ON THE COMPACTNESS FOR TWO DIMENSIONAL SCALAR CONSERVATION LAW WITH DISCONTINUOUS FLUX

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Abstract. We prove that a family of solutions to a Cauchy problem for a two dimensional scalar conservation law with a discontinuous smoothed flux and the vanishing viscosity is strongly $L^1_{\rm loc}$– precompact under a new genuine nonlinearity condition, weaker than in previous works on the subject.

1. Introduction

We consider the following Cauchy problem for two dimensional scalar conservation law

$$\begin{aligned} u_t + \operatorname{div} f(x,y,u) = 0, \\ u(0,x,y) = u_0(x,y), \end{aligned} \tag{1}$$

where $u = u(t,x,y)$, $x,y \in \mathbf{R}$, $t \in \mathbf{R}^+$ and $f = (f_1, f_2) : \mathbf{R}^3 \to \mathbf{R}^2$ (divergence is taken with respect to x and y). For the initial data u_0 we assume that

$$u_0 \in (BV \cap L^\infty)(\mathbf{R}^2), \quad a \le u_0(x,y) \le b, \quad x,y \in \mathbf{R}, \tag{2}$$

where $a, b \in \mathbf{R}$ are constants. The flux function $f = (f_1, f_2)$ has the following properties:

$$f_i(\cdot,\cdot,\lambda) \in (BV \cap L^\infty)(\mathbf{R}^2) \text{ for every } \lambda \in \mathbf{R}, \tag{3}$$

$$f_i(x,y,\cdot) \in \mathrm{Lip}(\mathbf{R}) \text{ for every } (x,y) \in \mathbf{R}^2, \tag{4}$$

$$\max_{a\le u\le b} |f_i(\cdot,\cdot,u)|_{BV(\mathbf{R}^2)} < \infty, \ \max_{a\le u\le b} |f_i(\cdot,\cdot,u)| \in L^q_{\rm loc}(\mathbf{R}^2) \text{ for some } q > 2, \tag{5}$$

$$0 = f_i(x,y,b) = f_i(x,y,a), \ i = 1,2, \text{ for every} (x,y) \in \mathbf{R}^2, \tag{6}$$

where $\mathrm{Lip}(\mathbf{R})$ denotes a space of Lipschitz continuous functions.

In recent years, problems of this kind received lots of attention since they model many physical phenomena. As examples of special importance we emphasize applications in flow in porous media, sedimentation processes, traffic flow, radar shape-from-shading problems, blood flow, and gas flow in a variable duct.

If f_1 and f_2 are smooth functions, then the existence and uniqueness of an entropy solution is provided by the well known method of doubling of variables due to Kružkov [16], or by using the measure valued concept by DiPerna [8]. It is well known, cf. [6, 16], that for the Lipschitz continuous flux, the family of solutions

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to the vanishing viscosity regularization of (1) (see (8)-(9) below) converges to a weak solution of (1) in the strong $L^1_{\rm loc}(\mathbf{R}^+ \times \mathbf{R}^2)$ topology. However, if the flux is discontinuous with respect to $(x, y) \in \mathbf{R}^2$, we can not apply a classical approach.

The existence of a weak solution for the problem of type (1) was settled in [12] for a flux in a slightly less general form satisfying the following genuine nonlinearity condition: Let $S^2 \subset \mathbf{R}^3$ denotes the unit sphere. We say that the flux (f_1, f_2) satisfy a genuine nonlinearity condition if

$$\begin{gathered}\text{for almost every } (x,y) \in \mathbf{R}^2 \text{ and every } \xi \in S^2 \text{ the mapping}\\ \lambda \mapsto \xi_0\lambda + f_1(x,y,\lambda)\xi_1 + f_2(x,y,\lambda)\xi_2\\ \text{is not constant in } \lambda \text{ on any nontrivial interval.}\end{gathered} \tag{7}$$

The existence was obtained as a consequence of the strong $L^1_{\rm loc}(\mathbf{R}^+ \times \mathbf{R}^2)$-precompactness of a family of solutions to (8). The proof is based on a two dimensional variant [19] of the method of compensated compactness [20].

More precisely, the following regularization of problem (1) was considered (here and in the sequel Δ stands for the Laplacian $\Delta u = u_{xx} + u_{yy}$):

$$\partial_t u^{\varepsilon,\delta} + \operatorname{div} f^\delta(x, y, u^{\varepsilon,\delta}) = \varepsilon \Delta u^{\varepsilon,\delta} \tag{8}$$

$$u^{\varepsilon,\delta}|_{t=0} = u_0^\delta, \tag{9}$$

where the approximations f_i^δ and u_0^δ are constructed in the following manner. Let $\omega : \mathbf{R} \to \mathbf{R}$ be arbitrary smooth function such that $\omega(\xi) = 0$ for $|\xi| \geq 1$, and $\int_{\mathbf{R}} \omega(\xi)\, d\xi = 1$. We define (we consider λ as a parameter below)

$$f_i^\delta(x, y, \lambda) = \frac{1}{\delta^2}\left(\omega(x/\delta)\omega(y/\delta)\right) \star f_i(x, y, \lambda)$$

and

$$u_0^\delta(x, y) = \frac{1}{\delta^2}\left(\omega(x/\delta)\omega(y/\delta)\right) \star (u_0\chi_\delta)(x, y),$$

where $\chi_\delta(x, y) = 1$ for $(x, y) \in B(0, 1/\delta) \subset \mathbf{R}^2$ and zero otherwise. Here, $\star$ stands for the convolution operator.

The case of an arbitrary dimension space was completed by Panov [18], using another method of Tartar – H-measures [21] (introduced independently by Gerard [10] who named them microlocal defect measures). Similarly as in [12], in [18], it was proved that a family of solutions to equation (1) with the regularized flux is strongly $L^1_{\rm loc}$-precompact under a multidimensional variant of genuine nonlinearity condition (7).

We stress that in the one dimensional case, one does not need any nonlinearity condition in order to prove the existence of a weak solution to a scalar conservation law with a flux discontinuous in the space variable. More precisely, using the compensated compactness argument [14], it is not difficult to prove that a family of entropy admissible solutions [12, 18] of the one dimensional variant of (8) converges weakly along a subsequence to a solution of the one dimensional variant of (1).

However, we can not state anything about strong $L^1_{\rm loc}$-precompactness of the family $(u^{\varepsilon,\delta})_{\varepsilon,\delta}$ (see [13, Remark 2.3]) which is of essential importance since a strong $L^1_{\rm loc}$-limit along a subsequence of $(u^{\varepsilon,\delta})_{\varepsilon,\delta}$ satisfy admissibility conditions (see [14, Definition 1.2]). On the other hand, such conditions provide a stability of solutions for problems of type (1) (see e.g. [5, 14]).

In this paper, we shall prove that under a relaxed genuine nonlinearity condition (see (10) below), a family of solution to (8) is strongly precompact in $L^1_{\rm loc}(\mathbf{R}^+\times\mathbf{R}^2)$. This will provide the existence of a weak solution to (1) when the flux f is not necessarily genuinely nonlinear which is actually the main contribution of the paper. As a consequence, in two physically relevant one dimensional situations of the problem, we are able to prove strong $L^1_{\rm loc}$-precompactness of the family $(u^{\varepsilon,\delta})_{\varepsilon,\delta}$ merely assuming that the initial data belong to the BV-class (which actually proves the existence of an entropy admissible weak solution to the one-dimensional variant of (1) without any additional assumptions on the flux). For the latter, see Section 3.

In order to get the result, we shall use estimates derived in [12] and the following theorem.

Theorem 1 ([18], Corollary 2)**.** *Let $\Omega\subset\mathbf{R}^n$ be an open set. Assume that the vector $\phi(x,u)\in(C(\mathbf{R}_u;BV(\Omega)))^n$ is genuinely nonlinear, i.e. for a.e. $x\in\Omega$ and for all $\xi\in\mathbf{R}^n$, $\xi\neq 0$, the map $(a,b)\ni u\mapsto(\xi,\phi(x,u))\neq$ constant on any nontrivial interval.*

Then, each bounded sequence $(u_k(x))_k\in L^\infty(\Omega)$, $a\leq u_k(x)\leq b$, satisfying for the Heaviside function H,

$$\mathrm{div}_x\Big[H(u_k(x)-p)(\phi(x,u_k(x))-\phi(x,p))\Big]\ \textit{is precompact in}\ W^{-1,2}_{\rm loc}(\Omega),$$

contains a subsequence that is convergent in $L^1_{\rm loc}(\Omega)$.

The key point of our procedure is the fact that for the family of solutions $(u^{\varepsilon,\delta})_{\varepsilon,\delta}$ of (8), (9), we have $\|u_t^{\varepsilon,\delta}(t,\cdot,\cdot)\|_{L^1(\mathbf{R}^2)}$ bound for every $t>0$. Therefore, we can replace $u_t^{\varepsilon,\delta}$ by a function $(h(x,y,u^{\varepsilon,\delta}))_t$ (actually, $u_t^{\varepsilon,\delta}$ will end up on the right hand side) without affecting the precompactness framework. This means that we can replace $\xi_0\lambda$ from (7) by $\xi_0 h(x,y,\lambda)$ where h is chosen so that conditions of Theorem 1 are satisfied (more precisely, in (7) the summand $\xi_0\lambda$ is replaced by $\xi_0 h(x,y,\lambda)$).

2. New genuine nonlinearity condition and the main result

At the beginning of the section, we introduce a generalization of nonlinearity condition (7) which we will use in the proof of Theorem 7.

Definition 2. We say that the vector $(f_1,f_2)\in\big(C^1(\mathbf{R}_\lambda;L^\infty(\mathbf{R}_x\times\mathbf{R}_y))\big)^2$ satisfies generalized genuine nonlinearity conditions if there exists a function $h(x,y,\lambda)\in C^1(\mathbf{R}_\lambda;L^\infty(\mathbf{R}_x\times\mathbf{R}_y))$ such that for all $\xi\in S^2$ (sphere in $\mathbf{R}^3$),

$$\begin{aligned}\xi_0\cdot h(x,y,\lambda)+\xi_1\cdot f_1(x,y,\lambda)+\xi_2\cdot f_2(x,y,\lambda)\neq \text{constant function in } \lambda\\ \text{on any nontrivial interval.}\end{aligned}\tag{10}$$

In the sequel, we denote $\Pi=(0,\infty)\times\mathbf{R}^2=\mathbf{R}^+\times\mathbf{R}^2$. Furthermore, we denote by $W^{-1,2}_{\rm c,loc}(\Pi)$ families of functions that are precompact in $W^{-1,2}_{\rm loc}(\Pi)$, and by $\mathcal{M}_{\rm b,loc}(\Pi)$ families of functions that are locally bounded in the space of Radon measures $\mathcal{M}(\Pi)$.

We recall Murat's lemma.

Lemma 3 ([9])**.** Assume that the family (Q_ε) is bounded in $L^p(\Omega)$, $p>2$, $\Omega\subset\mathbf{R}^d$ is an open set. Then,

$$(\mathrm{div}\,Q_\varepsilon)_\varepsilon\in W^{-1,2}_{\rm c,loc}(\Omega)\ \text{ if }\ \mathrm{div}\,Q_\varepsilon=p_\varepsilon+q_\varepsilon,$$

with $(q_\varepsilon)_\varepsilon \in W^{-1,2}_{\mathrm{c,loc}}(\Omega)$ and $(p_\varepsilon)_\varepsilon \in \mathcal{M}_{\mathrm{b,loc}}(\Omega)$.

We will also need the following a priory estimates (Lemmas 4–6), essentially proved in [12].

Lemma 4. [12, Lemma 4.1] (L^∞-bound) There exists constant $c_1 > 0$ such that for all $t > 0$,

$$\|u^{\varepsilon,\delta}(t,\cdot,\cdot)\|_{L^\infty(\mathbf{R}^2)} \leq c_1.$$

Lemma 5. [12, Lemma 4.2] (Lipschitz regularity in time) If $\delta = c\varepsilon$, for a constant $c > 0$, then there exists constant c_2, independent of ε and δ, such that for all $t > 0$,

$$\iint_{\mathbf{R}^2} |\partial_t u^{\varepsilon,\delta}(\cdot,\cdot,t)|\, dxdy \leq c_2.$$

Lemma 6. [12, Lemma 4.3] (Entropy dissipation bound) There exists a constant c_3, independent from ε and δ, such that for all $t > 0$

$$\varepsilon \iint_{\mathbf{R}^2} \left(u^{\varepsilon,\delta}_x(t,\cdot,\cdot)\right)^2 + \left(u^{\varepsilon,\delta}_y(t,\cdot,\cdot)\right)^2 dxdy \leq c_3.$$

Now, we can formulate the main theorem of the paper:

Theorem 7. *Assume that the flux function (f_1, f_2) from* (1) *satisfy the generalized genuine nonlinearity conditions from Definition 2, and conditions* (3)-(6)*. If $\varepsilon = c\delta$, then the family of solutions $(u^\varepsilon)_\varepsilon \equiv (u^{\varepsilon,\delta})_{\varepsilon,\delta}$ to* (8) *is strongly precompact in $L^1_{\mathrm{loc}}(\Pi)$.*

Proof: In order to use (10), we rewrite (8) as

$$\begin{aligned} h(x,y,u^{\varepsilon,\delta})_t + f_1^\delta(x,y,u^{\varepsilon,\delta})_x + f_2^\delta(x,y,u^{\varepsilon,\delta})_y \\ = h(x,y,u^{\varepsilon,\delta})_t - u^{\varepsilon,\delta}_t + \varepsilon(u^{\varepsilon,\delta}_{xx} + u^{\varepsilon,\delta}_{yy}). \end{aligned} \tag{11}$$

Denote $\eta'(\lambda) = H(\lambda - k)$, for some constant k (here H stands for the Heaviside step function) and define the corresponding entropy fluxes:

$$\begin{aligned} q_0(x,y,\lambda) &= H(\lambda-k)(h(x,y,\lambda) - h(x,y,k)), \\ q_i(x,u,\lambda) &= H(\lambda-k)(f_i(x,y,\lambda) - f_i(x,y,k)), \quad i=1,2, \\ q_i^\delta(x,y,\lambda) &= H(\lambda-k)(f_i^\delta(x,y,\lambda) - f_i^\delta(x,y,k)), \quad i=1,2. \end{aligned}$$

We multiply (11) by $\eta'(u^{\varepsilon,\delta})$ and add $\partial_x q_1(x,y,u^{\varepsilon,\delta})$ and $\partial_y q_2(x,y,u^{\varepsilon,\delta})$ on both side of equality (11) to obtain

$$\begin{aligned} &\partial_t q_0(x,y,u^{\varepsilon,\delta}) + \partial_x q_1(x,y,u^{\varepsilon,\delta}) + \partial_y q_2(x,y,u^{\varepsilon,\delta}) \\ &= H(u^{\varepsilon,\delta} - k)\left(\partial_t h(x,y,u^{\varepsilon,\delta}) - D_x f_1^\delta(x,y,k) - D_y f_2^\delta(x,y,k) - u^{\varepsilon,\delta}_t\right) \\ &+ \varepsilon(\partial_x(u^{\varepsilon,\delta}_x \eta'(u^{\varepsilon,\delta})) - (u^{\varepsilon,\delta}_x)^2 \eta''(u^{\varepsilon,\delta}) + \partial_y(u^{\varepsilon,\delta}_y \eta'(u^{\varepsilon,\delta})) - (u^{\varepsilon,\delta}_y)^2 \eta''(u^{\varepsilon,\delta})) \\ &+ \partial_x(q_1 - q_1^\delta)(x,y,u^{\varepsilon,\delta}) + \partial_y(q_2 - q_2^\delta)(x,y,u^{\varepsilon,\delta}) \\ &\leq H(u^{\varepsilon,\delta} - k)\left(\partial_t h(x,y,u^{\varepsilon,\delta}) - D_x f_1^\delta(x,y,k) - D_y f_2^\delta(x,y,k) - u^{\varepsilon,\delta}_t\right) \\ &+ \varepsilon(\partial_x(u^{\varepsilon,\delta}_x \eta'(u^{\varepsilon,\delta})) + \partial_y(u^{\varepsilon,\delta}_y \eta'(u^{\varepsilon,\delta}))) \\ &+ \partial_x(q_1 - q_1^\delta)(x,y,u^{\varepsilon,\delta}) + \partial_y(q_2 - q_2^\delta)(x,y,u^{\varepsilon,\delta}) \qquad \text{in } \mathcal{D}'(\Pi). \end{aligned} \tag{12}$$

In order to use Theorem 1 we have to show that

$$\mathrm{div}_{(t,x,y)}\left[(q_0,q_1,q_2)(x,y,u^\varepsilon)\right] \in W^{-1,2}_{\mathrm{c,loc}}(\Pi). \tag{13}$$

From (12) and the Schwartz lemma on nonnegative distributions, it follows that there exists Radon measure $\mu_k^{\varepsilon,\delta} \in \mathcal{M}(\Pi)$ such that

$$\begin{aligned}
&\partial_t q_0(x,y,u^\varepsilon) + \partial_x q_1(x,y,u^\varepsilon) + \partial_y q_2(x,y,u^\varepsilon) \\
&\quad = H(u^\varepsilon - k)\left(\partial_t h(x,y,u^{\varepsilon,\delta}) - D_x f_1^\delta(x,y,k) - D_y f_2^\delta(x,y,k) - u_t^\varepsilon\right) \\
&\quad + \partial_x(q_1 - q_1^\delta)(x,y,u^\varepsilon) + \partial_y(q_2 - q_2^\delta)(x,y,u^\varepsilon) \\
&\quad + \varepsilon(\partial_x(u_x^\varepsilon \eta'(u^\varepsilon)) + \partial_y(u_y^\varepsilon \eta'(u^\varepsilon))) + \mu_k^{\varepsilon,\delta}(t,x,y).
\end{aligned} \tag{14}$$

Now, we use Lemma 3 to see that (13) holds. Indeed, from Lemma 5 we obtain that

$$H(u^\varepsilon - k)\left(\partial_\lambda h(x,y,u^\varepsilon)\partial_t u^\varepsilon - \partial_t u^\varepsilon\right) \in \mathcal{M}_{\mathrm{b,loc}}(\Pi). \tag{15}$$

Lemma 6 implies

$$\partial_x(\varepsilon \partial_x u^\varepsilon H(u^\varepsilon - k)) + \partial_y(\varepsilon \partial_y u^\varepsilon H(u^\varepsilon - k)) \in W^{-1,2}_{\mathrm{c,loc}}(\Pi), \tag{16}$$

provided that

$$\varepsilon \partial_x u^\varepsilon H(u^\varepsilon - k) \to 0, \text{ in } L^2_{\mathrm{loc}}(\Pi),$$

and

$$\int_\Pi |\varepsilon \partial_x u^\varepsilon H(u^\varepsilon - k)|^2\, dxdydt \le \varepsilon^2 \int_\Pi |\partial_x u^\varepsilon|^2 dxdydt \le Tc\varepsilon \to 0, \quad \varepsilon \to 0.$$

Furthermore,

$$\left(D_x f_1^\delta(x,y,k) + D_y f_2^\delta(x,y,k)\right) H(u^\varepsilon - k)) \in \mathcal{M}_{\mathrm{b,loc}}(\Pi), \tag{17}$$

since $f_i^\delta \in BV(\Pi)$. Finally,

$$\partial_x(q_1 - q_1^\delta), \partial_y(q_2 - q_2^\delta) \in W^{-1}_{\mathrm{c,loc}}(\Pi), \tag{18}$$

since, according to (5),

$$\begin{aligned}
|q_i - q_i^\delta| &\le |f_i^\delta(x,y,u^\varepsilon) - f_i(x,y,u^\varepsilon)| + |f_i^\delta(x,y,k) - f_i(x,y,k)| \\
&\le 2 \max_{a \le p \le b} |f_i^\delta(x,y,p) - f_i(x,y,p)| \to 0, \text{ in } L^2_{\mathrm{loc}}(\mathbf{R}^2).
\end{aligned}$$

To prove that $\mu_k^{\varepsilon,\delta}$ belongs to $\mathcal{M}_{\mathrm{b,loc}}(\Pi)$, it is enough to prove that for any compact $K \subset \Pi$, it holds $\mu_k^{\varepsilon,\delta}(K) < c_4$, for a constant c_4 independent on ε and δ. To prove the latter, fix a function $\varphi_K \in C_0^1(\Pi)$ such that $\varphi_K \ge 0$, $|\nabla \varphi_K| \le c_5$ for a constant $c_5 > 0$ independent on ε, δ, and $\varphi_K(t,x,y) = 1$ for $(t,x,y) \in K$. Then, since $\mu_k^{\varepsilon,\delta}$ are positive Radon measures for every $\delta, \varepsilon > 0$, it follows from (14) that

$$\mu_k^{\varepsilon,\delta}(K) \le \int_\Pi \varphi_K(t,x,y) d\mu_k^{\varepsilon,\delta} \le c_4 |\nabla \varphi_K| \le c_6, \tag{19}$$

where c_6 depends on the set K, the constants c_4, c_5, and the constants c_i, $i = 1,2,3$, from Lemmas 4-6, but not on $\varepsilon, \delta > 0$.

Collecting (14-19), from Murat's lemma we obtain (13). Applying Theorem 1 we conclude the proof. ❏

3. Examples

In this section we give three examples which can not be dealt by using the known results.

a) First, we shall apply Theorem 7 on the following problem

$$\begin{aligned} u_t + (f(x,u))_x =&0, \\ u|_{t=0} = u_0(x) \in& BV(\mathbf{R}), \quad a \leq u_0(x) \leq b, \end{aligned}$$

where one-dimensional variant of (3)-(6) are satisfied. We assume that for almost every $x \in \mathbf{R}$ the mapping

$$[a,b] \ni \lambda \mapsto f(x,\lambda), \tag{20}$$

is different from a constant on any nontrivial interval.

Corollary 8. *A family of solutions $(u^\varepsilon)_\varepsilon$ of the problem*

$$\begin{aligned} u^\varepsilon_t + (f^\varepsilon(x,u^\varepsilon))_x =&\varepsilon u^\varepsilon_{xx}, \\ u^\varepsilon|_{t=0} =& u^\varepsilon_0(x), \end{aligned}$$

where the notation is taken from (8)-(9)*, is strongly precompact in $L^1_{\rm loc}(\mathbf{R}^+ \times \mathbf{R})$.*

Proof: According to the previous theorem, it is enough to find a function $h(x,\lambda)$ such that the mapping

$$\lambda \mapsto h(x,\lambda)\xi_0 + f(x,\lambda)\xi_1 \tag{21}$$

is different from a constant on any nontrivial interval. Taking

$$h(x,\lambda) = f^2(x,\lambda)$$

we conclude that (21) will not be satisfied only if there exists a nonzero set $\Omega \subset \mathbf{R}$ such that for $x \in \Omega$ there exists $(\xi_0,\xi_1) \in \mathbf{R}^2 \backslash \{0\}$ satisfying

$$f(x,\lambda) = \frac{-\xi_1 \pm \sqrt{\xi_1^2 + 4\xi_0 c}}{2\xi_0},$$

for a constant c, contradicting (20). ❏

b) Now, we consider the following example of a one dimensional conservation law with discontinuous flux,

$$\begin{cases} \partial_t u + (H(x)f(u) + (1-H(x))g(u))_x = 0, & (t,x) \in \mathbf{R}^+ \times \mathbf{R}, \\ u|_{t=0} = u_0(x) \in BV(\mathbf{R}), \quad a \leq u_0(x) \leq b, & x \in \mathbf{R}, \end{cases} \tag{22}$$

where $f, g \in C^1(\mathbf{R})$, $f(a) = f(b) = g(a) = g(b) = 0$. The problem has been thoroughly investigated in recent past (the following list is very incomplete [1, 2, 3, 4, 5, 7, 11, 13, 14, 15]). In the following Corollary 9, we shall prove that $(u^\varepsilon)_\varepsilon$ is strongly precompact in $L^1_{\rm loc}(\mathbf{R}^+ \times \mathbf{R})$ without any structural assumptions on the flux (such as genuine nonlinearity, crossing condition, convexity, a single crossing point of f and g, etc.). This is very important fact since appropriate limit function along a subsequence is not only a weak solution to problem (22), but at the same time satisfies entropy admissibility conditions (see e.g. [14, Definition 1.2]). Remark that the existence of an entropy admissible weak solution to (22) is established in [5] also with no structural conditions on the flux.

Corollary 9. *Let $\omega \in C_0^1(\mathbf{R})$ be a positive, compactly supported function with total mass one. Let $H_\varepsilon(z) = \int_{-\infty}^{z/\varepsilon} \omega(z')dz'$ represents a regularization of the Heaviside function. A family of solutions $(u^\varepsilon)_\varepsilon$ of the problem*

$$\begin{aligned} u_t^\varepsilon + \partial_x \left(H_\varepsilon(x)f(u) + (1 - H_\varepsilon(x))g(u)\right)_x &= \varepsilon u_{xx}^\varepsilon, \\ u^\varepsilon|_{t=0} &= u_0^\varepsilon(x), \end{aligned}$$

is strongly precompact in $L^1_{\rm loc}(\mathbf{R}^+ \times \mathbf{R})$.

Proof: If

$$\lambda \mapsto f(\lambda), \quad \lambda \mapsto g(\lambda), \tag{23}$$

are non-constant on any interval $(\alpha, \beta) \subset (a, b)$ then we can apply Corollary 8.

Otherwise, we take disjoint intervals $(\alpha_i^f, \beta_i^f) \subset (a, b)$, $i = 1, \dots, d_1$, $d_1 \in \mathbf{N} \cup \{\infty\}$, where the function f is constant, and the disjoint intervals $(\alpha_i^g, \beta_i^g) \subset (a, b)$, $i = 1, \dots, d_2$, $d_2 \in \mathbf{N} \cup \{\infty\}$, where the function g is constant. Then, we take the functions $\hat{f}, \hat{g} \in {\rm Lip}(\mathbf{R})$:

$$\hat{f}(\lambda) = \begin{cases} 0, & \lambda \notin (\alpha_i^f, \beta_i^f), \\ (\lambda - \alpha_i^f)(\lambda - \beta_i^f), & \lambda \in (\alpha_i^f, \beta_i^f), \end{cases} \quad i = 1, \dots, d_1,$$

$$\hat{g}(\lambda) = \begin{cases} 0, & \lambda \notin (\alpha_i^g, \beta_i^g), \\ (\lambda - \alpha_i^g)(\lambda - \beta_i^g), & \lambda \in (\alpha_i^g, \beta_i^g), \end{cases} \quad i = 1, \dots, d_2.$$

Now, the vector $(H(x)(f^2+\hat{f})(u)+(1-H(x))(g^2+\hat{g})(u), H(x)f(u)+(1-H(x))g(u))$ is genuinely nonlinear implying that $(u, H(x)f(u) + (1 - H(x))g(u))$ satisfies the generalized genuine nonlinearity condition. Thus, we can apply Theorem 7 to complete the proof of the corollary. ❏

Remark 10. Procedure similar to the one from the proof of Corollary 9 is an important part of recent preprint [17]. In [17], one can find an attempt [1] to settle the existence and uniqueness of an admissible weak solution to (22) with no structural assumptions on the flux.

c) Consider the following Cauchy problem

$$\begin{aligned} &u_t + (k(x)g(u))_x + (l(y)f(u))_y = 0 \\ &u|_{t=0} = u_0(x, y) \in BV(\mathbf{R}^2) \end{aligned} \tag{24}$$

with $-1 \leq u_0(x, y) \leq 1$, and

$$g(u) = \begin{cases} 0, & \text{for } |u| \geq 1 \\ u + 1, & \text{for } -1 < u \leq 0 \\ 1 - u^2, & \text{for } 0 < u < 1 \end{cases}, \quad k(x) = \begin{cases} 3, & \text{for } x \geq 0 \\ 1, & \text{for } x < 0, \end{cases}$$

and

$$f(u) = \begin{cases} 0, & \text{for } |u| \geq 1 \\ 1 - u^2, & \text{for } -1 < u \leq 0 \\ 1 - u, & \text{for } 0 < u < 1 \end{cases}, \quad l(y) = \begin{cases} 4, & \text{for } y \geq 0 \\ 2, & \text{for } y < 0, \end{cases}$$

The flux vector $(k(x)g(u), l(x)f(u))$ does not satisfy classical genuine nonlinearity condition and we can not apply results from [12]. Indeed, for any $x \in (-1, 0)$,

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[1]The preprint is not published yet; therefore, we leave a possibility that it contains a mistake.

$y \in \mathbf{R}$ (and the set $(0,1)\times\mathbf{R}$ has infinite measure), choose $(\xi_0, -\xi_0, 0) \in S^2$. Mapping (7) becomes:

$$\lambda \mapsto \xi_0\lambda - \xi_0(\lambda+1) = -\xi_0,$$

and this is constant in λ.

Therefore, to state that a weak solution to (24) exists, we have to use generalized genuine nonlinearity condition (10). According to Theorem 7, the family $(u^\varepsilon)_\varepsilon$ of solutions to the equation

$$u^\varepsilon_t + (k_\varepsilon(x)g(u^\varepsilon))_x + (l_\varepsilon(y)f(u_\varepsilon))_y = \varepsilon(u^\varepsilon_{xx} + u^\varepsilon_{yy})$$

where

$$k_\varepsilon(x) = \begin{cases} 3, & \text{for } \ x \geq \varepsilon \\ \frac{x}{\varepsilon} + 2, & \text{for } \ -\varepsilon < x < \varepsilon \\ 1, & \text{for } \ x \leq -\varepsilon, \end{cases}$$

and

$$l_\varepsilon(y) = \begin{cases} 4, & \text{for } \ x \geq \varepsilon \\ \frac{x}{\varepsilon} + 3, & \text{for } \ -\varepsilon < x < \varepsilon \\ 2, & \text{for } \ x \leq -\varepsilon, \end{cases}$$

is strongly precompact in $L^1_{\rm loc}(\mathbf{R}^+ \times \mathbf{R}^2)$. Indeed, take h from (10) to be $h(x,u) = u^3$. In that case, the vector field $(h(x,u), k(x)g(u), l(y)f(u))$ satisfies conditions from Theorem 7, since $k, l \in BV$. Therefore, Theorem 7 provides strong $L^1_{\rm loc}$-precompactness of the family $(u^\varepsilon)_\varepsilon$. Clearly, a strong $L^1_{\rm loc}$-limit along a subsequence of $(u^\varepsilon)_\varepsilon$ will represent a weak solution to (24).

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