The linear appearance theorem for a class of non linear non homogeneous hyperbolic systems involving a transport equation

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Abstract The linear appearance theorem states that, for a wide class of non linear hyperbolic systems, when a source term occurs, some solutions are also solutions to a linear homogeneous system, which means that the corresponding profiles are simply translated with a constant velocity. This allows to solve some problems by combining a sequence of such profiles separated by shock waves. Several examples are reported, such as the roll waves in hydraulics, acoustics waves as solution of gas dynamics systems in a duct, or a rarefaction wave in fluids, seen as limit of kinds of saw waves towards a Riemann invariant, as for the nonlinear homogeneous case. Some new numerical schemes adapted to the source terms are presented, and tested on examples.

1 The class of hyperbolic systems

Let $q > 0$ a quantity transported by a flux m in a one dimension context, which is ruled by the transport equation

$$q_t + m_x = 0. \tag{1.1}$$

The vector (q, m) is named an admissible state, belonging to some set Ω of the phase plane.
We look for hyperbolic systems whose first equation is (1.1). The second equation has the form

$$m_t + Aq_x + Bm_x + S(q, m, x, t) = 0 \ ,$$

with A and B such that the eigenvalues of the flux matrix $\begin{pmatrix} 0 & 1 \\ A & B \end{pmatrix}$, denoted by λ_1 and λ_2, are real and different (we choose $\lambda_2 > \lambda_1$), and S a given source term. Let $c = \frac{\lambda_2 - \lambda_1}{2}$ (> 0) , $u = \frac{\lambda_1 + \lambda_2}{2}$, then $A = -\lambda_1\lambda_2 = c^2 - u^2$ and $B = \lambda_1 + \lambda_2 = 2u$, and the second equation reads

$$m_t + \left(c^2 - u^2\right) q_x + 2um_x + S(q, m, x, t) = 0 \ . \tag{1.2}$$

We shall restrict the study to the case that u, c and S depend only on the states (q, m), (that is, not on (x, t) directly)

Definition 1.1 *The source term S is essentially non-zero when in any open set of Ω there is a state (q, m) for which $S(q, m) \neq 0$.*

In the next section the theorem of linear appearence is proved in a general case, and the restriction to the conservative form is studied in Section3. Section 4 is devoted to shock waves analysis and relaxation by contact discontinuities, and Section 5 to some elementary waves, as these provided by the "bow patterns" similar to wavelets, or complex waves, similar to saw waves. The other sections deal with examples: the roll waves in hydraulics in Section 6, the similarity between acoustics and fluid dynamics in a duct in Section 7, and the shock tube with friction in Section 8, a fundamental example of a Riemann invariant becoming the limit of tiny saw waves. Then Section 9 is devoted to te construction of new numerical schemes and some tests.

2 The Theorem of Linear Appearance

We consider the waves for which q and m are linked by e relation of the form $m = m(q)$, which seems a restrictive hypothesis. Indeed, this boils down to suppose that some changing of variables $\xi(x,t)$, $\eta(x,t)$ exists and both q and m depends on ξ only. For instance, in the homogeneous case, the variable $\xi = (x - x_0)/(t - t_0)$ does work. We shall see later (Remark 3.7) that this hypothesis is really relevant from the physical point of view.

Proposition 2.1 *When $m = m(q)$, the quantity $A(q) = m'(q)$ satisfies the Burgers equation*

$$A_t + AA_x = 0. \tag{2.1}$$

Proof Since $q_t + m(q)_x = 0$, we multiply by $m''(q)$ and notice that

$$m''(q)q_t = A_t, \ \ m''(q)m'(q)q_x = AA_x.$$

This remark will be used later in a numerical scheme

Theorem 2.2 *Let $c, u, S \in C^0(\Omega)$, with S essentially non-zero on Ω. Then for any regular wave for which $m = m(q)$, q and m are locally solution to the same advection equation*

$$q_t + Aq_x = 0 \quad , \quad m_t + Am_x = 0 \ , \tag{2.2}$$

and linked together by the relation $m = Aq - B$, with some constants A and B to be determined from the context.

Proof Since the wave is regular, and using the relation $m = m(q)$, we get

$$m_x = m'(q)q_x \quad , \quad m_t = m'(q)q_t = -m'(q)m_x = -m'(q)^2 q_x \ .$$

Thus (1.2) becomes $\left(c^2 - (m'(q) - u)^2\right) q_x + S(q, m) = 0$, or

$$\frac{(m'(q) - u)^2 - c^2}{S(q, m)} \, q_x = 1 \ . \tag{2.3}$$

Since $u = u(q, m(q))$ and $c = c(q, m(q))$, we can introduce a function $\psi(q)$ whose derivative is given by

$$\psi'(q) = \frac{(m'(q) - u(q, m(q)))^2 - c(q, m(q))^2}{S(q, m(q))} \quad , \tag{2.4}$$

and replace (2.3) by

$$\psi'(q)\ q_x = 1\ . \tag{2.5}$$

Integrating with respect to x gives $\psi(q) = x - K(t)$, with $K(t)$ depending on the second variable t. Then the derivation with respect to t gives $\psi'(q)\ q_t = -K'(t)$, where $q_t = -m'(q)q_x$. Using (2.3) it remains $m'(q) = K'(t)$, and the derivation with respect to x gives $m''(q)\ q_x\ =\ 0$. Since from (2.3) $q_x \neq 0$, one gets $m''(q) = 0$, that is

$$m(q) = Aq - B\ ,\ \ K(t) = At - C \tag{2.6}$$

for some constants A, B and C. Using $m = Aq - B$ in (1.1) gives the two advection equations (2.2).

Corollary 2.3 *The profile of the solution is determined by*

$$\psi(q) = x - At + C\ , \tag{2.7}$$

to be inverted to get the profiles of q and $m = Aq - B$ with respect to x at each time t.

The proof is obvious.

Remark 2.4 *The relation $m = Aq - B$ tells us that in the phase plane the authorized values of the state (q, m) belongs to the same straight line. Following [2] the corresponding waves are called* **source waves***.*

3 The conservative form

Definition 3.1 *Equation (1.2) is* **conservative** *when it can be written as*

$$m_t + F(q, m)_x + S(q, m) = 0 \tag{3.1}$$

with $F \in C^2(\Omega)$, called the **conservative flux***.*

As a consequence, $\dfrac{\partial F}{\partial q} = c^2 - u^2$, $\dfrac{\partial F}{\partial m} = 2u$, which implies $\dfrac{\partial^2 F}{\partial q \partial m} = 2c\dfrac{\partial c}{\partial m} - 2u\dfrac{\partial u}{\partial m} = 2\dfrac{\partial u}{\partial q}$, and reads also

$$\frac{\partial u}{\partial q} + u\frac{\partial u}{\partial m} = c\frac{\partial c}{\partial m}\ . \tag{3.2}$$

The **Galilean Invariance** is checked by using the following substitution of variables: $s = t,\ y = x - at$, with a constant, corresponding to the velocity of a new observer. Equations (1.1) and (1.2) become

$$q_s + (m - aq)_y = 0 \quad , \quad m_t + \left(c^2 - u^2\right) q_y + (2u - a)\, m_y + S(q,m) = 0 \ .$$

The eigenvalues of the new matrix of flux are $\mu_1 = u - c - a$ and $\mu_2 = u + c - a$, which preserves $c = \frac{1}{2}(\mu_2 - \mu_1)$ but change u into $u = \frac{1}{2}(\mu_1 + \mu_2) - a$. We get $\frac{\partial c}{\partial a} = 0$, $\frac{\partial u}{\partial a} = -1$, which also reads

$$\frac{\partial c}{\partial a} = \frac{\partial c}{\partial q}\frac{\partial q}{\partial a} + \frac{\partial c}{\partial m}\frac{\partial m}{\partial a} = 0 \quad , \quad \frac{\partial u}{\partial a} = \frac{\partial u}{\partial q}\frac{\partial q}{\partial a} + \frac{\partial u}{\partial m}\frac{\partial m}{\partial a} = -1 \ . \tag{3.3}$$

The Galilean Invariance ensures that this substitution does not change the profile of the wave, which makes for instance $\frac{\partial q}{\partial a} = 0$. In this case, (3.3) becomes $\frac{\partial c}{\partial m}\frac{\partial m}{\partial a} = 0$, $\frac{\partial u}{\partial m}\frac{\partial m}{\partial a} = -1$,which implies $\frac{\partial c}{\partial m} = 0$. We propose the following definition for the Galilean Invariance.

Definition 3.2 *The system (1.1), (1.2) has the property of* **Galilean Invariance** *when* $\frac{\partial c}{\partial m} = 0$.

Remark 3.3 *When c is not a constant, this is equivalent to $\frac{\partial q}{\partial a} = 0$, and has the advantage to implicate no substitution of variables. When c is a constant, this definition must be adapted, for instance by taking the limit of a sequence of $q-$depending $c_n(q)$.*

The Galilean invariance together with the conservativity leads to

$$\frac{\partial u}{\partial q} + u\frac{\partial u}{\partial m} = 0 \ ,$$

which is the well known Burgers equation with x replaced by m and t replaced by q.

Definition 3.4 *The system (1.1), (1.2) has the property of the* **zero-flux vacuum** *when*

$$q = 0 \Longrightarrow m = 0.$$

Theorem 3.5 *Suppose the hypotheses of Theorem 2.2 fullfilled, that the system (1.1), (1.2) is conservative and has the properties of Galilean invariance and zero-flux vacuum. Then*

$$m = qu \ , \tag{3.4}$$

the profile function $\psi'(q)$ in (2.4) becomes

$$\psi'(q) = \frac{B^2 - q^2 c(q)^2}{q^2\ S(q, Aq - B)} \quad , \tag{3.5}$$

and the conservative flux is given by

$$F(q,m) = \frac{m^2}{q} + \int_0^q c(\xi)^2\ d\xi \ . \tag{3.6}$$

Proof From the Galilean invariance (3.2) reduces to the Burgers equation in the phase plane, to be solved along a characteristic coming from the origin $(0,0)$ by the zero-flux vacuum property. This characteristic is a straight line of equation $m = qu$. Then the profile equation becomes (3.5) since $u = A - B/Q$, and

$$\psi'(q) = \frac{(A - A + B/q)^2 - c(q)^2}{S(q, Aq - B)} = \frac{B^2 - q^2 c(q)^2}{q^2 \ S(q, Aq - B)} \ .$$

Next, $\dfrac{\partial F}{\partial q} = c(q)^2 - \dfrac{m^2}{q^2}$, $\dfrac{\partial F}{\partial m} = 2\dfrac{m}{q}$, which leads to (3.6) (the integration constant is chosen equal to zero in order to preserve the zero-flux vacuum when u is bounded).

Definition 3.6 *The primitive $P(q) = \int_0^q c(\xi)^2 d\xi$ is called the* **pressure***.*

Remark 3.7 *The hypothesis $m = m(q)$ in Section 2 has lead to the relation $u = m/q$ which is expected in all the physical apllications. Since by construction, u depends only on q and m, this relation reads $m = q\ u(q,m)$, which implies locally a relation of the form $m = m(q)$. Thus this hypothesis is not in the least a restrictive one, since it appears now as a necessary condition.*

From now on, we suppose that the hypotheses of Theorem 3.5 are fullfilled.

4 The shock waves

Let (Σ) a discontiuity curve of q and m in the conservative case (equations (1.1) and (3.1)), whose equation has the form $x = x(t)$ in the $x - t$ plane. Integrating (1.1) and (3.1) on any set crossed by (Σ) and using the Green-Riemann formula gives the two Rankine Hugoniot conditions

$$x'(t)\ \Delta q = \Delta m \quad , \quad x'(t)\ \Delta m = \Delta F(q,m) \ ,$$

by denoting $\Delta q = q_2 - q_1$ the jump of q along (Σ), $\Delta m = m_2 - m_1$ the jump of m and $\Delta F(q,m) = F(q_2, m_2) - F(q_1, m_1)$ the jump of the conservative flux $F(q,m)$. This involves a compatibility condition:

$$(\Delta m)^2 = \Delta q\ \Delta F(q,m) \ ,$$

which reads

$$\Delta P \Delta q = (\Delta m)^2 - \Delta q \Delta\,(mu) = \left(\sqrt{\frac{q_1}{q_2}} m_2 - \sqrt{\frac{q_2}{q_1}} m_1\right)^2$$

in the case of the hypotheses of Theorem 3.5. This ensures $\Delta P \Delta q \geq 0$, which allows to write the **jump condition**

$$\Delta u = \pm \sqrt{\frac{\Delta P \Delta q}{q_1 q_2}} \ , \tag{4.1}$$

where the sign is linked to the velocity of the wave. As a matter of fact, we have

$$x'(t) = \frac{\Delta m}{\Delta q} = \frac{u_1 + u_2}{2} + \frac{q_1 + q_2}{2}\frac{\Delta u}{\Delta q} = \frac{u_1 + u_2}{2} \pm \frac{q_1 + q_2}{2\sqrt{q_1 q_2}} \sqrt{\frac{\Delta P}{\Delta q}} \ .$$

The sign $+$ involves a velocity of the form $\dfrac{u_1+u_2}{2}+\overline{c(q_1,q_2)}$, with $\overline{c(q_1,q_2)}=\dfrac{q_1+q_2}{2\sqrt{q_1q_2}}\sqrt{\dfrac{\Delta P}{\Delta q}}$ which is positive and has the dimension and the form of the wave velocity λ_2 (named a λ_2-shock wave). The sign $-$ involves a velocity of the form $\dfrac{u_1+u_2}{2}-\overline{c(q_1,q_2)}$, corresponding to the wave velocity λ_1 (named a λ_1-shock wave).

The Entropy condition: All shock waves are not allowed to develop. Since a shock wave is created by the intersection of the characteristics coming from the initial data. The entropy condition states the principle that no new characteristic can be created after the initial time. In other words, the characteristics must enter a shock waves, never take out from it. This corresponds for a λ_1-shock wave to the condition$u_1-c(q_1)\geq x'(t)\geq u_2-c(q_2)$ and, for a λ_2-shock wave to $u_1+c(q_1)\geq x'(t)\geq u_2+c(q_2)$, by assigning the index 1 to the left side of the shock wave and the index 2 to the right side.

When $qc(q)$ is an increasing function of q, a λ_1-shock wave must correspond to an increasing part of the profile of q and a λ_2-shock wave must correspond to a decreasing part of this profile.

4.1 The relaxation by contact discontinuities

Since $P_t=c(q)^2q_t$ and $P_x=c(q)^2q_x$, multiplying (1.1) by $c(q)^2$ provides a third equation

$$P_t+uP_x+qc(q)^2u_x=0 \tag{4.2}$$

called the **Hooke Law** similarly to the equation for strains in materials. The new system of three equations made of (1.1), (3.1) transformed as

$$q_t+uq_x+qu_x=0,\quad u_t+uu_x+\frac{1}{q}P_x+\frac{S(q,qu)}{q}=0,$$

and the Hooke law (4.2) has a flux matrix with three differenteigenvalues, namely $u-c(q)$, u and $u+c(q)$. This allows the possibility of another kind of wave, of velocity u, called a **contact discontinuity** which may relax the system in some too strained configuration though it cannot appear in the two equation context. Such contact discontinuities are characterized by the conditions $\Delta u=0,\ \Delta q\neq 0$, thus $\Delta P=0$ from (4.1). Note that equation (4.2) is not different from the tranport equation (1.1), and no new equation has been added. We simply have used it twice to get this new wave and forgotten the linkage $P(q)=\int_0^q c(\xi)^2d\xi$ between P and $c(q)$. The shock conditions for contact discontinuities (that is $\Delta u=0,\ \Delta P=0,\ \Delta q\neq 0$) can be derived even when the system is not written under conservative form (see [3]). However a conservative form can always be derived from (4.2), using (1.1) and (3.1), when $c(q)$ has the form $c(q)=C_0q^\nu$, with C_0 and ν given positive constants, and introducing a new variable corresponding to the total energy in practice.

5 Sequences of elementary waves

In the sequence we suppose that $qc(q)$ is an increasing function of q. The main difference with the usual homogeneous case is that the constant solutions do not exist when $S\neq 0$. The profile of the

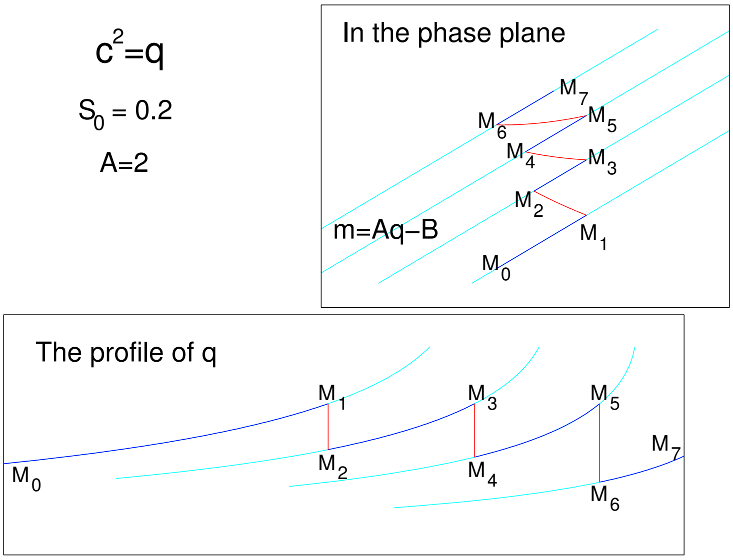

Figure 1: The profile of a sequence of waves

solution is made of a sequence of source waves, given by (2.5), with $\psi'(q)$ from (3.5), separated by shock waves. Figure 1 below presents a sequence of 4 source waves separated by 3 shock waves.

The source waves M_0M_1, M_2M_3, M_4M_5 and M_6M_5 propagate with the same velocity $A = 2$, and the profiles are computed with $c = \sqrt{q}$ and $S(q,m) = S_0 = 0.2$ (*constant*). The velocities of the shock waves (M_1M_2, M_3M_4 and M_5M_6) are all different, and also different from A. The profile corresponds to a path in the phase plane, from the left to the right side of the sequence of curves. Indeed, the positions of the shocks at each time are fixed by the mass conservation (1.1).

5.1 The possible connexions to a given state: the source waves

Let $M_0 = (q_0, m_0)$ an admissible state in the phase plane. We set $u_0 = m_0/q_0$, $c_0 = c(q_0)$. We look for the set of all the attainable states from M_0 by either a source wave (that is, with $m = Aq - B$) or a shock wave. We propose to name **elementary waves** those waves. The set of attainable states by a shock wave is

$$RH_{\pm}(M_0) = \{\ M = (q,m)\ \mid\ m = qu_0 \pm \sqrt{\frac{q}{q_0}\Delta P \Delta q}\ \} \tag{5.1}$$

corresponding to the Rankine-Hugoniot compatibility condition, with the sign $-$ for a λ_1-shock wave and the sign $+$ for a λ_2-shock wave. This set is represented by a concave (λ_1-wave) or a convex (λ_2-wave) curve passing through M_0 on Figure 2. The tangent straight lines to $RH_{\pm}(M_0)$ at $M_0 = (q_0, m_0)$ are

$$D_+(M_0) = \{m = (u_0 + c_0)\,q - q_0c_0\},\ D_-(M_0) = \{m = (u_0 - c_0)\,q + q_0c_0\} \tag{5.2}$$

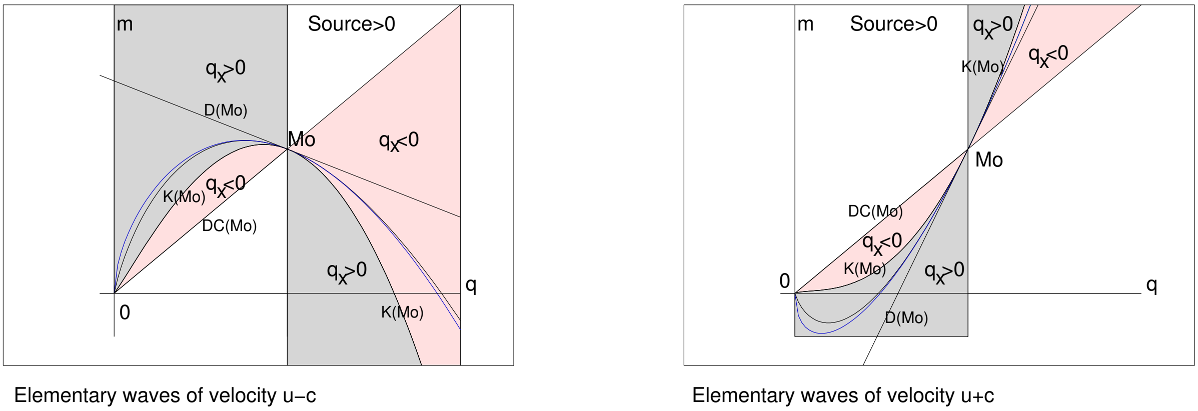

Figure 2: Attainable states from to a given state

according to the sign. The set $D_+(M_0)$ describes the states M such that the resulting source wave will move with the velocity $u_0 + c_0$ (a λ_2-source wave). The set $D_-(M_0)$ describes the states M such that the resulting source wave will move with the velocity $u_0 - c_0$ (a λ_1-source wave).

Therefore all admissible states M such that $M_0 \in D_\pm(M)$ are attainable with a source wave and we set

$$K_\pm(M_0) = \{M \ \mid \ M_0 \in D_\pm(M)\}. \tag{5.3}$$

These sets are described on Figure 2, and corresponds to a concave curve for a $\lambda_1 - wave$ and to a convex curve for a $\lambda_2 - wave$. Both pass throught the origin and throught M_0 with the same tangent $D_\pm(M_0)$ according to the sign.

For any $M \in K_\pm(M_0)$ the whole straight line $D_\pm(M)$ belongs to the set of attainable states from M_0. Therefore this set is the union of the two subsets

$$\{(q,m) \mid m \geq u_0 q,\ 0 \leq q < q_0\}\ ,\ \{(q,m) \mid m \leq u_0 q,\ q > q_0\}$$

and the point M_0 itself, for the λ_1-waves, and the two subsets

$$\{(q,m) \mid m \leq u_0 q,\ 0 \leq q < q_0\}\ ,\ \{(q,m) \mid m \geq u_0 q,\ q > q_0\}$$

with the point M_0 for the λ_2-waves. In each subset the curve $K_\pm(M_0)$ separates the states corresponding to an increasing $q-$profile from the states corresponding to a decreasing $q-$profile. This is shown on Figure 2, with different levels of grey, when the source term is positive. When the source term is negative the sign of q_x must be changed.

Since $D_\pm(M_0)$ corresponds to the equation $\frac{dm}{dq} = u_0 \pm c_0$, we consider the envelope of these straight lines whose equation is $\frac{dm}{dq} = \frac{m}{q} \pm c(q)$, passing throught M_0. Solving these equations gives $m = qu_0 \pm q \int_{q_0}^{q} \frac{c(\xi)}{\xi} d\xi$, which are the equations (according to the sign) of the Riemann Invariants of the homogeneous version of our model. We introduce the two sets

$$RI_\pm(M_0) = \{\ M = (q,m)\ \mid\ m = qu_0 \pm q \int_{q_0}^{q} \frac{c(\xi)}{\xi} d\xi\ \ \}\ , \tag{5.4}$$

which are the two **Riemann Invariants** passing throught M_0, considered here as the envelope of the straight lines $D_\pm(M)$.

Among the source waves, the **stationary waves** correpond to the waves of null velocity, that is when $A = 0$, and the equation reduces to $m = -B$, a constant.

5.2 The "bow pattern"

We consider now the case of a source term becoming null along a curve Σ on the phase plane.

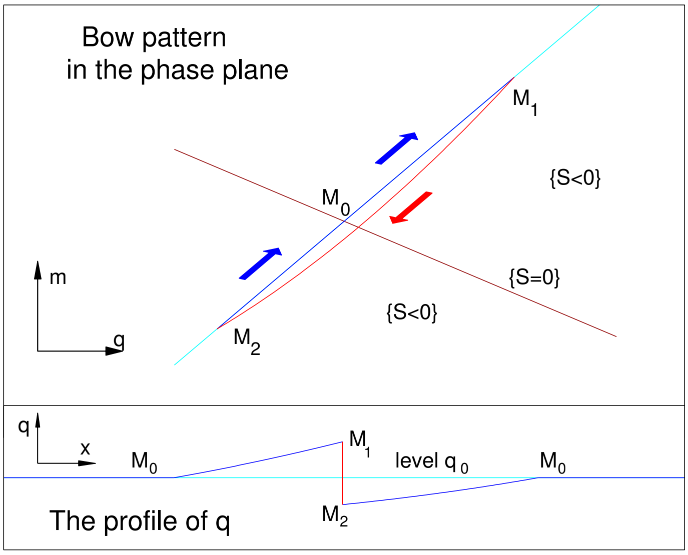

Figure 3: A bow pattern and the associate wave

We consider a λ_2-wave. The case of a λ_1-wave is similar. Let $M_0 \in \Sigma$ such as the line $D_+(M_0)$ crosses once Σ (the tangent case is ruled out). Then the expression of $\psi'(q)$ in (3.5) may be defined for $q = q_0$ since the numerator and the denaminator have both a single root. We get

$$\psi'(q_0) = -\frac{2c'(q_0)q_0 + 2c(q_0)^2}{q_0\left(\frac{\partial S}{\partial q}(q_0, Aq_0 - B) + A\frac{\partial S}{\partial m}(q_0, Aq_0 - B)\right)}\ .$$

For other lines crossing Σ in M_0 than $D_+(M_0)$ (or $D_-(M_0)$) $\psi'q)$ is not defined for $q = q_0$. Since the numerator and the denaminator in $\psi'(q)$ change their sign in q_0, the expression of $\psi'(q)$ keeps its sign. Let $M_1 = (q_1, m_1) \in D_+(M_0)$ with $q_1 > q_0$ and suppose for instance that M_1 belongs to $\Sigma_- = \{M = q, m) \mid S(q, m) < 0\}$ as presented on Figure 3.
Then for $q_0 < q < q_1$, we have $q_0^2 c(q_0)^2 - q^2 c(q)^2 < 0$ and $\psi'(q) > 0$, which implies $q_x > 0$, the corresponding profile of q in the associate source wave in increasing. On the other side, for $M_2 = (q_2, m_2) \in D_+(M_0)$, with $q_2 < q_0$, belonging to $\Sigma_+ = \{M = q, m) \mid S(q, m) > 0\}$ we also have an increasing profile of q for $q_2 < q < q_0$. It is also possible to draw the Rankine-Hugoniot curve

$RH_+(M_1)$ (see (5.1)) which cut $D_+(M_0)$ in some point M_2, with $q_2 < q_0$, in some applications. That way, we get a profile made of two source waves separated by a shock wave, propagating with the same constant velocity $A = m_0/q_0 + c(q_0)$, and allways keeping the same shape. The wave (see Figure 3, down) seems to be of null sum, which is not true and makes the difference from the wavelets. In the phase plane the straight section M_2M_1 looks like the string of a bow shaped by the section of $RH_+(M_1)$ linking M_1 back to M_2. The state M_0 does not correspond to the middle of the straight segment M_2M_1 in general. An example of this phenomenon arises in hydraulics, known as the roll waves, and is detailed in the next section.

5.3 The Riemann invariant as limit of saw waves

In the phase plane, the Riemann invariant is the set of the admissible values for a regular wave in the homogeneous case.

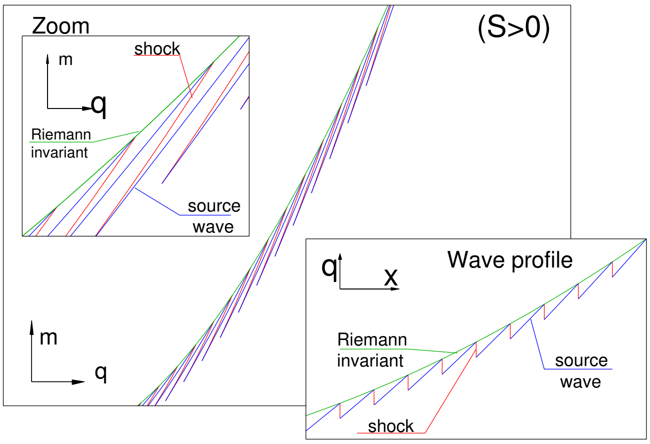

Figure 4: A saw wave

For a problem with initial data, in the non homogeneous case, the influence of the source term comes gradually and is small for the first times. That way, the starting values are the same as for the homogeneous case, and described by the Riemann invariant in the phase plane. Each admissible point of the Riemann invariant is the starting point of a regular wave, according to (3.5). However, the transport equation (1.1) implies the conservation of the quantity q (which is the mass when q is a density) and the development of the regular wave will modify this mass, which is not possible from physics laws. The reaction will be the immediate emergence of a shock wave. It is possible to construct a sequence of shocks and source waves along the Riemann invariant as shown on Figure 4, which respects the Entropy principle, even with a high frequency. The shocks may be also contact discontinuities. Such a wave is called a **saw wave**, whose values are close to the Riemann invariant. Thus the Riemann invariant appears as the limit of these saw waves.

In such waves the source term seems to have no action, which is not true. Indeed, the action of the source term is immediately tranformed in entropy and lost in the shock wave. An example of such saw wave is proposed in Section 8, for the shock tube with friction in gas dynamics, whose solution seems to be close to the one for the homogeneous case.

6 The Saint-Venant model and the Roll Waves

The one dimension flow of a river corresponds to the transport of the height of water h with a flux m which leads to a system of the form (1.1), (1.2). By taking a constant ratio c^2/h, whose value is the gravity constant g we get $c = c(h) = \sqrt{gh}$ and the Galilean invariance is fullfilled. Next, we claim that the system is conservative and that the zero flux vacuum property is true, from obvious physical principles, and both theorems 2.1 and 3.5 can apply. As a consequance, we get that the ratio $m/h = u$, the velocity. We get the One dimension Saint-Venant model (SV1)

$$h_t + m_x = 0 \ , \ m_t + \left(qu^2 + g\frac{h^2}{2}\right)_x + S(h,m) = 0 \ , \quad (m = hu) \ . \tag{6.1}$$

Remark that this construction does not need any argument of shallowness, thus the model SV1 is applicable even for deep water when the domain is sufficiently large.

We consider here the case of a river of constant bottom slope $p < 0$ and a constant Strickler friction coefficient $k > 0$, that is

$$S(h,m) = ghp + k\,|u|\,u \ ,$$

where $u > 0$ is expected.This source term is null along the curve $ghp + ku^2 = 0$. By introducing the parameter $\lambda = \sqrt{-\frac{p}{k}}$, this equation becomes $\lambda = \pm\frac{u}{c}$, the Froude number.

Theorem 6.1 *In permanent regime the Froude number $|u|/c$ of a river satisfies*

$$\frac{|u|}{c} = \lambda = \sqrt{-\frac{p}{k}} \quad . \tag{6.2}$$

The proof is obvious since the permanent regime necessarily corresponds to $S(h,m) = 0$. The formula (6.2) has a very important application: the value of the Strickler's coefficient k is easily obtained from the values of the river slope p, the water velocity u and the height of water h, since $c = \sqrt{gh}$, which are all easy to measure.

Let Σ_0 the curve corresponding to $S(h,m) = 0$ in the phase plane (h,m) . Let $M_0 = (h_0, m_0) \in \Sigma_0$ and $u_0 = m_0/h_0,\ c_0 = \sqrt{gh_0}$. We look for possible bow patterns for λ_2-waves, drawn from M_0. Since a decreasing shock wave is expected, the straight line $D_+(M_0)$ must cut Σ_0 by going from the set $\{S(h,m) > 0\}$ towards the set $\{S(h,m) < 0\}$. Since the slope of $D_+(M_0)$ is $u_0 + c_0 = c_0\,(1+\lambda)$ and the tangent slope of Σ_0 at M_0 is $3\lambda c_0/2$, the crossing is possible only for $c_0\,(1+\lambda) < 3\lambda c_0/2$, which reads $\lambda > 2$. We have proved the following

Theorem 6.2 *A necessary condition for a bow pattern in the SV1-model is $\lambda > 2$.*

This condition requires a fast torrential flood.

6.1 The profile of the Roll wave

From (3.5), we have here

$$\psi'(h) = \frac{gh_0^3 - gh^3}{h^2\,(ghp + ku^2)} = \frac{1}{k}\frac{h_0^3 - h^3}{((1+\lambda)h - h_0)^2 - \lambda^2 h^3} \tag{6.3}$$

and since $h = h_0$ is a root of both numerator and denominator, we get

$$\psi'(h) = \frac{1}{k\lambda^2}\left(1 + \frac{h_0^2 + h_0 h_{r1} + h_{r1}^2}{(h-h_{r1})(h_{r1}-h_{r2})} + \frac{h_0^2 + h_0 h_{r2} + h_{r2}^2}{(h-h_{r2})(h_{r2}-h_{r1})}\right) \;,$$

with $h_{r1} = \dfrac{1+2\lambda-\sqrt{1+4\lambda}}{2\lambda^2}h_0\ (>0)$ and $h_{r2} = \dfrac{1+2\lambda+\sqrt{1+4\lambda}}{2\lambda^2}h_0$. After integration we get

$$\psi(h) = \frac{1}{k\lambda^2}\left(h + \frac{h_0^2 + h_0 h_{r1} + h_{r1}^2}{h_{r1}-h_{r2}} ln\,|h-h_{r1}| + \frac{h_0^2 + h_0 h_{r2} + h_{r2}^2}{h_{r2}-h_{r1}}\; ln\,|h-h_{r2}|\right) \tag{6.4}$$

to be inserted into (2.7) and inverted to have h as a function of x. The profiles of source waves presented on Figure 5 are computed that way.

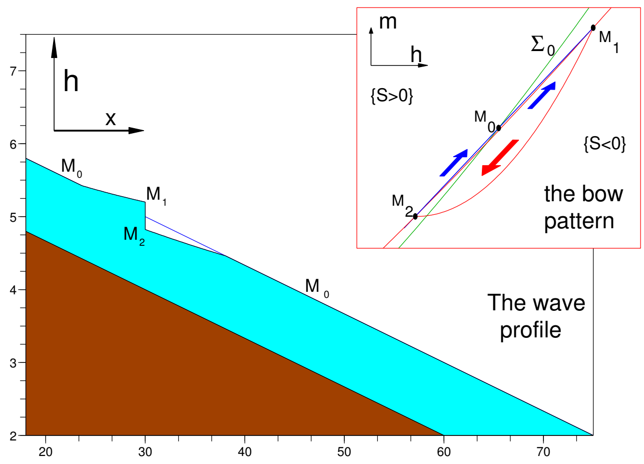

Figure 5: A Roll wave

In the inserted window, the distance between $RH_+(M_1)$ and $D_+(M_0)$ has been strongly emphasized. Otherwise the difference is imperceptible.

The simplification by $(h - h_0)$ in (6.3) was already done by Dressler in [1], together with the condition $\lambda > 2$. However the constant velocity of the waveseems to appear for the first time in [2]. A Roll wave is a combination of two source waves (6.4), one for $h < h_0$ followed by another with $h > h_0$ and separated by a shock wave whose velocity is necessarily the same, $A = (1 + \lambda)\, c_0$. A concatenation of a sequence of Roll waves is possible, even with different amplitudes, and travelling with the same velocity A.

7 From Gas dynamics to Acoustics: a wind instrument model

A sound is expected to correspond to the solution to a linear wave equation. However the Euler equations in Gas dynamics compose a non linear hyperbolic system with a source term corresponding to the friction and the geometry of the flow. We consider here the example of a sound tube and deduce the better shapes which provide sounds.

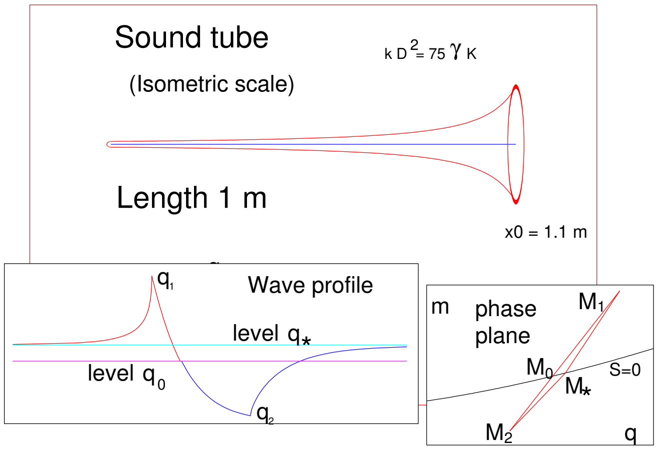

Figure 6: The shape and the wave

The tube is supposed to have a cylindrical symetry and a length $L > 0$, with a cross section denoted $a(x)$ for $0 \leq x \leq L$. The equation of conservation of the mass reads

$$a(x)\rho_t + (a(x)\rho u)_x = 0 \ , \tag{7.1}$$

where ρ is the density of the gas and u the velocity, both supposed to be uniform on a cross section. By writing $q = a(x)\rho$ and $m = a(x)\rho u$ the equation (7.1) is the same as the transport equation (1.1). The system is completed by an equation of the form (3.1) with a source term S given by

$$S(q,m,x) = k\,|u|\,u - \gamma K \frac{a'(x)}{a(x)^\gamma} q^\gamma \ , \tag{7.2}$$

where $\gamma = 1.4$ (adiabatic constant), $K = 69259.5\ MKS\ units$ and k a friction coefficient (depending on the nature of the material of the tube, wood or brass for example). This source term becomes independent on x when

$$\frac{a'(x)}{a(x)^\gamma} = Constant\ (denoted\ k\frac{D^2}{\gamma K}) \ ,$$

with D constant. Then the shape must be of the form

$$a(x) = \left(\frac{\gamma K}{(\gamma - 1)kD^2(x_0 - x)} \right)^{\frac{1}{\gamma-1}} \ , \tag{7.3}$$

where x_0 is another constant, to be chosen $> L$. This shape depends on two parameters x_0 and D and is drawn on Figure 6. The usual shape of wind instruments is recognizable, for clarinets or horns (in this case x is the curvilinear abscissa).

Equation (3.1) reads here

$$m_t + \left(\frac{m^2}{q} + c_0^2 q\right)_x + k\left(u^2 - D^2 q^\gamma\right) = 0 \tag{7.4}$$

by taking $c(q) = c_0$ whose value is given by $c_0 = \sqrt{\gamma K_0 T}$ with $K_0 = 287.06$ MKS *units* and T the ambient temperature in ^0K. The profiles of the source waves are computed from (2.5) which is here written as

$$\frac{c_0^2\left(q_0^2 - q^2\right)}{k\left(\left((1+\lambda)q - q_0\right)^2 c_0^2 - D^2 q^{\gamma+2}\right)} q_x = 1 \ ,$$

with $\lambda = \frac{m_0}{q_0 c_0}$, for a given state $M_0 = (q_0, m_0)$. The roots of the source term belong to the set Σ_0 of equation $m = Dq^{1+\gamma/2}$ and correspond to the stationary states. Figure 6 presents a sequence of 3 source waves built from two near states M_0 and $M_* = (q_*, m_*)$ of Σ_0, with $q_0 < q_*$ and $M_1,\ M_2 \in D_+(M_0)$. It is also possible to build a bow pattern whose string is on $D_+(M_0)$ when $M_0 \in \Sigma_0$ satisfies $u_0 + c_0 \leq D\left(1 + \frac{\gamma}{2}\right) q_0^{\gamma/2}$,with $u_0 = m_0/q_0$. The bow connects $M_1 = (q_1, m_1) \in D_+(M_0)$ with $q_1 > q_0$, to $M_2 \in D_+(M_0)$ by the shock curve $RH_+(M_1)$ of equation $m = \frac{q}{q_1} m_1 + \sqrt{\frac{q}{q_1}}\left(q - q_1\right) c_0$. In this case the profile of the wave is similar to a roll wave as on Figure 3.

8 The shock tube with friction

The Euler equations with a Strickler friction are

$$\rho_t + m_x = 0 \ , \ m_t + \left(\frac{m^2}{\rho} + P_0 \rho^\gamma\right)_x + k\left|u\right| u = 0 \ , \tag{8.1}$$

where ρ is the density, u the velocity, $m = \rho u$ the flux, and the three conctant: P_0 a reference pressure, k the Strickler friction coefficient and $\gamma = 1.4$ the adiabatic constant. The sound velocity is $c = c(\rho) = \sqrt{\gamma P_0 \rho^{\gamma-1}}$.

The problem for the shock tube consists into solving (8.1) for $t > 0,\ x \in \mathbb{R}$ by starting from the initial condition

$$\rho(x,0) = \begin{cases} \rho_L & for\ x < 0 \ , \\ \rho_R & for\ x > 0 \ , \end{cases} \quad , \quad u(x,0) = 0 \ , \tag{8.2}$$

with $\rho_L > \rho_R > 0$. The two states $M_L = (\rho_L, m_L)$ and $M_R = (\rho_R, m_R)$ are constant stationary states as roots of the source term $S(\rho, m) = k\left|u\right| u = k\left|m\right| m/\rho^2$. The shaping of two waves is expected, a front wave propagating forwards, on the right direction, and a rarefaction wave propagating backwards, on the left direction.

The action of the source term is perceptible only after a while and the solution is first shaped as the solution of the homogeneous problem with the shock curve $RH_+(M_R)$ intersecting the Riemann invariant $RI_-(M_L)$ at a state $M_0 = (\rho_0, m_0)$ with $m_0 > 0$. At the next time, each state $M = (\rho, m)$ of $RI_-(M_L)$ with $\rho_0 < \rho \leq \rho_l$ will generate a source wave, immediately broken by a shock wave, which corresponds to a saw wave. On the front side, let D_R be the secant to $RH_+(M_R)$ which extends the straight segment $M_R M_0$. A source wave valued on D_R , with $\rho > \rho_0$, will move at the same velocity $A_R = \frac{m_0}{\rho_0 - \rho_R}$ as the front shock wave. It remains to connect a state from D_R to a state of $RI_-(M_L)$ to get the whole solution. This can be done by combining a stationary wave and a contact discontinuity, whose position is controlled by the mass conservation principle.

The stationary wave links $RI_-(M_L)$ to M_0 and beyond, with values along the line $m = m_0$. Thus

$$\psi'(\rho) = \frac{m_0^2 - \gamma P_0 \rho^{\gamma+1}}{k\ m_0^2} \quad , \quad \psi(\rho) = \frac{\rho}{k} - \frac{\gamma P_0}{k\ m_0^2(\gamma+2)} \rho^{\gamma+2} \ ,$$

and from (2.7), ρ is obtained by solving $\psi(\rho) = x + \psi(\rho_0)$.

Since the equation of D_R is $m = A_R(\rho - \rho_R)$, with $A_R = \frac{m_0}{\rho_0 - \rho_R}$, we have

$$\psi'(\rho) = \frac{A_R^2 \rho_R^2 - \gamma P_0 \rho^{\gamma+1}}{k\ A_R^2 \left(\rho - \rho_R\right)^2} \ ,$$

to be integrated numerically to get $\psi(\rho)$. We obtain $\psi(\rho) = \psi(\rho_0) + x - A_R t$, since the wave propagates with the velocity A_R.

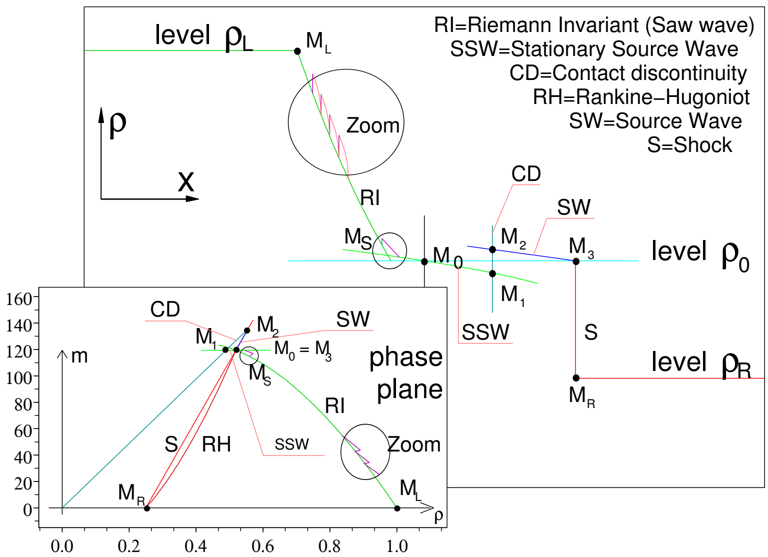

Figure 7: The shock tube with friction (t=0.1, k=0.00125)

It remains to join a state M_1 of the stationary wave to a state M_2 of D_R, at some point x determined such that the mass conservation is preserved, as shown on Figure 7 .

The profile is made, from left to right, of a constant stationary level ρ_L, then a saw wave from M_L to M_s, a combination of a source wave and a contact discontinuity (see the small circle on Figure 7) to link M_s to M_0, a stationary level from M_0 to M_1, a contact discontinuity from M_1 to

M_2 , then a source wave from M_2 to M_3 and a shock wave from M_3 to M_R where another constant stationary level ρ_R is found. Between M_0 and M_3 the mass over the level ρ_0 balances the lack of mass under this level, which determines the position of the contact discontinuity.

The saw wave is the limit odf a sequence of small contact discontinuities separated by small shocks, with a profile analogue to Figure 4.

Remark: This example gives some ideas for the construction of new Riemann solvers for non homogeneous systems, since the states M_0 (bottom of the rarefaction wave) and M_3 (top of the shock) coincide. A Riemann solver can be described as follows. First, solve the Riemann problem for the homogeneous equations, which give $M_0 = M_3$. Next, draw the source wave associated to the straight line of slope A_R (shock velocity) passing through $M_3(= M_0)$ and the stationary source wave (slope zero in the phase plane) passing through $M_0(= M_3)$, and then compute the position of the contact discontinuity linking these two waves in such a way that the mass balance is preserved. This solver is different from the ones proposed in [4] for Well Balanced (WB) schemes.

9 Two numerical methods

9.1 The method of stationary profiles (SP-scheme)

The previous example tells us that the usual Riemann solver (that is the one for the homogeneous case) can be used even for the non homogeneous case. The Godunov method can be adapted this way. However the constants are no more solutions and the idea of the **stationary profiles method, or SP-scheme** consists into interpreting the solution in each cell as a stationary solution. By denoting q_*, m_* the data in the middle x_* of the considered cell, the profile solution is shaped such that $m = m_*$, *constant*, and $\psi(q) = \psi(q_*) + x - x_*$, with x in the cell.

Consider a uniform mesh with cells of length Δx, centered at $x_j = j\Delta x$, $j \in \mathbb{Z}$, and two adjacent cells of index j and $j+1$. We first compute q_{jR} such that $\psi(q_{jR}) = \psi(q_j) + \frac{\Delta x}{2}$, then we compute $q_{j+1,L}$such that $\psi(q_{j+1,L}) = \psi(q_{j+1}) - \frac{\Delta x}{2}$ and next we solve the Riemann problem with (q_{jR}, m_j) as the left data and $(q_{j+1,L}, m_{j+1})$ as the right data along the line $x = x_j + \frac{\Delta x}{2}$. We get that way the two constants $q_{j+1/2}$ and $m_{j+1/2}$. Eventually we approximate the projections on the cells at the next time (n is the index for the time) by the Godunov formulae:

$$q_j^{n+1} = q_j^n - \frac{\Delta t}{\Delta x}\left(m_{j+1/2}^n - m_{j-1/2}^n\right) \ , \ m_j^{n+1} = m_j^n - \frac{\Delta t}{\Delta x}\left(F_{j+1/2}^n - F_{j-1/2}^n\right) - \Delta t S_j^n, \tag{9.1}$$

with $F_{j+1/2}^n = F(q_{j+1/2}^n, m_{j+1/2}^n)$ and S_j^n an approximation of $S(s_j^n, m_j^n)$ corresponding to a numerical scheme consistant with the differential equation $m' = -S(q, m)$.

Figure 9 represents a computation with the SP scheme, for $c = \sqrt{q}$ and $S(q, m) = k\,|u|\,u$ ($k = 0.2$). The discretisation of the differential equation $m' = -k\,|u|\,u = -k\frac{|m|m}{q^2}$ used the exact scheme $y_n + 1 = \dfrac{y_n}{1 + k\,|y_n|\,\Delta t}$ for the model equation $y' = -k\,|y|\,y$. The CFL number is rather small at the end of the computation and then, combined with the friction, the front waves are smoothed. We notice that the static equilibrium is restored behind the wave, which is a property of stability, as for the well balanced schemes (see [4]) though the method is very different here.

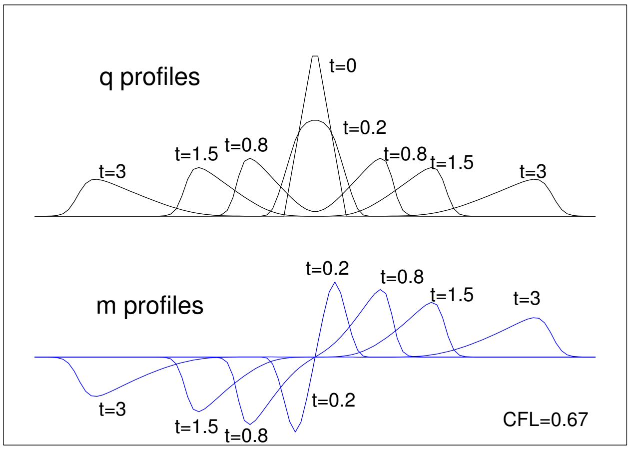

Figure 8: Profiles of q and m from the SP scheme

9.2 The method of dynamical profiles (DP-scheme)

The previous method of stationary profiles is unable to catch some waves, as the ones produced by a bow pattern for instance. We propose now a new scheme, called the **dynamical profiles method**, or **DP scheme**, which computes the value of the wave velocity A at each step, stores them in each cell to use them for the next time step.

The main idea is to use the velacity field $A = m'(q)$ in each cell at each time step, denoted A_j^n with the same numerical notations as before. We suppose that the initial values A_j^0 are available, then the Godunov method for the Burgers equation (2.1) works. At the level time $n\Delta t$, we first use the Riemann solver

$$A_{j+1/2}^n = \begin{cases} 0 & if\ A_j^n < 0 < A_{j+1}^n, \\ A_j^n & if\ A_j^n > 0,\ A_j^n + A_{j+1}^n > 0, \\ A_{j+1^n} & if\ A_{j+1}^n < 0,\ A_j^n + A_{j+1}^n < 0\ . \end{cases} \tag{9.2}$$

Next we compute the values at the time level $(n+1)\Delta t$ by

$$A_j^{n+1} = A_j^n - \frac{\Delta t}{\Delta x} \frac{\left(A_{j+1/2}^n\right)^2 - \left(A_{j-1/2}^n\right)^2}{2}\ . \tag{9.3}$$

This scheme requires the CFL condition

$$Max\left|A_j^n\right| \frac{\Delta t}{\Delta x} \leq 1\ . \tag{9.4}$$

Next, the values q_j^n and m_j^n are obtained by using the velocity field A_j^n. Since $\psi(q)$ satisfies $\psi'q)q_x = 1$, it is often easier to project $\psi(q)$ which is linearly depending on q than q itself. By multiplying the tranport equation (1.1) by $\psi'(q)$ we get either $\psi_t + A = 0$, or $\psi_t + A\psi_x = 0$, since $A = m'(q)$. Using the advection equation, we get the ψ_j^n by using the following scheme

$$\psi_j^{n+1} = \frac{\psi_j^n - \frac{\Delta t}{\Delta x}\left(A_{j+1/2}^n \psi_{j+1/2}^n - A_{j-1/2}^n \psi_{j-1/2}^n\right)}{1 - \frac{\Delta t}{\Delta x}\left(A_{j+1/2}^n - A_{j-1/2}^n\right)} \tag{9.5}$$

where

$$\psi_{j+1/2}^n = \begin{cases} \psi_j^n & if\ A_{j+1/2}^n = A_j^n, \\ \psi_{j+1}^n & if\ A_{j+1/2}^n = a_{j+1}^n \\ unused,\ for\ example\ 0 & if\ A_{j+1/2}^n = 0\ . \end{cases}$$

Then the values q_j^n are obtained by solving the implicit equations

$$\psi(q_j^n) = \psi_j^n\ . \tag{9.6}$$

The fluxes m are also advected by the velocity field A_j^n and are obtained by the same scheme as (9.5) for ψ. If necessary the values B_j^n are deduced from $B_j^n = A_j^n q_j^n - m_j^n$.

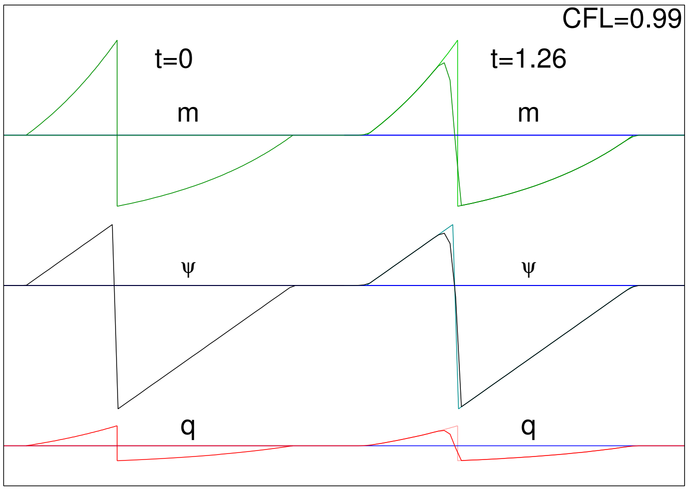

Figure 9: A roll wave computed by the DP scheme

Figure 10 presents a computation for the case of a bow pattern, with $c = \sqrt{q}$ and $S(q,m) = pq + k\,|u|\,u$, for $p = -1.40625$ and $k = 0.1$. This example is especially simple since A $(= 4.75)$ is a constant. Then (9.5) reduces to the usual donner cell scheme

$$\psi_j^{n+1} = \psi_j^n - \frac{\Delta t}{\Delta x} A\left(\psi_j^n - \psi_{j-1}^n\right)\ . \tag{9.7}$$

The same scheme works for tthe computation of the m_j^n; we can also use the relation $m_j^n = Aq_j^n - B$, since B is here too a constant. We notice a strong diffusion effect though the CFL condition was fullfilled close to one.

The scheme (9.5) is refered as the scheme weighted by the deformation rates. It preserves the L^∞ norm and the variation. The results can be greatly improved by using an antidiffusion technique.Note that when $A_{j+1/2}^n = 0$ the DP scheme coincides with the SP scheme.

10 Conclusions

First, remind that the hypothesis $m = m(q)$ is really not a restriction (Remark 3.7), which provides to this study a very large field of applications. The modelling of physical phenomenons involving a transport equation can be strongly reduced by using very simple and obvious hypotheses together with a mathematical reasoning, for example here for the Saint-Venant model in hydraulics. The apparent linearity of waves has often conduced the modelizers to write linear models, sometimes in opposition against supporters of nonlinearity. The present results will reduce this revalry between the two opinions since both can be right with the same model...

However, the examples of the shock tube and the saw waves proved that a priori estimates are probably difficult to settle, as well as uniqueness. It appears that to work in the phase plane can often help for the construction of solutions to a wide class of non homogeneous hyperbolic systems. The equilibria are hardly maintained during computations with usual schemes. Indeed, as shown here, the action of the source term is sometimes strongly reduced by the shocks, mainly in some behaviour as the saw waves or the relaxation by contact discontinuities. Solving locally, as for the Riemann solvers, will filter some phenomenons which must be considered in their wholeness, in order to preserve the established equilibria. This is the case when a bow pattern accurs, that usual schemes are often unable to capture.. The source waves have many other applications in environment modelling, for hydraulic phenemenons (Rogue wave or Tsunami) and also for shock waves in fluids or solids (earthquakes) or atmospheric applications (winds and hurricanes) and the theorem of linear appearence applies, even in two or three dimensions. Several papers on such applications are available on the Conservation Law Server (2004/014, 2005/048, 2008/019).

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