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Convergence of the Space-Time Expansion Discontinuous Galerkin Method for Scalar Conservation Laws

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ABSTRACT

In this paper we analyse a class of fully discrete Space-Time Expansion Discontinuous-Galerkin methods for scalar conservation laws. This method has been introduced in [11, 17, 18] for a specific expansion relying on the Cauchy-Kovaleskaya technique. We introduce a general concept of admissible expansions which in particular allows us to prove an error estimate for smooth solutions. The result applies for ansatz functions of arbitrary polynomial order $k \in \mathbb{N}$ provided the time step is sufficiently small. It gives a convergence rate of order $k + \frac{1}{2}$ in space and time. Finally we show that the Cauchy-Kovaleskaya technique leads to an admissible exansion. Furthermore we introduce two new expansions and prove that one of them, the characteristic expansion, is also admissible.

Keywords: Space-Time Expansion; Discontinuous-Galerkin method; conservation laws; error estimate

1. INTRODUCTION

The class of Discontinuous Galerkin (DG) schemes has become one of the most important discretization techniques for evolution equations, in particular in the field of nonlinear hyperbolic conservation laws. Although it has been suggested as early as 1974 [16] the breakthrough came with a series of papers by Cockburn&Shu and co-workers in the nineties [6–8]. The major advantage of the DG method is the cellwise use of polynomial ansatz functions of order $k \in \mathbb{N}_0$ without introducing a wider and wider stencil for increasing k as it is the case for e.g. Finite-Difference schemes. The tool of numerical flux functions, widely developped since the eighties, can then be applied. By now the literature on DG schemes is too voluminous to be cited completely. Let us refer to textbooks/reviews like [4, 12] and cites therein.

We consider in this paper a scalar conservation law in one space dimension. Precisely with $\Omega = (0, 1)$ (for simplicity) and $T > 0$ we search for a function $u : \Omega \times (0, T) \to \mathbb{R}$ such that

$$u_t + F(u)_x = 0 \qquad \text{in } \Omega \times (0, T), \tag{1a}$$

$$u(\cdot, 0) = u_0 \qquad \text{in } \Omega, \tag{1b}$$

$$u(0, t) = u(1, t) \qquad (t \in [0, T)) \tag{1c}$$

holds. Here $F : \mathbb{R} \to \mathbb{R}$ is the, in general nonlinear, flux and $u_0 : \Omega \to \mathbb{R}$ the initial datum.

The DG method for (1a) combines features of the Finite-Volume as well as the Finite-Element schemes. Therefore techniques from both worlds can be used. In [9] it has been shown that the DG-method as a method of lines together with the TVB-Runge-Kutta time discretization from [19] is total-variation bounded. This implies – roughly speaking– the convergence of the approximate solutions in L^1. Also convergence rates have been verified [5]. Most notably the analysis covers the case of (discontinuous) entropy solutions for (1a). However, increasing the formal order of the scheme is not reflected by this analysis in the spirit of techniques for Finite-Volume methods. We also outline the work [13] which ensures that a cell entropy inequality holds for semi-discrete DG-methods independent of the polynomial order.

Results which reflect the polynomial order in a L^2-error estimate have been obtained by Finite-Element-like

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techniques [22] provided the solution u of (1a)+(1b)+(1c) is assumed to be smooth enough. This assumption is also the basis of the analysis here (see the precise assumptions in Theorem 4.1 below) but let us note that classical solutions of (1a)+(1b)+(1c) can only be expected to exist for small times (except in trivial cases). In this situation L^2-error estimates for semi-discrete DG-methods of the type

$$\|u(\cdot,t) - u_h(\cdot,t)\|_2 \leq \mathcal{O}(h^{k+\frac{1}{2}}) \qquad (t \in (0,T)) \tag{2}$$

have been obtained in [10]. Here h is the mesh parameter, u_h the DG approximation using ansatz functions of order $k \in \mathbb{N}_0$, and $\|\cdot\|_2$ the L^2-norm on Ω.
Turning to fully discrete schemes less is known than in the semi-discrete case. Moreover the DG concept allows for various different methods to discretize in time which complicates the situation. If the flux F in (1a) is linear a fully discrete ansatz with space-time ansatz functions was proven to be convergent in [23] with orders as in (2). Zhang&Shu analyzed the DG-method together with the second order TVB-Runge-Kutta time discretization. They have been successfull to prove for monotone numerical fluxes the estimate

$$\|u(\cdot,n\tau) - u_h(\cdot,n\tau)\|_2 = \mathcal{O}(h^{k+1/2}) + \mathcal{O}(\tau^2) \qquad (k \in \mathbb{N},\, n = 0,\ldots,N), \tag{3}$$

where $\tau > 0$ is the time step and $N \in \mathbb{N}$ such that $N\tau = T$. As stability bound for the time step they assumed for some constant $\gamma > 0$ (and $k \geq 2$)

$$\tau \leq \gamma h^{4/3}.$$

A completely different fully-discrete approach has been developped by Lörcher et al. in [11, 17, 18]: the so-called Space-Time Expansion Discontinuous-Galerkin method (STE-DG). A nice feature of this approach is its high efficiency, which can be achieved through local time stepping (The time step of one cell is only restricted by the values of nearest neighbours). If one uses hp-adaptation techniques together with the STE-DG approach, one profits even more by local time stepping. The basic idea of the STE-DG aproach is to expand in each spatial cell I_j the approximate solution $u_h(\cdot,n\tau)$ to the complete space-time cell $I_j \times [n\tau,(n+1)\tau)$. The expansion is then used to evaluate the flux integrals in time. There is no unique way to define the expansions but in order to maintain the given polynomial order of the ansatz functions in time the expansion has to be constructed carefully. We introduce here the class of **admissible expansions of order** k. The original Cauchy-Kovaleskaya expansion from [11, 17, 18] belongs to this class (Section 5.1). Moreover we suggest a new characteristic-based expansion that is also admissible (Section 5.2). In the main part of the paper we exploit then the concept of admissible expansions to show in Theorem 4.1 the estimate

$$\|u(\cdot,n\tau) - u_h(\cdot,n\tau)\|_2 = \mathcal{O}(h^{k+1/2}) \qquad (k \in \mathbb{N},\, n = 0,\ldots,N). \tag{4}$$

Let us note that this result is not restricted to a certain order in time like (3) but covers all temporal orders. To obtain this generality we have to sharpen the time step restriction in the sense that

$$\tau \leq \gamma h^2$$

has to hold. The technique of proofs relies on the approach as in [23], a careful treatment of the flux integrals evaluated for the expansions, and of course the admissibility concept. Up to our knowledge the estimate (4) is the first convergence result that has been established for the STE-DG method. It is important to note that the estimates in (4) (as well as in (3)) can be improved to the optimal order $\mathcal{O}(h^{k+1})$ if instead of a monotone flux an upwind formulation is chosen (see also Remark 4.2 below).
To conclude we give an outline of the rest of the paper. In Section 2 we recall basic notations. Section 3 is devoted to the numerical method. The notion of admissible expansions is introduced. Three examples of expansions, the Cauchy-Kovaleskaya expansion, the new characteristic expansion and a new Riemann-solver-like expansion are discussed. In Section 4 the main theorem 4.1 is given and proven. Finally in Section 5 the admissibilty of the Cauchy-Kovaleskaya expansion and of the new characteristic expansion are verified.

2. NOTATIONS AND FUNCTION SPACES

In the following section we will introduce basic notations we need to describe the method in Section 3. Let

$$\Gamma_h = \left\{0 = x_{\frac{1}{2}} < x_{\frac{3}{2}} < \cdots < x_{J+\frac{1}{2}} = 1\right\} \tag{5}$$

be a (not necessarily equidistant) grid and $J \in \mathbb{N}$ the number of mesh cells. Define for each $j \in \{1,\dots,J\}$ the mesh cell

$$I_j := (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}),$$

with centre

$$x_j := \frac{1}{2}(x_{j-\frac{1}{2}} + x_{j+\frac{1}{2}})$$

and the local mesh parameter

$$\Delta x_j := x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}.$$

Next define the grid parameter h as

$$h := \max_{1\le j\le J} \Delta x_j.$$

In the same way we introduce a time grid and for simplicity we choose a constant time-step $\tau > 0$ and define for $n \in \{0,\dots,N\}$ the time levels t^n by $t^n = n\tau$. Choose $N \in \mathbb{N}$ and τ such that $t^N = N\tau = T$.
Define for all functions $v : \Omega \times [0,T] \to \mathbb{R}$

$$v^n(x) := v(x, t^n).$$

In the following we will need a number of function spaces:

Definition 2.1 (Function Spaces).
Let $k \in \mathbb{N}_0$ and Γ_h be a grid as in (5).

1. *Define the **space C^k_Γ of piecewise k-times differentiable functions** on Γ by*

$$C^k_\Gamma := \left\{ f : [0,1] \to \mathbb{R} \,\middle|\, f|_{I_j} \in C^k(I_j), j \in \{1,\dots,J\} \right\}.$$

2. *Let $V^k_\Gamma \subset C^k_\Gamma$ be the **space of piecewise polynomial functions** of order k on Γ, defined by*

$$V^k_\Gamma := \left\{ v_h : [0,1] \to \mathbb{R} \,\middle|\, v_h|_{I_j} \in \mathbf{P}^k(I_j), j \in \{1,\dots,J\} \right\},$$

where $\mathbf{P}^k(I_j)$ is the space of polynomials up to degree k on I_j.

3. *A norm on C^k_Γ is given by*

$$\|v\|_{h,k} := \sum_{0\le l\le k-1} \left\| {\partial_x}^l v \right\|_\infty + h^{\frac{1}{2}} \left\| {\partial_x}^k v \right\|_\infty, \qquad (v \in C^k_\Gamma).$$

*For $M > 0$, define the **set of bounded piecewise polynomial functions** on Γ by*

$$V^k_{\Gamma,M} := \left\{ v \in V^k_\Gamma \,\middle|\, \|v\|_{h,k} \le M \right\}.$$

By $\|\cdot\|_\infty$ we denote the maximum norm on Ω.

All elements of these function spaces allow discontinuities at cell boundaries. So there is a need for the following notation: Define for all $v \in C^k_\Gamma$ and $x \in [0,1]$

$$v^+(x) := \lim_{y\downarrow x} v(y) \quad \text{and} \quad v^-(x) := \lim_{y\uparrow x} v(y), \tag{6}$$
$$\overline{v}(x) := \frac{1}{2}\left(v^+(x) + v^-(x)\right) \quad \text{and} \quad [v](x) := v^+(x) - v^-(x).$$

Furthermore we define for any $j \in \{1,\dots,J\}$ with obvious notation

$$(v)_{j+\frac{1}{2}} := v(x_{j+\frac{1}{2}}) \qquad \text{and} \qquad [v]_{j+\frac{1}{2}} := [v](x_{j+\frac{1}{2}}).$$

We also need different norms and semi-norms on these function spaces. We use $\|\cdot\|_2$ to name the L^2-norm. Moreover we define for $v \in C_\Gamma^k$ the semi-norm

$$|v|_{\Gamma_h} := \left(\sum_{1 \le j \le J} \left(v^+(x_{j+\frac{1}{2}}) \right)^2 + \left(v^-(x_{j+\frac{1}{2}}) \right)^2 \right)^{\frac{1}{2}}. \tag{7}$$

By $C > 0$ we denote a generic constant, that may change from line to line, but does not depend on the mesh parameter h.

3. THE NUMERICAL METHOD

We introduce the STE-DG method in this section. Our presentation covers the original STE-DG method as it has been suggested in [11, 17, 18], but is more general: We present a number of new expansions to which our analysis applies. But first we recall the definition of a numerical flux function.

3.1 Numerical Flux Functions

The use of numerical fluxes is standard for the d.... of hyperbolic conservation laws. Following e.g. the notion in [15] we define:

Definition 3.1 (Numerical Flux Function).
A function $H\colon \mathbb{R}^2 \to \mathbb{R}$ is called ***numerical flux function*** *(for F), if the following properties hold:*

- *H is consistent to F, i.e. $H(w,w) = F(w)$ for all $w \in \mathbb{R}$.*
- *H is locally Lipschitz continuous, i.e.*

$$\forall M > 0 \,\exists\, C_M > 0 \,\forall v_0, v_1, w_0, w_1 \in B_M(0) : |H(v_0,w_0) - H(v_1,w_1)| \le C_M \left(|v_0 - v_1| + |w_0 - w_1| \right).$$

We call H a ***monotone numerical flux function*** *(for F), if further H is non-decreasing in the first argument and non-increasing in the second argument.*

As we will see in the proof of convergence, we have to deal with a term depending on the numerical flux we have chosen. To capture this term we introduce for a given numerical flux H and $p = (p^-, p^+) \in \mathbb{R}^2$ the quantities

$$\alpha(H;p) := \begin{cases} [p]^{-1} \left(F(\overline{p}) - H(p^-,p^+) \right) & \text{falls } [p] \ne 0, \\ |F'(\overline{p})| & \text{falls } [p] = 0. \end{cases} \tag{8}$$

We define - using the same notation - for $p = (p^-, p^+) : [0,T] \to \mathbb{R}^2$

$$\alpha_n(H;p) := \frac{1}{t^{n+1} - t^n} \int_{t^n}^{t^{n+1}} \alpha\left(H; \left(p^-(t), p^+(t) \right) \right) dt, \qquad (0 \le n \le N).$$

To simplify the notation we also introduce

$$\bar{\alpha}_n(H;p)[q]^2 := \sum_{1 \le j \le J} \left(\alpha_n(H;p)_{j+\frac{1}{2}} \right) [q]^2_{j+\frac{1}{2}}, \qquad (0 \le n \le N). \tag{9}$$

3.2 Expansions: Definition and Examples

The key element in the STE-DG method is the expansion. One can see it as a local approximation to the solution of (1a) on $[t^n, t^{n+1}) \times I_j$ only using the information $u_h(\cdot, t^n)|_{I_j}$, where $u_h \in V_h^k$ is the approximate solution (see Definition 3.7 below). In particular this implies that the exact solution generally does not provide a suitable example, as the exact weak solution depends on the values of $u_h(\cdot, t^n)$ in at least one of the neighbouring cells $I_{j\pm1}$ (, if the flux is non trivial). We use the following general framework.

DEFINITION 3.2 (EXPANSION).
*We define the **space of time-dependend piecewise polynomial functions** by*

$$\mathcal{V}_\Gamma^k := \left\{ v_h \in L^\infty([0,1] \times [0,\tau], \mathbb{R}) \;\middle|\; v_h(\cdot, t) \in V_h^k, 0 \le t \le \tau \right\}.$$

*A function $E : V_\Gamma^k \to \mathcal{V}_\Gamma^k$ is called **expansion**, if the equation*

$$E[v_h](\cdot, 0) = v_h$$

holds for all $v_h \in V_\Gamma^k$.
For any $s \in [0, T]$ define further E_s as

$$E_s[v_h](x,t) := E[v_h](x, t-s)$$

for any $x \in \Omega$ and $t \in [s, s+\tau]$.

REMARK 3.3. *Note that we do not require the expansion to be a linear operator. In general the expansion is nonlinear. For a specific example we refer to Section 3.2.1.*

Before we introduce the method, let us present some relevant choices for expansions. We present three different expansions. The first one was introduced by Lörcher et al. in [17]; the second one is new. These two expansions have clear algorithmical importance. The last expansion - an extension of the Godunov ideas - seems to be more theoretical, but illustrates nicely the fundamentals of the STE-DG approach.

3.2.1 Example 1: Cauchy-Kovalevskaya Expansion

Lörcher et al. introduced the STE-DG method in [17] together with the so-called Cauchy-Kovalevskaya (CK) expansion. To get this expansion one has to perform a Taylor expansion of the exact solution in space and time for each grid cell I_j. Then the time derivatives are replaced by space derivatives using the differential equation (1a). We need the following auxiliary notation.

DEFINITION 3.4 (DERIVATIONAL OPERATORS).
Let $F \in C^k(\mathbb{R})$ in (1a).
*For any function $v \in C_\Gamma^k$ define the **derivational operators** in I_j, $1 \le j \le J$, associated with (1a) by*

$$\begin{aligned}
\tilde{\partial}_x v &:= v_x, \\
\tilde{\partial}_t v &:= -F(v)_x = -F'(v)v_x, \\
\tilde{\partial}_{xt} v &:= -(\tilde{\partial}_x F(v))_x = -(F'(v)v_x)_x = -F''(v)v_x^2 - F'(v)v_{xx}, \\
\tilde{\partial}_{tt} v &:= -(\tilde{\partial}_t F(v))_x = -(F'(v)\tilde{\partial}_t v)_x = 2F'(v)F''(v)v_x^2 + (F'(v))^2 v_{xx}, \\
&\dots
\end{aligned}$$

Note that the derivatives $v_x, v_{xx}, \dots$ are well defined (only) for inner points of each cell. In the following we will not use this notation at cell boundaries of Γ_h.

Definition 3.5 (Cauchy-Kovalevskaya Expansion).
*The **Cauchy-Kovalevskaya expansion** $E^{CK}: V_\Gamma^k \to \mathcal{V}_\Gamma^k$ on I_j, $1 \le j \le J$, is defined by*

$$E^{CK}[v_h](y,s) := \sum_{l=0}^{k} \frac{1}{l!} \left((y-x_j)\tilde{\partial}_x + s\tilde{\partial}_t \right)^l v_h(x) \Big|_{x=x_j}$$

for $y \in I_j$ and $s \in [0,\tau]$.

Let us discuss the cases $k=0$ and $k=1$ for the sake of illustration.
In the case $k=0$ we obtain

$$E^{CK}[v_h](y,s) = v_h(y).$$

As we will see below the STE-DG method together with the CK-expansion reduces then to the standard finite volume scheme.
Take $k=1$ for instance, then the CK-expansion on I_j will simplify to

$$\begin{aligned} E^{CK}[v](y,s) &= \left(v(x) + (y-x_j)\tilde{\partial}_x v(x) + s\tilde{\partial}_t v(x) \right)\Big|_{x=x_j} \\ &= \left(v(x) + (y-x_j)v_x(x) - sF'(v(x))v_x(x) \right)\big|_{x=x_j} \\ &= v(x_j) + (y-x_j)v_x(x_j) - sF'(v(x_j))v_x(x_j) \\ &= v(y) - sF'(v(x_j))v_x(x_j). \end{aligned} \tag{10}$$

For the last equation we used $v \in \mathbf{P}^1$. As mentioned in Remark 3.3 this expansion is nonlinear in v, if F is nonlinear.

3.2.2 Example 2: Characteristic Expansion

Another way to get an expansion is to use characteristics. Up to our knowledge this choice has not been discussed in the literature before.
To illustrate this concept we define $k+1$ points in each cell I_j, $1 \le j \le J$, by

$$y_{j,l} := x_{j-\frac{1}{2}} + \Delta x_j \frac{l}{k}, \qquad (l = 0, \dots, k).$$

For some given $v \in V_\Gamma^k$ and all $t \in [0,\tau]$ we define $p_j^t \in \mathbf{P}^k(I_j)$ as the polynomial that satisfies

$$p_j^t \left(y_{j,l} + tF'\left(v(y_{j,l}) \right) \right) = v(y_{j,l}), \qquad (l = 0, \dots, k).$$

Note that p_j^t is well defined, if the set

$$\{ y_{j,l} + tF'\left(v(y_{j,l}) \right) \}_{l=0,\dots,k}$$

contains exactly $k+1$ elements. This is guaranteed, if

$$t \le \tau \le \frac{h}{(2k+1)\left\| F'(v) \right\|_\infty}, \tag{11}$$

holds. Thus we need here already a CFL-like condition to ensure, that the expansion is well-defined. With this notation we define:

Definition 3.6 (Characteristic Expansion).
*The **characteristic expansion** $E^{Ch}: V_\Gamma^k \to \mathcal{V}_\Gamma^k$ is defined by*

$$E^{Ch}[v](x,t) := p_j^t(x) \qquad \text{for } x \in I_j \text{ and } t \in [0,\tau].$$

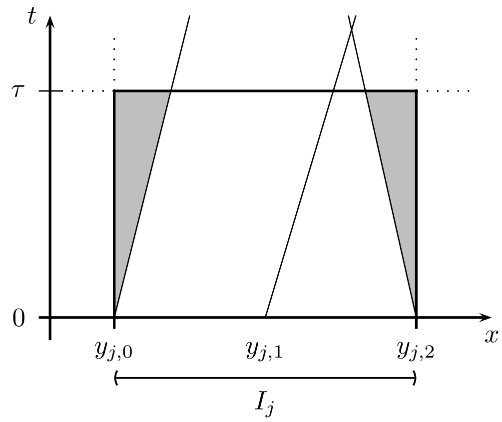

Figure 1. Region of good approximation (white) and bad approximation (shadowed).

For arbitrary $k \in \mathbb{N}$ we get on I_j, $1 \le j \le J$, the formulation

$$E^{Ch}[v_h](x,t) = \sum_{0 \le l \le k} v_l \prod_{\substack{0 \le m \le k \\ m \ne l}} \frac{x - y_{j,m} - tF'(v_m)}{y_{j,l} - y_{j,m} + t(F'(v_l) - F'(v_m))}, \quad (x \in I_j, 0 \le t \le \tau). \tag{12}$$

Here $v_l = v_h(y_{j,l})$. For $k = 0$ we obtain again the standard finite volume scheme.
We already got the CFL condition (11) by just examining when the characteristic expansion is well-defined. We observe, that the characteristic expansion only uses information inside the cell, such that we get a good approximation for the solution in the mid of the cell. This is visualised in Figure 1. Depending on the characteristic velocity the approximation gets worse closer to the cell boundary. A similar behaviour can be observed for the CK-expansion.

3.2.3 Example 3: Expansion with Exact Solution

It is also possible to use not only the information of one cell, but also the neighboring cells. In a Godunov-like ansatz we may solve generalised Riemann problems for discontinuous initial data to get the unique entropy solution, which we can use as an expansion. So suppose that we have a function $v_h \in V_h^k$. Let $v \in L^\infty\left(\Omega \times (0,\tau)\right)$ be the unique entropy solution of

$$\begin{aligned} v_t + F(v)_x &= 0 && \text{in } \Omega \times (0,\tau), \\ v(\cdot,0) &= v_h && \text{in } \Omega, \end{aligned}$$

assuming periodic boundary conditions. Then we are able to define the exact expansion by

$$E^{Ex}[v_h] := v.$$

Formally this is no expansion is the sense of Definition 3.2, because $E^{Ex}[v_h] \notin \mathcal{V}_\Gamma^k$. But one may weaken this assumption or just project v to $\mathcal{V}_\Gamma^k$. For more information on the use of generalised Riemann problems for numerical purposes see for example [1, 2, 20].

3.3 Definition of the STE-DG Method

With the definition of an expansion we are able to define the Space-Time Expansion DG method (STE-DG).

Definition 3.7 (Space-Time Expansion DG Method).
Let $E : V_\Gamma^k \to \mathcal{V}_\Gamma^k$ be an expansion, $H : \mathbb{R}^2 \to \mathbb{R}$ a numerical flux function for F and Γ_h a given grid.
A function $u_h : \Omega \times [0,T] \mapsto \mathbb{R}$ is called ***solution of the STE-DG method of order $k \in \mathbb{N}_0$****, if the following properties hold:*

1. *For each* $n \in \{0, \ldots, N\}$ *it holds, that*
$$u_h^n := u_h(\cdot, t^n) \in V_\Gamma^k$$
and
$$u_h = E_{t^n}[u_h^n] \quad \text{on } \Omega \times (t^n, t^{n+1}).$$

2. *For each* $v_h \in V_\Gamma^k$ *it holds, that*
$$\int_\Omega u_h^0 \, v_h \, dx = \int_\Omega u_0 \, v_h \, dx. \tag{13}$$

3. *For each* $j \in \{1, \ldots, J\}$*,* $n \in \{0, \ldots, N\}$ *and* $v_h \in V_\Gamma^k$ *it holds, that*
$$\int_{I_j} (u_h^{n+1} - u_h^n) v_h \, dx = \mathcal{H}_j^n[u_h, v_h]. \tag{14}$$

Here $\mathcal{H}_j^n(p, v_h)$ *is defined for* $p \in C^0([t^n, t^{n+1}], V_\Gamma^k)$ *and* $v_h \in V_\Gamma^k$ *through*
$$\mathcal{H}_j^n[p, v_h] = \int_{t^n}^{t^{n+1}} \left(\int_{I_j} F(p) \partial_x v_h \, dx - \left(H(p) v_h^-\right)_{j+\frac{1}{2}} + \left(H(p) v_h^+\right)_{j-\frac{1}{2}} \right) dt.$$

Remark 3.8. *If the integral can not be calculated exactly, one has to use quadrature rules. For more details - for example, which order one should use - see [17]. We assume throughout the paper that the integral can be computed exactly.*

3.4 Admissible Expansions

For the convergence analysis we need more requirements on the expansion.
In the first step we will define some error functions. In the proof of convergence we have to analyse the difference between $E[u_h^n]$ and $E[u^n]$. But $E[u^n]$ is only well defined if $u^n \in V_\Gamma^k$. So let $\mathbb{Q}$ be a projection from C_Γ^k to V_Γ^k and look at
$$\mathbb{P}[u] := E_{t^n}[\mathbb{Q}[u^n]] \quad \text{on } \Omega \times [t^n, t^{n+1}),$$
instead of $E[u^n]$. With the projection $\mathbb{Q}$ we can define the error functions we need:

Definition 3.9 (Error Functions).
Let u *be the exact solution of (1a)+(1b)+(1c) and* u_h *the solution of the STE-DG method of order* $k \in \mathbb{N}$ *from Definition 3.7.*

(i) The function
$$e : \begin{cases} \Omega \times [0, T] & \to & \mathbb{R} \\ (x, t) & \mapsto & e(x, t) := u_h(x, t) - u(x, t). \end{cases}$$
is called ***error function*** *of the discrete solution* u_h *to* u*.*

(ii) The function
$$\eta : \begin{cases} \Omega \times [0, T] & \to & \mathbb{R} \\ (x, t) & \mapsto & \eta(x, t) := \mathbb{P}[u](x, t) - u(x, t). \end{cases}$$
is called ***projection error****.*

(iii) The function

$$\epsilon : \begin{cases} \Omega \times [0,T] & \rightarrow & \mathbb{R} \\ (x,t) & \mapsto & \epsilon(x,t) := \mathbb{P}[u](x,t) - u_h(x,t). \end{cases}$$

is called ***approximation error****.*

With these notations we can define the properties, which an expansion has to satisfy, in order to guarantee convergence:

Definition 3.10 (Admissible Expansion).
An expansion $E : V_\Gamma^k \rightarrow \mathcal{V}_\Gamma^k$ is called an ***admissible expansion of order k****, if the following properties hold:*

(i) For all $M > 0$ there is a constant $C > 0$ independent of h, such that

$$\|(E[v_h](\cdot,t) - v_h) - (E[w_h](\cdot,t) - w_h)\|_2 \leq Ch \|v_h - w_h\|_2 \tag{15}$$

hold for $t \in [0, \tau)$ and all $v_h, w_h \in V_{\Gamma,M}^k$.

(ii) There is a projection $\mathbb{Q} : C_\Gamma^k \rightarrow V_\Gamma^k$ and a constant $C > 0$ independent of h, which satisfy for all $t \in [0,T]$, $0 \leq n < N$ and $v_h \in V_\Gamma^k$ the inequalities

$$\|\eta(\cdot,t)\|_2 + h^{\frac{1}{2}} |\eta(\cdot,t)|_{\Gamma_h} + h \|\eta(\cdot,t)\|_\infty \leq Ch^{k+1}, \tag{16a}$$

$$\left\|\eta(\cdot,t^{n+1}) - \eta(\cdot,t^n)\right\|_2 \leq C\tau h^{k+1} \tag{16b}$$

and

$$\sum_{1 \leq j \leq J} \left| \int_{I_j} \eta(x,t)\, \partial_x v_h \, dx \right| \leq Ch^{k+1} \|v_h\|_2 . \tag{16c}$$

Remark 3.11. *There is always a natural choice of $\mathbb{Q}$ if an expansion is given. Usually an interpolation is used, which keeps the values of the function or its derivatives at selected points. It is also possible, that global properties like integral means are preserved. For example the CK-expansion uses the space derivatives up to order k in the centre of the cell. So a suitable projection $\mathbb{Q}$ on a cell I_j is just the Taylor expansion of order k in the centre x_j.*

Remark 3.12. *Condition (i) can be seen as a stability restriction and (ii) as a consistency restriction, which corresponds to the order of the method.*

Remark 3.13. *The two expansions introduced in Section 3.2.1 and 3.2.2 are admissible expansions. In Section 5 we give the proof, that both expansions are admissible.*

3.5 Consequences for Admissible Expansions

With help of the inverse properties (see Lemma A.1), we get the same inequality as in (15), but with a different norm:

$$|(E[v_h](\cdot,t) - v_h) - (E[w_h](\cdot,t) - w_h)|_{\Gamma_h} \leq Ch^{\frac{1}{2}} \|v_h - w_h\|_2 . \tag{17}$$

Furthermore we present some consequences formulated in the following lemma:

Lemma 3.14 (Consequences for Admissible Expansions).
Let E be an admissible expansion of order k defined in Definition 3.10. Define

$$\Upsilon : \left\{ \begin{array}{ccl} \Omega \times [0,T] & \to & \mathbb{R}, \\ (x,t) & \mapsto & \Upsilon(x,t) := \epsilon(x,t) - \epsilon^n(x), \end{array} \right.$$

where $n = \lfloor \frac{t}{\tau} \rfloor$. Let $M > 0$. Then there is a constant $C > 0$ independent of h satisfying the following. If $u_h \in V^k_{h,M}$, then for all $t \in [t^n, t^{n+1})$ the inequalities

$$|\Upsilon(\cdot,t)|_{\Gamma_h} \leq Ch^{\frac{1}{2}} \|\epsilon^n\|_2 , \qquad \|\Upsilon(\cdot,t)\|_2 \leq Ch \|\epsilon^n\|_2 \tag{18a}$$

and

$$\|\epsilon(\cdot,t)\|_2 \leq C \|\epsilon^n\|_2 . \tag{18b}$$

hold.

Proof: The proof is an elementary consequence of (15) and (17). □

To get more compact formulas we define also

$$\omega(x,t) := e(x,t) + \epsilon^n(x) \qquad (x \in \Omega), \tag{19}$$

where $n = \lfloor \frac{t}{\tau} \rfloor$ and $0 \leq t \leq T$.

4. THEOREM OF CONVERGENCE

With the notations and definitions developed in the previous sections, we are able to formulate the main theorem:

Theorem 4.1 (Convergence).
Let $k \geq 1$. Let $u \in C^{k+2}(\Omega \times [0,T])$ be a classical solution of (1a)+(1b)+(1c) with $F \in C^{k+2}(\mathbb{R})$.
Let $E : V^k_\Gamma \to \mathcal{V}^k_\Gamma$ be an admissible expansion of order k, $H : \mathbb{R}^2 \to \mathbb{R}$ a monotone numerical flux function for F and Γ_h a given grid.
For any $\gamma > 0$, there is a constant $C > 0$, such that the inequality

$$\max_{0 \leq n \leq N} \|e^n\|_2^2 + \frac{\tau}{2} \sum_{0 \leq n < N} \bar{\alpha}_n(H; u_h)[u_h^n]^2 \leq Ch^{2k+1} \tag{20}$$

holds, provided we have h small enough and

$$\tau \leq \gamma h^2. \tag{21}$$

Remark 4.2. *The convergence rate in (20) can be improved if we restrict ourselves to upwind numerical flux functions for F. Precisely we get as in [23] the rate*

$$\max_{0 \leq n \leq N} \|e^n\|_2 \leq Ch^{k+1}.$$

By upwind numerical flux we mean, as in [23], a monotone numerical flux function satisfying

$$H(p^-,p^+) = \left\{ \begin{array}{lll} F(p^-) & \text{if } F'(q) \geq 0 & \forall q \in [\min(p^-,p^+), \max(p^-,p^+)], \\ F(p^+) & \text{if } F'(q) < 0 & \forall q \in [\min(p^-,p^+), \max(p^-,p^+)]. \end{array} \right.$$

Remark 4.3. *The STE-DG scheme is for the case $k = 0$ of piecewise constant ansatz functions nothing but a standard finite-volume scheme evolving cell averages. If the flux functions are e.g. monotone in the sense of Definition 3.1 extensive convergence results –including the case of discontinuous entropy solutions– are available. It is worth mentioning that estimate (20) gives the same L^2-rate that is already known for finite-volume methods provided the solution is smooth [14, 21]. Moreover the result of Shu&?? applies here.*

Remark 4.4. *The time step restriction* (21) *is not the one expected for first-order equations but rather for (explicit) schemes for parabolic evolution. The numerical experiments in [18] clearly show that convergence of the STE-DG scheme can be expected provided just a classical CFL-conditon of the form $\tau \leq \gamma h$ for some $\gamma > 0$ depending on F' controls the time step. Thus our proof appears not to be optimal. This seems to be an unsolved problem, in fact it is exactly the same reason why Zhang&Shu introduce in [23] for a second-order RK-DG method a restriction of type $\tau \leq \gamma h^{4/3}$.*

4.1 Proof of Theorem 4.1

In the following we will first assume, that

$$F \text{ and its derivatives up to order } k+2 \text{ are bounded.} \tag{22}$$

Later on we will see, that this assumption can be skipped.
The classical solution u of (1a)+(1b)+(1c) satisfies the following equation

$$\int_{I_j} (u^{n+1} - u^n) v_h \, dx = \mathcal{H}_j^n[u, v_h] \tag{23}$$

for all $v_h \in V_\Gamma^k$. By subtracting (23) from (14) and summing over all $1 \leq j \leq J$, we get

$$\sum_{1 \leq j \leq J} \int_{I_j} \left(\epsilon^{n+1} - \epsilon^n\right) v_h \, dx = \mathcal{K}^n[u_h, v_h]. \tag{24}$$

Here $\mathcal{K}^n$ is defined by

$$\mathcal{K}^n[u_h, v_h] := \sum_{1 \leq j \leq J} \left(\int_{I_j} \left(\eta^{n+1} - \eta^n\right) v_h \, dx + \mathcal{H}_j^n[u, v_h] - \mathcal{H}_j^n[u_h, v_h] \right). \tag{25}$$

The terms ϵ^n and η^n have been defined in Definition 3.9 and $\mathcal{H}_j^n$ was defined in Definition 3.7.
By choosing $v_h = \epsilon^n \in V_\Gamma^k$ in (24) we get the error equation

$$\left\|\epsilon^{n+1}\right\|_2^2 - \|\epsilon^n\|_2^2 = \left\|\epsilon^{n+1} - \epsilon^n\right\|_2^2 + 2\mathcal{K}^n[u_h, \epsilon^n]. \tag{26}$$

In the following lemmata the terms on the right hand side of (26) are estimated:

Lemma 4.5 (Estimate for $\mathcal{K}^n[u_h, \epsilon^n]$).
Let the assumptions of Theorem 4.1 and (22) be true.
For each $M > 0$ there is a constant $C > 0$ independent of h, such that the following holds:
If $u_h^n \in V_{h,M}^k$ and $\|e^n\|_2 \leq h^k$ for some $n \in \{1, \ldots, N\}$, then

$$\mathcal{K}^n[u_h, \epsilon^n] \leq C\tau(\|\epsilon^n\|_2^2 + h^{2k+1}) - \frac{\tau}{4}\bar{\alpha}_n(H; u_h)[\epsilon^n]^2.$$

Lemma 4.6 (Estimate for $\left\|\epsilon^{n+1} - \epsilon^n\right\|_2^2$).
Let the assumptions of Theorem 4.1 and (22) be true.
For each $M > 0$ there is a constant $C > 0$ independent of h, such that the following holds:
If $u_h^n \in V_{h,M}^k$ and $\|e^n\|_2 \leq h^k$ for some $n \in \{1, \ldots, N\}$, then

$$\left\|\epsilon^{n+1} - \epsilon^n\right\|_2^2 \leq C\tau(\|\epsilon^n\|_2^2 + h^{2k+2}).$$

The proofs of the Lemmata 4.5 and 4.6 are postponed to Section 4.2 and 4.3, respectively.
In the following we will use induction with respect to $n = 0, \dots, N$ to prove

$$\|e^n\|_2^2 + \frac{\tau}{2} \sum_{0 \le m < n} \bar{\alpha}_m(H; u_h)[u_h^m]^2 \le C h^{2k+1} \tag{27}$$

with a constant $C > 0$ independent of h (and n).
First check (27) for $n = 0$. Since u_h^0 is the L^2-projection of u_0 (27) is clearly satisfied.
Next we make the induction step $n \mapsto n+1$. Since $\bar{\alpha}_n(H; u_h)[u_h^m]^2$ is nonnegative (see Lemma A.2), (27) implies that

$$\|e^n\|_2 \le h^k \tag{28}$$

for h small enough. To verify that the constant C in (27) does not grow from one induction step to the next, we will only use (28) instead of (27) to prove the induction step.
Since u is $\|\cdot\|_\infty$-bounded and (28) holds, we get, that u_h is also $\|\cdot\|_\infty$-bounded, uniformly in h. So we can change the function F in such a way, that F itself and its derivatives are bounded and $F(u_h)$ is unchanged. So we see, that the assumption (22), we made at the beginning of the proof, was not necessary.
By using (16a) and (28) and inverse properties, from Lemma A.1, we get a constant $M > 0$ independent of h, such that

$$u_h^n \in V_{h,M}^k. \tag{29}$$

The error equation (24) together with estimates from Lemmata 4.5 and 4.6 yield

$$\left\|\epsilon^{n+1}\right\|_2^2 - \|\epsilon^n\|_2^2 + \frac{\tau}{2}\bar{\alpha}_n(H; u_h)[\epsilon^n]^2 \le C\tau(\|\epsilon^n\|_2^2 + h^{2k+1}). \tag{30}$$

By the Gronwall inequality we finally get

$$\left\|\epsilon^{n+1}\right\|_2^2 + \frac{\tau}{2} \sum_{0 \le m \le n} \bar{\alpha}_m(H; u_h)\,[\epsilon^m]^2 \le C h^{2k+1}. \tag{31}$$

In the last step we substitute ϵ to e by using (16a). This finishes the mathematical induction and the proof of Theorem 4.1. □

4.2 Proof of Lemma 4.5

To prove this lemma we first use the periodicity of the solution and rewrite $\mathcal{K}^n$ from (25) as

$$\begin{aligned}
\mathcal{K}^n[u_h, \epsilon^n] = &\sum_{1 \le j \le J} \int_{I_j} \left(\eta^{n+1} - \eta^n\right) \epsilon^n \, dx \\
&+ \sum_{1 \le j \le J} \int_{I_j} \left(\int_{t^n}^{t^{n+1}} (F(u) - F(u_h)) \, dt \right) \partial_x \epsilon^n \, dx \\
&+ \sum_{1 \le j \le J} \int_{t^n}^{t^{n+1}} (F(u) - F(\overline{u_h}))_{j+\frac{1}{2}} \, dt \, [\epsilon^n]_{j+\frac{1}{2}} \\
&+ \sum_{1 \le j \le J} \int_{t^n}^{t^{n+1}} (F(\overline{u_h}) - H(u_h))_{j+\frac{1}{2}} \, dt \, [\epsilon^n]_{j+\frac{1}{2}} \\
=:& \, W_1 + W_2 + W_3 + W_4.
\end{aligned} \tag{32}$$

In the following we will estimate all W_i's separately. To estimate W_1 we use Young's inequality and (16b) to get

$$W_1 \le \frac{1}{4}\tau^{-1} \left\|\eta^{n+1} - \eta^n\right\|_2^2 + \tau \|\epsilon^n\|_2^2 \le C\tau h^{2k+2} + \tau \|\epsilon^n\|_2^2 .$$

Next we estimate the term W_4. First we note that, because u is continuous,

$$[u_h] = [u + e] = [e] = [\omega - \epsilon^n]. \tag{33}$$

Now we use the definition of α from (8) at each cell boundary $x_{j+\frac{1}{2}}$ to get with (33) the estimate

$$\begin{aligned}(F(\overline{u_h}) - H(u_h))\,[\epsilon^n] &\le \alpha(H; u_h)[u_h]\,[\epsilon^n] \\ &= \alpha(H; u_h)\left([\omega]\,[\epsilon^n] - [\epsilon^n]^2\right) \\ &\le \alpha(H; u_h)\left([\omega]^2 - \frac{3}{4}\,[\epsilon^n]^2\right).\end{aligned} \tag{34}$$

For the last line we used Young's inequality. Next, we use the fact, that the expansion is admissible, and the definition of $\bar{\alpha}_n$ from (9) to get

$$\begin{aligned}W_4 &\le -\frac{3}{4}\tau\bar{\alpha}_n(H; u_h)[\epsilon^n]^2 + C\tau h\,|\eta|^2_{\Gamma_h} + C\tau h^{-1}\,|\Upsilon|^2_{\Gamma_h} \\ &\le -\frac{3}{4}\tau\bar{\alpha}_n(H; u_h)[\epsilon^n]^2 + C\tau h^{2k+2} + C\tau\,\|\epsilon^n\|_2^2\,.\end{aligned}$$

For a closer look at W_2 and W_3, we use Taylor expansions of F at u and get

$$\begin{aligned}F(u) - F(u_h) &= F'(u)\epsilon^n - \frac{1}{2}F''(u)(\epsilon^n)^2 - F'(u)\omega + F''(u)\omega\epsilon^n \\ &\quad - \frac{1}{2}F''(u)\omega^2 - \frac{1}{6}\bar{F}'''(\omega - \epsilon^n)^3 \\ &=: \phi_1 + \phi_2 + \phi_3 + \phi_4 + \phi_5 + \phi_6\end{aligned}$$

and

$$\begin{aligned}F(u) - F(\overline{u_h}) &= F'(u)\overline{\epsilon^n} - \frac{1}{2}F''(u)(\overline{\epsilon^n})^2 - F'(u)\overline{\omega} + F''(u)\overline{\omega}\overline{\epsilon^n} \\ &\quad - \frac{1}{2}F''(u)\overline{\omega}^2 - \frac{1}{6}\tilde{F}'''(\overline{\omega} - \overline{\epsilon^n})^3 \\ &=: \psi_1 + \psi_2 + \psi_3 + \psi_4 + \psi_5 + \psi_6.\end{aligned} \tag{35}$$

Here $\bar{F}'''$ and $\tilde{F}'''$ denote remainder terms in Taylor expansions of the bounded function F'''. Define for $1 \le i \le 6$

$$X_i := \sum_{1\le j\le J} \int_{I_j} \left(\int_{t^n}^{t^{n+1}} \phi_i(x,t)\,dt\right) \partial_x \epsilon^n(x)\,dx$$

and

$$Y_i := \sum_{1\le j\le J} \int_{t^n}^{t^{n+1}} \psi_i(x_{j+\frac{1}{2}}, t)\,dt\,[\epsilon^n]_{j+\frac{1}{2}}.$$

Formula (32) implies

$$W_2 = X_1 + \cdots + X_6 \quad \text{und} \quad W_3 = Y_1 + \cdots + Y_6.$$

In the remaining part of the proof we will estimate $X_i + Y_i$ for each i separately. From integration by parts we get

$$\begin{aligned}X_1 + Y_1 &= -\frac{1}{2}\sum_{1\le j\le J}\int_{t^n}^{t^{n+1}}\left(\int_{I_j} F'(u)\partial_x\,(\epsilon^n)^2\;dx + \left(F'(u)\,\left[(\epsilon^n)^2\right]\right)_{j+\frac{1}{2}}\right)\,dt \\ &= -\frac{1}{2}\sum_{1\le j\le J}\int_{t^n}^{t^{n+1}}\int_{I_j}\partial_x F'(u)\,(\epsilon^n)^2\;dx\,dt.\end{aligned} \tag{36}$$

Here we used $\epsilon^n \partial_x \epsilon^n = \frac{1}{2}\partial_x (\epsilon^n)^2$ and $\overline{\epsilon^n}[\epsilon^n] = \frac{1}{2}[(\epsilon^n)^2]$. So we get as a result

$$X_1 + Y_1 \leq C\tau \|\epsilon^n\|_2^2 .$$

To estimate $X_2 + Y_2$ we use also integration by parts and

$$\sum_{1\leq j\leq J} \left(-\frac{1}{6}(\epsilon^n)^3 \Big|_{x=x^+_{j-\frac{1}{2}}}^{x=x^-_{j+\frac{1}{2}}} - \frac{1}{2}(\overline{\epsilon^n_{j+\frac{1}{2}}})^2[\epsilon^n]_{j+\frac{1}{2}} \right) = \sum_{1\leq j\leq J} \frac{1}{24}[\epsilon^n]^3_{j+\frac{1}{2}}$$

to get

$$X_2 + Y_2 = \sum_{1\leq j\leq J} \int_{t^n}^{t^{n+1}} \left(\left(\frac{1}{24} F''(u)[\epsilon^n]^3 \right)_{j+\frac{1}{2}} + \frac{1}{6}\int_{I_j} \partial_x F''(u)\ (\epsilon^n)^3\ dx \right) dt. \tag{37}$$

By using the Taylor expansion

$$F''(u)[\epsilon^n] = (F''(\overline{u_h}) + \tilde{F}'''\overline{e})[\epsilon^n] = -F''(\overline{u_h})[u_h] + F''(\overline{u_h})[\omega] + \tilde{F}'''[\epsilon^n]\overline{e} \tag{38}$$

and (16a), as well as the Lemmata 3.14 and A.2 we get

$$\begin{aligned}\frac{1}{24}F''(u)[\epsilon^n] \leq& \frac{1}{3}\alpha(H;u_h) + C(\|\eta(\cdot,t)\|_\infty + |\Upsilon(\cdot,t)|_{\Gamma_h} + \|e(\cdot,t)\|_\infty^2) \\ \leq& \frac{1}{3}\alpha(H;u_h) + C(h + h^{\frac{1}{2}}\|\epsilon^n\|_2 + \|e(\cdot,t)\|_\infty^2).\end{aligned}$$

Finally (28) leads to

$$\begin{aligned}X_2 + Y_2 &\leq \frac{\tau}{3}\bar{\alpha}_n(H;u_h)[\epsilon^n]^2 + C\tau(1 + h^{-\frac{1}{2}}\|\epsilon^n\|_2 + h^{-1}\|e^n\|_\infty^2 + \|\epsilon^n\|_\infty)\|\epsilon^n\|_2^2 \\ &\leq \frac{\tau}{3}\bar{\alpha}_n(H;u_h)[\epsilon^n]^2 + C\tau\|\epsilon^n\|_2^2 .\end{aligned}$$

Rewriting X_3 we get

$$X_3 = -\sum_{1\leq j\leq J} F'(\tilde{u}_j)\int_{I_j}\int_{t^n}^{t^{n+1}} (\eta - \Upsilon)\,\partial_x\epsilon^n\,dt\,dx.$$

Here $\tilde{u}_1, \dots, \tilde{u}_J$ are the values we get by using the mean value theorem for integration. With (16c), Lemma 3.14 and the inverse properties from Lemma A.1 we get the estimate

$$X_3 \leq C\tau h^{k+1}\|\epsilon^n\|_2 + C\tau\|\Upsilon\|_2\|\partial_x\epsilon^n\|_2 \leq C\tau\left(\|\epsilon^n\|_2^2 + h^{2k+2}\right).$$

To handle Y_3 we use

$$|F'(u(x_{j+\frac{1}{2}},\cdot))| \leq 2\alpha(H;u_h)_{j+\frac{1}{2}} + C\|e^n\|_\infty ,$$

which follows immediately from (51a) in Lemma A.2 in the appendix. By using Young's inequality, (16a) and Lemma 3.14 we get

$$\begin{aligned}Y_3 &= \sum_{1\leq j\leq J}\left(\int_{t^n}^{t^{n+1}} F'(u)\overline{\Upsilon}\,dt\ [\epsilon^n] - \int_{t^n}^{t^{n+1}} F'(u)\overline{\eta}\,dt\ [\epsilon^n]\right)_{j+\frac{1}{2}} \\ &\leq C\tau\left(h^{-1}|\Upsilon|_{\Gamma_h}^2 + h|\epsilon^n|_{\Gamma_h}^2 + \|e^n\|_\infty\|\epsilon^n\|_\infty|\eta|_{\Gamma_h}\right) + \sum_{1\leq j\leq J}\int_{t^n}^{t^{n+1}}(2\alpha(H;u_h)[\epsilon^n]\,\overline{\eta})_{j+\frac{1}{2}}\,dt \\ &\leq C\tau\left(\|\epsilon^n\|_2^2 + |\eta|_{\Gamma_h}^2 + h^{-1}\|e^n\|_\infty^2\|\epsilon^n\|_2^2 + |\eta|_{\Gamma_h}^2\right) + \frac{\tau}{6}\bar{\alpha}_n(H;u_h)[\epsilon^n]^2 \\ &\leq C\tau\left(\|\epsilon^n\|_2^2 + h^{2k+1}\right) + \frac{\tau}{6}\bar{\alpha}_n(H;u_h)[\epsilon^n]^2.\end{aligned} \tag{39}$$

In a similar way we get the remaining inequalities

$$
\begin{aligned}
X_4 + Y_4 &\le C\frac{\tau}{h}\left\|\eta^n\right\|_\infty \left\|\epsilon^n\right\|_2^2 \le C\tau\left\|\epsilon^n\right\|_2^2,\\
X_5 + Y_5 &\le C\tau\left(\left\|\epsilon^n\right\|_2^2 + h^{2k+2}\right),\\
X_6 + Y_6 &\le Ch^{-1}\left\|e^n\right\|_\infty^2\left(\tau\left\|\epsilon^n\right\|_2^2 + C\tau h^{2k+2}\right) \le C\tau\left(\left\|\epsilon^n\right\|_2^2 + h^{2k+2}\right).
\end{aligned}
$$

By summing up over all terms we get the result of the lemma. □

REMARK 4.7. *The estimation of Y_3 has to be improved to get a convergence rate of $k+1$ instead of $k+\frac{1}{2}$. The term h^{2k+1}*

4.3 Proof of Lemma 4.6

First we rewrite $\left\|\epsilon^{n+1} - \epsilon^n\right\|_2^2$ by using (24) and $v_h = \epsilon^{n+1} - \epsilon^n$ and get

$$
\left\|\epsilon^{n+1} - \epsilon^n\right\|_2^2 = \mathcal{K}^n[u_h, \epsilon^{n+1} - \epsilon^n]. \tag{40}
$$

In the remaining proof we deduce an estimate for $\mathcal{K}^n[u_h, v_h]$ and each $v_h \in V_\Gamma^k$. $\mathcal{K}^n[u_h, v_h]$ we can rewrite it as

$$
\begin{aligned}
\mathcal{K}^n[u_h, v_h] = &\sum_{1\le j\le J}\int_{I_j}(\eta^{n+1} - \eta^n)v_h\,dx\\
&+ \sum_{1\le j\le J}\int_{t^n}^{t^{n+1}}\int_{I_j}\Pi(u,u_h)\,\partial_x v_h\,dx\,dt\\
&+ \sum_{1\le j\le J}\int_{t^n}^{t^{n+1}}\hat{\Pi}_{j+\frac{1}{2}}(u,u_h)[v_h]_{j+\frac{1}{2}}\,dt\\
=:&\ \theta_1(v_h) + \theta_2(v_h) + \theta_3(v_h).
\end{aligned}
$$

Here Π and $\hat{\Pi}_{j+\frac{1}{2}}$ are defined by

$$
\Pi(u,u_h) = F(u) - F(u_h) \quad \text{and} \quad \hat{\Pi}_{j+\frac{1}{2}}(u,u_h) = (F(u) - H(u_h))_{j+\frac{1}{2}}.
$$

Let δ be a fixed but arbitrary positive number. The first term $\theta_1(v_h)$ can easily be estimated by using Young's inequality and (16b) by

$$
|\theta_1(v_h)| \le C\tau^2 h^{2k+2} + \delta\left\|v_h\right\|_2^2. \tag{41}
$$

To estimate $\theta_2(v_h)$ we make a Taylor expansion of Π and get

$$
\Pi(u,u_h) = F(u) - F(u_h) = \bar{F}'e = \bar{F}'\epsilon - \bar{F}'\eta.
$$

Here $\bar{F}' \in \mathbb{R}$ is the number arising when using the mean value theorem. Altogether we get by Young's inequality, (16b) and (18b)

$$
\begin{aligned}
|\theta_2(v_h)| &\le \delta\left\|v_h\right\|_2^2 + Ch^{-2}\left\|\int_{t^n}^{t^{n+1}}\epsilon(\cdot,t) - \eta(\cdot,t)\,dt\right\|_2^2\\
&\le \delta\left\|v_h\right\|_2^2 + C\frac{\tau^2}{h^2}\left(\left\|\epsilon(\cdot,\xi_t^1)\right\|_2^2 + \left\|\eta(\cdot,\xi_t^2)\right\|_2^2\right)\\
&\le \delta\left\|v_h\right\|_2^2 + C\frac{\tau^2}{h^2}\left(\left\|\epsilon^n\right\|_2^2 + h^{2k+2}\right).
\end{aligned} \tag{42}
$$

Here $\xi_t^1, \xi_t^2 \in (t^n, t^{n+1})$ are values, which arise from applying the mean value theorem in time. To handle $\theta_3(v_h)$ we split it and make a Taylor expansion to get

$$\begin{aligned}\hat{\Pi}(u, u_h) &= F(u) - F(\overline{u_h}) + F(\overline{u_h}) - H(u_h) \\ &= \left(\tilde{F}'\overline{\eta} - \tilde{F}'\overline{\epsilon}\right) + \left(F(\overline{u_h}) - H(u_h)\right) \\ &=: Q_1 + Q_2.\end{aligned}$$

Here $\tilde{F}' \in \mathbb{R}$ is again a number arising from a Taylor expansion. Define

$$T_i := \sum_{1 \le j \le J} \int_{t^n}^{t^{n+1}} (Q_i)_{j+\frac{1}{2}} \, dt[v_h]_{j+\frac{1}{2}} \qquad (i = 1, 2).$$

The term T_1 is estimated by the same type of arguments as θ_2:

$$\begin{aligned}|T_1| &\le \delta \|v_h\|_2^2 + Ch^{-1} \sum_{1 \le j \le J} \left| \int_{t^n}^{t^{n+1}} \overline{\epsilon}(x_{j+\frac{1}{2}}, t) - \overline{\eta}(x_{j+\frac{1}{2}}, t) \, dt \right|^2 \\ &\le \delta \|v_h\|_2^2 + C\frac{\tau^2}{h} \left(\left|\epsilon(\cdot, \xi_t^1)\right|_{\Gamma_h}^2 + \left|\eta(\cdot, \xi_t^2)\right|_{\Gamma_h}^2 \right) \\ &\le \delta \|v_h\|_2^2 + C\frac{\tau^2}{h^2} \left(\|\epsilon^n\|_2^2 + h^{2k+2} \right).\end{aligned} \tag{43}$$

Here $\xi_t^1, \xi_t^2 \in (t^n, t^{n+1})$ are values, which arise from applying mean value theorem. With the definition of α from (8) we get

$$|Q_2| \le \alpha(H; u_h)|[u_h]| \le \alpha(H; u_h) \left(|[\epsilon]| + |[\eta]|\right).$$

Together with Lemma A.1 this gives

$$\begin{aligned}|T_2| &\le \delta \|v_h\|_2^2 + C\frac{\tau^2}{h} \bar{\alpha}_n^2(H; u_h)[\epsilon^n]^2 + C\tau^2 h^{2k} \\ &\le \delta \|v_h\|_2^2 + C\frac{\tau^2}{h^2} \|\epsilon^n\|_2^2 + C\tau^2 h^{2k}.\end{aligned} \tag{44}$$

By adding (41), (42), (43) and (44) we get

$$\mathcal{K}^n[u_h, v_h] \le 4\delta \|v_h\|_2^2 + C\frac{\tau^2}{h^2} \|\epsilon^n\|_2^2 + C\tau^2 h^{2k}.$$

Now choose $\delta = \frac{1}{8}$ and use (40) and (21) to get the final result

$$\left\|\epsilon^{n+1} - \epsilon^n\right\|_2^2 \le C\tau \left(\|\epsilon^n\|_2^2 + h^{2k+2} \right).$$

This completes the proof of this lemma. □

5. ADMISSIBLE EXPANSIONS

Theorem 4.1 applies to the STE-DG method if the expansion is what we called admissible (cf. Definition 3.10). We introduced this notion to abstain from the special choice for expansions, and to work out the essential conditions for obtaining convergence. Whether an expansion is admissible has to be checked by a case-by-case study. In this section we verify the admissibility for the Cauchy-Kovaleskaya expansion and the characteistic expansion.

5.1 Cauchy-Kovalevskaya Expansion

In this section we will outline the proof of the fact, that the CK-expansion from Definition 3.5 is an admissible expansion of order k in the sense of Definition 3.10. To do this we first define the projection $\mathbb{Q}^{CK}$ by

$$\mathbb{Q}^{CK}:\left\{\begin{array}{lll} C_\Gamma^k & \to & V_\Gamma^k \\ v & \mapsto & \mathbb{Q}^{CK}[v] := \mathbf{T}^k[v;x_j] \quad \text{on } I_j, \qquad \forall 1\le j\le J. \end{array}\right. \tag{45}$$

Here $\mathbf{T}^k[v;x_j]$ denotes the Taylor expansion of order k of the function v in x_j.
In the first step we prove two auxiliary lemmata. The first describes derivational operators; the second relates time derivatives of the expansion and Taylor expansions of derivational operators.

Lemma 5.1 (Form of Derivational Operators).
Let the assumptions in Theorem 4.1 hold. Let T_l, $l\in\mathbb{N}$, denote the set of multi-indices given by

$$T_l := \left\{\underline{j}=(j_1,\dots,j_l)\,:\, j_1+2\,j_2+\dots+l\,j_l = l\right\}.$$

Let the assumptions of Theorem 4.1 hold. Then there are functions $K_{\underline{i}}$, $i\in T_{l+p}$, such that for all $l,p\in\mathbb{N}_0$ the derivational operator from Definition 3.4 applied to a function $v\in C_\Gamma^k$ has the form

$$\tilde{\partial}_x{}^p\,\tilde{\partial}_t{}^l\,v(x) = \sum_{\underline{i}\in T_{l+p}} K_{\underline{i}}(v(x)) \prod_{m=1}^{l+p} \left(\partial_x{}^m v(x)\right)^{i_m}. \tag{46}$$

For all $\underline{i}\in T_{l+p}$ we have $K_{\underline{i}}\in C^{k-l-p+1}(\mathbb{R})$ and $K_{\underline{i}}$ is bounded and has bounded derivatives.

Proof: This is a straightforward consequence of the definition of derivational operators and the regularity assumptions on F made in Theorem 4.1. $\square$

Lemma 5.2 (Derivational Operators and Taylor Expansions).
For $v\in V_\Gamma^k$ and $0\le l\le k$ the following representation formula holds:

$$\partial_t{}^l\,E^{CK}[v]\big|_{t=0} = \mathbf{T}^{k-l}\left[\tilde{\partial}_t{}^l v;x_j\right] = \tilde{\partial}_t{}^l v - \mathbf{R}^{k-l}\left[\tilde{\partial}_t{}^l v;x_j\right] \quad \textit{on } I_j. \tag{47}$$

Proof: This is a straightforward consequence of the definitions of the CK-expansion (see Definition 3.5) and of derivational operators. $\square$

Define $\mathbf{W} := (E^{CK}[w]-w)-(E^{CK}[v]-v)$. By using the Lemmata 5.1 and 5.2 we get the following representation

$$\mathbf{W} = \sum_{0\le m\le k}\Big(t^{\min(m,1)}\partial_x{}^m(w-v)B_m(x,t,w,v) + t(x-x_j)^k(\partial_x{}^m(w-v))(\xi_x)D_m(x,t,w,v)\Big).$$

Here $B_m,\, D_m : [0,1]\times[0,\tau]\times V_{h,M}^k\times V_{h,M}^k\to\mathbb{R}$ are uniformly bounded with respect to h.
Finally we get (i) in Definition 3.10 by using inverse properties (see A.1)

$$\|\mathbf{W}(\cdot,t)\|_2^2 \le C\sum_{0\le m\le k}\left(\frac{\tau^{2\min(m,1)}}{h^{2m}} + \frac{\tau^2 h^{2k-1}}{h^{2m+1}}\right)\|w-v\|_2^2 \le Ch^2\,\|w-v\|_2^2\,.$$

Note that we used the time step restriction $\tau\le\gamma h^2$ from (21) here.
To prove (ii) we note, that

$$E_{t^n}^{CK}\left[\mathbb{Q}^{CK}[u^n]\right] = \tilde{\mathbf{T}}^k\left[u;(x_j,t^n)\right] \quad \text{on } I_j\times[t^n,t^{n+1}], \qquad \forall 1\le j\le J,\ 0\le n\le N, \tag{48}$$

and use approximation arguments of the Taylor expansion. Here u denotes the exact solution and $\tilde{\mathbf{T}}^k$ is the Taylor expansion in space and time.

5.2 Characteristic Expansion

In this section we want to prove, that the characteristic expansion from Section 3.2.2 is an admissible expansion of order k. We use here the time-step restriction

$$\tau \leq \gamma h^{\frac{5}{2}}. \tag{49}$$

By using the basic formulation (12), we derive the following lemma:

LEMMA 5.3 (ERROR REPRESENTATION).
Let $v,\ w \in V_\Gamma^k$. For each $1 \leq j \leq J$ and $0 \leq l \leq k$, there is a term $T_{j,l} = T_{j,l}(x,t,v,w)$ with support in $I_j \times [0,\tau] \times V_h^k \times V_h^k$, such that for $0 \leq t \leq \tau$

$$\mathbf{W} := (E^{Ch}[w] - w) - (E^{Ch}[v] - v) = \sum_{1\leq j\leq J} \sum_{0\leq l\leq k} T_{j,l}\,(w-v)(y_{j,l})$$

and

$$|T_{j,l}| \leq C\frac{\tau}{h} \tag{50}$$

holds.

Proof: To prove this result we first rewrite (12) and get for $x \in I_j$, $j \in \{1,\ldots,J\}$,

$$E^{Ch}[v](x,t) = \sum_{0\leq l\leq k} v_l \prod_{\substack{0\leq m\leq k\\ m\neq l}} g^{l,m}(x,t,v_l,v_m),$$

where

$$g^{l,m}(x,t,a,b) = \frac{x - y_{j,m} - tF'(b)}{y_{j,l} - y_{j,m} + t(F'(a) - F'(b))}$$

and

$$v_l = v(y_{j,l}).$$

In the next step we rewrite $\mathbf{W}(x,t) = s\left(\partial_t E^{Ch}[w] - \partial_t E^{Ch}[v]\right)(x,s)\big|_{s=\xi_t}$ with $0 \leq \xi_t \leq t$, in such a way, that it is factorised with one factor $(w-v)(y_{j,l})$. By using the estimates

$$\left|g^{l,m}\right| \leq C, \qquad \left|\partial_t g^{l,m}\right| \leq C\frac{1}{h}, \qquad \left|\partial_a g^{l,m}\right|, \left|\partial_b g^{l,m}\right| \leq C\frac{\tau}{h}, \qquad \left|\partial_{ta} g^{l,m}\right|, \left|\partial_{tb} g^{l,m}\right| \leq C\frac{1}{h}$$

we get the final result (50). These estimates hold for $x \in I_j$, $t \in [0,\tau]$ and $a, b \in \mathbb{R}$. $\square$

With Lemma 5.3 and (49), we can easily prove, that (i) in Definition 3.10 holds.
To prove (ii) of Definition 3.10 we define $\mathbb{Q}^{Ch}$ in such a way, that the properties $\mathbb{Q}^{Ch}[v] \in V_\Gamma^k$ and $\mathbb{Q}^{Ch}[v](y_{j,l}) = v(y_{j,l},0)$ hold for each $1 \leq j \leq J$, $0 \leq l \leq k$ and $v \in C_\Gamma^k$. Having the projection $\mathbb{Q}^{Ch}$ we only need standard interpolation properties to prove (ii).

6. CONCLUSION

Last we want to summarise the results of this paper:
We introduces the Space-Time Expansion DG method in a abstract setting and proof a convergence result only using the few properties of an admissible expansion. We also introduced two different admissible expansions namely the Cauchy-Kovalevskaya and the characteristic expansion. We achieved a convergence rate of $k+\frac{1}{2}$ in theorem 4.1, which in not optimal, but can be improved to $k+1$ by assuming that the numerical flux is a upwind flux as in [23]. Looking at experiments done by Munz et al. (see [11, 17]) the order $k+1$ seems to be optimal. But the same experiments suggest that the time step restriction $\tau \leq \gamma h^2$ is not optimal. A time step restriction of the form $\tau \leq \gamma h$, with $\gamma > 0$ small enough, should be possible.

Furthermore we did not mention, that the STE-DG method can be extended to use local time stepping (see [11, 17]). Local time stepping saves a lot of computation time. To get a convergence result for the local time stepping variant of the STE-DG method there is still some work to do.

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APPENDIX A.

We collect well-known results from the literature, which are used in the main part of the paper.

LEMMA A.1 (INVERSE PROPERTIES).
There is a constant $C > 0$ independent of h, such that the following inequalities hold for all $v_h \in V_\Gamma^k$:

$$\|\partial_x v_h\|_2 \leq Ch^{-1} \|v_h\|_2 , \quad |v_h|_{\Gamma_h} \leq Ch^{-\frac{1}{2}} \|v_h\|_2 , \quad \|v_h\|_\infty \leq Ch^{-\frac{1}{2}} \|v_h\|_2 .$$

The seminorm was defined in (7).

Proof: See [3]. □

We also need some information about α and $\hat{\alpha}$ defined in (8) and (9):

LEMMA A.2 (PROPERTIES OF THE NUMERICAL FLUX).
Let $H : \mathbb{R}^2 \to \mathbb{R}$ be a monotone, numerical flux for F.
Then the quantities $\alpha(H; p)$ and $\hat{\alpha}_n(H; p)$ are nonnegative and bounded for all $p \in \mathbb{R}^2$. Moreover there is a constant $C > 0$, such that

$$\frac{1}{2} |F'(\overline{p})| \leq \alpha(H; p) + C| [p] |, \tag{51a}$$

$$-\frac{1}{8} F''(\overline{p}) [p] \leq \alpha(H; p) + C| [p] |^2. \tag{51b}$$

Proof: See [23]. □

REFERENCES

[1] BEN-ARTZI, M.: The generalized Riemann problem in compressible duct flow. In: *Nonstrictly hyperbolic conservation laws (Anaheim, Calif., 1985)* Bd. 60. Providence, RI : Amer. Math. Soc., 1987, S. 11–18

[2] BEN-ARTZI, M. ; FALCOVITZ, J.: Generalized Riemann problems: from the scalar equation to multidimensional fluid dynamics. In: *Recent advances in computational sciences*. World Sci. Publ., Hackensack, NJ, 2008, S. 1–49

[3] CIARLET, P. G.: *The finite element method for elliptic problems*. Amsterdam : North-Holland Publishing Co., 1978. – Studies in Mathematics and its Applications, Vol. 4

[4] COCKBURN, B.: Discontinuous Galerkin methods. In: *ZAMM Z. Angew. Math. Mech.* 83 (2003), Nr. 11, S. 731–754

[5] COCKBURN, B. ; GREMAUD, P.-A.: Error estimates for finite element methods for scalar conservation laws. In: *SIAM J. Numer. Anal.* 33 (1996), Nr. 2, S. 522–554

[6] COCKBURN, B. ; HOU, S. ; SHU, C.-W.: The Runge-Kutta local projection discontinuous Galerkin finite element method for conservation laws. IV. The multidimensional case. In: *Math. Comp.* 54 (1990), Nr. 190, S. 545–581

[7] COCKBURN, B. ; LIN, S. Y. ; SHU, C.-W.: TVB Runge-Kutta local projection discontinuous Galerkin finite element method for conservation laws. III. One-dimensional systems. In: *J. Comput. Phys.* 84 (1989), Nr. 1, S. 90–113

[8] Cockburn, B. ; Shu, C.-W.: TVB Runge-Kutta local projection discontinuous Galerkin finite element method for conservation laws. II. General framework. In: *Math. Comp.* 52 (1989), Nr. 186, S. 411–435

[9] Cockburn, B. ; Shu, C.-W.: The Runge-Kutta local projection P^1-discontinuous-Galerkin finite element method for scalar conservation laws. In: *RAIRO Modél. Math. Anal. Numér.* 25 (1991), Nr. 3, S. 337–361

[10] Cockburn, B. ; Shu, C.-W.: The local discontinuous Galerkin method for time-dependent convection-diffusion systems. In: *SIAM J. Numer. Anal.* 35 (1998), Nr. 6, S. 2440–2463 (electronic)

[11] Gassner, G. ; Lörcher, F. ; Munz, C.-D.: A discontinuous Galerkin scheme based on a space-time expansion. II. Viscous flow equations in multi dimensions. In: *J. Sci. Comput.* 34 (2008), Nr. 3, S. 260–286

[12] Hesthaven, J. S. ; Warburton, T.: *Texts in Applied Mathematics.* Bd. 54: *Nodal discontinuous Galerkin methods*. New York : Springer, 2008. – Algorithms, analysis, and applications

[13] Jiang, G. S. ; Shu, C.-W.: On a cell entropy inequality for discontinuous Galerkin methods. In: *Math. Comp.* 62 (1994), Nr. 206, S. 531–538

[14] Jovanović, V. ; Rohde, C.: Error estimates for finite volume approximations of classical solutions for nonlinear systems of hyperbolic balance laws. In: *SIAM J. Numer. Anal.* 43 (2006), Nr. 6, S. 2423–2449 (electronic)

[15] Kröner, D.: *Numerical schemes for conservation laws.* Chichester : John Wiley & Sons Ltd., 1997 (Wiley-Teubner Series Advances in Numerical Mathematics)

[16] Lasaint, P. ; Raviart, P.-A.: On a finite element method for solving the neutron transport equation. In: *Mathematical aspects of finite elements in partial differential equations (Proc. Sympos., Math. Res. Center, Univ. Wisconsin, Madison, Wis., 1974).* Math. Res. Center, Univ. of Wisconsin-Madison, Academic Press, New York, 1974, S. 89–123. Publication No. 33

[17] Lörcher, F. ; Gassner, G. ; Munz, C.-D.: A discontinuous Galerkin scheme based on a space-time expansion. I. Inviscid compressible flow in one space dimension. In: *J. Sci. Comput.* 32 (2007), Nr. 2, S. 175–199

[18] Lörcher, F. ; Gassner, G. ; Munz, C.-D.: An explicit discontinuous Galerkin scheme with local time-stepping for general unsteady diffusion equations. In: *J. Comput. Phys.* 227 (2008), Nr. 11, S. 5649–5670

[19] Osher, S. ; Shu, C.-W.: High-order essentially nonoscillatory schemes for Hamilton-Jacobi equations. In: *SIAM J. Numer. Anal.* 28 (1991), Nr. 4, S. 907–922

[20] Toro, E. F. ; Titarev, V. A.: Solution of the generalized Riemann problem for advection-reaction equations. In: *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.* 458 (2002), Nr. 2018, S. 271–281

[21] Vila, J.-P.: *Lecture given at the workshop Finite-Volume Methods, Freiburg, Germany, December, 2000.*

[22] Ying, L. A. ; Ji, X. M. ; Deng, J.: A second-order explicit finite element scheme for multi-dimensional systems of conservation laws. In: *Math. Numer. Sin.* 23 (2001), Nr. 3, S. 321–332

[23] Zhang, Q. ; Shu, C.-W.: Error estimates to smooth solutions of Runge-Kutta discontinuous Galerkin methods for scalar conservation laws. In: *SIAM J. Numer. Anal.* 42 (2004), Nr. 2, S. 641–666 (electronic)

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