\mathbb{R})$. Then, there exists a unique entropy solution of (1.1). Moreover,*

i) $\|u(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq \|u_0\|_{L^\infty(\mathbb{R})}$,
ii) $\|u(\cdot,t)\|_{L^1(\mathbb{R})} \leq \|u_0\|_{L^1(\mathbb{R})}$,
iii) $|u(\cdot,t)|_{BV(\mathbb{R})} \leq |u_0|_{BV(\mathbb{R})}$.

Proof. Uniqueness follows from an easy modification of the results in [24]. Alternatively, Theorem 4.5 can be used: indeed, assume, by contradiction, that u, v are both entropy solutions; then, by adding and subtracting the numerical solution $\tilde{u}$,

$$\|u(\cdot,t) - v(\cdot,t)\|_{L^1(\mathbb{R})} \leq c_t\sqrt{\Delta x}$$

for all $\Delta x > 0$. Existence and stability follow from [2, 24]. □

The following lemma generalizes to our nonlocal setting a result due to Kuznetsov (see [26]). We will use it in the proof of Theorem 4.5.

Lemma 4.4. *Let $u_0 \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R}) \cap BV(\mathbb{R})$, $u : \mathbb{R} \times [0,T] \to \mathbb{R}$ be the entropy solution of (1.1), and $\tilde{u} : \mathbb{R} \times [0,T] \to \mathbb{R}$ the solution of the numerical method (4.1)-(4.2). Let us introduce the function*

$$\varphi(x,x',t,t') = \omega_\epsilon(x-x')\omega_\delta(t-t')$$

where $\omega_\alpha \in C_c^\infty(\mathbb{R})$, $\alpha > 0$, can be built as follows: choose $\omega \in C_c^\infty(\mathbb{R})$ such that $0 \le \omega(x) \le 1$ for all $x \in \mathbb{R}$, $\omega(x) = 0$ for all $|x| > 1$ and $\int_{\mathbb{R}} \omega(x)dx = 1$; then define $\omega_\alpha(x) = \omega(x/\alpha)/\alpha$. Moreover, starting from (4.4), let us call

$$\Lambda_{\epsilon,\delta}[\tilde{u}, u] = \int_0^T \int_{\mathbb{R}} \Lambda[\tilde{u}, \varphi(\cdot, x', \cdot, t'), u(x', t')]dx'dt'.$$

Then, for all $\epsilon > 0$ and $0 < \delta < T$, there exists $c > 0$ such that

$$\|u(\cdot, T) - \tilde{u}(\cdot, T)\|_{L^1(\mathbb{R})} \le c(\epsilon + \delta + \Delta x) - \Lambda_{\epsilon,\delta}[\tilde{u}, u].$$

Proof. See appendix. □

Theorem 4.5. (Convergence) *Let $u_0 \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R}) \cap BV(\mathbb{R})$, $u : \mathbb{R} \times [0, T] \to \mathbb{R}$ be the entropy solution of (1.1), and $\tilde{u} : \mathbb{R} \times [0, T] \to \mathbb{R}$ the solution of the numerical method (4.1)-(4.2). Then, there exist $c_T > 0$ such that*

$$\|u(\cdot, T) - \tilde{u}(\cdot, T)\|_{L^1(\mathbb{R})} \le c_T\sqrt{\Delta x}. \tag{4.5}$$

The above result generalizes the convergence rate obtained by Kuznetsov in [26] for difference methods for pure conservation laws to the nonlocal numerical method (4.1)-(4.2) for fractional conservation laws.

Proof. Recall Lemma 4.4: to obtain (4.5), $-\Lambda_{\epsilon,\delta}[\tilde{u}, u]$ has to be estimated.

First part. Introduce the notation $a \wedge b = \min\{a, b\}$, $a \vee b = \max\{a, b\}$. Fix $(x', t') \in \mathbb{R} \times (0, T)$. Call $k = u(x', t')$, $\eta_i^n = |U_i^n - k|$ and $q_i^n = f(U_i^n \vee k) - f(U_i^n \wedge k)$. Note that $-\Lambda_{\epsilon,\delta}[\tilde{u}, u]$ can be rewritten as

$$\begin{aligned} -\Lambda_{\epsilon,\delta}[\tilde{u}, u] = \int_0^T \int_{\mathbb{R}} \Bigg\{ \sum_{i\in\mathbb{Z}} \sum_{n=0}^{N-1} \Big[(\eta_i^{n+1} - \eta_i^n) \int_{x_i}^{x_{i+1}} \varphi(x, t_{n+1})dx \\ + (q_i^n - q_{i-1}^n) \int_{t_n}^{t_{n+1}} \varphi(x_i, t)dt \Big] \\ - \sum_{i\in\mathbb{Z}} \sum_{n=0}^{N-1} \eta_i^n \int_{t_n}^{t_{n+1}} \int_{x_i}^{x_{i+1}} g[\varphi(x, t)]dxdt \Bigg\} dx'dt'. \end{aligned} \tag{4.6}$$

Indeed,

$$\begin{aligned} &- \sum_{i\in\mathbb{Z}} \Bigg\{ \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \int_{x_i}^{x_{i+1}} \eta_i^n \varphi_t(x, t) + q_i^n \varphi_x(x, t)dxdt \\ &\qquad + \eta_i^0 \int_{x_i}^{x_{i+1}} \varphi(x, 0)dx - \eta_i^N \int_{x_i}^{x_{i+1}} \varphi(x, T)dx \Bigg\} \\ &= - \sum_{i\in\mathbb{Z}} \Bigg\{ \sum_{n=0}^{N-1} \eta_i^n \int_{x_i}^{x_{i+1}} \big[\varphi(x, t_{n+1},) - \varphi(x, t_n,)\big]dx \\ &\qquad + \sum_{n=0}^{N-1} q_i^n \int_{t_n}^{t_{n+1}} \big[\varphi(x_{i+1}, t) - \varphi(x_i, t)\big]dt \\ &\qquad + \eta_i^0 \int_{x_i}^{x_{i+1}} \varphi(x, 0)dx - \eta_i^N \int_{x_i}^{x_{i+1}} \varphi(x, T)dx \Bigg\} \\ &= \sum_{i\in\mathbb{Z}} \sum_{n=0}^{N-1} \Big[(\eta_i^{n+1} - \eta_i^n) \int_{x_i}^{x_{i+1}} \varphi(x, t_{n+1})dx \\ &\qquad + (q_i^n - q_{i-1}^n) \int_{t_n}^{t_{n+1}} \varphi(x_i, t)dt \Big]. \end{aligned}$$

Second part. Use monotonicity to get

$$U_i^{n+1} \vee k \leq U_i^n \vee k - \frac{\Delta t}{\Delta x}\Big[F(U_i^n \vee k, U_{i+1}^n \vee k) - F(U_{i-1}^n \vee k, U_i^n \vee k)\Big] + \Delta t g\langle U^n \vee k\rangle_i,$$

$$U_i^{n+1} \wedge k \geq U_i^n \wedge k - \frac{\Delta t}{\Delta x}\Big[F(U_i^n \wedge k, U_{i+1}^n \wedge k) - F(U_{i-1}^n \wedge k, U_i^n \wedge k)\Big] + \Delta t g\langle U^n \wedge k\rangle_i.$$

Call

$$Q_i^n = F(U_i^n \vee k, U_{i+1}^n \vee k) - F(U_i^n \wedge k, U_{i+1}^n \wedge k).$$

Since $|a-b| = a \vee b - a \wedge b$, subtracting $U_i^{n+1} \wedge k$ from $U_i^{n+1} \vee k$ yields

$$\eta_i^{n+1} - \eta_i^n + \frac{\Delta t}{\Delta x}(Q_i^n - Q_{i-1}^n) - \Delta t g\langle \eta^n\rangle_i \leq 0.$$

Thus, using the above inequality in (4.6),

$$\begin{aligned}
-\Lambda_{\epsilon,\delta}[\tilde{u}, u] \leq \int_0^T \int_{\mathbb{R}} \Bigg\{ &\sum_{i\in\mathbb{Z}} \sum_{n=0}^{N-1} \Big[(q_i^n - q_{i-1}^n) \int_{t_n}^{t_{n+1}} \varphi(x_i, t)dt \\
&- \frac{\Delta t}{\Delta x}(Q_i^n - Q_{i-1}^n) \int_{x_i}^{x_{i+1}} \varphi(x, t_{n+1})dx\Big] \\
&+ \Delta t \sum_{i\in\mathbb{Z}} \sum_{n=0}^{N-1} g\langle \eta^n\rangle_i \int_{x_i}^{x_{i+1}} \varphi(x, t_{n+1})dx \\
&- \sum_{i\in\mathbb{Z}} \sum_{n=0}^{N-1} \eta_i^n \int_{t_n}^{t_{n+1}} \int_{x_i}^{x_{i+1}} g[\varphi(x,t)]dxdt \Bigg\} dx'dt'.
\end{aligned} \tag{4.7}$$

Third part. In order to proceed, the following result is needed. Its proof follows. Let $v \in l^\infty(\mathbb{Z})$ and $w \in l^1(\mathbb{Z})$. Then,

$$\sum_{i\in\mathbb{Z}} g\langle v\rangle_i w_i = \sum_{i\in\mathbb{Z}} g\langle w\rangle_i v_i. \tag{4.8}$$

As a consequence of Proposition 4.1, $g : l^\infty(\mathbb{R}) \to l^\infty(\mathbb{R})$, and, as shown in (4.3), $g : l^1(\mathbb{R}) \to l^1(\mathbb{R})$. Thus, both sides of (4.8) are finite. Moreover,

$$\sum_{i\in\mathbb{Z}} g\langle v\rangle_i w_i = \frac{1}{\Delta x}\sum_{i\in\mathbb{Z}}\sum_{j\in\mathbb{Z}} m_{j-i} v_j w_i = \frac{1}{\Delta x}\sum_{j\in\mathbb{Z}} v_j \sum_{i\in\mathbb{Z}} m_{i-j} w_i = \sum_{j\in\mathbb{Z}} g\langle w\rangle_j v_j$$

since $m_{-j} = m_j$ for all $j \in \mathbb{Z}$.

Fourth part. Write $\bar{\varphi}_i^n = ⨍_{x_i}^{x_{i+1}} \varphi(x, t_n)dx$ where $⨍_{x_i}^{x_{i+1}} = \frac{1}{\Delta x}\int_{x_i}^{x_{i+1}}$. By (4.8),

$$\sum_{i\in\mathbb{Z}} g\langle \eta^n\rangle_i \bar{\varphi}_i^n = \sum_{i\in\mathbb{Z}} g\langle \bar{\varphi}^n\rangle_i \eta_i^n.$$

Recall that $g\langle \bar{\varphi}^n\rangle_i = ⨍_{x_i}^{x_{i+1}} g[\bar{\varphi}(x, t_n)]dx$ where $\bar{\varphi}(\cdot, t_n) : \mathbb{R} \to \mathbb{R}$ is the step function associated with $\bar{\varphi}^n : \mathbb{Z} \to \mathbb{R}$. Thus, (4.7) turns into

$$\begin{aligned}
-\Lambda_{\epsilon,\delta}[\tilde{u}, u] \leq \int_0^T \int_{\mathbb{R}} \Bigg\{ &\sum_{i\in\mathbb{Z}} \sum_{n=0}^{N-1} \Big[(q_i^n - q_{i-1}^n) \int_{t_n}^{t_{n+1}} \varphi(x_i, t)dt \\
&- \frac{\Delta t}{\Delta x}(Q_i^n - Q_{i-1}^n) \int_{x_i}^{x_{i+1}} \varphi(x, t_{n+1})dx\Big] \\
&+ \Delta t \sum_{i\in\mathbb{Z}} \sum_{n=0}^{N-1} \eta_i^n \int_{x_i}^{x_{i+1}} g[\bar{\varphi}(x, t_{n+1})]dx \\
&- \sum_{i\in\mathbb{Z}} \sum_{n=0}^{N-1} \eta_i^n \int_{t_n}^{t_{n+1}} \int_{x_i}^{x_{i+1}} g[\varphi(x,t)]dxdt \Bigg\} dx'dt'.
\end{aligned}$$

From the book by Holden *et al.* (see [22], Example 3.14),

$$
\begin{aligned}
(4.9)\qquad \int_0^T\int_{\mathbb{R}}\Bigg\{\sum_{i\in\mathbb{Z}}\sum_{n=0}^{N-1}\Big[(q_i^n-q_{i-1}^n)\int_{t_n}^{t_{n+1}}\varphi(x_i,t)dt\\
-\frac{\Delta t}{\Delta x}(Q_i^n-Q_{i-1}^n)\int_{x_i}^{x_{i+1}}\varphi(x,t_{n+1})dx\Big]\Bigg\}dx'dt' \le c_T\left(\frac{\Delta x}{\epsilon}+\frac{\Delta x}{\delta}\right).
\end{aligned}
$$

Thus, $-\Lambda_{\epsilon,\delta}[\tilde{u},u]\le c_T\left(\frac{\Delta x}{\epsilon}+\frac{\Delta x}{\delta}\right)+\mathcal{I}$ where

$$
\mathcal{I}=\int_0^T\int_{\mathbb{R}}\Bigg\{\sum_{i\in\mathbb{Z}}\sum_{n=0}^{N-1}\eta_i^n\int_{t_n}^{t_{n+1}}\int_{x_i}^{x_{i+1}}g[\bar{\varphi}(x,t_{n+1})-\varphi(x,t)]dxdt\Bigg\}dx'dt'.
$$

Fifth part. The idea is to split the integral term $\mathcal{I}$ into the integrals $\mathcal{M}$ and $\mathcal{N}$. Integration by parts is needed to handle $\mathcal{M}$, the integral containing the singularity.

Write $g=H+h$ where

$$
\begin{aligned}
H[\varphi(x)]&=c_\lambda\int_{|z|<1}\frac{\varphi(x+z)-\varphi(x)}{|z|^{1+\lambda}}dz,\\
h[\varphi(x)]&=c_\lambda\int_{|z|>1}\frac{\varphi(x+z)-\varphi(x)}{|z|^{1+\lambda}}dz.
\end{aligned}
$$

Then, $\mathcal{I}=\mathcal{M}+\mathcal{N}$ where

$$
\begin{aligned}
(4.10)\quad \mathcal{M}=\int_0^T\int_{\mathbb{R}}\Bigg\{\sum_{i\in\mathbb{Z}}\sum_{n=0}^{N-1}\eta_i^n\int_{x_i}^{x_{i+1}}\int_{t_n}^{t_{n+1}}H[\bar{\varphi}(x,t_{n+1})-\varphi(x,t)]dxdt\Bigg\}dx'dt'\\
\le c_T\left(\frac{\Delta x}{\epsilon}+\frac{\Delta x}{\delta}\right),
\end{aligned}
$$

$$
\begin{aligned}
(4.11)\quad \mathcal{N}=\int_0^T\int_{\mathbb{R}}\Bigg\{\sum_{i\in\mathbb{Z}}\sum_{n=0}^{N-1}\eta_i^n\int_{x_i}^{x_{i+1}}\int_{t_n}^{t_{n+1}}h[\bar{\varphi}(x,t_{n+1})-\varphi(x,t)]dxdt\Bigg\}dx'dt'\\
\le c_T\left(\frac{\Delta x}{\epsilon}+\frac{\Delta x}{\delta}\right).
\end{aligned}
$$

The proofs of the estimates (4.10) and (4.11) can be found in the appendix.

Final part. Use (4.9), (4.10), (4.11) and Lemma 4.4 to get

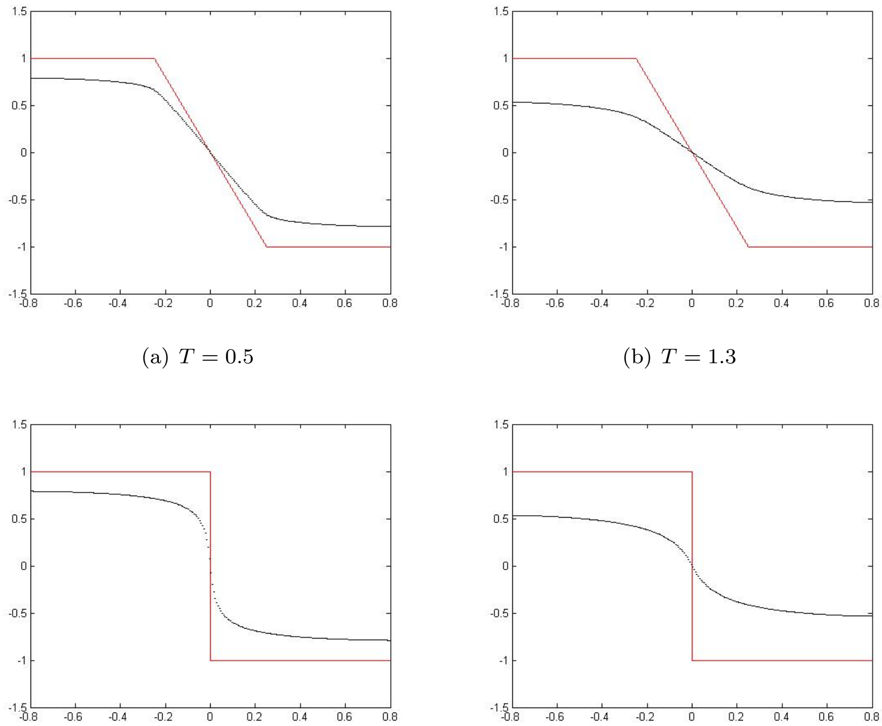
$$
\|u(\cdot,T)-\tilde{u}(\cdot,T)\|_{L^1(\mathbb{R})}\le c_T\left(\epsilon+\delta+\Delta x+\frac{\Delta x}{\epsilon}+\frac{\Delta x}{\delta}\right).
$$

Set $\epsilon=\delta=\sqrt{\Delta x}$ to conclude. □

5. Numerical experiments

We have implemented the numerical method (2.3)-(2.4) in the cases $k=0,1,2$. To perform computations, we have set the numerical solution to zero outside the region $\Omega=\{(x,t):|x|<1.5,t\ge 0\}$. In all the plots, the red solid line represents the initial datum while the black dotted one the numerical solution at $t=T$.

Remark 5.1. Due to infinite speed of propagation (see [2]), solutions of (1.1) do not have, in general, compact support. Therefore, the use of the region Ω introduces a new error which we have not considered in Theorem 3.3 and Theorem 4.5.



(a) $T = 0.5$ (b) $T = 1.3$

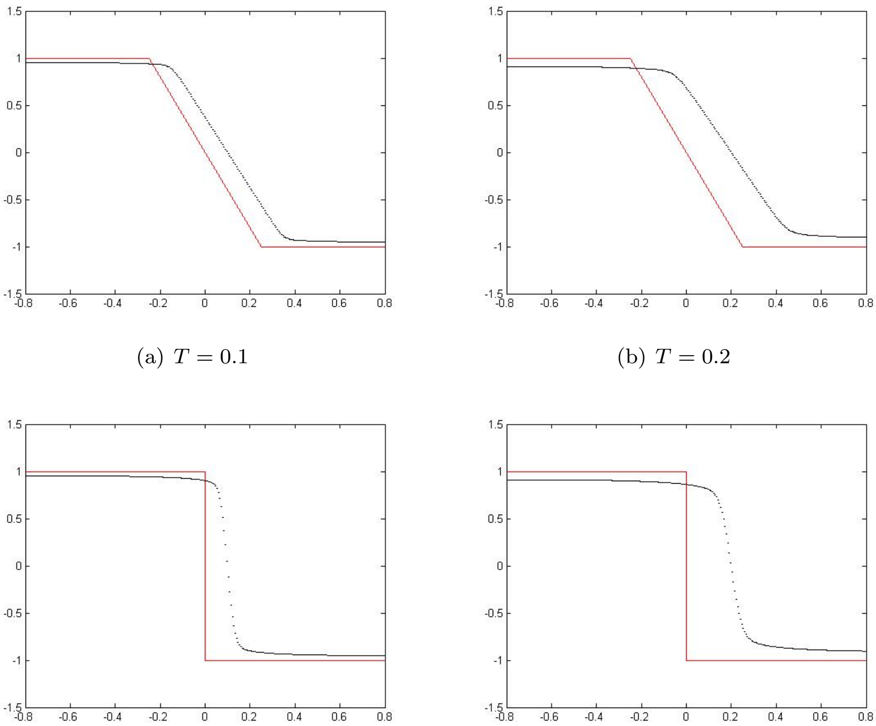
(c) $T = 0.5$ (d) $T = 1.3$

FIGURE 1. Solutions of the pure fractional equation ($\lambda = 0.5$) using the numerical method (4.1)-(4.2) with $\Delta x = 1/160$.

Example 5.1. Let us consider the pure fractional equation $\partial_t u = g[u]$. From e.g. [28], it follows that the solution of this equation is given by the convolution product $u(x,t) = (K * u_0)(x,t)$ where K is the kernel of g. Using the properties of the kernel, it can be shown that this equation has a regularizing effect on the initial datum (see e.g. [3]). This regularization appears clearly in the numerical experiments presented in FIGURE 1.

Example 5.2. Let us consider the fractional transport equation $\partial_t u + \partial_x u = g[u]$. Our numerical results suggest that, as done by $\partial_t u + \partial_x u = \partial_x^2 u$, this equation regularizes and transports the initial datum. Our numerical experiments are presented in FIGURE 2. The numerical flux (3.5) has been used.

Example 5.3. Let us consider the fractional Burgers' equation $\partial_t u + u\partial_x u = g[u]$. Our numerical experiments in FIGURE 3 confirm what has been shown by [3, 25]: this equation does not regularize the initial condition. Discontinuities in the initial datum can persist in the solution, and shocks can develop from smooth initial data. FIGURE 4 shows how the behavior of the solution changes with λ: as $\lambda \to 0$, our numerical solution approaches the solution of the pure Burgers' equation with a source, $\partial_t u + u\partial_x u = u$; as $\lambda \to 1$, our numerical solution approaches the smooth solution of the fractional Burgers' equation with $\lambda = 1$ (see [25]). FIGURE 5 clearly shows how a shock can develop and vanish in a finite time. FIGURE 6 shows how the accuracy of the semidiscrete method (2.3)-(2.4) improves with $k = 0, 1, 2$. A third order Runge-Kutta (RK3) time discretization and slope limiters (see [13])



(a) $T = 0.1$ (b) $T = 0.2$

(c) $T = 0.1$ (d) $T = 0.2$

FIGURE 2. Solutions of the fractional transport equation ($\lambda = 0.5$) using the numerical method (4.1)-(4.2) with $\Delta x = 1/160$.

have been deployed in FIGURE 6. We have used the Lax-Friedrichs flux

$$F(a,b) = \frac{1}{2}[f(a) + f(b) - c(b-a)] \quad c = \max\{|f'(a)| : |a| \leq \|u_0\|_{L^\infty(\mathbb{R})}\}$$

Let us note that the above numerical flux does not fulfill assumption *A1*. However, this assumption can be replaced with a milder one: it is enough to ask $F(a,b)$ to be Lipschitz continuous on $\{(a,b) : |a| \leq \|u_0\|_{L^\infty(\mathbb{R})}$ and $|b| \leq \|u_0\|_{L^\infty(\mathbb{R})}\}$.

To give an idea about the speed of convergence of our experiments, we have computed their rate of convergence in TABLE 1. We have measured the error

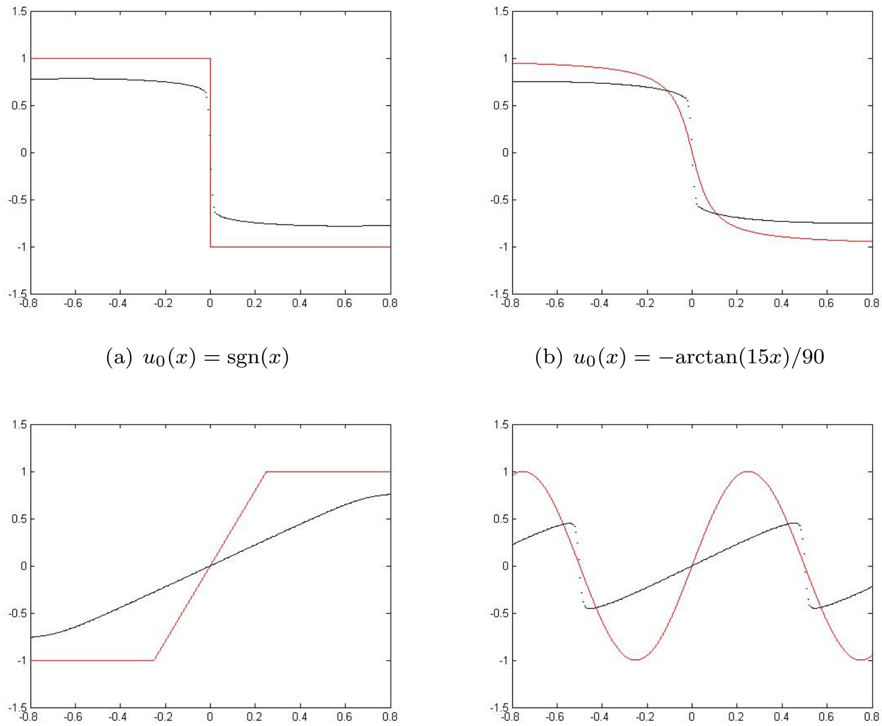
$$E_{\Delta x,p} = \|\tilde{u}_{\Delta x}(\cdot,T) - \tilde{u}_e(\cdot,T)\|_{L^p(\mathbb{R})}$$

($\tilde{u}_e$ is the numerical solution which has been computed using $\Delta x = 1/640$), the relative error $R_{\Delta x,p} = E_{\Delta x,p}/\|\tilde{u}_e(\cdot,T)\|_{L^p(\mathbb{R})}$ and the approximate rate of convergence $\alpha_{\Delta x,p} = c(\log E_{\Delta x,p} - \log E_{\Delta x/2,p})$ where $c = 1/\log 2$.

APPENDIX A. TECHNICAL LEMMAS

Lemma A.1. *Let $\varphi \in L^1(\mathbb{R}) \cap BV(\mathbb{R})$. Then, there exists $C_\lambda > 0$ such that*

$$\|g_\lambda[\varphi]\|_{L^1(\mathbb{R})} \leq c_\lambda \int_{\mathbb{R}} \int_{|z|>0} \frac{|\varphi(x+z) - \varphi(x)|}{|z|^{1+\lambda}} dz dx \leq C_\lambda \|\varphi\|_{L^1(\mathbb{R})}^{1-\lambda} |\varphi|_{BV(\mathbb{R})}^\lambda.$$



(a) $u_0(x) = \mathrm{sgn}(x)$

(b) $u_0(x) = -\arctan(15x)/90$

(c) $u_0(x) = -\mathrm{sgn}(x)$

(d) $u_0(x) = \sin(2\pi x)$

FIGURE 3. Solutions of the fractional Burgers' equation ($\lambda = 0.5$) for different initial data using (4.1)-(4.2); $T = 0.5$ and $\Delta x = 1/160$.

TABLE 1. Method (4.1)-(4.2) (left) as in FIGURE 3 (c) and RK3 method (2.3)-(2.4) (right) as in FIGURE 6 (b).

Δx	$E_{\Delta x,1}$	$R_{\Delta x,1}$	$\alpha_{\Delta x,1}$	$E_{\Delta x,2}$	$R_{\Delta x,2}$	$\alpha_{\Delta x,2}$
1/10	0.1990	0.1109	0.5726	0.4580	0.3765	1.0714
1/20	0.1338	0.0746	0.4711	0.2180	0.1792	1.2024
1/40	0.0965	0.0538	0.3964	0.0947	0.0779	1.1717
1/80	0.0734	0.0409	0.4399	0.0421	0.0346	1.0881
1/160	0.0541	0.0301	0.7235	0.0198	0.0163	
1/320	0.0327	0.0183				

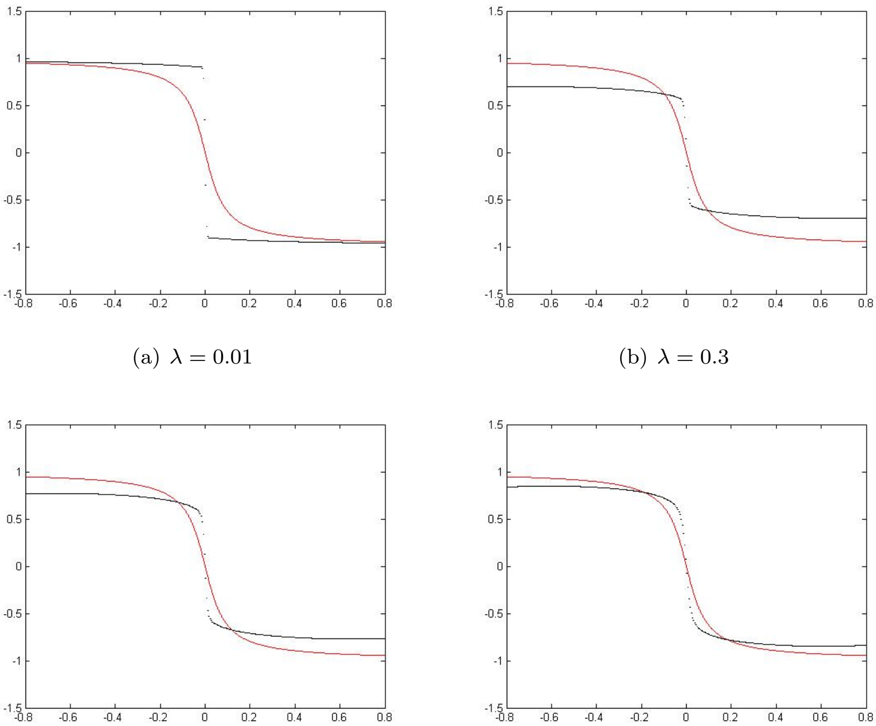
Proof. For all $\epsilon > 0$,

$$\int_{|z|<\epsilon}\int_{\mathbb{R}} \frac{|\varphi(x+z)-\varphi(x)|}{|z|^{1+\lambda}}dxdz \leq \epsilon^{1-\lambda}|\varphi|_{BV(\mathbb{R})}\int_{|z|<1}\frac{1}{|z|^{\lambda}}dz,$$
$$\int_{|z|>\epsilon}\int_{\mathbb{R}} \frac{|\varphi(x+z)-\varphi(x)|}{|z|^{1+\lambda}}dxdz \leq 2\epsilon^{-\lambda}\|\varphi\|_{L^1(\mathbb{R})}\int_{|z|>1}\frac{1}{|z|^{1+\lambda}}dz.$$

Set $\epsilon = \|\varphi\|_{L^1(\mathbb{R})}|\varphi|^{-1}_{BV(\mathbb{R})}$. □

Lemma A.2. *For all* $\varphi \in H^{\lambda}(\mathbb{R})$,

$$\|\varphi\|_{H^{\lambda}(\mathbb{R})} \leq \|\varphi\|^{1-\lambda}_{L^2(\mathbb{R})}\|\varphi\|^{\lambda}_{H^1(\mathbb{R})}.$$



(a) $\lambda = 0.01$ (b) $\lambda = 0.3$

(c) $\lambda = 0.7$ (d) $\lambda = 0.99$

FIGURE 4. Solutions of the fractional Burgers' equation for different values of λ using the method (4.1)-(4.2); $T = 0.5$ and $\Delta x = 1/200$.

Proof. Instead of (3.2), use the equivalent norm (see [21])

$$\|\varphi\|^2_{H^\lambda(\mathbb{R})} = \int_{\mathbb{R}} (1+\xi^2)^\lambda \hat{\varphi}^2(\xi) d\xi. \tag{A.1}$$

Call $A_\xi = (1+\xi^2)$. Then, for all $\epsilon > 0$,

$$\begin{aligned}\|\varphi\|^2_{H^\lambda(\mathbb{R})} &= \int_{\{\xi : A_\xi < \epsilon\}} A_\xi^\lambda \hat{\varphi}^2(\xi) d\xi + \int_{\{\xi : A_\xi > \epsilon\}} A_\xi^{\lambda-1} A_\xi \hat{\varphi}^2(\xi) d\xi \\ &\leq \epsilon^\lambda \int_{|\xi|<\epsilon} \hat{\varphi}^2(\xi) d\xi + \epsilon^{\lambda-1} \int_{|\xi|>\epsilon} A_\xi \hat{\varphi}^2(\xi) d\xi.\end{aligned}$$

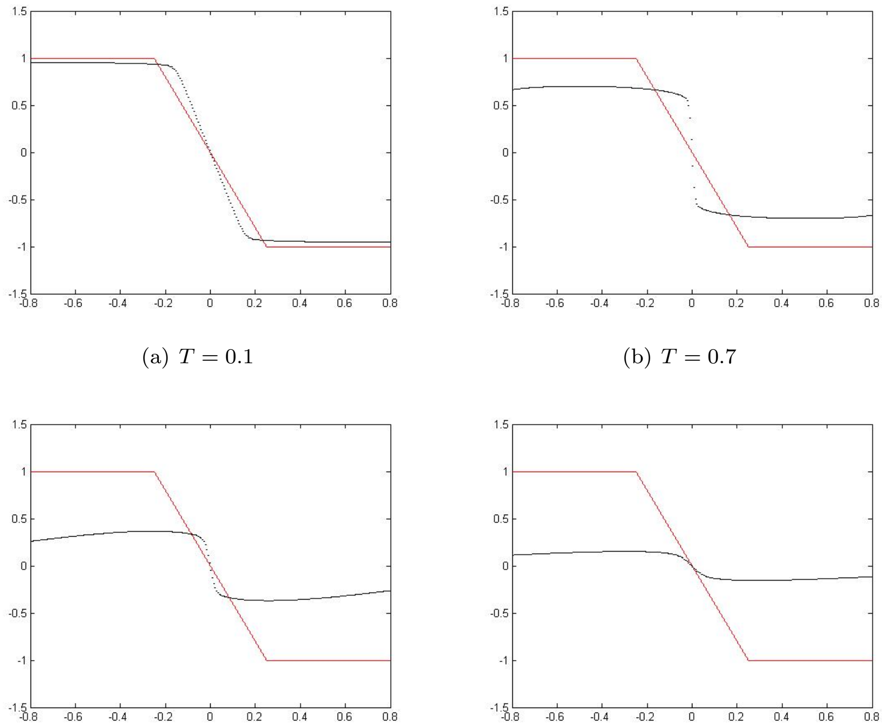
Set $\sqrt{\epsilon} = \|\varphi\|_{H^1(\mathbb{R})} \|\varphi\|^{-1}_{L^2(\mathbb{R})}$. □

Lemma A.3. *Let $\varphi, \phi \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R}) \cap BV(\mathbb{R})$. Then,*

$$\int_{\mathbb{R}} \varphi(x) g[\phi(x)] dx = \int_{\mathbb{R}} g[\varphi(x)] \phi(x) dx.$$

In particular,

$$\int_{\mathbb{R}} \varphi(x) g_\lambda[\varphi(x)] dx = -\frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{(\varphi(z) - \varphi(x))^2}{|z-x|^{1+\lambda}} dz dx.$$



(a) $T = 0.1$

(b) $T = 0.7$

(c) $T = 1.7$

(d) $T = 2.9$

FIGURE 5. Solutions of the fractional Burgers' equation ($\lambda = 0.5$) at different times T using the method (4.1)-(4.2) with $\Delta x = 1/200$.

Proof. By Lemma A.2, Fubini's theorem can be used to get

$$\int_{\mathbb{R}} \varphi(x) g[\phi(x)]dx = \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{(\phi(x)-\phi(z))(\varphi(z)-\varphi(x))}{|z-x|^{1+\lambda}} dzdx = \int_{\mathbb{R}} g[\varphi(x)]\phi(x)dx.$$

$\square$

Corollary A.4. *Lemma A.3 holds true for all $\varphi, \phi \in H^{\lambda/2}(\mathbb{R})$.*

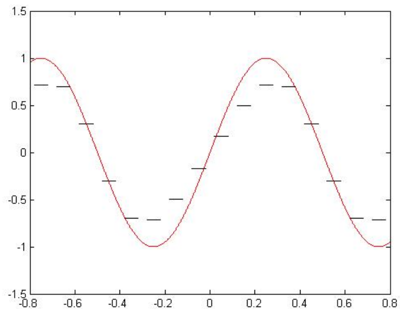
Proof. Lemma A.3 holds true, in particular, for all φ_n, ϕ_n step functions with compact support. Thus,

$$\int_{\mathbb{R}} \varphi_n(x) g[\phi_n(x)]dx = \int_{\mathbb{R}} g[\varphi_n(x)]\phi_n(x)dx. \tag{A.2}$$

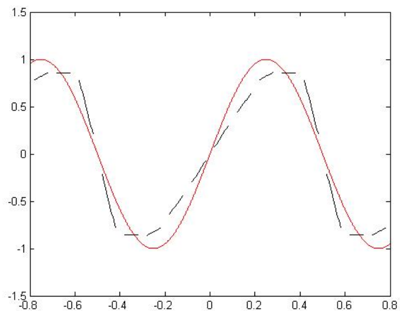
Choose, by density, $\varphi_n, \phi_n \to \varphi, \phi$ in $H^{\lambda/2}(\mathbb{R})$. Since $g[\varphi_n], g[\phi_n] \to g[\varphi], g[\phi]$ in $H^{-\lambda/2}(\mathbb{R})$, (A.2) still holds true in the limit. Indeed, using (1.2) and (A.1),

$$\begin{aligned}\|g[\varphi_n] - g[\phi]\|_{H^{-\lambda/2}(\mathbb{R})} &= \int_{\mathbb{R}} (1+\xi^2)^{-\lambda/2}\xi^{2\lambda}(\hat{\varphi}_n(\xi) - \hat{\varphi}(\xi))^2 d\xi \\ &\leq \int_{\mathbb{R}} (1+\xi^2)^{\lambda/2}(\hat{\varphi}_n(\xi) - \hat{\varphi}(\xi))^2(\xi) d\xi = \|\varphi_n - \phi\|_{H^{\lambda/2}(\mathbb{R})}\end{aligned}$$

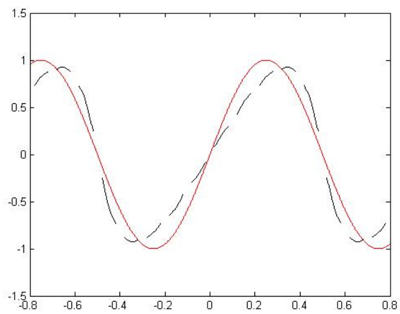
since $(1+\xi^2)^{-\lambda/2}\xi^{2\lambda} \leq (1+\xi^2)^{\lambda/2}$ for all $\xi \in \mathbb{R}$ (indeed, call $\xi^2 = x$, and multiply both sides by $(1+x)^{1-\lambda/2}$ to get $x^\lambda \leq (1+x)^\lambda$ which holds true for all $x \geq 0$). $\square$



(a) Method (4.1)-(4.2)



(b) RK3 (2.3)-(2.4) with $k = 1$



(c) RK3 (2.3)-(2.4) with $k = 2$

FIGURE 6. Solutions of the fractional Burgers' equation at $T = 1/10$ using different values of $k = 0, 1, 2$; $\Delta x = 1/10$.

APPENDIX B. PROOF OF PROPOSITION 4.1

Call $G_j^i = \int_{\mathbb{R}} \mathbf{1}_{I_i}(x) g[\mathbf{1}_{I_j}(x)]dx$. By Lemma A.3,

$$G_j^i = \int_{\mathbb{R}} \mathbf{1}_{I_i}(x) g[\mathbf{1}_{I_j}(x)]dx = \int_{\mathbb{R}} \mathbf{1}_{I_j}(x) g[\mathbf{1}_{I_i}(x)]dx = G_i^j$$

Thus, by Lemma A.1, $\sum_{j\in\mathbb{Z}} |G_j^i| \leq \int_{\mathbb{R}} |g[\mathbf{1}_{I_i}(x)]|dx < \infty$ and, by symmetry,

$$\sum_{j\in\mathbb{Z}} G_j^i = c_\lambda \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{\mathbf{1}_{I_i}(z) - \mathbf{1}_{I_i}(x)}{|z - x|^{1+\lambda}} dzdx = 0$$

All diagonal elements are equal and negative. Indeed,

$$G_i^i = c_\lambda \int_{I_i} \int_{|z|>0} \frac{\mathbf{1}_{I_i}(x+z) - \mathbf{1}_{I_i}(x)}{|z|^{1+\lambda}} dz dx = c_\lambda \int_{|z|>0} \frac{\xi(z)}{|z|^{1+\lambda}} dz$$

where

$$\xi(z) = \begin{cases} -|z| & z \in (-\Delta x, \Delta x) \\ -\Delta x & otherwise \end{cases}$$

Thus,

$$G_i^i = -c_\lambda \Bigg(\int_{|z|<1} \frac{1}{|z|^\lambda} dz + \int_{|z|>1} \frac{1}{|z|^{1+\lambda}} dz \Bigg) \Delta x^{1-\lambda}$$

All elements outside the diagonal are positive. Moreover, $G_{j+1}^{i+1} = G_j^i$ for all $i, j \in \mathbb{Z}$. Indeed, whenever $i \neq j$,

$$G_j^i = c_\lambda \int_{I_i} \int_{|z|>0} \frac{\mathbf{1}_{I_j}(x+z)}{|z|^{1+\lambda}} dz dx$$

Therefore, $(G_j^i)_{i,j\in\mathbb{Z}}$ can be built by repeatedly shifting $(G_j^0)_{j\in\mathbb{Z}}$ by one position. Thus, $(m_j)_{j\in\mathbb{Z}} = (G_j^0)_{j\in\mathbb{Z}}$ are the weights we were looking for.

Appendix C. Proof of Lemma 4.4

The idea is to use the symmetry of φ. Namely, $\varphi_t = -\varphi_{t'}$, $\varphi_x = -\varphi_{x'}$ and

$$\int_{|z|>0} \frac{\varphi(x+z, x') - \varphi(x, x')}{|z|^{1+\lambda}} dz = -\int_{|z|>0} \frac{\varphi(x, x'+z) - \varphi(x, x')}{|z|^{1+\lambda}} dz.$$

Thus,

$$\begin{aligned}
&\Lambda_{\epsilon,\delta}[\tilde{u}, u] = -\Lambda_{\epsilon,\delta}[u, \tilde{u}] \\
&- \int_0^T \int_{\mathbb{R}} \int_{\mathbb{R}} \varphi(x, x', t, T) \Bigg(|\tilde{u}(x, T) - u(x', t)| + |u(x', T) - \tilde{u}(x, t)| \Bigg) dx dx' dt \\
&+ \int_0^T \int_{\mathbb{R}} \int_{\mathbb{R}} \varphi(x, x', t, 0) \Bigg(|u(x', 0) - \tilde{u}(x, t)| + |\tilde{u}(x, 0) - u(x', t)| \Bigg) dx dx' dt.
\end{aligned}$$

Since u is an entropy solution of (1.1), $\Lambda_{\epsilon,\delta}[u, \tilde{u}] \geq 0$ and, from this point on, the proof follows the one contained in the book by Holden *et al.* ([22], Theorem 3.11). Proposition C.1 is needed there.

Proposition C.1. *Let $u_0 \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R}) \cap BV(\mathbb{R})$, $u : \mathbb{R} \times [0, T] \to \mathbb{R}$ be the entropy solution of (1.1) and $\tilde{u} : \mathbb{R} \times [0, T] \to \mathbb{R}$ the solution of the numerical method (4.1)-(4.2). Then, there exist $c > 0$ such that*

i) $\|u(\cdot, t + \delta) - u(\cdot, t)\|_{L^1(\mathbb{R})} \leq c\delta$

ii) $\sup_{|\tau|\leq\delta} \|\tilde{u}(\cdot, t + \tau) - \tilde{u}(\cdot, t)\|_{L^1(\mathbb{R})} \leq c(\Delta x + \delta)$

Proof. Item *i).* Choose $0 < a < b < T$ and let $\mathbf{1}^\epsilon_{[a,b]} : \mathbb{R} \to \mathbb{R}$ be a smooth approximation of $\mathbf{1}_{[a,b]}$. Call $\varphi^\epsilon(x, t) = \phi(x)\mathbf{1}^\epsilon_{[a,b]}(t)$ where $\phi \in C_c^\infty(\mathbb{R})$. Thus,

$$\int_0^T \int_{\mathbb{R}} u\varphi_t^\epsilon + f(u)\varphi_x^\epsilon + ug[\varphi^\epsilon] dx dt = 0$$

since u is an entropy solution of (1.1) (and so a weak solution, see [2] for more details about the above equality). The limit for $\epsilon \to 0$ is, cf. [22] (Theorem 7.10),

$$\int_{\mathbb{R}} \phi(x) \Big(u(x, a) - u(x, b) \Big) dx + \int_a^b \int_{\mathbb{R}} f(u)\phi_x + ug[\phi] dx dt = 0$$

As shown in [22], Theorem 7.10,

$$\begin{aligned}\|u(\cdot,b)-u(\cdot,a)\|_{L^1(\mathbb{R})} &= \sup_{|\phi|\le 1}\int_{\mathbb{R}}\phi(x)\Big(u(x,b)-u(x,a)\Big)dx\\ &= \sup_{|\phi|\le 1}\left\{-\int_a^b\int_{\mathbb{R}}(f(u)\phi_x+ug[\phi])dxdt\right\}\\ &\le c|u_0|_{BV(\mathbb{R})}\|f\|_{\mathrm{Lip}}(b-a)+\sup_{|\phi|\le 1}\left\{-\int_a^b\int_{\mathbb{R}}ug[\phi]dxdt\right\}\end{aligned}$$

In order to conclude the proof, the following estimate is needed:

$$\begin{aligned}\sup_{|\phi|\le 1}\left\{-\int_a^b\int_{\mathbb{R}}ug[\phi]dxdt\right\} = \sup_{|\phi|\le 1}\left\{-\int_a^b\int_{\mathbb{R}}\phi g[u]dxdt\right\} &\le \int_a^b\int_{\mathbb{R}}\big|g[u]\big|dxdt\\ &\le c(b-a)|u_0|^{\lambda}_{BV(\mathbb{R})}\|u_0\|^{1-\lambda}_{L^1(\mathbb{R})}\end{aligned}$$

where Lemma A.3, Lemma A.1, and Theorem 4.2 have been used. Use the same ideas in the proof of *ii)*. Start from the numerical method (4.1)-(4.2) to get

$$\|\tilde u(\cdot,t_{n+1})-\tilde u(\cdot,t_n)\|_{L^1(\mathbb{R})}\le c\left\{\|F\|_{\mathrm{Lip}}|u_0|_{BV(\mathbb{R})}+|u_0|^{\lambda}_{BV(\mathbb{R})}\|u_0\|^{1-\lambda}_{L^1(\mathbb{R})}\right\}\Delta x.$$

□

Appendix D. Proof of estimate (4.10)

Recall that $\bar\varphi(\cdot,t):\mathbb{R}\to\mathbb{R}$ is the step function generated by $\bar\varphi^t:\mathbb{Z}\to\mathbb{R}$, $\bar\varphi^t_i = \fint_{x_i}^{x_{i+1}}\varphi(x,t)dx$. That is to say,

$$\bar\varphi(x,x',t,t')=\bar\omega_\epsilon(x-x')\omega_\delta(t-t')$$

where $\bar\omega_\epsilon:\mathbb{R}\to\mathbb{R}$ is the step function generated by $\bar\omega_\epsilon:\mathbb{Z}\to\mathbb{R}$, $\bar\omega^i_\epsilon=\fint_{x_i}^{x_{i+1}}\omega_\epsilon(x)dx$. The following estimate is going to be used:

$$\|\omega_\epsilon-\bar\omega_\epsilon\|_{L^1(\mathbb{R})}\le c|\omega_\epsilon|_{BV(\mathbb{R})}\Delta x\le c\frac{\Delta x}{\epsilon}. \tag{D.1}$$

In the above expression and in what follows, $c>0$ is a large number.

By adding and subtracting identical terms, $\mathcal{M}=\mathcal{M}_1+\mathcal{M}_2$ where

$$\begin{aligned}\mathcal{M}_1 &= \int_0^T\int_{\mathbb{R}}\left\{\sum_{i\in\mathbb{Z}}\sum_{n=0}^{N-1}\eta_i^n\int_{t_n}^{t_{n+1}}\int_{x_i}^{x_{i+1}}H[\bar\varphi(x,t_{n+1})-\phi(x,t_{n+1})]dxdt\right\}dx'dt',\\ \mathcal{M}_2 &= \int_0^T\int_{\mathbb{R}}\left\{\sum_{i\in\mathbb{Z}}\sum_{n=0}^{N-1}\eta_i^n\int_{t_n}^{t_{n+1}}\int_{x_i}^{x_{i+1}}H[\phi(x,t_{n+1})-\varphi(x,t)]dxdt\right\}dx'dt'.\end{aligned}$$

The function $\phi(x,x',t,t')=\nu(x-x')\omega_\delta(t-t')$ where ν is a smooth approximation of the step function $\bar\omega_\epsilon$ such that

$$\|\bar\omega_\epsilon-\nu\|_{L^1(\mathbb{R})}\le c|\omega_\epsilon|_{BV(\mathbb{R})}\Big(\frac{\Delta x}{\epsilon}\Big)^{\frac{1}{1-\lambda}}. \tag{D.2}$$

The strategy to estimate $\mathcal{M}_2$ is the following: integration by parts is performed to pass a derivative from H to η_i^n (which becomes $\eta_{i+1}^n-\eta_i^n$); then, use the inequality

$$\sum_{i\in\mathbb{Z}}|\eta_{i+1}^n-\eta_i^n|\le\sum_{i\in\mathbb{Z}}|U_{i+1}^n-U_i^n|\le|u_0|_{BV(\mathbb{R})}.$$

However, integration by parts can be used only with smooth functions; thus, the auxiliary smooth function ν, which approximate the step function $\bar\omega_\epsilon$, is introduced.

In order to use Lemma A.1 to estimate the remaining $\mathcal{M}_1$, the smooth approximation ν has to be chosen suitably close to $\bar{\omega}_\epsilon$ in L^1-norm (cf. (D.2)).

Estimate of $\mathcal{M}_1$. $\mathcal{M}_1 \leq \mathcal{M}_{1,1} + \mathcal{M}_{1,2}$ where

$$\mathcal{M}_{1,1} = \int_0^T \int_{\mathbb{R}} \left\{ \sum_{i\in\mathbb{Z}} \sum_{n=0}^{N-1} |U_i^n| \int_{t_n}^{t_{n+1}} \int_{x_i}^{x_{i+1}} \big|H[\bar{\varphi}(x,t_{n+1}) - \phi(x,t_{n+1})]\big| dxdt \right\} dx'dt',$$

$$\mathcal{M}_{1,2} = \int_0^T \int_{\mathbb{R}} \left\{ \sum_{i\in\mathbb{Z}} \sum_{n=0}^{N-1} |u| \int_{t_n}^{t_{n+1}} \int_{x_i}^{x_{i+1}} \big|H[\bar{\varphi}(x,t_{n+1}) - \phi(x,t_{n+1})]\big| dxdt \right\} dx'dt'.$$

First, for all $x \in (x_i, x_{i+1})$,

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}} \big|H[\bar{\varphi}(x,x',t_{n+1},t') - \phi(x,x',t_{n+1},t')]\big| dx'dt' \\
&= \int_0^T \omega_\delta(t_{n+1}-t')dt' \int_{\mathbb{R}} \int_{|z|<1} \frac{|\bar{\omega}_\epsilon(x+z-x') - \nu(x-x')|}{|z|^{1+\lambda}} dzdx' \\
&\leq c\|\bar{\omega}_\epsilon - \nu\|_{L^1(\mathbb{R})}^{1-\lambda} |\bar{\omega}_\epsilon - \nu|_{BV(\mathbb{R})}^{\lambda}
\end{aligned}$$

by Lemma A.1. Thus, by (D.2) and Theorem 4.2,

$$\mathcal{M}_{1,1} \leq c\|\bar{\omega}_\epsilon - \nu\|_{L^1(\mathbb{R})}^{1-\lambda} |\bar{\omega}_\epsilon - \nu|_{BV(\mathbb{R})}^{\lambda} \sum_{i\in\mathbb{Z}} \sum_{n=0}^{N-1} |U_i^n| \Delta x \Delta t \leq cT\frac{\Delta x}{\epsilon}\|u_0\|_{L^1(\mathbb{R})}.$$

Second, for all $(x',t') \in \mathbb{R} \times (0,T)$, we similarly find

$$\begin{aligned}
&\sum_{i\in\mathbb{Z}} \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \int_{x_i}^{x_{i+1}} \big|H[\bar{\varphi}(x,x',t_{n+1},t') - \phi(x,x',t_{n+1},t')]\big| dx \\
&= \Delta t \sum_{n=0}^{N-1} \omega_\delta(t_{n+1}-t') \int_{\mathbb{R}} \int_{|z|<1} \frac{|\bar{\omega}_\epsilon(x+z-x') - \nu(x-x')|}{|z|^{1+\lambda}} dzdx \\
&\leq c\|\bar{\omega}_\epsilon - \nu\|_{L^1(\mathbb{R})}^{1-\lambda} |\bar{\omega}_\epsilon - \nu|_{BV(\mathbb{R})}^{\lambda}.
\end{aligned}$$

Thus, $\mathcal{M}_{1,2} \leq c\|\bar{\omega}_\epsilon - \nu\|_{L^1(\mathbb{R})}^{1-\lambda} |\bar{\omega}_\epsilon - \nu|_{BV(\mathbb{R})}^{\lambda} \int_0^T \int_{\mathbb{R}} |u(x',t')|dx'dt' \leq cT\frac{\Delta x}{\epsilon}\|u_0\|_{L^1(\mathbb{R})}$.

Estimate of $\mathcal{M}_2$. Note that, for all $\varphi \in C_c^\infty(\mathbb{R})$, integration by parts yields

$$\begin{aligned}
H[\varphi(x)] &= -\frac{c_\lambda \operatorname{sgn}(z)}{\lambda|z|^\lambda}[\varphi(x+z) - \varphi(x)]\Big|_{|z|=1} + \frac{c_\lambda}{\lambda}\int_{|z|<1} \frac{\varphi'(x+z)}{|z|^\lambda}\operatorname{sgn}(z)dz \\
&= -\frac{c_\lambda}{\lambda}[\varphi(x+1) + \varphi(x-1) - 2\varphi(x)] + \frac{d}{dx}\rho[\varphi(x)]
\end{aligned}$$

where $\rho[\varphi(x)] = \frac{c_\lambda}{\lambda}\int_{|z|<1} \frac{\varphi(x+z)}{|z|^\lambda}\operatorname{sgn}(z)dz$. Thus, $\mathcal{M}_2 \leq \mathcal{M}_{2,1} + \mathcal{M}_{2,2}$ where

$$\mathcal{M}_{2,1} = c\int_0^T \int_{\mathbb{R}} \left\{ \sum_{i\in\mathbb{Z}} \sum_{n=0}^{N-1} \eta_i^n \int_{t_n}^{t_{n+1}} \int_{x_i}^{x_{i+1}} |\phi(x,t_{n+1}) - \varphi(x,t)| dxdt \right\} dx'dt',$$

$$\mathcal{M}_{2,2} = \int_0^T \int_{\mathbb{R}} \left\{ \sum_{i\in\mathbb{Z}} \sum_{n=0}^{N-1} \eta_i^n \int_{t_n}^{t_{n+1}} \int_{x_i}^{x_{i+1}} \partial_x\rho[\phi(x,t_{n+1}) - \varphi(x,t)] dxdt \right\} dx'dt'.$$

To estimate $\mathcal{M}_{2,1}$ proceed as done in the proof of the estimate (4.11).

Estimate of $\mathcal{M}_{2,2}$. Integration by parts has made the term $\mathcal{M}_{2,2}$ suitable for summation by parts. Indeed,

$$\mathcal{M}_{2,2} = \int_0^T \int_{\mathbb{R}} \left\{ \sum_{i\in\mathbb{Z}} \sum_{n=0}^{N-1} (\eta_{i+1}^n - \eta_i^n) \int_{t_n}^{t_{n+1}} \rho[\phi(x_i,t_{n+1}) - \varphi(x_i,t)] dt \right\} dx'dt'.$$

Thus, $\mathcal{M}_{2,2} \le \mathcal{M}_{2,2,1} + \mathcal{M}_{2,2,2}$ where

$$\mathcal{M}_{2,2,1} = \int_0^T \int_{\mathbb{R}} \left\{ \sum_{i\in\mathbb{Z}} \sum_{n=0}^{N-1} |\eta_{i+1}^n - \eta_i^n| \int_{t_n}^{t_{n+1}} \big|\rho[\phi(x_i,t_{n+1}) - \phi(x_i,t)]\big| dt \right\} dx' dt',$$

$$\mathcal{M}_{2,2,2} = \int_0^T \int_{\mathbb{R}} \left\{ \sum_{i\in\mathbb{Z}} \sum_{n=0}^{N-1} |\eta_{i+1}^n - \eta_i^n| \int_{t_n}^{t_{n+1}} \big|\rho[\phi(x_i,t) - \varphi(x_i,t)]\big| dt \right\} dx' dt'.$$

First, for all $t \in (t_n, t_{n+1})$,

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}} \big|\rho[\phi(x_i,x',t_{n+1},t') - \phi(x_i,x',t,t')]\big| dx' dt' \\
&= \int_0^T |\omega_\delta(t_{n+1}-t') - \omega_\delta(t-t')| dt' \int_{\mathbb{R}} \int_{|z|<1} \frac{\nu(x_i+z-x')}{|z|^\lambda} \operatorname{sgn}(z) dz dx' \\
&\le c\Delta x |\omega_\delta|_{BV(\mathbb{R})}.
\end{aligned}$$

Thus, using $|\eta_{i+1}^n - \eta_i^n| \le |U_{i+1}^n - U_i^n|$ (D.1) and Theorem 4.2,

$$\mathcal{M}_{2,2,1} \le c\Delta x |\omega_\delta|_{BV(\mathbb{R})} \Delta t \sum_{n=0}^{N-1} \sum_{i\in\mathbb{Z}} |U_{i+1}^n - U_i^n| \le cT \frac{\Delta x}{\delta} |u_0|_{BV(\mathbb{R})}.$$

Second, for all $t \in (t_n, t_{n+1})$,

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}} \big|\rho[\phi(x_i,x',t,t') - \varphi(x_i,x',t,t')]\big| dx' dt' \\
&= \int_0^T \omega_\delta(t-t') dt' \int_{\mathbb{R}} \int_{|z|<1} \frac{|(\nu-\omega_\epsilon)(x_i+z-x') - (\nu-\omega_\epsilon)(x_i-x')|}{|z|^\lambda} \operatorname{sgn}(z) dz dx' \\
&\le c\|\omega_\epsilon - \nu\|_{L^1(\mathbb{R})}.
\end{aligned}$$

Thus, using (D.1) (D.2) and Theorem 4.2,

$$\mathcal{M}_{2,2,2} \le c\|\omega_\epsilon - \nu\|_{L^1(\mathbb{R})} \Delta t \sum_{n=0}^{N-1} \sum_{i\in\mathbb{Z}} |U_{i+1}^n - U_i^n| \le cT \frac{\Delta x}{\epsilon} |u_0|_{BV(\mathbb{R})}.$$

Appendix E. Proof of estimate (4.11)

By adding and subtracting identical terms, $\mathcal{N} \le \mathcal{N}_1 + \mathcal{N}_2 + \mathcal{N}_3 + \mathcal{N}_4$ where

$$\mathcal{N}_1 = \int_0^T \int_{\mathbb{R}} \left\{ \sum_{i\in\mathbb{Z}} \sum_{n=0}^{N-1} |U_i^n| \int_{t_n}^{t_{n+1}} \int_{x_i}^{x_{i+1}} \big|h[\bar\varphi(x,t_{n+1}) - \bar\varphi(x,t)]\big| dx dt \right\} dx' dt',$$

$$\mathcal{N}_2 = \int_0^T \int_{\mathbb{R}} \left\{ \sum_{i\in\mathbb{Z}} \sum_{n=0}^{N-1} |u| \int_{t_n}^{t_{n+1}} \int_{x_i}^{x_{i+1}} \big|h[\bar\varphi(x,t_{n+1}) - \bar\varphi(x,t)]\big| dx dt \right\} dx' dt',$$

$$\mathcal{N}_3 = \int_0^T \int_{\mathbb{R}} \left\{ \sum_{i\in\mathbb{Z}} \sum_{n=0}^{N-1} |U_i^n| \int_{t_n}^{t_{n+1}} \int_{x_i}^{x_{i+1}} \big|h[\bar\varphi(x,t) - \varphi(x,t)]\big| dx dt \right\} dx' dt',$$

$$\mathcal{N}_4 = \int_0^T \int_{\mathbb{R}} \left\{ \sum_{i\in\mathbb{Z}} \sum_{n=0}^{N-1} |u| \int_{t_n}^{t_{n+1}} \int_{x_i}^{x_{i+1}} \big|h[\bar\varphi(x,t) - \varphi(x,t)]\big| dx dt \right\} dx' dt'.$$

In what follows, $c > 0$ is a large number.

First, for all $(x,t)\in(x_i,x_{i+1})\times(t_n,t_{n+1})$,

$$\begin{aligned}
&\int_0^T\int_{\mathbb{R}}\big|h[\bar\varphi(x,x',t_{n+1},t')-\bar\varphi(x,x',t,t')]\big|dx'dt'\\
&=\int_0^T|\omega_\delta(t_{n+1}-t')-\omega_\delta(t-t')|dt'\int_{\mathbb{R}}\int_{|z|>1}\frac{|\bar\omega_\epsilon(x+z-x')-\bar\omega_\epsilon(x-x')|}{|z|^{1+\lambda}}dzdx'\\
&\le c\Delta x|\omega_\delta|_{BV(\mathbb{R})}.
\end{aligned}$$

Thus, by (D.1) and Theorem 4.2,

$$\mathcal{N}_1\le c\Delta x|\omega_\delta|_{BV(\mathbb{R})}\sum_{i\in\mathbb{Z}}\sum_{n=0}^{N-1}|U_i^n|\Delta x\Delta t\le cT\frac{\Delta x}{\delta}\|u_0\|_{L^1(\mathbb{R})}.$$

Second, for all $(x',t')\in\mathbb{R}\times(0,T)$, we similarly find

$$\sum_{i\in\mathbb{Z}}\sum_{n=0}^{N-1}\int_{t_n}^{t_{n+1}}\int_{x_i}^{x_{i+1}}\big|h[\bar\varphi(x,x',t_{n+1},t')-\bar\varphi(x,x',t,t')]\big|dxdt\le c\Delta x|\omega_\delta|_{BV(\mathbb{R})}.$$

Thus, $\mathcal{N}_2\le c\Delta x|\omega_\delta|_{BV(\mathbb{R})}\int_0^T\int_{\mathbb{R}}|u(x',t')|dx'dt'\le cT\frac{\Delta x}{\delta}\|u_0\|_{L^1(\mathbb{R})}$.

Third, for all $(x,t)\in(x_i,x_{i+1})\times(t_n,t_{n+1})$,

$$\begin{aligned}
&\int_0^T\int_{\mathbb{R}}\big|h[\bar\varphi(x,x',t,t')-\varphi(x,x',t,t')]\big|dx'dt'\\
&=\int_0^T\omega_\delta(t-t')dt'\int_{\mathbb{R}}\int_{|z|>1}\frac{|(\bar\omega_\epsilon-\omega_\epsilon)(x+z-x')-(\bar\omega_\epsilon-\omega_\epsilon)(x-x')|}{|z|^{1+\lambda}}dzdx'\\
&\le c\|\bar\omega_\epsilon-\omega_\epsilon\|_{L^1(\mathbb{R})}.
\end{aligned}$$

Thus, by (D.1) and Theorem 4.2,

$$\mathcal{N}_3\le c\|\bar\omega_\epsilon-\omega_\epsilon\|_{L^1(\mathbb{R})}\sum_{i\in\mathbb{Z}}\sum_{n=0}^{N-1}|U_i^n|\Delta x\Delta t\le cT\frac{\Delta x}{\epsilon}\|u_0\|_{L^1(\mathbb{R})}.$$

Fourth, for all $(x',t')\in\mathbb{R}\times(0,T)$, we similarly find

$$\int_0^T\int_{\mathbb{R}}\big|h[\bar\varphi(x,x',t,t')-\varphi(x,x',t,t')]\big|dxdt\le c\|\bar\omega_\epsilon-\omega_\epsilon\|_{L^1(\mathbb{R})}.$$

Thus, $\mathcal{N}_4\le c\|\bar\omega_\epsilon-\omega_\epsilon\|_{L^1(\mathbb{R})}\int_0^T\int_{\mathbb{R}}|u(x',t')|dx'dt'\le cT\frac{\Delta x}{\epsilon}\|u_0\|_{L^1(\mathbb{R})}$.

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