A ROBUST WELL-BALANCED SCHEME FOR MULTI-LAYER SHALLOW WATER EQUATIONS

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ABSTRACT. The numerical resolution of the multi-layer shallow water system encounters two additional difficulties with respect to the one-layer system. The first is that the system involves nonconservative terms, and the second is that it is not always hyperbolic. A splitting scheme has been proposed by Bouchut and Morales, that enables to ensure a discrete entropy inequality and the well-balanced property, without any theoretical difficulty related to the loss of hyperbolicity. However, this scheme has been shown to often give wrong solutions. We introduce here a variant of the splitting scheme, that has the overall property of being conservative in the total momentum. It is based on a source-centered hydrostatic scheme for the one-layer shallow water system, a variant of the hydrostatic scheme. The final method enables to treat an arbitrary number m of layers, with arbitrary densities $\rho_1, \ldots, \rho_m$, and arbitrary topography. It has no restriction concerning complex eigenvalues, it is well-balanced and it is able to treat vacuum, it satisfies a semi-discrete entropy inequality. The scheme is fast to execute, as is the one-layer hydrostatic method.

1. INTRODUCTION

The multi-layer shallow water model is used to describe incompressible flows in the shallow water regime, in the situation where several layers with different densities can be identified. With rotating Coriolis force, the model can be used to describe large scale atmosphere or ocean flows. In the setting of this model, the following assumptions are needed: shallowness of the layers with respect to horizontal scales, small viscosity, hydrostatics, small slope for the topography, almost uniform velocities across each layer, uniform densities for each layer.

Suitable numerical methods for solving one-layer shallow water models have been developed in the last fifteen years, and in particular for getting the so called well-balanced property, that enables to exactly

resolve steady states at rest. Well-recognized methods are those of [8]
and [4] (that are indeed identical at first order), in the context of the
Roe solver, and the hydrostatic reconstruction method of [3].

The two-layer shallow water model have been studied more recently,
and involves much more difficulties, because of nonconservative terms
and nonhyperbolicity. The only works that deal with it are [8], [2], [1],
[6].

Here we introduce a variant of the splitting method of [6], that en-
ables to correct the problem of wrong solutions with unexpected dis-
continuities found in [6]. The scheme involves a source-centered hydro-
static reconstruction scheme for the one-layer system, a variant of the
hydrostatic reconstruction scheme. In contrast to the previous works,
this new scheme allows to treat nonhyperbolicity, arbitrary number
of layers, and arbitrary densities in a very robust manner, it is well-
balanced and satisfies a semi-discrete entropy inequality.

2. One-layer shallow water

Before writing the multi-layer model, let us first recall the main
features of the one-layer shallow water model.

2.1. **The system.** The equations of the one-dimensional one-layer shal-
low water problem are:

$$\begin{cases} \partial_t h + \partial_x(hu) = 0, \\ \partial_t(hu) + \partial_x(hu^2 + gh^2/2) + hg\partial_x z = 0, \end{cases} \tag{1}$$

where $h(t,x) \geq 0$ is the water height, $u(t,x) \in \mathbb{R}$ is the velocity, $z(x)$
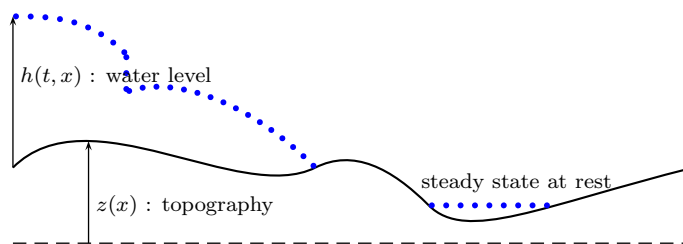is the topography, and $g > 0$ (see Figure 1). The main properties are

$h(t,x)$: water level

$z(x)$: topography

steady state at rest

FIGURE 1. One layer

that

- The system has a convex entropy $\widehat{\eta} = hu^2/2 + gh^2/2 + hgz$ with
  entropy flux $\widehat{G} = (\widehat{\eta} + gh^2/2)u$.

- The steady states are characterized by

$$hu = cst, \qquad u^2/2 + g(h+z) = cst. \tag{2}$$

  The steady states at rest are those for which $u = 0$ and $h + z = cst$.
- The system is hyperbolic, except at resonant points (crossing of eigenvalues), defined by $u = \pm\sqrt{gh}$.

2.2. Numerical difficulties.
The desirable properties of a numerical method for solving the one-layer shallow water system are of:

- keeping the water height h nonnegative,
- being able to compute dry areas where $h = 0$,
- preserving the total mass,
- maintaining steady states at rest (well-balanced property),
- satisfying a discrete entropy inequality,
- producing stable computations (no oscillations) for all data, including transcritical cases.

2.3. Hydrostatic reconstruction scheme.
The hydrostatic reconstruction scheme proposed in [3] satisfies all of the above properties (except the entropy inequality which is only semi-discrete), and is cheap computationally. With $U = (h, hu)$ it can be written with classical notations (see [7])

$$U_i^{n+1} - U_i + \frac{\Delta t}{\Delta x_i}\left(F_{i+1/2-} - F_{i-1/2+}\right) = 0, \tag{3}$$

$$F_{i+1/2-} = \mathcal{F}_l(U_i, U_{i+1}, z_i, z_{i+1}), \qquad F_{i+1/2+} = \mathcal{F}_r(U_i, U_{i+1}, z_i, z_{i+1}), \tag{4}$$

$$\mathcal{F}_l(U_l, U_r, z_l, z_r) = \mathcal{F}(U_l^*, U_r^*) + \begin{pmatrix} 0 \\ p(h_l) - p(h_l^*) \end{pmatrix},$$
$$\mathcal{F}_r(U_l, U_r, z_l, z_r) = \mathcal{F}(U_l^*, U_r^*) + \begin{pmatrix} 0 \\ p(h_r) - p(h_r^*) \end{pmatrix}, \tag{5}$$

where $p(h) = gh^2/2$, $\mathcal{F}(U_l, U_r)$ is a consistent numerical flux for the shallow water problem without source ($z = cst$), and the reconstructed states U_l^*, U_r^* are defined by

$$U_l^* = (h_l^*, h_l^* u_l), \qquad U_r^* = (h_r^*, h_r^* u_r),$$
$$h_l^* = \max(0, h_l + z_l - z^*), \qquad h_r^* = \max(0, h_r + z_r - z^*), \tag{6}$$
$$z^* = \max(z_l, z_r).$$

3. Multi-layer shallow water system

The one-dimensional multi-layer shallow water system can be written

$$
\begin{cases}
\partial_t h_j + \partial_x(h_j u_j) = 0, \\
\partial_t(h_j u_j) + \partial_x(h_j u_j^2 + g h_j^2/2) + g h_j \partial_x \left(z + \sum_{k>j} h_k + \sum_{k<j} \frac{\rho_k}{\rho_j} h_k \right) = 0,
\end{cases}
\tag{7}
$$

where $h_j \geq 0$, $j = 1, \ldots, m$ are the fluid depths, u_j are the velocities, and $z(x)$ is the topography. The constants g,

$$
0 < \rho_1 \leq \cdots \leq \rho_m
\tag{8}
$$

are respectively the gravity and the densities of the fluids. The fluids $1, \ldots, m$ are labeled from top to bottom. The situation is represented on Figure 2 for two layers. This system admits a convex entropy, and
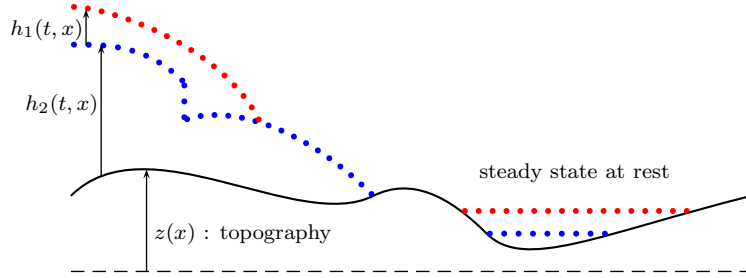

FIGURE 2. Two layers

thus we are looking for entropy solutions, satisfying

$$
\partial_t \left(\sum_j \rho_j \left(h_j u_j^2/2 + g h_j^2/2 + h_j g z \right) + g \sum_{j,k,k<j} \rho_k h_k h_j \right)
$$
$$
+ \partial_x \left(\sum_j \rho_j u_j \left(h_j u_j^2/2 + g h_j^2 + h_j g \left(z + \sum_{k>j} h_k + \sum_{k<j} \frac{\rho_k}{\rho_j} h_k \right) \right) \right) \leq 0.
\tag{9}
$$

The system has the steady states at rest

$$
u_j = 0, \qquad \partial_x \left(h_j + z + \sum_{k>j} h_k + \sum_{k<j} \frac{\rho_k}{\rho_j} h_k \right) = 0, \text{ for } j = 1, \ldots, m.
\tag{10}
$$

Notice that if $\rho_1 < \cdots < \rho_m$, this reduces to

$$
u_j = 0, \qquad \partial_x(z + h_m) = 0, \qquad \partial_x h_j = 0 \text{ for } j < m,
\tag{11}
$$

while if $\rho_1 = \cdots = \rho_m$, (10) reduces to

$$u_j = 0, \qquad \partial_x(z + h_1 + \cdots + h_m) = 0. \qquad (12)$$

As for the one-layer shallow water system ($m = 1$), the numerical difficulties related to this system are positivity of the depths h_j, with the possibility of treating out- or in-cropping ("drying" of one ore more layers), the exact preservation of the steady states at rest (well-balanced property), and the property to have a discrete entropy inequality. Overall, the multi-layer system has extra difficulties which are the non-conservativity of the system (even for smooth topography z) that implies the lack of appropriate Rankine-Hugoniot relations, and the possibility of having complex eigenvalues (the system is not everywhere hyperbolic). This last property raises the question of the mathematical well-posedness of the Cauchy problem. The answer is not clear, but some hope is possible, see in particular [11] for a study on a related system.

In physical terms the loss of hyperbolicity corresponds to the onset of the well-known Kelvin - Helmholtz (KH) instability, i.e. instability due to the strong velocity shear between adjacent fluid layers. It was shown in [10] that criteria of hyperbolicity loss and of KH instability coincide in two-layer shallow water with a rigid lid. This argument can be easily extended to the (present) free-surface case. The same line of argument is applicaple to the sheared *continously stratified* hydro-static flows, as was shown recently in [9], where the so-called Howard - Miles criterion of instability coincides with that of hyperbolicity loss. The shear instabilities persist even in the presence of dissipation, i.e. friction between the layers. In the multi-layer model, the mixing is replaced by numerical dissipation mechanisms, however of unknown nature since the system is not hyperbolic. One can hope that such dissipative modelling gives a rough description of the missing mixing, in the same spirit as the idea that a shock replaces multivalued solutions in the case of wave-breaking in fluid dynamics. Thus, the mathematical difficulty of having a mixed system corresponds to well-established physical phenomenon, which reinforces the hope of properly dealing with the Cauchy problem.

Several attempts have been made in order to solve the multi-layer shallow water system. In [8] and subsequent papers of the Malaga-Sevilla school, the two-layer case is treated by a Roe type method. A special treatment is done in order to recover positivity, and a special treatment is performed for complex eigenvalues, making the scheme inconsistent in this case. A relaxation method is proposed in [1], with similar properties, but which is not able to treat outcropping,

nor complex eigenvalues. In [2], the m-layer system is treated in the case $\rho_1 = \ldots \rho_m$, without restrictions on the eigenvalues and including drying. However, topography is not included, and the scheme is not extensible to arbitrary densities. None of these solvers is able to treat the general case with complex eigenvalues in a consistent way.

4. SPLITTING METHOD

In [6] we introduced the operator splitting method for the multilayer shallow water system, that leads to solving successively (7) for each j, with time independent h_k for $k \neq j$. This means that we solve successively m one-layer shallow water systems for $U_j = (h_j, h_j u_j)$, with topography

$$z^j = z + \sum_{k>j} h_k + \sum_{k<j} \frac{\rho_k}{\rho_j} h_k. \tag{13}$$

Applying a one-layer solver with numerical fluxes $\mathcal{F}_l$, $\mathcal{F}_r$ gives

$$U_i^{j,n+1} - U_i^j + \frac{\Delta t}{\Delta x_i} \left(\mathcal{F}_l(U_i^j, U_{i+1}^j, z_i^j, z_{i+1}^j) - \mathcal{F}_r(U_{i-1}^j, U_i^j, z_{i-1}^j, z_i^j) \right) = 0. \tag{14}$$

The advantages of the method are that

- The scheme is well-balanced if the resolution of each subsystem is performed with a well-balanced scheme. This is true even in the case of constant densities $\rho_1 = \cdots = \rho_m$.
- The scheme is nonnegative in heights and entropy satisfying if the resolution of each subsystem is performed with an entropy satisfying scheme, because the entropy of the system is an entropy for each of the subsystems, as explained in [6].
- No estimate of the (possibly complex) eigenvalues of the full system is necessary. In particular the scheme is always consistent, and one can use the CFL condition associated to each subsystem.

4.1. **Sum scheme.** A variant of the splitting scheme is the "sum scheme", that can be defined as follows for two operators. Consider an ordinary differential equation

$$\frac{dU}{dt} + A(U) + B(U) = 0. \tag{15}$$

Assume that we have two schemes for solving $dU/dt + A(U) = 0$ and $dU/dt + B(U) = 0$ respectively, given by

$$U^{n+1} - U^n + \Delta t\, \mathcal{A}(U^n) = 0, \qquad U^{n+1} - U^n + \Delta t\, \mathcal{B}(U^n) = 0. \tag{16}$$

Then the splitting scheme can be written

$$U^{n+1/2} - U^n + \Delta t\, \mathcal{A}(U^n) = 0, \qquad U^{n+1} - U^{n+1/2} + \Delta t\, \mathcal{B}(U^{n+1/2}) = 0. \tag{17}$$

while the sum scheme is simply

$$U^{n+1} - U^n + \Delta t\, (\mathcal{A}(U^n) + \mathcal{B}(U^n)) = 0. \tag{18}$$

In [6], the splitting scheme (17) was used. The application of the sum scheme gives the same properties (entropy inequality, well-balanced property, nonnegativity...), and give very similar results.

Note that in our context, applying the splitting scheme means that we solve (14) *successively* for each j, while applying the sum scheme means that we solve (14) *simultaneously* for all j.

4.2. **The problem of momentum conservation.** The application of the splitting method of [6], using the one-layer hydrostatic solver, leads to wrong solutions, as seen on Figure 4. It happens also with the sum scheme. In [6] we thought the problem was due to only a semidiscrete entropy inequality. Here we show that the problem is in fact due to non-conservativity of total momentum.

By summing up the momenta equations in (7) with weights ρ_j, we deduce the total momentum balance

$$\partial_t \Big(\sum_j \rho_j h_j u_j \Big) + \partial_x \Big(\sum_j \rho_j (h_j u_j^2 + g h_j^2/2) + \sum_{j,k,j<k} g \rho_j h_j h_k \Big) \\ + \Big(\sum_j \rho_j g h_j \Big) \partial_x z = 0, \tag{19}$$

that follows from the identity

$$\sum_j \rho_j \Big(h_j \partial_x \Big(z + \sum_{k>j} h_k + \sum_{k<j} \frac{\rho_k}{\rho_j} h_k \Big) \Big) \\ = \partial_x \Big(\sum_{j,k,j<k} \rho_j h_j h_k \Big) + \Big(\sum_j \rho_j h_j \Big) \partial_x z. \tag{20}$$

The equation (19) is conservative as soon as z is smooth. Since this conservative equation is physically meaningful, one would like to use a scheme that is also conservative in total momentum.

At the discrete level, this means that one would like a discrete analogue of (20), telling that the discrete nonconservative terms add up to give a conservative contribution. A natural way of achieving this is by relying on the quadratic identity

$$\frac{h_l^1 + h_r^1}{2}(h_r^2 - h_l^2) + \frac{h_l^2 + h_r^2}{2}(h_r^1 - h_l^1) = h_r^1 h_r^2 - h_l^1 h_l^2, \tag{21}$$

where indices l and r refer to left and right values with respect to a given interface between two cells, and h^1 and h^2 denote two different layers. Applying this several times leads to

$$
\begin{aligned}
&\sum_j \rho_j \frac{h_l^j + h_r^j}{2}\left(\Delta z + \sum_{k>j}(h_r^k - h_l^k) + \sum_{k<j}\frac{\rho_k}{\rho_j}(h_r^k - h_l^k)\right) \\
&= \Delta z \sum_j \rho_j \frac{h_l^j + h_r^j}{2} + \sum_{j,k,j<k} \rho_j(h_r^j h_r^k - h_l^j h_l^k),
\end{aligned}
\tag{22}
$$

which is the desired discrete analogue of (20) (we denote $\Delta z = z_r - z_l$). Since the right-hand side of (22) is in conservative form (in the limit $\Delta z \to 0$, which corresponds to the assumption of continuous $z(x)$), it is enough in order to get the discrete conservation that the nonconservative term in the momentum equation (7) is discretized for each j as

$$
\begin{aligned}
&g\frac{h_l^j + h_r^j}{2}\left(\Delta z + \sum_{k>j}(h_r^k - h_l^k) + \sum_{k<j}\frac{\rho_k}{\rho_j}(h_r^k - h_l^k)\right) \\
&= g\frac{h_l^j + h_r^j}{2}\Delta z^j,
\end{aligned}
\tag{23}
$$

where $\Delta z^j = z_r^j - z_l^j$ and

$$
z^j = z + \sum_{k>j} h^k + \sum_{k<j}\frac{\rho_k}{\rho_j}h^k
\tag{24}
$$

is the topography seen by the j's layer in the splitting scheme. However, in order not to have different times involved, such as $t_{n+1/2}$ in (17) (that would break the conservation), we need to apply the sum scheme (18), that involves only the data at time n.

One can observe that the one-layer hydrostatic reconstruction method (3)-(6) gives a nonconservative contribution $\mathcal{F}_l - \mathcal{F}_r \neq (0, g\frac{h_l+h_r}{2}\Delta z)$, explaining the failure of conservation of total momentum.

Note that the centered discretization (23) is consistent with the following Rankine-Hugoniot condition for the j's momentum

$$
\left[h_j u_j^2 + gh_j^2/2\right] + g\frac{h_l^j + h_r^j}{2}\left[z^j\right] - s\left[h_j u_j\right] = 0,
\tag{25}
$$

where as usual $[\ldots]$ denotes the jump of a quantity through a space-time discontinuity, s denotes the speed of the discontinuity, and indices l and r refer to the values on each side. These Rankine-Hugoniot conditions are also those corresponding to the scheme of [8]. Thus we expect to get the same solution, even if no rigorous proof of convergence

to generalized Rankine-Hugoniot conditions exists for nonconservative systems.

5. Source-centered hydrostatic reconstruction

As explained in the previous section, the sum scheme for the multi-layer shallow water system satisfies the discrete conservation of total momentum whenever the one-layer solver used for $U = (h, hu)$

$$U_i^{n+1} - U_i + \frac{\Delta t}{\Delta x_i} \left(F_{i+1/2-} - F_{i-1/2+} \right) = 0, \qquad (26)$$

$$F_{i+1/2-} = \mathcal{F}_l(U_i, U_{i+1}, z_i, z_{i+1}), \qquad F_{i+1/2+} = \mathcal{F}_r(U_i, U_{i+1}, z_i, z_{i+1}), \qquad (27)$$

with left/right numerical fluxes $\mathcal{F}_l$, $\mathcal{F}_r$, verifies the source-centered identity

$$\mathcal{F}_r(U_l, U_r, z_l, z_r) - \mathcal{F}_l(U_l, U_r, z_l, z_r) = \left(0, -g\frac{h_l + h_r}{2}(z_r - z_l) \right). \quad (28)$$

We can observe that this property (28) is true for Roe type solvers like those of [4], [8], because in this context the source discretization can be chosen arbitrarily. However this property is not true for the hydrostatic reconstruction solver, that we would like to keep because of its robustness. We provide below a correction to it in order to enforce the source-centered property, but without breaking the semi-discrete entropy inequality.

5.1. **Dispersive correction.** In order to provide some intuition, at the continuous level, the idea is to find a dispersive correction to the shallow water system, that is neutral at the level of entropy. A parametrization of possible modified systems is as follows,

$$\begin{cases} \partial_t h + \partial_x(hu) - \partial_x\left(\mu u \partial_x w \right) = 0, \\ \partial_t(hu) + \partial_x(hu^2 + gh^2/2) + hg\partial_x z - \mu(\partial_x w)g\partial_x(h + z) \\ \qquad\qquad\qquad\qquad - \partial_x\left(\mu u^2 \partial_x w \right) = 0, \end{cases} \qquad (29)$$

with $\partial_t z = 0$, and $\mu(t, x)$, $w(t, x)$ arbitrary functions. Then the entropy $\widehat{\eta} = hu^2/2 + gh^2/2 + hgz$ satisfies

$$\partial_t \widehat{\eta} + \partial_x\left((\widehat{\eta} + gh^2/2)u \right) - \partial_x\left(\left(g(h + z) + \frac{u^2}{2} \right)\mu u \partial_x w \right) = 0, \quad (30)$$

showing that the correction is conservative in entropy. It also keeps steady states at rest.

The aim is to find a correction $\mu\partial_x w$ of order Δx (it will be indeed of order Δz), in such a way that the nonconservative part of the momentum equation $-\mu(\partial_x w)g\partial_x(h+z)$ exactly balances the noncentered part of the numerical source coming from the hydrostatic reconstruction scheme.

At the discrete level, omitting the standard shallow water terms, a dispersive correction can be performed with $U = (h, hu)$ as

$$U_i^{n+1} - U_i + \frac{\Delta t}{\Delta x_i}\left(J_{i+1/2-} - J_{i-1/2+}\right) = 0, \qquad (31)$$

$$J_{i+1/2-} = \mathcal{J}_l(U_i, U_{i+1}, z_i, z_{i+1}), \qquad J_{i+1/2+} = \mathcal{J}_r(U_i, U_{i+1}, z_i, z_{i+1}). \qquad (32)$$

Denoting $J_{i+1/2\pm} = (J_{i+1/2\pm}^0, J_{i+1/2\pm}^1)$, we require conservativity in h

$$J_{i+1/2-}^0 = J_{i+1/2+}^0 \equiv J_{i+1/2}^0, \qquad (33)$$

and we take the momentum formulas

$$\begin{aligned} J_{i+1/2-}^1 &= \delta_{i+1/2-} + J_{i+1/2}^1, \\ J_{i+1/2+}^1 &= -\delta_{i+1/2+} + J_{i+1/2}^1, \end{aligned} \qquad (34)$$

where

$$J_{i+1/2}^1 = u_i(J_{i+1/2}^0)_+ + u_{i+1}(J_{i+1/2}^0)_- \qquad (35)$$

and $X_+ \equiv \max(0, X)$, $X_- \equiv \min(0, X)$. The scalars $\delta_{i+1/2-}$, $\delta_{i+1/2+}$ parametrize the nonconservative part $\delta_{i+1/2-} + \delta_{i+1/2+}$ of the momentum correction.

Writing (31) by interface,

$$U_i^{n+1} = \frac{1}{2}\left(U_i - 2\frac{\Delta t}{\Delta x_i}J_{i+1/2-}\right) + \frac{1}{2}\left(U_i + 2\frac{\Delta t}{\Delta x_i}J_{i-1/2+}\right), \qquad (36)$$

we deduce sufficient nonnegativity conditions by interface for h,

$$2\frac{\Delta t}{\Delta x_i}J_{i+1/2}^0 \le h_i, \qquad -2\frac{\Delta t}{\Delta x_{i+1}}J_{i+1/2}^0 \le h_{i+1}. \qquad (37)$$

Let us now write down discrete entropy inequalities. We decompose the entropy $\widehat{\eta} = hu^2/2 + gh^2/2 + hgz$ as $\widehat{\eta}(U, z) = K(U) + L(U, z)$ with

$$K(U) = \frac{1}{2}hu^2, \qquad L(U, z) = \frac{1}{2}gh^2 + hgz. \qquad (38)$$

Let us define $J_{i+1/2} = (J_{i+1/2}^0, J_{i+1/2}^1)$, and

$$U_i' = U_i - 2\frac{\Delta t}{\Delta x_i}J_{i+1/2}, \qquad U_i'' = U_i + 2\frac{\Delta t}{\Delta x_i}J_{i-1/2}. \qquad (39)$$

Then (37) say that U_i' and U_i'' have nonnegative first component. Writing

$$
\begin{aligned}
h_i' &= h_i - 2\frac{\Delta t}{\Delta x_i}J^0_{i+1/2}, \\
h_i'u_i' &= h_iu_i - 2\frac{\Delta t}{\Delta x_i}J^1_{i+1/2} \\
&= u_i\Big(h_i - 2\frac{\Delta t}{\Delta x_i}(J^0_{i+1/2})_+\Big) - 2u_{i+1}\frac{\Delta t}{\Delta x_i}(J^0_{i+1/2})_-,
\end{aligned}
\tag{40}
$$

we deduce that $u_i' = (1-\theta)u_i + \theta u_{i+1}$, for some $0 \le \theta \le 1$. Thus $(u_i')^2 \le (1-\theta)u_i^2 + \theta u_{i+1}^2$, or in other words

$$
h_i'(u_i')^2 \le u_i^2\Big(h_i - 2\frac{\Delta t}{\Delta x_i}(J^0_{i+1/2})_+\Big) - 2u_{i+1}^2\frac{\Delta t}{\Delta x_i}(J^0_{i+1/2})_-.
\tag{41}
$$

A similar computation leads to

$$
h_{i+1}''(u_{i+1}'')^2 \le u_{i+1}^2\Big(h_{i+1} + 2\frac{\Delta t}{\Delta x_{i+1}}(J^0_{i+1/2})_-\Big) + 2u_i^2\frac{\Delta t}{\Delta x_{i+1}}(J^0_{i+1/2})_+.
\tag{42}
$$

Therefore, one gets

$$
\begin{aligned}
K(U_i') - K(U_i) + 2\frac{\Delta t}{\Delta x_i}\vartheta^K_{i+1/2} &\le 0, \\
K(U_{i+1}'') - K(U_{i+1}) - 2\frac{\Delta t}{\Delta x_{i+1}}\vartheta^K_{i+1/2} &\le 0,
\end{aligned}
\tag{43}
$$

with

$$
\vartheta^K_{i+1/2} = \frac{u_i^2}{2}\left(J^0_{i+1/2}\right)_+ + \frac{u_{i+1}^2}{2}\left(J^0_{i+1/2}\right)_-.
\tag{44}
$$

Using (34) and the fact that K is quadratic in hu at fixed h, (43) yields

$$
\begin{aligned}
&K\Big(U_i - 2\frac{\Delta t}{\Delta x_i}J_{i+1/2-}\Big) - K(U_i) + 2\frac{\Delta t}{\Delta x_i}\vartheta^K_{i+1/2} \\
&\le \frac{1}{2h_i'}\Big(-4\frac{\Delta t}{\Delta x_i}(h_iu_i - 2\frac{\Delta t}{\Delta x_i}J^1_{i+1/2})\delta_{i+1/2-} + 4(\frac{\Delta t}{\Delta x_i}\delta_{i+1/2-})^2\Big), \\
&K\Big(U_{i+1} + 2\frac{\Delta t}{\Delta x_{i+1}}J_{i+1/2+}\Big) - K(U_{i+1}) - 2\frac{\Delta t}{\Delta x_{i+1}}\vartheta^K_{i+1/2} \\
&\le \frac{1}{2h_{i+1}''}\Big(-4\frac{\Delta t}{\Delta x_{i+1}}(h_{i+1}u_{i+1} + 2\frac{\Delta t}{\Delta x_{i+1}}J^1_{i+1/2})\delta_{i+1/2+} \\
&\qquad\qquad\qquad + 4(\frac{\Delta t}{\Delta x_{i+1}}\delta_{i+1/2+})^2\Big).
\end{aligned}
\tag{45}
$$

Next, we treat $L(U, z)$, which is easier since it does only depend on h and z. We have

$$
\begin{aligned}
&L\Big(h_i - 2\frac{\Delta t}{\Delta x_i}J_{i+1/2}^0, z_i\Big) - L(h_i, z_i) + 2\frac{\Delta t}{\Delta x_i}\vartheta_{i+1/2}^L \\
&= -2\frac{\Delta t}{\Delta x_i}gh_iJ_{i+1/2}^0 + 2g(\frac{\Delta t}{\Delta x_i}J_{i+1/2}^0)^2 - 2\frac{\Delta t}{\Delta x_i}gz_iJ_{i+1/2}^0 + 2\frac{\Delta t}{\Delta x_i}\vartheta_{i+1/2}^L, \\
&L\Big(h_{i+1} + 2\frac{\Delta t}{\Delta x_{i+1}}J_{i+1/2}^0, z_{i+1}\Big) - L(h_{i+1}, z_{i+1}) - 2\frac{\Delta t}{\Delta x_{i+1}}\vartheta_{i+1/2}^L \\
&= 2\frac{\Delta t}{\Delta x_{i+1}}gh_{i+1}J_{i+1/2}^0 + 2g(\frac{\Delta t}{\Delta x_{i+1}}J_{i+1/2}^0)^2 \\
&\qquad\qquad\qquad + 2\frac{\Delta t}{\Delta x_{i+1}}gz_{i+1}J_{i+1/2}^0 - 2\frac{\Delta t}{\Delta x_{i+1}}\vartheta_{i+1/2}^L,
\end{aligned}
\tag{46}
$$

for some $\vartheta_{i+1/2}^L$ to be determined. Adding up (45) and (46) gives

$$
\begin{aligned}
&\widehat{\eta}\Big(U_i - 2\frac{\Delta t}{\Delta x_i}J_{i+1/2-}, z_i\Big) - \widehat{\eta}(U_i, z_i) + 2\frac{\Delta t}{\Delta x_i}(\vartheta_{i+1/2}^K + \vartheta_{i+1/2}^L) \le \text{error}, \\
&\widehat{\eta}\Big(U_{i+1} + 2\frac{\Delta t}{\Delta x_{i+1}}J_{i+1/2+}, z_{i+1}\Big) - \widehat{\eta}(U_{i+1}, z_{i+1}) \\
&\qquad\qquad\qquad - 2\frac{\Delta t}{\Delta x_{i+1}}(\vartheta_{i+1/2}^K + \vartheta_{i+1/2}^L) \le \text{error}.
\end{aligned}
\tag{47}
$$

Now, assume for a moment that the errors in (47) are both nonpositive. Then, from (36) we get

$$
\begin{aligned}
&\widehat{\eta}(U_i^{n+1}, z_i) \\
&\le \frac{1}{2}\widehat{\eta}\Big(U_i - 2\frac{\Delta t}{\Delta x_i}J_{i+1/2-}, z_i\Big) + \frac{1}{2}\widehat{\eta}\Big(U_i + 2\frac{\Delta t}{\Delta x_i}J_{i-1/2+}, z_i\Big) \\
&\le \widehat{\eta}(U_i, z_i) - \frac{\Delta t}{\Delta x_i}\Big(\vartheta_{i+1/2}^K + \vartheta_{i+1/2}^L - \vartheta_{i-1/2}^K - \vartheta_{i-1/2}^L\Big),
\end{aligned}
\tag{48}
$$

showing that a discrete entropy inequality is satisfied with numerical entropy flux

$$
\vartheta_{i+1/2} = \vartheta_{i+1/2}^K + \vartheta_{i+1/2}^L.
\tag{49}
$$

But since requiring nonpositive errors in (47) is too restrictive, we shall only look for a semi-discrete entropy inequality, which is the limit of (48) divided by Δt when $\Delta t \to 0$. For this to hold it is enough that the errors in (47) divided by Δt give nonpositive contributions in the limit $\Delta t \to 0$. These errors coming from (45) and (46), this yields the sufficient conditions

$$
\begin{aligned}
-u_i\delta_{i+1/2-} - g(h_i + z_i)J_{i+1/2}^0 + \vartheta_{i+1/2}^L &\le 0, \\
-u_{i+1}\delta_{i+1/2+} + g(h_{i+1} + z_{i+1})J_{i+1/2}^0 - \vartheta_{i+1/2}^L &\le 0,
\end{aligned}
\tag{50}
$$

or in other words

$$g(h_{i+1}+z_{i+1})J^0_{i+1/2}-u_{i+1}\delta_{i+1/2+} \leq \vartheta^L_{i+1/2} \leq g(h_i+z_i)J^0_{i+1/2}+u_i\delta_{i+1/2-}. \tag{51}$$

In order to be able to find some $\vartheta^L_{i+1/2}$ satisfying (51), it is necessary and sufficient that the lower bound is less than the upper bound, which is

$$g(h_{i+1}+z_{i+1}-h_i-z_i)J^0_{i+1/2} \leq u_i\delta_{i+1/2-}+u_{i+1}\delta_{i+1/2+}. \tag{52}$$

Then one can choose for $\vartheta^L_{i+1/2}$ the half sum of the lower and upper bounds in (51),

$$\vartheta^L_{i+1/2}=\frac{1}{2}g(h_i+z_i+h_{i+1}+z_{i+1})J^0_{i+1/2}+\frac{1}{2}(u_i\delta_{i+1/2-}-u_{i+1}\delta_{i+1/2+}). \tag{53}$$

Noticing that the CFL conditions for nonnegativity (37) are satisfied in the semi-discrete limit, we conclude the following.

Proposition 1. *The scheme (31)-(32) with numerical fluxes* $\mathcal{J}_{l/r}(U_l,U_r,z_l,z_r)$ *satisfying* $\mathcal{J}_{l/r}=(\mathcal{J}^0,\mathcal{J}^1_{l/r})$,

$$\begin{aligned} \mathcal{J}^1_l &= \delta_l + \mathcal{J}^1, \\ \mathcal{J}^1_r &= -\delta_r + \mathcal{J}^1, \end{aligned} \tag{54}$$

and

$$\mathcal{J}^1 = u_l(\mathcal{J}^0)_+ + u_r(\mathcal{J}^0)_-, \tag{55}$$

is semi-discrete entropy satisfying for the entropy $\widehat{\eta}=hu^2/2+gh^2/2+hgz$ *as soon as*

$$g(h_r+z_r-h_l-z_l)\mathcal{J}^0 \leq u_l\delta_l+u_r\delta_r. \tag{56}$$

Then the entropy flux can be taken as

$$\vartheta = \frac{u_l^2}{2}(\mathcal{J}^0)_+ + \frac{u_r^2}{2}(\mathcal{J}^0)_- + \frac{1}{2}g(h_l+z_l+h_r+z_r)\mathcal{J}^0 + \frac{1}{2}(u_l\delta_l-u_r\delta_r). \tag{57}$$

5.2. **SCHR scheme.** Let us now explain how to build the source-centered hydrostatic reconstruction (SCHR) scheme. Denoting by $\mathcal{F}^{HR}_{l/r}$ the numerical fluxes of the original hydrostatic reconstruction scheme (5)-(6), we take

$$\mathcal{F}_l = \mathcal{F}^{HR}_l + \mathcal{J}_l, \qquad \mathcal{F}_r = \mathcal{F}^{HR}_r + \mathcal{J}_r, \tag{58}$$

with $\mathcal{J}_{l/r}=(\mathcal{J}^0,\mathcal{J}^1_{l/r})$ defined by (54), (55). We have to take δ_l, δ_r so that $\mathcal{J}^1_r - \mathcal{J}^1_l = -(\delta_l+\delta_r)$ will kill the source error between the hydrostatic source and the desired centered source,

$$\delta_l + \delta_r = (\mathcal{F}^{HR}_r - \mathcal{F}^{HR}_l)^1 + g\frac{h_l+h_r}{2}(z_r-z_l), \tag{59}$$

that gives (28). Then we define $\mathcal{J}^0$ in order to have equality in (56). Notice that even if the construction (58)-(59) looks possible for any scheme instead of $\mathcal{F}_{l/r}^{HR}$, the issue is in the possibility to compute $\mathcal{J}^0$ from (56), which somehow involves the property that $h_r + z_r - h_l - z_l$ is in factor in δ_l and δ_r, and thus must be factorizable from the right-hand side of (59). Indeed, the well-balanced property ensures that whenever $u_l = u_r = 0$ and $h_r + z_r - h_l - z_l = 0$, the right-hand side of (59) vanishes. But this does not imply that $h_r + z_r - h_l - z_l$ can be factorized, unless $(\mathcal{F}_r^{HR} - \mathcal{F}_l^{HR})^1$ is independent of u_l, u_r. This is wrong in general, but this is true for the hydrostatic solver. Indeed one can check that

$$(\mathcal{F}_r^{HR} - \mathcal{F}_l^{HR})^1 + g\frac{h_l + h_r}{2}(z_r - z_l) = g(h_r - h_l + \Delta z)\widehat{\kappa}, \qquad (60)$$

with

$$\widehat{\kappa} = \frac{1}{2}|\underline{\Delta z}| + \begin{cases} \dfrac{1}{2}\dfrac{(\Delta z - h_l)h_r}{h_r + \Delta z - h_l} & \text{if } \Delta z > h_l, \\[2mm] 0 & \text{if } -h_r \leq \Delta z \leq h_l, \\[2mm] \dfrac{1}{2}\dfrac{-(h_r + \Delta z)h_l}{h_l - (h_r + \Delta z)} & \text{if } \Delta z < -h_r, \end{cases} \qquad (61)$$

$$\underline{\Delta z} = \begin{cases} \min(\Delta z, h_l) & \text{if } \Delta z \geq 0, \\ \max(\Delta z, -h_r) & \text{if } \Delta z \leq 0. \end{cases} \qquad (62)$$

One can observe that

$$0 \leq \widehat{\kappa} \leq \min\left(\frac{1}{2}|\Delta z|, \frac{h_l + h_r}{2}\right). \qquad (63)$$

Thus one would like to take $\delta_l = \frac{1}{2}(1 + \theta)g(h_r - h_l + \Delta z)\widehat{\kappa}$, $\delta_r = \frac{1}{2}(1 - \theta)g(h_r - h_l + \Delta z)\widehat{\kappa}$, for some $-1 \leq \theta \leq 1$, and $\mathcal{J}^0 = \frac{1}{2}\big((1 + \theta)u_l + (1 - \theta)u_r\big)\widehat{\kappa}$. However, in order for the CFL condition (37) to be finite, we need that $|\mathcal{J}^0| \leq C\min(h_l, h_r)$. But one can check that this is not true, and only (63) is valid.

Therefore, our choice is the following. We define

$$\kappa = \min\left(\widehat{\kappa}, \frac{5}{2}\min(h_l, h_r)\right), \qquad (64)$$

and take

$$\begin{aligned} \delta_l &= \frac{1}{2}(1 + \theta)g(h_r - h_l + \Delta z)\kappa, \\ \delta_r &= \frac{1}{2}(1 - \theta)g(h_r - h_l + \Delta z)\kappa, \\ \mathcal{J}^0 &= \frac{1}{2}\big((1 + \theta)u_l + (1 - \theta)u_r\big)\kappa, \end{aligned} \qquad (65)$$

so that the semi-discrete entropy condition (56) is satisfied. The value of θ is not very important, and we have chosen an upwind formula,

$$\theta = \min\left(1, \frac{(u_l)_+}{\sqrt{gh_l}}\right) - \min\left(1, \frac{(-u_r)_+}{\sqrt{gh_r}}\right). \tag{66}$$

Now, denote by $a^{HR}(U_l, U_r, \Delta z)$ the speed involved in the CFL condition for nonnegativity of the hydrostatic reconstruction scheme, and by $a^{\mathcal{J}}(U_l, U_r, \Delta z)$ the one associated to the correction scheme (31)-(32). Indeed, according to (37), one can take

$$a^{\mathcal{J}}(U_l, U_r, \Delta z) = 2\left(\frac{(\mathcal{J}^0)_+}{h_l} + \frac{(-\mathcal{J}^0)_+}{h_r}\right). \tag{67}$$

We recall that according to [3], $a^{HR}(U_l, U_r, \Delta z) = a^{\mathcal{F}}(U_l^*, U_r^*)$ where $a^{\mathcal{F}}$ stands for the speed associated to the homogeneous solver $\mathcal{F}$.

Then, the CFl condition for nonnegativity for the scheme (58) is

$$\frac{\Delta t}{\min(\Delta x_i, \Delta x_{i+1})} a^{SCHR}(U_i, U_{i+1}, z_{i+1} - z_i) \leq 1, \tag{68}$$

with

$$a^{SCHR}(U_l, U_r, \Delta z) = a^{HR}(U_l, U_r, \Delta z) + a^{\mathcal{J}}(U_l, U_r, \Delta z). \tag{69}$$

Theorem 1. *The source-centered hydrostatic reconstruction scheme (SCHR) (26)-(27) with numerical fluxes (58), (54), (55), (65), (66), (64), (61), (62) is*

(i) consistent with the one-layer shallow water system (1),

(ii) semi-discrete entropy satisfying,

(iii) well-balanced,

(iv) conservative in h, and nonnegative under the CFL condition (68), (69), (67),

(v) source-centered away from sharp variations of h close to vacuum.

Proof. Recall that the hydrostatic reconstruction scheme satisfies (i)-(iv). Then, according to (63), (64), (65) one has

$$\delta_l = O\Big(g\Delta z(h_r - h_l + \Delta z)\Big), \qquad \delta_r = O\Big(g\Delta z(h_r - h_l + \Delta z)\Big),$$
$$\mathcal{J}^0 = O\Big((|u_l| + |u_r|)\Delta z\Big), \tag{70}$$

which yields (i) and (iii). Property (ii) follows from the above construction (by applying Proposition 1), as well as property (iv). Finally, for (v), we observe that $\delta_l + \delta_r = g(h_r - h_l + \Delta z)\kappa$. Thus, whenever $\kappa = \widehat{\kappa}$, the identity (60) gives (59), proving (28). Therefore, the only restriction is that $\kappa = \widehat{\kappa}$, which according to (64), (63), holds as soon as

$\frac{1}{2}(h_l + h_r) \leq \frac{5}{2}\min(h_l, h_r)$. This inequality is what we mean by "away from sharp variations of h close to vacuum". $\qquad\square$

In practice, this scheme gives numerical results for the one-layer system that are almost identical to the usual hydrostatic reconstruction scheme, except that it needs slightly smaller timesteps because of the restricted CFL condition. This can be improved by diminishing a bit the constant $5/2$ in (64) (take for example $3/2$), at the cost of losing a bit of source-centering for sharp variations of h close to vacuum.

In order to treat the one-layer shallow water system with transverse velocity v solving

$$\partial_t(hv) + \partial_x(huv) = 0, \tag{71}$$

one has to apply also the correction $\mathcal{J}$. This means that we have to complete (31) with the passive advection scheme

$$h_i^{n+1} v_i^{n+1} - h_i v_i + \frac{\Delta t}{\Delta x_i} \left(J_{i+1/2}^2 - J_{i-1/2}^2 \right) = 0, \tag{72}$$

with

$$J_{i+1/2}^2 = v_i (J_{i+1/2}^0)_+ + v_{i+1}(J_{i+1/2}^0)_-. \tag{73}$$

Then the discrete entropy inequality associated to the entropy $hu^2/2 + hv^2/2 + gh^2/2 + hgz$ holds with an additional numerical entropy flux

$$\vartheta_{i+1/2}^2 = \frac{v_i^2}{2}(J_{i+1/2}^0)_+ + \frac{v_{i+1}^2}{2}(J_{i+1/2}^0)_-. \tag{74}$$

Equation (71), completed with (1), corresponds to the one and half dimensional one-layer shallow water model. For treating further zero-order right-hand sides in the momentum equation (such as Coriolis force) in a well-balanced setting, one can apply the apparent topography method of [5], see [7] for details.

5.3. **Sum scheme for the multi-layer system.** From Theorem 1 we get directly

Theorem 2. *For the multi-layer shallow water system, the sum scheme (14), (13) using the source-centered hydrostatic reconstruction scheme on each layer is*

(i) consistent for smooth solutions,

(ii) semi-discrete entropy satisfying,

(iii) well-balanced,

(iv) conservative in h_j for each layer j, and nonnegative under the CFL conditions (68), (69), (67) for each j,

(v) conservative in total momentum away from sharp variations of h_j close to vacuum.

It is noticeable to remark that for a sum scheme

$$U^{n+1} - U^n + \Delta t \Big(\mathcal{A}^1(U^n) + \cdots + A^m(U^n) \Big) = 0, \qquad (75)$$

if each scheme $\mathcal{A}^j$ is stable (in the sense of invariant domain or fully discrete entropy inequality) under a CFL condition $\Delta t \, a^j(U^n) \leq 1$, then the sum scheme is stable under the CFL condition

$$\Delta t \sum_{j=1}^{m} a^j(U^n) \leq 1. \qquad (76)$$

However, since in our case the height h_j evolves only through the term $\mathcal{A}^j$ (and not through $\mathcal{A}^k$ for $k \neq j$), the nonnegativity is true under the weak CFL condition

$$\Delta t \max_{j=1,\ldots,m} a^j(U^n) \leq 1, \qquad (77)$$

as stated in (iv), while by nature the semi-discrete entropy inequality does not need any CFL condition (because it involves only the limit $\Delta t \to 0$). Therefore, even if one would expect that for a really stable sum scheme, the CFL condition (76) is needed, in practice all our computations with a large number of layers show that (77) is sufficient, at the cost of putting a factor $1/2$ in the right-hand side, or a factor $1/4$ in two dimensions.

Note that the one and half dimensional multi-layer model can be treated similarly by applying the sum scheme, leading to additional resolutions of (71) for each layer. Then, fully two-dimensional finite volume simulations can be performed as usual by applying interface one and half dimensional solvers.

5.4. **Upwinding dispersive correction.** The $SCHR$ scheme obtained in the previous paragraph for the multi-layer shallow water system shows good numerical results, except for "upwind data" for which all eigenvalues have the same sign. For these we observe some important oscillations, as shown on Figure 3. These oscillations do not increase when refining, thus are not instabilities, but rather dispersive errors. An explanation of this phenomenon is that in the upwind situation, the scheme differs from the standard upwind scheme evaluating all the data on the upwind side, because the apparent topography $z^j = z + \sum_{k>j} h_k + \sum_{k<j} \frac{\rho_k}{\rho_j} h_k$ is treated as a steady variable (instead of upwind).

A correction can be done at the level of the one-layer solver, by changing the definition (64) of κ into

$$\kappa = \begin{cases} \widetilde{\kappa} & \text{if } |\widetilde{\kappa}| \le \dfrac{5}{2}\min(h_l, h_r), \\[2ex] \dfrac{5}{2}\min(h_l, h_r)\dfrac{\widetilde{\kappa}}{|\widetilde{\kappa}|} & \text{if } |\widetilde{\kappa}| > \dfrac{5}{2}\min(h_l, h_r), \end{cases} \tag{78}$$

with

$$\begin{aligned} \widetilde{\kappa} = \widehat{\kappa} &+ \left((\underline{\Delta z})_+ - \widehat{\kappa}\right)\min\left(1, \frac{1}{4}\frac{(u_l)_+}{\sqrt{gh_l}}\right) \\ &+ \left((-\underline{\Delta z})_+ - \widehat{\kappa}\right)\min\left(1, \frac{1}{4}\frac{(-u_r)_+}{\sqrt{gh_r}}\right). \end{aligned} \tag{79}$$

The idea is that for Froude number $Fr = u/\sqrt{gh}$ small, $\widetilde{\kappa} = \widehat{\kappa}$ and the scheme is as previously, while for large Froude number, $\widetilde{\kappa} = (\underline{\Delta z})_+$ (Fr positive, i.e. positive eigenvalues), or $\widetilde{\kappa} = (-\underline{\Delta z})_+$ (Fr negative, i.e. negative eigenvalues).

The reason for putting these values is as follows. Consider the case of positive eigenvalues. Then the HR scheme (5)-(6) gives the numerical fluxes, if $\Delta z \ge 0$,

$$\begin{aligned} \mathcal{F}_l^{HR} &= \left(h_l^* u_l, h_l^* u_l^2 + gh_l^2/2\right), \\ \mathcal{F}_r^{HR} &= \left(h_l^* u_l, h_l^* u_l^2 + g(h_l^*)^2/2\right), \\ h_l^* &= (h_l - \Delta z)_+ = h_l - \underline{\Delta z}, \end{aligned} \tag{80}$$

while if $\Delta z \le 0$,

$$\begin{aligned} \mathcal{F}_l^{HR} &= \left(h_l u_l, h_l u_l^2 + gh_l^2/2\right), \\ \mathcal{F}_r^{HR} &= \left(h_l u_l, h_l u_l^2 + gh_l^2/2 + gh_r^2/2 - g(h_r^*)^2/2\right), \\ h_r^* &= (h_r + \Delta z)_+ = h_r + \underline{\Delta z}. \end{aligned} \tag{81}$$

These values have to be compared to the "ideal" upwind values

$$\begin{aligned} \mathcal{F}_l^{uw} &= \left(h_l u_l, h_l u_l^2 + gh_l^2/2\right), \\ \mathcal{F}_r^{uw} &= \left(h_l u_l, h_l u_l^2 + gh_l^2/2 - g\frac{h_l + h_r}{2}\Delta z\right). \end{aligned} \tag{82}$$

Taking into account (60), this yields the errors

$$\begin{aligned} \mathcal{F}_l^{HR} - \mathcal{F}_l^{uw} &= -(\underline{\Delta z})_+(u_l, u_l^2), \\ \mathcal{F}_r^{HR} - \mathcal{F}_r^{uw} &= -(\underline{\Delta z})_+(u_l, u_l^2) + (0, g(h_r - h_l + \Delta z)\widehat{\kappa}). \end{aligned} \tag{83}$$

In (83) we can neglect $g(h_r - h_l + \Delta z)\widehat{\kappa}$ since it is of higher order (recall that $0 \le \widehat{\kappa} \le |\Delta z|/2$). Now it remains to take into account the contribution of the correction fluxes (54)-(55) into (58). Since in (65) δ_l and δ_r can also be neglected, it remains $\mathcal{J}^0 = \kappa u_l$, and (55) gives

$\mathcal{J}^1 = \kappa u_l^2$. Thus $(\mathcal{J}^0, \mathcal{J}^1) = \kappa(u_l, u_l^2)$. In order for this to compensate (83), we need to have $\kappa = (\underline{\Delta z})_+$, which justifies (79). The case of negative eigenvalues is similar.

The obtained upwinding dispersive correction hydrostatic reconstruction (UDCHR) scheme, defined by (58), (54), (55), (65), (66), (78), (79), (61), (62), satisfies the points (i)-(iv) of Theorem 1, and is in some sense "weakly source-centered". The use of this scheme in the context of Theorem 2 has given the best results, even if property (v) is satisfied in a very weak sense.

6. Numerical tests

We only intend to show here that our approach works on the basic tests proposed in [6]. More involved tests (non-hyperbolic regime, more than two layers, second-order, two dimensions, Coriolis force) will be reported in other places.

In all our tests we take $g = 9.81$ with two layers, the one-layer UDCHR scheme of Subsection 5.4 being used into the sum scheme. The homogeneous solver $\mathcal{F}$ of (5) is taken as the Suliciu relaxation solver (the HLL solver gives similar results). A second-order two-step scheme in time is applied (Heun scheme). The weak CFL condition (77) is implemented with right-hand side $1/2$. We use the first-order scheme in space, but nevertheless for Tests 1, 2, 4, 5, a reference solution computed at second-order with a very fine grid is plotted. Except for Test 1, the SCHR scheme gives similar results as the UDCHR scheme.

6.1. **Test 1.** It is an upwind Riemann problem with $\rho_1/\rho_2 = 0.98$, $z = 0$, $u_1 = u_2 = 2.5$,

$$h_l^1 = 0.5, \quad h_r^1 = 0.55, \quad h_l^2 = 0.5, \quad h_r^2 = 0.45, \qquad (84)$$

$x \in (0,1)$ and the discontinuity is taken at $x = 0.5$. The final time is $t = 0.05$ and we use 100 points. Figure 3 shows the interface $z + h_2$. For comparison the $SCHR$ scheme is plotted also (dispersive oscillations).

6.2. **Test 2.** It is a centered Riemann problem, $\rho_1/\rho_2 = 0.7$, $z = 0$, $u_1 = u_2 = 0$,

$$h_l^1 = 1.8, \quad h_r^1 = 0.2, \quad h_l^2 = 0.2, \quad h_r^2 = 1.8, \qquad (85)$$

$x \in (0,10)$ and the discontinuity is taken at $x = 5$. The final time is $t = 1$ and we use 500 points. Figure 4 shows the interface. The result with the HR scheme is plotted also (unphysical discontinuity). The UDCHR scheme gives the same solution as the Roe method, indeed this test was proposed in [8].
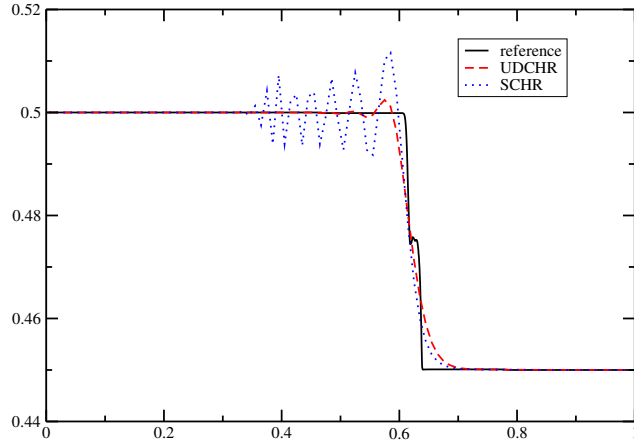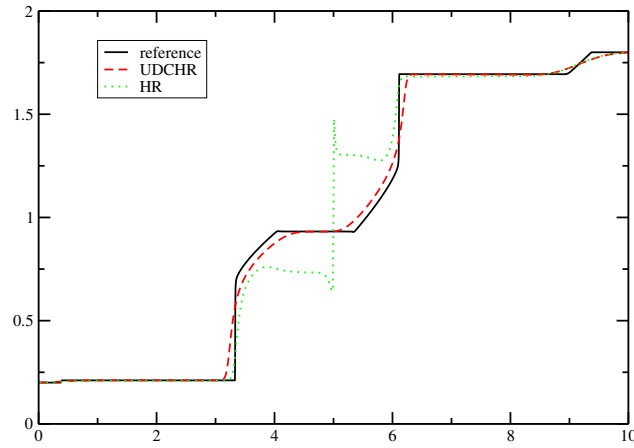
FIGURE 3. Test 1: upwind flow

FIGURE 4. Test 2: centered Riemann problem

6.3. **Test 3.** Small perturbation of a constant state with complex eigen-values. We take $\rho_1/\rho_2 = 0.98$, $z = 0$, initial data are $u_1 = 0.6$, $u_2 = -0.6$,

$$h_2(x) = \begin{cases} 0.5 + 0.01 \left(1 + \cos \left(\frac{(x-5)\pi}{0.1} \right) \right) & \text{if } |x - 5| \leq 0.1, \\ 0.5 & \text{if } |x - 5| \geq 0.1, \end{cases} \tag{86}$$

$h_1 + h_2 = 1$, $x \in (0, 10)$, the final time is $t = 1$ and we use 1000 points. Figures 5 and 6 show the benefit of using the UDCHR scheme.
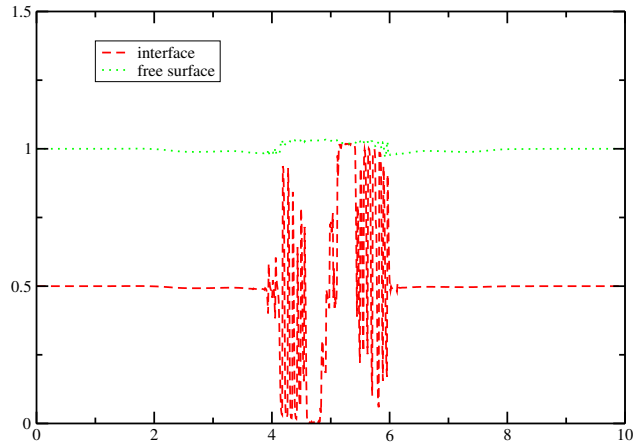
FIGURE 5. Test 3: complex eigenvalues, HR scheme
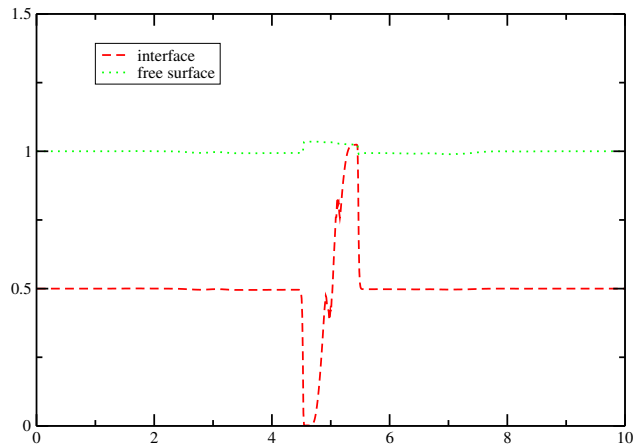
FIGURE 6. Test 3: complex eigenvalues, UDCHR scheme

6.4. **Test 4.** Mixing of two pure layers. We take $\rho_1/\rho_2 = 0.85$, $z = 0$, with initial data $u_1 = u_2 = 0$,

$$h_l^1 = 0, \quad h_r^1 = 1, \quad h_l^2 = 1, \quad h_r^2 = 0, \tag{87}$$

$x \in (0, 1)$ and the discontinuity is taken at $x = 0.5$. We use Neumann boundary conditions, the final time is $t = 0.32$ and we use 100 points. The interface is plotted on Figure 7.
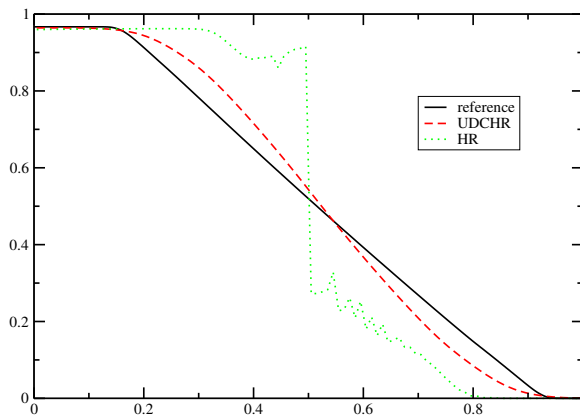
FIGURE 7. Test 4: mixing of two pure layers

6.5. **Test 5.** Drying and topography. We take $\rho_1/\rho_2 = 0.95$,

$$z(x) = \begin{cases} 0 & \text{if } x \leq 0.5, \\ 4(x - 1/2) & \text{if } x \geq 0.5, \end{cases} \tag{88}$$

with initial data $u_1 = u_2 = 0$,

$$\begin{aligned} h_2(x) &= \begin{cases} 0.5 & \text{if } x < 0.25, \\ 0 & \text{if } x > 0.25, \end{cases} \\ h_1(x) &= (1 - h_2(x) - z(x))_+ , \end{aligned} \tag{89}$$

$x \in (0, 1)$ with 100 points. Wall boundary conditions are used. Figure 9 shows the result for $t = 0.5$, while Figure 10 is for $t = 50$.

7. CONCLUSION

We have proposed a simple uncoupled method to solve the multilayer shallow water system. Its features are:

- It keeps the water heights nonnegative, and it is able to compute with dry areas,
- It is well-balanced,
- It satisfies a semi-discrete entropy inequality,
- It works for an arbitrary number of layers, arbitrary densities, and arbitrary topography,
- It can deal with complex eigenvalues consistently, indeed the eigenvalues of the system are never computed,
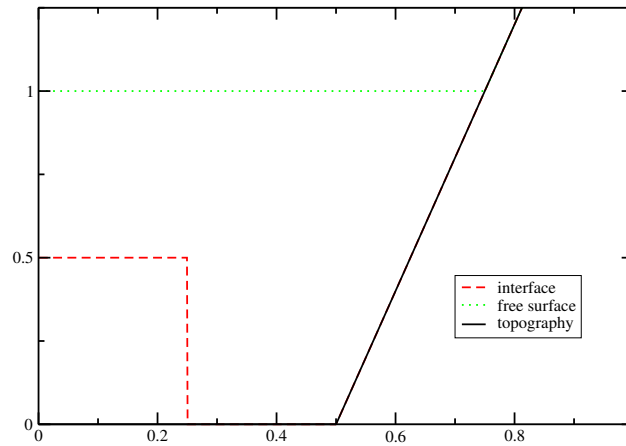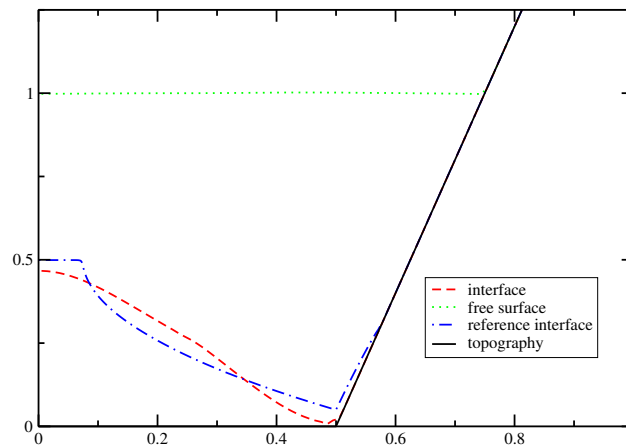- It runs very fast since it needs only to run a one-layer solver for each layer.

FIGURE 8. Test 5: initial

FIGURE 9. Test 5: lower layer reaches the shore ($t = 0.5$)

We have performed simple tests in order to validate the method, that gave good results, showing the exceptional robustness of the scheme. More involved tests will be presented in other publications, including [12].

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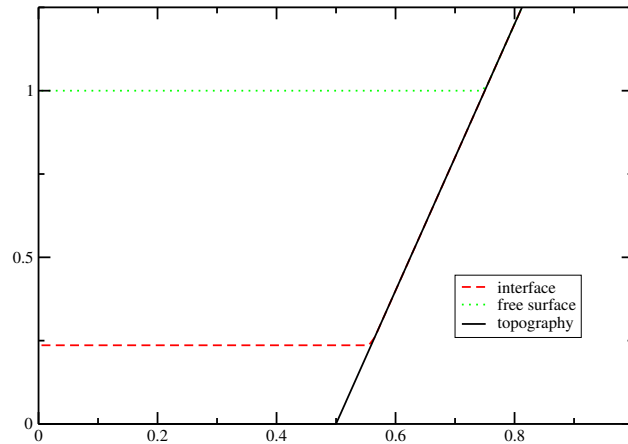
FIGURE 10. Test 5: long time ($t = 50$)

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