STRONG COMPACTNESS OF APPROXIMATE SOLUTIONS TO DEGENERATE ELLIPTIC-HYPERBOLIC EQUATIONS WITH DISCONTINUOUS FLUX FUNCTION

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ABSTRACT. Under a non-degeneracy condition on the nonlinearities we show that sequences of approximate entropy solutions of mixed elliptic-hyperbolic equations are strongly precompact in the general case of a Caratheodory flux vector. The proofs are based on deriving localization principles for H-measures associated to sequences of measure-valued functions. This main result implies existence of solutions to degenerate parabolic convection-diffusion equations with discontinuous flux. Moreover, it provides a framework in which one can prove convergence of various types of approximate solutions, such as those generated by the vanishing viscosity method and numerical schemes.

1. INTRODUCTION

Let Ω be an open subset of $\mathbf{R}^n$. In the domain Ω we consider the quasilinear elliptic equation

$$\operatorname{div}_x \varphi(x,u) - D^2 \cdot B(u) + \psi(x,u) = 0, \tag{1}$$

where $D^2 \cdot B(u) = \partial^2_{x_i x_j} b_{ij}(u)$ (we use the conventional rule of summation over repeated indexes), $B(u) = \{b_{ij}(u)\}_{i,j=1}^n$ is a symmetric matrix. We shall assume that this matrix is only continuous: $b_{ij}(u) \in C(\mathbf{R})$, $i,j = 1,\dots,n$. In this case the ellipticity of (1) is understood in the following sense

$$B(u_1) - B(u_2) \geq 0, \quad u_1, u_2 \in \mathbf{R},\ u_1 > u_2, \tag{2}$$

that is, for all $\xi \in \mathbf{R}^n$ we have $(B(u_1) - B(u_2))\xi \cdot \xi \geq 0$ (here $u \cdot v$ denotes the scalar product of vectors $u, v \in \mathbf{R}^n$).

We suppose that $\varphi(x,u) = (\varphi_1(x,u), \dots, \varphi_n(x,u))$ is a Caratheodory vector (i.e., it is continuous with respect to u and measurable with respect to x) such that the functions

$$\alpha_M(x) = \max_{|u| \leq M} |\varphi(x,u)| \in L^2_{\mathrm{loc}}(\Omega) \tag{3}$$

for all $M > 0$ (here and below $|\cdot|$ stands for the Euclidean norm of a finite-dimensional vector). We also assume that for all $p \in \mathcal{P}$, where $\mathcal{P} \subset \mathbf{R}$ is a set of full measure, the distribution

$$\operatorname{div}_x \varphi(x,p) = \gamma_p \in \mathrm{M}_{\mathrm{loc}}(\Omega), \tag{4}$$

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where $\mathrm{M}_{\mathrm{loc}}(\Omega)$ denotes the space of locally finite Borel measures on Ω equipped with the standard locally convex topology generated by semi-norms $p_\Phi(\mu) = \mathrm{Var}\,(\Phi\mu)$, with $\Phi = \Phi(x) \in C_0(\Omega)$.

The function $\psi(x,u)$ is assumed to be a Caratheodory function on $\Omega \times \mathbf{R}$, and

$$\beta_M(x) = \max_{|u|\le M} |\psi(x,u)| \in L^1_{\mathrm{loc}}(\Omega) \quad \text{for all } M > 0. \tag{5}$$

Let $\gamma_p = \gamma_p^r + \gamma_p^s$ be the decomposition of the γ_p into the sum of regular and singular measures, so that $\gamma_p^r = \omega_p(x)dx$, $\omega_p \in L^1_{\mathrm{loc}}(\Omega)$, and γ_p^s is a singular measure (supported on a set of zero Lebesgue measure). We denote by $|\gamma_p^s|$ the variation of the measure γ_p^s, which is a non-negative locally finite Borel measure on Ω.

As usual we denote

$$\mathrm{sign}(u) = \begin{cases} 1, & u > 0, \\ -1, & u < 0, \\ 0, & u = 0. \end{cases}$$

Now, we introduce a notion of entropy solution of (1).

Definition 1. A measurable function $u(x)$ on Ω is called an entropy solution of equation (1) if $\varphi_i(x,u(x)), b_{ij}(u(x)), \psi(x,u(x)) \in L^1_{\mathrm{loc}}(\Omega)$, $i,j = 1,\dots,n$, and for almost all $p \in \mathcal{P}$ the Kružkov-type entropy inequality (see [10])

$$\begin{aligned} &\mathrm{div}_x\,(\mathrm{sign}(u(x)-p)(\varphi(x,u(x)) - \varphi(x,p))) \\ &\quad - D^2 \cdot (\mathrm{sign}(u(x)-p)(B(u(x)) - B(p))) \\ &\quad + \mathrm{sign}(u(x)-p)[\omega_p(x) + \psi(x,u(x))] - |\gamma_p^s| \le 0 \end{aligned} \tag{6}$$

holds in the sense of distributions on Ω (in the space $\mathcal{D}'(\Omega)$); that is, for all non-negative functions $f(x) \in C_0^\infty(\Omega)$

$$\begin{aligned} \int_\Omega \mathrm{sign}(u(x)-p)\Big[&(\varphi(x,u(x)) - \varphi(x,p)) \cdot \nabla f(x) + (B(u(x)) - B(p)) \cdot D^2 f \\ &-(\omega_p(x) + \psi(x,u(x)))f(x)\Big]dx + \int_\Omega f(x)d|\gamma_p^s|(x) \ge 0. \end{aligned}$$

We use the notation $D^2 f$ for the matrix $\{\partial^2_{x_i x_j} f\}_{i,j=1}^n$ and

$$P \cdot Q = \mathrm{Tr} PQ = \sum_{i,j=1}^n p_{ij} q_{ij}$$

denotes scalar product of symmetric matrices $P = \{p_{ij}\}_{i,j=1}^n$, $Q = \{q_{ij}\}_{i,j=1}^n$. In particular,

$$(B(u(x)) - B(p)) \cdot D^2 f = (b_{ij}(u) - b_{ij}(p))\partial^2_{x_i x_j} f.$$

In the case when the second-order term is absent ($B(u) \equiv 0$) our definition extends the notion of the entropy solution for first-order balance laws introduced for the case of one space variable in [6, 8], see also [7] for one-dimensional degenerate convection-diffusion equations.

We also notice that we do not require $u(x)$ to be a weak solution of (1). If $u(x) \in L^\infty(\Omega)$ and $\gamma_p^s = 0$ for $p \in \mathcal{P}$, then any entropy solution $u(x)$ satisfies (1) in $\mathcal{D}'(\Omega)$, i.e., $u(x)$ is a weak solution of (1). Indeed, this follows from (6) with $p > \|u\|_\infty$ and $p < -\|u\|_\infty$. But in general, entropy solutions are not weak solutions, even in the case when the singular measures γ_p^s are absent. For instance, as is easily verified, $u(x) = \mathrm{sign}\, x\, |x|^{-1/2}$ is an entropy solution of the first-order equation $(xu^2)_x = 0$ on the line $\Omega = \mathbf{R}$, but it does not satisfy this equation in $\mathcal{D}'(\mathbf{R})$.

We assume that equation (1) is non-degenerate in the following sense:

Definition 2. Equation (1) is said to be *non-degenerate* if for almost all $x \in \Omega$ for all $\xi \in \mathbf{R}^n$, $\xi \neq 0$ the functions $\lambda \mapsto \xi \cdot \varphi(x, \lambda)$, $\lambda \mapsto B(\lambda)\xi \cdot \xi$ are not constant simultaneously on non-degenerate intervals.

In this paper, we establish the strong precompactness property for sequences of entropy solutions. This result generalizes previous results of [12, 13, 14, 15, 17] to the case of quasi-linear elliptic equations.

Theorem 3. *Suppose that u_k, $k \in \mathbf{N}$, is a sequence of entropy solutions of the non-degenerate equation* (1) *such that*

$$|\varphi(x, u_k(x))| + |\psi(x, u_k(x))| + |B(u_k(x))| + m(u_k(x))$$

is bounded in $L^1_{\mathrm{loc}}(\Omega)$, where $m(u)$ is a nonnegative super-linear function[1]. *Then there exists a subsequence of u_k, which converges in $L^1_{\mathrm{loc}}(\Omega)$ to an entropy solution $u(x)$ of* (1).

We use here and everywhere below the notation $|B|$ for the Euclidean norm of a symmetric matrix B, that is $|B|^2 = B \cdot B$.

More generally, we establish the strong precompactness of approximate sequences $u_k(x)$ for non-degenerate equation (1). The only assumption we need is that the sequence of distributions

$$\mathrm{div}_x \varphi(x, s_{a,b}(u_k(x))) - D^2 \cdot B(s_{a,b}(u_k(x)))$$

is precompact in the Sobolev space $W^{-1}_{d,\mathrm{loc}}(\Omega)$ for some $d > 1$, for each $a, b \in \mathbf{R}$, $a < b$ (see relation (78) below). Throughout this paper we use $s_{a,b}(u)$ to denote the cut-off function

$$s_{a,b}(u) = \max(a, \min(u, b)) = \begin{cases} a, & u < a, \\ u, & a \le u \le b, \\ b, & u > b. \end{cases}$$

Observe that the non-degeneracy condition is essential for the statement of Theorem 3. In the case of the equation $\mathrm{div}\varphi(u) - D^2 \cdot B(u) = 0$ this condition is necessary for strong precompactness property. For instance, if $\xi \cdot \varphi(u)$ and $B(u)\xi \cdot \xi$ are constant on the segment $[a, b]$ with $\xi \in \mathbf{R}^n$, $\xi \neq 0$ then the sequence $u_k(x) = [a + b + (b - a)\sin(k\xi \cdot x)]/2$ of entropy solutions does not contain strongly convergent subsequences.

We also stress that for sequences of weak solutions (without additional entropy constraints) the statement of Theorem 3 does not hold. For example, the sequence $u_k = \mathrm{sign} \sin kx$ consists of weak solutions for the Burgers equation $u_t + (u^2)_x = 0$ (as well as for the corresponding stationary equation $(u^2)_x = 0$) and converges only weakly, while the non-degeneracy condition is evidently satisfied.

Theorem 3 will be proved in the last section. The proof is based on general localization properties for ultra-parabolic H-measures corresponding to bounded sequences of measure-valued functions. It also follows from these properties the strong convergence of various approximate solutions for equation (1).

We describe below one useful approximation procedure. We assume for simplicity that $\psi(x, u) \equiv 0$, $b_{ij}(u) \in C^1(\mathbf{R})$, $i, j = 1, \ldots, n$. As shown in [17], there exists a sequence $\varphi_m(x, u) \in C^\infty(\Omega \times \mathbf{R})$ such that $\varphi_m(x, u) \underset{m\to\infty}{\to} \varphi(x, u)$ in $L^2_{\mathrm{loc}}(\Omega, C(\mathbf{R}, \mathbf{R}^n))$ while for each $p \in \mathcal{P}$, $\mathrm{div}_x \varphi_m(x, p) = \gamma^m_{pr}(x) + \gamma^m_{ps}(x)$, where $\gamma^m_{pr}(x) \underset{m\to\infty}{\to} \omega_p(x)$ in $L^1_{\mathrm{loc}}(\Omega)$, $|\gamma^m_{ps}(x)| \underset{m\to\infty}{\to} |\gamma^s_p|$ weakly in $M_{\mathrm{loc}}(\Omega)$.

[1] A nonnegative super-linear function m satisfies $m(u)/u \to \infty$ as $u \to \infty$.

By the ellipticity assumption $A(u) = B'(u) \geq 0$, we can choose a sequence of smooth symmetric matrices $A_m(u) = \{a_{ij}^m(u)\}_{i,j=1}^n$ such that $A_m \geq \varepsilon_m I$, $\varepsilon_m > 0$ (here I is the identity matrix), and for each $M > 0$

$$\varepsilon_m^{-1/2} \max_{|u| \leq M} |A_m(u) - A(u)| \underset{m\to\infty}{\to} 0.$$

Then we have the limit relation

$$|(A_m(u) - A(u))(A_m(u))^{-1/2}| \to 0 \text{ in } C(\Omega).$$

Moreover, passing to a subsequence of A_m if necessary, we may achieve that for each $M > 0$ and every compact $K \subset \Omega$

$$\max_{|u| \leq M} |(A_m(u) - A(u))(A_m(u))^{-1/2}| \underset{m\to\infty}{=} o\left(I_m(K, M+1)^{-1/2}\right), \tag{7}$$

where

$$I_m(K, M) = 1 + \int_K \int_{-M}^{M} |\mathrm{div}_x \varphi_m(x,p)| dp dx.$$

In general, the sequence $I_m(K, M)$ may tend to infinity as $m \to \infty$. We consider the approximate equation

$$\mathrm{div}_x[\varphi_m(x,u) - A_m(u)\nabla u] = 0 \tag{8}$$

and suppose that $u = u_m(x)$ is a bounded weak solution of (8) (for instance, we can take $u = u_m(x)$ being a weak solution to the Dirichlet problem with a bounded data at $\partial\Omega$). This means (see [11, Chapter 4]) that $u \in L^\infty(\Omega) \cap W^1_{2,\mathrm{loc}}(\Omega)$, where $W^1_{2,\mathrm{loc}}(\Omega)$ is the Sobolev space consisting of functions whose generalized derivatives are in $L^2_{\mathrm{loc}}(\Omega)$, and the following standard integral identity is satisfied: For all $f = f(x) \in C^1_0(\Omega)$ we have

$$\int_\Omega [\varphi_m(x, u(x)) - A_m(u)\nabla u(x)] \cdot \nabla f(x) dx = 0. \tag{9}$$

We also assume that the sequence u_m is bounded in $L^\infty(\Omega)$. Under the above assumptions we establish the strong convergence of the approximations.

Theorem 4. *Suppose that equation* (1) *is non-degenerate. Then the sequence* $u_m(x) \underset{m\to\infty}{\to} u(x)$ *in* $L^1_{\mathrm{loc}}(\Omega)$, *where* $u = u(x)$ *is an entropy and a distributional solution of* (1).

We remark that Theorem 4 allows to establish the existence of entropy solutions of boundary value problems for equation (1) (as well as initial or initial boundary value problems for evolutionary equations of the kind (1)).

For example, in [17] we use approximations and the strong precompactness property in order to prove the existence of entropy solutions to the Cauchy problem for an evolutionary hyperbolic equation with discontinuous multidimensional flux. This extends results of [9], where the two-dimensional case is treated by the compensated compactness method.

We also remark that another approach to prove the strong precompactness property for equation (1) based on the kinetic formulation and averaging lemmas was developed in [21]. But this approach can be applied only when the flux $\varphi = \varphi(u)$ does not depend on $x \in \mathbf{R}^n$, and when the flux vector as well as the diffusion matrix are sufficiently regular.

In Sections 2, 3 we describe the main concepts, in particular the concept of measure-valued functions, and introduce a notion of the H-measure. Most of the statements in the sections are taken from [16]. For completeness we also reproduce the proofs of these statements. In [16] we considered the strong pre-compactness property for the general ultra-parabolic equation

$$\mathrm{div}\varphi(x,u) - D^2 \cdot B(x,u) + \psi(x,u) = 0, \tag{10}$$

where it is assumed that $B(x,u)$ is a Caratheodory matrix-valued function, which satisfies the ellipticity condition $\mathrm{sign}(u_1-u_2)(B(x,u_1)-B(x,u_2)) \ge 0$, and degenerates on a fixed subspace X (that is, $X \subset \ker(B(x,u)-B(x,u_0))$).

We have more complicated situation in (1) since the diffusion matrix $B = B(u)$, $u \in \mathbf{R}$, degenerates on a subspace $X = X(u)$ depending on $u \in \mathbf{R}$. Still, since the matrix $B = B(p)$, $p \in \mathbf{R}$, is continuous, we will be able to reduce our investigation on the behavior of the corresponding H-measure in a neighborhood of a fixed point $p_0 \in \mathbf{R}$ (see the statement of Theorem 25). Therefore, we will be able to use techniques from [16] (of course, in a rather nontrivial manner).

Observe that results analogous to Theorems 3 and 4 were proved in [16] for equation (10) under the stronger non-degeneracy assumption:

For almost all $x \in \Omega$ and for all $\tilde{\xi} \in X$, $\bar{\xi} \in X^\perp$ such that $\tilde{\xi} \ne 0$, $\bar{\xi} \ne 0$, the functions $\lambda \mapsto \tilde{\xi} \cdot \varphi(x,\lambda)$, $\lambda \mapsto B(x,\lambda)\bar{\xi} \cdot \bar{\xi}$ are not constant on non-degenerate intervals.

Here $X^\perp$ denotes the orthogonal complement to the subspace X.

In Section 4 we prove the localization property for the above defined H-measures corresponding to sequences of measure-valued functions. Finally, in the last Section 5, these results are applied to prove our main theorems.

2. Main concepts

Recall (see [2, 3, 22]) that a *measure-valued* function on Ω is a weakly measurable map $x \mapsto \nu_x$ of the set Ω into the space of probability Borel measures with compact support in $\mathbf{R}$. The weak measurability of ν_x means that for each continuous function $f(\lambda)$ the function $x \mapsto \int f(\lambda) d\nu_x(\lambda)$ is Lebesgue measurable on Ω.

Remark 5. If ν_x is a measure-valued function, then, as was shown in [13], the functions $\int g(\lambda) d\nu_x(\lambda)$ are measurable in Ω for all bounded Borel functions $g(\lambda)$. More generally, if $f(x,\lambda)$ is a Caratheodory function and $g(\lambda)$ is a bounded Borel function then the function $\int f(x,\lambda) g(\lambda) d\nu_x(\lambda)$ is measurable. This follows from the fact that any Caratheodory function is strongly measurable as a map $x \mapsto f(x,\cdot) \in C(\mathbf{R})$ (see [5, Ch. 2]) and, therefore, is a pointwise limit of step functions $f_m(x,\lambda) = \sum_i g_{mi}(x) h_{mi}(\lambda)$ with measurable functions $g_{mi}(x)$ and continuous $h_{mi}(\lambda)$ so that for $x \in \Omega$ $f_m(x,\cdot) \underset{m\to\infty}{\to} f(x,\cdot)$ in $C(\mathbf{R})$.

A measure-valued function ν_x is said to be bounded if there exists $M > 0$ such that $\mathrm{supp}\,\nu_x \subset [-M, M]$ for almost all $x \in \Omega$. We denote by $\|\nu_x\|_\infty$ the smallest value of M with this property.

Finally, measure-valued functions of the form $\nu_x(\lambda) = \delta(\lambda - u(x))$, where $\delta(\lambda - u)$ is the Dirac measure concentrated at u are said to be *regular*; we identify them with the corresponding functions $u(x)$. Thus, the set $MV(\Omega)$ of bounded measure-valued functions on Ω contains the space $L^\infty(\Omega)$. Note that for a regular measure-valued function $\nu_x(\lambda) = \delta(\lambda - u(x))$ the value $\|\nu_x\|_\infty = \|u\|_\infty$. Extending the concept of boundedness in $L^\infty(\Omega)$ to measure-valued functions, we shall say that a subset A of $MV(\Omega)$ is *bounded* if $\sup_{\nu_x \in A} \|\nu_x\|_\infty < \infty$.

Below we define the weak and the strong convergence of sequences of measure-valued functions.

Definition 6. Let $\nu_x^k \in MV(\Omega)$, $k \in \mathbf{N}$, and let $\nu_x \in MV(\Omega)$. Then

(1) the sequence ν_x^k converges weakly to ν_x if for each $f(\lambda) \in C(\mathbf{R})$,

$$\int f(\lambda) d\nu_x^k(\lambda) \underset{k\to\infty}{\to} \int f(\lambda) d\nu_x(\lambda) \text{ weak star in } L^\infty(\Omega);$$

(2) the sequence ν_x^k converges to ν_x *strongly* if for each $f(\lambda) \in C(\mathbf{R})$,

$$\int f(\lambda) d\nu_x^k(\lambda) \underset{k\to\infty}{\to} \int f(\lambda) d\nu_x(\lambda) \text{ in } L^1_{\mathrm{loc}}(\Omega).$$

The next result was proved in [22] for regular functions ν_x^k. The proof can be easily extended to the general case, as was done in [13].

Theorem T. *Let ν_x^k, $k \in \mathbf{N}$, be a bounded sequence of measure-valued functions. Then there exist a subsequence $\nu_x^r = \nu_x^k$, $k = k_r$, and a measure-valued function $\nu_x \in MV(\Omega)$ such that $\nu_x^r \to \nu_x$ weakly as $r \to \infty$.*

Theorem T shows that bounded sets of measure-valued functions are weakly precompact. If $u_k(x) \in L^\infty(\Omega)$ is a bounded sequence, treated as a sequence of regular measure-valued functions, and $u_k(x)$ converges weakly to a measure-valued function ν_x then ν_x is regular, $\nu_x(\lambda) = \delta(\lambda - u(x))$, if and only if $u_k(x) \to u(x)$ in $L^1_{\mathrm{loc}}(\Omega)$ (see [22]). Obviously, if $u_k(x)$ converges to ν_x strongly then $u_k(x) \to u(x) = \int \lambda d\nu_x(\lambda)$ in $L^1_{\mathrm{loc}}(\Omega)$ and then $\nu_x(\lambda) = \delta(\lambda - u(x))$.

We shall study the strong precompactness property using Tartar's technique of H-measures.

Let

$$F(u)(\xi) = \int e^{-2\pi i \xi\cdot x} u(x) dx, \qquad \xi \in \mathbf{R}^n,$$

be the Fourier transform extended as unitary operator on the space $u(x) \in L^2(\mathbf{R}^n)$, and let $S = S^{n-1} = \{\xi \in \mathbf{R}^n : |\xi| = 1\}$ be the unit sphere in $\mathbf{R}^n$. Denote complex conjugation of $u \in \mathbf{C}$ by $\overline{u}$.

The concept of H-measure associated to a sequence of vector-valued functions bounded in $L^2(\Omega)$ was introduced by Tartar [23] and Gerárd [4] on the basis of the following result. For a fixed $l \in \mathbf{N}$, let $U_k(x) = \big(U_k^1(x), \dots, U_k^l(x)\big) \in L^2(\Omega, \mathbf{R}^l)$ be a sequence weakly convergent to the zero vector as $k \to \infty$.

Proposition 7 ([23, Thm. 1.1])**.** *There is a family of complex Borel measures $\mu = \left\{\mu^{ij}\right\}_{i,j=1}^l$ on $\Omega \times S$ and a subsequence $U_r(x) = U_k(x)$, $k = k_r$, such that*

$$\langle \mu^{ij}, \Phi_1(x)\overline{\Phi_2(x)}\psi(\xi)\rangle = \lim_{r\to\infty} \int_{\mathbf{R}^n} F(U_r^i \Phi_1)(\xi)\overline{F(U_r^j\Phi_2)(\xi)}\psi\left(\frac{\xi}{|\xi|}\right) d\xi, \tag{11}$$

for all $\Phi_1(x), \Phi_2(x) \in C_0(\Omega)$ and $\psi(\xi) \in C(S)$.

The family $\mu = \left\{\mu^{ij}\right\}_{i,j=1}^l$ is called the H-measure associated to $U_r(x)$.

Here, we shall need more general variant of the H measures developed in [16] and based on the concept of the parabolic H-measures recently introduced in [1].

Suppose that $X \subset \mathbf{R}^n$ is a linear subspace, $X^\perp$ is its orthogonal complement, P_1, P_2 are orthogonal projections on X, $X^\perp$, respectively. For $\xi \in \mathbf{R}^n$, we write

$$\tilde{\xi} = P_1\xi, \quad \bar{\xi} = P_2\xi,$$

so that $\tilde{\xi} \in X$, $\bar{\xi} \in X^\perp$, $\xi = \tilde{\xi} + \bar{\xi}$. Let

$$S_X = \left\{\xi \in \mathbf{R}^n : |\tilde{\xi}|^2 + |\bar{\xi}|^4 = 1\right\}.$$

Then S_X is a compact smooth manifold of codimension 1. In the case when $X = \{0\}$ or $X = \mathbf{R}^n$ it coincides with the unit sphere $S = \{\xi \in \mathbf{R}^n : |\xi| = 1\}$. Let us define the projection $\pi_X : \mathbf{R}^n \setminus \{0\} \to S_X$ by

$$\pi_X(\xi) = \frac{\tilde{\xi}}{(|\tilde{\xi}|^2 + |\bar{\xi}|^4)^{1/2}} + \frac{\bar{\xi}}{(|\tilde{\xi}|^2 + |\bar{\xi}|^4)^{1/4}}.$$

Observe that in the case when $X = \{0\}$ or $X = \mathbf{R}^n$,

$$\pi_X(\xi) = \xi/|\xi|.$$

We denote

$$p(\xi) = \left(|\tilde{\xi}|^2 + |\bar{\xi}|^4\right)^{1/4}.$$

The following useful property of the projection holds:

Lemma 8 ([16, Lemma 1])**.** *Let* $\xi, \eta \in \mathbf{R}^n$, $\max(p(\xi), p(\eta)) \geq 1$. *Then*

$$|\pi_X(\xi) - \pi_X(\eta)| \leq \frac{6|\xi - \eta|}{\max(p(\xi), p(\eta))}.$$

Proof. We define for $\xi \in \mathbf{R}^n$, $\alpha > 0$ $\xi_\alpha = \alpha^2\tilde{\xi} + \alpha\bar{\xi}$. Observe that for all $\alpha > 0$ $\pi_X(\xi_\alpha) = \pi_X(\xi)$. Without lose of generality we may suppose that $p(\xi) \geq p(\eta)$, and, in particular, $p(\xi) \geq 1$. Remark that $\pi_X(\xi) = \xi_\alpha$, $\pi_X(\eta) = \eta_\beta$, where $\alpha = 1/p(\xi)$, $\beta = 1/p(\eta)$. Therefore,

$$\begin{aligned}
|\pi_X(\xi) - \pi_X(\eta)| &= |\xi_\alpha - \eta_\beta| \\
&\leq |\xi_\alpha - \eta_\alpha| + |\eta_\alpha - \eta_\beta| \\
&= \left(\alpha^4|\tilde{\xi} - \tilde{\eta}|^2 + \alpha^2|\bar{\xi} - \bar{\eta}|^2\right)^{1/2} + \left((\beta^2 - \alpha^2)^2|\tilde{\eta}|^2 + (\beta - \alpha)^2|\bar{\eta}|^2\right)^{1/2} \\
&\leq \alpha|\xi - \eta| + (\beta - \alpha)\left((\beta + \alpha)^2|\tilde{\eta}|^2 + |\bar{\eta}|^2\right)^{1/2}.
\end{aligned} \tag{12}$$

Here we take into account that $\alpha \leq 1$ and therefore $\alpha^4 \leq \alpha^2$. Since

$$(\beta + \alpha)^2 \leq 4\beta^2 = 4(|\tilde{\eta}|^2 + |\bar{\eta}|^4)^{-1/2} \leq 4/|\tilde{\eta}|,$$

we have the estimate

$$(\beta + \alpha)^2|\tilde{\eta}|^2 + |\bar{\eta}|^2 \leq 4(|\tilde{\eta}| + |\bar{\eta}|^2) \leq 4\left(2(|\tilde{\eta}|^2 + |\bar{\eta}|^4)\right)^{1/2} \leq 6(p(\eta))^2. \tag{13}$$

Concerning the term $\beta - \alpha$, we estimate it as follows

$$\begin{aligned}
\beta - \alpha &= \frac{p(\xi) - p(\eta)}{p(\xi)p(\eta)} \\
&= \frac{|\tilde{\xi}|^2 - |\tilde{\eta}|^2 + |\bar{\xi}|^4 - |\bar{\eta}|^4}{p(\xi)p(\eta)(p(\xi) + p(\eta))((p(\xi))^2 + (p(\eta))^2)} \\
&\leq \frac{(|\tilde{\xi}| + |\tilde{\eta}|)|\tilde{\xi} - \tilde{\eta}| + (|\bar{\xi}| + |\bar{\eta}|)(|\bar{\xi}|^2 + |\bar{\eta}|^2)|\bar{\xi} - \bar{\eta}|}{p(\xi)p(\eta)(p(\xi) + p(\eta))((p(\xi))^2 + (p(\eta))^2)} \\
&\leq \frac{|\tilde{\xi}| + |\tilde{\eta}| + (|\bar{\xi}| + |\bar{\eta}|)(|\bar{\xi}|^2 + |\bar{\eta}|^2)}{p(\xi)p(\eta)(p(\xi) + p(\eta))((p(\xi))^2 + (p(\eta))^2)}|\xi - \eta| \\
&\leq \frac{(p(\xi))^2 + (p(\eta))^2 + (p(\xi) + p(\eta))((p(\xi))^2 + (p(\eta))^2)}{p(\xi)p(\eta)(p(\xi) + p(\eta))((p(\xi))^2 + (p(\eta))^2)}|\xi - \eta| \\
&\leq \frac{1 + p(\xi) + p(\eta)}{p(\xi) + p(\eta)}\frac{|\xi - \eta|}{p(\xi)p(\eta)} \\
&\leq \frac{2|\xi - \eta|}{p(\xi)p(\eta)}.
\end{aligned} \tag{14}$$

Here we use that $\tilde{\xi} \leq (p(\xi))^2$, $\bar{\xi} \leq p(\xi)$, $\tilde{\eta} \leq (p(\eta))^2$, $\bar{\eta} \leq p(\eta)$, and that $p(\xi)+p(\eta) \geq 1$. Now it follows from (12), (13), (14) that

$$|\pi_X(\xi) - \pi_X(\eta)| \leq \frac{|\xi - \eta|}{p(\xi)} + \frac{2\sqrt{6}|\xi - \eta|}{p(\xi)} \leq \frac{6|\xi - \eta|}{p(\xi)} = \frac{6|\xi - \eta|}{\max(p(\xi), p(\eta))},$$

as was to be proved. □

Let $b(x) \in C_0(\mathbf{R}^n)$, $a(z) \in C(S_X)$. Then we can define pseudo-differential operators $\mathcal{B}$, $\mathcal{A}$ with symbols $b(x)$, $a(\pi_X(\xi))$, respectively. These operators are multiplication operators

$$\mathcal{B}u(x) = b(x)u(x), \quad F(\mathcal{A}u)(\xi) = a(\pi_X(\xi))F(u)(\xi).$$

Obviously, the operators $\mathcal{B}, \mathcal{A}$ are welldefined and bounded in L^2. As was proved in [23], in the case when $S_X = S$, $\pi_X(\xi) = \xi/|\xi|$ the commutator $[\mathcal{A}, \mathcal{B}] = \mathcal{A}\mathcal{B} - \mathcal{B}\mathcal{A}$ is a compact operator. Using the assertion of Lemma 8 one can easily extend this result to the general case (when $\dim X = 1$ this was done in [1]). For completeness we give below the details for the general setting.

Lemma 9 ([16, Lemma 2])**.** *The operator $[\mathcal{A}, \mathcal{B}]$ is compact in L^2.*

Proof. We can find sequences $a_k(z) \in C^\infty(S_X)$, $b_k(x) \in C^\infty(\mathbf{R}^n)$, $k \in \mathbf{N}$, of symbols with the following properties: $F(b_k)(\xi) \in C_0^\infty(\mathbf{R}^n)$ and, as $k \to \infty$, $a_k(z) \to a(z)$, $b_k(x) \to b(x)$ uniformly on S_X, $\mathbf{R}^n$, respectively. Then the sequences of the operators $\mathcal{A}_k$, $\mathcal{B}_k$ with symbols $a_k(\pi_X(\xi))$, $b_k(x)$ converge as $k \to \infty$ to the operators $\mathcal{A}$, $\mathcal{B}$, respectively (in the operator norm). Therefore, $[\mathcal{A}_k, \mathcal{B}_k] \to [\mathcal{A}, \mathcal{B}]$ and it is sufficient to prove that the operators $[\mathcal{A}_k, \mathcal{B}_k]$ are compact for all $k \in \mathbf{N}$ (then $[\mathcal{A}, \mathcal{B}]$ is also compact operator as a limit of compact operators).

Let $u = u(x) \in L^2(\mathbf{R}^n)$. Then by the well-known property

$$F(bu)(\xi) = F(b) * F(u)(\xi) = \int F(b)(\xi - \eta)F(u)(\eta)d\eta,$$

we obtain

$$\begin{aligned} F([\mathcal{A}_k, \mathcal{B}_k]u)(\xi) &= F(\mathcal{A}_k\mathcal{B}_k u)(\xi) - F(\mathcal{B}_k\mathcal{A}_k u)(\xi) \\ &= a_k(\pi_X(\xi))F(b_k u)(\xi) - F(b_k\mathcal{A}_k u)(\xi) \\ &= \int_{\mathbf{R}^n} (a_k(\pi_X(\xi)) - a_k(\pi_X(\eta)))F(b_k)(\xi - \eta)F(u)(\eta)d\eta. \end{aligned}$$

We have to prove that the integral operator $Kv(\xi) = \int_{\mathbf{R}^n} k(\xi, \eta)v(\eta)d\eta$ with the kernel $k(\xi, \eta) = (a_k(\pi_X(\xi)) - a_k(\pi_X(\eta)))F(b_k)(\xi - \eta)$ is compact on $L^2(\mathbf{R}^n)$.

Since $a_k \in C^\infty(S_X)$, Lemma 8 implies

$$|a_k(\pi_X(\xi)) - a_k(\pi_X(\eta))| \leq C\frac{|\xi - \eta|}{\max(p(\xi), p(\eta))},$$

for $\max(p(\xi), p(\eta)) \geq 1$. Thus for all $\xi, \eta \in \mathbf{R}^n$ such that $\max(p(\xi), p(\eta)) > m > 1$ and $\xi - \eta \in \operatorname{supp} F(b_k)$

$$|a_k(\pi_X(\xi)) - a_k(\pi_X(\eta))| \leq \frac{C}{m}|\xi - \eta|. \tag{15}$$

Let $\chi_m(\xi, \eta)$ be the indicator function of $\big\{(\xi, \eta) \in \mathbf{R}^{2n} : \max(p(\xi), p(\eta)) \leq m\big\}$, and

$$\begin{aligned} k_m(\xi, \eta) &= \chi(\xi, \eta)\,(a_k(\pi_X(\xi)) - a_k(\pi_X(\eta)))\,F(b_k)(\xi - \eta), \\ r_m(\xi, \eta) &= (1 - \chi(\xi, \eta))\,(a_k(\pi_X(\xi)) - a_k(\pi_X(\eta)))\,F(b_k)(\xi - \eta). \end{aligned}$$

Then $k(\xi, \eta) = k_m(\xi, \eta) + r_m(\xi, \eta)$ and $K = K_m + R_m$, where K_m, R_m are integral operators with the kernels $k_m(\xi, \eta)$, $r_m(\xi, \eta)$, respectively. Since the function $k_m(\xi, \eta)$ is bounded and compactly supported, the operator K_m is a Hilbert–Schmidt operator, which is compact. On the other hand, in view of (15),

$$|R_m v(\xi)| \leq \frac{C}{m}\int_{\mathbf{R}^n} |(\xi - \eta)F(b_k)(\xi - \eta)||v(\eta)|d\eta = \frac{C}{m}[|\xi F(b_k)| * |v|](\xi)$$

and, by Young's inequality,

$$\|R_m v\|_2 \le \frac{C}{m}\|\xi F(b_k)\|_1 \|v\|_2, \quad v \in L^2(\mathbf{R}^n).$$

Therefore, $\|R_m\| \le \mathrm{const}/m$ and $R_m \to 0$ as $m \to \infty$. We conclude that $K_m \to K$ and therefore K is a compact operator, as a limit of compact operators. This completes the proof. □

Now we fix a space $X \subset \mathbf{R}^n$. An ultra-parabolic H-measure μ^{ij}, $i,j = 1,\dots,l$, corresponding to a sequence $U_r(x) \in L^2(\Omega, \mathbf{R}^l)$ is defined on $\Omega \times S_X$ by the relation similar to (11), namely, for all $\Phi_1(x), \Phi_2(x) \in C_0(\Omega)$, $\psi(\xi) \in C(S_X)$,

$$\left\langle \mu^{ij}, \Phi_1(x)\overline{\Phi_2(x)}\psi(\xi)\right\rangle = \lim_{r\to\infty}\int_{\mathbf{R}^n} F(\Phi_1 U_r^i)(\xi)\overline{F(\Phi_2 U_r^j)(\xi)}\psi(\pi_X(\xi))d\xi. \qquad (16)$$

The existence of an H-measure μ^{ij} is proved exactly in the same way as in [23], using Lemma 2. This H-measure satisfies the same properties as the "usual" H-measure μ^{pq} (corresponding to the case $X = \{0\}$ or $X = \mathbf{R}^n$).

The concept of an H-measure was extended in [13] (see also [14, 15]) to sequences of measure-valued functions. A similar extension can be provided for ultra-parabolic H-measures. We study the properties of such H-measures in the next section.

3. Ultra-parabolic H-measures corresponding to bounded sequences of measure-valued functions

Let $\nu_x^k \in MV(\Omega)$ be a bounded sequence of measure-valued functions weakly convergent to a measure-valued function $\nu_x^0 \in MV(\Omega)$. For $x \in \Omega$ and $p \in \mathbf{R}$ we introduce the distribution functions

$$u_k(x,p) = \nu_x^k((p,+\infty)), \quad u_0(x,p) = \nu_x^0((p,+\infty)).$$

Then, as mentioned in Remark 5, for $k \in \mathbf{N}\cup\{0\}$ and $p \in \mathbf{R}$, the functions $u_k(x,p)$ are measurable in $x \in \Omega$; thus, $u_k(x,p) \in L^\infty(\Omega)$ and $0 \le u_k(x,p) \le 1$.

Let

$$E = E(\nu_x^0) = \left\{p_0 \in \mathbf{R} : u_0(x,p) \underset{p\to p_0}{\to} u_0(x,p_0) \text{ in } L^1_{\mathrm{loc}}(\Omega)\right\}.$$

We have the following result, whose proof can be found in [13].

Lemma 10. *The complement $\bar{E} = \mathbf{R}\setminus E$ is at most countable and if $p \in E$ then $u_k(x,p) \underset{k\to\infty}{\to} u_0(x,p)$ weak star in $L^\infty(\Omega)$.*

By Lemma 10, as $k \to \infty$,

$$U_k^p(x) := u_k(x,p) - u_0(x,p) \to 0 \quad \text{weak star in } L^\infty(\Omega), \text{ for } p \in E.$$

Let X be a linear subspace of $\mathbf{R}^n$. The next result, similar to Proposition 7, was also established in [13] in the case $X = \mathbf{R}^n$. The general case of arbitrary X was proved exactly in the same way.

Proposition 11. *(1) There exists a family of locally finite complex Borel measures $\{\mu^{pq}\}_{p,q\in E}$ in $\Omega \times S_X$ and a subsequence $U_r(x) = \{U_r^p(x)\}_{p\in E}$, $U_r^p(x) = U_k^p(x)$, $k = k_r$, such that for all $\Phi_1(x), \Phi_2(x) \in C_0(\Omega)$, $\psi(\xi) \in C(S_X)$,*

$$\left\langle \mu^{pq}, \Phi_1(x)\overline{\Phi_2(x)}\psi(\xi)\right\rangle = \lim_{r\to\infty}\int_{\mathbf{R}^n} F(\Phi_1 U_r^p)(\xi)\overline{F(\Phi_2 U_r^q)(\xi)}\psi(\pi_X(\xi))d\xi. \qquad (17)$$

(2) The correspondence $(p,q) \mapsto \mu^{pq}$ is a continuous map from $E \times E$ into the space $\mathrm{M}_{\mathrm{loc}}(\Omega \times S)$.

We call the family of measures $\{\mu^{pq}\}_{p,q\in E}$ the ultra-parabolic H-measure corresponding to the subsequence $\nu_x^r = \nu_x^k$, $k = k_r$.

Remark 12. We can replace the function $\psi(\pi_X(\xi))$ in relation (17) (and in (16)) by a function $\tilde{\psi}(\xi) \in C(\mathbf{R}^n)$, which equals $\psi(\pi_X(\xi))$ for large $|\xi|$. Indeed, since $U_r^q \underset{r\to\infty}{\to} 0$ weak star in $L^\infty(\Omega)$, we have $F(\Phi_2 U_r^q)(\xi) \underset{r\to\infty}{\to} 0$ pointwise and in $L^2_{\rm loc}(\mathbf{R}^n)$ (in view of the bound $|F(\Phi_2 U_r^q)(\xi)| \le \|\Phi_2 U_r^q\|_1 \le$ const). Taking into account that $\chi(\xi) = \tilde{\psi}(\xi) - \psi(\pi_X(\xi))$ is bounded and has a compact support, we conclude

$$\overline{F(\Phi_2 U_r^q)(\xi)}\chi(\xi) \underset{r\to\infty}{\to} 0 \ \text{ in } L^2(\mathbf{R}^n).$$

This implies that

$$\lim_{r\to\infty} \int_{\mathbf{R}^n} F(\Phi_1 U_r^p)(\xi)\overline{F(\Phi_2 U_r^q)(\xi)}\chi(\xi)d\xi = 0.$$

Therefore

$$\begin{aligned}
&\lim_{r\to\infty} \int_{\mathbf{R}^n} F(\Phi_1 U_r^p)(\xi)\overline{F(\Phi_2 U_r^q)(\xi)}\tilde{\psi}(\xi)d\xi \\
&\quad = \lim_{r\to\infty} \int_{\mathbf{R}^n} F(\Phi_1 U_r^p)(\xi)\overline{F(\Phi_2 U_r^q)(\xi)}\psi(\pi_X(\xi))d\xi = \left\langle \mu^{pq}, \Phi_1(x)\overline{\Phi_2(x)}\psi(\xi)\right\rangle,
\end{aligned}$$

as required.

We point out the following important properties of an H-measure.

Lemma 13 ([16, Lemma 4])**.** *(i)* $\mu^{pp} \ge 0$ *for each* $p \in E$;
(ii) $\mu^{pq} = \overline{\mu^{qp}}$ *for all* $p, q \in E$;
(iii) for $p_1, \dots, p_l \in E$, $g_1, \dots, g_l \in C_0(\Omega \times S_X)$, *the matrix* A *with components* $a_{ij} = \langle \mu^{p_i p_j}, g_i \overline{g_j}\rangle$, $i, j = 1, \dots, l$, *is Hermitian and positive-definite.*

Proof. We begin by proving (iii). First, let the functions $g_i = g_i(x,\xi)$ be finite sums of functions of the form $\Phi(x)\psi(\xi)$, where $\Phi(x) \in C_0(\Omega)$ and $\psi(\xi) \in C(S_X)$. Then it follows from (17) that

$$a_{ij} = \lim_{r\to\infty} \int_{\mathbf{R}^n} H_r^i(\xi)\overline{H_r^j(\xi)}d\xi, \tag{18}$$

where $H_r^i(\xi) = F(g_i(\,\cdot\,, \pi_X(\xi))U_r^{p_i})(\xi)$. Hence, setting

$$g_i(x,\xi) = g(x,\xi) = \sum_{k=1}^m \Phi_k(x)\psi_k(\xi),$$

we obtain

$$H_r^i(\xi) = \sum_{k=1}^m F(\Phi_k U_r^{p_i})(\xi)\psi_k(\pi_X(\xi)).$$

It immediately follows from (18) that $a_{ji} = \overline{a_{ij}}$, $i, j = 1, \dots, l$, which shows that A is a Hermitian matrix. Furthermore, for $\alpha_1, \dots, \alpha_l \in \mathbf{C}$, we have

$$\sum_{i,j=1}^l a_{ij}\alpha_i\overline{\alpha_j} = \lim_{r\to\infty} \int_{\mathbf{R}^n} |H_r(\xi)|^2 d\xi \ge 0, \quad H_r(\xi) = \sum_{i=1}^l H_r^i(\xi)\alpha_i,$$

which means that A is positive-definite.

In the general case when $g_i \in C_0(\Omega \times S_X)$, one carries out the proof of (iii) by approximating the functions g_i, $i = 1, \dots, l$, in the uniform norm by finite sums of functions of the form $\Phi(x)\psi(\xi)$.

Assertions (i) and (ii) are easy consequences of (iii). Indeed, setting $l = 1$, $p_1 = p$ and $g_1 = g$, we obtain the relation $\langle \mu^{pp}, |g|^2\rangle \ge 0$, which holds for all $g \in C_0(\Omega \times S_X)$, thus showing that μ^{pp} is real and non-negative. To prove (ii) we represent an arbitrary function $g = g(x,\xi)$ with compact support in the form $g = g_1\overline{g_2}$. Let $l = 2$, $p_1 = p$ and $p_2 = q$. In view of (iii),

$$\langle \mu^{pq}, g\rangle = \langle \mu^{pq}, g_1\overline{g_2}\rangle = \overline{\langle \mu^{qp}, g_2\overline{g_1}\rangle} = \overline{\langle \mu^{qp}, \overline{g}\rangle} = \langle \overline{\mu^{qp}}, g\rangle$$

and $\mu^{pq} = \overline{\mu^{qp}}$. The proof is complete. □

We consider now a countable dense index subset $D \subset E$.

Proposition 14 ([16, Proposition 3])**.** *There exists a family of complex finite Borel measures μ_x^{pq} on S_X with $p,q \in D$, $x \in \Omega'$, where Ω' is a subset of Ω of full measure, such that $\mu^{pq} = \mu_x^{pq}dx$, that is, for all $\Phi(x,\xi) \in C_0(\Omega \times S_X)$ the function*

$$x \mapsto \langle \mu_x^{pq}(\xi), \Phi(x,\xi)\rangle = \int_{S_X} \Phi(x,\xi)d\mu_x^{pq}(\xi)$$

is Lebesgue measurable on Ω, bounded, and

$$\langle \mu^{pq}, \Phi(x,\xi)\rangle = \int_\Omega \langle \mu_x^{pq}(\xi), \Phi(x,\xi)\rangle dx.$$

Moreover, $\operatorname{Var}\mu_x^{pq} \le 1$ *for all $p,q \in D$.*

Proof. We claim that $\operatorname{pr}_\Omega \operatorname{Var}\mu^{pq} \le \operatorname{meas}$ for $p,q \in E$, where meas is the Lebesgue measure on Ω. Assume first that $p = q$. By Lemma 13, the measure μ^{pp} is nonnegative. Next, in view of (17) with $\Phi_1(x) = \Phi_2(x) = \Phi(x) \in C_0(\Omega)$ and $\psi(\xi) \equiv 1$,

$$\begin{aligned}\left\langle \mu^{pp}, |\Phi(x)|^2\right\rangle &= \lim_{r\to\infty} \int_{\mathbf{R}^n} F(\Phi U_r^p)(\xi)\overline{F(\Phi U_r^p)(\xi)}d\xi \\ &= \lim_{r\to\infty} \int_\Omega |U_r^p(x)|^2|\Phi(x)|^2dx \le \int_\Omega |\Phi(x)|^2dx\end{aligned}$$

(we use here Plancherel's equality and the estimate $|U_r^p(x)| \le 1$). Thus, we see that that $\operatorname{pr}_\Omega\mu^{pp} \le \operatorname{meas}$.

Let $p,q \in E$, A be a bounded open subset of Ω, and $g = g(x,\xi) \in C_0(A \times S_X)$, $|g| \le 1$. Let also $g_1 = g/\sqrt{|g|}$ (we set $g_1 = 0$ for $g = 0$) and $g_2 = \sqrt{|g|}$. Then $g_1, g_2 \in C_0(A \times S_X)$, $g = g_1\overline{g_2}$, $|g_1|^2 = |g_2|^2 = |g|$, and the matrix

$$\begin{pmatrix} \langle \mu^{pp}, |g|\rangle & \langle \mu^{pq}, g\rangle \\ \overline{\langle \mu^{pq}, g\rangle} & \langle \mu^{qq}, |g|\rangle \end{pmatrix}$$

is positive-definite by Lemma 13; in particular,

$$|\langle \mu^{pq}, g\rangle| \le (\langle \mu^{pp}, |g|\rangle \langle \mu^{qq}, |g|\rangle)^{1/2} \le (\mu^{pp}(A \times S_X)\mu^{qq}(A \times S_X))^{1/2} \le \operatorname{meas}(A).$$

We take into account the inequalities $\operatorname{pr}_\Omega\mu^{pp} \le \operatorname{meas}$ and $\operatorname{pr}_\Omega\mu^{qq} \le \operatorname{meas}$ to obtain the last estimate. Since g can be an arbitrary function in $C_0(A \times S_X)$, $|g| \le 1$, we obtain the inequality $\operatorname{Var}\mu^{pq}(A \times S_X) \le \operatorname{meas}(A)$. The measure μ^{pq} is regular, therefore this estimate holds for all Borel subsets A of Ω and

$$\operatorname{pr}_\Omega \operatorname{Var}\mu^{pq} \le \operatorname{meas}. \tag{19}$$

It follows from (19) that for all $\psi(\xi) \in C(S_X)$ we have

$$\operatorname{Var}\operatorname{pr}_\Omega(\psi(\xi)\mu^{pq}(x,\xi)) \le \|\psi\|_\infty \operatorname{pr}_\Omega \operatorname{Var}\mu^{pq} \le \|\psi\|_\infty \operatorname{meas}. \tag{20}$$

In view of (20) the measures $\operatorname{pr}_\Omega(\psi(\xi)\mu^{pq}(x,\xi))$ are absolutely continuous with respect to the Lebesgue measure, and the Radon–Nikodym theorem shows that

$$\operatorname{pr}_\Omega(\psi(\xi)\mu^{pq}(x,\xi)) = h_\psi^{pq}(x)\operatorname{meas},$$

where the densities $h_\psi^{pq}(x)$ are measurable on Ω and, as seen from (20),

$$\|h_\psi^{pq}(x)\|_\infty \le \|\psi\|_\infty. \tag{21}$$

We now choose a non-negative function $K(x) \in C_0^\infty(\mathbf{R}^n)$ with support in the unit ball such that $\int K(x)dx = 1$ and set $K_m(x) = m^n K(mx)$ for $m \in \mathbf{N}$. Clearly, the sequence of K_m converges in $\mathcal{D}'(\mathbf{R}^n)$ to the Dirac δ-function.

Let $\operatorname{B}\lim\limits_{m\to\infty} c_m$ be a generalized Banach limit on the space l_∞ of bounded sequences $c = \{c_m\}_{m\in\mathbf{N}}$, i.e., $L(c) = \operatorname{B}\lim\limits_{m\to\infty} c_m$ is a linear functional on l_∞ with the property:

$$\varliminf_{m\to\infty} c_m \le L(c) \le \varlimsup_{m\to\infty} c_m$$

(for convergent sequences $c = \{c_m\}$, $L(c) = \lim\limits_{m\to\infty} c_m$). For any complex sequence $c_m = a_m + ib_m$, the Banach limits are defined by complexification:

$$\operatorname{B}\lim_{m\to\infty} c_m = L(a) + iL(b),$$

where $a = \{a_m\}$, $b = \{b_m\}$ are real and imaginary parts, respectively, of the sequence $c = \{c_m\}$. Modifying the densities $h_\psi^{pq}(x)$ on subsets of measure zero, for instance, replacing them by the functions

$$\operatorname{B}\lim_{m\to\infty} \int_\Omega h_\psi^{pq}(y) K_m(x-y)dy$$

(obviously, the value $h_\psi^{pq}(x)$ does not change for any Lebesgue point x of the function h_ψ^{pq}), we shall assume that for all $x \in \Omega$

$$h_\psi^{pq}(x) = \operatorname{B}\lim_{m\to\infty} \int_\Omega h_\psi^{pq}(y) K_m(x-y)dy. \tag{22}$$

Let Ω' be the set of common Lebesgue points of the functions

$$h_\psi^{pq}(x), \quad u_0(x,p) = \nu_x^0((p,+\infty)), \quad u_0^-(x,p) = \nu_x^0([p,+\infty)) = \lim_{q\to p-} u_0(x,q),$$

where $p, q \in D$ and ψ belongs to some countable dense subset F of $C(S_X)$. The family of (p,q,ψ) is countable, therefore Ω' is of full measure.

The dependence of h_ψ^{pq} on ψ, regarded as a map from $C(S_X)$ into $L^\infty(\Omega)$, is clearly linear and continuous (in view of (21)), therefore it follows from the density of F in $C(S_X)$ that $x \in \Omega'$ is a Lebesgue point of the functions $h_\psi^{pq}(x)$ for all $\psi(\xi) \in C(S_X)$ and $p, q \in D$ (here we also take (22) into account).

For $p, q \in D$ and $x \in \Omega'$ the equality $l(\psi) = h_\psi^{pq}(x)$ defines a continuous linear functional in $C(S_X)$; moreover, $\|l\| \le 1$ in view of (21). By the Riesz–Markov theorem this functional can be defined by integration with respect to some complex Borel measure $\mu_x^{pq}(\xi)$ in S_X and $\operatorname{Var} \mu_x^{pq} = \|l\| \le 1$. Hence

$$h_\psi^{pq}(x) = \langle \mu_x^{pq}(\xi), \psi \rangle = \int_{S_X} \psi(\xi) d\mu_x^{pq}(\xi), \quad \psi(\xi) \in C(S_X). \tag{23}$$

Equality (23) shows that the functions $x \mapsto \int_S \psi(\xi) d\mu_x^{pq}(\xi)$ are bounded and measurable for all $\psi(\xi) \in C(S_X)$. Next, for $\Phi(x) \in C_0(\Omega)$ and $\psi(\xi) \in C(S_X)$ we have

$$\begin{aligned} \int_\Omega \left(\int_{S_X} \Phi(x)\psi(\xi) d\mu_x^{pq}(\xi) \right) dx &= \int_\Omega \Phi(x) h_\psi^{pq}(x) dx = \int_\Omega \Phi(x) d\mathrm{pr}_\Omega\left(\psi(\xi)\mu^{pq}\right) \\ &= \int_{\Omega\times S_X} \Phi(x)\psi(\xi) d\mu^{pq}(x,\xi). \end{aligned} \tag{24}$$

Approximating an arbitrary function $\Phi(x,\xi) \in C_0(\Omega \times S_X)$ in the uniform norm by linear combinations of functions of the form $\Phi(x)\psi(\xi)$ we derive from (24) that the integral $\int_{S_X} \Phi(x,\xi) d\mu_x^{pq}(\xi)$ is Lebesgue-measurable with respect to $x \in \Omega$, bounded, and

$$\int_\Omega \left(\int_{S_X} \Phi(x,\xi) d\mu_x^{pq}(\xi) \right) dx = \int_{\Omega\times S_X} \Phi(x,\xi) d\mu^{pq}(x,\xi),$$

that is, $\mu^{pq}=\mu_x^{pq}dx$. Recall that $\operatorname{Var}\mu_x^{pq}\le 1$. The proof is complete. $\square$

The assumption that $x\in\Omega'$ are Lebesgue points of the functions $u_0(x,p)$, $u_0^-(x,p)$ for all $p\in D$, will be used later. Observe that since $p\in D\subset E$ is a continuity point of the map $p\mapsto u_0(x,p)$ in $L^1_{\mathrm{loc}}(\Omega)$, then $u_0^-(x,p)=u_0(x,p)$ a.e. in Ω. By construction $x\in\Omega'$ is a common Lebesgue point of the functions $u_0(x,p)$, $u_0^-(x,p)$, therefore

$$\nu_x^0(\{p\})=u_0^-(x,p)-u_0(x,p)=0,\quad p\in D. \tag{25}$$

Remark 15. (a) Since the H-measure is absolutely continuous with respect to x-variables (17) is satisfied for $\Phi_1(x),\Phi_2(x)\in L^2(\Omega)$. Indeed, by Proposition 14 we can rewrite this identity in the following form: For all $\Phi_1(x),\Phi_2(x)\in C_0(\Omega)$, $\psi(\xi)\in C(S_X)$

$$\int_\Omega \Phi_1(x)\overline{\Phi_2(x)}\langle\psi(\xi),\mu_x^{pq}(\xi)\rangle dx=\lim_{r\to\infty}\int_{\mathbf{R}^n}F(\Phi_1U_r^p)(\xi)\overline{F(\Phi_2U_r^q)(\xi)}\psi(\pi_X(\xi))d\xi. \tag{26}$$

Both sides of this identity are continuous with respect to $(\Phi_1(x),\Phi_2(x))$ in $L^2(\Omega)\times L^2(\Omega)$ and since $C_0(\Omega)$ is dense in $L^2(\Omega)$ we conclude that (26) is satisfied for each $\Phi_1(x),\Phi_2(x)\in L^2(\Omega)$;

(b) if $x\in\Omega'$ is a Lebesgue point of a function $\Phi(x)\in L^2(\Omega)$, then

$$\Phi(x)\langle\mu_x^{pq},\psi(\xi)\rangle=\lim_{m\to\infty}\lim_{r\to\infty}\int_{\mathbf{R}^n}F(\Phi\Phi_mU_r^p)(\xi)\overline{F(\Phi_mU_r^q)(\xi)}\psi(\pi_X(\xi))d\xi \tag{27}$$

for all $\psi(\xi)\in C(S_X)$, where

$$(\Phi\Phi_mU_r^p)(y)=\Phi(y)\Phi_m(x-y)U_r^p(y),\quad (\Phi_mU_r^q)(y)=\Phi_m(x-y)U_r^q(y),$$

and $\Phi_m(x-y)=\sqrt{K_m(x-y)}$, the sequence K_m is defined in the proof of Proposition 14.

Indeed, it follows from (26) that

$$\lim_{r\to\infty}\int_{\mathbf{R}^n}F(\Phi\Phi_mU_r^p)(\xi)\overline{F(\Phi_mU_r^q)(\xi)}\psi(\pi_X(\xi))d\xi=\int_\Omega h_\psi^{pq}(y)\Phi(y)K_m(x-y)dy. \tag{28}$$

Now, since $x\in\Omega'$ is a Lebesgue point of the functions $h_\psi^{pq}(y)$ and $\Phi(y)$, and the function $h_\psi^{pq}(y)$ is bounded, x is also a Lebesgue point for the product of these functions. Therefore,

$$\lim_{m\to\infty}\int_\Omega h_\psi^{pq}(y)\Phi(y)K_m(x-y)dy=\Phi(x)h_\psi^{pq}(x)=\Phi(x)\langle\mu_x^{pq},\psi(\xi)\rangle,$$

and (27) follows from (28) in the limit as $m\to\infty$;

(c) for $x\in\Omega'$ and each family $p_i\in D$, $\psi_i(\xi)\in C(S_X)$, $i=1,\dots,l$, the matrix $\left\langle\mu_x^{p_ip_j},\psi_i\overline{\psi_j}\right\rangle$, $i,j=1,\dots,l$, is positive definite. Indeed, as follows from Lemma 13(iii), for $\alpha_1,\dots,\alpha_l\in\mathbf{C}$

$$\sum_{i,j=1}^l\left\langle\mu_x^{p_ip_j},\psi_i\overline{\psi_j}\right\rangle\alpha_i\overline{\alpha_j}$$
$$=\lim_{m\to\infty}\sum_{i,j=1}^l\left\langle\mu^{p_ip_j}(y,\xi),\Phi_m(x-y)\psi_i(\xi)\overline{\Phi_m(x-y)\psi_j(\xi)}\right\rangle\alpha_i\overline{\alpha_j}\ge 0.$$

Taking in the above property $l=2$, $p_1=p$, $p_2=q$, $\psi_1(\xi)=\psi(\xi)/\sqrt{|\psi(\xi)|}$ ($\psi_1=0$ for $\psi=0$) and $\psi_2(\xi)=\sqrt{|\psi(\xi)|}$, $\psi(\xi)\in C(S_X)$, we obtain, as in the proof of Proposition 14, that the matrix $\begin{pmatrix}\langle\mu_x^{pp},|\psi|\rangle & \langle\mu_x^{pq},\psi\rangle\\ \overline{\langle\mu_x^{pq},\psi\rangle} & \langle\mu_x^{qq},|\psi|\rangle\end{pmatrix}$ is positive definite. In particular,

$$|\langle\mu_x^{pq},\psi\rangle|\le\left(\langle\mu_x^{pp},|\psi|\rangle\cdot\langle\mu_x^{qq},|\psi|\rangle\right)^{1/2},$$

and this easily implies that for any Borel set $A \subset S_X$

$$\operatorname{Var} \mu_x^{pq}(A) \le \left(\mu_x^{pp}(A)\mu_x^{qq}(A)\right)^{1/2}. \tag{29}$$

Denote by $\theta(\lambda)$ the Heaviside function:

$$\theta(\lambda) = \begin{cases} 1, & \lambda > 0, \\ 0, & \lambda \le 0. \end{cases}$$

Below we shall frequently use the following simple estimate.

Lemma 16 ([16, Lemma 5]). *Let $p_0, p \in D$, $\chi(\lambda) = \theta(\lambda - p_0) - \theta(\lambda - p)$,*

$$V_r(y) = \int |\chi(\lambda)| d(\nu_y^r(\lambda) + \nu_y^0(\lambda)),$$

$\Phi(y) \in L^2(\Omega)$, $x \in \Omega'$ is a Lebesgue point of $(\Phi(y))^2$. Then

$$\varlimsup_{m\to\infty} \varlimsup_{r\to\infty} \|\Phi_m(x-y)\Phi(y)V_r(y)\|_2 \le 2|\Phi(x)||u_0(x,p_0) - u_0(x,p)|^{1/2} \underset{p\to p_0}{\to} 0.$$

Proof. It is clear that

$$\begin{aligned} V_r(y) &= |u_r(y,p_0) - u_r(y,p) + u_0(y,p_0) - u_0(y,p)| \\ &= \operatorname{sign}(p - p_0)(u_r(y,p_0) - u_r(y,p) + u_0(y,p_0) - u_0(y,p)) \le 2 \end{aligned}$$

and, in particular, $(V_r(y))^2 \le 2V_r(y)$. Therefore,

$$\begin{aligned} &\|\Phi_m(x-y)\Phi(y)V_r(y)\|_2^2 \\ &\le 2\operatorname{sign}(p-p_0) \int (\Phi(y))^2 K_m(x-y)(u_r(y,p_0) - u_r(y,p) + u_0(y,p_0) - u_0(y,p))dy. \end{aligned}$$

Since $p_0, p \in D \subset E$, $u_r(y,p_0) - u_r(y,p) \to u_0(y,p_0) - u_0(y,p)$ as $r \to \infty$ weak star in $L^\infty(\Omega)$ and we derive from the above inequality that

$$\begin{aligned} &\varlimsup_{r\to\infty} \|\Phi_m(x-y)\Phi(y)V_r(y)\|_2^2 \\ &\qquad \le 4\operatorname{sign}(p-p_0) \int (\Phi(y))^2 K_m(x-y)(u_0(y,p_0) - u_0(y,p))dy. \end{aligned}$$

Now, passing to the limit as $m \to \infty$ and taking into account that $x \in \Omega'$ is a Lebesgue point of the bounded function $u_0(y,p_0) - u_0(y,p)$ as well as the function $(\Phi(y))^2$ (therefore, x is a Lebesgue point of the product of these functions), we find

$$\varlimsup_{m\to\infty} \varlimsup_{r\to\infty} \|\Phi_m(x-y)\Phi(y)V_r(y)\|_2^2 \le 4(\Phi(x))^2|u_0(x,p_0) - u_0(x,p)|.$$

This implies the required relation

$$\varlimsup_{m\to\infty} \varlimsup_{r\to\infty} \|\Phi_m(x-y)\Phi(y)V_r(y)\|_2 \le 2|\Phi(x)||u_0(x,p_0) - u_0(x,p)|^{1/2}.$$

To complete the proof it only remains to observe that, in view of (25), $\nu_x^0(\{p_0\}) = 0$ and therefore $u_0(x,p) \to u_0(x,p_0)$ as $p \to p_0$. □

The following statement is rather well-known.

Lemma 17. *Let $U_r(x)$ be a sequence bounded in $L^2(\mathbf{R}^n) \cap L^1(\mathbf{R}^n)$ and weakly convergent to zero, $a(\xi)$ be a bounded function on $\mathbf{R}^n$ such that $a(\xi) \to 0$ as $|\xi| \to \infty$. Then $a(\xi)F(U_r)(\xi) \underset{r\to\infty}{\to} 0$ in $L^2(\mathbf{R}^n)$.*

Proof. First, observe that by the assumption $a(\xi) \to 0$ at infinity for any $\varepsilon > 0$ we can choose $R > 0$ such that $|a(\xi)| < \varepsilon$ for $|\xi| > R$. Then

$$\int_{|\xi|>R} |a(\xi)|^2 |F(U_r)(\xi)|^2 d\xi \le \varepsilon^2 \|F(U_r)\|_2 = \varepsilon^2 \|U_r\|_2 \le C\varepsilon^2, \tag{30}$$

where $C = \sup_{r\in\mathbf{N}} \|U_r\|_2$ is a constant independent of r.

Furthermore, by our assumption $U_r \to 0$ as $r \to \infty$ weakly in L^1. This implies that $F(U_r)(\xi) \to 0$ pointwise as $r \to \infty$. Moreover, $|F(U_r)(\xi)| \le \|U_r\|_1 \le \text{const}$. Hence, using the Lebesgue dominated convergence theorem, we find that

$$\int_{|\xi|\le R} |a(\xi)|^2 |F(U_r)(\xi)|^2 d\xi \to 0 \tag{31}$$

as $r \to \infty$. It follows from (30), (31) that

$$\varlimsup_{r\to\infty} \int_{\mathbf{R}^n} |a(\xi)|^2 |F(U_r)(\xi)|^2 d\xi \le C\varepsilon^2.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that

$$\lim_{r\to\infty} \int_{\mathbf{R}^n} |a(\xi)|^2 |F(U_r)(\xi)|^2 d\xi = 0,$$

that is, $a(\xi)F(U_r)(\xi) \underset{r\to\infty}{\to} 0$ in $L^2(\mathbf{R}^n)$. The proof is complete. □

We now fix $x \in \Omega'$, $p_0, p \in D$. Let $L(p) \subset \mathbf{R}^n$ be the smallest linear subspace containing $\operatorname{supp} \mu_x^{pp_0}$, and $L = L(p_0)$. As follows from (29), $\operatorname{supp} \mu_x^{pp_0} \subset \operatorname{supp} \mu_x^{p_0p_0}$ and therefore $L(p) \subset L$.

Suppose that $f(y, \lambda)$ is a Caratheodory vector-function on $\Omega \times \mathbf{R}$ such that

$$\|f(x, \cdot)\|_{M,\infty} = \max_{|\lambda|\le M} |f(x, \lambda)| \le \alpha_M(x) \in L^2_{\text{loc}}(\Omega), \tag{32}$$

for all $M > 0$. Since the space $C(\mathbf{R}, \mathbf{R}^n)$ is separable with respect to the standard locally convex topology generated by seminorms $\|\cdot\|_{M,\infty}$, then, by the Pettis theorem (see [5, Ch. 3]), the map $x \mapsto F(x) = f(x, \cdot) \in C(\mathbf{R}, \mathbf{R}^n)$ is strongly measurable and in view of estimate (32) we see that $F(x) \in L^2_{\text{loc}}(\Omega, C(\mathbf{R}, \mathbf{R}^n))$, $|F(x)|^2 \in L^1_{\text{loc}}(\Omega, C(\mathbf{R}))$. In particular (see [5, Ch. 3]), the set Ω_f of common Lebesgue points of the maps $F(x), |F(x)|^2$ has full measure. For $x \in \Omega_f$ we have

$$\lim_{m\to\infty} \int K_m(x - y)\|F(x) - F(y)\|_{M,\infty} dy = 0,$$

$$\lim_{m\to\infty} \int K_m(x - y)\||F(x)|^2 - |F(y)|^2\|_{M,\infty} dy = 0$$

for all $M > 0$. Since, evidently,

$$\|F(x) - F(y)\|^2_{M,\infty} \le 2\|F(x) - F(y)\|_{M,\infty}\|F(x)\|_{M,\infty} + \||F(x)|^2 - |F(y)|^2\|_{M,\infty},$$

it follows from the above limit relations that for $x \in \Omega_f$

$$\lim_{m\to\infty} \int K_m(x - y)\|F(x) - F(y)\|^2_{M,\infty} dy = 0, \tag{33}$$

for all $M > 0$. Clearly, each $x \in \Omega_f$ is a Lebesgue point of all functions $x \to f(x, \lambda)$, $\lambda \in \mathbf{R}$. Let $\Omega'' = \Omega' \cap \Omega_f$, $\gamma_x^r = \nu_x^r - \nu_x^0$. Suppose that $x \in \Omega''$, $p_0 \in D$, $\chi(\lambda) = \theta(\lambda - p_1) - \theta(\lambda - p_2)$, where $p_1, p_2 \in D$.

For a vector-function $h(y, \lambda)$ on $\Omega \times \mathbf{R}$, which is Borel and locally bounded with respect to the second variable, we denote $I_r(h)(y) = \int h(y, \lambda) d\gamma_y^r(\lambda)$. In view of the strong measurability of $F(x)$ and (32) we see that the sequence $I_r = I_r(f \cdot \chi)(y)$ is bounded in $L^2_{\text{loc}}(\Omega)$ (also see Remark 5). Moreover, this sequence weakly converges to zero as $r \to \infty$. The latter easily follows from the fact that $f\chi(y)$ can be pointwise approximated by finite sums of functions of the kind $h(y, \lambda) = g(y)\theta(\lambda - p)$, where $g(y) \in L^2_{\text{loc}}(\Omega)$ and $p \in D$. Since $I_r(h)(y) = g(y)U_r^p(y)$ we see that $I_r(y)$ is approximated in $L^2_{\text{loc}}(\Omega)$ by finite sums of the indicated functions $g(y)U_r^p(y)$. By Lemma 10 the functions $g(y)U_r^p(y) \to 0$ as $r \to \infty$ weakly in $L^2_{\text{loc}}(\Omega)$ and we conclude, by the approximation arguments, that the same remains true for the original sequence $I_r(y)$.

Let X be the subspace from the definition of ultra-parabolic H-measure, $X^\perp$ be the orthogonal complement to X. We denote by $\tilde{L}$, $\bar{L}$ the spaces obtained by orthogonal projections of L on the subspaces X, $X^\perp$, respectively: $\tilde{L} = P_1(L)$, $\bar{L} = P_2(L)$.

Proposition 18 ([16, Prop. 4])**.** *Assume that $f(x,\lambda) \in \tilde{L}^\perp$, and $\rho(\xi) \in C^\infty(\mathbf{R}^n)$ is a function such that $0 \le \rho(\xi) \le 1$ and $\rho(\xi) = 0$ for $|\tilde{\xi}|^2 + |\bar{\xi}|^4 \le 1$, $\rho(\xi) = 1$ for $|\tilde{\xi}|^2 + |\bar{\xi}|^4 \ge 2$. Then for all $\psi(\xi) \in C(S_X)$*

$$\lim_{m\to\infty} \overline{\lim_{r\to\infty}} \left| \int_{\mathbf{R}^n} \frac{\rho(\xi)\xi \cdot F(\Phi_m I_r(f\cdot\chi))(\xi)}{(|\tilde{\xi}|^2 + |\bar{\xi}|^4)^{1/2}} \overline{F(\Phi_m U_r^{p_0})(\xi)} \psi(\pi_X(\xi)) d\xi \right| = 0.$$

Here $\Phi_m = \Phi_m(x-y) = \sqrt{K_m(x-y)}$ and $I_r(f\cdot\chi)$ are supposed to be functions of the variable $y \in \Omega$.

Proof. Note that

$$|I_r(y)| \le \int |f(y,\lambda)||\chi(\lambda)| d\mathrm{Var}\,\gamma_y^r(\lambda) \le 2\alpha_M(y), \tag{34}$$

where $M = \sup \|\nu_x^r\|_\infty$. Let us first show that for each $m \in \mathbf{N}$

$$\lim_{r\to\infty} \int_{\mathbf{R}^n} \frac{\rho(\xi)\bar{\xi} \cdot F(\Phi_m I_r)(\xi)}{(|\tilde{\xi}|^2 + |\bar{\xi}|^4)^{1/2}} \overline{F(\Phi_m U_r^{p_0})(\xi)} \psi(\pi_X(\xi)) d\xi = 0. \tag{35}$$

For that, it is sufficient to demonstrate that

$$\frac{\rho(\xi)|\bar{\xi}|}{(|\tilde{\xi}|^2 + |\bar{\xi}|^4)^{1/2}} |F(\Phi_m U_r^{p_0})(\xi)| \underset{r\to\infty}{\to} 0 \text{ in } L^2(\mathbf{R}^n). \tag{36}$$

Remark that the sequence $\Phi_m U_r^{p_0}$, $r \in \mathbf{N}$ is bounded in $L^2(\mathbf{R}^n)$ and in $L^1(\mathbf{R}^n)$ (since $\operatorname{supp}\Phi_m$ is compact) and weakly converges to zero. Hence, (36) follows from Lemma 17. We only need to demonstrate that the function

$$a(\xi) = \frac{\rho(\xi)|\bar{\xi}|}{(|\tilde{\xi}|^2 + |\bar{\xi}|^4)^{1/2}}$$

satisfies the assumptions of this lemma. First, we show that $a(\xi) \le 1$. Indeed, for $|\tilde{\xi}|^2 + |\bar{\xi}|^4 \le 1$ the value $\rho(\xi) = 0$ while in the case $|\tilde{\xi}|^2 + |\bar{\xi}|^4 > 1$ we have $\dfrac{\rho(\xi)|\bar{\xi}|}{(|\tilde{\xi}|^2 + |\bar{\xi}|^4)^{1/2}} \le \min(|\bar{\xi}|, 1/|\bar{\xi}|) \le 1$.

Then, observe that for $|\tilde{\xi}|^2 + |\bar{\xi}|^4 \ge R^4 > 0$

$$a(\xi) \le \frac{|\bar{\xi}|}{(|\tilde{\xi}|^2 + |\bar{\xi}|^4)^{1/2}} \le (|\tilde{\xi}|^2 + |\bar{\xi}|^4)^{-1/4} \le R^{-1}.$$

Therefore, $a(\xi) \to 0$ as $|\xi| \to \infty$. Thus, assumptions of Lemma 17 are satisfied and by Lemma 17 we conclude that (36), (35) hold.

In view of (35),

$$\begin{aligned}
&\lim_{m\to\infty} \overline{\lim_{r\to\infty}} \left| \int_{\mathbf{R}^n} \frac{\rho(\xi)\xi \cdot F(\Phi_m I_r)(\xi)}{(|\tilde{\xi}|^2 + |\bar{\xi}|^4)^{1/2}} \overline{F(\Phi_m U_r^{p_0})(\xi)} \psi(\pi_X(\xi)) d\xi \right| \\
&\quad = \lim_{m\to\infty} \overline{\lim_{r\to\infty}} \left| \int_{\mathbf{R}^n} \frac{\rho(\xi)\tilde{\xi} \cdot F(\Phi_m I_r)(\xi)}{(|\tilde{\xi}|^2 + |\bar{\xi}|^4)^{1/2}} \overline{F(\Phi_m U_r^{p_0})(\xi)} \psi(\pi_X(\xi)) d\xi \right|.
\end{aligned} \tag{37}$$

Let $g(\lambda) = f(x,\lambda)$, $I_r' = I_r(g\chi)(y) = \int g(\lambda)\chi(\lambda) d\gamma_y^r(\lambda)$, $M = \sup \|\nu_y^r\|_\infty$. Then

$$|I_r - I_r'| \le \int |f(y,\lambda) - f(x,\lambda)| d\mathrm{Var}\,\gamma_y^r(\lambda) \le 2\|F(y) - F(x)\|_{M,\infty}.$$

This and the Plancherel identity imply that

$$\left|\int_{\mathbf{R}^n} \frac{\rho(\xi)\tilde{\xi}\cdot F(\Phi_m(I_r-I'_r))(\xi)}{(|\tilde{\xi}|^2+|\bar{\xi}|^4)^{1/2}}\overline{F(\Phi_m U_r^{p_0})(\xi)}\psi(\pi_X(\xi))d\xi\right|$$
$$\le \|\psi\|_\infty \|F(\Phi_m(I_r-I'_r))\|_2 \|F(\Phi_m U_r^{p_0})\|_2 \le \|\psi\|_\infty \|\Phi_m(I_r-I'_r)\|_2$$
$$\le 2\|\psi\|_\infty \left(\int_{\mathbf{R}^n} K_m(x-y)\|F(y)-F(x)\|^2_{M,\infty}dy\right)^{1/2}.$$

It follows from the above estimate and (33) that

$$\lim_{m\to\infty}\overline{\lim_{r\to\infty}}\left|\int_{\mathbf{R}^n} \frac{\rho(\xi)\tilde{\xi}\cdot F(\Phi_m I_r)(\xi)}{(|\tilde{\xi}|^2+|\bar{\xi}|^4)^{1/2}}\overline{F(\Phi_m U_r^{p_0})(\xi)}\psi(\pi_X(\xi))d\xi \right.$$
$$\left. - \int_{\mathbf{R}^n} \frac{\rho(\xi)\tilde{\xi}\cdot F(\Phi_m I'_r)(\xi)}{(|\tilde{\xi}|^2+|\bar{\xi}|^4)^{1/2}}\overline{F(\Phi_m U_r^{p_0})(\xi)}\psi(\pi_X(\xi))d\xi\right|$$
$$\le \lim_{m\to\infty}\overline{\lim_{r\to\infty}}\left|\int_{\mathbf{R}^n} \frac{\rho(\xi)\tilde{\xi}\cdot F(\Phi_m(I_r-I'_r))(\xi)}{(|\tilde{\xi}|^2+|\bar{\xi}|^4)^{1/2}}\overline{F(\Phi_m U_r^{p_0})(\xi)}\psi(\pi_X(\xi))d\xi\right| = 0$$

and, in view of this relation and (37), it is sufficient to prove that

$$\lim_{m\to\infty}\overline{\lim_{r\to\infty}}\left|\int_{\mathbf{R}^n} \frac{\rho(\xi)\tilde{\xi}\cdot F(\Phi_m I'_r)(\xi)}{(|\tilde{\xi}|^2+|\bar{\xi}|^4)^{1/2}}\overline{F(\Phi_m U_r^{p_0})(\xi)}\psi(\pi_X(\xi))d\xi\right| = 0. \tag{38}$$

The vector-function $g(\lambda)$ is continuous and does not depend on y. Therefore for any $\varepsilon > 0$ there exists a vector-valued function $h(\lambda)$ of the form $h(\lambda) = \sum_{i=1}^k v_i\theta(\lambda - p_i)$, where $v_i \in \tilde{L}^\perp$ and $p_i \in D$ such that $\|g\cdot\chi - h\|_\infty \le \varepsilon$ on $\mathbf{R}$.

Using again Plancherel's identity and the fact that

$$|I'_r - I_r(h)| = \left|\int (g\cdot\chi - h)(\lambda)d\gamma_y^r(\lambda)\right| \le \int |(g\cdot\chi-h)(\lambda)|d\mathrm{Var}\,(\gamma_y^r)(\lambda) \le 2\varepsilon,$$

we obtain

$$\begin{aligned}\left|\int_{\mathbf{R}^n} \frac{\rho(\xi)\tilde{\xi}\cdot F(\Phi_m I'_r)(\xi)}{(|\tilde{\xi}|^2+|\bar{\xi}|^4)^{1/2}}\overline{F(\Phi_m U_r^{p_0})(\xi)}\psi(\pi_X(\xi))d\xi \right.\\ \left. - \int_{\mathbf{R}^n} \frac{\rho(\xi)\tilde{\xi}\cdot F(\Phi_m I_r(h))(\xi)}{(|\tilde{\xi}|^2+|\bar{\xi}|^4)^{1/2}}\overline{F(\Phi_m U_r^{p_0})(\xi)}\psi(\pi_X(\xi))d\xi\right| \\ \le \|\psi\|_\infty\|\Phi_m I_r(g\cdot\chi - h)\|_2 \le 2\varepsilon\|\psi\|_\infty\|\Phi_m\|_2 = 2\varepsilon\|\psi\|_\infty.\end{aligned} \tag{39}$$

Since

$$I_r(h)(y) = \int\left(\sum_{i=1}^k v_i\theta(\lambda-p_i)\right)d\gamma_y^r(\lambda) = \sum_{i=1}^k v_i U_r^{p_i}(y),$$

it follows from (27) the limit relation

$$\begin{aligned}\lim_{m\to\infty}\lim_{r\to\infty}\int_{\mathbf{R}^n} \frac{\rho(\xi)\tilde{\xi}\cdot F(\Phi_m I_r(h))(\xi)}{(|\tilde{\xi}|^2+|\bar{\xi}|^4)^{1/2}}\overline{F(\Phi_m U_r^{p_0})(\xi)}\psi(\pi_X(\xi))d\xi \\ = \sum_{i=1}^k \left\langle \mu_x^{p_i p_0}, (v_i\cdot\tilde{\xi})\psi(\xi)\right\rangle.\end{aligned} \tag{40}$$

Here we also take into account Remark 12. Since $\rho(\xi)\psi(\pi_X(\xi)) = \psi(\pi_X(\xi))$ for large $|\xi|$ then, by this remark, for $i = 1,\ldots,k$,

$$\lim_{r\to\infty}\int_{\mathbf{R}^n} \frac{\rho(\xi)\tilde{\xi}\cdot v_i F(\Phi_m U_r^{p_i})(\xi)}{(|\tilde{\xi}|^2+|\bar{\xi}|^4)^{1/2}}\overline{F(\Phi_m U_r^{p_0})(\xi)}\psi(\pi_X(\xi))d\xi$$

$$= \lim_{r\to\infty}\int_{\mathbf{R}^n} \frac{\tilde{\xi}\cdot v_i F(\Phi_m U_r^{p_i})(\xi)}{(|\tilde{\xi}|^2+|\bar{\xi}|^4)^{1/2}}\overline{F(\Phi_m U_r^{p_0})(\xi)}\psi(\pi_X(\xi))d\xi$$
$$= \left\langle \mu^{p_ip_0}(y,\xi), K_m(x-y)(v_i\cdot\tilde{\xi})\psi(\xi)\right\rangle.$$

Now observe that $\operatorname{supp}\mu_x^{p_ip_0}\subset L(p_0)=L$, and for each $\xi\in L$ $v_i\cdot\tilde{\xi}=0$ because $\tilde{\xi}\in\tilde{L}$ while $v_i\perp\tilde{L}$. Hence $\sum_{i=1}^k\langle\mu_x^{p_ip_0},(v_i\cdot\tilde{\xi})\psi(\xi)\rangle=0$, and it follows from (40) that

$$\lim_{m\to\infty}\lim_{r\to\infty}\int_{\mathbf{R}^n}\frac{\rho(\xi)\tilde{\xi}\cdot F(\Phi_m I_r(h))(\xi)}{(|\tilde{\xi}|^2+|\bar{\xi}|^4)^{1/2}}\overline{F(\Phi_m U_r^{p_0})(\xi)}\psi(\pi_X(\xi))d\xi=0.$$

This relation together with (39) yields

$$\overline{\lim_{m\to\infty}}\,\overline{\lim_{r\to\infty}}\left|\int_{\mathbf{R}^n}\frac{\rho(\xi)\tilde{\xi}\cdot F(\Phi_m I_r')(\xi)}{(|\tilde{\xi}|^2+|\bar{\xi}|^4)^{1/2}}\overline{F(\Phi_m U_r^{p_0})(\xi)}\psi(\pi_X(\xi))d\xi\right|\le 2\varepsilon\|\psi\|_\infty,$$

and since $\varepsilon>0$ is arbitrary we claim that (38) holds. This completes the proof. $\square$

Let $Q(\lambda)$ be a continuous matrix-valued function, which ranges in the space Sym_n of symmetric matrices of order n, and $Q(\lambda)\xi\cdot\xi=0$ for all $\xi\in\bar{L}=P_2(L)$ (recall that P_2 is the orthogonal projection onto $X^\perp$). Let $p_1,p_2\in D$, $\chi(\lambda)=\theta(\lambda-p_1)-\theta(\lambda-p_2)$, $J_r(y)=J_r(Q)(y)=\int\chi(\lambda)Q(\lambda)d\gamma_y^r(\lambda)$, and let $\rho(\xi)$ be a function as in Proposition 18.

Proposition 19 ([16, Prop. 5])**.** *Under the above notation for each $\psi(\xi)\in C(S_X)$*

$$\lim_{m\to\infty}\overline{\lim_{r\to\infty}}\left|\int_{\mathbf{R}^n}\frac{\rho(\xi)F(\Phi_m J_r)(\xi)\bar{\xi}\cdot\bar{\xi}}{(|\tilde{\xi}|^2+|\bar{\xi}|^4)^{1/2}}\overline{F(\Phi_m U_r^{p_0})(\xi)}\psi(\pi_X(\xi))d\xi\right|=0. \tag{41}$$

Proof. Since the space Y of symmetric matrices A, satisfying the property $A\xi\cdot\xi=0$ for $\xi\in\bar{L}$, is linear, for every $\varepsilon>0$ one can find a step function $H(\lambda)=\sum_{i=1}^k\theta(\lambda-p_i)Q_i$, where $p_i\in D$, $Q_i\in Y$ for each $i=1,\dots,k$ such that $|\chi(\lambda)Q(\lambda)-H(\lambda)|<\varepsilon$ for all $\lambda\in\mathbf{R}$. We denote $J_r'(y)=\int H(\lambda)d\gamma_y^r(\lambda)$ and observe that

$$J_r'(y)=\sum_{i=1}^k U_r^{p_i}(y)Q_i, \tag{42}$$
$$|J_r(y)-J_r'(y)|\le\int|Q(\lambda)-H(\lambda)||\chi(\lambda)|d\mathrm{Var}\,\gamma_y^r(\lambda)\le 2\varepsilon. \tag{43}$$

We also remark that

$$\left|\frac{F(\Phi_m J_r)(\xi)\bar{\xi}\cdot\bar{\xi}}{(|\tilde{\xi}|^2+|\bar{\xi}|^4)^{1/2}}\right|\le\frac{|F(\Phi_m J_r)(\xi)||\bar{\xi}|^2}{(|\tilde{\xi}|^2+|\bar{\xi}|^4)^{1/2}}\le|F(\Phi_m J_r)(\xi)|.$$

The latter estimate and (43) imply that

$$\begin{aligned}
&\left|\int_{\mathbf{R}^n} \frac{\rho(\xi)F(\Phi_m J_r)(\xi)\bar\xi\cdot\bar\xi}{(|\tilde\xi|^2+|\bar\xi|^4)^{1/2}}\overline{F(\Phi_m U_r^{p_0})(\xi)}\psi(\pi_X(\xi))d\xi \right.\\
&\qquad\qquad \left. - \int_{\mathbf{R}^n} \frac{\rho(\xi)F(\Phi_m J'_r)(\xi)\bar\xi\cdot\bar\xi}{(|\tilde\xi|^2+|\bar\xi|^4)^{1/2}}\overline{F(\Phi_m U_r^{p_0})(\xi)}\psi(\pi_X(\xi))d\xi\right|\\
&= \left|\int_{\mathbf{R}^n} \frac{\rho(\xi)F(\Phi_m (J_r-J'_r))(\xi)\bar\xi\cdot\bar\xi}{(|\tilde\xi|^2+|\bar\xi|^4)^{1/2}}\overline{F(\Phi_m U_r^{p_0})(\xi)}\psi(\pi_X(\xi))d\xi\right|\\
&\le \|\psi\|_\infty \|F(\Phi_m(J_r-J'_r))\|_2\|F(\Phi_m U_r^{p_0})\|_2\\
&= \|\psi\|_\infty \|\Phi_m(J_r-J'_r)\|_2\|\Phi_m U_r^{p_0}\|_2\\
&\le \|\psi\|_\infty \|\Phi_m(J_r-J'_r)\|_2\\
&= \|\psi\|_\infty \left(\int_{\mathbf{R}^n} K_m(x-y)|J_r(y)-J'_r(y)|^2dy\right)^{1/2}\\
&\le 2\varepsilon\|\psi\|_\infty .
\end{aligned} \tag{44}$$

We also use that $|U_r^{p_0}|\le 1$ and therefore $\|\Phi_m U_r^{p_0}\|_2\le 1$. In view of (42)

$$\begin{aligned}
&\int_{\mathbf{R}^n} \frac{\rho(\xi)F(\Phi_m J'_r)(\xi)\bar\xi\cdot\bar\xi}{(|\tilde\xi|^2+|\bar\xi|^4)^{1/2}}\overline{F(\Phi_m U_r^{p_0})(\xi)}\psi(\pi_X(\xi))d\xi\\
&\quad = \sum_{i=1}^k \int_{\mathbf{R}^n} \frac{\rho(\xi)F(\Phi_m U_r^{p_i})(\xi)Q_i\bar\xi\cdot\bar\xi}{(|\tilde\xi|^2+|\bar\xi|^4)^{1/2}}\overline{F(\Phi_m U_r^{p_0})(\xi)}\psi(\pi_X(\xi))d\xi,
\end{aligned}$$

and by relation (27) and Remark 12 we find

$$\begin{aligned}
&\lim_{m\to\infty}\lim_{r\to\infty}\int_{\mathbf{R}^n} \frac{\rho(\xi)F(\Phi_m J'_r)(\xi)\bar\xi\cdot\bar\xi}{(|\tilde\xi|^2+|\bar\xi|^4)^{1/2}}\overline{F(\Phi_m U_r^{p_0})(\xi)}\psi(\pi_X(\xi))d\xi\\
&\quad = \sum_{i=1}^k \left\langle \mu_x^{p_ip_0}\psi(\xi)Q_i\bar\xi\cdot\bar\xi\right\rangle = 0,
\end{aligned} \tag{45}$$

because $\operatorname{supp}\mu_x^{p_ip_0}\subset L$ and therefore $Q_i\bar\xi\cdot\bar\xi=0$ on $\operatorname{supp}\mu_x^{p_ip_0}$ (recall that $Q_i\bar\xi\cdot\bar\xi=0$ for any $\bar\xi\in\bar L$).

By (44) and (45) we obtain the relation

$$\overline{\lim_{m\to\infty}}\;\overline{\lim_{r\to\infty}}\left|\int_{\mathbf{R}^n} \frac{\rho(\xi)F(\Phi_m J_r)(\xi)\bar\xi\cdot\bar\xi}{(|\tilde\xi|^2+|\bar\xi|^4)^{1/2}}\overline{F(\Phi_m U_r^{p_0})(\xi)}\psi(\pi_X(\xi))d\xi\right| \le 2\varepsilon\|\psi\|_\infty$$

and since $\varepsilon>0$ is arbitrary, we conclude that (41) holds. The proof is complete. □

In the sequel we will need the following simple result.

Lemma 20. *Let $\{\ \xi_k\ :\ k=1,\dots,l\ \}\subset L$ be a basis in L. Then there exists a positive constant C such that for every $v\in\mathbf{R}^n$, $Q\in\mathrm{Sym}_n$*

$$|v_1|+|Q_1|\le C\max_{k=1,\dots,l}|iv\cdot\tilde\xi_k+Q\bar\xi_k\cdot\bar\xi_k|,$$

where $v_1=\tilde Pv$, $Q_1=\bar PQ\bar P$, $\tilde P$, and $\bar P$ are orthogonal projections on the spaces $\tilde L$, $\bar L$, respectively, and $i=\sqrt{-1}$.

Proof. We introduce the linear spaces $\bar S=\{\ Q\in\mathrm{Sym}_n\ :\ Q=\bar PQ\bar P\ \}$, $H=\tilde L\oplus\bar S$ and remark that $p(v,Q)=\max\limits_{k=1,\dots,l}|iv\cdot\tilde\xi_k+Q\bar\xi_k\cdot\bar\xi_k|$ is a norm in H. Indeed, it is clear that p is a seminorm. To prove that p is a norm, suppose that $p(v,Q)=0$. Then $v\cdot\tilde\xi_k=Q\bar\xi_k\cdot\bar\xi_k=0$ and since vectors $\tilde\xi_k$, $\bar\xi_k$ generate spaces $\tilde L$, $\bar L$, respectively, we claim that $v\tilde{\xi} = 0$ for all $\xi \in \tilde{L}$ and $Q\xi \cdot \xi = 0$ for all $\xi \in \bar{L}$. Since $v \in \tilde{L}$ we see that $v = 0$. Furthermore, since $Q \in \bar{S}$ we find that for every $\xi \in \mathbf{R}^n$

$$Q\xi \cdot \xi = \bar{P}Q\bar{P}\xi \cdot \xi = Q\bar{P}\xi \cdot \bar{P}\xi = 0,$$

and we conclude that $Q = 0$. It is well-known that any two norms in a finite-dimensional space are equivalent. Applying this property to the norms $p(v, Q)$ and $p_1(v, Q) = |v| + |Q|$ and using the relations

$$v \cdot \tilde{\xi}_k = v_1 \cdot \tilde{\xi}_k, \ Q\bar{\xi}_k \cdot \bar{\xi}_k = Q\bar{P}\bar{\xi}_k \cdot \bar{P}\bar{\xi}_k = Q_1\bar{\xi}_k \cdot \bar{\xi}_k, \ k = 1, \dots l,$$

we find that for some constant $C > 0$

$$|v_1| + |Q_1| \le C \max_{k=1,\dots,l} |iv_1 \cdot \tilde{\xi}_k + Q_1\bar{\xi}_k \cdot \bar{\xi}_k| = C \max_{k=1,\dots,l} |iv \cdot \tilde{\xi}_k + Q\bar{\xi}_k \cdot \bar{\xi}_k|,$$

as was to be proved. □

Corollary 21. *There exist functions $\psi_k(\xi) \in C(S_X)$, $k = 1, \dots, l = \dim L$ and a constant $C > 0$ such that, in the notation of Lemma 20, for all $v \in \mathbf{R}^n$, $Q \in \mathrm{Sym}_n$ such that $Q \ge 0$*

$$|v_1| + |Q_1| \le C \max_{k=1,\dots,l} |\langle \mu_x^{p_0 p_0}, (iv \cdot \tilde{\xi} + Q\bar{\xi} \cdot \bar{\xi})\psi_k(\xi)\rangle|. \tag{46}$$

Proof. We remark that the measure $\mu_x^{p_0 p_0} \ge 0$. If $\mu_x^{p_0 p_0} = 0$, then both sides of the inequality (46) equal zero, and this inequality is evidently satisfied. Thus, suppose that $\mu_x^{p_0 p_0}(S_X) > 0$. Since L is a linear span of $\operatorname{supp} \mu_x^{p_0 p_0}$, we can choose functions $\psi_k(\xi) \in C(S_X)$, $k = 1, \dots, l$ such that $\psi_k(\xi) \ge 0$, $\int \psi_k(\xi) d\mu_x^{p_0 p_0} = 1$ for all $k = 1, \dots, l$, and the family $\xi_k = \int \xi\psi_k(\xi) d\mu_x^{p_0 p_0}$, $k = 1, \dots, l$, is a basis in L. By Lemma 20 there exists a constant $C > 0$ such that for all $v \in \mathbf{R}^n$, $Q \in \mathrm{Sym}_n$

$$|v_1| + |Q_1| \le C \max_{k=1,\dots,l} |iv \cdot \tilde{\xi}_k + Q\bar{\xi}_k \cdot \bar{\xi}_k|, \tag{47}$$

where $v_1 = \tilde{P}v$, $Q_1 = \bar{P}Q\bar{P}$. Now, we observe that

$$\tilde{\xi}_k = \int \tilde{\xi}\psi_k(\xi) d\mu_x^{p_0 p_0}(\xi), \quad \bar{\xi}_k = \int \bar{\xi}\psi_k(\xi) d\mu_x^{p_0 p_0}(\xi).$$

Therefore,

$$v \cdot \tilde{\xi}_k = \int v \cdot \tilde{\xi}\psi_k(\xi) d\mu_x^{p_0 p_0}(\xi),$$

and if $Q \ge 0$ then

$$Q\bar{\xi}_k \cdot \bar{\xi}_k = Q \int \bar{\xi}\psi_k(\xi) d\mu_x^{p_0 p_0}(\xi) \cdot \int \bar{\xi}\psi_k(\xi) d\mu_x^{p_0 p_0}(\xi) \le \int Q\bar{\xi} \cdot \bar{\xi}\psi_k(\xi) d\mu_x^{p_0 p_0}(\xi)$$

by Jensen's inequality applied to the convex function $\xi \to Q\bar{\xi} \cdot \bar{\xi}$. In view of the above relation, (46) readily follows from (47) (we also take into account that for real a the function $f(x) = |ia + x|$ increases on $[0, +\infty)$). The proof is complete. □

4. Localization principle and strong precompactness of bounded sequences of measure-valued functions

In this section we need some results about Fourier multipliers in spaces L^d, $d > 1$. Recall that a function $a(\xi) \in L^\infty(\mathbf{R}^n)$ is a Fourier multiplier in L^d if the pseudo-differential operator $\mathcal{A}$ with the symbol $a(\xi)$, defined as $F(\mathcal{A}u)(\xi) = a(\xi)F(u)(\xi)$, $u = u(x) \in L^2(\mathbf{R}^n) \cap L^d(\mathbf{R}^n)$, can be extended as a bounded operator on $L^d(\mathbf{R}^n)$, that is,

$$\|\mathcal{A}u\|_d \le C\|u\|_d, \quad u \in L^2(\mathbf{R}^n) \cap L^d(\mathbf{R}^n),$$

for a constant C. We denote by M_d the space of Fourier multipliers in L^d. We also denote

$$\dot{\mathbf{R}}^n = (\mathbf{R} \setminus \{0\})^n = \{\ \xi = (\xi_1, \dots, \xi_n)\ :\ \prod_{k=1}^{n} \xi_k \neq 0\ \}.$$

The following statement readily follows from the known Marcinkiewicz multiplier theorem (see [20, Ch. 4]).

Theorem 22. *Suppose that $a(\xi) \in C^n(\dot{\mathbf{R}}^n)$ is a function such that for some constant C*

$$|\xi^\alpha D^\alpha a(\xi)| \leq C, \quad \xi \in \dot{\mathbf{R}}^n, \tag{48}$$

for every multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$ such that $|\alpha| = \alpha_1 + \cdots + \alpha_n \leq n$. Then $a(\xi) \in M_d$ for all $d > 1$.

Here we use the notation $\xi^\alpha = \prod_{i=1}^n (\xi_i)^{\alpha_i}$, $D^\alpha = \prod_{i=1}^n \left(\frac{\partial}{\partial \xi_i}\right)^{\alpha_i}$. Actually, it is sufficient to require that (48) is satisfied for multi-indices α such that $\alpha_i \in \{0, 1\}$, $i = 1, \dots, n$.

We will use the statement of Theorem 22 for symbols of special type. Namely, assume that X is a linear subspace of $\mathbf{R}^n$, and $\pi_X : \mathbf{R}^n \to S_X$ be the projection defined in Section 2.

Corollary 23. *If $\psi \in C^n(S_X)$, then $\psi(\pi_X(\xi)) \in M_d$ for every $d > 1$.*

Proof. Using an orthogonal transform, we can assume that $X = \mathbf{R}^k = \{\xi \in \mathbf{R}^n\ :\ \xi = (y_1, \dots, y_k, 0, \dots, 0)\ \}$ while $X^\perp = \{\xi \in \mathbf{R}^n\ :\ \xi = (0, \dots, 0, z_1, \dots, z_{n-k})\ \}$. By the definition of π_X we have $\pi_X(t^2 y, tz) = \pi_X(y, z)$ for each $t > 0$ and $\xi = (y, z) \in \mathbf{R}^n$, $\xi \neq 0$. The function $a(y, z) = \psi(\pi_X(y, z))$ satisfies the same property $a(t^2 y, tz) = a(y, z)$. As is easy to see $a(y, z) \in C^n(\mathbf{R}^n \setminus \{0\})$ and it follows from the above homogeneity relation that

$$D_y^\alpha D_z^\beta a(y, z) = D_y^\alpha D_z^\beta a(t^2 y, tz) = t^{2|\alpha| + |\beta|} (D_y^\alpha D_z^\beta a)(t^2 y, tz). \tag{49}$$

Here $\alpha = (\alpha_1, \dots, \alpha_k)$, $\beta = (\beta_1, \dots, \beta_{n-k})$ are multi-indices corresponding to variables $y \in X$, $z \in X^\perp$, respectively, and $|\alpha| + |\beta| \leq n$.

Putting $t = (|y|^2 + |z|^4)^{-1/4}$ in (49), we obtain that

$$D_y^\alpha D_z^\beta a(y, z) = D_y^\alpha D_z^\beta a(t^2 y, tz) = (|y|^2 + |z|^4)^{-|\alpha|/2 - |\beta|/4} (D_y^\alpha D_z^\beta a)(y', z'), \tag{50}$$

where $(y', z') = \pi_X(y, z)$. Since the derivatives $|D_y^\alpha D_z^\beta a|$ are bounded on S_X it follows from (50) that for some constant $C > 0$

$$\begin{aligned} &|y^\alpha z^\beta D_y^\alpha D_z^\beta a(y, z)| \\ &\quad \leq C \frac{|y|^{|\alpha|} |z|^{|\beta|}}{(|y|^2 + |z|^4)^{|\alpha|/2 + |\beta|/4}} \leq C \frac{|y|^{|\alpha|}}{(|y|^2 + |z|^4)^{|\alpha|/2}} \frac{|z|^{|\beta|}}{(|y|^2 + |z|^4)^{|\beta|/4}} \leq C, \end{aligned}$$

for all multi-indices (α, β) such that $|\alpha| + |\beta| \leq n$. By Theorem 22 we conclude that $a(\xi) \in M_d$ for every $d > 1$. The proof is complete. □

Now we consider the symbol $a(\xi) = \dfrac{\rho(\xi)\sqrt{(1 + |\xi|^2)}}{(|\tilde{\xi}|^2 + |\bar{\xi}|^4)^{1/2}}$, where $\rho(\xi) \in C^\infty(\mathbf{R}^n)$ is a function with the properties indicated in Proposition 14, namely: $0 \leq \rho(\xi) \leq 1$, $\rho(\xi) = 0$ for $|\tilde{\xi}|^2 + |\bar{\xi}|^4 \leq 1$, $\rho(\xi) = 1$ for $|\tilde{\xi}|^2 + |\bar{\xi}|^4 \geq 2$. Another consequence of Theorem 22 is the following result.

Corollary 24. *$a(\xi) \in M_d$ for every $d > 1$.*

Proof. Obviously, $a(\xi) \in C^\infty(\mathbf{R}^n \setminus \{0\})$. As in the proof of Corollary 23, we can suppose that $X = \mathbf{R}^k = \{\xi \in \mathbf{R}^n \mid \xi = (y_1, \dots, y_k, 0, \dots, 0)\ \}$. Then for $\xi = (y, z) \in X \times X^\perp$, $\xi \neq 0$

$$a(y,z) = \frac{\rho(y,z)(1+|y|^2+|z|^2)^{1/2}}{(|y|^2+|z|^4)^{1/2}} = \rho(y,z)a_1(y,z)a_2(y,z),$$

where we denote

$$a_1(y,z) = (1+|y|^2+|z|^2)^{1/2}, \quad a_2(y,z) = (|y|^2+|z|^4)^{-1/2}.$$

In correspondence with (48) we have to show that for all α, β, $|\alpha| + |\beta| \le n$

$$|y^\alpha z^\beta D_y^\alpha D_z^\beta (a_1(y,z)a_2(y,z))| \le C \tag{51}$$

in the domain $|y|^2 + |z|^4 \ge 1$ (here we take into account the properties of $\rho(\xi)$) for some constant C. In order to prove (51), we estimate derivatives of functions a_1, a_2. Evidently,

$$\begin{aligned} |D_y^\alpha D_z^\beta a_1(y,z)| &\le A_m(1+|y|^2+|z|^2)^{(1-|\alpha|-|\beta|)/2} \\ &\le A_m(1+|y|^2+|z|^2)^{1/2}|y|^{-|\alpha|}|z|^{-|\beta|}, \end{aligned} \tag{52}$$

where A_m is a constant depending only on $m = |\alpha| + |\beta|$. Furthermore, we observe that the function $a_2(y,z)$ satisfies the homogeneity relation $a_2(t^2y, tz) = t^{-2}a_2(y,z)$. It follows from this relation that

$$D_y^\alpha D_z^\beta a_2(y,z) = t^2 D_y^\alpha D_z^\beta a_2(t^2y, tz) = t^{2|\alpha|+|\beta|+2}(D_y^\alpha D_z^\beta a_2)(t^2y, tz).$$

Taking in this equality $t = (|y|^2+|z|^4)^{-1/4}$, we arrive at

$$D_y^\alpha D_z^\beta a_2(y,z) = (|y|^2+|z|^4)^{-(2|\alpha|+|\beta|+2)/4}(D_y^\alpha D_z^\beta a_2)(y', z'), \quad (y', z') = \pi_X(y,z) \in S_X.$$

Since the derivatives $D_y^\alpha D_z^\beta a_2$ are bounded on S_X the latter equality yields the estimates

$$\begin{aligned} |D_y^\alpha D_z^\beta a_2(y,z)| &\le B_m(|y|^2+|z|^4)^{-(2|\alpha|+|\beta|+2)/4} \\ &\le B_m(|y|^2+|z|^4)^{-1/2}|y|^{-|\alpha|}|z|^{-|\beta|}, \end{aligned} \tag{53}$$

where the constants B_m depend on $m = |\alpha| + |\beta|$. By the Leibniz formula we derive from (52) and (53) the estimates

$$|D_y^\alpha D_z^\beta a_1(y,z)a_2(y,z)| \le C_m(1+|y|^2+|z|^2)^{1/2}(|y|^2+|z|^4)^{-1/2}|y|^{-|\alpha|}|z|^{-|\beta|}, \tag{54}$$

where C_m is a constant. As is easily verified, in the domain $|y|^2 + |z|^4 \ge 1$

$$\frac{1+|y|^2+|z|^2}{|y|^2+|z|^4} \le 1 + \frac{|y|^2}{|y|^2+|z|^4} + \frac{|z|^2}{|y|^2+|z|^4} \le 2 + \min(|z|^2, |z|^{-2}) \le 3$$

and by (54) we conclude that in this domain for each α, β, $|\alpha| + |\beta| \le n$

$$|D_y^\alpha D_z^\beta a_1(y,z)a_2(y,z)| \le C|y|^{-|\alpha|}|z|^{-|\beta|},$$

C being a constant. It is clear that this implies (51). Hence, the requirements of Theorem 22 are satisfied. Therefore, $a(\xi) \in M_d$ for all $d > 1$. The proof is complete. □

Now we consider the bounded sequence of measure-valued functions $\nu_x^k \in \mathrm{MV}(\Omega)$ and suppose that for some $d > 1$ and each $a, b \in \mathbf{R}$, $a < b$ the sequence of distributions

$$\mathrm{div}_x \int \varphi(x, s_{a,b}(\lambda))d\nu_x^k(\lambda) - D^2 \cdot \int B(s_{a,b}(\lambda))d\nu_x^k(\lambda) \text{ is precompact in } W_{d,\mathrm{loc}}^{-1}(\Omega). \tag{55}$$

Here $s_{a,b}(u) = \max(a, \min(u,b))$ is the cut-off function and $W_{d,\mathrm{loc}}^{-1}(\Omega)$ denotes the locally convex space of distributions $u(x)$ such that $uf(x)$ belongs to the Sobolev space $W_d^{-1}(\mathbf{R}^n)$ for all $f(x) \in C_0^\infty(\Omega)$. The topology in $W_{d,\mathrm{loc}}^{-1}(\Omega)$ is generated by the family of semi-norms $u \mapsto \|uf\|_{W_d^{-1}}$, $f(x) \in C_0^\infty(\Omega)$.

We choose the subsequence $\nu_x^r = \nu_x^k$, $k = k_r$ weakly convergent to a bounded measure-valued function ν_x^0 such that the H-measure μ^{pq}, $p, q \in E$ is welldefined.

Fix $p_0 \in E$ and choose a countable dense subset $D \subset E$ such that $p_0 \in D$. Now define a linear subspace X being the maximal among linear subspaces of $Y \subset \mathbf{R}^n$ such that for some positive δ for every $p \in [p_0, p_0 + \delta]$

$$(B(p) - B(p_0))\xi \cdot \xi = 0$$

for all $\xi \in Y$. Since for $p \geq p_0$ the matrices $(B(p) - B(p_0)) \geq 0$, the space X can be expressed as follows

$$X = \bigcup_{\delta > 0} \bigcap_{0 \leq p - p_0 \leq \delta} \ker(B(p) - B(p_0)). \tag{56}$$

It is possible that the space $X = \{0\}$. Passing to a subsequence of ν_x^r if necessary we can suppose that the ultra-parabolic H-measure $\mu^{pq} = \mu_X^{pq}$ is welldefined. By Proposition 14 this H-measure can be represented in the form $\mu^{pq} = \mu_x^{pq} dx$, $p, q \in D$, $x \in \Omega'$, where $\Omega' \subset \Omega$ is a set of full measure indicated in the proof of Proposition 14. Define the set of full measure Ω_φ consisting of common Lebesgue points of the maps $F(x) = \varphi(x, \cdot) \in C(\mathbf{R}, \mathbf{R}^n)$, $|F(x)|^2 = |\varphi(x, \cdot)| \in C(\mathbf{R})$ and fix $x \in \Omega'' = \Omega' \cap \Omega_\varphi$.

Under the above assumptions we have the following localization principle.

Theorem 25. *Let L be a linear span of $\operatorname{supp} \mu_x^{p_0 p_0}$. Then $L \subset X$ and there exists $\delta_1 > 0$ such that $(\varphi(x, \lambda) - \varphi(x, p_0)) \cdot \xi = 0$ for all $\xi \in L$, $\lambda \in [p_0, p_0 + \delta_1]$.*

Proof. By the definition of the space X for some $\delta > 0$ and each $p \in [p_0, p_0 + \delta]$, $(B(p) - B(p_0))\xi \cdot \xi = 0$ for all $\xi \in X$. Let $V = V_\delta = [p_0, p_0 + \delta] \cap D$ and $p \in V$. As follows from (55) and the weak convergence $\nu_y^r \to \nu_y^0$,

$$\begin{aligned} \mathcal{L}_p^r(y) = \operatorname{div}_y \int \varphi(y, s_{p_0,p}(\lambda)) d\gamma_y^r(\lambda) \\ - D^2 \cdot \int B(s_{p_0,p}(\lambda)) d\gamma_y^r(\lambda) \underset{r \to \infty}{\to} 0 \quad \text{in } W_{d,\mathrm{loc}}^{-1}(\Omega), \end{aligned} \tag{57}$$

where $\gamma_y^r = \nu_y^r - \nu_y^0$. As is easy to compute,

$$\begin{aligned} \varphi(y, s_{p_0,p}(\lambda)) &= \varphi(y, p_0) + (\varphi(y, p) - \varphi(y, p_0))\theta(\lambda - p_0) - (\varphi(y, p) - \varphi(y, \lambda))\chi(\lambda), \\ B(s_{p_0,p}(\lambda)) &= B(p_0) + (B(p) - B(p_0))\theta(\lambda - p_0) - (B(p) - B(\lambda))\chi(\lambda), \end{aligned}$$

where $\chi(\lambda) = \theta(\lambda - p_0) - \theta(\lambda - p)$ is the indicator function of the interval $(p_0, p]$. Therefore, $\mathcal{L}_p^r = \operatorname{div}_y(P_r(y)) - D^2 \cdot Q_r(y)$ where the vector $P_r(y)$ and the matrix $Q_r(y) = \{(Q_r)_{kl}(y)\}_{kl=1}^n$ are as follows (notice that $\int d\gamma_y^r(\lambda) = 0$):

$$\begin{aligned} P_r(y) &= \int (\varphi(y, p) - \varphi(y, p_0))\theta(\lambda - p_0) d\gamma_y^r(\lambda) \\ &\qquad - \int (\varphi(y, p) - \varphi(y, \lambda))\chi(\lambda) d\gamma_y^r(\lambda) \\ &= (\varphi(y, p) - \varphi(y, p_0)) U_r^{p_0}(y) - \int (\varphi(y, p) - \varphi(y, \lambda))\chi(\lambda) d\gamma_y^r(\lambda); \end{aligned} \tag{58}$$

$$Q_r(y) = U_r^{p_0}(y)(B(p) - B(p_0)) - \int (B(p) - B(\lambda))\chi(\lambda) d\gamma_y^r(\lambda). \tag{59}$$

In particular, it follows from (59) and the choice of the space X that $X \subset \ker Q_r$.

For $\Phi(y) \in C_0^\infty(\Omega)$ we consider the sequence

$$L_r = \operatorname{div}_y(\Phi(y) P_r(y)) + 2\operatorname{div}(Q_r(y) \nabla \Phi(y)) - D^2 \cdot (\Phi(y) Q_r(y))$$

$$= \operatorname{div}_y(\Phi(y)P_r(y)) + 2(\Phi_{y_l}(Q_r)_{kl}(y))_{y_k} - \partial^2_{y_k y_l}(\Phi(y)(Q_r)_{kl}(y))$$
$$= \Phi(y)\mathcal{L}^r_p(y) + P_r(y)\cdot\nabla\Phi(y) + D^2\Phi(y)\cdot Q_r(y).$$

Since the sequence $P_r(y)\cdot\nabla\Phi(y) + D^2\Phi(y)\cdot Q_r(y)$ is bounded in L^2 and weakly converges to zero as $r\to\infty$, this sequence converges to zero in W_d^{-1} (we can suppose that $d\le 2$). Besides, in view of (57), $\Phi(y)\mathcal{L}^r_p(y) \underset{r\to\infty}{\to} 0$ in W_d^{-1} as well, and we claim that $L_r \underset{r\to\infty}{\to} 0$ in W_d^{-1}. Introduce the vector $G_r(y,\lambda) = 2Q_r(y)\nabla\Phi(y)$ with components $(G_r)_k(y) = 2\Phi_{y_l}(Q_r)_{kl}(y)$, $k=1,\dots,n$. Then the distributions L_r can be represented in the form $L_r = \operatorname{div}_y(\Phi P_r + G_r) - D^2\cdot(\Phi Q_r)$. Hence,

$$\operatorname{div}_y(\Phi P_r + G_r) - D^2\cdot(\Phi Q_r) \underset{r\to\infty}{\to} 0 \ \text{ in } W_d^{-1}.$$

Applying the Fourier transform to this relation and then multiplying the result by $\rho(\xi)(|\tilde\xi|^2+|\bar\xi|^4)^{-1/2}$, we arrive at

$$\begin{aligned}\frac{\rho(\xi)(2\pi i\xi\cdot F(\Phi P_r+G_r)(\xi) + 4\pi^2 F(\Phi Q_r)(\xi)\xi\cdot\xi)}{(|\tilde\xi|^2+|\bar\xi|^4)^{1/2}}&\\ = F(l_r)(\xi), \qquad \text{where } l_r \underset{r\to\infty}{\to} 0 \quad &\text{in } L^d(\mathbf{R}^n)\end{aligned} \tag{60}$$

(the function $\rho(\xi)$ is indicated in Proposition 18). Indeed, (60) follows from the representation

$$\rho(\xi)(|\tilde\xi|^2+|\bar\xi|^4)^{-1/2} = \frac{\rho(\xi)(1+|\xi|^2)^{1/2}}{(|\tilde\xi|^2+|\bar\xi|^4)^{1/2}}(1+|\xi|^2)^{-1/2},$$

the statement of Corollary 24 and the definition of W_d^{-1}. Let $\psi(\xi)\in C^n(S_X)$. Then by Corollary 23 we see that the sequence $F(\Phi U_r^{p_0})(\xi)\overline{\psi(\pi_X(\xi))} = F(h_r)$, where h_r is bounded in $L^{d'}(\mathbf{R}^n)$, $d' = d/(d-1)$. This and (60) imply the relation

$$\begin{aligned}\int_{\mathbf{R}^n}\frac{\rho(\xi)(2\pi i\xi\cdot F(\Phi P_r+G_r)(\xi) + 4\pi^2F(\Phi Q_r)(\xi)\xi\cdot\xi)}{(|\tilde\xi|^2+|\bar\xi|^4)^{1/2}}&\overline{F(\Phi U_r^{p_0})(\xi)}\psi(\pi_X(\xi))d\xi\\ = \int_{\mathbf{R}^n} l_r(x)\overline{h_r(x)}dx \underset{r\to\infty}{\to} 0.&\end{aligned} \tag{61}$$

Now, we remark that the sequences $\Phi(y)U_r^{p_0}(y)$ is bounded in $L^2\cap L^1$ and weakly converges to zero. By Lemma 17 we have

$$\frac{\overline{F(\Phi U_r^{p_0})(\xi)}\rho(\xi)\bar\xi}{(|\tilde\xi|^2+|\bar\xi|^4)^{1/2}} \underset{r\to\infty}{\to} 0 \ \text{ in } L^2(\mathbf{R}^n,\mathbf{R}^n) \tag{62}$$

because

$$a(\xi) = \frac{\rho(\xi)|\bar\xi|}{(|\tilde\xi|^2+|\bar\xi|^4)^{1/2}} \le \frac{|\bar\xi|}{(|\tilde\xi|^2+|\bar\xi|^4)^{1/4}} \le 1$$

and evidently $a(\xi)\to 0$ as $|\xi|\to\infty$. Besides,

$$\begin{aligned}\tilde\xi\cdot F(G_r)(\xi) &= 2\int_{\mathbf{R}^n} e^{-2\pi i\xi\cdot y}(Q_r)_{kl}(y)\Phi_{y_l}(y)\tilde\xi_k dy\\ &= \int_{\mathbf{R}^n} e^{-2\pi i\xi\cdot y}Q_r(y)\tilde\xi\cdot\nabla\Phi(y)dy = 0,\end{aligned} \tag{63}$$

$$F(\Phi Q_r)(\xi)\tilde\xi = \int_{\mathbf{R}^n} e^{-2\pi i\xi\cdot y}\Phi(y)Q_r(y)\tilde\xi dy = 0, \tag{64}$$

since $\tilde{\xi} \in X \subset \ker Q_r$. Taking into account relations (62), (63), (64), and the boundedness of the sequence $F(\Phi U_r^{p_0})(\xi)$ in $L^2(\mathbf{R}^n)$, we derive from (61) that

$$\int_{\mathbf{R}^n} \frac{\rho(\xi)(2\pi i\tilde{\xi}\cdot F(\Phi P_r)(\xi) + 4\pi^2 F(\Phi Q_r)(\xi)\bar{\xi}\cdot\bar{\xi})}{(|\tilde{\xi}|^2+|\bar{\xi}|^4)^{1/2}} \overline{F(\Phi U_r^{p_0})(\xi)}\psi(\pi_X(\xi))d\xi \underset{r\to\infty}{\to} 0. \tag{65}$$

Taking into account representations (58) and (59) we can rewrite the last relation as follows

$$\begin{aligned} \lim_{r\to\infty} \Bigg\{ \int_{\mathbf{R}^n} & \frac{\rho(\xi)(2\pi i\tilde{\xi}\cdot F(\Phi f U_r^{p_0})(\xi) + 4\pi^2 F(\Phi U_r^{p_0})(\xi)(B(p)-B(p_0))\bar{\xi}\cdot\bar{\xi})}{(|\tilde{\xi}|^2+|\bar{\xi}|^4)^{1/2}} \\ & \times \overline{F(\Phi U_r^{p_0})(\xi)}\psi(\pi_X(\xi))d\xi \\ - \int_{\mathbf{R}^n} & \frac{\rho(\xi)(i\tilde{\xi}\cdot F(\Phi V_r^p)(\xi) + F(\Phi G_r^p)(\xi)\bar{\xi}\cdot\bar{\xi})}{(|\tilde{\xi}|^2+|\bar{\xi}|^4)^{1/2}} \overline{F(\Phi U_r^{p_0})(\xi)}\psi(\pi_X(\xi))d\xi \Bigg\} = 0, \end{aligned} \tag{66}$$

where

$$f(y) = \varphi(y,p) - \varphi(y,p_0), \quad V_r^p(y) = 2\pi \int (\varphi(y,p)-\varphi(y,\lambda))\chi(\lambda)d\gamma_y^r(\lambda) \in \mathbf{R}^n,$$

$$G_r^p(y) = 4\pi^2 \int (B(p)-B(\lambda))\chi(\lambda)d\gamma_y^r(\lambda) \in \mathrm{Sym}_n\,.$$

In (66) we set $\Phi(y) = \Phi_m(x-y) = \sqrt{K_m(x-y)}$, where the functions K_m were defined in section 3 in the proof of Proposition 14, and pass to the limit as $m\to\infty$. By Remark 15 (see (27)) we obtain

$$\begin{aligned} \lim_{m\to\infty}\lim_{r\to\infty} \int_{\mathbf{R}^n} & \frac{\rho(\xi)(2\pi i\tilde{\xi}\cdot F(f\Phi_m U_r^{p_0})(\xi) + 4\pi^2 F(\Phi_m U_r^{p_0})(\xi)(B(p)-B(p_0))\bar{\xi}\cdot\bar{\xi})}{(|\tilde{\xi}|^2+|\bar{\xi}|^4)^{1/2}} \\ & \times \overline{F(\Phi_m U_r^{p_0})(\xi)}\psi(\pi_X(\xi))d\xi \\ = & \left\langle \mu_x^{p_0p_0}, (2\pi i\tilde{\xi}\cdot f(x) + 4\pi^2(B(p)-B(p_0))\bar{\xi}\cdot\bar{\xi})\psi(\xi)\right\rangle, \end{aligned}$$

and therefore

$$\begin{aligned} & \left\langle \mu_x^{p_0p_0}, (2\pi i\tilde{\xi}\cdot f(x) + 4\pi^2(B(p)-B(p_0))\bar{\xi}\cdot\bar{\xi})\psi(\xi)\right\rangle \\ & = \lim_{m\to\infty}\lim_{r\to\infty}\int_{\mathbf{R}^n} \frac{\rho(\xi)(i\tilde{\xi}\cdot F(\Phi_m V_r^p)(\xi) + F(\Phi_m G_r^p)(\xi)\bar{\xi}\cdot\bar{\xi})}{(|\tilde{\xi}|^2+|\bar{\xi}|^4)^{1/2}} \\ & \qquad \times \overline{F(\Phi_m U_r^{p_0})(\xi)}\psi(\pi_X(\xi))d\xi. \end{aligned} \tag{67}$$

Since the space $C^n(S_X)$ is dense in $C(S_X)$, it is clear that (67) holds for each $\psi(\xi) \in C(S_X)$. Let $g(y,\lambda) = \tilde{P}\varphi(y,\lambda)$, $B_1(\lambda) = \bar{P}B(\lambda)\bar{P}$, where $\tilde{P}$ and $\bar{P}$ are orthogonal projections on the spaces $\tilde{L} = P_1(L)$ and $\bar{L} = P_2(L)$, respectively, L being the linear span of $\operatorname{supp}\mu_x^{p_0p_0}$ (see the notation of Section 3). Obviously,

$$\begin{aligned} & \left\langle \mu_x^{p_0p_0}, (2\pi i\tilde{\xi}\cdot f(x) + 4\pi^2(B(p)-B(p_0))\bar{\xi}\cdot\bar{\xi})\psi(\xi)\right\rangle \\ & = \left\langle \mu_x^{p_0p_0}, (2\pi i\tilde{\xi}\cdot(g(x,p)-g(x,p_0)) + 4\pi^2(B_1(p)-B_1(p_0))\bar{\xi}\cdot\bar{\xi})\psi(\xi)\right\rangle. \end{aligned} \tag{68}$$

We denote $h(y,\lambda) = \varphi(y,\lambda) - g(y,\lambda)$, $B_2(\lambda) = B(\lambda) - B_1(\lambda)$,

$$V_{r1}^p(y) = 2\pi \int (g(y,p)-g(y,\lambda))\chi(\lambda)d\gamma_y^r(\lambda),$$

$$V_{r2}^p(y) = 2\pi \int (h(y,p)-h(y,\lambda))\chi(\lambda)d\gamma_y^r(\lambda),$$

$$G_{r1}^p(y) = 4\pi^2 \int (B_1(p)-B_1(\lambda))\chi(\lambda)d\gamma_y^r(\lambda),$$

$$G^p_{r2}(y) = 4\pi^2 \int (B_2(p) - B_2(\lambda))\chi(\lambda) d\gamma^r_y(\lambda).$$

Since $\xi \cdot h(y,\lambda) = 0$ for all $\xi \in \tilde{L}$, $B_2(\lambda)\xi \cdot \xi = 0$ for all $\xi \in \bar{L}$, and in the notation of Propositions 18 and 19, $V^p_{r2}(y) = I_r(f\chi)(y)$ with $f(y,\lambda) = 2\pi(h(y,p) - h(y,\lambda))$, $G^p_{r2}(y) = J_r(Q)(y)$ with $Q(\lambda) = 4\pi^2(B_2(p) - B_2(\lambda))$, it follows from Propositions 18 and 19 that

$$\lim_{m\to\infty} \overline{\lim_{r\to\infty}} \left| \int_{\mathbf{R}^n} \frac{\rho(\xi)(i\tilde{\xi} \cdot F(\Phi_m V^p_{r2})(\xi) + F(\Phi_m G^p_{r2})(\xi)\bar{\xi} \cdot \bar{\xi})}{(|\tilde{\xi}|^2 + |\bar{\xi}|^4)^{1/2}} \times \overline{F(\Phi_m U^{p_0}_r)(\xi)} \psi(\pi_X(\xi)) d\xi \right| = 0,$$

and in view of (67) we find that

$$\begin{aligned} &\left| \left\langle \mu^{p_0 p_0}_x, (2\pi i \tilde{\xi} \cdot (g(x,p) - g(x,p_0)) + 4\pi^2 (B_1(p) - B_1(p_0))\bar{\xi} \cdot \bar{\xi})\psi(\xi) \right\rangle \right| \\ &\le \overline{\lim_{m\to\infty}} \; \overline{\lim_{r\to\infty}} \left| \int_{\mathbf{R}^n} \frac{\rho(\xi)(i\tilde{\xi} \cdot F(\Phi_m V^p_{r1})(\xi) + F(\Phi_m G^p_{r1})(\xi)\bar{\xi} \cdot \bar{\xi})}{(|\tilde{\xi}|^2 + |\bar{\xi}|^4)^{1/2}} \times \overline{F(\Phi_m U^{p_0}_r)(\xi)} \psi(\pi_X(\xi)) d\xi \right|. \end{aligned} \tag{69}$$

Here we also use relation (68). Now we observe that

$$\left| \frac{\rho(\xi)(i\tilde{\xi} \cdot F(\Phi_m V^p_{r1})(\xi) + F(\Phi_m G^p_{r1})(\xi)\bar{\xi} \cdot \bar{\xi})}{(|\tilde{\xi}|^2 + |\bar{\xi}|^4)^{1/2}} \right| \le |F(\Phi_m V^p_{r1})(\xi)| + |F(\Phi_m G^p_{r1})(\xi)|,$$

and therefore

$$\begin{aligned} &\left\| \frac{\rho(\xi)(i\tilde{\xi} \cdot F(\Phi_m V^p_{r1})(\xi) + F(\Phi_m G^p_{r1})(\xi)\bar{\xi} \cdot \bar{\xi})}{(|\tilde{\xi}|^2 + |\bar{\xi}|^4)^{1/2}} \right\|_2 \\ &\le \|F(\Phi_m V^p_{r1})\|_2 + \|F(\Phi_m G^p_{r1})\|_2 = \|\Phi_m V^p_{r1}\|_2 + \|\Phi_m G^p_{r1}\|_2, \end{aligned} \tag{70}$$

by Plancherel's equality. Since $|U^{p_0}_r| \le 1$,

$$\|F(\Phi_m U^{p_0}_r)\|_2 = \|\Phi_m U^{p_0}_r\|_2 \le 1,$$

and we derive from (69) with the help of Cauchy–Bunyakovsky–Schwarz inequality and (70) that

$$\begin{aligned} &\left| \left\langle \mu^{p_0 p_0}_x, (2\pi i \tilde{\xi} \cdot (g(x,p) - g(x,p_0)) + 4\pi^2 (B_1(p) - B_1(p_0))\bar{\xi} \cdot \bar{\xi})\psi(\xi) \right\rangle \right| \\ &\le \|\psi\|_\infty \overline{\lim_{m\to\infty}} \; \overline{\lim_{r\to\infty}} \left(\|\Phi_m V^p_{r1}\|_2 + \|\Phi_m G^p_{r1}\|_2\right). \end{aligned} \tag{71}$$

Next, for $M_p(y) = \max_{\lambda \in [p_0,p]} |g(y,p) - g(y,\lambda)|$

$$\begin{aligned} |V^p_{r1}(y)| &\le 2\pi M_p(y) \int \chi(\lambda) d\left(\nu^r_y(\lambda) + \nu^0_y(\lambda)\right) \\ &= 2\pi M_p(y)(u_r(y,p_0) - u_r(y,p) + u_0(y,p_0) - u_0(y,p)), \end{aligned}$$

and by Lemma 16

$$\overline{\lim_{m\to\infty}} \; \overline{\lim_{r\to\infty}} \|\tilde{V}^p_{r1}\Phi_m\|_2 \le 4\pi M_p(x)(u_0(x,p_0) - u_0(x,p))^{1/2}. \tag{72}$$

Here we bear in mind that x is a Lebesgue point of the function $(M_p(y))^2$ (which easily follows from the fact that $x \in \Omega_\varphi$ is a Lebesgue point of the maps $y \mapsto \varphi(y,\cdot)$, $y \mapsto |\varphi(y,\cdot)|^2$ into the spaces $C(\mathbf{R}, \mathbf{R}^n)$, $C(\mathbf{R})$, respectively). Furthermore, the matrix $0 \le B_1(p) - B_1(\lambda) \le B_1(p) - B_1(p_0)$ for each $\lambda \in [p_0, p]$ (since the matrix

$B_1(\lambda)-B_1(p_0)$ is positive definite). This implies the corresponding inequality for the Euclidean norms $|B_1(p)-B_1(\lambda)|\le|B_1(p)-B_1(p_0)|$. Therefore

$$\begin{aligned}|G^p_{r1}(y)|&\le 4\pi^2\int|B_1(p)-B_1(\lambda)|\chi(\lambda)d\left(\nu^r_y(\lambda)+\nu^0_y(\lambda)\right)\\&\le 4\pi^2|B_1(p)-B_1(p_0)|(u_r(y,p_0)-u_r(y,p)+u_0(y,p_0)-u_0(y,p)).\end{aligned}$$

By Lemma **??** again we claim that

$$\overline{\lim_{m\to\infty}}\ \overline{\lim_{r\to\infty}}\|G^p_{r1}\Phi_m\|_2\le 8\pi^2|B_1(p)-B_1(p_0)|(u_0(x,p_0)-u_0(x,p))^{1/2}.\tag{73}$$

In view of (72) and (73) we derive from (71) that

$$\begin{aligned}&\left|\left\langle\mu^{p_0p_0}_x,(2\pi i\tilde{\xi}\cdot(g(x,p)-g(x,p_0))+4\pi^2(B_1(p)-B_1(p_0))\bar{\xi}\cdot\bar{\xi})\psi(\xi)\right\rangle\right|\\&\qquad\le c\|\psi\|_\infty(M_p(x)+|B_1(p)-B_1(p_0)|)\omega(p),\end{aligned}\tag{74}$$

where c is a constant, and $\omega(p)=(u_0(x,p_0)-u_0(x,p))^{1/2}\underset{p\to p_0}{\to}0$ (recall that $p_0\in D$ is a continuity point of $p\to u_0(x,p)$ for $x\in\Omega'$). Next, by Corollary 21, we can choose functions $\psi_k(\xi)\in C(S_X)$, $k=1,\dots,l$, such that

$$\begin{aligned}&|g(x,p)-g(x,p_0)|+|B_1(p)-B_1(p_0)|\\&\le C\max_{k=1,\dots,l}\left|\left\langle\mu^{p_0p_0}_x,(i\tilde{\xi}\cdot(g(x,p)-g(x,p_0))+(B_1(p)-B_1(p_0))\bar{\xi}\cdot\bar{\xi})\psi_k(\xi)\right\rangle\right|\\&\le c(M_p(x)+|B_1(p)-B_1(p_0)|)\omega(p),\end{aligned}\tag{75}$$

where C,c are positive constants.

We choose $\delta_1\in(0,\delta)$ such that $2c\omega(p)\le\varepsilon<1$ for all $p\in[p_0,p_0+\delta_1]\cap D$. Then, in view of (75),

$$\begin{aligned}&|g(x,p)-g(x,p_0)|+|B_1(p)-B_1(p_0)|\\&\le\frac{\varepsilon}{2}\left(\max_{\lambda\in[p_0,p]}|g(x,p)-g(x,\lambda)|+|B_1(p)-B_1(p_0)|\right),\end{aligned}\tag{76}$$

and since $g(x,p)$, $B_1(p)$ are continuous with respect to p and the set D is dense, estimate (76) holds for all $p\in[p_0,p_0+\delta_1]$.

Now we claim that $g(x,p)=g(x,p_0)$, $B_1(p)=B_1(p_0)$ for $p\in[p_0,p_0+\delta_1]$. Indeed, assume that for $p'\in[p_0,p_0+\delta_1]$

$$|g(x,p')-g(x,p_0)|=\max_{\lambda\in[p_0,p_0+\delta_1]}|g(x,\lambda)-g(x,p_0)|.$$

Then for $\lambda\in[p_0,p']$ we have

$$\begin{aligned}|g(x,p')-g(x,\lambda)|&\le|g(x,\lambda)-g(x,p_0)|+|g(x,p')-g(x,p_0)|\\&\le 2|g(x,p')-g(x,p_0)|\end{aligned}$$

and

$$\max_{\lambda\in[p_0,p']}|g(x,p')-g(x,\lambda)|\le 2|g(x,p')-g(x,p_0)|.$$

We derive from (76) with $p=p'$ that

$$|g(x,p')-g(x,p_0)|+|B_1(p')-B_1(p_0)|\le\varepsilon(|g(x,p')-g(x,p_0)|+|B_1(p')-B_1(p_0)|),$$

and since $\varepsilon<1$, this implies that

$$|g(x,p')-g(x,p_0)|=\max_{\lambda\in[p_0,p_0+\delta_1]}|g(x,\lambda)-g(x,p_0)|=0.$$

This means that $g(x,\lambda)=g(x,p_0)$ for $\lambda\in[p_0,p_0+\delta_1]$. Then, (76) takes the form

$$|B_1(p)-B_1(p_0)|\le\frac{\varepsilon}{2}|B_1(p)-B_1(p_0)|,\quad\varepsilon<1.$$

Hence $B_1(p) = B_1(p_0)$ for every $p \in [p_0, p_0 + \delta_1]$. By the definition of $B_1(p)$ we see that $(B(p) - B(p_0))\bar{P} = 0$, that is $\bar{L} \subset \ker(B(p) - B(p_0))$ for all $p \in [p_0, p_0 + \delta_1]$. Taking into account that also $X \subset \ker(B(p) - B(p_0))$ for such p we claim that $X \oplus \bar{L} \subset \ker(B(p) - B(p_0))$ for all $p \in [p_0, p_0 + \delta_1]$. By the maximality of the space X we conclude that $\bar{L} = \{0\}$, that is, $L \subset X$. Then $\tilde{L} = L$ and the relation $\tilde{P}(\varphi(x, \lambda) - \varphi(x, p_0)) = g(x, \lambda) - g(x, p_0) = 0$ on $[p_0, p_0 + \delta_1]$ implies that for all $\xi \in L$ $\xi \cdot (\varphi(x, \lambda) - \varphi(x, p_0)) = 0$ on the segment $\lambda \in [p_0, p_0 + \delta_1]$. The proof is complete. □

Under the non-degeneracy condition, indicated in Definition 2, Theorem 25 yields the following result.

Theorem 26. *Suppose the non-degeneracy condition is satisfied. Then any sequence ν_x^k weakly converging as $k \to \infty$ to ν_x^0 and satisfying* (55) *strongly converges to ν_x^0.*

Proof. Let $\nu_x^r = \nu_x^k$, $k = k_r$, be a subsequence such that the H-measure $\{\tilde{\mu}^{pq}\}_{p,q \in E}$, corresponding to the trivial subspace $X = \{0\}$, is welldefined. We fix $p_0 \in E$ and define the subspace X as in (56). Selecting a subsequence, if necessary, we can assume that the ultra-parabolic H-measure μ^{pq} corresponding to X is also welldefined. This H-measure admits the representation $\mu^{pq} = \mu_x^{pq} dx$ and, as directly follows from the assertion of Theorem 25 and non-degeneracy condition in Definition 2, $\mu_x^{p_0 p_0} = 0$ for a.e. $x \in \Omega$. Therefore, $\mu^{p_0 p_0} = \mu_x^{p_0 p_0} dx \equiv 0$. By relation (17) with $\psi \equiv 1$ we see that $\mathrm{pr}_\Omega \tilde{\mu}^{p_0 p_0} = \mathrm{pr}_\Omega \mu^{p_0 p_0} = 0$. Hence, $\tilde{\mu}^{p_0 p_0} = 0$. Since $p_0 \in E$ is arbitrary we conclude that $\tilde{\mu}^{pp} = 0$ for all $p \in E$. This implies that

$$u_r(x, p) \to u_0(x, p) \quad \text{in } L^2_{\text{loc}}(\Omega),$$

as $r \to \infty$. Indeed, it follows from the definition of an H-measure and Plancherel's equality that

$$\lim_{r \to \infty} \|U_r^p \Phi\|_2^2 = \left\langle \tilde{\mu}^{pp}, |\Phi(x)|^2 \right\rangle = 0$$

for all $\Phi(x) \in C_0(\Omega)$ and $p \in E$. Thus, for $p \in E$ we have

$$\int \theta(\lambda - p) d\nu_x^r(\lambda) \underset{r \to \infty}{\to} \int \theta(\lambda - p) d\nu_x^0(\lambda) \quad \text{in } L^2_{\text{loc}}(\Omega). \tag{77}$$

Any continuous function can be uniformly approximated on any compact subset by finite linear combinations of functions $\lambda \mapsto \theta(\lambda - p)$, $p \in E$. Hence, it follows from (77) that for all $f(\lambda) \in C(\mathbf{R})$ we have

$$\int f(\lambda) d\nu_x^r(\lambda) \underset{r \to \infty}{\to} \int f(\lambda) d\nu_x^0(\lambda) \quad \text{in } L^2_{\text{loc}}(\Omega),$$

and therefore also in $L^1_{\text{loc}}(\Omega)$, that is, the subsequence ν_x^r strongly converges to ν_x^0. Finally, for each admissible choice of the subsequence ν_x^r the limit measure-valued function is uniquely defined, therefore the original sequence ν_x^k is also strongly convergent to ν_x^0. The proof is complete. □

Taking into account Theorem T, one can give another formulation of Theorem 26: each bounded sequence of measure-valued functions satisfying (55) is precompact in the sense of strong convergence. Observe that in the regular case $\nu_x^k(\lambda) = \delta(\lambda - u_k(x))$ condition (55) has the form: for some $d > 1$ and each $a, b \in \mathbf{R}$, $a < b$

$$\mathrm{div}_x \varphi(x, s_{a,b}(u_k(x))) - D^2 \cdot B(s_{a,b}(u_k(x))) \quad \text{is precompact in } W^{-1}_{d,\text{loc}}(\Omega). \tag{78}$$

In this case Theorem 26 yields the following result.

Corollary 27. *Under the non-degeneracy condition, each bounded sequence $u_k(x) \in L^\infty(\Omega)$ satisfying* (78) *contains a subsequence convergent in $L^1_{\text{loc}}(\Omega)$.*

Proof. We only need to note that if the sequence $u_k(x)$ converges to a measure-valued function ν_x^0 strongly in $MV(\Omega)$, then by the definition of strong convergence

$$u_k(x) \underset{k\to\infty}{\to} u_0(x) = \int \lambda d\nu_x^0(\lambda) \text{ in } L^1_{\rm loc}(\Omega)$$

(which also shows that $\nu_x^0(\lambda) = \delta(\lambda - u_0(x))$ is regular in Ω). □

The statements of Theorems 25 and 26 remain true for sequences of unbounded measure-valued (or usual) functions. For the proof we should apply the cut-off functions $s_{a,b}(u) = \max(a, \min(u, b))$, $a, b \in \mathbf{R}$ and derive that bounded sequences of measure-valued functions $s^*_{a,b}\nu_x^k$ ($s^*_{a,b}\nu_x^k$ is the image of ν_x^k under the map $s_{a,b}$) satisfy (55). Then, under the non-degeneracy condition, we obtain the strong precompactness property for these sequences.

For instance, consider the sequence $u_k(x)$, $k \in \mathbf{N}$ of measurable functions on Ω. Suppose that condition (78) and the non-degeneracy condition hold. Let $\alpha, \beta \in \mathbf{R}$, $\alpha < \beta$, $v_k = s_{\alpha,\beta}(u_k) = \max(\alpha, \min(u_k, \beta))$. Then $v_k = v_k(x)$ is a bounded sequence in $L^\infty(\Omega)$ and for each $a, b \in \mathbf{R}$, $a < b$

$$\begin{aligned}\operatorname{div}_x\varphi(x, s_{a,b}(v_k(x))) &- D^2 \cdot B(s_{a,b}(v_k(x))) \\ &= \operatorname{div}_x\varphi(x, s_{a',b'}(u_k(x))) - D^2 \cdot B(s_{a',b'}(u_k(x)))\end{aligned}$$

where $a' = s_{a,b}(\alpha)$, $b' = s_{a,b}(\beta)$. It follows from this identity and (78) that the sequence $\operatorname{div}_x\varphi(x, s_{a,b}(v_k(x))) - D^2 \cdot B(s_{a,b}(v_k(x)))$ is precompact in $H^{-1}_{d,\rm loc}(\Omega)$. By Corollary 27 the sequences $v_k(x) = s_{\alpha,\beta}(u_k)$ are precompact in $L^1_{\rm loc}(\Omega)$ for every $\alpha, \beta \in \mathbf{R}$, $\alpha < \beta$. Using a standard diagonal argument, we can choose a subsequence $u_r(x) = u_{k_r}(x)$ such that for each $m \in \mathbf{N}$ the sequence $s_{-m,m}(u_r)$ converges as $r \to \infty$ to some function $w_m(x)$ in $L^1_{\rm loc}(\Omega)$. Obviously, a.e. in Ω

$$|w_m(x)| \le m, \quad w_m(x) = s_{-m,m}(w_l(x))$$

for all $l > m$. This allows to define a unique (up to equality a.e.) measurable function $u(x) \in \mathbf{R} \cup \{\pm\infty\}$ such that $w_m(x) = s_{-m,m}(u(x))$ a.e. on Ω. If $a, b \in \mathbf{R}$, $a < b$, then for $m > \max(|a|, |b|)$

$$\begin{aligned}s_{a,b}(u_r) = s_{a,b}(s_{-m,m}(u_r)) &\underset{r\to\infty}{\to} s_{a,b}(w_m) \\ &= s_{a,b}(s_{-m,m}(u)) = s_{a,b}(u) \quad \text{in } L^1_{\rm loc}(\Omega).\end{aligned}$$

In fact, we proved the following general statement:

Theorem 28. *Suppose that the sequence of measurable functions $u_k(x)$ satisfies (78) and the nondegeneracy condition holds. Then*

(a) there exists a measurable function $u(x) \in \mathbf{R} \cup \{\pm\infty\}$ such that, after extraction of a subsequence u_r, $r \in \mathbf{N}$, $s_{a,b}(u_r) \to s_{a,b}(u)$ as $r \to \infty$ in $L^1_{\rm loc}(\Omega)$ for all $a, b \in \mathbf{R}$, $a < b$.

(b) If, in addition, the following estimates are satisfied

$$\int_K m(u_k(x))dx \le C_K, \tag{79}$$

for each compact set $K \subset \Omega$, where $m(u)$ is a positive Borel function, such that $m(u)/u \underset{u\to\infty}{\to} \infty$, then $u(x) \in L^1_{\rm loc}(\Omega)$ and $u_r \to u$ in $L^1_{\rm loc}(\Omega)$ as $r \to \infty$.

Proof. We only need to prove (b). Observe that, extracting a subsequence, if necessary, we can assume that $s_{-m,m}(u_r) \to s_{-m,m}(u)$ as $m \to \infty$ a.e. in Ω for every $m \in \mathbf{N}$. This implies that $u_r \to u$ a.e. in Ω and by the Fatou lemma it follows from (79) that

$$\int_K m(u(x))dx \le C_K.$$

In particular, $u(x) \in L^1_{\mathrm{loc}}(\Omega)$. Now, fix a compact $K \subset \Omega$ and $\varepsilon > 0$. By the assumption $m(u)/u \underset{u\to\infty}{\to} \infty$ we can choose $l \in \mathbf{N}$ such that $|u|/m(u) \le \varepsilon/(2C_K)$ for $|u| > l$. Then

$$\begin{aligned}
\int_K |u_r(x) - u(x)|dx &\le \int_K |s_{-l,l}(u_r(x)) - s_{-l,l}(u(x))|dx \\
&\qquad + \int_K |u_r(x)|\theta(|u_r(x)| - l)dx + \int_K |u(x)|\theta(|u(x)| - l)dx \\
&\le \int_K |s_{-l,l}(u_r(x)) - s_{-l,l}(u(x))|dx \\
&\qquad + \frac{\varepsilon}{2C_K}\left(\int_K m(u_r(x))dx + \int_K m(u(x))dx\right) \\
&\le \int_K |s_{-l,l}(u_r(x)) - s_{-l,l}(u(x))|dx + \varepsilon.
\end{aligned}$$

This implies $\overline{\lim\limits_{r\to\infty}} \int_K |u_r(x) - u(x)|dx \le \varepsilon$ and since $\varepsilon > 0$ is arbitrary we conclude $\lim\limits_{r\to\infty} \int_K |u_r(x) - u(x)|dx = 0$ for any compact $K \subset \Omega$, i.e., $u_r \to u$ in $L^1_{\mathrm{loc}}(\Omega)$. □

Remark 29. As is easy to see from the proof of Theorems 25 and 26, the statement of Theorem 28 remains valid under the requirement that condition (78) is satisfied for almost all $a, b \in \mathbf{R}$. Indeed, by Theorems 25 and 26 we can claim that the H-measure $\tilde{\mu}^{pp}$ indicated in the proof of Theorem 26 vanishes for almost all $p \in E$. By the continuity of $\tilde{\mu}^{pp}$ with respect to $p \in E$, we conclude that $\tilde{\mu}^{pp} \equiv 0$ and this yields the strong precompactness property and all its consequences, including Theorem 28.

5. Proofs of Theorems 3 and 4

We need the following simple result.

Lemma 30. *Suppose $u = u(x)$ is an entropy solution of* (1). *Then for almost all $a, b \in \mathbf{R}$, $a < b$,*

$$\operatorname{div}_x \varphi(x, s_{a,b}(u)) - D^2 \cdot B(s_{a,b}(u)) = \zeta_{a,b} \quad \text{in } \mathcal{D}'(\Omega), \tag{80}$$

where $\zeta_{a,b} \in \mathrm{M}_{\mathrm{loc}}(\Omega)$. Moreover, for each compact set $K \subset \Omega$ we have $\operatorname{Var}\zeta_{a,b}(K) \le C(K, a, b, I)$, where $I = I(x) = |\varphi(x, u(x))| + |\psi(x, u(x))| + |B(u(x))| \in L^1_{\mathrm{loc}}(\Omega)$ and the map $I \mapsto C(K, a, b, I)$ is bounded on bounded sets in $L^1_{\mathrm{loc}}(\Omega)$.

Proof. By the known representation property for non-negative distributions we derive from (6) that for $p \in P$, $P \subset \mathcal{P}$ being a set of full measure,

$$\begin{aligned}
\operatorname{div}_x[\operatorname{sign}(u(x) - p)(\varphi(x, u(x)) - \varphi(x, p))] - D^2 \cdot [\operatorname{sign}(u(x) - p)(B(u(x)) - B(p))] \\
+ \operatorname{sign}(u(x) - p)[\omega_p(x) + \psi(x, u(x))] - |\gamma_p^s| = -\kappa_p \quad \text{in } \mathcal{D}'(\Omega),
\end{aligned}$$

where $\kappa_p \in \mathrm{M}_{\mathrm{loc}}(\Omega)$, $\kappa_p \ge 0$. Furthermore, for a compact set $K \subset \Omega$ we choose a non-negative function $f_K(x) \in C_0^\infty(\Omega)$, which equals 1 on K. Then we have the estimate

$$\begin{aligned}
\kappa_p(K) &\le \int f_K(x) d\kappa_p(x) \\
&= \int_\Omega \Big[\operatorname{sign}(u(x) - p)(\varphi(x, u(x)) - \varphi(x, p)) \cdot \nabla f_K(x) \\
&\qquad + \operatorname{sign}(u(x) - p)(B(u(x)) - B(p)) \cdot D^2 f_K(x) \\
&\qquad - \operatorname{sign}(u(x) - p)(\omega_p(x) + \psi(x, u(x)))f_K(x)\Big]dx + \int_\Omega f_K(x) d|\gamma_p^s|(x)
\end{aligned}$$

$$\le A(K,p,I) := \int_\Omega \Big[I(x)\max(|f_K(x)|, |\nabla f_K(x)|, |D^2 f_K(x)|) \\ + |\varphi(x,p)|\cdot|\nabla f_K(x)| + |B(p)|\cdot|D^2 f_K(x)| \\ + |\omega_p(x)||f_K(x)\Big] dx + \int_\Omega f_K(x)d|\gamma_p^s|(x).$$

Hence,

$$\begin{aligned} &\mathrm{div}_x[\mathrm{sign}(u(x)-p)(\varphi(x,u(x))-\varphi(x,p))] \\ &\quad - D^2\cdot[\mathrm{sign}(u(x)-p)(B(u(x))-B(p))] = \zeta_p, \end{aligned} \tag{81}$$

where

$$\zeta_p = |\gamma_p^s| - \kappa_p - \mathrm{sign}(u(x)-p)[\omega_p(x) + \psi(x,u(x))] \in \mathrm{M}_{\mathrm{loc}}(\Pi).$$

In particular, taking into account the equality $|\gamma_p^s| + |\omega_p(x)|dx = |\gamma_p|$, we obtain the following estimates for the measure ζ_p: $|\zeta_p| \le \kappa_p + |\gamma_p| + |\psi(x,u(x))|dx$.

Furthermore, notice that for $a,b \in P$,

$$\begin{aligned} &\varphi(x, s_{a,b}(u)) = (\varphi(x,a)+\varphi(x,b))/2 \\ &\qquad + \big(\mathrm{sign}(u-a)(\varphi(x,u)-\varphi(x,a)) - \mathrm{sign}(u-b)(\varphi(x,u)-\varphi(x,b))\big)/2; \\ &B(s_{a,b}(u)) = (B(a)+B(b))/2 + \big(\mathrm{sign}(u-a)(B(u)-B(a)) \\ &\qquad - \mathrm{sign}(u-b)(B(u)-B(b))\big)/2, \end{aligned}$$

and it follows from (81) that relation (80) holds with $\zeta_{a,b} = (\zeta_a - \zeta_b + \gamma_a + \gamma_b)/2$. Moreover, we have

$$\mathrm{Var}\,\zeta_{a,b}(K) \le C(K,a,b,I) = (A(K,a,I) + A(K,b,I))/2 \\ + |\gamma_a|(K) + |\gamma_b|(K) + \int_K |\psi(x,u(x))|dx.$$

To complete the proof, it remains to note that for fixed K,a,b the constant $C(K,a,b,I)$ is bounded on bounded sets of $I(x) \in L^1_{\mathrm{loc}}(\Omega)$. □

5.1. **Proof of Theorem 3.** Taking into account that the sequence

$$I_k(x) = |\varphi(x,u_k(x))| + |\psi(x,u_k(x))| + |B(u_k(x))|$$

is bounded in $L^1_{\mathrm{loc}}(\Omega)$, we derive from Lemma 30 that for almost all $a,b\in\mathcal{P}$

$$\mathrm{div}\varphi(x, s_{a,b}(u_k)) - D^2\cdot B(s_{a,b}(u_k)) = \zeta^k_{a,b} \quad \text{in } \mathcal{D}'(\Omega),$$

where $\zeta^k_{a,b}$ is a bounded sequence in $\mathrm{M}_{\mathrm{loc}}(\Omega)$. Since $\mathrm{M}_{\mathrm{loc}}(\Omega)$ is compactly embedded in $W^{-1}_{d,\mathrm{loc}}(\Omega)$ for each $d\in[1, n/(n-1))$ we see that condition (78) is satisfied for almost all a,b. By our assumption condition (79) is also satisfied. By Theorem 28 and Remark 29 we conclude that some subsequence u_r converges as $r\to\infty$ to a limit function u in $L^1_{\mathrm{loc}}(\Omega)$. Extracting a subsequence if necessary, we can assume that $u_r \underset{r\to\infty}{\to} u$ a.e. in Ω. Passing to the limit as $r\to\infty$ in relation (6) with $u=u_r$, we claim that the limit function $u=u(x)$ satisfies this relation for all $p\in P$ such that the level set $u^{-1}(p)$ has zero measure (then $\mathrm{sign}(u_r-p)\to\mathrm{sign}(u-p)$ as $r\to\infty$ a.e. in Ω). Since the set of such p has full measure, we conclude that $u(x)$ is an entropy solution of (1). □

5.2. **Proof of Theorem 4.** To simplify the notation, we temporarily drop the index m in equation (8), and stress that the flux $\varphi(x,u)$ in this equation is smooth.

First we show that a weak solution $u = u(x)$ of equation (8) is an entropy solution in the sense of Definition 1. For this observe that in relation (9) we can choose test functions $f(x) \in W_2^1(\Omega)$, with compact support in Ω. In particular, for $\eta(u) \in C^2(\mathbf{R})$, $f = f(x) \in C_0^\infty(\Omega)$ the function $\eta'(u)f$, $u = u(x)$, is an admissible test function, and we derive from (9) that

$$\begin{aligned} 0 &= -\int_\Omega [\varphi(x,u)\cdot\nabla\eta'(u)f - A(u)\nabla(u)\cdot\nabla\eta'(u)f]\,dx \\ &= \int_\Omega [(\mathrm{div}\varphi(x,u))\eta'(u)f + \eta''(u)fA(u)\nabla u\cdot\nabla u + A(u)\eta'(u)\nabla u\cdot\nabla f]dx. \end{aligned} \tag{82}$$

Introduce the vector $q(x,u)$ such that $q'_u(x,u) = \eta'(u)\varphi'_u(x,u)$. This vector is determined by the above equality up to an additive constant $c = c(x)$. We also introduce the symmetric matrix $Q(u)$ defined, up to an additive matrix constant, by the equality $Q'(u) = A(u)\eta'(u) = \eta'(u)B'(u)$. Now we can transform the terms $\mathrm{div}\varphi(x,u)\eta'(u)f$, $A(u)\eta'(u)\nabla u\cdot\nabla f$ as follows:

$$\begin{aligned} \mathrm{div}\varphi(x,u)\eta'(u)f &= (\mathrm{div}_x\varphi(x,u) + \varphi'_u(x,u)\cdot\nabla u)\eta'(u)f \\ &= (\eta'(u)\mathrm{div}_x\varphi(x,u))f + (q'_u(x,u)\cdot\nabla u)f \\ &= f\mathrm{div}q(x,u) + (\eta'(u)\mathrm{div}_x\varphi(x,u) - \mathrm{div}_x q(x,u))f; \\ A(u)\eta'(u)\nabla u\cdot\nabla f &= Q'(u)\nabla u\cdot\nabla f = Q'_{ij}(u)u_{x_j}f_{x_i} = (Q_{ij}(u))_{x_j}f_{x_i} \end{aligned}$$

(here Q_{ij}, $i,j = 1,\dots,n$, denote the components of the matrix Q). Putting these equalities into (82) and integrating by parts, we obtain that

$$\begin{aligned} \int_\Omega \Big[& q(x,u)\cdot\nabla f + (\mathrm{div}_x q(x,u) - \eta'(u)\mathrm{div}_x\varphi(x,u))f \\ & + Q(u)\cdot D^2 f - \eta''(u)fA(u)\nabla u\cdot\nabla u\Big]dx = 0. \end{aligned} \tag{83}$$

We shall assume that $\eta''(u)$ has a compact support in $\mathbf{R}$. Let $R > 0$ be such that $\mathrm{supp}\,\eta''(u) \subset (-R,R)$ and $L = (\eta'(-R) + \eta'(R))/2$ (evidently, L does not depend on R). Then we can choose $q(x,u)$ in the following way

$$q(x,u) = \frac{1}{2}\int \mathrm{sign}(u-p)(\varphi(x,u) - \varphi(x,p))d\eta'(p) + L\varphi(x,u). \tag{84}$$

Indeed, taking $R > |u|$ and integrating by parts, we obtain the equality

$$\begin{aligned} &\int \mathrm{sign}(u-p)(\varphi(x,u) - \varphi(x,p))d\eta'(p) \\ &\quad = \int_{-R}^R \mathrm{sign}(u-p)(\varphi(x,u) - \varphi(x,p))d\eta'(p) \\ &\quad = \int_{-R}^u (\varphi(x,u) - \varphi(x,p))d\eta'(p) - \int_u^R (\varphi(x,u) - \varphi(x,p))d\eta'(p) \\ &\quad = \int_{-R}^u \varphi'_u(x,p)\eta'(p)dp - \int_u^R \varphi'_u(x,p)\eta'(p)dp \\ &\qquad\qquad - 2L\varphi(x,u) + \varphi(x,-R)\eta'(-R) + \varphi(x,R)\eta'(R). \end{aligned}$$

We see that, up to a function which does not depend on u,

$$\frac{1}{2}\int \mathrm{sign}(u-p)(\varphi(x,u) - \varphi(x,p))d\eta'(p) + L\varphi(x,u)$$

$$= \frac{1}{2}\left(\int_{-R}^{u} \varphi'_u(x,p)\eta'(p)dp - \int_{u}^{R} \varphi'_u(x,p)\eta'(p)dp\right)$$

and therefore

$$\frac{\partial}{\partial u}\left(\frac{1}{2}\int \operatorname{sign}(u-p)(\varphi(x,u)-\varphi(x,p))d\eta'(p) + L\varphi(x,u)\right) = \eta'(u)\varphi'_u(x,u),$$

as required. In the similar way we find that, up to an additive matrix constant,

$$Q(u) = \frac{1}{2}\int \operatorname{sign}(u-p)(B(u)-B(p))d\eta'(p) + LB(u). \tag{85}$$

Furthermore, the function $\eta'(u)\mathrm{div}_x\varphi(x,u) - \mathrm{div}_x q(x,u)$ admits the representation

$$\eta'(u)\mathrm{div}_x\varphi(x,u) - \mathrm{div}_x q(x,u) = \frac{1}{2}\int \operatorname{sign}(u-p)\mathrm{div}_x\varphi(x,p)d\eta'(p). \tag{86}$$

Indeed, in view of (84), we see that for sufficiently large R

$$\begin{aligned}
2q(x,u) &= \int_{-R}^{u}(\varphi(x,u)-\varphi(x,p))d\eta'(p) - \int_{u}^{R}(\varphi(x,u)-\varphi(x,p))d\eta'(p) + 2L\varphi(x,u)\\
&= \varphi(x,u)(\eta'(u)-\eta'(-R)) - \int_{-R}^{u}\varphi(x,p)d\eta'(p) - \varphi(x,u)(\eta'(R)-\eta'(u))\\
&\quad + \int_{u}^{R}\varphi(x,p)d\eta'(p) + 2L\varphi(x,u)\\
&= 2\eta'(u)\varphi(x,u) - \int \operatorname{sign}(u-p)\varphi(x,p)d\eta'(p),
\end{aligned}$$

where we use the equality $2L = \eta'(R) + \eta'(-R)$. Applying the operator div_x to the above equality, we arrive at (86).

Now, we suppose that $\eta''(u) \ge 0$. We transform (83), using equalities (84), (85), (86) and the identity

$$\int_\Omega \{\varphi(x,u)\cdot\nabla f + (B(u)\cdot D^2 f)\}dx = 0, \tag{87}$$

following from (83) with $\eta(u) \equiv u$. We find that for each $f = f(x) \in C_0^\infty(\Omega)$, $f \ge 0$

$$\begin{aligned}
\int\int_\Omega &\operatorname{sign}(u-p)[(\varphi(x,u)-\varphi(x,p))\cdot\nabla f - f\mathrm{div}_x\varphi(x,p)\\
&\qquad + (B(u)-B(p))\cdot D^2 f]\eta''(p)dxdp\\
&= 2\int_\Omega \eta''(u) f A(u)\nabla u\cdot\nabla u \ge 0,
\end{aligned}$$

and since $\eta''(p)$ is an arbitrary finite continuous non-negative function on $\mathbf{R}$ we arrive at

$$\begin{aligned}
I(p) := \int_\Omega \operatorname{sign}(u-p)\Big[&(\varphi(x,u)-\varphi(x,p))\cdot\nabla f - f\mathrm{div}_x\varphi(x,p)\\
&+ (B(u)-B(p))\cdot D^2 f)\Big]dx \ge 0
\end{aligned} \tag{88}$$

for all $p \in P$, where the set P consists of points p such that the level set $u^{-1}(p)$ has null Lebesgue measure. We use the fact that the function $I(p)$ is continuous at any point of P. In view of (88) for all $p \in P$

$$\begin{aligned}
&\mathrm{div}[\operatorname{sign}(u-p)(\varphi(x,u)-\varphi(x,p))]\\
&\quad + \operatorname{sign}(u-p)\mathrm{div}_x\varphi(x,p) - D^2\cdot[\operatorname{sign}(u-p)(B(u)-B(p))] \le 0
\end{aligned} \tag{89}$$

in $\mathcal{D}'(\Omega)$. Since the set P has full measure and therefore is dense, for an arbitrary $p \in \mathbf{R}$ we can choose sequences $p_r^- < p < p_r^+$, $p_r^\pm \in P$, $r \in \mathbf{N}$ convergent to p.

Taking a sum of relations (89) with $p = p_r^-$ and $p = p_r^+$ and passing to the limit as $r \to \infty$, in view of the pointwise relation $\operatorname{sign}(u-p_r^-)+\operatorname{sign}(u-p_r^+) \underset{r\to\infty}{\to} 2\operatorname{sign}(u-p)$ and continuity of $\operatorname{div}_x\varphi(x,p)$, we obtain that (89) holds for all $p \in \mathbf{R}$, i.e., $u(x)$ is an entropy solution of (8), moreover condition (6) is satisfied for all $p \in \mathbf{R}$.

We also need an a priori estimate of ∇u. Choose $M \geq \|u\|_\infty$ and a function $\eta(u) \in C_0^2(\mathbf{R})$ such that $\eta(u) = u^2/2$ on the segment $[-M, M]$ and $\operatorname{supp}\eta(u) \in [-M-1, M+1]$. Then for $u = u(x)$, $\eta''(u) = 1$ a.e. in Ω and we derive from (83) that for each $f = f(x) \in C_0^\infty(\Omega)$, $f \geq 0$

$$\begin{aligned}&\int_\Omega fA(u)\nabla u \cdot \nabla u dx \\ &\leq \left|\int_\Omega \Big[q(x,u)\cdot\nabla f + (\operatorname{div}_x q(x,u) - \eta'(u)\operatorname{div}_x\varphi(x,u))f + Q(u)\cdot D^2 f\Big]dx\right|.\end{aligned} \tag{90}$$

It follows from (84), (85), and (86) that

$$|q(x,u)| \leq C\max_{|u|\leq M+1}|\varphi(x,u)|, \quad |Q(u)| \leq C\max_{|u|\leq M+1}|B(u)|,$$

$$|\operatorname{div}_x q(x,u) - \eta'(u)\operatorname{div}_x\varphi(x,u)| \leq C\int_{-M-1}^{M+1}|\operatorname{div}_x\varphi(x,p)|dp,$$

where C is the constant depending only on the fixed function η. Putting these estimates into (90), we get

$$\begin{aligned}\int_\Omega fA(u)\nabla u\cdot\nabla u dx \leq C\int_\Omega\Big[\max_{|u|\leq M+1}|\varphi(x,u)||\nabla f| + \max_{|u|\leq M+1}|B(u)||D^2 f|\Big]dx \\ + C\int_\Omega\int_{-M-1}^{M+1}|\operatorname{div}_x\varphi(x,p)|f(x)dpdx.\end{aligned} \tag{91}$$

By our assumptions, $\varphi_m(x,u)$, $B_m(u)$ converge as $m \to \infty$ in $L^2_{\rm loc}(\Omega, C(\mathbf{R},\mathbf{R}^n))$ and in $C^1(\mathbf{R}, \mathrm{Sym}_n)$, respectively. Therefore, the sequence

$$\int_\Omega\Big[\max_{|u|\leq M+1}|\varphi_m(x,u)||\nabla f| + \max_{|u|\leq M+1}|B_m(u)||D^2 f|\Big]dx$$

is bounded by a constant depending only on f. Here we take $M \geq \sup_m\|u_m\|_\infty$. It follows from estimate (91) that

$$\int_\Omega fA_m(u_m)\nabla u_m\cdot\nabla u_m dx \leq C_f I_m(K, M+1), \tag{92}$$

with $K = \operatorname{supp} f$, where the sequence

$$I_m(K,M) = 1 + \int_K\int_{-M}^M|\operatorname{div}_x\varphi_m(x,p)|dpdx$$

was mentioned in introduction. Now we take $a, b \in \mathbf{R}$, $a < b$. Let us demonstrate that the sequence

$$L_m = \operatorname{div}\varphi(x, s_{a,b}(u_m)) - D^2\cdot B(s_{a,b}(u_m))$$

is precompact in $W^{-1}_{d,\rm loc}$ with some $d > 1$. For that, recall that $u_m(x)$ is an entropy solution of (8) and by Lemma 30 (also see the proof of this lemma)

$$\operatorname{div}\varphi_m(x, s_{a,b}(u_m)) - D^2\cdot B_m(s_{a,b}(u_m)) = \xi_m$$

where ξ_m is a bounded sequence in the space $\mathrm{M}_{\rm loc}(\Omega)$, which is compactly embedded in $W^{-1}_{d,\rm loc}(\Omega)$ for each $d \in [1, n/(n-1))$. Furthermore, we have $L_m = L_{1m}+L_{2m}+\xi_m$, where

$$L_{1m} = \operatorname{div}(\varphi(x, s_{a,b}(u_m)) - \varphi_m(x, s_{a,b}(u_m))),$$

$$L_{2m} = D^2 \cdot (B_m(s_{a,b}(u_m)) - B(s_{a,b}(u_m))) .$$

In view of the estimate

$$|\varphi(x, s_{a,b}(u_m)) - \varphi_m(x, s_{a,b}(u_m))| \le \max_{|u|\le M} |\varphi_m(x,u) - \varphi(x,u)|$$

and the condition $\varphi_m(x,u) \underset{m\to\infty}{\to} \varphi(x,u)$ in $L^2_{\rm loc}(\Omega, C(\mathbf{R}, \mathbf{R}^n))$ we have

$$\varphi(x, s_{a,b}(u_m)) - \varphi_m(x, s_{a,b}(u_m)) \underset{m\to\infty}{\to} 0 \ \text{ in } L^2_{\rm loc}(\Omega, \mathbf{R}^n).$$

Hence $L_{1m} \to 0$ in $W^{-1}_{2,\rm loc}(\Omega)$. Concerning the sequence L_{2m}, we first remark that by the chain rule a.e. in Ω

$$D^2 \cdot B_m(s_{a,b}(u_m)) = \operatorname{div}[\chi(u_m) A_m(u_m) \nabla u_m],$$
$$D^2 \cdot B(s_{a,b}(u_m)) = \operatorname{div}[\chi(u_m) A(u_m) \nabla u_m],$$

where $\chi(u)$ is the indicator function of the segment $[a,b]$. Therefore,

$$L_{2m} = \operatorname{div}[\chi(u_m)(A_m(u_m) - A(u_m))\nabla u_m]. \tag{93}$$

Since

$$\begin{aligned} &|(A_m(u_m) - A(u_m))\nabla u_m|^2 \\ &\quad \le |(A_m(u_m) - A(u_m))(A_m(u_m))^{-1/2}|^2 |(A_m(u_m))^{1/2}\nabla u_m|^2 \\ &\quad = |(A_m(u_m) - A(u_m))(A_m(u_m))^{-1/2}|^2 (A_m(u_m)\nabla u_m \cdot \nabla u_m) \\ &\quad \le \max_{|u|\le M} |(A_m(u) - A(u))(A_m(u))^{-1/2}|^2 (A_m(u_m)\nabla u_m \cdot \nabla u_m) \end{aligned} \tag{94}$$

then for every $f = f(x) \in C_0^\infty(\Omega)$, $f \ge 0$

$$\begin{aligned} &\int_\Omega |\chi(u_m)(A_m(u_m) - A(u_m))\nabla u_m|^2 f dx \\ &\quad \le \max_{|u|\le M} |(A_m(u) - A(u))(A_m(u))^{-1/2}|^2 \int_\Omega (A_m(u_m)\nabla u_m \cdot \nabla u_m) f(x) dx. \end{aligned}$$

Taking into account relation (7) and estimate (92) we derive that

$$\int_\Omega |\chi(u_m)(A_m(u_m) - A(u_m))\nabla u_m|^2 f dx \underset{m\to\infty}{\to} 0,$$

i.e. $\chi(u_m)(A_m(u_m) - A(u_m))\nabla u_m \to 0$ in $L^2_{\rm loc}(\Omega; \mathbf{R}^n)$. In view of (93) this implies that $L_{2m} \to 0$ in $W^{-1}_{2,\rm loc}(\Omega)$. We conclude that $L_m = L_{1m} + L_{2m} + \xi_m$ is precompact in $W^{-1}_{d,\rm loc}(\Omega)$ with some $d > 1$. Hence, assumption (78) is satisfied. By Corollary 27 we see that the sequence u_m converges in $L^1_{\rm loc}(\Omega)$ to some function $u = u(x) \in L^\infty(\Omega)$. Obviously, $\|u\|_\infty \le M$. It only remains to demonstrate that u is an entropy solution of (1). By relation (88) for each $p \in \mathbf{R}$, $f = f(x) \in C_0^\infty(\Omega)$, $f \ge 0$

$$\begin{aligned} \int_\Omega \Big(&\operatorname{sign}(u_m - p)[(\varphi_m(x,u_m) - \varphi_m(x,p)) \cdot \nabla f - f \operatorname{div}_x \varphi_m(x,p)] \\ &+ \operatorname{sign}(u_m - p)(B_m(u_m) - B_m(p)) \cdot D^2 f \Big) dx \ge 0. \end{aligned}$$

Since $\operatorname{div}_x \varphi_m(x,p) = \gamma^m_{pr}(x) + \gamma^m_{ps}(x)$, $p \in \mathcal{P}$, the above relation implies that for all $p \in \mathcal{P}$

$$\begin{aligned} \int_\Omega \Big(&\operatorname{sign}(u_m - p)[(\varphi_m(x,u_m) - \varphi_m(x,p)) \cdot \nabla f - f \gamma^m_{pr}(x)] \\ &+ f|\gamma^m_{ps}(x)| + \operatorname{sign}(u_m - p)(B_m(u_m) - B_m(p)) \cdot D^2 f \Big) dx \ge 0. \end{aligned} \tag{95}$$

Passing to a subsequence, we may assume that $u_m(x) \to u(x)$ as $m \to \infty$ a.e. in Ω. Then

$$\begin{aligned}
&\operatorname{sign}(u_m - p)(\varphi_m(x, u_m) - \varphi_m(x, p)) \underset{m\to\infty}{\to} \operatorname{sign}(u - p)(\varphi(x, u) - \varphi(x, p)), \\
&\operatorname{sign}(u_m - p)(B_m(u_m) - B_m(p)) \underset{m\to\infty}{\to} \operatorname{sign}(u - p)(B(u) - B(p)), \\
&\operatorname{sign}(u_m - p) \underset{m\to\infty}{\to} \operatorname{sign}(u - p)
\end{aligned}$$

a.e. in Ω and, as a consequence, in $L^1_{\mathrm{loc}}(\Omega)$. The latter relation holds for $p \in \mathcal{P}$ such that the level set $u^{-1}(p)$ has zero Lebesgue measure. Besides, by our assumptions $\gamma^m_{pr}(x) \underset{m\to\infty}{\to} \omega_p(x)$ in $L^1_{\mathrm{loc}}(\Omega)$, $|\gamma^m_{ps}(x)| \underset{m\to\infty}{\to} |\gamma^s_p|$ weakly in $M_{\mathrm{loc}}(\Omega)$. Taking into account the above limit relations, we can pass to the limit in (95) and obtain that

$$\begin{aligned}
\int_\Omega \Big(&\operatorname{sign}(u - p)[(\varphi(x, u_m) - \varphi(x, p)) \cdot \nabla f - f\omega_p(x)] \\
&+ \operatorname{sign}(u - p)(B(u) - B(p)) \cdot D^2 f \Big) dx + \int_\Omega f(x) d|\gamma^s_p|(x) \geq 0,
\end{aligned} \tag{96}$$

for almost all $p \in \mathcal{P}$, i.e. $u(x)$ is an entropy solution of (8). Finally, passing to the limit as $m \to \infty$ in relation (87)

$$\int_\Omega \Big(\varphi_m(x, u_m) \cdot \nabla f + (B_m(u_m) \cdot D^2 f) \Big) dx = 0,$$

we obtain that for all $f = f(x) \in C^\infty_0(\Omega)$

$$\int_\Omega \Big(\varphi(x, u) \cdot \nabla f + (B(u) \cdot D^2 f) \Big) dx = 0.$$

Hence, $u = u(x)$ is a distributional solution of (1). This completes the proof of Theorem 4. □

Remark 31. We bring this paper to an end by mentioning that the strong precompactness property for Graetz–Nusselt type equations

$$\operatorname{div}_x (\varphi(x, u) - A(x)\nabla g(u)) + \psi(x, u) = 0$$

was studied in [19, 18]. In particular, Theorems 3 and 4 were proved in [18] for such equations.

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