GLOBAL SEMIGROUP OF CONSERVATIVE SOLUTIONS OF THE NONLINEAR VARIATIONAL WAVE EQUATION

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Abstract. We prove the existence of a global semigroup for conservative solutions of the nonlinear variational wave equation $u_{tt} - c(u)(c(u)u_x)_x = 0$. We allow for initial data $u|_{t=0}$ and $u_t|_{t=0}$ that contain measures. We assume that $0 < \kappa^{-1} \leq c(u) \leq \kappa$. Solutions of this equation may experience concentration of the energy density $(u_t^2 + c(u)^2 u_x^2)dx$ into sets of measure zero. The solution is constructed by introducing new variables related to the characteristics, whereby singularities in the energy density become manageable. Furthermore, we prove that the energy may only focus on a set of times of zero measure or at points where $c'(u)$ vanishes. A new numerical method to construct conservative solutions is provided and illustrated on examples.

Contents

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1. Introduction

The nonlinear variational wave equation (NVW), which was first introduced by Saxton in [11], is given by the following nonlinear partial differential equation on the line

$$u_{tt} - c(u)(c(u)u_x)_x = 0 \tag{1.1}$$

with initial data

$$u|_{t=0} = u_0, \ u_t|_{t=0} = u_1. \tag{1.2}$$

The equation can be derived from the variational principle applied to the functional

$$\iint \left(u_t^2 - c^2(u)u_x^2\right) \, dxdt.$$

We are interested in the analysis of conservative solutions of this initial value problem for $u_0, u_1 \in L^2(\mathbb{R})$. It is well known that solutions of this equation develop singularities in finite time, even for smooth initial data, see, e.g., [8]. The continuation past singularities is highly nontrivial, and allows for several distinct solutions. Thus additional information or requirements are needed to select a unique solution, and stability of solutions becomes a particularly delicate issue. We here study the conservative case where one in addition to the solution u itself, requires that the energy is conserved. For smooth solutions the energy is given by $\mathcal{E}(t) = \int_{\mathbb{R}}(u_t^2+c^2u_x^2)(t,x)\,dx$. However, as energy may focus in isolated points, one has to look at energy density in the sense of measures such that the absolutely continuous part of the measure corresponds to the usual energy density. The analysis resembles to a large extent recent work done on the Camassa–Holm equation and the Hunter–Saxton equation (see, e.g., [9, 3, 13, 10] and references therein). Our main result is the proof of the existence of a global semigroup for conservative solutions of the NVW equation, allowing for concentration of the energy density on sets of zero measure.

The NVW equation has been extensively studied by Zhang and Zheng [12, 13, 14, 15, 16, 17, 18]. However, our approach is closely related to the approach by Bressan and Zheng [5], in that we introduce new variables based on the characteristics, thereby, loosely speaking, separating waves going in positive and negative direction.

It is difficult to illustrate the ideas in this paper as there are no elementary and explicit solutions available, except for the trivial case where c is constant, which yields the classical linear wave equation. Thus one is forced to illustrate ideas numerically. Traditional finite difference schemes will not yield conservative solutions, but rather dissipative solutions due to the intrinsic numerical diffusion in these methods. Hence it is a challenge of separate interest to compute numerically conservative solutions of this equation to display some of the intricacies. This question is addressed and analyzed in Section 9.

Let us now turn to a more precise description of the content of this paper. We consider the variables R and S defined as

$$\begin{cases} R = u_t + c(u)u_x, \\ S = u_t - c(u)u_x. \end{cases} \tag{1.3}$$

By (1.1), we have

$$\begin{cases} R_t - cR_x = \dfrac{c'}{4c}(R^2 - S^2), \\ S_t + cS_x = \dfrac{c'}{4c}(S^2 - R^2), \end{cases} \tag{1.4}$$

or, on conservative form,

$$\begin{cases} (R^2 + S^2)_t - (c(R^2 - S^2))_x = 0, \\ (\dfrac{1}{c}(R^2 - S^2))_t - (R^2 + S^2)_x = 0. \end{cases} \tag{1.5}$$

Let $\mathcal{E}(t)$ denote the total energy of the system at time t, i.e.,

$$\mathcal{E}(t) = \int_{\mathbb{R}} (u_t^2 + c^2 u_x^2)(t,x)\,dx = \int_{\mathbb{R}} (R^2 + S^2)\,dx. \tag{1.6}$$

We assume that the initial total energy that we denote $\mathcal{E}_0$ is finite and that u is bounded in L^∞. For smooth solutions of (1.1) we have $\frac{d\mathcal{E}}{dt} = 0$. We also assume that $c \in C^1(\mathbb{R})$ and $c\colon \mathbb{R} \to [\kappa^{-1}, \kappa]$ for some constant $\kappa > 0$.

From (1.6) we see that we need that the functions R and S belong to $L^2(\mathbb{R})$. It turns out that, as time evolves, the functions R^2 and S^2 can concentrate on sets of measure zero. The example presented in Figure 1, see Section 8.2, illustrates this phenomenon. In this case, we have a nontrivial solution u for t nonzero, which is however is identically equal to one at $t = 0$ and $u_t(0,x) = 0$. However, when we analyze this example closer, we see that the energy concentrates at the origin, indeed

$$\lim_{t\to 0} R^2(t,x)\,dx = \delta \quad \text{and} \quad \lim_{t\to 0} S^2(t,x)\,dx = 2\delta$$

where δ is Dirac's delta function. Clearly this complicates the existence and uniqueness

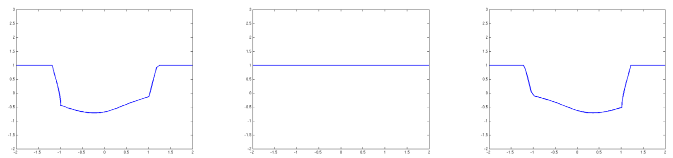

FIGURE 1. Plot of $u(t,x)$ for $t = -3$ (left), $t = 0$ (center), $t = 3$ (right).

question for this equation. As we want to construct a semigroup of solutions for this type of solutions, we have to know the location and amount of backward (R^2) and forward (S^2) energy that has concentrated on sets of zero measure, an information which is not given by the function u itself. (In our example, since, at $t = 0$, the function u is identically one and its time derivative u_t identically zero, we cannot infer where the energy has concentrated.) Thus we introduce the set $\mathcal{D}$ whose elements, in addition to u, R, S, contain two measures, μ and ν, corresponding to forward and backward energy density. More precisely, the measures are nonnegative Radon measures that satisfy

$$\mu_{\text{ac}} = \frac{1}{4}R^2\,dx, \quad \nu_{\text{ac}} = \frac{1}{4}S^2\,dx.$$

Our main contribution in this article is to present a rigorous construction of the semigroup of conservative solutions in $\mathcal{D}$. Note that the set $\mathcal{D}$ is the natural set of solutions for conservative solutions, and the semigroup property can only be established in $\mathcal{D}$, as illustrated by the example of Figure 1. Furthermore, by incorporating the energy measures as independent variables the formation of singularities is natural, and it allows for more general initial data. The present approach also provides a natural numerical method for conservative solutions.

As in [5], the construction of the solutions is achieved via a change of variables into a new coordinate system (X, Y) that straightens the characteristics. Even if we use different variables, the solutions we obtain are the same, but by extending the solutions to the set $\mathcal{D}$, we are able to establish that the solutions we construct satisfy the semigroup property. We have to study in details the change of variables mapping — from the original variables to the new variables and vice versa — because, in order to prove the semigroup property, we have to establish that the two sets of variables match in an appropriate way. Compared to the variables used in [5], we prefer variables with a more direct physical interpretation. Namely, the variables we are considering are time, $t(X,Y)$, space, $x(X,Y)$, the solution function $U(X,Y)$, which formally satisfies $u(t(X,Y),x(X,Y)) = U(X,Y)$ and the energy potentials J and K. The definition of the energy potentials J and K follows from (1.5), which says that the forms $\frac{1}{4}(R^2+S^2)\,dx + \frac{1}{4}c(u)(R^2-S^2)\,dt$ and $\frac{1}{4c(u)}(R^2-S^2)\,dx + \frac{1}{4}(R^2+S^2)dt$ are closed, so that, by Poincaré's lemma, there exist functions, here denoted the energy potentials J and K, whose differentials are equal to the given forms. Thus the new set of variables we will be considering equals $Z = (t, x, U, J, K)$ and, after rewriting the governing equations (1.3) and (1.4) in the new coordinate system (X, Y), we get a system of equations of the form

$$Z_{XY} = F(Z)(Z_X, Z_Y) \tag{1.7}$$

where $F(Z)\colon \mathbb{R}^5 \times \mathbb{R}^5 \to \mathbb{R}^5$ is a bi-linear and symmetric operator, which depends only on U, cf. (2.13).

In the new coordinates, the initial data corresponds to the set $\Gamma_0 = \{(X,Y) \in \mathbb{R}^2 \mid t(X,Y) = 0\}$. In the smooth case, Γ_0 will be a strictly monotone curve. However, in our setting, Γ_0 may not even be a curve, and even if it is a curve, it may not be continuous nor strictly monotone. Indeed, it may contain horizontal and vertical segments, and furthermore, rectangular boxes corresponding to the situation where both μ and ν are singular at the same point. If Γ_0 is a curve with no vertical or horizontal parts and the initial data is bounded in L^∞ (by initial data, we mean the values of Z, Z_X and Z_Y on Γ_0), then the existence and uniqueness of solutions to (1.7) is a classical result, see, for example, [7, Ch. 4]. In the present paper we have to deal with unbounded data in $\mathcal{D}$ (u_x and u_t are unbounded in L^∞). The new coordinates (X, Y) are given by

$$dx - c(u)dt = 0 \text{ if and only if } dY = 0$$

and

$$dx + c(u)dt = 0 \text{ if and only if } dX = 0,$$

that is, the characteristics are mapped to horizontal and vertical lines. We denote by $\mathbf{L}$ the mapping from the possible initial data in $\mathcal{D}$ to the set $\mathcal{F}$ defined by Γ_0 and the value of the initial data on Γ_0, thus $\mathbf{L}\colon \mathcal{D} \to \mathcal{F}$, see Definition 3.8. From Γ_0 we have to select one curve that can be used as initial data for the equation (1.7). There is a certain nonuniqueness due to fact that Γ_0 may not be curve. Let $\mathcal{G}_0$ denote the set of all curves, including the information about the initial data. We let $\mathbf{C}$ denote the mapping that from a set Γ_0 selects one possible curve, that is, $\mathbf{C}\colon \mathcal{F} \to \mathcal{G}_0$, see Definition 3.5. The inverse map that from curve determines the corresponding set in $\mathcal{F}$ is denoted $\mathbf{D}$, see Definition 3.7. Once we have a curve with the initial data, we can in principle compute the solution by solving (1.7). To show the existence of a global solution we use the the bi-linearity of (1.7) and an a priori bound on the energy potentials J and K, see Section 4. We let the set of all possible solutions be denoted by $\mathcal{H}$, and let $\mathbf{S}\colon \mathcal{G} \to \mathcal{H}$ denote the map that computes the solution that passes through the curve in $\mathcal{G}$, see Theorem 4.15. Here $\mathcal{G}$ is defined as $\mathcal{G}_0$ without the constraint that $t = 0$, see Definition 3.2. Recall that as t now is a dependent variable, it does not make sense to compute the solution up to a specific time, but rather we determine the global solution for all times. Thus we need a mapping that extracts the solution Z for a given time T, that is, the intersection of the solution in $\mathcal{H}$ with the set where $t(X,Y) = T$. Let $\mathbf{E}\colon \mathcal{H} \to \mathcal{G}_0$ denote the map that from any given solution in $\mathcal{H}$ extracts the solution at $t = 0$, that is, in $\mathcal{G}_0$, see Definition 5.1. Next we define the operator $\mathbf{t}_T\colon \mathcal{H} \to \mathcal{H}$ that shifts time in a solution in $\mathcal{H}$ by a given time T, see Definition 5.2. Now we can define the map $S_T\colon \mathcal{F} \to \mathcal{F}$ by $S_T = \mathbf{D} \circ \mathbf{E} \circ \mathbf{t}_T \circ \mathbf{S} \circ \mathbf{C}$, see Definition 5.4. A key result is that S_T is a semigroup on $\mathcal{F}$, see Theorem 5.5. Next we need to return to the original variables. Let $\mathbf{M}\colon \mathcal{F} \to \mathcal{D}$ denote that map, see Definition 6.1. Thus the solution operator $\bar{S}_T\colon \mathcal{D} \to \mathcal{D}$ is defined by (Definition 6.4)

$$\bar{S}_T = \mathbf{M} \circ S_T \circ \mathbf{L}. \tag{1.8}$$

It remains to show that $\bar{S}_T$ is a semigroup. However, since $\mathbf{M}$ is not inverse of $\mathbf{L}$, as $\mathbf{L} \circ \mathbf{M} \neq \mathrm{Id}_{\mathcal{F}}$, the semigroup property of $\bar{S}_T$ still does not follow from (1.8). This fact is explained as follows. When changing variables, we have introduced a degree of freedom that we now want to eliminate. This degree of freedom can be identified precisely with the action of the group G^2, where G denotes the group of diffeomorphisms of the real line. Indeed, by simply using the bi-linearity of (1.7), one can check that if Z is a solution to (1.7), then $\bar{Z}(X,Y) = Z(f(X), g(Y))$, where $(f,g) \in G^2$, is also a solution to the same equation. The transformation $(X,Y) \mapsto (f(X), g(Y))$ corresponds to a stretching of the plane $\mathbb{R}^2$ in the X and Y directions. Note that this transformation maps horizontal (resp. vertical) lines to horizontal (resp. vertical) lines and therefore preserves the directions of the characteristics. Moreover, this transformation does not affect the solution in the original coordinates. To illustrate this we ignore for the moment for the sake of simplicity, the energies μ and ν in the definition of $\mathcal{D}$. The solution $u(t,x)$ can be seen as the surface in $\mathbb{R}^3$ given by $(t, x, u(t,x))$ where $(t,x) \in \mathbb{R}^2$ are parameters. Through our change of variables, we obtain another parametrization of the same surface, namely,

$$(t(X,Y), x(X,Y), U(X,Y)) \tag{1.9}$$

where $(X,Y) \in \mathbb{R}^2$ are the new parameters. Additional properties of the solution $Z = (t, x, U, J, K)$ which are contained in the definition of $\mathcal{H}$ guarantee that the surface defined by (1.9) does not fold over itself so that it is in fact a graph. It is then clear from (1.9) that the transformation $(X,Y) \mapsto (f(X), g(Y))$ is simply a re-parametrization of the same surface, which defines $u(t,x)$ uniquely. At the level of the set $\mathcal{F}$, which corresponds to a parametrization of the initial data in the new coordinates, we can also define the action of the group G^2 that denote $\psi \times (f,g) \mapsto \psi \cdot (f,g)$ for any $\psi \in \mathcal{F}$ and $(f,g) \in G^2$. We prove that two elements which are equivalent correspond to the same element in $\mathcal{D}$, that is,

$$\mathbf{M}(\bar{\psi}) = \mathbf{M}(\psi) \tag{1.10}$$

where $\bar{\psi} = \psi \cdot (f,g)$ for some $(f,g) \in G^2$. From (1.10), it is now clear why $\mathbf{L} \circ \mathbf{M} \neq \mathrm{Id}_{\mathcal{F}}$ as, in general, $\bar{\psi}$ and ψ are distinct. We introduce a subset $\mathcal{F}_0$ of $\mathcal{F}$ which corresponds to a section of $\mathcal{F}$ with respect to the action of the group G^2, which means that the set $\mathcal{F}_0$ contains only one representative of each equivalence class so that $\mathcal{F}/G^2$ and $\mathcal{F}_0$ are in bijection. The system (1.7) preserves the strict positivity of the quantities $x_X + J_X$ and $x_Y + J_Y$ and the set $\mathcal{F}$ somehow inherits this property which makes it possible to define the projection $\Pi\colon \mathcal{F} \to \mathcal{F}_0$. The projection Π associates to any element in $\mathcal{F}$ its unique representative in $\mathcal{F}_0$ which belongs to the same equivalence class. As expected, since we have now eliminated the degree of freedom we introduced by changing variables, we obtain that $\mathcal{F}_0$ and $\mathcal{D}$ are in bijection. We are then able to prove that $\bar{S}_t$ is a semigroup.

Our main result, Theorem 7.9, reads as follows:

Theorem. *Given $(u_0, R_0, S_0, \mu_0, \nu_0) \in \mathcal{D}$, let us denote $(u, R, S, \mu, \nu)(t) = \bar{S}_t(u_0, R_0, S_0, \mu_0, \nu_0)$. Then u is a weak solution of the nonlinear variational wave equation* (1.1)*, that is,*

$$\int_{\mathbb{R}^2} (\phi_t - (c(u)\phi)_x) R \, dxdt + \int_{\mathbb{R}^2} (\phi_t + (c(u)\phi)_x) S \, dxdt = 0 \tag{1.11}$$

for all smooth functions ϕ with compact support and where

$$R = u_t + c(u) u_x, \quad S = u_t - c(u) u_x. \tag{1.12}$$

Moreover, the measures $\mu(t)$ and $\nu(t)$ satisfy the following equations in the sense of distribution

$$(\mu + \nu)_t - (c(\mu - \nu))_x = 0 \tag{1.13a}$$

and

$$(\frac{1}{c}(\mu - \nu))_t - (\mu + \nu)_x = 0. \tag{1.13b}$$

The mapping $\bar{S}_T : \mathcal{D} \to \mathcal{D}$ is a semigroup, that is,

$$\bar{S}_{t+t'} = \bar{S}_t \circ \bar{S}_{t'}$$

for all positive t and t'.

Furthermore, we note the following important result (Theorem 7.10):

Theorem. *The solution satisfies the following properties:*

(i) *For all $t \in \mathbb{R}$*

$$\mu(t)(\mathbb{R}) + \nu(t)(\mathbb{R}) = \mu_0(\mathbb{R}) + \nu_0(\mathbb{R}). \tag{1.14}$$

(ii) *For almost every $t \in \mathbb{R}$, the singular part of $\mu(t)$ and $\nu(t)$ are concentrated on the set where $c'(u) = 0$.*

In this article, we do not study the stability of the solutions. However, since the solutions we obtain coincide with the ones obtained in [5] for initial data which do not contain any singular measure, the solutions in that case also satisfy the stability result stated in [5, Theorem 2]. To obtain a continuous semigroup of solution in $\mathcal{D}$, we would like to follow the approach developed in [9], [4] for the Camassa–Holm equation and Hunter–Saxton equations. In these two papers, the conservative solutions are also obtained via a change of variables which is invariant with respect to relabeling (i.e., with respect to the action of the group of diffeomorphisms G). We define a distance between equivalence classes in the new coordinates. This distance is then mapped back to the original set of coordinates so that we obtain a continuous semigroup for this metric. In the case of the nonlinear wave equation, in particular, because of the truly two dimensional nature of the problem, it is not so easy to formulate a stability result in the new coordinates which holds when mapping back to the original set of variables. In Lemma 4.12 we present a result in that direction.

There is a lack of explicit solutions to NVW. In this paper we consider two explicit examples. The first example, see Section 8.1, is the simplest possible, namely the linear wave equation (with c constant), but with general initial data. We recover as expected the familiar d'Alembert solution. The energy measures are transported with velocity $\pm c$. The second example, see Section 8.2, is a truly nonlinear case with velocity given by (1.19). However, here we choose the simplest nontrivial initial data with energy concentration initially for both measures. The corresponding equation (1.7) is solved numerically, and the result is illustrated on Figs. 1, 9–12.

The numerical method that yields conservative solutions is described in Section 9.

1.1. **Physical motivation for the nonlinear variational wave equation.** The NVW equation was first derived in the context of nematic liquid crystals, see [11, 10]. More precisely, a nematic crystal can be described, when we ignore the motion of the fluid, by the dynamics of the so-called director field $\mathbf{n} = \mathbf{n}(x, y, z, t) \in \mathbb{R}^3$ describing the orientation of rod-like molecules. Thus $|\mathbf{n}| = 1$. The Oseen–Franck strain-energy potential is given by

$$W(\mathbf{n}, \nabla\mathbf{n}) = \alpha\,|\mathbf{n} \times (\nabla \times \mathbf{n})|^2 + \beta(\nabla \cdot \mathbf{n})^2 + \gamma(\mathbf{n} \cdot \nabla \times \mathbf{n})^2, \tag{1.15}$$

where α, β, γ are constitutive constants. Consider next the highly simplified case of director fields of the type

$$\mathbf{n} = \mathbf{n}(x, t) = \cos(u(t, x))\mathbf{e}_x + \sin(u(t, x))\mathbf{e}_y \tag{1.16}$$

where $\mathbf{e}_x$ and $\mathbf{e}_y$ are unit vectors in the x and y direction, respectively. In this case the functional $W(\mathbf{n}, \nabla\mathbf{n})$ vastly simplifies to

$$W(\mathbf{n}, \nabla\mathbf{n}) = (\beta \cos^2 u + \alpha \sin^2 u)u_x^2, \tag{1.17}$$

and $|\mathbf{n}_t|^2 = u_t^2$. The dynamics is described by the variational principle

$$\frac{\delta}{\delta u} \iint \left(u_t^2 - c^2(u)u_x^2\right) dx dt = 0, \tag{1.18}$$

where

$$c^2(u) = \beta \cos^2 u + \alpha \sin^2 u, \tag{1.19}$$

which results in the nonlinear variational wave equation

$$u_{tt} - c(u)(c(u)u_x)_x = 0. \tag{1.20}$$

2. Equivalent system for the NVW equation

In this section, we assume the existence of a smooth solution $u = u(t,x)$ to (1.1). We introduce the change of variables $(t,x) \mapsto (X,Y)$ which straightens out the characteristics: The forward characteristics, which are given by the solutions of $\frac{dx}{dt} = c(u(t,x(t)))$, are mapped to the horizontal lines while the backward characteristics, which are given by the solutions of $\frac{dx}{dt} = -c(u(t,x(t)))$, are mapped to the vertical lines. Formally, we can rewrite these conditions as

$$dx - c(u)dt = 0 \text{ if and only if } dY = 0 \tag{2.1}$$

and

$$dx + c(u)dt = 0 \text{ if and only if } dX = 0. \tag{2.2}$$

Our goal now is to rewrite the governing equation (1.1) in terms of the new variables (X,Y). The variables (t,x) become functions of (X,Y) that we denote $t(X,Y)$ and $x(X,Y)$. We set

$$U(X,Y) = u(t,x). \tag{2.3}$$

Since

$$dx = x_X dX + x_Y dY \quad \text{and} \quad dt = t_X dX + t_Y dY,$$

we obtain from (2.1) and (2.2) that

$$x_X = c(U)t_X \quad \text{and} \quad x_Y = -c(U)t_Y. \tag{2.4}$$

From (1.5), we infer that the forms $\frac{1}{4}(R^2+S^2)dx + \frac{c}{4}(R^2-S^2)dt$ and $\frac{1}{4c}(R^2-S^2)dx + \frac{1}{4}(R^2+S^2)dt$ are closed. Therefore, by Poincaré's lemma, we infer the existence of two functions J and K for which these forms are the differentials, that is,

$$dJ = \frac{1}{4}(R^2+S^2)dx + \frac{c}{4}(R^2-S^2)dt \tag{2.5}$$

and

$$dK = \frac{1}{4c}(R^2-S^2)dx + \frac{1}{4}(R^2+S^2)dt. \tag{2.6}$$

We have, after using (2.4),

$$\begin{aligned} dJ &= \frac{1}{4}(R^2+S^2)dx + \frac{c}{4}(R^2-S^2)dt \\ &= \frac{1}{4}(R^2+S^2)(x_X dX + x_Y dY) + \frac{c}{4}(R^2-S^2)(t_X dX + t_Y dY) \\ &= \frac{1}{2}R^2 x_X dX + \frac{1}{2}S^2 x_Y dY, \end{aligned}$$

and, similarly, we get

$$dK = \frac{R^2}{2c}x_X dX - \frac{S^2}{2c}x_Y dY$$

so that

$$J_X = c(U)K_X \quad \text{and} \quad J_Y = -c(U)K_Y \tag{2.7}$$

hold. Note the similarity between the relations (2.7) for the pair (J,K) and the relations (2.4) for the pair (t,x). We want to compute the mixed second derivatives of our new variables, namely, t, x, U, J and K. By (2.4), we obtain

$$dt = t_X dX + t_Y dY = \frac{1}{c} x_X dX - \frac{1}{c} x_Y dY.$$

By expressing the fact that the form dt is closed (since it is exact), we get

$$\frac{\partial}{\partial Y}\left(\frac{1}{c} x_X\right) = -\frac{\partial}{\partial X}\left(\frac{1}{c} x_Y\right)$$

which implies

$$x_{XY} = \frac{c'}{2c}\left(u_Y x_X + u_X x_Y\right).$$

Similarly, since the form

$$dx = x_X dX + x_Y dY = c t_X dX - c t_Y dY$$

is closed, we obtain

$$\frac{\partial}{\partial Y}\left(c t_X\right) = -\frac{\partial}{\partial X}\left(c t_Y\right)$$

which implies

$$t_{XY} = -\frac{c'}{2c}\left(u_X t_Y + u_Y t_X\right).$$

By using the relations (2.7), the form dK can be rewritten as

$$dK = \frac{1}{c} J_X dX - \frac{1}{c} J_Y dY$$

and, expressing the fact that dK is a closed, we obtain

$$\frac{\partial}{\partial Y}\left(\frac{1}{c} J_X\right) = \frac{\partial}{\partial X}\left(-\frac{1}{c} J_Y\right)$$

which yields

$$J_{XY} = \frac{c'}{2c}\left(J_X U_Y + J_Y U_X\right).$$

Similarly, we can rewrite the form dJ as

$$dK = cK_X dX - cK_Y dY$$

and, expressing the fact dK is closed, we get

$$K_{XY} = -\frac{c'}{2c}\left(K_X U_Y + K_Y U_X\right).$$

Let us consider the forms

$$\omega_1 = \frac{R}{2c} dx + \frac{1}{2} R dt \tag{2.8}$$

and

$$\omega_2 = \frac{S}{2c} dx - \frac{1}{2} S dt. \tag{2.9}$$

In the new variables, these forms rewrite

$$\omega_1 = \frac{u_t + cu_x}{2c}(x_X dX + x_Y dY) + \frac{1}{2}(u_t + cu_x)(t_X dX + t_Y dY)$$

$$= (u_t t_X + u_x x_X)dX \quad \text{(after using (2.4))}$$
$$= U_X dX, \tag{2.10}$$

and, similarly, we find

$$\omega_2 = -U_Y dY. \tag{2.11}$$

From (2.8), by using (1.4), we obtain

$$\begin{aligned} d\omega_1 &= (\frac{R_t}{2c} - \frac{c'R}{2c^2}u_t)dt \wedge dx + \frac{1}{2}R_x dx \wedge dt \\ &= \frac{R_t - cR_x}{2c}dt \wedge dx - \frac{c'R}{2c^2}\frac{1}{2}(R+S)dt \wedge dx \\ &= \Big(\frac{c'(R^2 - S^2)}{8c^2} - \frac{c'R}{2c^2}\frac{1}{2}(R+S)\Big)\, dt \wedge dx \\ &= \frac{c'}{2c^2}\left(\frac{1}{2}(R+S)\right)^2 dx \wedge dt = \frac{c'}{2c^2}u_t^2 dx \wedge dt, \end{aligned}$$

and, furthermore, we obtain

$$\begin{aligned} d\omega_1 &= \frac{c'}{2c^2}\Big(\frac{1}{4}(R^2+S^2)dx \wedge dt + \frac{1}{2}RSdx \wedge dt\Big) \\ &= \frac{c'}{2c^2}dJ \wedge dt - \frac{c'}{2c}\omega_1 \wedge \omega_2 \end{aligned} \tag{2.12}$$

because

$$\omega_1 \wedge \omega_2 = -\frac{RS}{2c}dx \wedge dt.$$

We rewrite (2.12) in the new set of variables

$$\begin{aligned} d\omega_1 &= \frac{c'}{2c^2}(J_X dX + J_Y dY) \wedge (t_X dX + t_Y dY) - \frac{c'}{2c}\omega_1 \wedge \omega_2 \\ &= -\frac{c'}{2c^3}(J_X x_Y + J_Y x_X)dX \wedge dY + \frac{c'}{2c}U_X U_Y dX \wedge dY. \end{aligned}$$

At the same time, by (2.10), we have $d\omega_1 = -U_{XY} dX \wedge dY$, and therefore it follows that

$$U_{XY} = \frac{c'}{2c^3}\left(J_X x_X + J_Y x_X\right) - \frac{c'}{2c}U_X U_Y.$$

Finally, we obtain following system of equations

$$t_{XY} = -\frac{c'}{2c}\left(U_X t_Y + U_Y t_X\right), \tag{2.13a}$$

$$x_{XY} = \frac{c'}{2c}\left(U_Y x_X + U_X x_Y\right), \tag{2.13b}$$

$$U_{XY} = \frac{c'}{2c^3}\left(x_Y J_X + J_Y x_X\right) - \frac{c'}{2c}U_Y U_X, \tag{2.13c}$$

$$J_{XY} = \frac{c'}{2c}\left(J_X U_Y + J_Y U_X\right), \tag{2.13d}$$

$$K_{XY} = -\frac{c'}{2c}\left(K_X U_Y + K_Y U_X\right). \tag{2.13e}$$

Let Z denote the vector (t, x, U, J, K). The system (2.13) then rewrites as

$$Z_{XY} = F(Z)(Z_X, Z_Y) \tag{2.14}$$

where $F(Z)$ is a bi-linear and symmetric tensor from $\mathbb{R}^5 \times \mathbb{R}^5$ to $\mathbb{R}^5$. Due to the relations (2.4), either one of the equations (2.13a) and (2.13b) is redundant: one could remove one of them, and the system would remain well-posed, and one retrieves t or x by using (2.4). Similarly, either one of the equations (2.13d) and (2.13e) becomes redundant by (2.7). However, we find it convenient to work with the complete set of variables, that is, $Z = (t, x, U, J, K)$. We will see later that the solutions of the system (2.13) preserve these conditions.

To prove the existence of solutions to (2.13), we use a fixed point argument. The argument is similar to the one that can be found for example in [7] and in [5]. However, in order to take into account the non-regularity of the data (u_{0x} and u_{0t} are in L^2 and the energy can concentrate on sets of zero measure), we have to consider, in the new set of coordinates, data given on curves which have parts which are parallel to the characteristic directions. In particular, the curves are not given as graphs of a function. We are looking for a solution that satisfies a given initial condition at time $t = 0$. In the (X, Y) plane, the set of points which correspond to initial time, that is, $t(X, Y) = 0$, may be a curve, $(\mathcal{X}(s), \mathcal{Y}(s)) \in \mathbb{R}^2$, parametrized by $s \in \mathbb{R}$, but it may also be a more complicated set, see Figure 3 that we will comment on later. We consider curves of the following type.

Definition 2.1. *We denote by $\mathcal{C}$ the set of curves in the plane $\mathbb{R}^2$ parametrized by $(\mathcal{X}(s), \mathcal{Y}(s))$ with $s \in \mathbb{R}$, such that*

$$\mathcal{X} - \mathrm{Id},\ \mathcal{Y} - \mathrm{Id} \in W^{1,\infty}(\mathbb{R}), \tag{2.15a}$$

$$\dot{\mathcal{X}} \geq 0, \quad \dot{\mathcal{Y}} \geq 0 \tag{2.15b}$$

and the normalization

$$\frac{1}{2}(\mathcal{X}(s) + \mathcal{Y}(s)) = s, \text{ for all } s \in \mathbb{R}. \tag{2.15c}$$

We set

$$\|(\mathcal{X}, \mathcal{Y})\|_{\mathcal{C}} = \|\mathcal{X} - \mathrm{Id}\|_{L^\infty} + \|\mathcal{X} - \mathrm{Id}\|_{L^\infty}. \tag{2.16}$$

From the initial data (u_0, R_0, S_0), we want to define the curve $\Gamma_0 = (\mathcal{X}(s), \mathcal{Y}(s))$ in $\mathcal{C}$ which corresponds to the initial time and the value of Z on this curve. To solve the governing equations (2.13), we need to know the values of Z, Z_X and Z_Y on the curve Γ_0. In total, we have to determine 17 unknown functions. Given the initial data (u_0, R_0, S_0), there is no unique way to define the curve $(\mathcal{X}(s), \mathcal{Y}(s))$ and the values of Z on this curve in order to obtain to the desired solution. This fact is due to the relabeling symmetry, a degree of freedom which is embedded in the set of equations (2.13) that we precisely identify in Section 7. For now, the goal is to use this degree of freedom to construct an initial data which is bounded in $L^\infty(\mathbb{R})$ on the curve. Let us now explain how we proceed for an initial data $(u_0, R_0, S_0) \in [L^2(\mathbb{R})]^3$ for which energy has not concentrated and we will see later how to extend this construction to initial data containing singular measures. In this case, the function $\mathcal{X}$ and $\mathcal{Y}$ are invertible and, slightly abusing notation, we denote by $Z(s)$, $Z_X(X)$ and $Z_Y(Y)$ the values of $Z(\mathcal{X}(s),\mathcal{Y}(s))$, $Z_X(X,\mathcal{Y}(\mathcal{X}^{-1}(X)))$ and $Z_Y(\mathcal{X}(\mathcal{Y}^{-1}(Y)),Y)$, respectively. By definition, we have

$$t(s)=0, \tag{2.17}$$

and, naturally, we set

$$U(s)=u_0(x(s)). \tag{2.18}$$

From the formal derivation of the previous section, we have the following relations

$$J_X(\mathcal{X})=c(u)K_X(\mathcal{X})=\frac{1}{2}R_0^2(x)x_X(\mathcal{X}),\quad J_Y(\mathcal{Y})=-c(u)K_Y(\mathcal{Y})=\frac{1}{2}S_0^2(x)x_Y(\mathcal{Y}), \tag{2.19}$$

$$U_X(\mathcal{X})=\frac{R_0(x)}{c(u(x))}x_X(\mathcal{X}),\qquad U_Y(\mathcal{Y})=-\frac{S_0(x)}{c(u(x))}x_Y(\mathcal{Y}). \tag{2.20}$$

We have the compatibility condition

$$\dot{Z}(s)=Z_X(\mathcal{X}(s))\dot{\mathcal{X}}(s)+Z_Y(\mathcal{Y}(s))\dot{\mathcal{Y}}(s). \tag{2.21}$$

We have 17 unknowns ($\mathcal{X}$,$\mathcal{Y}$,Z,Z_X,Z_Y) and 15 equations, namely (2.17)–(2.21), (2.15c) and (2.4). We use the two degrees of freedom that remain in order to obtain Z_X and Z_Y bounded. We set

$$2x_X(X)+J_X(X)=1\quad\text{and}\quad 2x_Y(Y)+J_Y(Y)=1. \tag{2.22}$$

Since x_X and J_X are positive, it follows from (2.22) that these two quantities are bounded. From the fact that $2x_XJ_X=(c(U)U_X)^2$, it also follows that U_X is bounded so that Z_X is bounded. The same conclusion holds for Z_Y. The normalisation (2.22) is convenient but arbitrary, see Section 7. In particular, the coefficient 2 in front of x_X and x_Y in (2.22) does not have any importance; it is used here to make the definition compatible with the normalization we will introduce in Section 3 for the general case. From (2.21), (2.20) and (2.19), we get

$$x_X(\mathcal{X})=\frac{2}{4+R_0^2}(x)\quad\text{and}\quad x_Y(\mathcal{Y})=\frac{2}{4+S_0^2}(x), \tag{2.23}$$

$$J_X(\mathcal{X})=\frac{1}{c}K_X(\mathcal{X})=\frac{R_0^2}{4+R_0^2}(x),\qquad J_Y(\mathcal{Y})=-\frac{1}{c}K_Y(\mathcal{Y})=\frac{S_0^2}{4+S_0^2}(x), \tag{2.24}$$

$$U_X(\mathcal{X})=\frac{2R_0}{c(4+R_0^2)}(x),\qquad U_Y(\mathcal{Y})=-\frac{2S_0}{c(4+S_0^2)}(x). \tag{2.25}$$

Equation (2.17) implies

$$0=t_X(\mathcal{X})\dot{\mathcal{X}}+t_Y(\mathcal{Y})\dot{\mathcal{Y}}=x_X(\mathcal{X})\dot{\mathcal{X}}-x_Y(\mathcal{Y})\dot{\mathcal{Y}}$$

and, at the same time, we have by the chain rule

$$\dot{x}(s)=x_X(\mathcal{X})\dot{\mathcal{X}}+x_Y(\mathcal{Y})\dot{\mathcal{Y}}$$

and therefore

$$x_X(\mathcal{X})\dot{\mathcal{X}}=x_Y(\mathcal{Y})\dot{\mathcal{Y}}=\frac{\dot{x}}{2}. \tag{2.26}$$

Hence, by (2.15c), (2.23) and (2.26), we get

$$2 = \dot{\mathcal{X}} + \dot{\mathcal{Y}} = \left(2 + \frac{1}{4}(R_0^2 + S_0^2)(x)\right)\dot{x}$$

and we define $x(s)$ implicitly as

$$2x(s) + \int_{-\infty}^{x(s)} \frac{1}{4}(R_0^2 + S_0^2)\,dx = 2s. \tag{2.27}$$

We have

$$2\dot{x} + \dot{J} = 2x_X(\mathcal{X})\dot{\mathcal{X}} + J_X(\mathcal{X})\dot{\mathcal{X}} + 2x_Y(\mathcal{Y})\dot{\mathcal{Y}} + J_Y(\mathcal{Y})\dot{\mathcal{Y}} = 2$$

because of (2.21) and (2.15c) so that $2x + J = 2s$. Hence,

$$J(s) = \frac{1}{8}\int_{-\infty}^{x(s)} (R_0^2 + S_0^2)\,dx \tag{2.28}$$

and

$$K(s) = \int_{-\infty}^{x(s)} \frac{R_0^2 - S_0^2}{8c}\,dx, \tag{2.29}$$

which are also defined as the integrals of the forms dJ and dK given by (2.5) and (2.6) on the line $(t, x) = \{0\} \times (-\infty, x(s))$. From (2.26) and (2.23), it follows that

$$\dot{\mathcal{X}}(s) = \dot{x}(s)(1 + \frac{1}{4}R_0^2)(x(s)) \quad \text{and} \quad \dot{\mathcal{Y}}(s) = \dot{x}(s)(1 + \frac{1}{4}S_0^2)(x(s)), \tag{2.30}$$

and we set

$$\mathcal{X}(s) = x(s) + \frac{1}{4}\int_{-\infty}^{x(s)} R_0^2\,dx \quad \text{and} \quad \mathcal{Y}(s) = x(s) + \frac{1}{4}\int_{-\infty}^{x(s)} S_0^2\,dx. \tag{2.31}$$

3. The initial data

In order to construct a semigroup of conservative solutions, we have to take into account the part of the energy which has concentrated in sets of measure zero and we need to consider initial data in the set $\mathcal{D}$ that we now define.

Definition 3.1. *The set $\mathcal{D}$ consists of the elements (u, R, S, μ, ν) such that*

$$(u, R, S) \in [L^2(\mathbb{R})]^3,$$

$u_x = \frac{1}{2c}(R - S)$ *and μ and ν are finite positive Radon measures with*

$$\mu_{\text{ac}} = \frac{1}{4}R^2\,dx, \quad \nu_{\text{ac}} = \frac{1}{4}S^2\,dx. \tag{3.1}$$

The measures μ and ν correspond to the left and right traveling energy densities, respectively. Given the initial data $(u_0, R_0, S_0, \mu_0, \nu_0)$, we have defined an element in $\mathcal{G}_0$ where the set $\mathcal{G}_0$ is defined below and which correspond to a parametrization of the initial data in the new system of coordinates. Elements of $\mathcal{G}$ consists of a curve $(\mathcal{X}(s), \mathcal{Y}(s))$ (for $\mathcal{G}_0$, this curve corresponds to time equal to zero) and three variables, $\mathcal{Z}$, $\mathcal{V}$ and $\mathcal{W}$, that we now introduce. These functions correspond to the data that matches the solution Z to (2.13) on the curve $(\mathcal{X}, \mathcal{Y})$ in the sense that

$$\mathcal{Z}(s) = Z(\mathcal{X}(s), \mathcal{Y}(s)) \tag{3.2a}$$

and

$$\mathcal{V}(\mathcal{X}(s)) = Z_X(\mathcal{X}(s),\mathcal{Y}(s)) \text{ and } \mathcal{W}(\mathcal{X}(s)) = Z_Y(\mathcal{X}(s),\mathcal{Y}(s)) \tag{3.2b}$$

It is then convenient to introduce the following notation: To any triplet $(\mathcal{Z},\mathcal{V},\mathcal{W})$ of five dimensional vector functions (we write $\mathcal{Z} = (\mathcal{Z}_1,\mathcal{Z}_2,\mathcal{Z}_3,\mathcal{Z}_4,\mathcal{Z}_5)$, etc), we associate the triplet $(\mathcal{Z}^a,\mathcal{V}^a,\mathcal{W}^a)$ given by

$$\mathcal{Z}_2^a = \mathcal{Z}_2 - \mathrm{Id}, \quad \mathcal{V}_2^a = \mathcal{V}_2 - \frac{1}{2}, \quad \mathcal{W}_2^a = \mathcal{W}_2 - \frac{1}{2} \tag{3.3a}$$

and

$$\mathcal{Z}_i^a = \mathcal{Z}_i, \quad \mathcal{V}_i^a = \mathcal{V}_i, \quad \mathcal{W}_i^a = \mathcal{W}_i \tag{3.3b}$$

for $i \in \{1,3,4,5\}$.

Definition 3.2. *The set $\mathcal{G}$ is the set of all elements which consist of a curve $(\mathcal{X}(s),\mathcal{Y}(s))$ and three vector valued functions from $\mathbb{R}$ to $\mathbb{R}^5$ denoted $\mathcal{Z}(s),\mathcal{V}(X),\mathcal{W}(Y)$. We denote $\Theta = (\mathcal{X},\mathcal{Y},\mathcal{Z},\mathcal{V},\mathcal{W})$ and set*

$$\|\Theta\|_{\mathcal{G}} = \|U\|_{L^2(\mathbb{R})} + \|\mathcal{V}^a\|_{L^2} + \|\mathcal{W}^a\|_{L^2} \tag{3.4}$$

where we denote $U = \mathcal{Z}_3$ and

$$|||\Theta|||_{\mathcal{G}} = \|(\mathcal{X},\mathcal{Y})\|_{\mathcal{C}} + \|\frac{1}{\mathcal{V}_2+\mathcal{V}_4}\|_{L^\infty(\mathbb{R})} + \|\frac{1}{\mathcal{W}_2+\mathcal{W}_4}\|_{L^\infty(\mathbb{R})} + \|\mathcal{Z}^a\|_{L^\infty} + \|\mathcal{V}^a\|_{L^\infty} + \|\mathcal{W}^a\|_{L^\infty}. \tag{3.5}$$

The element $\Theta \in \mathcal{G}$ if

(i)

$$\|\Theta\|_{\mathcal{G}} < \infty \quad \textit{and} \quad |||\Theta|||_{\mathcal{G}} < \infty;$$

(ii)

$$\mathcal{V}_2,\mathcal{W}_2,\mathcal{V}_4,\mathcal{W}_4 \geq 0; \tag{3.6}$$

(iii) *for almost every s, we have*

$$\dot{\mathcal{Z}}(s) = \mathcal{V}(\mathcal{X}(s))\dot{\mathcal{X}}(s) + \mathcal{W}(\mathcal{Y}(s))\dot{\mathcal{Y}}(s); \tag{3.7}$$

(iv) *for almost every X and Y, we have*

$$2\mathcal{V}_4(\mathcal{X})\mathcal{V}_2(\mathcal{X}) = (c(U)\mathcal{V}_3(\mathcal{X}))^2, \qquad 2\mathcal{W}_4(\mathcal{Y})\mathcal{W}_2(\mathcal{Y}) = (c(U)\mathcal{W}_3(\mathcal{Y}))^2, \tag{3.8a}$$

$$\mathcal{V}_2(\mathcal{X}) = c(U)\mathcal{V}_1(\mathcal{X}), \qquad \mathcal{W}_2(\mathcal{Y}) = -c(U)\mathcal{W}_1(\mathcal{Y}), \tag{3.8b}$$

$$\mathcal{V}_4(\mathcal{X}) = c(U)\mathcal{V}_5(\mathcal{X}), \qquad \mathcal{W}_4(\mathcal{Y}) = -c(U)\mathcal{W}_5(\mathcal{Y}). \tag{3.8c}$$

(v) *We require*

$$\lim_{s\to-\infty} J(s) = 0 \tag{3.9}$$

where we denote $J(s) = \mathcal{Z}_4(s)$.

We denote by $\mathcal{G}_0$ the subset of $\mathcal{G}$ which parametrizes data at time $t = 0$, that is,

$$\mathcal{G}_0 = \{\Theta \in \mathcal{G} \mid \mathcal{Z}_1 = 0\}.$$

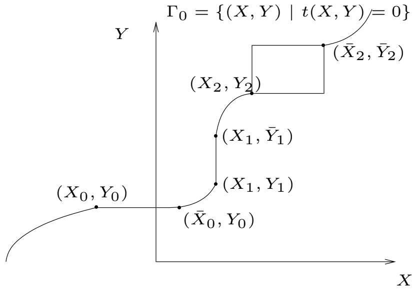

FIGURE 2. The domain $t(X,Y) = 0$ in the X, Y plane consists of the union of a graph of a strictly increasing function, vertical and horizontal segments and rectangular boxes.

The requirement (3.9) corresponds to a normalization of the energy potential (or cumulative energy) to zero at minus infinity. The variables $\mathcal{Z}$, $\mathcal{V}$ and $\mathcal{W}$ are not independent of one another as it can be seen from (3.7), (3.8b), (3.8c) but selecting a set of independent variables will require an arbitrary choice that we prefer to avoid and that is why we consider all the variables at the same level. For $\Theta \in \mathcal{G}_0$, we get by using (3.7) and (3.8b), that

$$\mathcal{V}_2(\mathcal{X}(s))\dot{\mathcal{X}}(s) = \mathcal{W}_2(\mathcal{Y}(s))\dot{\mathcal{Y}}(s). \tag{3.10}$$

By using the normalization (2.15c), we obtain that

$$\dot{\mathcal{X}} = \frac{2\mathcal{W}_2(\mathcal{Y})}{\mathcal{V}_2(\mathcal{X}) + \mathcal{W}_2(\mathcal{Y})}, \qquad \dot{\mathcal{Y}} = \frac{2\mathcal{V}_2(\mathcal{X})}{\mathcal{V}_2(\mathcal{X}) + \mathcal{W}_2(\mathcal{Y})} \tag{3.11}$$

and, in principle, by integrating (3.11), we recover $\mathcal{X}$ and $\mathcal{Y}$. However, there are two obstacles to that: The function $\mathcal{V}_2$ and $\mathcal{W}_2$ are in general not Lipschitz so that we cannot use the standard existence theorems for the solutions to (3.11) and, in addition, both $\mathcal{V}_2$ and $\mathcal{W}_2$ may vanish (it is what happens in the case of a box) and (3.11) does not make sense any more. Given $(u_0, R_0, S_0, \mu_0, \nu_0)$, in the case where $\mu_0 = (\mu_0)_{\text{ac}}$ and $\nu_0 = (\nu_0)_{\text{ac}}$, we have defined $\Theta = (\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{V}, \mathcal{W}) \in \mathcal{G}_0$ by (2.17), (2.18), (2.27), (2.28), (2.29), (2.23), (2.31) and

$$\mathcal{V}_4(\mathcal{X}(s)) = c(U)\mathcal{V}_5(\mathcal{X}(s)) = \frac{R_0^2}{4 + R_0^2}(x(s)),$$

$$\mathcal{W}_4(\mathcal{X}(s)) = -c(U)\mathcal{W}_5(\mathcal{X}(s)) = \frac{S_0^2}{4 + S_0^2}(x(s)),$$

$$\mathcal{V}_3(\mathcal{X}(s)) = \frac{2R_0}{c(4 + R_0^2)}(x(s)), \qquad \mathcal{W}_3(\mathcal{Y}(s)) = -\frac{2S_0}{c(4 + S_0^2)}(x(s)).$$

We do not prove here that, for this definition, we indeed have $(\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{V}, \mathcal{W}) \in \mathcal{G}_0$ because it will be done later in more generality, see Definitions 3.8 and 3.5. In the next section we consider $(\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{V}, \mathcal{W}) \in \mathcal{G}_0$ and construct solutions of (2.13) which satisfy (3.2). However, the set $\mathcal{G}_0$ is not adequate when it comes to parametrize initial data. In the case where there is no concentration of the measures, that is, $\mu_0 = (\mu_0)_{\text{ac}}$ and $\nu_0 = (\nu_0)_{\text{ac}}$, we can see from (2.31) and (2.27) that $\dot{\mathcal{X}} > 0$ and $\dot{\mathcal{Y}} > 0$ almost everywhere so that the curve does not contain strictly vertical or horizontal regions. This property is *not* preserved by the equation. In particular it means that at a later time, say $T > 0$, we can find a curve $(\bar{\mathcal{X}}, \bar{\mathcal{Y}}) \in \mathcal{C}$ such that $t(\bar{\mathcal{X}}(s), \bar{\mathcal{Y}}(s)) = T$ and $\dot{\mathcal{X}}(s) = 0$ or $\dot{\mathcal{Y}}(s) = 0$ on an interval $[s_l, s_r]$, with $s_l < s_r$. In general, the set of points

$$\Gamma_T = \{(X, Y) \in \mathbb{R}^2 \mid t(X, Y) = T\}$$

is not a curve but a domain which consists of the union of a graph of a strictly increasing function, vertical and horizontal segments and rectangular boxes, see Figure 3. If Γ_T contains regions with boxes or vertical or horizontal lines, it means that part of the energy of the solution is concentrated at time T in sets of zero measure, see Section 6. We want to parametrize domains Γ_0 (or Γ_T) depicted in Figure 3, which give the solution at time zero (or a given time T) and which may contain boxes. The set $\mathcal{G}_0$ defined above is inappropriate. When considering an element in $\mathcal{G}_0$, we choose a curve and in the case of a box, the choice of the curve which joins the two diagonal corners of the box while remaining inside the box is arbitrary. Thus we introduce an unwanted degree of freedom in the parametrization of the initial data. The domain Γ_0 depicted in Figure 3 can be parametrized by using two nondecreasing functions $x_1(X)$ and $x_1(Y)$ and by considering the set $\{(X, Y) \in \mathbb{R} \mid x_1(X) = x_2(Y)\}$. Such sets consist exactly of the union of the graph of a strictly increasing function (when $x_1' > 0$ and $x_2' > 0$), a horizontal segment (when $x_1'(X) = 0$ for $X \in [X_0, \bar{X}_0]$ and $x_2'(Y_0) > 0$), a vertical segment (when $x_1'(X_1) > 0$ and $x_2'(Y) = 0$ for $Y \in [Y_1, \bar{Y}_1]$) and a rectangular box (when $x_1'(X) = x_2'(Y) = 0$ for $X \in [X_2, \bar{X}_2]$ and $Y \in [Y_2, \bar{Y}_2]$), see Figure 3. This observation (partially) justifies the definition of the set $\mathcal{F}$ which is introduced below. The set $\mathcal{F}$ can be considered as a consistent way to parametrize initial data. However, to construct the solutions, we need to choose a curve and we use the description of the initial data given by $\mathcal{G}_0$ so that, finally, both sets are needed. To define $\mathcal{F}$, we have to introduce the group G of diffeomorphisms with some regularity conditions.

Definition 3.3. *The group G is given by all invertible functions f such that*

$$f - \mathrm{Id} \ \textit{and}\ f^{-1} - \mathrm{Id} \ \textit{both belong to}\ W^{1,\infty}(\mathbb{R}), \tag{3.12}$$

and

$$(f - \mathrm{Id})' \in L^2(\mathbb{R}).$$

We can now define the set $\mathcal{F}$.

Definition 3.4. *We define the set $\mathcal{F}$ consisting of all function $\psi = (\psi_1, \psi_2)$ such that*

$$\psi_1(X) = (x_1(X), U_1(X), V_1(X), J_1(X), K_1(X))$$
$$\textit{and} \quad \psi_2(Y) = (x_2(Y), U_2(Y), V_2(Y), J_2(Y), K_2(Y))$$

satisfy the following regularity and decay conditions

$$x_1 - \mathrm{Id},\ x_2 - \mathrm{Id},\ J_1,\ J_2,\ K_1,\ K_2 \in W^{1,\infty}(\mathbb{R}), \tag{3.13a}$$

$$x_1' - 1,\ x_2' - 1,\ J_1',\ J_2',\ K_1',\ K_2' \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R}), \tag{3.13b}$$

$$U_1,\ U_2 \in H^1(\mathbb{R}), \tag{3.13c}$$

$$V_1,\ V_2 \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R}), \tag{3.13d}$$

and which satisfy the additional conditions that

$$x_1', x_2', J_1', J_2' \geq 0, \tag{3.14}$$

$$J_1' = c(U_1)K_1', \quad J_2' = -c(U_2)K_2', \tag{3.15}$$

$$x_1'J_1' = (c(U_1)V_1)^2, \quad x_2'J_2' = (c(U_2)V_2)^2, \tag{3.16}$$

$$x_1 + J_1, \quad x_2 + J_2 \in G, \tag{3.17}$$

$$\lim_{X\to-\infty} J_1(X) = \lim_{Y\to-\infty} J_2(Y) = 0 \tag{3.18}$$

and, for any curve $(\mathcal{X}, \mathcal{Y}) \in \mathcal{C}$ *such that*

$$x_1(\mathcal{X}(s)) = x_2(\mathcal{Y}(s)) \text{ for all } s \in \mathbb{R},$$

we have

$$U_1(\mathcal{X}(s)) = U_2(\mathcal{Y}(s)) \tag{3.19a}$$

for all $s \in \mathbb{R}$ *and*

$$U_1'(\mathcal{X}(s))\dot{\mathcal{X}}(s) = U_2'(\mathcal{Y}(s))\dot{\mathcal{Y}}(s) = V_1(\mathcal{X}(s))\dot{\mathcal{X}}(s) + V_2(Y(s))\dot{\mathcal{Y}}(s) \tag{3.19b}$$

for almost all $s \in \mathbb{R}$.

We show in Section 5 that, given a solution Z of (2.13), there exists a unique element $\psi \in \mathcal{F}$ which describes in a unique way the set $\Gamma_0 = \{(X,Y) \in \mathbb{R}^2 \mid t(X,Y) = 0\}$ and the values of Z, Z_X and Z_Y on this set. The functions x_1 and x_2 define the set Γ_0 by $\Gamma_0 = \{(X,Y) \in \mathbb{R}^2 \mid x_1(X) = x_2(Y)\}$. It means in particular that, for any curve $(\mathcal{X}, \mathcal{Y}) \in \mathcal{C}$ such that $x_1(\mathcal{X}(s)) = x_2(\mathcal{Y}(s))$, we have $t(\mathcal{X}(s), \mathcal{Y}(s)) = 0$. The functions U_1 and U_2 give the value of $U(X,Y)$ on the set Γ_0, as a function of X and Y. To be more concrete, let us consider the example where x_1 and x_2 are smooth, invertible and the inverses are also smooth. In that case, which in fact corresponds to the case where u_0, R_0 and S_0 are smooth and there is no concentration of energy, i.e., $\mu_0 = (\mu_0)_{\text{ac}}$ and $\nu_0 = (\nu_0)_{\text{ac}}$, the set Γ_0 is the graph of a strictly increasing function (there is no rectangular box and no vertical or horizontal segments). The curve Γ_0 is given by either $Y = x_1^{-1} \circ x_2(X)$ or $X = x_2^{-1} \circ x_1(Y)$ and, just for this paragraph, for the sake of simplicity, we denote $Y(X) = x_1^{-1} \circ x_2(X)$ and $X(Y) = x_2^{-1} \circ x_1(Y)$. Then, we have

$$U_1(X) = U(X, Y(X)) \text{ and } U_2(Y) = U(X(Y), Y). \tag{3.20}$$

The functions of V_1 and V_2 give the partial derivative of U. We have

$$V_1(X) = U_X(X, Y(X)) \text{ and } V_2(Y) = U_Y(X(Y), Y). \tag{3.21}$$

As we can see in this example, the functions U_1, U_2, V_1 and V_2 are not independent from one another, and the way they depend one another is given by (3.19a) and (3.19b). The function $J_1(X)$ gives the amount of forward energy contained on the curve $Y = Y(X)$ between $-\infty$ and X, that is,

$$J_1(X) = \int_{-\infty}^{X} J_X(X, Y(X))\, dX.$$

In the original set of coordinates, it gives $J_1(X) = \frac{1}{4}\int_{-\infty}^{x_1(X)} R_0^2\,dx$. Similarly, the function $J_2(Y)$ gives the amount of backward energy which is contained on the same curve between $-\infty$ and Y, that is,

$$J_2(Y) = \int_{-\infty}^{Y} J_Y(X(Y), Y)\,dY.$$

In the original set of coordinates, it gives $J_2(Y) = \frac{1}{4}\int_{-\infty}^{x_2(Y)} S_0^2\,dx$. We recall that these expressions hold only for smooth initial data with no concentration of energy. Still in this case, the functions x_1 and x_2 are strictly increasing so that $x_1' > 0$ and $x_2' > 0$ and the conditions (3.16) entirely determine the energy densities, which are given J_1' and J_2' in the new sets of coordinates. We have

$$\mu_0 = (\mu_0)_{\mathrm{ac}} = \frac{1}{4}R_0^2(x)\,dx = \frac{J_1'}{x_1'}\circ x_1^{-1}(x)\,dx$$

and the corresponding expression for ν_0. In the case where there is concentration of energy, the functions x_1' or x_2' vanish. The set where x_1 (respectively x_2) vanishes corresponds to the region where the energy density μ_0 (respectively ν_0) has a singular part. On those sets, the energy densities J_1' and J_2' cannot be retrieved from (3.16). It is consistent with the fact that the singular parts of the energy μ and ν cannot be recovered by the knowledge of the function u, R and S, as illustrated in the example presented in the introduction. As we will see in Section 6, the relations (3.16) correspond to a reformulation in the new coordinate system of (3.1).

We define a mapping $\mathbf{C}$ which to any given initial data $\psi \in \mathcal{F}$ associate the corresponding data $\Theta = (\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{V}, \mathcal{W}) \in \mathcal{G}_0$.

Definition 3.5. *For any $\psi = (\psi_1, \psi_2) \in \mathcal{F}$, we define*

$$\mathcal{X}(s) = \sup\{X \in \mathbb{R} \mid x_1(X') < x_2(2s - X') \text{ for all } X' < X\} \tag{3.22}$$

and set $\mathcal{Y}(s) = 2s - \mathcal{X}(s)$. We have

$$x_1(\mathcal{X}(s)) = x_2(\mathcal{Y}(s)). \tag{3.23}$$

We define

$$t(s) = 0, \tag{3.24a}$$

$$x(s) = x_1(\mathcal{X}(s)) = x_2(\mathcal{Y}(s)), \tag{3.24b}$$

$$U(s) = U_1(\mathcal{X}(s)) = U_2(\mathcal{Y}(s)), \tag{3.24c}$$

$$J(s) = J_1(\mathcal{X}(s)) + J_2(\mathcal{Y}(s)), \tag{3.24d}$$

$$K(s) = K_1(\mathcal{X}(s)) + K_2(\mathcal{Y}(s)) \tag{3.24e}$$

and

$$\mathcal{V}_1(X) = \frac{1}{2c(U_1(X))}x_1'(X), \qquad \mathcal{W}_1(Y) = -\frac{1}{2c(U_2(Y))}x_2'(Y),$$

$$\mathcal{V}_2(X) = \frac{1}{2}x_1'(X), \qquad \mathcal{W}_2(Y) = \frac{1}{2}x_2'(Y),$$

$$\mathcal{V}_3(X) = V_1(X), \qquad \mathcal{W}_3(Y) = V_2(Y),$$

$$\mathcal{V}_4(X) = J_1'(X), \qquad \mathcal{W}_4(Y) = J_2'(Y),$$

$$\mathcal{V}_5(X) = K_1'(X), \qquad \mathcal{W}_5(Y) = K_2'(Y).$$

Let $\mathbf{C}$ *be the mapping from* $\mathcal{F}$ *to* $\mathcal{G}_0$ *which to any* $\psi \in \mathcal{F}$ *associates the element* $(\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{V}, \mathcal{W})$ *defined above.*

Proof of the well-posedness of Definition 3.5. Let us prove that $\mathcal{X}$ is increasing. Given $\bar{s} > s$, we consider a sequence X_i which converges to $\mathcal{X}(s)$ with $X_i < \mathcal{X}(s)$. We have $x_1(X_i) < x_2(2s - X_i)$ which implies $x_1(X_i) < x_2(2\bar{s} - X_i)$ because x_2 is increasing. Hence $X_i < \mathcal{X}(\bar{s})$. By letting i tend to infinity, we get that $\mathcal{X}(s) \leq \mathcal{X}(\bar{s})$. By continuity of x_1 and x_2, we have $x_1(\mathcal{X}(s)) = x_2(\mathcal{Y}(s))$. We claim that $\mathcal{X}$ is Lipschitz with a Lipshitz constant no bigger than 2, i.e.,

$$|\mathcal{X}(\bar{s}) - \mathcal{X}(\bar{s})| \leq 2\,|\bar{s} - s|\,. \tag{3.25}$$

Let us assume without loss of generality that $\bar{s} > s$. If (3.25) does not hold, we have

$$\mathcal{X}(\bar{s}) - \mathcal{X}(s) > 2(\bar{s} - s) \tag{3.26}$$

for some s and $\bar{s}$ in $\mathbb{R}$. It implies $\mathcal{Y}(\bar{s}) < \mathcal{Y}(s)$. Then, by monotonicity of x_2,

$$x_1(\mathcal{X}(s)) = x_2(\mathcal{Y}(s)) \geq x_2(\mathcal{Y}(\bar{s})) = x_1(\mathcal{X}(\bar{s})),$$

and therefore $x_1(\mathcal{X}(s)) = x_1(\mathcal{X}(\bar{s}))$ because x_1 is an increasing function and $\mathcal{X}(s) < \mathcal{X}(\bar{s})$. It follows that x_1 is constant on $[\mathcal{X}(s), \mathcal{X}(\bar{s})]$. Similarly, one proves that x_2 is constant on $[\mathcal{Y}(\bar{s}), \mathcal{Y}(s)]$. Let us consider the point (X, Y) given by $Y = \mathcal{Y}(s)$ and $X = 2\bar{s} - \mathcal{Y}(s)$. We have

$$\mathcal{X}(s) = 2s - \mathcal{Y}(s) < X < 2\bar{s} - \mathcal{Y}(\bar{s}) = \mathcal{X}(\bar{s})$$

so that $x_1(X) = x_1(\mathcal{X}(s)) = x_2(\mathcal{Y}(s)) = x_2(2\bar{s} - Y)$ and $X < \mathcal{X}(\bar{s})$, which contradicts the definition of $\mathcal{X}(\bar{s})$. Hence, (3.26) cannot hold and we have proved (3.25). Let us prove that $\mathcal{X} - \mathrm{Id} \in L^\infty$. We have

$$\mathcal{X}(s) - s = \frac{1}{2}(\mathcal{X}(s) - \mathcal{Y}(s)) = \frac{1}{2}(\mathcal{X}(s) - x_1(\mathcal{X}(s)) + x_2(\mathcal{Y}(s)) - \mathcal{Y}(s))$$

which is bounded as $x_1 - \mathrm{Id}$ and $x_2 - \mathrm{Id}$ belong to L^∞. Let

$$B = \{s \in \mathbb{R} \mid \dot{\mathcal{X}}(s) \geq 1\}.$$

Since $\dot{\mathcal{X}} + \dot{\mathcal{Y}} = 2$, we have $\dot{\mathcal{Y}} \geq 1$ on B^c. Hence,

$$\begin{aligned}
\int_{\mathbb{R}} U^2(s)\,ds &= \int_B U^2(s)\,ds + \int_{B^c} U^2(s)\,ds \\
&\leq \int_B U_1^2(\mathcal{X}(s))\dot{\mathcal{X}}(s)\,ds + \int_{B^c} U_2^2(\mathcal{Y}(s))\dot{\mathcal{Y}}(s)\,ds \\
&\leq \|U_1\|_{L^2}^2 + \|U_2\|_{L^2}^2\,.
\end{aligned}$$

It is then straightforward to check that the remaining properties that enter in the definition of $\mathcal{G}_0$ are fulfilled by $(\mathcal{Z}, \mathcal{V}, \mathcal{W})$. To check that (3.17) is fulfilled, we use Lemma 3.6 which is stated below. □

Lemma 3.6. *Let* $\alpha \geq 0$*. If* f *satisfies* (3.12)*, then* $1/(1+\alpha) \leq f_\xi \leq 1 + \alpha$ *almost everywhere. Conversely, if* f *is absolutely continuous,* $f - \mathrm{Id} \in L^\infty(\mathbb{R})$ *and there exists* $c \geq 1$ *such that* $1/c \leq f_\xi \leq c$ *almost everywhere, then* f *satisfies* (3.12) *and*

$$\|f - \mathrm{Id}\|_{W^{1,\infty}(\mathbb{R})} + \left\|f^{-1} - \mathrm{Id}\right\|_{W^{1,\infty}(\mathbb{R})} \leq \alpha$$

for some α depending only on c and $\|f-\mathrm{Id}\|_{L^\infty(\mathbb{R})}$.

The proof of this short lemma is given in [9]. In the opposite direction, to any element $(\mathcal{X},\mathcal{Y},\mathcal{Z},\mathcal{V},\mathcal{W})\in\mathcal{G}_0$, there corresponds an element $(\psi_1,\psi_2)\in\mathcal{F}$ given by the mapping $\mathbf{D}$ that we define next.

Definition 3.7. *Given $(\mathcal{X},\mathcal{Y},Z,\mathcal{V},\mathcal{W})\in\mathcal{G}_0$, let $\psi_1=(x_1,U_1,J_1,K_1,V_1)$ and $\psi_2=(x_2,U_2,J_2,K_2,V_2)$ be defined as*

$$x_1(\mathcal{X}(s))=x_2(\mathcal{Y}(s))=x(s) \tag{3.27}$$

where we denote $x(s)=\mathcal{Z}_2(s)$ and

$$U_1(\mathcal{X}(s))=U_2(\mathcal{Y}(s))=U(s) \tag{3.28}$$

where we denote $U(s)=\mathcal{Z}_3(s)$ and

$$J_1(\mathcal{X}(s))=\int_{-\infty}^{s}\mathcal{V}_4(\mathcal{X}(s))\dot{\mathcal{X}}(s)\,ds,\qquad J_2(\mathcal{Y}(s))=\int_{-\infty}^{s}\mathcal{W}_4(\mathcal{Y}(s))\dot{\mathcal{Y}}(s)\,ds, \tag{3.29}$$

$$K_1(\mathcal{X}(s))=\int_{-\infty}^{s}\mathcal{V}_5(\mathcal{X}(s))\dot{\mathcal{X}}(s)\,ds,\qquad K_2(\mathcal{Y}(s))=\int_{-\infty}^{s}\mathcal{W}_5(\mathcal{Y}(s))\dot{\mathcal{Y}}(s)\,ds, \tag{3.30}$$

and

$$V_1=\mathcal{V}_3,\quad V_2=\mathcal{W}_3. \tag{3.31}$$

We denote by $\mathbf{D}$ the mapping from $\mathcal{G}_0$ to $\mathcal{F}$ which to any $(\mathcal{X},\mathcal{Y},\mathcal{Z},\mathcal{V},\mathcal{W})\in\mathcal{G}_0$ associates the element ψ as defined above.

Well-posedness of Definition 3.7. Since $t(s)=0$ we have

$$0=\dot{t}(s)=\mathcal{V}_1(\mathcal{X}(s))\dot{\mathcal{X}}(s)+\mathcal{W}_1(\mathcal{Y}(s))\dot{\mathcal{Y}}(s)$$

which implies that

$$\mathcal{V}_2(\mathcal{X}(s))\dot{\mathcal{X}}(s)=\mathcal{W}_2(\mathcal{Y}(s))\dot{\mathcal{Y}}(s) \tag{3.32}$$

by (3.8b). We check the well-posedness of (3.27) and (3.28). Let us consider s and $\bar{s}$ such that $\mathcal{X}(s)=\mathcal{X}(\bar{s})$. Since $\mathcal{X}$ is increasing, it implies $\dot{\mathcal{X}}(\tilde{s})=0$ and $\dot{\mathcal{Y}}(\tilde{s})=2$ for all $\tilde{s}\in[s,\bar{s}]$. From (3.32), it follows that $\mathcal{W}_2(\mathcal{Y}(\tilde{s}))=0$ for all $\tilde{s}\in[s,\bar{s}]$. Hence,

$$\dot{x}(\tilde{s})=\mathcal{V}_2(\mathcal{X}(\tilde{s}))\dot{\mathcal{X}}(\tilde{s})+\mathcal{W}_2(\mathcal{Y}(\tilde{s}))\dot{\mathcal{Y}}(\tilde{s})=0$$

and $x(s)=x(\bar{s})$ so that the definition (3.27) is well-posed. For $\tilde{s}\in[s,\bar{s}]$, we have $\mathcal{W}_3(\mathcal{Y}(\tilde{s}))=0$, by (3.8a) and the fact that $\mathcal{W}_2(\mathcal{Y}(\tilde{s}))=0$. Hence,

$$\dot{U}(\tilde{s})=\mathcal{V}_3(\mathcal{X}(\tilde{s}))\dot{\mathcal{X}}(\tilde{s})+\mathcal{W}_3(\mathcal{Y}(\tilde{s}))\dot{\mathcal{Y}}(\tilde{s})=0$$

and $U(s)=U(\bar{s})$ so that the definition (3.28) is well-posed. Let us prove that x_1 is Lipschitz. We have

$$\begin{aligned}
x_1(\mathcal{X}(s))-x_1(\mathcal{X}(\bar{s}))&=x(s)-x(\bar{s})\\
&=\int_{\bar{s}}^{s}\dot{x}(s)\,ds\\
&=\int_{\bar{s}}^{s}\mathcal{V}_2(\mathcal{X}(s))\dot{\mathcal{X}}(s)+\mathcal{W}_2(\mathcal{Y}(s))\dot{\mathcal{Y}}(s)\,ds
\end{aligned}$$

$$
\begin{aligned}
&= 2\int_{\bar{s}}^{s} \mathcal{V}_2(\mathcal{X}(s))\dot{\mathcal{X}}(s)\,ds && \text{(by (3.32))}\\
&\leq \|\mathcal{V}_2\|_{L^\infty} |\mathcal{X}(s) - \mathcal{X}(\bar{s})|\,.
\end{aligned}
$$

Hence, x_1 is Lipschitz. One proves in the same way that x_2 is Lipschitz. Since

$$0 \leq \mathcal{V}_4(\mathcal{X}(s))\dot{\mathcal{X}}(s) \leq \mathcal{V}_4(\mathcal{X}(s))\dot{\mathcal{X}}(s) + \mathcal{W}_4(\mathcal{Y}(s))\dot{\mathcal{Y}}(s) = \dot{J}(s)$$

the function $\mathcal{V}_4(\mathcal{X}(s))\dot{\mathcal{X}}(s)$ belongs to $L^1(\mathbb{R})$. Assume that there exists an $s < \bar{s}$ such that $\mathcal{X}(s) = \mathcal{X}(\bar{s})$. Since $\mathcal{X}$ is increasing, it implies that $\dot{\mathcal{X}}(s) =$ for all $s \in [s, \bar{s}]$ and therefore $\int_{-\infty}^{s} \mathcal{V}_4(\mathcal{X}(s))\dot{\mathcal{X}}(s)\,ds = \int_{-\infty}^{\bar{s}} \mathcal{V}_4(\mathcal{X}(s))\dot{\mathcal{X}}(s)\,ds$ and the definition (3.29) of J_1 is well-posed. The same results hold for J_2. Let us prove that U_1 is absolutely continuous on any compact set. We consider $X_1 < \cdots < X_N$ and s_i such $\mathcal{X}(s_i) = X_i$. We have

$$
\begin{aligned}
\sum_{i=1}^{N} |U_1(X_{i+1}) - U_1(X_i)| &= \sum_{i=1}^{N} |U_1(s_{i+1}) - U_1(s_i)|\\
&\leq \int_{\cup_i(s_i,s_{i+1})} \left|\dot{U}_1(s)\right|\,ds\\
&\leq \int_{\cup_i(s_i,s_{i+1})} (\mathcal{V}_3(\mathcal{X})\dot{\mathcal{X}} + \mathcal{W}_3(\mathcal{Y})\dot{\mathcal{Y}})\,ds\\
&\leq \|\mathcal{V}_3\|_{L^\infty} \int_{\cup_i(s_i,s_{i+1})} \dot{\mathcal{X}}\,ds\\
&\quad + \operatorname{meas}(\cup_i(s_i, s_{i+1}))^{1/2} \Big(\int_{\cup_i(s_i,s_{i+1})} \mathcal{W}_3(\mathcal{Y})^2\dot{\mathcal{Y}}^2\,ds\Big)^{1/2}.
\end{aligned}
$$

By (3.8a), we get $\mathcal{W}_3^2 \leq 2\kappa \|\mathcal{W}_4\|_{L^\infty(\mathbb{R})} \mathcal{W}_2$, and therefore $\mathcal{W}_3^2(\mathcal{Y})\dot{\mathcal{Y}}^2 \leq C\mathcal{W}_2(\mathcal{Y})\dot{\mathcal{Y}}^2 = C\mathcal{V}_2(\mathcal{X})\dot{\mathcal{X}}\dot{\mathcal{Y}}$, by (3.32), for some constant C. Hence,

$$
\begin{aligned}
\int_{\cup_i(s_i,s_{i+1})} \mathcal{W}_3(\mathcal{Y})^2\dot{\mathcal{Y}}^2\,ds &\leq C\int_{\cup_i(s_i,s_{i+1})} \mathcal{V}_2(\mathcal{X})\dot{\mathcal{X}}\,ds\\
&\leq C\int_{\cup_i(s_i,s_{i+1})} \dot{\mathcal{X}}\,ds = C\operatorname{meas}(\cup_i(X_i, X_{i+1}))
\end{aligned}
$$

for some constant C. Finally,

$$\sum_{i=1}^{N} |U_1(X_{i+1}) - U_1(X_i)| \leq C(\operatorname{meas}(\cup_i(X_i, X_{i+1})) + \operatorname{meas}(\cup_i(X_i, X_{i+1}))^{1/2})$$

and U_1 is absolutely continuous. After differentiating (3.28), we get

$$U_1'(\mathcal{X})\dot{\mathcal{X}} = \mathcal{V}_3(\mathcal{X})\dot{\mathcal{X}} + \mathcal{W}_3(\mathcal{Y})\dot{\mathcal{Y}}$$

and, after taking the square of this expression, we obtain

$$U_1'(\mathcal{X})^2\dot{\mathcal{X}}^2 \leq 2(\mathcal{V}_3(\mathcal{X})^2\dot{\mathcal{X}}^2 + \mathcal{W}_3(\mathcal{Y})^2\dot{\mathcal{Y}}^2). \tag{3.33}$$

Since

$$\mathcal{W}_3(\mathcal{Y})^2\dot{\mathcal{Y}}^2 = c\mathcal{W}_2\mathcal{W}_4\dot{\mathcal{Y}}^2 = c\mathcal{V}_2\dot{\mathcal{X}}\mathcal{W}_4\dot{\mathcal{Y}} \text{ (by (3.32))},$$

we have that (3.33) implies

$$\begin{aligned} U_1'(\mathcal{X})^2\dot{\mathcal{X}} &\le C(\mathcal{V}_3(\mathcal{X})^2\dot{\mathcal{X}} + \mathcal{W}_4(\mathcal{Y})\dot{\mathcal{Y}}) \\ &\le C(\mathcal{V}_3(\mathcal{X})^2\dot{\mathcal{X}} + \dot{J}) \end{aligned}$$

and, after a change of variables, we obtain

$$\|U_1'\|_{L^2}^2 \le C(\|\mathcal{V}_3\|_{L^2}^2 + \|J\|_{L^\infty}) < \infty.$$

Hence, U_1' belongs to L^2. Similarly one proves that U_2 is absolutely continuous on any compact set and $U_2' \in L^2(\mathbb{R})$. To prove that the property (3.17) is fulfilled, we use Lemma 3.6 and the fact that $1/(\mathcal{V}_2 + \mathcal{V}_4), 1/(\mathcal{W}_2 + \mathcal{W}_4) \in L^\infty(\mathbb{R})$. The other properties of $\mathcal{F}$ that ψ has to fulfill can be checked more or less directly from the definition of $\mathcal{G}_0$. □

The sets $\mathcal{F}$ and $\mathcal{G}_0$ are not in bijection; otherwise we would not have introduced $\mathcal{F}$, and indeed one can show that $\mathbf{C} \circ \mathbf{D} \neq \mathrm{Id}_{\mathcal{G}_0}$. However, we have $\mathbf{D} \circ \mathbf{C} = \mathrm{Id}_{\mathcal{F}}$, as we will see in Lemma 5.3.

Now we define how, from any initial data in $\mathcal{D}$, that is, in the set of original coordinates, we define the corresponding element in $\mathcal{F}$.

Definition 3.8. *We define the mapping* $\mathbf{L}\colon \mathcal{D} \to \mathcal{F}$ *where, for any* $(u, R, S, \mu, \nu) \in \mathcal{D}$, $\psi = (\psi_1, \psi_2) = \mathbf{L}(u, R, S, \mu, \nu)$ *is defined as follows. We set*

$$x_1(X) = \sup\{x \in \mathbb{R} \mid x' + \mu(-\infty, x') < X \text{ for all } x' < x\}, \tag{3.34a}$$

$$x_2(Y) = \sup\{x \in \mathbb{R} \mid x' + \nu(-\infty, x') < Y \text{ for all } x' < x\} \tag{3.34b}$$

and

$$J_1(X) = X - x_1(X), \quad J_2(Y) = Y - x_2(Y) \tag{3.34c}$$

and

$$U_1(X) = u(x_1(X)), \quad U_2(Y) = u(x_2(Y)) \tag{3.34d}$$

and

$$V_1(X) = \left[\frac{R}{2c(U_1)}\right](x_1(X))x_1'(X),\ V_2(Y) = -\left[\frac{S}{2c(U_2)}\right](x_2(Y))x_2'(Y) \tag{3.34e}$$

and

$$K_1(X) = \int_{-\infty}^{X} \frac{J_1'(\bar{X})}{c(U_1(\bar{X}))}\,d\bar{X}, \quad K_2(Y) = -\int_{-\infty}^{Y} \frac{J_2'(\bar{Y})}{c(U_2(\bar{Y}))}\,d\bar{Y}. \tag{3.34f}$$

Before proving the well-posedness of this definition, we check that we end up with the same initial data that was obtained at the end of Section 2, where μ and ν were assumed to be absolutely continuous with respect to the Lebesgue measure. Then, the functions

$$\mu(-\infty, x') = \int_{-\infty}^{x'} \frac{1}{4}R^2\,dx \quad \text{and} \quad \nu(-\infty, x') = \int_{-\infty}^{x'} \frac{1}{4}S^2\,dx$$

are continuous, and furthermore, (3.34a) and (3.34b) rewrite as

$$x_1(X) + \int_{-\infty}^{x_1(X)} \frac{1}{4}R^2\,dx = X \quad \text{and} \quad x_2(Y) + \int_{-\infty}^{x_2(Y)} \frac{1}{4}S^2\,dx = Y.$$

We sum these two equalities, and, since $x_1(\mathcal{X}(s)) = x_2(\mathcal{Y}(s)) = x(s)$ and $\mathcal{X}+\mathcal{Y} = 2s$, we get

$$2x(s) + \int_{-\infty}^{x(s)} \frac{1}{4}(R^2+S^2)\,dx = 2s$$

and recover (2.27). In the definitions (3.34), we use the degree of freedom we have in the new set of coordinates to set the values of x_1 and x_2 in such a way that their derivatives, x_1' and x_2', are bounded.

Proof of well-posedness of Definition 3.8. Clearly, the definition of x_1 yields an increasing function and $\lim_{X\to\pm\infty} x_1(X) = \pm\infty$. For any $z > x_1(X)$, we have $X \leq z + \mu((-\infty, z))$. Hence, $X - z \leq \mu(\mathbb{R})$ and, since we can choose z arbitrarily close to $x_1(X)$, we get $X - x_1(X) \leq \mu(\mathbb{R})$. It is not hard to check that $x_1(X) \leq X$. Hence,

$$|x_1(X) - X| \leq \mu(\mathbb{R}) \tag{3.35}$$

and $\|x_1 - \mathrm{Id}\|_{L^\infty} \leq \mu(\mathbb{R})$. Let us prove that x_1 is Lipschitz with Lipschitz constant at most one. We consider X, X' in $\mathbb{R}$ such that $X < X'$ and $x_1(X) < x_1(X')$. It follows from the definition that there exists an increasing sequence, z_i', and a decreasing one, z_i, such that $\lim_{i\to\infty} z_i = x_1(X)$, $\lim_{i\to\infty} z_i' = x_1(X')$ with $\mu((-\infty, z_i')) + z_i' < X'$ and $\mu((-\infty, z_i)) + z_i \geq X$. Combining the these two inequalities, we obtain

$$\mu((-\infty, z_i')) - \mu((-\infty, z_i)) + z_i' - z_i < X' - X. \tag{3.36}$$

For j large enough, since by assumption $x_1(X) < x_1(X')$, we have $z_i < z_i'$ and therefore $\mu((-\infty, z_i')) - \mu((-\infty, z_i)) = \mu([z_i, z_i')) \geq 0$. Hence, $z_i' - z_i < X' - X$. Letting i tend to infinity, we get $x_1(X') - x_1(X) \leq X' - X$. Hence, x_1 is Lipschitz with Lipschitz constant bounded by one and, by Rademacher's theorem, differentiable almost everywhere. Following [6], we decompose μ into its absolute continuous, singular continuous and singular part, denoted μ_{ac}, μ_{sc} and μ_{s}, respectively. We have $\mu_{\mathrm{ac}} = \frac{1}{4}R^2\,dx$. The support of μ_{s} consists of a countable set of points. Let $H(x) = \mu((-\infty, x))$, then H is lower semi-continuous and its points of discontinuity exactly coincide with the support of μ_{s} (see [6]). Let A denote the complement of $x_1^{-1}(\mathrm{supp}(\mu_{\mathrm{s}}))$. We claim that for any $X \in A$, we have

$$\mu((-\infty, x_1(X))) + x_1(X) = X. \tag{3.37}$$

From the definition of $x_1(X)$ follows the existence of an increasing sequence z_i which converges to $x_1(X)$ and such that $H(z_i) + z_i < X$. Since H is lower semi-continuous, $\lim_{i\to\infty} H(z_i) = H(x_1(X))$ and therefore

$$H(x_1(X)) + x_1(X) \leq X. \tag{3.38}$$

Let us assume that $H(x_1(X)) + x_1(X) < X$. Since $x_1(X)$ is a point of continuity of H, we can then find an x such that $x > x_1(X)$ and $H(x) + x < X$. This contradicts the definition of $x_1(X)$ and proves our claim (3.37). In order to check that (3.16) is satisfied, we have to compute the derivative of x_1. We define the set B_1 as

$$B_1 = \left\{ x \in \mathbb{R} \mid \lim_{\rho\downarrow 0} \frac{1}{2\rho}\mu((x-\rho, x+\rho)) = \frac{1}{4}R^2(x) \right\}.$$

Since $\frac{1}{4}R^2(x)\,dx$ is the absolutely continuous part of μ, we have, from Besicovitch's derivation theorem (see [1]), that $\mathrm{meas}(B_1^c) = 0$. Given $X \in x_1^{-1}(B_1)$, we denote $x =$

$x_1(X)$. We claim that for all $i \in \mathbb{N}$, there exists $0 < \rho < \frac{1}{i}$ such that $x - \rho$ and $x + \rho$ both belong to $\operatorname{supp}(\mu_{\mathrm{s}})^c$. Assume namely the opposite. Then for any $z \in (x - \frac{1}{i}, x + \frac{1}{i}) \setminus \operatorname{supp}(\mu_{\mathrm{s}})$, we have that $z' = 2x - z$ belongs to $\operatorname{supp}(\mu_{\mathrm{s}})$. Thus we can construct an injection between the uncountable set $(x - \frac{1}{i}, x + \frac{1}{i}) \setminus \operatorname{supp}(\mu_{\mathrm{s}})$ and the countable set $\operatorname{supp}(\mu_{\mathrm{s}})$. This is impossible, and our claim is proved. Hence, since x_1 is surjective, we can find two sequences X_i and X_i' in A such that $\frac{1}{2}(x_1(X_i) + x_1(X_i')) = x_1(X)$ and $x_1(X_i') - x_1(X_i) < \frac{1}{i}$. We have, by (3.37), since $x_1(X_i)$ and $x_1(X_i')$ belong to A,

$$\mu([x_1(X_i), x_1(X_i'))) + x_1(X_i') - x_1(X_i) = X_i' - X_i. \tag{3.39}$$

Since $x_1(X_i) \notin \operatorname{supp}(\mu_{\mathrm{s}})$, we infer that $\mu(\{x_1(X_i)\}) = 0$ and $\mu([x_1(X_i), x_1(X_i'))) = \mu((x_1(X_i), x_1(X_i')))$. Dividing (3.39) by $X_i' - X_i$ and letting i tend to ∞, we obtain

$$x_1'(X)\frac{1}{4}R^2(x_1(X)) + x_1'(X) = 1 \tag{3.40}$$

where x_1 is differentiable in $x_1^{-1}(B_1)$, that is, almost everywhere in $x_1^{-1}(B_1)$. We will use several times this short lemma whose proof can be found in [9].

Lemma 3.9. *Given an increasing Lipschitz function $f\colon \mathbb{R} \to \mathbb{R}$, for any set B of measure zero, we have $f' = 0$ almost everywhere in $f^{-1}(B)$.*

We apply Lemma 3.9 to B_1^c and get, since $\operatorname{meas}(B_1^c) = 0$, that $x_1' = 0$ almost everywhere on $x_1^{-1}(B_1^c)$. On $x_1^{-1}(B_1)$, we proved that x_1' satisfies (3.40). It follows that $0 \le x_1' \le 1$ almost everywhere, which implies, since $J_1' = 1 - x_1'$, that $J_1' \ge 0$. From (3.40), we get

$$x_1(X)'J_1'(X) = x_1'(X)^2\frac{1}{4}R^2(x_1(X)) = (c(U_1(X))V_1(X))^2.$$

Let us prove that U_1 is absolutely continuous on any bounded interval. We consider a partition $X_1 \le \cdots \le X_N$. We have

$$\sum_{i=1}^{N} |U_1(X_{i+1}) - U_1(X_i)| \le \int_{\cup_{i=1}^{N}(x_1(X_i), x_1(X_{i+1}))} |u_x|\, dx.$$

Given $M > 0$, for any $\varepsilon > 0$, there exists δ such that for any set $A \subset [-M, M]$, we have that $\operatorname{meas}(A) < \delta$ implies $\int_A |u_x|\, dx < \varepsilon$, because $u_x \in L^1_{\mathrm{loc}}$. We have

$$\operatorname{meas}(\cup_{i=1}^{N}(x_1(X_i), x_1(X_{i+1}))) \le \left\|x_1'\right\|_{L^\infty} \operatorname{meas}(\cup_{i=1}^{N}(X_i, X_{i+1}))$$

and since $x_1' \in L^\infty$, it follows that for any partition such that $\sum_i^N |X_i - X_{i+1}| < \delta$, we have $\sum_{i=1}^N |U_1(X_{i+1}) - U_1(X_i)| < \varepsilon$ and U_1 is absolutely continuous in any compact. Let us consider a curve $(\mathcal{X}, \mathcal{Y}) \in \mathcal{C}$ such that $x_1(\mathcal{X}(s)) = x_2(\mathcal{Y}(s))$ (such curves exist, see Definition 3.5). We differentiate $U_1(\mathcal{X})$ and obtain

$$\begin{aligned}
U_1'(\mathcal{X})\dot{\mathcal{X}} = u_x(x_1(\mathcal{X}))x_1'(\mathcal{X})\dot{\mathcal{X}} &= \frac{(R-S)(x_1(\mathcal{X}))}{2c(U_1(\mathcal{X}))}x_1'(\mathcal{X})\dot{\mathcal{X}} \\
&= \frac{R(x_1(\mathcal{X}))}{2c(U_1(\mathcal{X}))}x_1'(\mathcal{X})\dot{\mathcal{X}} - \frac{S(x_2(\mathcal{Y}))}{2c(U_2(\mathcal{Y}))}x_2'(\mathcal{Y})\dot{\mathcal{Y}} \\
&= V_1(\mathcal{X})\dot{\mathcal{X}} + V_2(\mathcal{Y})\dot{\mathcal{Y}}.
\end{aligned}$$

Here we have use the fact that

$$x_1'(\mathcal{X})\dot{\mathcal{X}} = x_2'(\mathcal{Y})\dot{\mathcal{Y}}$$

which follows from $x_1(\mathcal{X}) = x_2(\mathcal{Y})$. From (3.35), we obtain $\|J_1\|_{L^\infty} \leq \mu(\mathbb{R})$. We have, since $J_1' \geq 0$,

$$\left\|J_1'\right\|_{L^2}^2 \leq \left\|J_1'\right\|_{L^\infty} \left\|J_1'\right\|_{L^1} \leq \|J_1\|_{L^\infty} \leq \mu(\mathbb{R}).$$

After a change of variables, we get

$$\int_{\mathbb{R}} V_1(X)^2\,dX \leq \frac{\kappa^2}{4}\int_{\mathbb{R}} R^2(x)\,dx < \infty$$

and

$$\begin{aligned}\int_{\mathbb{R}} U_1'^2(X)\,dX &= \int_{\mathbb{R}} u_x^2(x_1(X))x_1'^2(X)\,dX\\ &\leq \int_{\mathbb{R}} u_x^2(x_1(X))x_1'(X)\,dX = \int_{\mathbb{R}} u_x^2(x)\,dx < \infty,\end{aligned}$$

so that both V_1 and U_1' belong to L^2. Similarly, one proves that V_2 and U_2' belong to L^2. Let $B_3 = \{\xi \in \mathbb{R} \mid x_1' < \frac{1}{2}\}$. Since $x_1' - 1 \geq 0$, $B_3 = \{\xi \in \mathbb{R} \mid |x_1' - 1| > \frac{1}{2}\}$, and, after using the Chebychev inequality, as $x_1' - 1 = -J_1' \in L^2$, we obtain $\operatorname{meas}(B_3) < \infty$. Hence,

$$\begin{aligned}\int_{\mathbb{R}} U_1^2(X)\,dX &= \int_{B_3} U_1^2(X)\,dX + \int_{B_3^c} U_1^2(X)\,dX\\ &\leq \operatorname{meas}(B_3)\,\|u\|_{L^\infty}^2 + 2\int_{B_3^c} (u\circ x_1)^2 x_1'\,dX\\ &\leq \operatorname{meas}(B_3)\,\|u\|_{L^\infty}^2 + 2\,\|u\|_{L^2}^2\,,\end{aligned}$$

after a change of variables, and $U_1 \in L^2$. Similarly, one proves that $U_2 \in L^2$. □

In this section we have shown how to construct, from a given initial data in $\mathcal{D}$, an element in $\mathcal{F}$ (via the mapping $\mathbf{L}$) and then, from an element in $\mathcal{F}$, and element in $\mathcal{G}_0$ (via the mapping $\mathbf{C}$). From an element in $\mathcal{G}_0$, we can finally construct the corresponding solution of (2.13).

Now, we turn to the existence of solution to (2.13) for given data in $\mathcal{G}$.

4. Existence of solution for the equivalent system

4.1. **Short-range existence.** We first establish the short-range existence of solutions to (2.13). The difficulty here consists of taking into account initial data defined on a curve which may be parallel to the characteristic curves $X = \text{constant}$ or $Y = \text{constant}$. In the following, we will denote by Ω any rectangular domain of the type

$$\Omega = [X_l, X_r] \times [Y_l, Y_r],$$

and we denote $s_l = \frac{1}{2}(X_l + Y_l)$ and $s_r = \frac{1}{2}(X_r + Y_r)$. We define curves in Ω as follows.

Definition 4.1. *Given $\Omega = [X_l, X_r] \times [Y_l, Y_r]$, we denote by $\mathcal{C}(\Omega)$ the set of curves in Ω given by $(\mathcal{X}(s), \mathcal{Y}(s))$ for $s \in [s_l, s_r]$ which match the diagonal points of Ω, that is, $\mathcal{X}(s_l) = X_l$, $\mathcal{X}(s_r) = X_r$, $\mathcal{Y}(s_l) = Y_l$, $\mathcal{Y}(s_r) = Y_r$, and such that*

$$\mathcal{X} - \mathrm{Id},\ \mathcal{Y} - \mathrm{Id} \in W^{1,\infty}([s_l, s_r]), \tag{4.1a}$$

$$\dot{\mathcal{X}} \geq 0, \quad \dot{\mathcal{Y}} \geq 0, \tag{4.1b}$$

$$\frac{1}{2}(\mathcal{X}(s) + \mathcal{Y}(s)) = s, \text{ for all } s \in \mathbb{R}. \tag{4.1c}$$

We set

$$\|(\mathcal{X}, \mathcal{Y})\|_{\mathcal{C}(\Omega)} = \|\mathcal{X} - \mathrm{Id}\|_{L^\infty([s_l,s_r])} + \|\mathcal{Y} - \mathrm{Id}\|_{L^\infty([s_l,s_r])}.$$

In this subsection we will construct solutions on small rectangular domains Ω. We introduce the set $\mathcal{G}(\Omega)$ which is the counterpart of $\mathcal{G}$ on bounded intervals. Elements of $\mathcal{G}(\Omega)$ correspond to a curve in $\mathcal{C}(\Omega)$ and data on this curve.

Definition 4.2. *Given a rectangular domain $\Omega = [X_l, X_r] \times [Y_l, Y_r]$, let $\Theta = (\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{V}, \mathcal{W})$ where $(\mathcal{X}, \mathcal{Y}) \in \mathcal{C}(\Omega)$ and $\mathcal{Z}(s), \mathcal{V}(X), \mathcal{W}(Y)$ are three five-dimensional vector-valued measurable functions. Using the same notation as in* (3.3)*, we set*

$$\|\Theta\|_{\mathcal{G}(\Omega)} = \|U\|_{L^2([s_l,s_r])} + \|\mathcal{V}^a\|_{L^2([X_l,X_r])} + \|\mathcal{W}^a\|_{L^2([Y_l,Y_r])}$$

where we denote $U = \mathcal{Z}_3$ and

$$\begin{aligned}|||\Theta|||_{\mathcal{G}(\Omega)} = \|(\mathcal{X}, \mathcal{Y})\|_{\mathcal{C}(\Omega)} + \|\frac{1}{\mathcal{V}_2 + \mathcal{V}_4}\|_{L^\infty([X_l,X_r])} + \|\frac{1}{\mathcal{W}_2 + \mathcal{W}_4}\|_{L^\infty([Y_l,Y_r])} \\ + \|\mathcal{Z}^a\|_{L^\infty([s_l,s_r])} + \|\mathcal{V}^a\|_{L^\infty([X_l,X_r])} + \|\mathcal{W}^a\|_{L^\infty([Y_l,Y_r])}.\end{aligned}$$

The element Θ belongs to $\mathcal{G}(\Omega)$ if the following four conditions hold:

(i)

$$|||\Theta|||_{\mathcal{G}(\Omega)} < \infty$$

and therefore $\|\Theta\|_{\mathcal{G}(\Omega)} < \infty$ because we here consider a bounded domain.

(ii)

$$\mathcal{V}_2, \mathcal{W}_2, \mathcal{Z}_4, \mathcal{V}_4, \mathcal{W}_4 \geq 0. \tag{4.2}$$

(iii) *For almost every s, we have*

$$\dot{\mathcal{Z}}(s) = \mathcal{V}(\mathcal{X}(s))\dot{\mathcal{X}}(s) + \mathcal{W}(\mathcal{Y}(s))\dot{\mathcal{Y}}(s). \tag{4.3}$$

(iv) *For almost every X and Y, we have*

$$2\mathcal{V}_4(\mathcal{X})\mathcal{V}_2(\mathcal{X}) = (c(U)\mathcal{V}_3(\mathcal{X}))^2, \qquad 2\mathcal{W}_4(\mathcal{Y})\mathcal{W}_2(\mathcal{Y}) = (c(U)\mathcal{W}_3(\mathcal{Y}))^2, \tag{4.4a}$$

$$\mathcal{V}_2(\mathcal{X}) = c(U)\mathcal{V}_1(\mathcal{X}), \qquad \mathcal{W}_2(\mathcal{Y}) = -c(U)\mathcal{W}_1(\mathcal{Y}), \tag{4.4b}$$

$$\mathcal{V}_4(\mathcal{X}) = c(U)\mathcal{V}_5(\mathcal{X}), \qquad \mathcal{W}_4(\mathcal{Y}) = -c(U)\mathcal{W}_5(\mathcal{Y}). \tag{4.4c}$$

We introduce the Banach spaces $W^{1,\infty}_X(\Omega)$ and $W^{1,\infty}_Y(\Omega)$ defined as

$$W^{1,\infty}_X(\Omega) = L^\infty([Y_l, Y_r], W^{1,\infty}([X_l, X_r])), \quad W^{1,\infty}_Y(\Omega) = L^\infty([X_l, X_r], W^{1,\infty}([Y_l, Y_r])), \tag{4.5}$$

and the Banach spaces $L^\infty_X(\Omega)$ and $L^\infty_Y(\Omega)$ defined as

$$L^\infty_X(\Omega) = L^\infty([Y_l, Y_r], C([X_l, X_r])), \quad L^\infty_Y(\Omega) = L^\infty([X_l, X_r], C([Y_l, Y_r])).$$

Let us consider $(\mathcal{X},\mathcal{Y},\mathcal{Z},\mathcal{V},\mathcal{W})\in\mathcal{G}(\Omega)$. By definition, the functions $\mathcal{X}$ and $\mathcal{Y}$ are increasing. To any increasing function, one can associate its generalized inverse, a concept which is exposed for example in Brenier [2]. More generally, an increasing function (not necessarily continuous) can be identified as the subdifferential of a convex function (which is a multivalued function). The generalized inverse is then the subdifferential of the conjugate of this convex function. We do not use this framework here and prove directly the results we need.

Definition 4.3. *Given $\Omega=[X_l,X_r]\times[Y_l,Y_r]$ and a curve $(\mathcal{X},\mathcal{Y})\in\mathcal{C}(\Omega)$, we define the generalized inverse of $\mathcal{X}$ and $\mathcal{Y}$, respectively, as*

$$\alpha(X)=\sup\{s\in[s_l,s_r]\mid\mathcal{X}(s)<X\}\text{ for }X\in(X_l,X_r],\tag{4.6}$$

$$\beta(Y)=\sup\{s\in[s_l,s_r]\mid\mathcal{Y}(s)<Y\}\text{ for }Y\in(Y_l,Y_r].\tag{4.7}$$

We denote $\mathcal{X}^{-1}=\alpha$ and $\mathcal{Y}^{-1}=\beta$.

The generalized inverse functions $\mathcal{X}^{-1}$ and $\mathcal{Y}^{-1}$ enjoy the following properties.

Lemma 4.4. *The functions $\mathcal{X}^{-1}$ and $\mathcal{Y}^{-1}$ are lower semicontinuous nondecreasing functions. We have*

$$\mathcal{X}\circ\mathcal{X}^{-1}=\mathrm{Id}\ \text{ and }\mathcal{Y}\circ\mathcal{Y}^{-1}=\mathrm{Id},\tag{4.8}$$

and

$$\mathcal{X}^{-1}\circ\mathcal{X}(s)=s\text{ for any }s\text{ such that }\dot{\mathcal{X}}(s)>0,\tag{4.9}$$

$$\mathcal{Y}^{-1}\circ\mathcal{Y}(s)=s\text{ for any }s\text{ such that }\dot{\mathcal{Y}}(s)>0.\tag{4.10}$$

Definition 4.3 extends naturally to curves in $\mathcal{C}$ and Lemma 4.4 still holds.

Proof. We prove the lemma only for $\mathcal{X}^{-1}$, as the results for $\mathcal{Y}^{-1}$ can be proved in the same way. Let us prove that α is nondecreasing. For any $X<\bar{X}$, there exists a sequence s_i such that $\lim_{i\to\infty}s_i=\alpha(X)$ and $\mathcal{X}(s_i)<X$. Hence, $\mathcal{X}(s_i)<\bar{X}$ which implies $s_i\leq\alpha(\bar{X})$, which after letting i tend to infinity, gives $\alpha(X)\leq\alpha(\bar{X})$. Let us prove that α is lower semicontinuous. Given a sequence X_i such that $\lim_{i\to\infty}X_i=X$, for any $\varepsilon>0$, there exists $s\in[s_l,s_r]$ such that

$$\alpha(X)>s>\alpha(X)-\varepsilon\tag{4.11}$$

because $\alpha(X)>s_l$ for all $X\in(X_l,X_r]$. It implies $\mathcal{X}(s)<X$ as, otherwise, $X\leq\mathcal{X}(s)$ would yield $\alpha(X)\leq s$, which contradicts (4.11). Thus, for large enough i, we have $\mathcal{X}(s)<X_i$ so that $s<\alpha(X_i)$. Combined with (4.11), it implies

$$\alpha(X)-\varepsilon<s\leq\liminf\alpha(X_i)$$

and, as ε is arbitrary, we get that α is lower semicontinuous. Let us prove (4.8). Given $X\in(X_l,X_r]$, we consider an increasing sequence s_i such that $\lim_{i\to\infty}s_i=\alpha(X)$ and $\mathcal{X}(s_i)<X$. Letting i tend to infinity, since $\mathcal{X}$ is continuous, we get $\mathcal{X}(\alpha(X))\leq X$. Assume that $\mathcal{X}(\alpha(X))<X$, since $\mathcal{X}$ is continuous, there exists s such that $\mathcal{X}(\alpha(X))<\mathcal{X}(s)$ and $\mathcal{X}(s)<X$. The latter inequality implies that $s\leq\alpha(X)$ which, by the monotonicity of $\mathcal{X}$, yields $\mathcal{X}(s)\leq\mathcal{X}(\alpha(X))$ and we obtain a contradiction. Let us prove (4.9). We denote $\mathcal{N}=\{s\in[s_l,s_r]\mid\dot{X}(s)>0\}$. We consider a fixed element $s_0\in\mathcal{N}$. We have

$$\alpha\circ\mathcal{X}(s_0)\leq s_0.\tag{4.12}$$

Indeed, by the monotonicity of $\mathcal{X}$, for any $s \in \{s \in [s_l, s_r] \mid \mathcal{X}(s) < \mathcal{X}(s_0)\}$, we have $s < s_0$ and therefore, after taking the supremum, we obtain (4.12). Let us assume that $\alpha \circ \mathcal{X}(s_0) < s_0$. We denote $s_1 = \alpha \circ \mathcal{X}(s_0)$. By (4.8), $\mathcal{X}(s_1) = \mathcal{X}(s_0)$, and from the monotonicity of $\mathcal{X}$, it follows that $\mathcal{X}(s) = \mathcal{X}(s_0) = \mathcal{X}(s_1)$ for all $s \in [s_0, s_1]$. It implies that $\dot{\mathcal{X}}(s_0) = 0$, which contradicts the fact that $s_0 \in \mathcal{N}$. $\square$

In the following and when there is no ambiguity, we will slightly abuse the notation and denote $\mathcal{Y} \circ \mathcal{X}^{-1}(X)$ and $\mathcal{X} \circ \mathcal{Y}^{-1}(Y)$ by $\mathcal{Y}(X)$ and $\mathcal{X}(Y)$, respectively. The curve $(\mathcal{X}(s), \mathcal{Y}(s))$ is *almost* a graph as it consists of the union of the graphs of the functions $X \mapsto \mathcal{Y}(X)$ and (after rotating the axes by $\frac{\pi}{2}$) $Y \mapsto \mathcal{X}(Y)$. We prove the existence of solutions to (2.13) on rectangular boxes. First we give the definition of solutions.

Definition 4.5. *We say that Z is solution to* (2.13) *in $\Omega = [X_l, X_r] \times [Y_l, Y_r]$ if*

(i) *we have*

$$Z \in W^{1,\infty}(\Omega), \quad Z_X \in W_Y^{1,\infty}(\Omega), \quad Z_Y \in W_X^{1,\infty}(\Omega);$$

(ii) *and for almost every $X \in [X_l, X_r]$,*

$$(Z_X(X,Y))_Y = F(Z)(Z_X, Z_Y)(X,Y); \tag{4.13}$$

and, for almost every $Y \in [Y_l, Y_r]$,

$$(Z_Y(X,Y))_X = F(Z)(Z_X, Z_Y)(X,Y). \tag{4.14}$$

We say that Z is a global solution to (2.13) *if Z is a solution to* (2.13) *as defined above, for any rectangular domain Ω.*

The regularity that we impose is also necessary to extract the relevant data on a curve from a function defined in the plane, as it is explained in the following lemma.

Lemma 4.6 (Extraction of data from a curve)**.** *We consider a five-dimensional vector function Z in $\mathbb{R}^2$ such that*

$$Z \in W^{1,\infty}(\Omega), \quad Z_X \in W_Y^{1,\infty}(\Omega), \quad Z_Y \in W_X^{1,\infty}(\Omega)$$

for any rectangular domain Ω. Then, given a curve $(\mathcal{X}, \mathcal{Y}) \in \mathcal{C}$, let $(\mathcal{Z}, \mathcal{V}, \mathcal{W})$ be defined as

$$\mathcal{Z}(s) = Z(\mathcal{X}(s), \mathcal{Y}(s)) \text{ for all } s \in \mathbb{R} \tag{4.15}$$

and

$$\mathcal{V}(X) = Z_X(X, \mathcal{Y}(X)) \text{ for a.e. } X \in \mathbb{R}, \tag{4.16a}$$

$$\mathcal{W}(Y) = Z_Y(\mathcal{X}(Y), Y) \text{ for a.e. } Y \in \mathbb{R}, \tag{4.16b}$$

or, equivalently,

$$\mathcal{V}(\mathcal{X}(s)) = Z_X(\mathcal{X}(s), \mathcal{Y}(s)) \text{ for a.e. } s \in \mathbb{R} \text{ such that } \dot{\mathcal{X}}(s) > 0,$$
$$\mathcal{W}(\mathcal{Y}(s)) = Z_Y(\mathcal{X}(s), \mathcal{Y}(s)) \text{ for a.e. } s \in \mathbb{R} \text{ such that } \dot{\mathcal{Y}}(s) > 0.$$

We have $(\mathcal{Z}, \mathcal{V}, \mathcal{W}) \in L^\infty_{\text{loc}}(\mathbb{R})$ and we denote $\Theta = (\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{V}, \mathcal{W})$ by

$$Z \bullet (\mathcal{X}, \mathcal{Y}).$$

Proof. We consider a domain $\Omega = [X_l, X_r] \times [Y_l, Y_r]$. We claim that for any $f \in W_Y^{1,\infty}(\Omega)$ then $\tilde{f}(X) = f(X, \mathcal{Y}(X))$ is measurable and $f(X, \mathcal{Y}(X)) \in L^\infty([X_l, X_r])$. It suffices to show that the linear mapping $f \mapsto \tilde{f}$ from $W_Y^{1,\infty}(\Omega)$ to $L^\infty([X_l, X_r])$ is well-defined on simple functions and continuous. We assume that f is a simple function, that is,

$$f(X, Y) = \sum_{j=1}^{N} g_j(Y)\chi_{A_j}(X)$$

where χ_A denotes the indicator function of the set A, A_j are disjoint measurable sets and $g_j \in W^{1,\infty}([Y_l, Y_r])$. Then, $\tilde{f}(X) = \sum_{j=1}^{N} g_j(\mathcal{Y}(X))\chi_{A_j}(X)$ is measurable (as $X \mapsto \mathcal{Y}(X)$ is lower semicontinuous) and

$$\begin{aligned}
\operatorname*{esssup}_{X\in[X_l,X_r]} \left|\tilde{f}(X)\right| &\leq \max_{j\in\{1,\ldots,N\}} \operatorname*{esssup}_{X\in[X_l,X_r]} |g_j(\mathcal{Y}(X))| \\
&\leq \max_{j\in\{1,\ldots,N\}} \|g_j\|_{W^{1,\infty}([Y_l,Y_r])} \\
&\leq \|f\|_{W_Y^{1,\infty}(\Omega)}
\end{aligned}$$

so that $\tilde{f} \in L^\infty([X_l, X_r])$. Note that we need $g_j \in W^{1,\infty}([Y_l, Y_r])$ as, if g_j only belongs to $L^\infty([Y_l, Y_r])$, we *do not have* in general $\operatorname{esssup}_{X\in[X_l,X_r]} |g_j(\mathcal{Y}(X))| \leq \|g_j\|_{L^\infty([Y_l,Y_r])}$ as the function $X \mapsto \mathcal{Y}(X)$ may send sets of strictly positive measure to a set of measure zero (for example if $\mathcal{Y}$ is constant on an interval). Therefore the continuity in the Y direction which is necessary to make meaning of (4.16). Using the same type of estimate, one gets that

$$\|\tilde{f}\|_{L^\infty([X_l,X_r])} \leq \|f\|_{W_Y^{1,\infty}(\Omega)},$$

which concludes the proof of the claim. Similarly one proves that, for any $f \in W_X^{1,\infty}(\Omega)$, the mapping $Y \mapsto f(\mathcal{X}(Y), Y)$ is measurable and belongs to $L^\infty([X_l, X_r])$. Hence, we get that $(\mathcal{Z}, \mathcal{V}, \mathcal{W}) \in L^\infty_{\text{loc}}(\mathbb{R})$. ☐

The decay of Z at infinity in the diagonal direction is more conveniently expressed in term of the function Z^a which we now define as

$$Z_2^a = Z_2 - \frac{1}{2}(X + Y) \text{ and } Z_i^a = Z_i \text{ for } i \in \{1, 3, 4, 5\}. \tag{4.17}$$

Even if we are not concerned yet with the behavior at infinity, it is convenient to introduce Z^a already here to write the estimate in a convenient way. We now introduce the set $\mathcal{H}(\Omega)$ of all solutions to (2.13) on rectangular domains, which satisfy additional properties.

Definition 4.7. *Given a rectangular domain* $\Omega = [X_l, X_r] \times [Y_l, Y_r]$*, let* $\mathcal{H}(\Omega)$ *be the set of all functions* Z *which are solutions to* (2.13) *in the sense of Definition 4.5 and which satisfy the following properties*

$$x_X = c(U)t_X, \qquad x_Y = -c(U)t_Y, \tag{4.18a}$$

$$J_X = c(U)K_X, \qquad J_Y = -c(U)K_Y, \tag{4.18b}$$

$$2J_X x_X = (c(U)U_X)^2, \qquad 2J_Y x_Y = (c(U)U_Y)^2, \tag{4.18c}$$

$$x_X \geq 0, \qquad J_X \geq 0, \tag{4.18d}$$

$$x_Y \geq 0, \qquad J_Y \geq 0, \tag{4.18e}$$

$$x_X + J_X > 0, \qquad\qquad y_Y + J_Y > 0. \tag{4.18f}$$

We have the following short-range existence theorem.

Theorem 4.8. *There exists an increasing function C such that, for any $\Omega = [X_l, X_r] \times [Y_l, Y_r]$ and $\Theta = (\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{V}, \mathcal{W})$ in $\mathcal{G}(\Omega)$, if $s_r - s_l \leq 1/C(|||\Theta|||_{\mathcal{G}(\Omega)})$, then there exists a unique solution $Z \in \mathcal{H}(\Omega)$ such that*

$$\Theta = Z \bullet (\mathcal{X}, \mathcal{Y}). \tag{4.19}$$

Proof. We use a Picard fixed-point argument. Define $\mathcal{B}$ as the set of elements (Z_h, Z_v, V, W) such that

$$Z_h \in [L^\infty_X]^5, \quad Z_v \in [L^\infty_Y]^5, \quad V \in [L^\infty_Y]^5, \quad W \in [L^\infty_X]^5$$

and

$$\sum_{i=1}^{5} (\left\|Z^a_{h,i}\right\|_{L^\infty_X} + \left\|Z^a_{v,i}\right\|_{L^\infty_Y} + \|V_i\|_{L^\infty_X} + \|W_i\|_{L^\infty_X}) \leq 2|||\Theta|||_{\mathcal{G}(\Omega)} \tag{4.20}$$

where we use for Z_h and Z_v the same notation given in (4.17) for Z. For the fixed point, the functions Z_h and Z_v coincide and are equal to the solution Z, see below, but it is convenient to define both quantities in this proof and keep the symmetry of the problem with respect to the X and Y variables. We introduce the mapping $\mathcal{P}$ given, for any $(Z_h, Z_v, V, W) \in \mathcal{B}$, by $\mathcal{P}(Z_h, Z_v, V, W) = (\bar{Z}_h, \bar{Z}_v, \bar{V}, \bar{W})$ where

$$\bar{Z}_h(X, Y) = \mathcal{Z}(\mathcal{Y}^{-1}(Y)) + \int_{\mathcal{X}(Y)}^{X} V(\tilde{X}, Y)\, d\tilde{X} \tag{4.21a}$$

for a.e. $Y \in [Y_l, Y_r]$ and all $X \in [X_l, X_r]$,

$$\bar{Z}_v(X, Y) = \mathcal{Z}(\mathcal{X}^{-1}(X)) + \int_{\mathcal{Y}(X)}^{Y} W(X, \tilde{Y})\, d\tilde{Y} \tag{4.21b}$$

for a.e. $X \in [X_l, X_r]$ and all $Y \in [Y_l, Y_r]$,

$$\bar{V}(X, Y) = \mathcal{V}(X) + \int_{\mathcal{Y}(X)}^{Y} F(Z_h)(V, W)(X, \tilde{Y})\, d\tilde{Y} \tag{4.21c}$$

for a.e. $X \in [X_l, X_r]$ and all $Y \in [Y_l, Y_r]$,

$$\bar{W}(X, Y) = \mathcal{W}(Y) + \int_{\mathcal{X}(Y)}^{X} F(Z_h)(V, W)(\tilde{X}, Y)\, d\tilde{X} \tag{4.21d}$$

for a.e. $Y \in [Y_l, Y_r]$ and all $X \in [X_l, X_r]$. Let us consider a solution Z to (2.13) which satisfies (4.19). For any $(X, Y) \in \Omega$, we have

$$Z(X, \mathcal{Y}(s)) = \mathcal{Z}(s) + \int_{\mathcal{X}(s)}^{X} Z_X(\tilde{X}, \mathcal{Y}(s))\, d\tilde{X}$$

which, after taking $s = \mathcal{Y}^{-1}(Y)$, yields

$$Z(X, Y) = \mathcal{Z}(\mathcal{Y}^{-1}(Y)) + \int_{\mathcal{X}(Y)}^{X} Z_X(\tilde{X}, Y)\, d\tilde{X},$$

by (4.8). Similarly, one proves that

$$Z(X,Y) = \mathcal{Z}(\mathcal{X}^{-1}(X)) + \int_{\mathcal{Y}(X)}^{Y} Z_Y(X,\tilde{Y})\,d\tilde{Y}.$$

For every almost every $X \in [X_l, X_r]$ and all $Y \in [Y_l, Y_r]$, we have

$$Z_X(X,Y) = Z_X(X,\mathcal{Y}(X)) + \int_{\mathcal{Y}(X)}^{Y} F(Z)(Z_X, Z_Y)(X,\tilde{Y})\,d\tilde{Y},$$

and therefore, by (4.19), we get

$$Z_X(X,Y) = \mathcal{V}(X) + \int_{\mathcal{Y}(X)}^{Y} F(Z)(Z_X, Z_Y)(X,\tilde{Y})\,d\tilde{Y}. \tag{4.22}$$

Similarly, we have

$$Z_Y(X,Y) = \mathcal{W}(Y) + \int_{\mathcal{X}(Y)}^{X} F(Z)(Z_X, Z_Y)(\tilde{X},Y)\,d\tilde{X} \tag{4.23}$$

for all $X \in [X_l, X_r]$ and a.e. $Y \in [Y_l, Y_r]$. Thus, if Z is a solution to (2.13) (in the sense of Definition 4.5) which satisfies (4.19) then (Z, Z, Z_X, Z_Y) is a fixed point of $\mathcal{P}$. Since $0 \le \dot{\mathcal{X}} \le 2$ and $0 \le \dot{Y} \le 2$, we have

$$|X - \mathcal{X}(Y)| = |\mathcal{X}(\alpha(X)) - \mathcal{X}(\alpha(Y))| \le \mathcal{X}(s_r) - \mathcal{X}(s_l) \le 2(s_l - s_r) \le 2\delta \tag{4.24}$$

and, similarly,

$$|Y - \mathcal{Y}(X)| \le 2(s_l - s_r) \le 2\delta. \tag{4.25}$$

We can choose δ small enough, depending only on $|||\Theta|||_{\mathcal{G}(\Omega)}$, such that the mapping $\mathcal{P}$ maps $\mathcal{B}$ into $\mathcal{B}$. Let us check this in more details only for the second component of Z_h. We have

$$\bar{Z}^a_{h,2} = \bar{Z}_{h,2} - \frac{1}{2}(X+Y) = \mathcal{Z}_2(\mathcal{Y}^{-1}(Y)) - \frac{1}{2}(X+Y) + \int_{\mathcal{X}(Y)}^{X} V(\tilde{X},Y)\,d\tilde{X}$$

and, denoting $s = \mathcal{Y}^{-1}(Y)$,

$$\begin{aligned}\mathcal{Z}_2(s) - \frac{1}{2}(X+Y) &= \mathcal{Z}_2(s) - \frac{1}{2}(X+\mathcal{Y}(s)) = \mathcal{Z}_2(s) - \frac{1}{2}(\mathcal{X}(s)+\mathcal{Y}(s)) + \frac{1}{2}(X - \mathcal{X}(s))\\ &= \mathcal{Z}^a_2(s) + \frac{1}{2}(X - \mathcal{X}(Y)).\end{aligned}$$

Hence,

$$\left\|\bar{Z}^a_{h,2}\right\|_{L^\infty(\Omega)} \le |||\mathcal{Z}^a_2|||_{\mathcal{G}(\Omega)} + \delta(1 + 2|||\Theta|||_{\mathcal{G}(\Omega)})$$

by (4.24) and (4.20). After doing the same for the other components, we get

$$\sum_{i=1}^{5}(\left\|\bar{Z}^a_{h,i}\right\|_{L^\infty_X} + \left\|\bar{Z}^a_{v,i}\right\|_{L^\infty_Y} + \left\|\bar{V}_i\right\|_{L^\infty_X} + \left\|\bar{W}_i\right\|_{L^\infty_X}) \le |||\Theta|||_{\mathcal{G}(\Omega)} + \delta C$$

for a constant C which depends only on $|||\Theta|||_{\mathcal{G}(\Omega)}$. Hence, by taking δ small enough, the mapping $\mathcal{P}$ maps $\mathcal{B}$ into $\mathcal{B}$. Using the fact that F is locally Lipschitz (because it is bi-linear with respect to the two last variables and depends smoothly on $U = Z_3$), we prove that $\mathcal{P}$ is contractive. Hence, $\mathcal{P}$ admits a unique fixed point that we denote

(Z_h, Z_v, V, W). Let us prove that $Z_h = Z_v$. It basically follows from the fact that $W_X = V_Y$. Let us now denote by $\mathcal{N}_X$ the set of points $X \in [X_l, X_r]$ for which (4.21b) and (4.21c) hold (by definition, the set $\mathcal{N}_X$ has full measure). Similarly we denote by $\mathcal{N}_Y$ the set of points $Y \in [Y_l, Y_r]$ for which (4.21a) and (4.21d) hold. We have $\operatorname{meas}([X_l, X_r] \setminus \mathcal{N}_X) = \operatorname{meas}([Y_l, Y_r] \setminus \mathcal{N}_Y) = 0$ so that $\operatorname{meas}(\Omega \setminus \mathcal{N}_X \times \mathcal{N}_Y) = 0$. For any $(X, Y) \in \mathcal{N}_X \times \mathcal{N}_Y$, we have

$$Z_h(X,Y) - Z_v(X,Y) = \mathcal{Z}(\mathcal{Y}^{-1}(Y)) + \int_{\mathcal{X}(Y)}^{X} V(\tilde{X}, Y)\, d\tilde{X} - \mathcal{Z}(\mathcal{X}^{-1}(X)) - \int_{\mathcal{Y}(X)}^{Y} W(X, \tilde{Y})\, d\tilde{Y}. \tag{4.26}$$

Since the terms involving F cancel, we obtain by (4.21c) and (4.21d) that

$$\int_{\mathcal{X}(Y)}^{X} V(\tilde{X}, Y)\, d\tilde{X} - \int_{\mathcal{Y}(X)}^{Y} W(X, \tilde{Y})\, d\tilde{Y} = \int_{\mathcal{X}(Y)}^{X} \mathcal{V}(\tilde{X})\, d\tilde{X} - \int_{\mathcal{Y}(X)}^{Y} \mathcal{W}(\tilde{Y})\, d\tilde{Y}.$$

Returning to the rigorous notation, we get

$$\int_{\mathcal{X}(Y)}^{X} V(\tilde{X}, Y)\, d\tilde{X} - \int_{\mathcal{Y}(X)}^{Y} W(X, \tilde{Y})\, d\tilde{Y} = \int_{\mathcal{X}(\mathcal{Y}^{-1}(Y))}^{\mathcal{X}(\mathcal{X}^{-1}(X))} \mathcal{V}(\tilde{X})\, d\tilde{X} - \int_{\mathcal{Y}(\mathcal{X}^{-1}(X))}^{\mathcal{Y}(\mathcal{Y}^{-1}(Y))} \mathcal{W}(\tilde{Y})\, d\tilde{Y} \tag{4.27}$$

where we have also used (4.8). We proceed with a change of variables in the two integrals on the right-hand side of (4.27) and get

$$\begin{aligned} \int_{\mathcal{X}(\mathcal{Y}^{-1}(Y))}^{\mathcal{X}(\mathcal{X}^{-1}(X))} \mathcal{V}(\tilde{X})\, d\tilde{X} &- \int_{\mathcal{Y}(\mathcal{X}^{-1}(X))}^{\mathcal{Y}(\mathcal{Y}^{-1}(Y))} \mathcal{W}(\tilde{Y})\, d\tilde{Y} \\ &= -\int_{\mathcal{X}^{-1}(X)}^{\mathcal{Y}^{-1}(Y)} \Big(\mathcal{W}(\mathcal{Y}(s))\dot{\mathcal{Y}}(s) + \mathcal{V}(\mathcal{X}(s))\dot{\mathcal{X}}(s) \Big)\, ds \\ &= -\int_{\mathcal{X}^{-1}(X)}^{\mathcal{Y}^{-1}(Y)} \dot{\mathcal{Z}}(s)\, ds \qquad \text{by (4.3)} \\ &= \mathcal{Z}(\mathcal{X}^{-1}(X)) - \mathcal{Z}(\mathcal{Y}^{-1}(Y)) \end{aligned} \tag{4.28}$$

and combining (4.26), (4.27) and (4.28), we get that $Z_h(X,Y) = Z_v(X,Y)$ for all $(X, Y) \in \mathcal{N}_X \times \mathcal{N}_Y$, that is, almost everywhere. We denote $Z = Z_h = Z_v$. For any (X, Y) and $(\bar{X}, \bar{Y})$ belonging to $\mathcal{N}_X \times \mathcal{N}_Y$, we get, by using (4.21a) and (4.21b), that

$$Z(X,Y) - Z(\bar{X}, \bar{Y}) = \int_{\bar{X}}^{X} V(\tilde{X}, Y)\, d\tilde{X} + \int_{\bar{Y}}^{Y} W(\bar{X}, \tilde{Y})\, d\tilde{Y}.$$

Hence, by using the bound (4.20),

$$\left| Z(X,Y) - Z(\bar{X}, \bar{Y}) \right| \leq 2 |||\Theta|||_{\mathcal{G}(\Omega)} (\left| X - \bar{X} \right| + \left| Y - \bar{Y} \right|)$$

and Z is Lipschitz in $\mathcal{N}_X \times \mathcal{N}_Y$. It implies that Z is uniformly continuous in $\mathcal{N}_X \times \mathcal{N}_Y$ and there exists a unique continuous extension of Z to the closure of $\mathcal{N}_X \times \mathcal{N}_Y$, that is, Ω. From (4.21a) and (4.21b), we get that

$$Z_X(X,Y) = V(X,Y) \tag{4.29}$$

for all $Y \in [Y_l, Y_r]$ and a.e. $X \in [X_l, X_r]$ and

$$Z_Y(X,Y) = W(X,Y) \tag{4.30}$$

for all $X \in [X_l, X_r]$ and a.e. $Y \in [Y_l, Y_r]$. By using the fact that (Z, Z, Z_X, Z_Y) is a fixed point in $\mathcal{B}$ and (4.29) and (4.30), we can check that $Z_X \in W^{1,\infty}_Y(\Omega)$ and $Z_Y \in W^{1,\infty}_X(\Omega)$. By density, we can prove that

$$Z(X,Y) - Z(\bar{X},Y) = \int_{\bar{X}}^{X} Z_X(\tilde{X},Y)\,dX, \tag{4.31}$$

not only for almost every $Y \in [Y_l, Y_r]$ as (4.21a) yields, but *for all* $Y \in [Y_l, Y_r]$. Indeed, for any $Y \in [Y_l, Y_r]$, there exists a sequence $Y_n \in \mathcal{N}_Y$ such that $\lim_{n\to\infty} Y_n = Y$ as $\operatorname{meas}([Y_l, Y_r] \setminus \mathcal{N}_Y) = 0$ and we have

$$Z(X,Y_n) - Z(\bar{X},Y_n) = \int_{\bar{X}}^{X} Z_X(\tilde{X},Y_n)\,dX. \tag{4.32}$$

Since $Z_X \in W^{1,\infty}_Y(\Omega)$, we have that $\|Z_X(\,\cdot\,,Y_n)\|_{L^\infty([X_l,X_r])} \leq \|Z_X\|_{W^{1,\infty}_Y(\Omega)} \leq 2|||\Theta|||_{\mathcal{G}(\Omega)}$ and, for a.e. $\tilde{X} \in [X_l, X_r]$, $\lim_{n\to\infty} Z_X(\tilde{X}, Y_n) = Z_X(\tilde{X}, Y)$ for almost every $\tilde{X}$. Hence, by Lebesgue dominated convergence theorem and the continuity of Z, (4.32) implies (4.31). It remains to check that Z satisfies (4.19). Since (Z, Z, Z_X, Z_Y) is a fixed point of $\mathcal{P}$, we have, by (4.21c) and (4.21d), that

$$Z_X(X, \mathcal{Y}(X)) = \mathcal{V}(X) \text{ and } Z_Y(\mathcal{X}(Y), Y) = \mathcal{W}(Y)$$

for a.e. $X \in [X_l, X_r]$ and $Y \in [Y_l, Y_r]$, respectively. It remains to check that Z satisfies (4.15). On one hand, we have that

$$Z(\mathcal{X}(s), \mathcal{Y}(s)) = \mathcal{Z}(\mathcal{X}^{-1}(\mathcal{X}(s))) \tag{4.33}$$

by (4.21b) and, by (4.9), it implies (4.15) for all $s \in [s_l, s_r]$ such that $\dot{\mathcal{X}}(s) > 0$. On the other hand we have

$$Z(\mathcal{X}(s), \mathcal{Y}(s)) = \mathcal{Z}(\mathcal{Y}^{-1}(\mathcal{Y}(s))) \tag{4.34}$$

by (4.21b) and, by (4.10), it implies (4.15) for all $s \in [s_l, s_r]$ such that $\dot{\mathcal{Y}}(s) > 0$. Since $\dot{\mathcal{X}} + \dot{\mathcal{Y}} = 2$, the set of all $s \in [s_l, s_r]$ such that $\dot{\mathcal{X}}(s) > 0$ or $\dot{\mathcal{Y}}(s) > 0$ has full measure and therefore, for almost every $s \in [s_l, s_r]$, (4.15) holds. By continuity, we infer that (4.15) holds for all $s \in [s_l, s_r]$. Hence, we have proved that Z is a solution to (2.13) which satisfies (4.19) if and only if it is a fixed point of $\mathcal{P}$. Since the fixed point exists and is unique, we have proved the existence and uniqueness of the solution. Let us define the functions $v \in W^{1,\infty}_Y(\Omega)$ and $w \in$ in $W^{1,\infty}_X(\Omega)$ as

$$v = x_X - c(U)t_X \quad \text{and} \quad w = x_Y + c(U)t_Y.$$

We want to prove that v and w are both zero. After some computations using the governing equations (2.13), we obtain

$$\begin{aligned} v_Y &= x_{XY} - c'(U)U_Y t_X - c(U)t_{XY} \\ &= \frac{c'(U)}{2c(U)}\,(U_Y v + U_X w) \end{aligned} \tag{4.35}$$

and

$$\begin{aligned} w_X &= x_{XY} + c'(U)U_X t_Y - c(U)t_{XY} \\ &= \frac{c'(U)}{2c(U)}\left(U_Y v + U_X w\right). \end{aligned} \tag{4.36}$$

It follows that

$$\begin{aligned} v(X,Y) &= \int_{\mathcal{Y}(X)}^{Y} \frac{c'(U)}{2c(U)}\left(U_Y v + U_X w\right), \\ w(X,Y) &= \int_{\mathcal{X}(X)}^{X} \frac{c'(U)}{2c(U)}\left(U_Y v + U_X w\right), \end{aligned}$$

which implies, after using (4.24), (4.25) and (4.20), that

$$\begin{aligned} \|v\|_{L^\infty_Y} &\leq \delta 2 |||\Theta|||_{\mathcal{G}(\Omega)} \max \frac{c'(U)}{2c(U)} (\|v\|_{L^\infty_Y} + \|w\|_{L^\infty_X}), \\ \|w\|_{L^\infty_X} &\leq \delta 2 |||\Theta|||_{\mathcal{G}(\Omega)} \max \frac{c'(U)}{2c(U)} (\|v\|_{L^\infty_Y} + \|w\|_{L^\infty_X}). \end{aligned}$$

Hence, by taking δ smaller if necessary, we get $\|v\|_{L^\infty_Y} = \|w\|_{L^\infty_X} = 0$. Let us now introduce $z \in W^{1,\infty}_Y(\Omega)$ as $z = 2J_X x_X - (c(U)U_X)^2$. We have

$$\begin{aligned} z_Y &= 2J_{XY}x_X + 2J_X x_{XY} - 2c(U)^2 U_X U_{XY} - 2c(U)c'(U)U_Y\left(U_X\right)^2 \\ &= \frac{c'(U)}{c(U)} U_Y z \end{aligned} \tag{4.37}$$

and $z(X,\mathcal{Y}(X)) = 0$ for $X \in [X_l, X_r]$, the unique solution to (4.37) is $z = 0$. One proves in the same way that $J_Y x_Y = 2\left(c(U)U_Y\right)^2$. Let us now prove (4.18f). Since the initial data belongs to $\mathcal{G}(\Omega)$, we have $\|1/(x_X + J_X)(X,\mathcal{Y}(X))\|_{L^\infty([X_l,X_r])} \leq |||\Theta|||_{\mathcal{G}(\Omega)}$ as $1/(x_X + J_X)(X,\mathcal{Y}(X)) = 1/(\mathcal{V}_2 + \mathcal{W}_2)(X)$ for a.e. $X \in [X_l, X_r]$. For all fixed X such that $1/(x_X + J_X)(X,\mathcal{Y}(X)) \leq |||\Theta|||_{\mathcal{G}(\Omega)}$, that is for almost every $X \in [X_l, X_r]$, we define

$$Y_* = \inf\{Y \in [Y_l, Y_r] \mid Y \leq \mathcal{Y}(X) \text{ and } (x_X + J_X)(X, Y') > 0 \text{ for all } Y' > Y\}$$

and similarly

$$Y^* = \sup\{Y \in [Y_l, Y_r] \mid Y \geq \mathcal{Y}(X) \text{ and } (x_X + J_X)(X, Y') > 0 \text{ for all } Y' < Y\}.$$

On (Y_*, Y^*), we have $(x_X + J_X)(X,Y) > 0$ and we define

$$q(Y) = \frac{1}{(x_X + J_X)(X,Y)}.$$

Let us assume that $Y_* < Y_r$ and therefore, by continuity,

$$(x_X + J_X)(X, Y^*) = 0. \tag{4.38}$$

On (Y_*, Y^*), we have $q(Y) \geq 0$ and, since $J_X x_X \geq 0$ by (4.18c), it implies that

$$x_X \geq 0 \text{ and } J_X \geq 0. \tag{4.39}$$

By using (2.13), we obtain

$$q_Y = -\frac{x_{XY} + J_{XY}}{(x_X + J_X)^2}$$

$$= -\frac{c'(U)}{2c(U)}\frac{U_Y(x_X+J_X)+(J_Y+x_Y)U_X}{(x_X+J_X)^2}. \tag{4.40}$$

From (4.18c), we infer that

$$|U_X| = \frac{1}{c(U)\sqrt{2}}\sqrt{J_X x_X} \leq \frac{1}{2c(U)\sqrt{2}}(J_X+x_X).$$

Hence, (4.40) yields

$$q_Y \leq \frac{|c'(U)|}{2c(U)}(|U_Y|+|J_Y|+|x_Y|)q \leq Cq$$

for some constant C which depends only

$$\operatorname*{esssup}_{Y\in[Y_l,Y_r]}(|U|(X,Y)+|Z_Y|(X,Y)), \tag{4.41}$$

that is $|||\Theta|||_{\mathcal{G}(\Omega)}$, by (4.20). By Gronwall's lemma, it follows that q cannot blow up in Y^*, contradicting (4.38) and

$$\frac{1}{x_X+J_X}(X,Y) \leq \frac{1}{\mathcal{V}_3+\mathcal{V}_4}(X)e^{C|Y-\mathcal{Y}(X)|} \tag{4.42}$$

for a.e. $X\in[X_l,X_r]$ and all $Y\in[Y_l,Y_r]$. In the same way, one proves that $Y_* = Y_l$. Hence, we have proved that, for almost every $X\in[X_l,X_r]$,

$$x_X(X,Y)\geq 0 \text{ and } J_X(X,Y)\geq 0 \text{ for all } Y$$

and $\|1/(x_X+J_X)(X,Y)\|_{W^\infty(Y_l,Y_r)}$ is bounded, for almost every $X\in[X_l,X_r]$, by a constant which is independent of X and therefore $1/(x_X+J_X)\in W^{1,\infty}_Y(\Omega)$. This concludes the proof of (4.18d) and the first identity (4.18f) while (4.18e) and the second identity in (4.18f) can be proven in the same way. ☐

4.2. **A priori estimates.** Given a positive constant L, we call domains of the type

$$D = \{(X,Y)\in\mathbb{R}^2 \mid |Y-X|<2L\}$$

for *strip domain.* Strip domains are correspond to domains where time is bounded. We have the following a priori estimates for the solution of (2.13). The energy $J(X,Y)$ is bounded in the whole plane while Z^a (that is, Z, up to a shift in the second component) and its derivatives are bounded in every strip domain.

Lemma 4.9. *Given $\Omega=[X_l,X_r]\times[Y_l,Y_r]$ and $\Theta=(\mathcal{X},\mathcal{Y},\mathcal{Z},\mathcal{V},\mathcal{W})\in\mathcal{G}(\Omega)$, let $Z\in\mathcal{H}(\Omega)$ be a solution to (2.13) such that $\Theta = Z\bullet(\mathcal{X},\mathcal{Y})$. Let $\mathcal{E}_0 = \|\mathcal{Z}_4\|_{L^\infty([s_l,s_r])}+\|\mathcal{Z}_5\|_{L^\infty([s_l,s_r])}$. Then the following statements hold:*

(i) *Global boundedness of the energy, more precisely,*

$$0\leq J(X,Y)\leq\mathcal{E}_0 \text{ for all } (X,Y)\in\Omega \quad\text{and}\quad \|K\|_{L^\infty(\Omega)}\leq(1+\kappa)\mathcal{E}_0 \tag{4.43}$$

where $J=Z_4$ and $K=Z_5$.

(ii) *The function Z and its derivatives remain uniformly bounded in strip domains. More precisely there exists a nondecreasing function $C_1=C_1(L,|||\Theta|||_{\mathcal{G}(\Omega)})$, such that, for any $L>0$ and any X and Y such that $|X-Y|\leq 2L$, we have*

$$|Z^a(X,Y)|\leq C_1,\quad |Z_X(X,Y)|\leq C_1,\quad |Z_Y(X,Y)|\leq C_1 \tag{4.44}$$

and

$$\frac{1}{x_X + J_X}(X,Y) \leq C_1, \quad \frac{1}{x_Y + J_Y}(X,Y) \leq C_1. \tag{4.45}$$

The condition (ii) above is equivalent to the following condition (iii):

(iii) *For any curve* $(\bar{\mathcal{X}}, \bar{\mathcal{Y}}) \in \mathcal{C}(\Omega)$*, we have*

$$|||Z \bullet (\bar{\mathcal{X}}, \bar{\mathcal{Y}})|||_{\mathcal{G}(\Omega)} \leq C_1 \tag{4.46}$$

where $C_1 = C_1(\|(\bar{\mathcal{X}}, \bar{\mathcal{Y}})\|_{\mathcal{C}(\Omega)}, |||\Theta|||_{\mathcal{G}(\Omega)})$ *is a given function which is increasing with respect to both its arguments.*

The inequalities in (4.44) hold in fact in $L^\infty(\Omega)$, $W^{1,\infty}_Y(\Omega)$ and $W^{1,\infty}_X(\Omega)$, respectively. The inequalities in (4.45) hold in $W^{1,\infty}_Y(\Omega)$ and $W^{1,\infty}_X(\Omega)$, respectively.

Proof. Given $P = (X,Y) \in \Omega$, let $s_0 = \mathcal{Y}^{-1}(Y)$ and $s_1 = \mathcal{X}^{-1}(X)$. We denote $P_0 = (X(s_0), Y(s_0))$ and $P_1 = (X(s_1), Y(s_1))$. We assume that $s_0 \leq s_1$ (the proof for the other case is very similar). Since $\dot{\mathcal{X}}$ and $\dot{\mathcal{Y}}$ are positive, it implies that $X = \mathcal{X}(s_1) \geq \mathcal{X}(s_0)$ and $Y = Y(s_0) \leq \mathcal{Y}(s_1)$. Then, because $J_X \geq 0$, $J_Y \geq 0$ and $\mathcal{Z}_4 \geq 0$, we have

$$0 \leq \mathcal{Z}_4(s_0) = J(P_0) \leq J(P) \leq J(P_1) = \mathcal{Z}_4(s_1) \leq \mathcal{E}_0 \tag{4.47}$$

which gives the first inequality in (4.43). By (4.18b), we get

$$|K(P) - K(P_0)| \leq \kappa(J(P) - J(P_0))$$

which implies

$$|K(P)| \leq |K(P_0)| + \kappa(J(P) - J(P_0)) \tag{4.48}$$

and

$$|K(P)| \leq (1+\kappa)\mathcal{E}_0$$

by (4.43). Since $x_X \geq 0$, we have

$$x(P) \geq x(P_0) = \mathcal{Z}_2(s_0) \geq -|||\Theta|||_{\mathcal{G}(\Omega)} + s_0. \tag{4.49}$$

Since $\frac{1}{2}(X+Y) = Y + \frac{1}{2}(X-Y) \leq Y(s_0) + L$, it follows that

$$x(P) - \frac{1}{2}(X+Y) \geq -|||\Theta|||_{\mathcal{G}(\Omega)} + s_0 - \mathcal{Y}(s_0) - L \geq -2|||\Theta|||_{\mathcal{G}(\Omega)} - L.$$

Similarly, using that $x_Y \geq 0$, we get

$$x(P) \leq x(P_1) = \mathcal{Z}_2(s_1) \leq |||\Theta|||_{\mathcal{G}(\Omega)} + s_1. \tag{4.50}$$

and

$$x(P) - \frac{1}{2}(X+Y) \leq 2|||\Theta|||_{\mathcal{G}(\Omega)} + L. \tag{4.51}$$

Hence, $\left|x(P) - \frac{1}{2}(X+Y)\right| \leq 2|||\Theta|||_{\mathcal{G}(\Omega)} + L$. We have

$$|t(P)| = \left|\int_Y^{\mathcal{Y}(s_1)} t_Y(X,\tilde{Y})\,d\tilde{Y}\right| \leq \int_Y^{\mathcal{Y}(s_1)} \frac{x_Y}{c(U)}\,d\tilde{Y} \leq \kappa(x(P_1) - x(P)). \tag{4.52}$$

Since

$$x(P_1) - x(P) \leq x(P_1) - x(P_0) = \mathcal{Z}_2(s_1) - \mathcal{Z}_2(s_0) \leq 2|||\Theta|||_{\mathcal{G}(\Omega)} + s_1 - s_0 \tag{4.53}$$

and

$$s_1 - s_0 = s_1 - \mathcal{Y}(s_1) - (s_0 - \mathcal{X}(s_0)) + Y - X \leq 2|||\Theta|||_{\mathcal{G}(\Omega)} + 2L, \tag{4.54}$$

it follows from (4.52) that $|t(P)| \leq \kappa(4|||\Theta|||_{\mathcal{G}(\Omega)} + 2L)$. We have

$$|U(P)| \leq |U(P_1)| + \int_Y^{\mathcal{Y}(s_1)} U_Y \, d\tilde{Y}.$$

By (4.18c), we have that

$$|U_Y| \leq \frac{\kappa}{2\sqrt{2}}(J_Y + x_Y). \tag{4.55}$$

Hence,

$$\begin{aligned} |U(P)| &\leq |U(P_1)| + \frac{\kappa}{2\sqrt{2}} \, |J(P_1) + x(P_1) - J(P) - x(P)| \\ &\leq |||\Theta|||_{\mathcal{G}(\Omega)} + \frac{\kappa}{2\sqrt{2}}(\mathcal{E}_0 + 4|||\Theta|||_{\mathcal{G}(\Omega)} + 2L) \text{ by (4.53) and (4.47).} \end{aligned} \tag{4.56}$$

To prove that Z_X and Z_Y remain bounded, we use the bi-linearity embedded in the governing equation (2.14). We use first the linearity of $F(Z)$ with respect to the first variable and for almost every $X \in [X_l, X_r]$, we get, after applying Gronwall's lemma, that

$$\begin{aligned} |Z_X(X,Y)| &\leq |Z_X(X, \mathcal{Y}(X))| \exp(\int_Y^{\mathcal{Y}^{-1}(X)} |F(Z)(\,\cdot\,, Z_Y)| \; d\tilde{Y}) \\ &= |\mathcal{V}(X)| \exp(\int_Y^{\mathcal{Y}^{-1}(X)} |F(Z)(\,\cdot\,, Z_Y)| \; d\tilde{Y}). \end{aligned} \tag{4.57}$$

Here $F(Z)(\cdot, W)$ denotes the matrix $V \mapsto F(Z)(V,W)$ and we use any matrix norm as they are all equivalent. We also assume (as earlier) that $Y \leq \mathcal{Y}^{-1}(X)$ (otherwise we have to interchange the bounds in the integral) and we denote $P_1 = (X, \mathcal{Y}^{-1}(X))$. We have $|\mathcal{V}(X)| \leq |||\Theta|||_{\mathcal{G}(\Omega)}$. After using (4.18a), (4.18b) and (4.55), we obtain that

$$|F(Z)(\cdot, Z_Y)| \leq C(|t_Y| + |x_Y| + |U_Y| + |J_Y| + |K_Y|)$$

and we have used here the linearity of $F(Z)$ with respect to its second variable. Hence,

$$\begin{aligned} |F(Z)(\cdot, Z_Y)| &= C\left(\frac{1}{c}(x_Y + J_Y) + x_Y + J_Y + |U_Y|\right) \\ &\leq C(x_Y + J_Y) \end{aligned}$$

for a constant C that depends only on $c(U)$ and therefore only on $|||\Theta|||_{\mathcal{G}(\Omega)}$ and L, by (4.56). Hence,

$$\begin{aligned} \int_Y^{\mathcal{Y}^{-1}(X)} |F(Z)(\,\cdot\,, Z_Y)| \; d\tilde{Y} &\leq C \int_Y^{\mathcal{Y}^{-1}(X)} (x_Y + J_Y) \; d\tilde{Y} \\ &= C(x(P_1) - x(P) + J(P_1) - J(P)) \\ &\leq C(\mathcal{E}_0 + 4|||\Theta|||_{\mathcal{G}(\Omega)} + 2L), \end{aligned}$$

by (4.47), (4.53) and (4.54). Combined with (4.57), it yields

$$|Z_X(X,Y)| \leq C$$

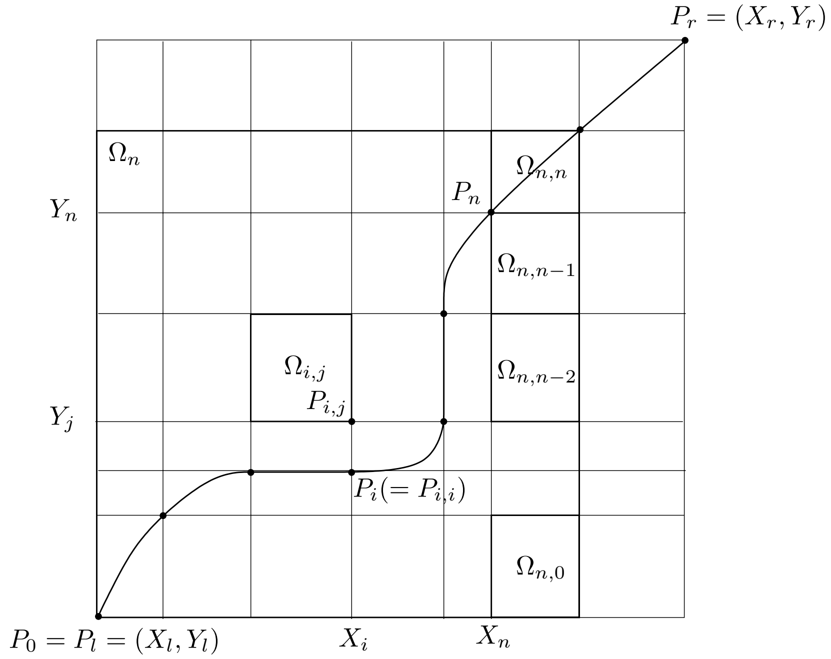

FIGURE 3. Construction of the global solution.

for some constant C which depends only on L and $|||\Theta|||_{\mathcal{G}(\Omega)}$. Similarly, one proves the bound on Z_Y. The estimate (4.45) follows from the estimate (4.42) in the proof of Theorem 4.8 as the constant C in (4.42) depends only on L and $|||\Theta|||_{\mathcal{G}(\Omega)}$, by (4.44), and

$$|Y - \mathcal{Y}(X)| = \left|Y - X + \mathcal{X}(\mathcal{X}^{-1}(X)) - \mathcal{Y}(\mathcal{X}^{-1}(X))\right| \leq L + |||\Theta|||_{\mathcal{G}(\Omega)}.$$

□

4.3. **Global existence.** We obtain the following existence and uniqueness lemma.

Lemma 4.10 (Existence and uniqueness on arbitrarily large rectangles)**.** *Given a rectangular domain $\Omega = [X_l, X_r] \times [Y_l, Y_r]$ and $\Theta = (\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{V}, \mathcal{W})$ in $\mathcal{G}(\Omega)$, there exists a unique $Z \in \mathcal{H}(\Omega)$ such that*

$$\Theta = Z \bullet (\mathcal{X}, \mathcal{Y}).$$

Proof. Let N denote an integer that we will set later and $\delta = \frac{s_r - s_l}{N}$. For $i = 0, \ldots, N$, let $s_i = i\delta + s_l$ and we consider the sequence of points $P_i = (X_i, Y_i) = (X(s_i), Y(s_i))$. For $i, j = 0, \ldots, N$, we construct a grid which consists of the points $P_{i,j} = (X_{i,j}, Y_{i,j})$ where $X_{i,j} = X_i$ and $Y_{i,j} = Y_j$, see Figure 3. We denote by by $\Omega_{i,j}$ the rectangle with diagonal points $P_{i,j}$ and $P_{i+1,j+1}$. Let Ω_n denote the rectangle with diagonal points given by (X_0, Y_0) and (X_n, Y_n). We prove by induction that there exists a unique $Z \in \mathcal{H}(\Omega_n)$ such $\Theta = Z \bullet (X, Y)$ (Here we use the same notation for $\Theta \in \mathcal{G}(\Omega)$ and $(\mathcal{X}, \mathcal{Y}) \in \mathcal{C}(\Omega)$ and their restriction to Ω_n which belong to $\mathcal{G}(\Omega_n)$ and $\mathcal{C}(\Omega_n)$, respectively). On Ω_1, we can choose N large enough and depending only on $|||\Theta|||_{\mathcal{G}(\Omega_1)} \leq |||\Theta|||_{\mathcal{G}(\Omega)}$ such that

$$s_1 - s_0 \leq \delta \leq C(|||\Theta|||_{\mathcal{G}(\Omega)})^{-1} \leq C(|||\Theta|||_{\mathcal{G}(\Omega_1)})^{-1},$$

and, by Theorem 4.8, there exists a unique solution $Z \in \mathcal{H}(\Omega_1)$ such that $\Theta = Z\bullet(X,Y)$. We assume that there exists a unique solution $Z \in \Omega_n$ and prove that there exists a solution on Ω_{n+1}. On $\Omega_{n,n}$, we get the existence of a unique solution by Theorem 4.8 as

$$s_{n+1} - s_n \leq \delta \leq C(|||\Theta|||_{\mathcal{G}(\Omega)})^{-1} \leq C(|||\Theta|||_{\mathcal{G}(\Omega_{n,n})})^{-1}.$$

For $j = n-1, \ldots, 0$, we construct iteratively the unique solution in $\Omega_{n,j}$ and $\Omega_{j,n}$ as follows. We treat only the case of $\Omega_{n,j}$. We assume the solution is known on $\Omega_{n,j+1}$, then we define $\tilde{\Theta} = (\tilde{\mathcal{X}}, \tilde{\mathcal{Y}}, \tilde{\mathcal{Z}}, \tilde{\mathcal{V}}, \tilde{\mathcal{W}}) \in \mathcal{G}(\Omega_{n,j})$ as follows: The curve $\tilde{\mathcal{X}}(s), \tilde{\mathcal{Y}}(s)$ is given by

$$\tilde{\mathcal{X}}(s) = X_n, \quad \tilde{\mathcal{Y}}(s) = 2s - X_n$$

for $\frac{1}{2}(Y_j + X_n) \leq s \leq \frac{1}{2}(Y_{j+1} + X_n)$,

$$\tilde{\mathcal{X}}(s) = 2s - Y_{j+1}, \quad \tilde{\mathcal{Y}}(s) = Y_{j+1}$$

for $\frac{1}{2}(Y_{j+1} + X_n) \leq s \leq \frac{1}{2}(Y_{j+1} + X_{n+1})$ and set

$$\begin{aligned}
\tilde{\mathcal{Z}}(s) &= Z(\tilde{\mathcal{X}}(s), \tilde{\mathcal{Y}}(s)) \text{ for } s \in [\frac{1}{2}(Y_j + X_n), \frac{1}{2}(Y_{j+1} + X_{n+1})],\\
\tilde{\mathcal{V}}(X) &= Z_X(X, Y_{j+1}) \text{ for a.e. } X \in [X_n, X_{n+1}],\\
\tilde{\mathcal{W}}(Y) &= Z_Y(X_n, Y) \text{ for a.e. } X \in [Y_j, Y_{j+1}].
\end{aligned}$$

Using Lemma 4.9, we can check that $|||\tilde{\Theta}|||_{\mathcal{G}(\Omega_{n,j})}$ is bounded by a constant C_2 that depends only on L and $|||\Theta|||_{\mathcal{G}(\Omega)}$. We have

$$\frac{1}{2}(Y_{j+1} + X_{n+1} - Y_j - X_n) = \frac{1}{2}(\mathcal{Y}(s_{j+1}) + \mathcal{X}(s_{n+1}) - \mathcal{Y}(s_j) - \mathcal{X}(s_n)) \leq 2\delta.$$

Here we have used that $\mathcal{X}$ and $\mathcal{Y}$ are Lipschitz with Lipschitz constant smaller than 2. By taking N large enough so that 2δ is smaller that $C(C_2)^{-1}$, we can apply Theorem 4.8 to $\Omega_{n,j}$ and obtain the existence of a unique solution in $\mathcal{H}(\Omega_{n,j})$. Similarly we get the existence of a unique solution in $\mathcal{H}(\Omega_{j,n})$. Since

$$\Omega_{n+1} = \Omega_n \cup (\cup_{j=0}^{n} \Omega_{j,n}) \cup (\cup_{j=0}^{n} \Omega_{n,j}),$$

we have proved the existence of a unique solution in Ω_{n+1}. □

In Lemma 4.9, we establish L^∞-bounds on the derivatives on a strip domain. It turns out that we can also establish L^2-bounds on the derivatives as stated in the next lemma. In this context, by L^2-bounds, we mean that we can bound the integrals of the differential forms $(Z_X^a)^2\,dX$ and $(Z_Y^a)^2\,dY$ along a curve in $\mathcal{C}$. It is useful to have in mind that, for any given time T, we can find a curve $(\mathcal{X}, \mathcal{Y}) \in \mathcal{C}$ which corresponds to this given time T, that is, $t(\mathcal{X}(s), \mathcal{Y}(s)) = T$ for all $s \in \mathbb{R}$. Thus the L^2-bound we now establish is fundamental to obtain L^2-bounds in the initial set of coordinates.

Lemma 4.11 (A Gronwall lemma for curves). *Given $\Omega = [X_l, X_r] \times [Y_l, Y_r]$, $Z \in \mathcal{H}(\Omega)$ and $(\mathcal{X}, \mathcal{Y}) \in \mathcal{C}(\Omega)$. Then, for any $(\bar{\mathcal{X}}, \bar{\mathcal{Y}}) \in \mathcal{C}(\Omega)$,*

$$\|Z \bullet (\bar{\mathcal{X}}, \bar{\mathcal{Y}})\|_{\mathcal{G}(\Omega)} \leq C \|Z \bullet (\mathcal{X}, \mathcal{Y})\|_{\mathcal{G}(\Omega)} \tag{4.58}$$

where $C = C(\|(\bar{\mathcal{X}}, \bar{\mathcal{Y}})\|_{\mathcal{C}(\Omega)}, |||Z \bullet (\mathcal{X}, \mathcal{Y})|||_{\mathcal{G}(\Omega)})$ is a given increasing function with respect to both its arguments.

Proof. Note that, for any function in $f \in W_Y^{1,\infty}(\Omega)$ (and respectively $g \in W_X^{1,\infty}(\Omega)$), the forms $f(X,Y)\,dX$ (respectively $g(X,Y)\,dY$) are well defined while the forms $f(X,Y)\,dY$ (respectively $g(X,Y)\,dX$) are not. Given $Z \in \mathcal{H}(\Omega)$, we can consider the forms $U^2\,dX$, $U^2\,dY$, $(Z_X^a)^2\,dX$ and $(Z_Y^a)^2\,dY$. For any curve $\bar\Gamma = (\bar{\mathcal{X}}, \bar{\mathcal{Y}}) \in \mathcal{C}(\Omega)$, $(\bar{\mathcal{X}}, \bar{\mathcal{Y}}, \bar{\mathcal{Z}}, \bar{\mathcal{V}}, \bar{\mathcal{W}}) = \Theta \in \mathcal{G}(\Omega)$, we have (by definition of the integral of a form along a curve and the definition of $Z \bullet (\bar{\mathcal{X}}, \bar{\mathcal{Y}})$)

$$\int_{\bar\Gamma} (U^2\,dX + U^2\,dY) = 2\int_{s_l}^{s_r} \bar{\mathcal{Z}}_3^2(s)\,ds \qquad (\text{as } \mathcal{X} + \mathcal{Y} = 2s),$$

and

$$\int_{\bar\Gamma} (Z_X^a)^2\,dX = \int_{X_l}^{X_r} \bar{\mathcal{V}}^a(X)^2 dX, \quad \int_{\bar\Gamma} (Z_Y^a)^2\,dY = \int_{Y_l}^{Y_r} \bar{\mathcal{W}}^a(Y)^2 dY.$$

We can rewrite

$$\left\| Z \bullet (\bar X, \bar Y) \right\|_{\mathcal{G}(\Omega)}^2 = \int_{\bar\Gamma} \left(\frac{1}{2} U^2\,(dX + dY) + (Z_X^a)^2\,dX + (Z_Y^a)^2\,dY \right).$$

Thus, we want to prove that

$$\begin{aligned} \int_{\bar\Gamma} &\left(\frac{1}{2} U^2\,(dX + dY) + (Z_X^a)^2\,dX + (Z_Y^a)^2\,dY \right) \\ &\leq C \int_{\Gamma} \left(\frac{1}{2} U^2\,(dX + dY) + (Z_X^a)^2\,dX + (Z_Y^a)^2\,dY \right). \end{aligned} \tag{4.59}$$

We decompose the proof into three steps.

Step 1. We first prove that (4.59) holds for small domains. We claim that there exist constants δ and C, which depend uniquely on $\left\|(\bar{\mathcal{X}}, \bar{\mathcal{Y}})\right\|_{\mathcal{C}(\Omega)}$ and $|||Z \bullet (\mathcal{X}, \mathcal{Y})|||_{\mathcal{G}(\Omega)}$ such that, for any rectangular domain $\Omega = [X_l, X_r] \times [Y_l, Y_r]$ with $s_r - s_l \leq \delta$, (4.59) holds. We denote by C a generic increasing function of $\left\|(\bar{\mathcal{X}}, \bar{\mathcal{Y}})\right\|_{\mathcal{C}(\Omega)}$ and $|||Z \bullet (\mathcal{X}, \mathcal{Y})|||_{\mathcal{G}(\Omega)}$. By Lemma 4.9, we have

$$\|U\|_{L^\infty(\Omega)} + \|Z_X^a\|_{L^\infty(\Omega)} + \|Z_Y^a\|_{L^\infty(\Omega)} \leq C.$$

Let

$$A = \sup_{\bar\Gamma} \int_{\Gamma} \left(\frac{1}{2} U^2\,(dX + dY) + (Z_X^a)^2\,dX + (Z_Y^a)^2\,dY \right)$$

where the supremum is taken over all $\bar\Gamma = (\bar{\mathcal{X}}, \bar{\mathcal{Y}}) \in \mathcal{C}(\Omega)$. Since Z is a solution of (2.13), we have for a.e. $X \in [X_l, X_r]$ and all $Y \in [Y_l, Y_r]$, that

$$\begin{aligned} (x_X - \frac{1}{2})^2(X,Y) &= (x_X - \frac{1}{2})^2(X, \mathcal{Y}(X)) + \int_{\mathcal{Y}(X)}^{Y} (x_X - \frac{1}{2}) x_{XY}\,d\bar Y \\ &= (x_X - \frac{1}{2})^2(X, \mathcal{Y}(X)) \\ &\quad + \int_{\mathcal{Y}(X)}^{Y} \frac{c'(U)}{c(U)} \big((x_X - \frac{1}{2})^2 (U_X + U_Y) + U_X (x_X - \frac{1}{2}) + U_Y (x_X - \frac{1}{2}) \big)\,d\bar Y \end{aligned}$$

and

$$\int_{\bar\Gamma}(x_X-\frac{1}{2})^2\,dX\le\int_{\Gamma}(x_X-\frac{1}{2})^2\,dX+C\int_{X_l}^{X_r}\int_{Y_l}^{Y_r}((Z_X^a)^2+(Z_Y^a)^2)dXdY. \tag{4.60}$$

For any $Y\in[Y_l,Y_r]$, the integral $\int_{X_l}^{X_r}(Z_X^a)^2(X,Y)\,dX$ can be seen as the integral of the form $(Z_X^a)^2\,dX$ on the piecewise linear path Γ going through the points (X_l,Y_l), (X_l,Y), (X_r,Y), (X_r,Y_r), so that $\int_{X_l}^{X_r}(Z_X^a)^2(X,Y)\,dX\le A$. Similarly, $\int_{Y_l}^{Y_r}(Z_X^a)^2(X,Y)\,dY\le A$, for any $X\in[X_l,X_r]$. Hence, (4.60) yields

$$\begin{aligned}\int_{\bar\Gamma}(x_X-\frac{1}{2})^2\,dX&\le\int_{\Gamma}(x_X-\frac{1}{2})^2\,dX+C(Y_r-Y_l+X_r-X_l)A\\&\le\int_{\Gamma}(x_X-\frac{1}{2})^2\,dX+C\delta A\end{aligned}$$

as $(Y_r-Y_l+X_r-X_l)=s_r-s_l$. By treating similarly the other components of Z_X^a and Z_Y^a, we get
(4.61)

$$\int_{\bar\Gamma}(Z_X^a)^2\,dX\le\int_{\Gamma}(Z_X^a)^2\,dX+6C\delta A\quad\text{and}\quad\int_{\bar\Gamma}(Z_Y^a)^2\,dY\le\int_{\Gamma}(Z_Y^a)^2\,dY+6C\delta A.$$

For the component U, we have

$$\begin{aligned}U^2(X,Y)&=U^2(X,\mathcal{Y}(X))+\int_{\mathcal{Y}(X)}^{Y}2UU_Y\,d\bar Y\\&\le U^2(X,\mathcal{Y}(X))+\int_{\mathcal{Y}(X)}^{Y}U^2\,d\bar Y+\int_{\mathcal{Y}(X)}^{Y}U_Y^2\,d\bar Y\end{aligned}$$

and it follows, as before, that

$$\int_{\bar\Gamma}U^2\,dX\le\int_{\Gamma}U^2\,dX+C\delta A. \tag{4.62}$$

Similarly, we obtain

$$\int_{\bar\Gamma}U^2\,dY\le\int_{\Gamma}U^2\,dY+C\delta A. \tag{4.63}$$

After adding (4.61), (4.62), (4.63) and recalling that $ds=\frac{1}{2}(dX+dY)$, we obtain

$$\begin{aligned}&\int_{\bar\Gamma}(\frac{1}{2}U^2\,(dX+dY)+(Z_X^a)^2\,dX+(Z_Y^a)^2\,dY)\\&\qquad\le\int_{\Gamma}(\frac{1}{2}U^2\,(dX+dY)+(Z_X^a)^2\,dX+(Z_Y^a)^2\,dY)+13C\delta A\end{aligned}$$

which yields, after taking the supremum over all curves $\bar\Gamma$,

$$(1-13C\delta)A\le\int_{\Gamma}(\frac{1}{2}U^2\,(dX+dY)+(Z_X^a)^2\,dX+(Z_Y^a)^2\,dY)$$

and (4.59) follows.

Step 2. For an arbitrarily large rectangular domain $\Omega = [X_l, X_r] \times [Y_l, Y_r]$, let us prove that (4.59) holds for the curves $\bar{\Gamma} = (\bar{\mathcal{X}}, \bar{\mathcal{Y}}) \in \mathcal{C}(\Omega)$ such that

$$\bar{\mathcal{Y}}(s) - \bar{\mathcal{X}}(s) > \mathcal{Y}(s) - \mathcal{X}(s) \text{ for all } s \in (s_l, s_r),$$

that is, the curve $\bar{\Gamma}$ is above Γ and intersects Γ only at the end points. Similarly one proves that (4.59) holds for curves $\bar{\Gamma} = (\bar{\mathcal{X}}, \bar{\mathcal{Y}}) \in \mathcal{C}(\Omega)$ such that $\bar{\mathcal{Y}}(s) - \bar{\mathcal{X}}(s) < \mathcal{Y}(s) - \mathcal{X}(s)$ for all $s \in [s_l, s_r]$. For a constant $K > 0$ that we will determine later, we have for a.e. $X \in [X_l, X_r]$, that

$$\begin{aligned} e^{-K(\bar{\mathcal{Y}}(X)-X)}(x_X - \frac{1}{2})^2(X, \bar{\mathcal{Y}}(X)) - e^{-K(\mathcal{Y}(X)-X)}(x_X - \frac{1}{2})^2(X, \mathcal{Y}(X)) \\ = \int_{\mathcal{Y}(X)}^{\bar{\mathcal{Y}}(X)} -Ke^{-K(Y-X)}(x_X - \frac{1}{2})^2\, dY + \int_{\mathcal{Y}(X)}^{\bar{\mathcal{Y}}(X)} e^{-K(X-Y)}(x_X - \frac{1}{2})x_{XY}\, dY \end{aligned}$$

which implies, since Z is solution and by the estimates of Lemma 4.9, that

$$\begin{aligned} \int_{\bar{\Gamma}} e^{-K(Y-X)}(x_X - \frac{1}{2})^2\, dX - \int_{\Gamma} e^{-K(Y-X)}(x_X - \frac{1}{2})^2\, dX \\ \leq \int_{X_l}^{X_r} \int_{\mathcal{Y}(X)}^{\bar{\mathcal{Y}}(X)} -Ke^{-K(Y-X)}(x_X - \frac{1}{2})^2\, dXdY \\ + C\int_{X_l}^{X_r} \int_{\mathcal{Y}(X)}^{\bar{\mathcal{Y}}(X)} e^{-K(Y-X)}((Z_X^a)^2 + (Z_Y^a)^2)\, dXdY. \end{aligned} \tag{4.64}$$

Note that (4.64) corresponds to an application of Stokes's theorem to the domain bounded by the curves Γ and $\bar{\Gamma}$. We treat in the same way each component of Z_X^a and obtain that

$$\begin{aligned} \int_{\bar{\Gamma}} e^{-K(Y-X)}(Z_X^a)^2\, dX - \int_{\Gamma} e^{-K(Y-X)}(Z_X^a)^2\, dX \\ \leq \int_{X_l}^{X_r} \int_{\mathcal{Y}(X)}^{\bar{\mathcal{Y}}(X)} -Ke^{-K(Y-X)}(Z_X^a)^2\, dXdY \\ + C\int_{X_l}^{X_r} \int_{\mathcal{Y}(X)}^{\bar{\mathcal{Y}}(X)} e^{-K(Y-X)}((Z_X^a)^2 + (Z_Y^a)^2)\, dXdY. \end{aligned} \tag{4.65}$$

As far as Z_Y^a is concerned, we get

$$\begin{aligned} \int_{\bar{\Gamma}} e^{-K(Y-X)}(Z_Y^a)^2\, dY - \int_{\Gamma} e^{-K(Y-X)}(Z_Y^a)^2\, dY \\ \leq -\int_{Y_l}^{Y_r} \int_{\bar{\mathcal{X}}(Y)}^{\mathcal{X}(Y)} Ke^{-K(Y-X)}(Z_X^a)^2\, dXdY \\ + C\int_{Y_l}^{Y_r} \int_{\bar{\mathcal{X}}(Y)}^{\mathcal{X}(Y)} e^{-K(Y-X)}((Z_X^a)^2 + (Z_Y^a)^2)\, dXdY. \end{aligned} \tag{4.66}$$

Let us prove that the sets

$$\mathcal{N}_1 = \begin{cases} X_l < X < X_r \\ \mathcal{Y}(X) < Y < \bar{\mathcal{Y}}(X) \end{cases} \quad \text{and} \quad \mathcal{N}_2 = \begin{cases} Y_l < Y < Y_r \\ \bar{\mathcal{X}}(Y) < X < \mathcal{X}(Y) \end{cases}$$

are equal up to a set of zero measure. Let us consider $(X,Y)\in\mathcal{N}_1$. We set $s_1=\mathcal{X}^{-1}(X)$, $s_2=\mathcal{Y}^{-1}(Y)$, $s_3=\bar{\mathcal{Y}}^{-1}(Y)$ and $s_4=\bar{\mathcal{X}}^{-1}(X)$. Since $\mathcal{Y}(X)<Y<\bar{\mathcal{Y}}(X)$, we get

$$\mathcal{Y}(X)=\mathcal{Y}(s_1)<\mathcal{Y}(s_2)=\bar{\mathcal{Y}}(s_3)=Y<\bar{\mathcal{Y}}(s_4)=\bar{\mathcal{Y}}(X).$$

Hence, $s_1<s_2$ and $s_3<s_4$, which implies

$$X=\mathcal{X}(s_1)\leq\mathcal{X}(s_2)=\mathcal{X}(Y),\quad \bar{\mathcal{X}}(Y)=\bar{\mathcal{X}}(s_3)\leq\bar{\mathcal{X}}(s_4)=X$$

and therefore $\bar{\mathcal{X}}(Y)\leq X\leq\mathcal{X}(Y)$. Thus we have prove that $\mathcal{N}_1\subset\mathcal{N}_2$ up to a set of zero measure. Similarly, one proves the reverse inclusion. Hence, by adding (4.65) and (4.66), we get

$$\begin{aligned}(4.67)\quad &\int_{\bar\Gamma} e^{-K(Y-X)}((Z_X^a)^2\,dX+(Z_Y^a)^2\,dY)-\int_{\Gamma} e^{-K(Y-X)}((Z_X^a)^2\,dX+(Z_Y^a)^2\,dY)\\ &\leq -K\int_{\mathcal{N}_1} e^{-K(Y-X)}((Z_X^a)^2+(Z_Y^a)^2)\,dXdY+C\int_{\mathcal{N}_1} e^{-K(Y-X)}((Z_X^a)^2+(Z_Y^a)^2)\,dXdY.\end{aligned}$$

As far as U is concerned, we proceed in the same way and get

$$\begin{aligned}&\int_{\bar\Gamma} e^{-K(Y-X)}U^2\,dX-\int_{\Gamma} e^{-K(Y-X)}U^2\,dX\\ &\qquad=\int_{X_l}^{X_r}\int_{\mathcal{Y}(X)}^{\bar{\mathcal{Y}}(X)} e^{-K(Y-X)}(-KU^2+2UU_Y)\,dXdY\\ (4.68)\quad &\qquad\leq\int_{\mathcal{N}_1} e^{-K(Y-X)}(-KU^2+U^2+U_Y^2)\,dXdY\end{aligned}$$

and

$$\begin{aligned}&\int_{\bar\Gamma} e^{-K(Y-X)}U^2\,dY-\int_{\Gamma} e^{-K(Y-X)}U^2\,dY\\ (4.69)\quad &\qquad\leq\int_{\mathcal{N}_2} e^{-K(Y-X)}(-KU^2+U^2+U_X^2)\,dXdY.\end{aligned}$$

Combining (4.65), (4.66), (4.68), (4.68), we get

$$\begin{aligned}(4.70)\quad &\int_{\bar\Gamma} e^{-K(Y-X)}(\frac{1}{2}U^2\,(dX+dY)+(Z_X^a)^2\,dX+(Z_Y^a)^2\,dY)\\ &\quad-\int_{\Gamma} e^{-K(Y-X)}(\frac{1}{2}U^2\,(dX+dY)+(Z_X^a)^2\,dX+(Z_Y^a)^2\,dY)\\ &\qquad\leq\int_{\mathcal{N}_1}(C-K)e^{-K(Y-X)}(U^2+(Z_X^a)^2+(Z_Y^a)^2)\,dXdY.\end{aligned}$$

We choose K sufficiently large so that the right-hand side in (4.70) is negative and we obtain that

$$\begin{aligned}&e^{-K\|\bar{\mathcal{X}}-\bar{\mathcal{Y}}\|_{L^\infty}}\int_{\bar\Gamma}(\frac{1}{2}U^2\,(dX+dY)+(Z_X^a)^2\,dX+(Z_Y^a)^2\,dY)\\ &\qquad\leq e^{K\|\mathcal{X}-\mathcal{Y}\|_{L^\infty}}\int_{\Gamma}(\frac{1}{2}U^2\,(dX+dY)+(Z_X^a)^2\,dX+(Z_Y^a)^2\,dY)\end{aligned}$$

and (4.59) follows.

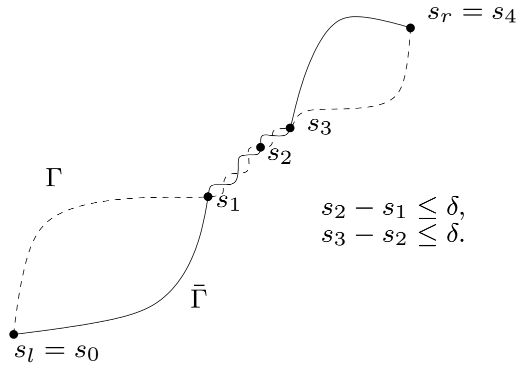

FIGURE 4. The interval $[s_l, s_r]$ is divided into *large* intervals (in this example $[s_0, s_1]$ and $[s_3, s_4]$) where one curve is over the other and *small* intervals of length smaller than δ (in this example $[s_1, s_2]$ and $[s_2, s_3]$) where the curves can cross an arbitrarily number of times.

Step 3. Given any rectangle $\Omega = [X_l, X_r] \times [Y_l, Y_r]$, we consider a sequence of rectangular domains $\Omega_i = [X_i, X_{i+1}] \times [Y_i, Y_{i+1}]$ for $i = 0, \ldots, N-1$ where X_i and Y_i are increasing and $X_0 = X_l$, $Y_0 = Y_l$, $X_N = X_r$, $Y_N = Y_r$ and such that $(\mathcal{X}, \mathcal{Y})$, $(\bar{\mathcal{X}}, \bar{\mathcal{Y}})$ belong to $\mathcal{G}(\Omega_i)$ for $s \in [s_i, s_{i+1}]$. We construct the sequence of rectangles such that either $s_{i+1} - s_i \leq \delta$ (and Step 1 applies) or $\bar{\mathcal{Y}}(s) - \bar{\mathcal{X}}(s) \leq \mathcal{Y}(s) - \mathcal{X}(s)$ or $\mathcal{Y}(s) - \mathcal{X}(s) \leq \bar{\mathcal{Y}}(s) - \bar{\mathcal{X}}(s)$ for $s \in [s_i, s_{i+1}]$ (and Step 2 applies). Hence,

$$\begin{aligned}
\left\|Z \bullet (\bar{\mathcal{X}}, \bar{\mathcal{Y}})\right\|_{\mathcal{G}(\Omega)}^2 &= \sum_{i=0}^{N-1} \left\|Z \bullet (\bar{\mathcal{X}}, \bar{\mathcal{Y}})\right\|_{\mathcal{G}(\Omega_i)}^2 \\
&\leq \sum_{i=0}^{N-1} C \left\|Z \bullet (\mathcal{X}, \mathcal{Y})\right\|_{\mathcal{G}(\Omega_i)}^2 \quad \text{(by steps 1 and 2)} \\
&\leq C \left\|Z \bullet (\mathcal{X}, \mathcal{Y})\right\|_{\mathcal{G}(\Omega)}^2 .
\end{aligned}$$

We can construct the sequence of rectangles as follows. Let $\bar{N}$ be an integer such that $\frac{s_l - s_r}{\bar{N}} \leq \frac{\delta}{2}$ and we set $\tilde{s}_j = s_l + j\frac{\delta}{2}$ for $j = 0, \ldots, \bar{N}$. We take $s_0 = s_l$ and define s_i iteratively: Given s_i and $j_i \in \{0, \ldots, \bar{N} - 1\}$ such that $j_i \geq i$, $\mathcal{X}(s_i) = \bar{\mathcal{X}}(s_i)$, $\mathcal{Y}(s_i) = \bar{\mathcal{Y}}(s_i)$ and $s_i \in [\tilde{s}_{j_i}, \tilde{s}_{j_i+1}]$. If $j_i + 1 = \bar{N}$, we set $N = i + 1$, $s_{i+1} = s_r$ and we are done. Otherwise, there exists an index $k \geq j_i + 1$ such that $\bar{\mathcal{Y}}(s) - \bar{\mathcal{X}}(s) < \mathcal{Y}(s) - \mathcal{X}(s)$ for all $s \in [\tilde{s}_{j+1}, \tilde{s}_k)$ or $\bar{\mathcal{Y}}(s) - \bar{\mathcal{X}}(s) > \mathcal{Y}(s) - \mathcal{X}(s)$ for all $s \in [\tilde{s}_{j+1}, \tilde{s}_k)$ and there exists $s \in [\tilde{s}_k, \tilde{s}_{k+1}]$ such that $\bar{\mathcal{Y}}(s) - \bar{\mathcal{X}}(s) = \mathcal{Y}(s) - \mathcal{X}(s)$ (which implies that $\mathcal{X}(s) = \bar{\mathcal{X}}(s)$ and $\mathcal{Y}(s) = \bar{\mathcal{Y}}(s)$). We then set $j_{i+1} = k$ and choose $s_{i+1} \in [\tilde{s}_k, \tilde{s}_{k+1}]$ such that $\mathcal{X}(s_{i+1}) = \bar{\mathcal{X}}(s_{i+1})$ and $\mathcal{Y}(s_{i+1}) = \bar{\mathcal{Y}}(s_{i+1})$. Since $j_i \geq i$, the iteration stops in a finite number of steps. □

Given two solutions Z and $\bar{Z}$, we want to compare along curves in $\mathcal{C}$ the forms $Z_X\,dX$ and $Z_XY\,dY$ with $\bar{Z}_X\,dX$ and $\bar{Z}_XY\,dY$, respectively. By using the same argument as in the proof above, we obtain the following stability result in L^2. This is a stronger result than the one that could be established from the fixed point argument in Lemma 3.9 as the latter would only hold in L^∞.

Lemma 4.12 (Stability in L^2)**.** *Given* $\Omega = [X_l, X_r] \times [Y_l, Y_r]$, $Z, \bar{Z} \in \mathcal{H}(\Omega)$ *and* $(\mathcal{X}, \mathcal{Y}) \in \mathcal{C}(\Omega)$. *Then, for any* $(\bar{\mathcal{X}}, \bar{\mathcal{Y}}) \in \mathcal{C}(\Omega)$,

$$\|(Z - \bar{Z}) \bullet (\bar{\mathcal{X}}, \bar{\mathcal{Y}})\|_{\mathcal{G}(\Omega)} \leq C \|(Z - \bar{Z}) \bullet (\mathcal{X}, \mathcal{Y})\|_{\mathcal{G}(\Omega)} \tag{4.71}$$

where $C = C(\|(\bar{\mathcal{X}}, \bar{\mathcal{Y}})\|_{\mathcal{C}(\Omega)}, |||Z \bullet (\mathcal{X}, \mathcal{Y})|||_{\mathcal{G}(\Omega)}), |||\bar{Z} \bullet (\mathcal{X}, \mathcal{Y})|||_{\mathcal{G}(\Omega)})$ *is a given increasing function with respect to both its arguments.*

In the definition below of global solutions we include a condition about the decay of the solutions along the diagonal. This condition is necessary to guarantee that, given a solution, the curves which correspond to a given time T belong to $\mathcal{C}$.

Definition 4.13 (Global solutions)**.** *Let* $\mathcal{H}$ *be the set of all functions* $Z \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^2)$ *such that*

(i) $Z \in \mathcal{H}(\Omega)$ *for all rectangular domains* Ω*; and*
(ii) *there exists a curve* $(\mathcal{X}, \mathcal{Y}) \in \mathcal{C}$ *such that* $Z \bullet (\mathcal{X}, \mathcal{Y}) \in \mathcal{G}$.

The condition (ii), which corresponds to a decay condition, does not depend on the particular curve for which it holds, as the next lemma shows. In particular, we can replace condition (iv) in Definition 4.13 by the requirement that $Z \bullet (\mathcal{X}_d, \mathcal{Y}_d) \in \mathcal{G}$ for the diagonal ($Y = X$), which is given by $\mathcal{X}_d(s) = \mathcal{Y}_d(s) = s$. We then denote

$$\|Z\|_{\mathcal{H}} = \|Z \bullet (\mathcal{X}_d, \mathcal{Y}_d)\|_{\mathcal{G}} \quad \text{and} \quad |||Z|||_{\mathcal{H}} = |||Z \bullet (\mathcal{X}_d, \mathcal{Y}_d)|||_{\mathcal{G}}.$$

Lemma 4.14. *Given* $Z \in \mathcal{H}$, *we have* $Z \bullet (\mathcal{X}, \mathcal{Y}) \in \mathcal{G}$ for any curve $(\mathcal{X}, \mathcal{Y}) \in \mathcal{C}$. *Moreover, the limit* $\lim_{s\to\infty} J(\mathcal{X}(s), \mathcal{Y}(s))$ *is independent of the curve* $(\mathcal{X}, \mathcal{Y}) \in \mathcal{C}$.

In this lemma we denote as before $\mathcal{Z}_4$ by J where $\Theta = (\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{V}, \mathcal{W}) = Z \bullet (\mathcal{X}, \mathcal{Y})$. Later, we will see that the limit of J at infinity corresponds to the total energy and the lemma would allow us to prove that the total energy is conserved.

Proof of Lemma 4.14. For any curve $(\bar{\mathcal{X}}, \bar{\mathcal{Y}}) \in \mathcal{C}$, we have to prove that

$$|||Z \bullet (\bar{\mathcal{X}}, \bar{\mathcal{Y}})|||_{\mathcal{G}} < \infty \quad \text{and} \quad \left\|Z \bullet (\bar{\mathcal{X}}, \bar{\mathcal{Y}})\right\|_{\mathcal{G}} < \infty \tag{4.72}$$

and

$$\lim_{s\to\infty} \bar{J}(s) = 0 \tag{4.73}$$

where $\bar{J} = \bar{\mathcal{Z}}_4$ with $(\bar{\mathcal{X}}, \bar{\mathcal{Y}}, \bar{\mathcal{Z}}, \bar{\mathcal{V}}, \bar{\mathcal{W}}) = Z \bullet (\bar{\mathcal{X}}, \bar{\mathcal{Y}})$. For any real number $\bar{s} \in \mathbb{R}$ that will eventually tend to infinity and denote $\Omega_{\bar{s}} = [\bar{\mathcal{X}}(-\bar{s}), \bar{\mathcal{X}}(\bar{s})] \times [\bar{\mathcal{Y}}(-\bar{s}), \bar{\mathcal{Y}}(\bar{s})]$. Let

$$s_{\max} = \begin{cases} \mathcal{Y}^{-1}(\bar{\mathcal{Y}}(\bar{s})) & \text{if } \mathcal{Y}(\bar{\mathcal{X}}(\bar{s})) \leq \bar{\mathcal{Y}}(\bar{s}), \\ \mathcal{X}^{-1}(\bar{\mathcal{X}}(\bar{s})) & \text{otherwise,} \end{cases} \tag{4.74}$$

and

$$s_{\min} = \begin{cases} \mathcal{Y}^{-1}(\bar{\mathcal{Y}}(-\bar{s})) & \text{if } \mathcal{Y}(\bar{\mathcal{X}}(-\bar{s})) \leq \bar{\mathcal{Y}}(-\bar{s}), \\ \mathcal{X}^{-1}(\bar{\mathcal{X}}(-\bar{s})) & \text{otherwise,} \end{cases} \tag{4.75}$$

see Figure 5 for an example. One can check that by construction $s_{\min} \leq -\bar{s} \leq \bar{s} \leq s_{\max}$ and we denote $\tilde{\Omega}_{\bar{s}} = [\mathcal{X}(s_{\min}), \mathcal{X}(s_{\max})] \times [\mathcal{Y}(s_{\min}), \mathcal{Y}(s_{\max})]$. We have $\Omega_{\bar{s}} \subset \tilde{\Omega}_{\bar{s}}$.

We construct the curve which consists of a (vertical or horizontal) straight line joining $(\mathcal{X}(s_{\min}),\mathcal{Y}(s_{\min}))$ and $(\mathcal{X}(\bar{s}),\mathcal{Y}(\bar{s}))$, the curve $(\bar{\mathcal{X}}(s),\bar{\mathcal{Y}}(s))$ for $s\in[-\bar{s},\bar{s}]$ and another (vertical or horizontal) straight line joining $(\bar{\mathcal{X}}(\bar{s}),\bar{\mathcal{Y}}(\bar{s}))$ and $(\mathcal{X}(s_{\max}),\mathcal{Y}(s_{\max}))$, see Figure 5. We denote by $(\tilde{\mathcal{X}},\tilde{\mathcal{Y}})$ this curve and we have that $(\tilde{\mathcal{X}},\tilde{\mathcal{Y}})$ and $(\mathcal{X},\mathcal{Y})$ belong to $\mathcal{G}(\tilde{\Omega}_{\bar{s}})$. By Lemma 4.9, we get

$$
\begin{aligned}
|||Z\bullet(\bar{\mathcal{X}},\bar{\mathcal{Y}})|||_{\mathcal{G}(\Omega_{\bar{s}})} &\leq |||Z\bullet(\bar{\mathcal{X}},\bar{\mathcal{Y}})|||_{\mathcal{G}(\tilde{\Omega}_{\bar{s}})}\\
&\leq C_1(\|(\bar{\mathcal{X}},\bar{\mathcal{Y}})\|_{\mathcal{C}(\tilde{\Omega}_{\bar{s}})},|||\Theta|||_{\mathcal{G}(\tilde{\Omega}_{\bar{s}})})\leq C_1(\|(\bar{\mathcal{X}},\bar{\mathcal{Y}})\|_{\mathcal{C}},|||\Theta|||_{\mathcal{G}})
\end{aligned}
$$

and by letting $\bar{s}$ tend to infinity, we get $|||Z\bullet(\bar{\mathcal{X}},\bar{\mathcal{Y}})|||_{\mathcal{G}}<\infty$. By Lemma 4.11, we get

$$
\left\|Z\bullet(\bar{\mathcal{X}},\bar{\mathcal{Y}})\right\|_{\mathcal{G}(\Omega_{\bar{s}})}\leq\left\|Z\bullet(\bar{\mathcal{X}},\bar{\mathcal{Y}})\right\|_{\mathcal{G}(\tilde{\Omega}_{\bar{s}})}\leq C\left\|Z\bullet(\mathcal{X},\mathcal{Y})\right\|_{\mathcal{G}(\tilde{\Omega}_{\bar{s}})}\leq C\left\|Z\bullet(\mathcal{X},\mathcal{Y})\right\|_{\mathcal{G}} \tag{4.76}
$$

where the constant C depends on $\left\|(\bar{X},\bar{Y})\right\|_{\mathcal{G}(\tilde{\Omega}_{\bar{s}})}$ and $|||Z\bullet(\mathcal{X},\mathcal{Y})|||_{\mathcal{G}(\tilde{\Omega}_{\bar{s}})}$, that is, on $\left\|(\bar{X},\bar{Y})\right\|_{\mathcal{G}}$ and $|||Z\bullet(\mathcal{X},\mathcal{Y})|||_{\mathcal{G}}$, which are independent on $\bar{s}$. By letting $\bar{s}$ tend to infinity in (4.76), we get $\left\|Z\bullet(\bar{\mathcal{X}},\bar{\mathcal{Y}})\right\|_{\mathcal{G}}<\infty$. It remains to prove (4.73). We know that $\bar{J}$ is positive. We denote $J(s)=\mathcal{Z}_4(s)$ with $(\mathcal{X},\mathcal{Y},\mathcal{Z},\mathcal{V},\mathcal{W})=Z\bullet(\mathcal{X},\mathcal{Y})$ and, slightly abusing the notation, we denote also by J, $J(X,Y)=Z_4(X,Y)$. For any $s\in\mathbb{R}$, let $s_1=\mathcal{X}^{-1}\bar{\mathcal{X}}(s)$ and $s_2=\mathcal{Y}^{-1}\bar{\mathcal{Y}}(s)$. If $s_1\leq s_2$, then $\bar{\mathcal{X}}(s)=\mathcal{X}(s_1)\leq\mathcal{X}(s_2)$ and $\mathcal{Y}(s_1)\leq\bar{\mathcal{Y}}(s)=\bar{\mathcal{Y}}(s_2)$. By the monotonicity of $J(X,Y)$, we get

$$
J(\mathcal{X}(s_1),\mathcal{Y}(s_1))\leq J(\bar{\mathcal{X}}(s),\bar{\mathcal{Y}}(s))\leq J(\mathcal{X}(s_2),\mathcal{Y}(s_2)).
$$

If $s_2\leq s_1$, we get a similar result so that, finally,

$$
\min\{J(\mathcal{X}(s_1),\mathcal{Y}(s_1)),J(\mathcal{X}(s_2),\mathcal{Y}(s_2))\}\leq J(\bar{\mathcal{X}}(s),\bar{\mathcal{Y}}(s)) \tag{4.77a}
$$

and

$$
J(\bar{\mathcal{X}}(s),\bar{\mathcal{Y}}(s))\leq\max\{J(\mathcal{X}(s_1),\mathcal{Y}(s_1)),J(\mathcal{X}(s_2),\mathcal{Y}(s_2))\}. \tag{4.77b}
$$

Since

$$
|s_1-s|\leq|\mathcal{X}(s_1)-s_1|+\left|\bar{\mathcal{X}}(s)-s\right|\leq\|(\mathcal{X},\mathcal{Y})\|_{\mathcal{C}}+\left\|(\bar{\mathcal{X}},\bar{\mathcal{Y}})\right\|_{\mathcal{C}},
$$

we have that $\lim_{s\to\pm\infty}s_1=\pm\infty$ similarly we obtain that $\lim_{s\to\pm\infty}s_2=\pm\infty$. Hence, (4.77) yields

$$
\lim_{s\to\pm\infty}J(\bar{\mathcal{X}}(s),\bar{\mathcal{Y}}(s))=\lim_{s\to\pm\infty}J(\mathcal{X}(s),\mathcal{Y}(s)).
$$

Thus, these limits are independent of the curve $\bar{\Gamma}$ which is chosen. The existence of the limits is guaranteed by the monotonicity and boudedness of J. The identity (4.73) follows from (3.9). □

From Lemma 4.10, we infer the following global existence theorem for the equivalent system.

Theorem 4.15 (Existence and uniqueness of global solutions). *For any initial data $(\mathcal{X},\mathcal{Y},\mathcal{Z},\mathcal{V},\mathcal{W})$ in $\mathcal{G}$, there exists a unique solution $Z\in\mathcal{H}$ such that $\Theta=Z\bullet(\mathcal{X},\mathcal{Y})$. We denote this solution mapping by $\mathbf{S}\colon\mathcal{G}\to\mathcal{H}$.*

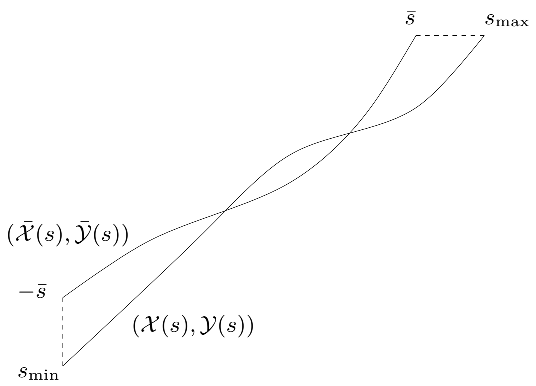

FIGURE 5. Prolongation of the curve $\bar{\Gamma}$.

5. SEMIGROUP OF SOLUTION S_T IN $\mathcal{F}$

From a solution function $Z \in \mathcal{H}$ in the whole plane, we want to extract the data at a given time. It is enough to do it at $t = 0$, and the definition below describes how we proceed.

Definition 5.1. *Given $Z \in \mathcal{H}$, we define*

$$\mathcal{X}(s) = \sup\{X \in \mathbb{R} \mid t(X', 2s - X') < 0 \quad \text{for all } X' < X\} \tag{5.1}$$

and $\mathcal{Y}(s) = 2s - \mathcal{X}(s)$. Then, we have $(\mathcal{X}(s), \mathcal{Y}(s)) \in \mathcal{C}$ and $Z \bullet (\mathcal{X}, \mathcal{Y}) \in \mathcal{G}_0$. We denote by $\mathbf{E}$ the mapping from $\mathcal{H}$ to $\mathcal{G}_0$ that associates to any $Z \in \mathcal{H}$ the element $Z \bullet (\mathcal{X}, \mathcal{Y}) \in \mathcal{G}_0$ as defined here.

Proof of the well-posedness of Definition 5.1. First we prove that $\mathcal{X}$ is increasing. Let X_n be a sequence such that $X_n < \mathcal{X}(s)$ and $X_n \to \mathcal{X}(s)$. We have $t(X_n, 2s - X_n) < 0$ for any $\bar{s} > s$. Since t is decreasing with respect to the second variable (as $t_Y \leq 0$), we have that $t(X_n, 2\bar{s} - X_n) < 0$ and therefore $X_n \leq \mathcal{X}(\bar{s})$. After letting n tend to infinity, we obtain $\mathcal{X}(s) \leq \mathcal{X}(\bar{s})$ so that $\mathcal{X}$ is an increasing function. Let us prove that $\mathcal{X}$ is Lipschitz with Lipschitz coefficient smaller than 2. Let us assume the opposite, i.e., there exists $\bar{s} > s$ such that

$$\mathcal{X}(\bar{s}) - \mathcal{X}(s) > 2(\bar{s} - s). \tag{5.2}$$

It implies that $\mathcal{Y}(s) > \mathcal{Y}(\bar{s})$. Since $t_X \geq 0$ and $t_Y \leq 0$, we have, for any $(X, Y) \in [\mathcal{X}(s), \mathcal{X}(\bar{s})] \times [\mathcal{Y}(\bar{s}), \mathcal{Y}(s)]$

$$0 = t(\mathcal{X}(s), \mathcal{Y}(s)) \leq t(X, \mathcal{Y}(s)) \leq t(X, Y) \leq t(X, \mathcal{Y}(\bar{s})) \leq t(\mathcal{X}(\bar{s}), \mathcal{Y}(\bar{s})) = 0$$

and, therefore $t(X, Y) = 0$ on $\Omega = [\mathcal{X}(s), \mathcal{X}(\bar{s})] \times [\mathcal{Y}(\bar{s}), \mathcal{Y}(s)]$. Let us consider the point $(X, Y) \in \Omega$ for $Y = \mathcal{Y}(s)$ and $X = 2\bar{s} - \mathcal{Y}(s)$. We have $t(X, Y) = 0$, $X + Y = 2\bar{s}$ and $X < \mathcal{X}(\bar{s})$, which contradict the definition of $\mathcal{X}$ at $\bar{s}$. Thus, we have proved that $\mathcal{X}$ is Lipschitz. To show that $(\mathcal{X}, \mathcal{Y}) \in \mathcal{C}$, it remains to prove that $\|\mathcal{X} - \mathcal{Y}\|_{L^\infty(\mathbb{R})} < \infty$. We claim that there there exists $\bar{L}$ such that

$$\liminf_{X \to \infty} t(X + L, X) \geq 1 \tag{5.3}$$

for any $L \geq \bar{L}$. Let us prove this claim. By using the fact that $x_X = c(U)t_X$ and $c(U) \geq \frac{1}{\kappa}$, we get

$$\begin{aligned} t(X+L,X) &= t(X,X) + \int_X^{X+L} t_X(\tilde{X},X)\,d\tilde{X} \\ &\geq -|||Z|||_{\mathcal{H}} + \frac{L}{2\kappa} + \frac{1}{\kappa}\int_X^{X+L}(x_X - \frac{1}{2})(\tilde{X},X)\,d\tilde{X}. \end{aligned} \tag{5.4}$$

We look at the domain $\Omega_{X,L} = [X, X+L]\times[X, X+L]$. We consider the curve $\mathcal{X}_d(s) = \mathcal{Y}_d(s) = s$ (the diagonal) and the curve $(\bar{\mathcal{X}}, \bar{\mathcal{Y}})$ which consists of a horizontal and a vertical segment given by

$$\begin{cases} \bar{\mathcal{X}}(s) = 2s - X, & \bar{\mathcal{Y}}(s) = X \qquad\qquad\quad \text{for } s \in [0, X + \frac{L}{2}] \\ \bar{\mathcal{X}}(s) = X + L, & \bar{\mathcal{Y}}(s) = 2s - (X+L) \quad \text{for } s \in [X+\frac{L}{2}, X+L]. \end{cases}$$

By Lemma 4.11 we get

$$\int_X^{X+L}(x_X - \frac{1}{2})^2(\tilde{X},X)\,d\tilde{X} \leq \left\|Z\bullet(\bar{X},\bar{\mathcal{Y}})\right\|^2_{\mathcal{G}(\Omega_{X,L})} \leq C\left\|Z\bullet(\mathcal{X}_d,\mathcal{Y}_d)\right\|^2_{\mathcal{G}(\Omega_{X,L})} \tag{5.5}$$

where the constant C depends on L and $|||Z\bullet(\mathcal{X}_d,\mathcal{Y}_d)|||_{\mathcal{G}(\Omega_{X,L})}$ which is bounded by $|||Z|||_{\mathcal{H}}$. We have

$$\lim_{X\to\infty}\left\|Z\bullet(\mathcal{X}_d,\mathcal{Y}_d)\right\|^2_{\mathcal{G}(\Omega_{X,L})} = \lim_{X\to\infty}\int_X^{X+L}(U^2(\tilde{X},\tilde{X}) + (Z_X^a)^2(\tilde{X},\tilde{X}) + (Z_Y^a)^2(\tilde{X},\tilde{X}))\,d\tilde{X} = 0,$$

and therefore (5.5) and (5.4) yield

$$\liminf_{X\to\infty} t(X+L,X) \geq -\left\|Z\right\|_{\mathcal{H}} + \frac{L}{2\kappa}$$

which, for L large enough, implies (5.3). Using the same type of argument, we prove that there exists L such that

$$\liminf_{X\to\infty} t(X+L,X) \geq 1, \qquad \limsup_{X\to\infty} t(X-L,X) \leq -1, \tag{5.6}$$

$$\liminf_{X\to-\infty} t(X+L,X) \geq 1, \qquad \liminf_{X\to-\infty} t(X-L,X) \leq -1. \tag{5.7}$$

Let us prove that $\limsup_{s\to\infty}(\mathcal{X}(s) - s) \leq \frac{L}{2}$. We assume the opposite and then, there exists $s \in \mathbb{R}$ such that $t(s+\frac{L}{2}, s-\frac{L}{2}) \geq 1$, by (5.6), and $\mathcal{X}(s) > s + \frac{L}{2}$. It implies that $\mathcal{Y}(s) = 2s - \mathcal{X}(s) \leq s - \frac{L}{2}$, and, using the monotonicity of t (that is $t_X \geq 0$ and $t_Y \leq 0$), we get $1 \leq t(s+\frac{L}{2}, s-\frac{L}{2}) \leq t(\mathcal{X}(s),\mathcal{Y}(s)) = 0$, which is a contradiction. Similarly one proves that $\liminf_{s\to\infty}(\mathcal{X}(s)-s) \geq -\frac{L}{2}$, $\limsup_{s\to-\infty}(\mathcal{X}(s)-s) \leq \frac{L}{2}$ and $\liminf_{s\to-\infty}(\mathcal{X}(s)-s) \geq -\frac{L}{2}$ and it follows that

$$\limsup_{s\to\pm\infty}|\mathcal{X}(s) - s| \leq \frac{L}{2}.$$

Hence, the condition (2.15a) is satisfied and $(\mathcal{X},\mathcal{Y}) \in \mathcal{C}$. By Lemma 4.14, we have $(\mathcal{X},\mathcal{Y},\mathcal{Z},\mathcal{V},\mathcal{W}) = Z\bullet(\mathcal{X},\mathcal{Y}) \in \mathcal{G}$ and by construction $\mathcal{Z}_1(s) = t(\mathcal{X}(s),\mathcal{Y}(s)) = 0$ so that $Z\bullet(\mathcal{X},\mathcal{Y}) \in \mathcal{G}_0$. □

Definition 5.2. *Given any T, let us introduce the mapping $\mathbf{t}_T\colon \mathcal{H} \to \mathcal{H}$ defined as follows. For any $Z \in \mathcal{H}$, let $\mathbf{t}_T(Z) = \bar{Z} \in \mathcal{H}$ be given by*

$$\bar{t}(X,Y) = t(X,Y) - T$$

and

$$\bar{x}(X,Y) = x(X,Y), \qquad \bar{U}(X,Y) = U(X,Y),$$
$$\bar{J}(X,Y) = J(X,Y), \qquad \bar{K}(X,Y) = K(X,Y).$$

We have

$$\mathbf{t}_{T+T'} = \mathbf{t}_T \circ \mathbf{t}_{T'}. \tag{5.8}$$

We have now defined the following mappings:

$$\mathcal{F} \underset{\mathbf{D}}{\overset{\mathbf{C}}{\rightleftarrows}} \mathcal{G}_0 \underset{\mathbf{E}}{\overset{\mathbf{S}}{\rightleftarrows}} \mathcal{H} \circlearrowright \mathbf{t}_T \tag{5.9}$$

The mapping $\mathbf{t}_T$ is used to extract the solution at any given time T by only using the operator $\mathbf{E}$, which is designed for time zero. Indeed, by taking $\mathbf{E} \circ \mathbf{t}_T$, we recover an element in $\mathcal{G}_0$ which corresponds to the solution at time T. In the following lemma, we prove that $\mathcal{F}$ and $\mathcal{H}$ are in bijection, which also justify the introduction of $\mathcal{F}$: It is a consistent way to parametrize initial data: To any element in $\mathcal{F}$, there corresponds a unique solution in $\mathcal{H}$, and vice versa. The $\mathcal{G}_0$ does not fit that role as $\mathcal{G}_0$ and $\mathcal{H}$ are not in bijection.

Lemma 5.3. *We have*

$$\mathbf{C} \circ \mathbf{D} \circ \mathbf{E} = \mathbf{E}, \quad \mathbf{D} \circ \mathbf{C} = \mathrm{Id} \tag{5.10}$$

and

$$\mathbf{E} \circ \mathbf{S} \circ \mathbf{C} = \mathbf{C}, \quad \mathbf{S} \circ \mathbf{E} = \mathrm{Id}\,. \tag{5.11}$$

It follows that $\mathbf{S} \circ \mathbf{C} = (\mathbf{D} \circ \mathbf{E})^{-1}$ and the sets $\mathcal{F}$ and $\mathcal{H}$ are in bijection.

Proof. **Step 1.** We prove (5.10). Given $Z \in \mathcal{H}$, let $(\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{V}, \mathcal{W}) = \mathbf{E}(Z)$, $(\psi_1, \psi_2) = \mathbf{D}(\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{V}, \mathcal{W})$ and $(\bar{\mathcal{X}}, \bar{\mathcal{Y}}, \bar{\mathcal{Z}}, \bar{\mathcal{V}}, \bar{\mathcal{W}}) = \mathbf{C}(\psi_1, \psi_2)$. We want to prove that $(\bar{\mathcal{X}}, \bar{\mathcal{Y}}, \bar{\mathcal{Z}}, \bar{\mathcal{V}}, \bar{\mathcal{W}}) = (\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{V}, \mathcal{W})$. We have to prove that $\mathcal{X} = \bar{\mathcal{X}}$, the rest will easily follow. For any $s \in \mathbb{R}$, we claim that for any couple (X, Y) such that $X < \mathcal{X}(s)$ and $X + Y = 2s$ then we have either

$$x_1(X) < x_1(\mathcal{X}(s)) \quad \text{or} \quad x_2(Y) > x_2(\mathcal{Y}(s)). \tag{5.12}$$

Let us assume the opposite, that is, there exist $\bar{s}$, $\bar{X}$ and $\bar{Y}$ such that $\bar{X} < \mathcal{X}(\bar{s})$, $\bar{X} + \bar{Y} = 2\bar{s}$ and

$$x_1(\bar{X}) = x_1(\mathcal{X}(\bar{s})) = x(\bar{s}) = x_2(\mathcal{Y}(\bar{s})) = x_2(\bar{Y}).$$

Here, $x(s)$ denotes $\mathcal{Z}_2(s)$, see (3.27). Let $s_0 = \mathcal{X}^{-1}(\bar{X})$ and $s_1 = \mathcal{Y}^{-1}(\bar{Y})$. Since $\bar{X} < \mathcal{X}(\bar{s})$ and $\bar{Y} > \mathcal{Y}(\bar{s})$, we have $s_0 < \bar{s} < s_1$. We have

$$x(s_0) = x_1(\mathcal{X}(s_0)) = x_1(\bar{X}) = x(\bar{s})$$

and, similarly, we obtain that $x(s_1) = x(\bar{s})$. We consider the rectangular domain $\Omega = [\mathcal{X}(s_0), \mathcal{X}(s_1)] \times [\mathcal{Y}(s_0), \mathcal{Y}(s_1)]$. Since $x(s) = x(s_0) = x(s_1)$ for all $s \in [s_0, s_1]$, we have

$\dot{x} = 0$ on $[s_0, s_1]$ because x is nondecreasing. We have $\dot{x} = 0 = \mathcal{V}_2(\mathcal{X})\dot{\mathcal{X}} + \mathcal{W}_2(\mathcal{Y})\dot{\mathcal{Y}}$ on $[s_0, s_1]$, which implies that $\mathcal{V}_2(X) = 0$ for a.e. $X \in [\mathcal{X}(s_0), \mathcal{X}(s_1)]$ and $\mathcal{W}_2(Y) = 0$ for a.e. $Y \in [\mathcal{Y}(s_0), \mathcal{Y}(s_1)]$. By (3.8b) and (3.8a) it implies $\mathcal{V}_1 = \mathcal{V}_2 = \mathcal{V}_3 = 0$ on $[\mathcal{X}(s_0), \mathcal{X}(s_1)]$ and $\mathcal{W}_1 = \mathcal{W}_2 = \mathcal{W}_3 = 0$ on $[\mathcal{Y}(s_0), \mathcal{Y}(s_1)]$. Then, we can check that $\tilde{Z}$ given by

$$\tilde{t}(X,Y) = 0, \quad \tilde{x}(X,Y) = x(\bar{s}), \quad \tilde{U}(X,Y) = U(\bar{s})$$

and

$$\tilde{J}(X,Y) = J_1(X) + J_2(Y), \quad \tilde{K}(X,Y) = K_1(X) + K_2(Y)$$

is a solution to (2.13) in Ω (that is, $\tilde{Z} \in \mathcal{H}(\Omega)$) and $\tilde{Z} \bullet (\mathcal{X}, \mathcal{Y}) = (\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{V}, \mathcal{W})$. By uniqueness of the solution, we get $\tilde{Z} = Z$. In particular, we have $t(\bar{X}, \bar{Y}) = 0$ such that $\bar{X} + \bar{Y} = 2s$ and $\bar{X} < \mathcal{X}(\bar{s})$, which contradicts the definition of $\mathcal{X}(s)$ given by (5.1). This concludes the proof of the claim (5.12). Since $x_1(\mathcal{X}(s)) = x_2(2s - \mathcal{X}(s))$ we get, by (3.27), that

$$\bar{\mathcal{X}}(s) \leq \mathcal{X}(s).$$

We have by the continuity of x_1 and x_2 that

$$x_1(\bar{\mathcal{X}}(s)) = x_2(\bar{\mathcal{Y}}(s)). \tag{5.13}$$

Let us assume that $\bar{\mathcal{X}}(s) < \mathcal{X}(s)$, then, by the claim (5.12) we have proved, we have either $x_1(\bar{\mathcal{X}}(s)) < x_1(\mathcal{X}(s))$ or $x_2(\bar{\mathcal{Y}}(s)) > x_2(\mathcal{Y}(s))$. If $x_1(\bar{\mathcal{X}}(s)) < x_1(\mathcal{X}(s))$, then, as $\bar{\mathcal{Y}}(s) > \mathcal{Y}(s)$

$$x_1(\bar{\mathcal{X}}(s)) < x_1(\mathcal{X}(s)) = x_2(\mathcal{Y}(s)) \leq x_2(\bar{\mathcal{Y}}(s))$$

which contradicts (5.13). Similarly, we check that if $x_2(\bar{\mathcal{Y}}(s)) > x_2(\mathcal{Y}(s))$ then we obtain a contradiction to (5.13). Hence, $\bar{\mathcal{X}} = \mathcal{X}$ and therefore $\bar{\mathcal{Y}} = \mathcal{Y}$. Then, $\bar{x}(s) = x_2(\bar{\mathcal{X}}(s)) = x_2(\mathcal{X}(s)) = x(s)$ and similarly, we treat the other components of $\bar{\mathcal{Z}}$. From the definitions of $\mathbf{C}$ and $\mathbf{D}$, we have that $\bar{\mathcal{V}} = \mathcal{V}$ and $\bar{\mathcal{W}} = \mathcal{W}$. Hence, we have proved that $\mathbf{C} \circ \mathbf{D} \circ \mathbf{E} = \mathbf{E}$. The fact that $\mathbf{D} \circ \mathbf{C} = \mathrm{Id}$ follows directly from the definitions of $\mathbf{C}$ and $\mathbf{D}$.

Step 2. We prove (5.11). Given $(\psi_1, \psi_2) \in \mathcal{F}$, let $(\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{V}, \mathcal{W}) = \mathbf{C}(\psi_1, \psi_2)$, $Z = \mathbf{S}(\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{V}, \mathcal{W})$ and $(\bar{\mathcal{X}}, \bar{\mathcal{Y}}, \bar{\mathcal{Z}}, \bar{\mathcal{V}}, \bar{\mathcal{W}}) = \mathbf{E}(Z)$. We have to prove that $\bar{\mathcal{X}} = \mathcal{X}$, the rest will easily follow. Since $Z \in \mathcal{H}$ is a solution with data $(\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{V}, \mathcal{W}) \in \mathcal{G}_0$, we have $t(\mathcal{X}(s), \mathcal{Y}(s)) = 0$. Hence, from the definition of $\mathbf{E}$, we get $\bar{\mathcal{X}}(s) \leq \mathcal{X}(s)$. Assume that there exists $s \in \mathbb{R}$ such that $\bar{\mathcal{X}}(s) < \mathcal{X}(s)$. By the definition of $\mathbf{E}$, we have $t(\bar{\mathcal{X}}(s), \bar{\mathcal{Y}}(s)) = 0$. Let

$$s_0 = \mathcal{X}^{-1}(\bar{\mathcal{X}}(s)) \quad \text{and} \quad s_1 = \mathcal{Y}^{-1}(\bar{\mathcal{Y}}(s)).$$

We have

$$\mathcal{X}(s_0) = \bar{\mathcal{X}}(s) < \mathcal{X}(s) \quad \text{and} \quad \mathcal{Y}(s_1) = \bar{\mathcal{Y}}(s) > \mathcal{Y}(s),$$

and therefore $s_0 < s < s_1$. Due to the monotonicity of $t(X,Y)$ (that is, $t_X \geq 0$ and $t_Y \leq 0$), since $t(\mathcal{X}(s_0), \mathcal{Y}(s_0)) = t(\bar{\mathcal{X}}(s), \bar{\mathcal{Y}}(s)) = t(\mathcal{X}(s_1), \mathcal{Y}(s_1)) = 0$, we get that $t(X,Y) = 0$ on the rectangle $\Omega = [\mathcal{X}(s_0), \mathcal{X}(s_1)] \times [\mathcal{Y}(s_0), \mathcal{Y}(s_1)]$. It implies that $x_X = c(U)t_X = 0$ and $x_Y = c(U)t_Y = 0$ on Ω. Hence, the function $x(X,Y)$ is constant on Ω and we have

$$x_1(\bar{\mathcal{X}}(s)) = x_1(\mathcal{X}(s_0)) = x(\mathcal{X}(s_0), \mathcal{Y}(s_0)) = x(\mathcal{X}(s_1), \mathcal{Y}(s_1)) = x_2(\mathcal{Y}(s_1)) = x_2(\bar{\mathcal{Y}}(s)).$$

However, the fact that $x_1(\bar{\mathcal{X}}(s)) = x_2(\bar{\mathcal{Y}}(s))$ and $\bar{\mathcal{X}}(s) < \mathcal{X}(s)$ contradicts the definition of $\mathcal{X}$ in (3.22), and therefore we have proved that $\bar{X} = \mathcal{X}$. Then, $\bar{\mathcal{Y}} = \mathcal{Y}$ and

$$\bar{\mathcal{Z}}(s) = Z(\bar{\mathcal{X}}(s), \bar{\mathcal{Y}}(s)) = Z(\mathcal{X}(s), \mathcal{Y}(s)) = \mathcal{Z}(s).$$

Similarly, one proves that $\bar{\mathcal{V}} = \mathcal{V}$ and $\bar{\mathcal{W}} = \mathcal{W}$. Thus we have proved that $\mathbf{E} \circ \mathbf{S} \circ \mathbf{C} = \mathbf{C}$. The fact that $\mathbf{S} \circ \mathbf{E} = \mathrm{Id}$ follows from the uniqueness of the solution for a given data. □

Definition 5.4. *For any $T \geq 0$, we define the mapping $S_T : \mathcal{F} \to \mathcal{F}$ by*

$$S_T = \mathbf{D} \circ \mathbf{E} \circ \mathbf{t}_T \circ \mathbf{S} \circ \mathbf{C}. \tag{5.14}$$

Theorem 5.5. *The mapping S_T is a semigroup.*

Proof. We have

$$\begin{aligned} S_T \circ S_{T'} &= \mathbf{D} \circ \mathbf{E} \circ \mathbf{t}_T \circ \mathbf{S} \circ \mathbf{C} \circ \mathbf{D} \circ \mathbf{E} \circ \mathbf{t}_{T'} \circ \mathbf{S} \circ \mathbf{C} \\ &= \mathbf{D} \circ \mathbf{E} \circ \mathbf{t}_T \circ \mathbf{t}_{T'} \circ \mathbf{S} \circ \mathbf{C} && \text{(by Lemma 5.3)} \\ &= \mathbf{D} \circ \mathbf{E} \circ \mathbf{t}_{T+T'} \circ \mathbf{S} \circ \mathbf{C} && \text{(by (5.8))} \\ &= S_{T+T'}. \end{aligned}$$

□

6. Returning to the original variables

Definition 6.1. *Given $\psi = (\psi_1, \psi_2) \in \mathcal{F}$, we define $(u, R, S, \mu, \nu) \in \mathcal{D}$ as*

$$u(x) = U_1(X) \text{ if } x_1(X) = x \tag{6.1a}$$

or, equivalently,

$$u(x) = U_2(X) \text{ if } x_2(X) = x \tag{6.1b}$$

and[1]

$$\mu = (x_1)_\#(J_1'(X)\,dX), \tag{6.1c}$$

$$\nu = (x_2)_\#(J_2'(Y)\,dY), \tag{6.1d}$$

$$R(x)\,dx = (x_1)_\# \left(2c(U_1(X))V_1(X)\,dX\right), \tag{6.1e}$$

$$S(x)\,dx = (x_2)_\# \left(-2c(U_2(Y))V_2(Y)\,dY\right). \tag{6.1f}$$

The relations (6.1e) *and* (6.1f) *are equivalent to*

$$R(x_1(X))x_1'(X) = 2c(U_1(X))V_1(X) \tag{6.2a}$$

and

$$S(x_2(Y))x_2'(Y) = 2c(U_2(Y))V_2(Y) \tag{6.2b}$$

for a.e. X and Y. We denote by $\mathbf{M} : \mathcal{F} \to \mathcal{D}$ the mapping that to any $\psi \in \mathcal{F}$ associates $(u, R, S, \mu, \nu) \in \mathcal{F}$ as defined above.

We have to prove that the measures $(x_1)_\# \left(c(U_1(X))V_1(X)\,dX\right)$ and $(x_2)_\# \left(-c(U_2(Y))V_2(Y)\,dY\right)$ are absolutely continuous with respect to the Lebesgue measure and that (u, R, S, μ, ν) belongs to $\mathcal{D}$ so that the definition is well-posed. It will be done in the proof of the following lemma where an equivalent definition of the mapping $\mathbf{M}$ is given.

[1] The push-forward of a measure λ by a function f is the measure $f_\#\lambda$ defined by $f_\#\lambda(B) = \lambda(f^{-1}(B))$ for Borel sets B.

Lemma 6.2. *Given $\psi = (\psi_1, \psi_2) \in \mathcal{F}$, let $(u, R, S, \mu, \nu) = \mathbf{M}(\psi_1, \psi_2)$ as defined in Definition 6.1. Then, for any $\Theta = (\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{V}, \mathcal{W}) \in \mathcal{G}_0$ such that $(\psi_1, \psi_1) = \mathbf{D}\Theta$, we have*

$$u(\bar{x}) = U(s) \text{ if } \bar{x} = x(s) \tag{6.3a}$$

and

$$\mu = x_{\#}(\mathcal{V}_4(\mathcal{X}(s))\dot{\mathcal{X}}(s)\,ds), \tag{6.3b}$$

$$\nu = x_{\#}(\mathcal{W}_4(\mathcal{Y}(s))\dot{\mathcal{Y}}(s)\,ds), \tag{6.3c}$$

$$R(x)\,dx = x_{\#}\left(2c(U(s))\mathcal{V}_3(\mathcal{X}(s))\dot{\mathcal{X}}(s)\,ds\right), \tag{6.3d}$$

$$S(x)\,dx = x_{\#}\left(-2c(U(s))\mathcal{W}_3(\mathcal{Y}(s))\dot{\mathcal{Y}}(s)\,ds\right). \tag{6.3e}$$

The relations (6.3d) *and* (6.3e) *are equivalent to*

$$R(x(s))\mathcal{V}_2(\mathcal{X}(s)) = c(U(s))\mathcal{V}_3(\mathcal{X}(s)) \text{ for any } s \text{ such that } \dot{\mathcal{X}}(s) > 0 \tag{6.4a}$$

and

$$S(x(s))\mathcal{W}_2(\mathcal{X}(s)) = -c(U(s))\mathcal{W}_3(\mathcal{X}(s)) \text{ for any } s \text{ such that } \dot{\mathcal{Y}}(s) > 0, \tag{6.4b}$$

respectively.

Proof. We decompose the proof into 5 steps.

Step 1. We prove that (6.1) imply (6.3). If $\bar{x} = x_1(X)$, let $s = \mathcal{X}^{-1}(X)$. Then, we have $\bar{x} = x_1(\mathcal{X}(s)) = x(s)$ and $U_1(X) = U_1(\mathcal{X}(s)) = U(s)$. Hence, (6.1a) implies (6.3a). For any measurable set A, we have

$$\begin{aligned}
\mu(A) &= \int_{x_1^{-1}(A)} J_1'(X)\,dX \\
&= \int_{(x_1\circ\mathcal{X})^{-1}(A)} J_1'(\mathcal{X}(s))\dot{\mathcal{X}}(s)\,ds && \text{(after a change of variables)} \\
&= \int_{x^{-1}(A)} \mathcal{V}_4(\mathcal{X}(s))\dot{\mathcal{X}}(s)\,ds && \text{(by the definition of } \mathbf{D})
\end{aligned}$$

and therefore $\mu = x_{\#}(\mathcal{V}_4(\mathcal{X}(s))\dot{\mathcal{X}}(s)\,ds)$. One proves in the same way the other identities in (6.3).

Step 2. We prove that u is a well-defined function L^2 that is Hölder continuous with exponent 1/2. Given $\bar{x}$ such that $\bar{x} = x(s_0) = x(s_1)$. Since x is nondecreasing, it implies that $\dot{x} = \mathcal{V}_2(\mathcal{X})\dot{\mathcal{X}} + \mathcal{W}_2(\mathcal{Y})\dot{\mathcal{Y}} = 0$ in $[s_0, s_1]$. Hence, $\mathcal{V}_2(\mathcal{X})\dot{\mathcal{X}} = \mathcal{W}_2(\mathcal{Y})\dot{\mathcal{Y}} = 0$ as both quantities are positive. By (3.8a), it implies that $\mathcal{V}_3(\mathcal{X})\dot{\mathcal{X}} = \mathcal{W}_3(\mathcal{Y})\dot{\mathcal{Y}} = 0$ and therefore

$$\dot{U} = \mathcal{V}_3(\mathcal{X})\dot{\mathcal{X}} + \mathcal{W}_3(\mathcal{Y})\dot{\mathcal{Y}} = 0$$

in $[s_0, s_1]$ and $U(s_0) = U(s_1)$. The definition of u is therefore well-posed. We have

$$\int_{\mathbb{R}} u^2(x)\,dx = \int_{\mathbb{R}} u^2(x(s))\dot{x}(s)\,ds \le \|U\|_{L^2}^2$$

and $u \in L^2(\mathbb{R})$. We have

$$u(x(s)) - u(x(\bar{s})) = \int_{\bar{s}}^{s} \dot{U}(s)\,ds = \int_{\bar{s}}^{s} (\mathcal{V}_3(\mathcal{X})\dot{\mathcal{X}} + \mathcal{W}_3(\mathcal{Y})\dot{\mathcal{Y}})\,ds. \tag{6.5}$$

Since $\Theta \in \mathcal{G}_0$, we have $t(s) = 0$ which implies that $\mathcal{V}_1(\mathcal{X})\dot{\mathcal{X}} = \mathcal{W}_1(\mathcal{Y})\dot{\mathcal{Y}}$ and, therefore, $\mathcal{V}_2(\mathcal{X})\dot{\mathcal{X}} = \mathcal{W}_2(\mathcal{Y})\dot{\mathcal{Y}}$, by (3.8b), and

$$\dot{x} = 2\mathcal{V}_2(\mathcal{X})\dot{\mathcal{X}} = 2\mathcal{W}_2(\mathcal{Y})\dot{\mathcal{Y}}. \tag{6.6}$$

By using the Cauchy–Schwarz inequality and (3.8a), we get

$$\begin{aligned}
\int_{\bar{s}}^{s} |\mathcal{V}_3(\mathcal{X})|\,\dot{\mathcal{X}}\,ds &\le \left(\int_{\bar{s}}^{s} \mathcal{V}_3^2(\mathcal{X})\dot{\mathcal{X}}\,ds\right)^{1/2} \left(\int_{\bar{s}}^{s} \dot{\mathcal{X}}\,ds\right)^{1/2} \\
&\le C\left(\int_{\bar{s}}^{s} \mathcal{V}_2^2(\mathcal{X})\dot{\mathcal{X}}\,ds\right)^{1/2} \\
&\le C\left(\int_{\bar{s}}^{s} \dot{x}\,ds\right)^{1/2} = C(x(s) - x(\bar{s}))^{1/2} \qquad (6.7)
\end{aligned}$$

where the constant C depends on $|||\Theta|||_{\mathcal{G}}$ and $|\mathcal{X}(s) - \mathcal{X}(\bar{s})|$. Similarly, one proves that $\int_{\bar{s}}^{s} |\mathcal{W}_3(\mathcal{Y})|\,\dot{\mathcal{Y}}\,ds \le C(x(s) - x(\bar{s}))^{1/2}$. Hence, (6.5) implies that u is locally Hölder continuous with exponent $1/2$.

Step 3. We show that the measures $x_{\#}\left(c(U(s))\mathcal{V}_3(\mathcal{X}(s))\dot{\mathcal{X}}(s)\,ds\right)$ and $x_{\#}\left(-c(U(s))\mathcal{W}_3(\mathcal{Y}(s))\dot{\mathcal{Y}}(s)\,ds\right)$ are absolutely continuous and (6.4a) and (6.4b) hold. The inequality (6.7) proves that the measure $\mathcal{V}_3(\mathcal{X})\dot{\mathcal{X}}\,ds$ is absolutely continuous with respect to $\dot{x}\,ds$. For any set A of zero measure, we have $\int_{x^{-1}(A)} \dot{x}\,ds = 0$ and therefore $\int_{x^{-1}(A)} \mathcal{V}_3(\mathcal{X})\dot{\mathcal{X}}\,ds = 0$. It follows that

$$x_{\#}\left(2c(U(s))\mathcal{V}_3(\mathcal{X}(s))\dot{\mathcal{X}}(s)\,ds\right)(A) = \int_{x^{-1}(A)} 2c(U)\mathcal{V}_3(\mathcal{X})\dot{\mathcal{X}}\,ds = 0$$

and the measure $x_{\#}\left(2c(U(s))\mathcal{V}_3(\mathcal{X}(s))\dot{\mathcal{X}}(s)\,ds\right)$ is absolutely continuous. In the same way, one proves that $x_{\#}\left(-2c(U(s))\mathcal{W}_3(\mathcal{Y}(s))\dot{\mathcal{Y}}(s)\,ds\right)$ is absolutely continuous and $R(x)$ and $S(x)$ as given by (6.3d) and (6.3e) are well-defined. We have

$$\int_{x^{-1}(A)} R(x(s))\dot{x}(s)\,ds = \int_{x^{-1}(A)} 2c(U(s))\mathcal{V}_3(\mathcal{X}(s))\dot{\mathcal{X}}(s)\,ds \tag{6.8}$$

for any measurable set A. For any measurable set B, we have the decomposition $x^{-1}(x(B)) = B \cup (B^c \cap x^{-1}(x(B)))$. Let prove that the set $B^c \cap x^{-1}(x(B))$ has measure zero with respect to $\dot{x}(s)\,ds$. We consider a point $\bar{s} \in B^c \cap x^{-1}(x(B))$, there exists $s \in B$ such that $x(\bar{s}) = x(s)$. Since x is increasing, it implies that $\dot{x}(\bar{s}) = 0$ and therefore $\int_{B^c \cap x^{-1}(x(B))} \dot{x}(s)\,ds = 0$ so that the set $B^c \cap x^{-1}(x(B))$ has zero measure with respect to $\dot{x}(s)\,ds$. Since $\mathcal{V}_3(\mathcal{X}(s))\dot{\mathcal{X}}(s)\,ds$ is absolutely continuous with respect to $\dot{x}(s)\,ds$, it implies that $\int_{B^c \cap x^{-1}(x(B))} \mathcal{V}_3(\mathcal{X}(s))\dot{\mathcal{X}}(s)\,ds = 0$. Hence, taking $A = x(B)$ in (6.8), we obtain

$$\int_B R(x(s))\dot{x}(s)\,ds = \int_B 2c(U(s))\mathcal{V}_3(\mathcal{X}(s))\dot{\mathcal{X}}(s)\,ds$$

for any Borel set B. Hence,

$$R(x(s))\dot{x}(s) = 2c(U(s))\mathcal{V}_3(\mathcal{X}(s))\dot{\mathcal{X}}(s) \tag{6.9}$$

which yields

$$R(x(s))\mathcal{V}_2(\mathcal{X}(s)) = c(U(s))\mathcal{V}_3(\mathcal{X}(s)) \tag{6.10}$$

after simplifying by $\dot{\mathcal{X}}(s)$. Similarly, we obtain (6.4b).

Step 4. We show that R and S belong to L^2 and $u_x = \frac{R-S}{2c}$. We have

$$\begin{aligned}
\int_{\mathbb{R}} R^2(x)\,dx &= \int_{\mathbb{R}} R^2(x(s))\dot{x}\,ds \\
&= 2\int_{\mathbb{R}} R^2(x(s))\mathcal{V}_2(\mathcal{X}(s))\dot{\mathcal{X}}\,ds \\
&= 2\int_{\{s\in\mathbb{R}\mid\mathcal{V}_2(\mathcal{X}(s))>0\}} \frac{(R(x(s))\mathcal{V}_2(\mathcal{X}(s)))^2}{\mathcal{V}_2(\mathcal{X}(s))}\dot{\mathcal{X}}\,ds \\
&\leq 4\int_{\mathbb{R}} \mathcal{V}_4(\mathcal{X})\dot{\mathcal{X}}\,ds \qquad \text{(by (6.4a) and (3.8a))} \\
&\leq 4\left\|J\right\|_{L^\infty(\mathbb{R})} \leq 4|||\Theta|||_{\mathcal{G}}
\end{aligned}$$

because $\dot{J}(s) = \mathcal{V}_4(\mathcal{X})\dot{\mathcal{X}} + \mathcal{W}_4(\mathcal{Y})\dot{\mathcal{Y}}$ and $\mathcal{V}_4$ and $\mathcal{W}_4$ are positive. Hence $R \in L^2$ and, similarly, one proves that $S \in L^2$. For any smooth function ϕ with compact support, we have

$$\int_{\mathbb{R}} u\phi_x\,dx = \int_{\mathbb{R}} u(x(s))\phi_x(x(s))\dot{x}(s)\,ds = \int_{\mathbb{R}} U(s)\phi(x(s))_s\,ds.$$

After integrating by parts, it yields

$$\begin{aligned}
\int_{\mathbb{R}} u\phi_x\,dx = -\int_{\mathbb{R}} \dot{U}(s)\phi(x(s))\,ds &= \int_{\mathbb{R}} (\mathcal{V}_3(\mathcal{X})\dot{\mathcal{X}} + \mathcal{W}_3(\mathcal{Y})\dot{\mathcal{Y}})\phi(x(s))\,ds \\
&= \int_{\mathbb{R}} \frac{1}{c(U)}(R(x(s))\mathcal{V}_2(\mathcal{X})\dot{\mathcal{X}} - S(x(s))\mathcal{W}_2(\mathcal{Y})\dot{\mathcal{Y}})\phi(x(s))\,ds \\
&= \int_{\mathbb{R}} \frac{1}{2c(U)}(R(x(s)) - S(x(s))\phi(x(s))\dot{x}\,ds \qquad \text{(by (6.6))} \\
&= \int_{\mathbb{R}} \frac{R-S}{2c(u)}\phi\,dx,
\end{aligned}$$

after a change of variables. Hence, $u_x = \frac{R-S}{2c(u)}$ in the sense of distribution.

Step 5. We show that $\mu_{\text{ac}} = \frac{1}{4}R^2 dx$ and $\nu_{\text{ac}} = \frac{1}{4}S^2 dx$. Let

$$A = \{s \mid \mathcal{V}_2(\mathcal{X}(s)) > 0\} \quad \text{and } B = (x(A^c))^c. \tag{6.11}$$

We have

$$\operatorname{meas}(B^c) = \int_{A^c} \dot{x}\,ds = 0$$

because $\dot{x} = 0$ almost everywhere on A^c, by (6.6). The set B has therefore full measure. We have $x^{-1}(B) \subset A$. Indeed, for any $s \in x^{-1}(B)$, we have $x(s) \neq x(\bar{s})$ for all $\bar{s} \in A^c$. For any measurable subset $B' \subset B$, we have, by definition of μ, that

$$\mu(B') = \int_{x^{-1}(B')} \mathcal{V}_4(\mathcal{X}(s))\dot{\mathcal{X}}(s)\,ds. \tag{6.12}$$

Hence, since $x^{-1}(B') \subset A$,

$$\begin{aligned}
\mu(B') &= \int_{x^{-1}(B')} \frac{\mathcal{V}_4(\mathcal{X})\mathcal{V}_2(\mathcal{X})}{\mathcal{V}_2(\mathcal{X})}\dot{\mathcal{X}}\,ds \\
&= \int_{x^{-1}(B')} \frac{c^2(U)\mathcal{V}_3^2(\mathcal{X})}{2\mathcal{V}_2(\mathcal{X})}\dot{\mathcal{X}}\,ds && \text{(by (3.8a))} \\
&= \frac{1}{4}\int_{x^{-1}(B')} R^2(x(s))\dot{x}(s)\,ds && \text{(by (6.4a) and (6.6))} \\
&= \frac{1}{4}\int_{B'} R^2\,dx
\end{aligned}$$

and therefore $\mu_{\rm ac} = \frac{1}{4}R^2\,dx$. Similarly, one proves that $\nu_{\rm ac} = \frac{1}{4}S^2\,dx$. □

Lemma 6.3. *Given $(u_0, R_0, S_0, \mu_0, \nu_0) \in \mathcal{D}$, let us denote $(u, R, S, \mu, \nu)(T) = \mathbf{M} \circ S_T \circ \mathbf{L}(u_0, R_0, S_0, \mu_0, \nu_0)$ and $Z = \mathbf{S} \circ \mathbf{L}(u_0, R_0, S_0, \mu_0, \nu_0)$. Then, we have*

$$u(t(X,Y), x(X,Y)) = U(X,Y) \tag{6.13}$$

for all $(X,Y) \in \mathbb{R}^2$, and

$$R(t(X,Y), x(X,Y))x_X(X,Y) = c(U(X,Y))U_X(X,Y), \tag{6.14a}$$

$$S(t(X,Y), x(X,Y))x_Y(X,Y) = -c(U(X,Y))U_Y(X,Y), \tag{6.14b}$$

for almost all $(X,Y) \in \mathbb{R}^2$ such that $x_X(X,Y) > 0$ and $x_Y(X,Y) > 0$. We have

$$u_t = \frac{1}{2}(R+S) \quad \text{and} \quad u_x = \frac{1}{2c(u)}(R-S) \tag{6.15}$$

in the sense of distributions.

Proof. We consider a solution $Z \in \mathcal{H}$. Given $(X,Y) \in \mathbb{R}^2$, let us denote $\bar{t} = t(X,Y)$ and $\bar{x} = x(X,Y)$. Let $(\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{V}, \mathcal{W}) = \mathbf{E} \circ \mathbf{t}_{\bar{t}}(Z)$. By definition, we have $t(\mathcal{X}(s), \mathcal{Y}(s)) = \bar{t}$, and, slightly abusing the notation, $x(s) = \mathcal{Z}_2(s) = x(\mathcal{X}(s), \mathcal{Y}(s))$ and $U(s) = \mathcal{Z}_3(s) = U(\mathcal{X}(s), \mathcal{Y}(s))$ for all $s \in \mathbb{R}$. By Lemma 6.2, we have $u(\bar{t}, \bar{x}) = U(s)$ for any s such that $x(s) = \bar{x}$. It implies that, for any $\bar{s}$ such that

$$t(\mathcal{X}(\bar{s}), \mathcal{Y}(\bar{s})) = \bar{t} = t(X,Y) \tag{6.16a}$$

and

$$x(\mathcal{X}(\bar{s}), \mathcal{Y}(\bar{s})) = \bar{x} = x(X,Y), \tag{6.16b}$$

we have

$$u(\bar{t}, \bar{x}) = U(\mathcal{X}(\bar{s}), \mathcal{Y}(\bar{s})).$$

Then, (6.13) will be proved once we have proved that

$$U(\mathcal{X}(\bar{s}), \mathcal{Y}(\bar{s})) = U(X,Y). \tag{6.17}$$

Let us prove that when (6.16) hold, then either $(X,Y)=(\mathcal{X}(\bar s),\mathcal{Y}(\bar s))$ or

$$x_X=x_Y=U_X=U_Y=0, \tag{6.18}$$

in the rectangle with corners at (X,Y) and $(\mathcal{X}(\bar s),\mathcal{Y}(\bar s))$, so that (6.17) holds in both cases. We consider first the case where $\mathcal{X}(\bar s)\leq X$ and $\mathcal{Y}(\bar s)\leq Y$. Since the function x is increasing in the X and Y directions, we must have $x_X=x_Y=0$ in the rectangle $[\mathcal{X}(\bar s),X]\times[\mathcal{Y}(\bar s),Y]$ and, by (4.18c), $U_X=U_Y=0$ in the same rectangle so that U is constant and we have proved (6.17). In the case where $\mathcal{X}(\bar s)\leq X$ and $\mathcal{Y}(\bar s)\geq Y$, since the function t is increasing in the X direction and decreasing in the Y direction, it follows that $t_X=t_Y=0$ in the rectangle $[\mathcal{X}(\bar s),X]\times[Y,\mathcal{Y}(\bar s)]$. Hence, $x_X=x_Y=0$ and, as before, we prove (6.17). The other cases can be treated in the same way and this concludes the proof of (6.17) and therefore (6.13) holds. Let us prove (6.14a). By (6.4a) and the definition of $\mathbf{E}$, we get

$$R(\bar t,x(s))x_X(\mathcal{X}(s),\mathcal{Y}(s))=c(U(\mathcal{X}(s),\mathcal{Y}(s)))U_X(\mathcal{X}(s),\mathcal{Y}(s))$$

so that

$$R(t(X,Y),x(X,Y))x_X(\mathcal{X}(\bar s),\mathcal{Y}(\bar s))=c(U(\mathcal{X}(\bar s),\mathcal{Y}(\bar s)))U_X(\mathcal{X}(\bar s),\mathcal{Y}(\bar s)) \tag{6.19}$$

for any $\bar s$ such that (6.16) holds. We have proved that when (6.16) is satisfied, then either $(X,Y)=(\mathcal{X}(\bar s),\mathcal{Y}(\bar s))$ or (6.18) holds. Hence, (6.14a) follows from (6.19). Similarly, one proves (6.14b). For any smooth function $\phi(t,x)$ with compact support, we have

$$\begin{aligned}\int_{\mathbb{R}^2}u(t,x)\phi_t(t,x)=\\ \int_{\mathbb{R}^2}u(t(X,Y),x(X,Y))\phi_t(t(X,Y),x(X,Y))(t_Xx_Y-x_Xt_Y)\,dXdY\end{aligned} \tag{6.20}$$

where we have used (4.18a) and (6.13). By differentiating the function $\phi(t(X,Y),x(X,Y))$ with respect to X and Y, we get that

$$\phi(t(X,Y),x(X,Y))_Xx_Y-\phi(t(X,Y),x(X,Y))_Yx_X=\phi_t(t(X,Y),x(X,Y))(t_Xx_Y-t_Yx_X).$$

We insert this identity in (6.20) and obtain, after integrating by parts,

$$\begin{aligned}\int_{\mathbb{R}^2}u(t,x)\phi_t(t,x)\,dtdx&=-\int_{\mathbb{R}^2}((Ux_Y)_X-(Ux_X)_Y)(X,Y)\phi(t(X,Y),x(X,Y))\,dXdY\\&=-\int_{\mathbb{R}^2}((U_Xx_Y-U_Yx_X)(X,Y)\phi(t(X,Y),x(X,Y))\,dXdY.\end{aligned}$$

We use (6.14) and get

$$\begin{aligned}\int_{\mathbb{R}^2}u(t,x)\phi_t(t,x)\,dtdx&=-\int_{\mathbb{R}^2}\Big(\big(\frac{R+S}{c(u)}\phi\big)\circ(t,x)\,x_Xx_Y\,dXdY\\&=-\int_{\mathbb{R}^2}\big(\frac{1}{2}(R+S)2\phi\big)\circ(t,x)\,(t_Xx_Y-tx_Yx_X)\,dXdY\\&=-\int_{\mathbb{R}^2}\frac{1}{2}(R+S)(t,x)\phi(t,x)\,dtdx.\end{aligned}$$

This proves the first identity in (6.15); the second one is proven in the same way. □

We can now define the semigroup mapping $\bar S_T$ on $\mathcal{D}$, the original set of variables.

Definition 6.4. *For any $T > 0$, let $\bar{S}_T : \mathcal{D} \to \mathcal{D}$ be defined as*

$$\bar{S}_T = \mathbf{M} \circ S_T \circ \mathbf{L}$$

Given $(u_0, R_0, S_0, \mu_0, \nu_0) \in \mathcal{D}$, let us denote $(u, R, S, \mu, \nu)(t) = \bar{S}_t(u_0, R_0, S_0, \mu_0, \nu_0)$. In the theorem that follows, we prove that $u(t, x)$ is a weak solution of the nonlinear wave equation. However it is not clear if $\bar{S}_T$ is a semigroup. Indeed, we have

$$\bar{S}_T \circ \bar{S}_{T'} = \mathbf{M} \circ S_T \circ \mathbf{L} \circ \mathbf{M} \circ S_{T'} \circ \mathbf{L}.$$

By the semigroup property of S_T, it would follow immediately that $\bar{S}_T$ is also a semigroup if we had $\mathbf{L} \circ \mathbf{M} = \mathrm{Id}$, but this identity does not hold in general. It is the aim of the last section to show that $\bar{S}_T$ is a semigroup.

7. Relabeling symmetry

We consider the set of transformations of the $\mathbb{R}^2$-plane given by

$$(X, Y) \mapsto (f(X), g(Y))$$

for any $(f, g) \in G^2$, where G is the group of diffeomorphisms on the line, see Definition 3.3. It is a subgroup of the group of diffeomorphisms of $\mathbb{R}^2$. Such transformations let the characteristics lines invariant. Indeed, vertical and horizontal lines, which correspond to the characteristics in our new sets of coordinates, remain vertical and horizontal lines through this mapping. In this section, we show that the subgroup G^2 plays an essential role by exactly capturing the degree of freedom we have introduced when changing coordinates and introduced the equivalent system (2.13). Given f and g in G, the $\mathbb{R}^2$ plane is stretched in the X and Y direction by the transformations $X \mapsto f(X)$ and $Y \mapsto g(Y)$. The solutions of (2.13) are preserved and we can define an action of G^2 on the set of solutions $\mathcal{H}$.

Definition 7.1. *For any $Z \in \mathcal{H}$, f and g in G, we define $\bar{Z} \in \mathcal{H}$ as*

$$\bar{Z}(X, Y) = Z(f(X), g(Y)). \tag{7.1}$$

The mapping from $\mathcal{H} \times G^2$ to $\mathcal{H}$ given by $Z \times (f, g) \mapsto \bar{Z}$ defines an action of the group G^2 on $\mathcal{H}$ and we denote $\bar{Z} = Z \cdot (f, g)$.

Proof of well-posedness of Definition 7.1. For any Ω, given $Z \in \mathcal{H}(\Omega)$ and $(f, g) \in G^2$, let $\bar{X}_l = f(X_l)$, $X_r = f(X_r)$, $\bar{Y}_l = g(Y_l)$, $\bar{Y}_r = g(Y_r)$ and

$$\bar{\Omega} = [\bar{X}_l, \bar{X}_r] \times [\bar{Y}_l, \bar{Y}_r].$$

Let us prove that $\bar{Z} \in \mathcal{H}(\bar{\Omega})$. We have

$$\bar{Z}_X(X, Y) = f'(X) Z_X(f(X), g(Y)), \quad \bar{Z}_Y(X, Y) = g'(Y) Z_X(f(X), g(Y)).$$

and

$$\bar{Z}_{XY}(X, Y) = f'(X) g'(Y) Z_{XY}(f(X), g(Y)).$$

By using the linearity of the mapping $F(Z)$ in (2.14), we get

$$\begin{aligned}
\bar{Z}_{XY} &= f' g' Z_{XY}(f, g) \\
&= f' g' F(Z(f, g))\big(Z_X(f, g), Z_Y(f, g)\big) \\
&= F(Z(f, g))\big(f' Z_X(f, g), g' Z_Y(f, g)\big) \qquad \text{(by the linearity of } F(Z))
\end{aligned}$$

$$= F(\bar{Z})(\bar{Z}_X, \bar{Z}_Y)$$

and $\bar{Z}$ is a solution of (2.13). Since f and g belong to G, there exists $\delta > 0$ such that $f'(X) > \delta$ for a.e. $X \in \mathbb{R}$ and $g'(Y) > \delta$ for a.e. $Y \in \mathbb{R}$, see Lemma 3.6. We have to check that $\bar{Z}$ fulfills (4.18). It is not difficult to do so once one has observed that the equalities and inequalities in (4.18) enjoy the required homogeneity properties. For example, we have

$$2\bar{J}_X\bar{x}_X = 2f'^2 J_X(f,g)x_X(f,g) = f'^2(c(U(f,g))U_X(f,g))^2 = (c(\bar{U})\bar{U}_X)^2$$

and

$$\bar{x}_X = f' x_X(f,g) \geq 0.$$

We will prove that $\bar{Z}$ fulfills the condition (ii) in Definition 4.13 after we have introduced the action of G^2 on $\mathcal{G}$. □

We can define an action on $\mathcal{C}$ as follows. This action corresponds to a stretching of the curve in the X and Y direction.

Definition 7.2. *Given $(\mathcal{X}, \mathcal{Y}) \in \mathcal{C}$, we define $(\bar{\mathcal{X}}, \bar{\mathcal{Y}}) \in \mathcal{C}$ such that*

$$\bar{\mathcal{X}} = f^{-1} \circ \mathcal{X} \circ h \qquad \bar{\mathcal{Y}} = g^{-1} \circ \mathcal{Y} \circ h \tag{7.2}$$

where h is the re-normalizing function which yields $\bar{\mathcal{X}} + \bar{\mathcal{Y}} = 2\,\mathrm{Id}$, that is,

$$(f^{-1} \circ \mathcal{X} + g^{-1} \circ \mathcal{Y}) \circ h = 2\,\mathrm{Id}\,. \tag{7.3}$$

We denote $(\bar{\mathcal{X}}, \bar{\mathcal{Y}}) = (\mathcal{X}, \mathcal{Y}) \cdot (f, g)$.

Proof of wellposedness of Definition 7.2. Let us denote $v = f^{-1} \circ \mathcal{X} + g^{-1} \circ \mathcal{Y}$. We have $v - \mathrm{Id} \in W^{1,\infty}(\mathbb{R})$ because $f^{-1} - \mathrm{Id}$, $g^{-1} - \mathrm{Id}$, $\mathcal{X} - \mathrm{Id}$ and $\mathcal{Y} - \mathrm{Id}$ all belong to $W^{1,\infty}(\mathbb{R})$. There exists $\delta > 0$ such that $(f^{-1})' \geq \delta$ and $(g^{-1})' \geq \delta$ a.e. and therefore $\dot{v} \geq \delta(\dot{\mathcal{X}} + \dot{\mathcal{Y}}) = 2\delta$. Hence, by Lemma 3.6, we have that v is invertible so that h exists and $h - \mathrm{Id} \in W^{1,\infty}(\mathbb{R})$. One proves then easily that $(\bar{\mathcal{X}}, \bar{\mathcal{Y}}) \in \mathcal{C}$. □

We define the action on $\mathcal{G}$ so that it commutes with the $\bullet$ operation and the actions on $\mathcal{H}$ and $\mathcal{C}$, that is,

$$(Z \bullet \Gamma) \cdot \phi = (Z \cdot \phi) \bullet (\Gamma \cdot \phi) \tag{7.4}$$

for any $Z \in \mathcal{H}$, $\Gamma = (\mathcal{X}, \mathcal{Y}) \in \mathcal{C}$ and $\phi = (f, g) \in G^2$. We obtain the following definition.

Definition 7.3. *For any $\Theta = (\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{V}, \mathcal{W}) \in \mathcal{G}$ and $f, g \in G$, we define $\bar{\Theta} = (\bar{\mathcal{X}}, \bar{\mathcal{Y}}, \bar{\mathcal{Z}}, \bar{\mathcal{V}}, \bar{\mathcal{W}}) \in \mathcal{G}$ as follows*

$$(\bar{\mathcal{X}}, \bar{\mathcal{Y}}) = (\mathcal{X}, \mathcal{Y}) \cdot (f, g)$$

and

$$\bar{\mathcal{V}}(X) = f'(X)\mathcal{V}(f(X)) \qquad \bar{\mathcal{W}}(Y) = g'(Y)\mathcal{W}(g(Y)) \tag{7.5}$$

and

$$\bar{\mathcal{Z}} = \mathcal{Z} \circ h \tag{7.6}$$

where h is given by (7.3). The mapping from $\mathcal{G} \times G^2$ to $\mathcal{G}$ given by $\Theta \times (f, g) \mapsto \bar{\Theta}$ defines an action of the group G^2 on $\mathcal{G}$ that we denote $\bar{\Theta} = \Theta \cdot (f, g)$.

To check that this definition is well-posed, we have to check that $\bar{\Theta} \in \mathcal{G}$. This can be done without any special difficulty and we omit the details here. Let us however prove (7.4) in details as we will use it several times in the following. For any $Z \in \mathcal{H}$, $\Gamma = (\mathcal{X}, \mathcal{Y}) \in \mathcal{C}$ and $\phi = (f, g) \in G^2$, we denote $\Theta = (\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{V}, \mathcal{W}) = Z \bullet \Gamma$, $\bar{\Theta} = (\bar{\mathcal{X}}, \bar{\mathcal{Y}}, \bar{\mathcal{Z}}, \bar{\mathcal{V}}, \bar{\mathcal{W}}) = \Theta \cdot \phi$, $\bar{\Gamma} = (\bar{\mathcal{X}}, \bar{\mathcal{Y}}) = \Gamma \cdot \phi$ and $\tilde{\Theta} = \bar{Z} \cdot \bar{\Gamma}$. We want to prove that $\tilde{\Theta} = \bar{\Theta}$. We have $\mathcal{Z}(s) = Z(\mathcal{X}(s), \mathcal{Y}(s))$ and, by (7.6), $\bar{\mathcal{Z}}(s) = Z(\mathcal{X} \circ h(s), \mathcal{Y} \circ h(s))$. Hence, by (7.2),

$$\tilde{\mathcal{Z}} = \bar{Z}(\bar{\mathcal{X}}, \bar{\mathcal{Y}}) = Z(f \circ \bar{\mathcal{X}}, g \circ \bar{\mathcal{Y}}) = Z(\mathcal{X} \circ h, \mathcal{Y} \circ h) = \bar{\mathcal{Z}}.$$

We have $\mathcal{V}(\mathcal{X}(s)) = Z_X(\mathcal{X}(s), \mathcal{Y}(s))$ and, by (7.5),

$$\begin{aligned}\bar{\mathcal{V}}(\bar{\mathcal{X}}(s)) = f'(\bar{\mathcal{X}}(s))\mathcal{V}(f \circ \bar{\mathcal{X}}(s)) = f'(\bar{\mathcal{X}}(s))\mathcal{V}(\mathcal{X} \circ h(s))& \\ = f'(\bar{\mathcal{X}}(s)) Z_X(\mathcal{X} \circ h(s), \mathcal{Y} \circ h(s))&\end{aligned}$$

and

$$\begin{aligned}\tilde{\mathcal{V}}(\bar{\mathcal{X}}(s)) = \bar{Z}_X(\bar{\mathcal{X}}(s), \bar{\mathcal{Y}}(s)) = f'(\mathcal{X}(s)) Z_X(f \circ \bar{\mathcal{X}}(s), g \circ \bar{\mathcal{Y}}(s))& \\ = f'(\bar{\mathcal{X}}(s)) Z_X(\mathcal{X} \circ h(s), \mathcal{Y} \circ h(s)).&\end{aligned}$$

Hence, $\tilde{\mathcal{V}} = \bar{\mathcal{V}}$. Similarly one proves that $\tilde{\mathcal{W}} = \bar{\mathcal{W}}$, which concludes the proof of (7.4).

End of proof of well-posedness of Definition 7.1. For any $\phi \in G^2$ and $Z \in \mathcal{H}$, it remains to prove that $\bar{Z}$, as defined by (7.1), fulfills the condition (ii) in Definition 4.13. By this same condition, for any $Z \in \mathcal{H}$, there exists a curve $\Gamma = (\mathcal{X}, \mathcal{Y}) \in \mathcal{G}$ such that $Z \bullet \Gamma \in \mathcal{G}$. From (7.4), it follows that, for the curve $\bar{\Gamma} = \Gamma \cdot \phi$, we have $\bar{Z} \bullet \bar{\Gamma} \in \mathcal{G}$ because $(Z \bullet \Gamma) \cdot \phi \in \mathcal{G}$ and therefore $\bar{Z}$ fulfills the condition (ii) in Definition 4.13. □

Definition 7.4. *For any $\psi = (\psi_1, \psi_2) \in \mathcal{F}$ and $f, g \in G$, we define $\bar{\psi} = (\bar{\psi}_1, \bar{\psi}_2) \in \mathcal{F}$ as follows*

$$\begin{aligned} &\bar{x}_1(X) = x_1(f(X)), \qquad &\bar{U}_1(X) = U_1(f(X)), \qquad &\bar{J}_1(X) = J_1(f(X)),\\ &\bar{x}_2(Y) = x_2(g(Y)), \qquad &\bar{U}_2(Y) = U_2(g(Y)), \qquad &\bar{J}_2(Y) = J_2(g(Y)),\end{aligned}$$

and

$$\bar{V}_1(X) = V_1(f(X))f'(X), \qquad \bar{V}_2(Y) = V_2(g(Y))g'(Y).$$

The mapping from $\mathcal{F} \times G^2$ to $\mathcal{F}$ given by $\psi \times (f, g) \mapsto \bar{\psi}$ defines an action of the group G^2 on $\mathcal{F}$, and we denote

$$\bar{\psi} = \psi \cdot \phi.$$

Proof of well-posedness of Definition 7.4. We have to check that $\bar{\psi} = (\bar{\psi}_1, \bar{\psi}_2) \in \mathcal{F}$. We will only check that the identities (3.19) in the definition 3.4 of $\mathcal{F}$ are fulfilled, as the other properties can be checked without difficulty. For any curve $(\bar{\mathcal{X}}, \bar{\mathcal{Y}}) \in \mathcal{C}$ such that $\bar{x}_1(\bar{\mathcal{X}}) = \bar{x}_2(\bar{\mathcal{Y}})$, let $(\mathcal{X}, \mathcal{Y}) = (\bar{\mathcal{X}}, \bar{\mathcal{Y}}) \cdot (f, g)$. We have

$$x_1(\mathcal{X}(s)) = \bar{x}_1 \circ f^{-1} \circ \mathcal{X}(s) = \bar{x}_1 \circ \bar{\mathcal{X}} \circ h^{-1}(s) = \bar{x}_2 \circ \bar{\mathcal{Y}} \circ h^{-1}(s) = \bar{x}_2 \circ f^{-1} \circ \mathcal{Y}(s) = x_2(\mathcal{Y}(s))$$

and therefore, since $\psi \in \mathcal{F}$, $U_1(\mathcal{X}(s)) = U_2(\mathcal{Y}(s))$ for all $s \in \mathbb{R}$, which implies

$$\bar{U}_1(\bar{\mathcal{X}}) = U_1 \circ \mathcal{X} \circ h = U_2 \circ \mathcal{Y} \circ h = \bar{U}_2(\bar{\mathcal{Y}})$$

and this proves (3.19a) for $\bar{\psi}$. Similarly, one proves that (3.19b) holds for $\bar{\psi}$. □

In the following lemma, we show that all the mappings given in (5.9) are equivariant with respect to the action of the group G^2.

Lemma 7.5. *The mappings* $\mathbf{E}$, $\mathbf{t}_T$, $\mathbf{C}$, $\mathbf{D}$ *and* $\mathbf{C}$ *are* G^2*-equivariant, that is, for all* $\phi = (f, g) \in G^2$,

$$\mathbf{E}(Z \cdot \phi) = \mathbf{E}(Z) \cdot \phi, \tag{7.7a}$$

$$\mathbf{t}_T(Z \cdot \phi) = \mathbf{t}_T(Z) \cdot \phi \tag{7.7b}$$

for all $Z \in \mathcal{H}$ *and*

$$\mathbf{S}(\Theta \cdot \phi) = \mathbf{S}(\Theta) \cdot \phi \tag{7.7c}$$

for all $\Theta \in \mathcal{G}$ *and*

$$\mathbf{D}(\Theta \cdot \phi) = \mathbf{D}(\Theta) \cdot \phi \tag{7.7d}$$

for all $\Theta \in \mathcal{G}_0$ *and*

$$\mathbf{C}(\psi \cdot \phi) = \mathbf{C}(\psi) \cdot \phi \tag{7.7e}$$

for $\psi \in \mathcal{F}$. *Therefore* S_T *is* G^2*-equivariant, that is,*

$$S_T(\psi \cdot \phi) = S_T(\psi) \cdot \phi \tag{7.8}$$

for all $\psi \in \mathcal{F}$.

Proof. Let us prove (7.7a). We denote $\Theta = (\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{V}, \mathcal{W}) = \mathbf{E}(Z)$, $\bar{Z} = Z \cdot \phi$, $\bar{\Theta} = (\bar{\mathcal{X}}, \bar{\mathcal{Y}}, \bar{\mathcal{Z}}, \bar{\mathcal{V}}, \bar{\mathcal{W}}) = \mathbf{E}(\bar{Z})$ and $\tilde{\Theta} = (\tilde{\mathcal{X}}, \tilde{\mathcal{Y}}, \tilde{\mathcal{Z}}, \tilde{\mathcal{V}}, \tilde{\mathcal{W}}) = \Theta \cdot \phi$. We want to prove that $\tilde{\Theta} = \bar{\Theta}$. First we prove that $\tilde{\Gamma} = \bar{\Gamma}$ where $\tilde{\Gamma} = (\tilde{\mathcal{X}}, \tilde{\mathcal{Y}})$ and $\bar{\Gamma} = (\bar{\mathcal{X}}, \bar{\mathcal{Y}})$. By definition (5.1), we have

$$\bar{\mathcal{X}}(s) = \sup\{X \in \mathbb{R} \mid t(f(X'), g(Y')) < 0 \quad \text{for all } X' < X \text{ and } Y' \text{ such that } X' + Y' = s\}.$$

We have

$$t(f(\tilde{\mathcal{X}}(s)), g(\tilde{\mathcal{Y}}(s))) = t(\mathcal{X} \circ h(s), \mathcal{Y} \circ h(s)) = 0,$$

by the definition of $(\mathcal{X}, \mathcal{Y})$ given by (5.1) which implies that $t(\mathcal{X}(s), \mathcal{Y}(s)) = 0$ for all $s \in \mathbb{R}$. Hence,

$$\bar{\mathcal{X}} \leq \tilde{\mathcal{X}}. \tag{7.9}$$

Assume that $\bar{\mathcal{X}}(s) < \tilde{\mathcal{X}}(s)$ for some point s. We have $t(f(\bar{\mathcal{X}}(s)), g(\bar{\mathcal{Y}}(s))) = 0$ and due to the monotonicity of t, it implies that $t(X, Y) = 0$ for all $(X, Y) \in [f \circ \bar{\mathcal{X}}(s), f \circ \tilde{\mathcal{X}}(s)] \times [g \circ \tilde{\mathcal{Y}}(s), g \circ \bar{\mathcal{Y}}(s)]$. If $f \circ \bar{\mathcal{X}}(s) \leq 2h(s) - g \circ \bar{\mathcal{X}}(s)$, we obtain a contradiction. Indeed, if we set $X' = 2h(s) - g \circ \bar{\mathcal{Y}}(s)$ and $Y' = g \circ \bar{\mathcal{Y}}(s)$, then we have

$$X' < 2h(s) - g \circ \tilde{\mathcal{Y}}(s) = 2h(s) - \mathcal{Y} \circ h(s) = \mathcal{X} \circ h(s)$$

so that $t(X', Y') = 0$ and $X' + Y' = 2h(s)$, which contradicts the definition (5.1) of $(\mathcal{X}, \mathcal{Y})$ at $h(s)$. If $f \circ \bar{\mathcal{X}}(s) > 2h(s) - g \circ \bar{\mathcal{X}}(s)$, then we set $X' = f \circ \bar{\mathcal{X}}(s) < \mathcal{X} \circ h(s) = f \circ \tilde{\mathcal{X}}(s)$ and $Y' = 2h(s) - f \circ \bar{\mathcal{X}}(s)$. We have $t(X', Y') = 0$ and $X' + Y' = 2h(s)$, which also leads to a contradiction of (5.1). Hence, we have proved that $\bar{\mathcal{X}} = \tilde{\mathcal{X}}$ and therefore $\bar{\Gamma} = \tilde{\Gamma}$. It means that

$$\mathbf{E}(Z \cdot \phi) = (Z \cdot \phi) \bullet \bar{\Gamma} = (Z \cdot \phi) \bullet \tilde{\Gamma} = (Z \cdot \phi) \bullet (\Gamma \cdot \phi).$$

Hence, by (7.4), it yields

$$\mathbf{E}(Z\cdot\phi)=(Z\bullet\Gamma)\cdot\phi=\mathbf{E}(Z)\cdot\phi,$$

and we have proved (7.7a). The identity (7.7b) follows directly from the definition of $\mathbf{t}_T$. Let us prove (7.7c). For any $\Theta=(\mathcal{X},\mathcal{Y},\mathcal{Z},\mathcal{V},\mathcal{W})$, we denote $Z=S(\Theta)$, $\bar{Z}=S(\Theta\cdot\phi)$, $\tilde{Z}=Z\cdot\phi$. We want to prove that $\tilde{Z}=\bar{Z}$. By definition of the solution operator S, we have

$$\bar{Z}\bullet(\Gamma\cdot\phi)=\Theta\cdot\phi,$$

and

$$\tilde{Z}\bullet(\Gamma\cdot\phi)=(Z\cdot\phi)\bullet(\Gamma\cdot\phi)=(Z\bullet\Gamma)\cdot\phi=\Theta\cdot\phi,$$

by (7.4). Hence, $\bar{Z}$ and $\tilde{Z}$ are solutions that match the same data on a curve. Since the solution is unique by Theorem 4.15, we get $\bar{Z}=\tilde{Z}$. The property (7.7d) follows directly from the definitions. Let us prove (7.7e). For any $\psi=(\psi_1,\psi_2)\in\mathcal{F}$, $\phi\in G^2$, we denote $\bar{\psi}=(\bar{\psi}_1,\bar{\psi}_2)=\psi\cdot\phi$, $\Theta=(\mathcal{X},\mathcal{Y},\mathcal{Z},\mathcal{V},\mathcal{W})=\mathbf{C}(\psi)$, $\bar{\Theta}=(\bar{\mathcal{X}},\bar{\mathcal{Y}},\bar{\mathcal{Z}},\bar{\mathcal{V}},\bar{\mathcal{W}})=\mathbf{C}(\bar{\psi})$ and $\tilde{\Theta}=\Theta\cdot\phi$. We want to prove that $\tilde{\Theta}=\bar{\Theta}$. First, we prove that $\tilde{\Gamma}=\bar{\Gamma}$ where $\tilde{\Gamma}=(\tilde{\mathcal{X}},\tilde{\mathcal{Y}})$ and $\bar{\Gamma}=(\bar{\mathcal{X}},\bar{\mathcal{Y}})$. By definition (5.1), we have

$$\begin{aligned}\bar{\mathcal{X}}(s)=\sup\{X\in\mathbb{R}\mid x_1\circ f(X')<x_2\circ g(Y'),&\\ \text{for all } X'<X \text{ and } Y' \text{ such that } X'+Y'=s\},&\end{aligned}\tag{7.10}$$

and

$$\mathcal{X}(s)=\sup\{X\in\mathbb{R}\mid x_1(X')<x_2(Y'),\text{for all } X'<X \text{ and } Y' \text{ such that } X'+Y'=s\}.$$

By continuity, we have $x_1(\mathcal{X}(s))=x_2(\mathcal{Y}(s))$ so that $x_1\circ\mathcal{X}\circ h=x_2\circ\mathcal{Y}\circ h$. Hence, $x_1\circ f\circ\tilde{\mathcal{X}}=x_2\circ g\circ\tilde{\mathcal{Y}}$ and it implies, by (7.10), that

$$\bar{\mathcal{X}}\leq\tilde{\mathcal{X}}.$$

The proof then resembles to what was done above after (7.9). Let us assume that $\bar{\mathcal{X}}(s)<\tilde{\mathcal{X}}(s)$ for some point s. We have $f(\bar{X}(s))<f(\tilde{\mathcal{X}}(s))$ and $g(\tilde{Y}(s))<g(\bar{\mathcal{Y}}(s))$. By using the monotonicity of x_1 and x_2, we get

$$x_1\circ f\circ\bar{\mathcal{X}}(s)\leq x_1\circ f\circ\tilde{\mathcal{X}}(s)=x_2\circ g\circ\tilde{\mathcal{Y}}(s)\leq x_2\circ g\circ\bar{\mathcal{Y}}(s)=x_1\circ f\circ\bar{\mathcal{X}}(s).$$

Hence, $x_1\circ f\circ\bar{\mathcal{X}}(s)=x_1\circ f\circ\tilde{\mathcal{X}}(s)$ and $x_2\circ g\circ\tilde{\mathcal{Y}}(s)=x_2\circ g\circ\bar{\mathcal{Y}}(s)$. Since x_1 and x_2 are decreasing, it follows that x_1 and x_2 are constant on $[f\circ\bar{\mathcal{X}}(s),f\circ\tilde{\mathcal{X}}(s)]$ and $[g\circ\tilde{\mathcal{Y}}(s),g\circ\bar{\mathcal{Y}}(s)]$, respectively. If $f\circ\bar{\mathcal{X}}(s)\leq 2h(s)-g\circ\bar{\mathcal{X}}(s)$, then we obtain a contradiction. Indeed, let us set $X'=2h(s)-g\circ\bar{\mathcal{Y}}(s)$ and $Y'=g\circ\bar{\mathcal{Y}}(s)$, we have

$$X'<2h(s)-g\circ\tilde{\mathcal{Y}}(s)=2h-\mathcal{Y}\circ h(s)=\mathcal{X}\circ h(s)$$

so that $x_1(X')=x_2(Y')=0$ and $X'+Y'=2h(s)$, which contradicts the definition (3.22) of $(\mathcal{X},\mathcal{Y})$ at $h(s)$. If $f\circ\bar{\mathcal{X}}(s)>2h(s)-g\circ\bar{\mathcal{X}}(s)$, then we have

$$x_1(f\circ\bar{\mathcal{X}}(s))=x_2(2h(s)-f\circ\bar{\mathcal{X}}(s)),$$

which, in the same way, leads to a contradiction of (3.22). Thus we have proved that $\tilde{\Gamma}=\bar{\Gamma}$. We then prove that $\tilde{\mathcal{Z}}=\bar{\mathcal{Z}}$, $\tilde{\mathcal{V}}=\bar{\mathcal{V}}$ and $\tilde{\mathcal{W}}=\bar{\mathcal{W}}$. It is just a matter of applying directly the definitions. For example, we have

$$\bar{U}(s)=\bar{U}_1\circ\bar{\mathcal{X}}(s)=U_1\circ f\circ\bar{\mathcal{X}}(s)=U_1\circ f\circ\tilde{\mathcal{X}}(s)=U_1\circ\mathcal{X}\circ h(s)=U\circ h(s)=\tilde{U}(s)$$

and

$$\tilde{\mathcal{V}}_1(X) = f'(X)\mathcal{V}_1(f(X)) = \frac{f'(X)x_1'(f(X))}{2c(U_1(f(X)))} = \frac{\bar{x}_1'(X)}{2c(\tilde{U}_1(X))} = \frac{\bar{x}_1'(X)}{2c(\bar{U}_1(X))} = \bar{\mathcal{V}}_1(X).$$

The equivariance property (7.8) of S_T follows directly from the definition of S_T and the equivariance properties (7.7). □

Definition 7.6. *We define by $\mathcal{F}/G^2$ the quotient of $\mathcal{F}$ with respect to the action of the group G^2 on $\mathcal{F}$, that is,*

$$\psi \sim \bar{\psi} \text{ if there exists } \phi \in G^2 \text{ such that } \bar{\psi} = \psi \cdot \phi.$$

Definition 7.7. *Let*

$$\mathcal{F}_0 = \{\psi = (\psi_1, \psi_2) \in \mathcal{F} \mid x_1 + J_1 = \mathrm{Id} \text{ and } x_2 + J_2 = \mathrm{Id}\}$$

and $\mathbf{\Pi}\colon \mathcal{F} \to \mathcal{F}_0$ be the projection on $\mathcal{F}_0$ given by $\bar{\psi} = (\bar{\psi}_1, \bar{\psi}_2) = \mathbf{\Pi}(\psi)$ where $\bar{\psi} \in \mathcal{F}_0$ is defined as follows. Let

$$f(X) = x_1(X) + J_1(X), \quad g(Y) = x_2(Y) + J_2(Y), \tag{7.11}$$

we set

$$\bar{\psi} = \psi \cdot \phi^{-1}.$$

Lemma 7.8. *The following statements hold:*

(i) *We have*

$$\psi \sim \bar{\psi} \quad \text{if and only if} \quad \mathbf{\Pi}(\psi) = \mathbf{\Pi}(\bar{\psi}) \tag{7.12}$$

so that the sets $\mathcal{F}/G^2$ and $\mathcal{F}_0$ are in bijection.

(ii) *We have*

$$\mathbf{M} \circ \mathbf{\Pi} = \mathbf{M} \tag{7.13}$$

and

$$\mathbf{L} \circ \mathbf{M}|_{\mathcal{F}_0} = \mathrm{Id}\,|_{\mathcal{F}_0} \quad \text{and} \quad \mathbf{M} \circ \mathbf{L} = \mathrm{Id} \tag{7.14}$$

so that the sets $\mathcal{D}$, $\mathcal{F}_0$ and $\mathcal{F}/G^2$ are in bijection.

(iii) *We have*

$$\mathbf{\Pi} \circ S_T \circ \mathbf{\Pi} = \mathbf{\Pi} \circ S_T. \tag{7.15}$$

Note that the first identity in (7.14) is equivalent to

$$\mathbf{L} \circ \mathbf{M} \circ \mathbf{\Pi} = \mathbf{\Pi}. \tag{7.16}$$

Proof. **Step 1.** We prove (7.12). If $\bar{\psi} \sim \psi$, then there exists $\tilde{\phi} \in G^2$ such that $\bar{\psi} = \psi \cdot \tilde{\phi}$. Let $\phi = (f, g)$ and $\bar{\phi} = (\bar{f}, \bar{g})$ be given by (7.11) for ψ and $\bar{\psi}$, respectively. One can check that $\bar{\phi} = \phi \circ \tilde{\phi}$ and therefore

$$\mathbf{\Pi}(\bar{\psi}) = \bar{\psi} \cdot (\bar{\phi})^{-1} = (\psi \cdot \tilde{\phi}) \cdot (\phi \circ \tilde{\phi})^{-1} = \psi \cdot (\tilde{\phi} \circ (\phi \circ \tilde{\phi})^{-1}) = \psi \cdot \phi^{-1} = \mathbf{\Pi}(\psi).$$

Conversely, if $\mathbf{\Pi}(\bar{\psi}) = \mathbf{\Pi}(\psi)$ then $\bar{\psi} \cdot \bar{\phi}^{-1} = \psi \cdot \phi^{-1}$ so that $\bar{\psi} = (\psi \cdot \phi^{-1}) \cdot \bar{\phi} = \psi \cdot (\phi^{-1} \circ \phi)$ and $\bar{\psi}$ and ψ are equivalent.

Step 2. We prove that $\mathbf{L}\circ\mathbf{M}=\mathrm{Id}_{\mathcal{F}_0}$. Given $\psi=(\psi_1,\psi_2)\in\mathcal{F}_0$, let us consider $(u,R,S,\mu,\nu)=\mathbf{L}(\psi_1,\psi_2)$ and $\bar\psi=(\bar\psi_1,\bar\psi_2)=\mathbf{M}(u,R,S,\mu,\nu)$. We want to prove that $\bar\psi=\psi$. Let

$$g(x)=\sup\{X\in\mathbb{R}\mid x_1(X)<x\}. \tag{7.17}$$

It is not hard to prove, using the fact that x_1 is increasing and continuous, that

$$x_1(g(x))=x \tag{7.18}$$

for all $x\in\mathbb{R}$ and $x_1^{-1}((-\infty,x))=(-\infty,g(x))$. For any $x\in\mathbb{R}$, we have, by (6.1c), that

$$\mu((-\infty,x))=\int_{x_1^{-1}((-\infty,x))}J_1'(X)\,dX=\int_{-\infty}^{g(x)}J_1'(X)\,dX=J_1(g(x)) \tag{7.19}$$

because $J_1(-\infty)=0$. Since $\psi\in\mathcal{F}_0$, $x_1+J_1=\mathrm{Id}$ and we get, by (7.18) and (7.19), that

$$\mu((-\infty,x))+x=g(x). \tag{7.20}$$

From the definition of $\bar x_1$, we then obtain that

$$\bar x_1(X)=\sup\{x\in\mathbb{R}\mid g(x)<X\}. \tag{7.21}$$

For any given $X\in\mathbb{R}$, let us consider an increasing sequence z_i tending to $\bar x_1(X)$ such that $g(z_i)<X$; such sequence exists by (7.21). Since x_1 is increasing and using (7.18), it follows that $z_i\le x_1(X)$. Letting i tend to ∞, we obtain $\bar x_1(X)\le x_1(X)$. Assume that $\bar x_1(X)<x_1(X)$. Then, there exists x such that $\bar x_1(X)<x<x_1(X)$ and (7.21) then implies that $g(x)\ge X$. On the other hand, $x=x_1(g(x))<x_1(X)$ implies $g(x)<X$ because x_1 is increasing, which gives us a contradiction. Hence, we have $\bar x_1=x_1$. It follows directly from the definitions, since $x_1+J_1=\mathrm{Id}$, that $\bar J_1=J_1$ and $\bar U_1=U_1$. It follows from the definition (3.34e) and (6.2) that $\bar V_1=V_1$ and $\bar V_2=V_2$. Thus we have proved that $\bar\psi_1=\psi_1$. In the same way, we prove that $\bar\psi_2=\psi_2$, which concludes the proof that $L\circ M=\mathrm{Id}_{\mathcal{F}_0}$.

Step 3. We prove that $\mathbf{M}\circ\mathbf{L}=\mathrm{Id}$. Given $(u,R,S,\mu,\nu)\in\mathcal{D}$, let $\psi=(\psi_1,\psi_2)=\mathbf{L}(u,R,S,\mu,\nu)$ and $(\bar u,\bar R,\bar S,\bar\mu,\bar\nu)=\mathbf{M}(\psi)$. We want to prove that $(\bar u,\bar R,\bar S,\bar\mu,\bar\nu)=(u,R,S,\mu,\nu)$. Let g be the function defined as before by (7.17). The same computation that leads to (7.20) now gives

$$\bar\mu((-\infty,x))+x=g(x). \tag{7.22}$$

Given $X\in\mathbb{R}$, we consider an increasing sequence x_i which converges to $x_1(X)$ and such that $\mu((-\infty,x_i))+x_i<X$. Passing to the limit and since $x\mapsto\mu((-\infty,x))$ is lower semi-continuous, we obtain $\mu((-\infty,x_1(X)))+x_1(X)\le X$. We take $X=g(x)$ and get

$$\mu((-\infty,x))+x\le g(x). \tag{7.23}$$

From the definition of g, there exists an increasing sequence X_i which converges to $g(x)$ such that $x_1(X_i)<x$. The definition (3.34a) of x_1 tells us that $\mu((-\infty,x))+x\ge X_i$. Letting i tend to infinity, we obtain $\mu((-\infty,x))+x\ge g(x)$ which, together with (7.23), yields

$$\mu((-\infty,x))+x=g(x). \tag{7.24}$$

Comparing (7.24) and (7.22) we get that $\bar\mu=\mu$. Similarly, one proves that $\bar\nu=\nu$. It is clear from the definitions that $\bar u=u$. The fact that $\bar R=R$ and $\bar S=S$ follow from (3.34e)

and (6.2). Hence, we have proved that $(\bar{u}, \bar{R}, \bar{S}, \bar{\mu}, \bar{\nu}) = (u, R, S, \mu, \nu)$ and $M \circ L = \mathrm{Id}_{\mathcal{D}}$.
Step 4. We prove (7.15). For any $\psi = (\psi_1, \psi_2) \in \mathcal{F}$, we denote $\psi_T = S_T\psi$. Let $\phi = (f, g) \in G^2$ and $\phi_T = (f_T, g_T) \in G^2$ be defined as in (7.11) so that $\mathbf{\Pi}\psi = \psi \cdot \phi^{-1}$ and $\mathbf{\Pi}\psi_T = \psi_T \cdot \phi_T^{-1}$. By using (7.8), we get

$$S_T \circ \mathbf{\Pi}(\psi) = S_T(\psi \cdot \phi^{-1}) = S_T(\psi) \cdot \phi^{-1}$$

and therefore $S_T \circ \mathbf{\Pi}(\psi)$ and $S_T(\psi)$ are equivalent. Then, (7.15) follows from (7.12). □

We now come to our main theorem.

Theorem 7.9. *Given $(u_0, R_0, S_0, \mu_0, \nu_0) \in \mathcal{D}$, let us denote $(u, R, S, \mu, \nu)(t) = \bar{S}_t(u_0, R_0, S_0, \mu_0, \nu_0)$. Then u is a weak solution of the nonlinear variational wave equation* (1.1)*, that is,*

$$\int_{\mathbb{R}^2} (\phi_t - (c(u)\phi)_x) R\, dxdt + \int_{\mathbb{R}^2} (\phi_t + (c(u)\phi)_x) S\, dxdt = 0 \tag{7.25}$$

for all smooth functions ϕ with compact support and where

$$R = u_t + c(u)u_x, \quad S = u_t - c(u)u_x. \tag{7.26}$$

Moreover, the measures $\mu(t)$ and $\nu(t)$ satisfy the following equations in the sense of distribution

$$(\mu + \nu)_t - (c(\mu - \nu))_x = 0 \tag{7.27a}$$

and

$$(\frac{1}{c}(\mu - \nu))_t - (\mu + \nu)_x = 0. \tag{7.27b}$$

The mapping $\bar{S}_T : \mathcal{D} \to \mathcal{D}$ is a semigroup, that is,

$$\bar{S}_{t+t'} = \bar{S}_t \circ \bar{S}_{t'}$$

for all positive t and t'.

Proof. From Lemma 6.3, we know that (7.26) is fulfilled. On can check that (7.25) is equivalent to

$$R_t - (c(u)R)_x + S_t + (c(u)S)_x + c'(u)\frac{(R - S)^2}{2c(u)} = 0 \tag{7.28}$$

in the sense of distributions, which makes senses, as R, S belong to L^2 and $c(u)$, $c'(u)$ are bounded. After a change of variables, we have

$$\begin{aligned}
\int_{\mathbb{R}^2} (R\phi_t - (c(u)R)\phi_x)(t, x)\, dtdx &= \int_{\mathbb{R}^2} (R(\phi_t - c(u)\phi_x))(t, x)(t_X x_Y - t_Y x_X)\, dXdY \\
&= 2\int_{\mathbb{R}^2} (c(u)R(\phi_t - c(u)\phi_x))(t, x) t_X x_Y\, dXdY \\
&= -2\int_{\mathbb{R}^2} c(U)U_X \phi(t, x)_Y\, dXdY
\end{aligned}$$

by (6.14a) and because $\phi(t, x)_Y = \phi_t(t, x)t_Y + \phi_x(t, x)x_Y = -\frac{x_Y}{c}(\phi_t - c\phi_x)(t, x)$. We integrate by parts and obtain

$$\int_{\mathbb{R}^2} (R\phi_t - (c(u)R)\phi_x)(t, x)\, dtdx = 2\int_{\mathbb{R}^2} (c(U)U_X)_Y \phi(t, x)\, dXdY$$

$$= 2\int_{\mathbb{R}^2} (c(U)U_{XY} + c'(U)U_XU_Y)\phi(t,x)\,dXdY. \tag{7.29}$$

In the same way, one proves that

$$\int_{\mathbb{R}^2} (S\phi_t + (c(u)S)\phi_x)(t,x)\,dtdx = 2\int_{\mathbb{R}^2} (c(U)U_{XY} + c'(U)U_XU_Y)\phi(t,x)\,dXdY. \tag{7.30}$$

We have, after a change of variables,

$$\int_{\mathbb{R}^2} \frac{R^2 - 2RS + S^2}{2c(u)} c'(u)\phi\,dtdx = 2\int_{\mathbb{R}^2} \Big(\frac{R^2 - 2RS + S^2}{2c(u)^2} c'(u)\phi\Big)(t,x)x_Xx_Y\,dXdY. \tag{7.31}$$

We introduce the set $A = A_1 \cup A_2$ where

$$A_1 = \{(X,Y) \in \mathbb{R}^2 \mid x_X(X,Y) = 0, \quad x_Y(X,Y) > 0 \text{ and } c'(U)(X,Y) \neq 0\} \tag{7.32}$$

and

$$A_2 = \{(X,Y) \in \mathbb{R}^2 \mid x_Y(X,Y) = 0, \quad x_X(X,Y) > 0 \text{ and } c'(U)(X,Y) \neq 0\}. \tag{7.33}$$

We claim that

$$\operatorname{meas}(A) = \operatorname{meas}(A_1) = \operatorname{meas}(A_2) = 0. \tag{7.34}$$

We prove this claim later. By using (7.34), we get

$$\begin{aligned}
\int_{\mathbb{R}^2} \frac{R^2 - 2RS + S^2}{2c(u)} c'(u)\phi\,dtdx &= 2\int_{A^c} \Big(\frac{R^2 - 2RS + S^2}{2c(u)^2} c'(u)\phi\Big)(t,x)x_Xx_Y\,dXdY \\
&= \int_{A^c} \Big(\frac{U_X^2}{x_X}x_Y + 2U_XU_Y + \frac{U_Y^2}{x_Y}x_X\Big)c'(U)\phi(t,x)\,dXdY \\
&= \int_{A^c} \Big(2\frac{J_Xx_Y}{c^2(U)} + 2U_XU_Y + \frac{J_Yx_X}{c^2(U)}\Big)c'(U)\phi(t,x)\,dXdY \qquad (7.35)\\
&= \int_{\mathbb{R}} \Big(2\frac{J_Xx_Y}{c^2(U)} + 2U_XU_Y + \frac{J_Yx_X}{c^2(U)}\Big)c'(U)\phi(t,x)\,dXdY. \qquad (7.36)
\end{aligned}$$

Note that (7.34) is necessary to get (7.36) from (7.35) as the integrand in (7.35) does not vanish on A. After combining (7.29), (7.30) and (7.36), and using the governing equations (2.13), we get

$$\begin{aligned}
&\int_{\mathbb{R}^2} (R+S)\phi_t - (c(u)(R-S))\phi_x - c'(u)\frac{(R-S)^2}{2c(u)}\phi\,dtdx \\
&\qquad = -\int_{\mathbb{R}^2} \Big(4c(U)U_{XY} - \frac{2c'}{c^2}(J_Xx_Y + J_Yx_X) + 2c'(U)U_YU_X\Big)\phi(t,x)\,dXdY \\
&\qquad = 0,
\end{aligned}$$

which proves (7.28) and therefore (7.25) holds. It remains to prove the claim (7.34). Let us introduce the set

$$A_1(X) = \{Y \in \mathbb{R} \mid (X,Y) \in A_1\}.$$

Let us prove that, for almost every $X \in \mathbb{R}$, $\operatorname{meas}(A_1(X)) = 0$ and therefore, by Fubini's theorem, $\operatorname{meas}(A_1) = 0$. We consider a point $Y_0 \in A_1(X)$ and a rectangle $\Omega = [X_l, X_r] \times [Y_l, Y_r]$ which contains (X, Y_0). Since $Z \in \mathcal{H}(\Omega)$, there exist $\delta > 0$ such that $(x_X + J_X)(X,Y) > \delta$ for almost every $X \in \mathbb{R}$ and all $Y \in \mathbb{R}$. Since $x_X \in W_Y^{1,\infty}(\Omega)$, the function x_X is continuous with respect to Y for almost any given $X \in \mathbb{R}$. Formally the argument goes as follows: We consider a fixed given $X \in \mathbb{R}$ and denote $f(Y) = x_X(X, Y)$. For any $Y_0 \in A_1(X)$, we have, by definition, $x_X(X, Y_0) = f(Y_0) = 0$. By using (2.13b), we get

$$f'(Y_0) = x_{XY}(X, Y) = 0$$

because $x_X = 0$ implies $U_X = 0$, see (4.18c). We do not have enough regularity to differentiate (2.13b); but if nevertheless we do so, then we formally obtain

$$\begin{aligned} f''(Y_0) = x_{XYY}(X, Y_0) &= \frac{c'}{2c}(U_Y x_{XY} + U_{XY} x_Y)(X, Y_0) \\ &= \frac{c'^2}{4c^4}(J_X x_Y^2)(X, Y_0) \end{aligned}$$

where we have used again the fact $x_X(X, Y_0) = U_X(X, Y_0) = 0$. We have $J_X(X, Y_0) = (x_X + J_X)(X, Y_0) \geq \delta$ and $x_Y(X, Y_0) > 0$, $c'^2(U(X, Y_0)) > 0$ because $(X, Y_0) \in A_1(X)$. Hence, $f''(Y_0) > 0$ and it implies that $f(Y) > 0$ for all Y different from Y_0 in a neighborhood of Y_0, so that the points in $A_1(X)$ are isolated. Let us now prove this result rigorously. Again, we consider $Y_0 \in A_1(X)$ and a rectangle $\Omega = [X_l, X_r] \times [Y_l, Y_r]$ which contains (X, Y_0). Without loss of generality, we assume that Y_0 is a Lebesgue point for the function $Y \mapsto x_Y(X, Y)$ and, therefore, since $x_Y(X, Y_0) > 0$, there exists $\delta > 0$ such that

$$\int_{Y_0}^{Y} x_Y(\bar{Y})\, d\bar{Y} > \delta'(Y - Y_0) \tag{7.37}$$

in a neighborhood of Y_0. We can choose $\delta' > 0$ such that, in addition,

$$c'(U(X, Y)) > \delta \quad \text{and} \quad J_X(X, Y) > \delta$$

in a neighborhood of Y_0 (we recall that U is continuous). We have, after using the governing equations (2.13),

$$\begin{aligned} x_X &= \int_{Y_0}^{Y} \frac{c'}{2c}(U_Y x_X + U_X x_Y)\, d\bar{Y} \\ &= \int_{Y_0}^{Y} \frac{c'}{2c}(U_Y \int_{Y_0}^{\bar{Y}} x_{XY}\, d\tilde{Y} + x_Y \int_{Y_0}^{\bar{Y}} U_{XY} d\tilde{Y})\, d\bar{Y} \\ &= \int_{Y_0}^{Y} \frac{c'}{2c}\Big(U_Y \int_{Y_0}^{\bar{Y}} x_{XY}\, d\tilde{Y} + x_Y \int_{Y_0}^{\bar{Y}} \big(\frac{c'}{2c^3}(x_Y J_X + J_Y x_X - \frac{c'}{2c} U_Y U_X)\big) d\tilde{Y}\Big)\, d\bar{Y}. \end{aligned}$$

Since x_{XY} and U_{XY} are bounded (by (2.13)) and $x_X(X, Y_0) = U_X(X, Y_0) = 0$, we have that $x_X(X, Y) \leq C\,|Y - Y_0|$ and $U_X(X, Y) \leq C\,|Y - Y_0|$ in a neighborhood of Y_0 for a constant C which depends only on $|||Z|||_{\mathcal{H}(\Omega)}$. Hence,

$$\begin{aligned} \left|\int_{Y_0}^{\bar{Y}} x_{XY}\, d\tilde{Y}\right| &= \left|\int_{Y_0}^{\bar{Y}} \frac{c'}{2c}(U_Y x_X + U_X x_Y)\, d\tilde{Y}\right| \\ &\leq C \int_{Y_0}^{\bar{Y}} \left|\tilde{Y} - Y_0\right| d\tilde{Y} \\ &\leq C(Y - Y_0)^2. \end{aligned}$$

Thus,

$$\left|\int_{Y_0}^{Y}\frac{c'}{2c}(U_Y\int_{Y_0}^{\bar Y}x_{XY}\,d\tilde Y)\,d\bar Y\right|\le C\left|Y-Y_0\right|^3. \tag{7.38}$$

In the same way, one proves that

$$\left|\int_{Y_0}^{Y}\frac{c'}{2c}x_Y\int_{Y_0}^{\bar Y}\Big(\frac{c'}{2c^3}(J_Yx_X-\frac{c'}{2c}U_YU_X)\Big)d\tilde Y)\,d\bar Y\right|\le C\left|Y-Y_0\right|^3.$$

For $Y>Y_0$, we have

$$\begin{aligned}\int_{Y_0}^{Y}\frac{c'}{2c}x_Y\Big(\int_{Y_0}^{\bar Y}\frac{c'}{2c^3}x_YJ_Xd\tilde Y\Big)\,d\bar Y&\ge\frac{\kappa^4\delta^3}{4}\int_{Y_0}^{Y}x_Y\Big(\int_{Y_0}^{\bar Y}x_Yd\tilde Y\Big)\,d\bar Y\\&=\frac{\kappa^4\delta^3}{4}\Big(\int_{Y_0}^{Y}x_Y\,d\bar Y\Big)^2\\&\ge\frac{\kappa^4\delta^5}{4}(Y-Y_0)^2\end{aligned}$$

in a neighborhood of Y_0. We can check that the same inequality holds for $Y<Y_0$. Finally, we obtain that, in a neighborhood of Y_0,

$$x_X(X,Y)\ge\frac{\kappa^4\delta^5}{4}(Y-Y_0)^2-C\left|Y-Y_0\right|^3\ge\frac{\kappa^4\delta^5}{8}(Y-Y_0)^2. \tag{7.39}$$

To complete the argument, we consider the sets

$$\begin{aligned}A_1^k(X)=\{Y_0\in A_1(X)\cap[Y_l,Y_r]\mid x_X(X,Y_0)=0\\ \text{and } x_X(X,Y)>0\text{ for all }Y\in[Y_0-\frac{1}{k},Y_0+\frac{1}{k}]\setminus\{Y_0\}\}\end{aligned}$$

for any integer k. By (7.39), we have

$$A_1(X)\cap[Y_l,Y_r]=\cup_{k>0}A_1^k(X).$$

At the same time, since $A_1^k(X)$ consists of points separated by a distance of at least $\frac{1}{k}$, we have $\operatorname{meas}(A_1^k(X))=0$. Hence, after taking sequences of Y_l and Y_r which tend to plus and minus infinity, respectively, we get $\operatorname{meas}(A_1(X))=0$ so that $\operatorname{meas}(A_1)=0$ and the proof of the claim (7.34) is complete. Let us prove (7.27a), that is,

$$\int_{\mathbb{R}^2}(\phi_t-c(u)\phi_x)\,d\mu dt+\int_{\mathbb{R}^2}(\phi_t+c(u)\phi_x)\,d\nu dt=0$$

for all smooth function ϕ with compact support. We have, after a change of variables, that

$$\begin{aligned}&\int_{\mathbb{R}}\Big(\int_{\mathbb{R}}(\phi_t-c(u)\phi_x)\,d\mu(t)\Big)dt\\&\quad=\int_{\mathbb{R}}\Big(\int_{\mathbb{R}}(\phi_t(t,x(t,s))-c(u(t,x(t,s)))\phi_x(t,x(t,s)))\mathcal{V}_4(t,\mathcal{X}(t,s))\mathcal{X}_s(t,s)\,ds\Big)dt,\end{aligned}$$

where we have added the dependence in t of the values of $\Theta(t)=\mathbf{L}(u,R,S,u,\mu,\nu)(t)$ (which gives $x(t,s)$, $\mathcal{V}_4(t,X)$, $u(t,s)$ and $\mathcal{X}(t,s)$ in the equation above). We proceed to the change of variables $(X,Y)\mapsto(t(X,Y),s=\frac{1}{2}(X+Y))$ whose Jacobian is equal to $\frac{x_X+x_Y}{2c(u)}$ and get

$$\int_{\mathbb{R}}\Big(\int_{\mathbb{R}}(\phi_t-c(u)\phi_x)\,d\mu(t)\Big)dt \\ =\int_{\mathbb{R}^2}(\phi_t(t,x)-c(u(t,x))\phi_x(t,x))J_X(X,Y)\mathcal{X}_s(t,s)\frac{(x_X+x_Y)(X,Y)}{2c(U(X,Y))}\,dXdY. \tag{7.40}$$

Since $t(\mathcal{X}(t,s),\mathcal{Y}(t,s))=t$, by definition, we get $t_X\mathcal{X}_s+t_Y\mathcal{Y}_s=0$ and, since $\mathcal{X}(s)+\mathcal{Y}(s)=2s$, we have $\mathcal{X}_s+\mathcal{Y}_s=2$. Hence, $(x_X+x_Y)\mathcal{X}_s(t,s)=2x_Y$ and (7.40) implies

$$\int_{\mathbb{R}}\Big(\int_{\mathbb{R}}(\phi_t-c(u)\phi_x)\,d\mu(t)\Big)dt \\ =\int_{\mathbb{R}^2}(\phi_t(t,x)-c(u(t,x))\phi_x(t,x))J_X(X,Y)\frac{x_Y}{c(U(X,Y))}\,dXdY.$$

Since $\phi(t,x)_Y=-\frac{x_Y}{c(u)}(\phi_t-c(u)\phi_x)(t,x)$, it yields

$$\int_{\mathbb{R}}\Big(\int_{\mathbb{R}}(\phi_t-c(u)\phi_x)\,d\mu(t)\Big)dt=-\int_{\mathbb{R}^2}\phi(t,x)_YJ_X\,dXdY.$$

Similarly, one proves that

$$\int_{\mathbb{R}}\Big(\int_{\mathbb{R}}(\phi_t+c(u)\phi_x)\,d\nu(t)\Big)dt=\int_{\mathbb{R}^2}\phi(t,x)_XJ_Y\,dXdY$$

so that

$$\int_{\mathbb{R}^2}(\phi_t-c(u)\phi_x)\,d\mu dt+\int_{\mathbb{R}^2}(\phi_t+c(u)\phi_x)\,d\nu dt=\int_{\mathbb{R}^2}(-\phi(t,x)_YJ_X+\phi(t,x)_XJ_Y)\,dXdY=0, \tag{7.41}$$

by integration by parts, as the support of ϕ is compact. Similarly one proves (7.27b). Note that the integrand in (7.41) is equal to the exact form $d(\phi dJ)$ and equation (7.27a) is actually equivalent to $ddJ=0$ while (7.27b) is equivalent to $ddK=0$. The proof of the semigroup property follows in a straightforward manner from the results that have been established in this section. We have

$$\begin{aligned}
\bar{S}_T\circ\bar{S}_{T'}&=\mathbf{M}\circ S_T\circ\mathbf{L}\circ\mathbf{M}\circ S_{T'}\circ\mathbf{L}\\
&=\mathbf{M}\circ\mathbf{\Pi}\circ S_T\circ\mathbf{L}\circ\mathbf{M}\circ\mathbf{\Pi}\circ S_{T'}\circ\mathbf{L} &&\text{by (7.13)}\\
&=\mathbf{M}\circ\mathbf{\Pi}\circ S_T\circ\mathbf{\Pi}\circ S_{T'}\circ\mathbf{L} &&\text{by (7.16)}\\
&=\mathbf{M}\circ\mathbf{\Pi}\circ S_T\circ S_{T'}\circ\mathbf{L} &&\text{by (7.15)}\\
&=\mathbf{M}\circ S_T\circ S_{T'}\circ\mathbf{L} &&\text{by (7.13)}\\
&=\mathbf{M}\circ S_{T+T'}\circ\mathbf{L} &&\text{by Theorem 5.5}\\
&=\bar{S}_{T+T'}.
\end{aligned}$$

□

The semigroup of solution we have constructed is conservative in the sense given by the following theorem.

Theorem 7.10. *Given* $(u_0, R_0, S_0, \mu_0, \nu_0) \in \mathcal{D}$*, let us denote* $(u, R, S, \mu, \nu)(t) = \bar{S}_t(u_0, R_0, S_0, \mu_0, \nu_0)$. *We have*

(i) *For all* $t \in \mathbb{R}$

$$\mu(t)(\mathbb{R}) + \nu(t)(\mathbb{R}) = \mu_0(\mathbb{R}) + \nu_0(\mathbb{R}). \tag{7.42}$$

(ii) *For almost every* $t \in \mathbb{R}$*, the singular part of* $\mu(t)$ *and* $\nu(t)$ *are concentrated on the set where* $c'(u) = 0$.

This theorem corresponds to Theorem 3 in [5]. We use a different proof based on the coarea formula.

Proof. Let us prove (i). We consider a given time that we denote τ (to avoid any confusion with the function $t(X, Y)$). As in the proof of the previous theorem, we add the dependence in time of the values of $\Theta(\tau) = \mathbf{L}(u, R, S, u, \mu, \nu)(\tau)$. In particular, the curve $(\mathcal{X}(\tau, s), \mathcal{Y}(\tau, s))$ corresponds to the curve where time is constant and equal to τ, that is, $t((\mathcal{X}(\tau, s), \mathcal{Y}(\tau, s))) = \tau$. By definition (see (6.3b), and Definition 5.1), we have, for any Borel set B,

$$\mu_\tau(B) = \int_{\{s \in \mathbb{R} \mid x(\tau, s) \in B\}} \mathcal{V}_4(\mathcal{X}(\tau, s))\dot{\mathcal{X}}(\tau, s)\, ds \tag{7.43}$$

$$= \int_{\{s \in \mathbb{R} \mid x(\mathcal{X}(\tau, s), \mathcal{Y}(\tau, s)) \in B\}} J_X(\mathcal{X}(\tau, s), \mathcal{Y}(\tau, s))\dot{\mathcal{X}}(\tau, s)\, ds. \tag{7.44}$$

Here, the notation may be confusing as x denotes two different but of course very related functions. In (7.43), we have $x(\tau, s) = \mathcal{Z}_2(\tau, s)$, which corresponds to the space variable $\mathcal{Z}_2$ parametrized by s at time τ while, in (7.44), $x(X, Y) = Z_2(X, Y)$, corresponds to the value of the space variable Z_2, where $Z(X, Y)$ is the solution of (2.13) on the whole $\mathbb{R}^2$ plane. We have $\mathcal{Z}_2(\tau, s) = Z_2(\mathcal{X}(\tau, s), \mathcal{X}(\tau, s))$ by (4.15) abd Definition 5.1. Correspondingly, we have

$$\nu_\tau(B) = \int_{\{s \in \mathbb{R} \mid x(\mathcal{X}(\tau, s), \mathcal{Y}(\tau, s)) \in B\}} J_Y(\mathcal{X}(\tau, s), \mathcal{Y}(\tau, s))\dot{\mathcal{Y}}(\tau, s)\, ds.$$

Hence,

$$\begin{aligned}
\mu_\tau(\mathbb{R}) + \nu_\tau(\mathbb{R}) &= \int_{\mathbb{R}} J_X(\mathcal{X}(\tau, s), \mathcal{Y}(\tau, s))\dot{\mathcal{X}}(\tau, s) + J_Y(\mathcal{X}(\tau, s), \mathcal{Y}(\tau, s))\dot{\mathcal{Y}}(\tau, s)\, ds \\
&= \lim_{s \to \infty} J(\mathcal{X}(\tau, s), \mathcal{Y}(\tau, s))\, ds \\
&= \lim_{s \to \infty} J(\mathcal{X}(0, s), \mathcal{Y}(0, s))\, ds, \qquad \text{by Lemma 4.14,} \\
&= \mu_0(\mathbb{R}) + \nu_0(\mathbb{R})
\end{aligned}$$

Let us prove (ii). Let $\mu_\tau = (\mu_\tau)_{\text{ac}} + (\mu_\tau)_{\text{sing}}$ be the Radon-Nykodin decomposition of μ_τ. We want to prove that, for allmost every time $\tau \in \mathbb{R}$, we have

$$(\mu_\tau)_{\text{sing}}(\{x \in \mathbb{R} \mid c'(u(\tau, x)) \neq 0\}) = 0. \tag{7.45}$$

Let us introduce

$$A_\tau = \{s \in \mathbb{R} \mid x_X(\mathcal{X}(\tau, s), \mathcal{Y}(\tau, s)) > 0\}.$$

The set A_τ corresponds to A in (6.11) in the proof of Lemma 6.2. In this same proof, we obtain that, for any Borel set B,

$$(\mu_\tau)_{\text{ac}}(B) = \mu_\tau(B \cap (x(\tau, (A_\tau)^c))^c)$$

so that

$$(\mu_\tau)_{\text{sing}}(B) = \mu_\tau(B \cap (x(\tau, (A_\tau)^c))),$$

because $\text{meas}(x(\tau, (A_\tau)^c)) = 0$. Hence,

$$(\mu_\tau)_{\text{sing}}(B) = \int_{\{s\in\mathbb{R} \mid x(\mathcal{X}(\tau,s),\mathcal{Y}(\tau,s))\in B\cap(A_\tau)^c\}} J_X(\mathcal{X}(\tau, s), \mathcal{Y}(\tau, s))\dot{\mathcal{X}}(s)\, ds. \tag{7.46}$$

We introduce the set

$$E = \{(X, Y) \in \mathbb{R}^2 \mid x_X(X, Y) = 0 \text{ and } c'(U(X, Y)) \neq 0\}.$$

By using (7.46), we get

$$\mu_\tau(\{x \in \mathbb{R} \mid c'(u(\tau, x)) \neq 0\}) = \int_{\{s\in\mathbb{R} \mid (\mathcal{X}(\tau,s),\mathcal{Y}(\tau,s))\in E\}} J_X(\mathcal{X}(\tau, s), \mathcal{Y}(\tau, s))\dot{\mathcal{X}}(s)\, ds \tag{7.47}$$

By the coarea formula, see [1], we get

$$\int_{\mathbb{R}} \mathcal{H}^1(E \cap t^{-1}(\tau))\, d\tau = \int_E \sqrt{t_X^2 + t_Y^2}\, dXdY = \int_E \frac{x_Y}{c(U)}\, dXdY = 0$$

because of (7.34). Here, $\mathcal{H}^1$ denotes the one-dimensional Hausdorff measure. Hence, we have that, for allmost every time $\tau \in \mathbb{R}$, the set $E \cap t^{-1}(\tau)$ has zero one-dimensional Hausdorff measure. We claim that, if $\mu_\tau(\{x \in \mathbb{R} \mid c'(u(\tau, x)) \neq 0\}) > 0$, then $\mathcal{H}^1(E \cap t^{-1}(\tau)) > 0$. Indeed, let us define, for a given τ, the mapping $\Gamma_\tau : s \mapsto (\mathcal{X}(\tau, s), \mathcal{Y}(\tau, s))$ from $\mathbb{R}$ to $\mathbb{R}^2$. We rewrite (7.47) as

$$\mu_\tau(\{x \in \mathbb{R} \mid c'(u(\tau, x)) \neq 0\}) = \int_{\Gamma_\tau^{-1}(E)} J_X(\mathcal{X}(\tau, s), \mathcal{Y}(\tau, s))\dot{\mathcal{X}}(s)\, ds.$$

In particular it implies that, if $\mu_\tau(\{x \in \mathbb{R} \mid c'(u(\tau, x)) \neq 0\}) > 0$, then $\text{meas}(\Gamma_\tau^{-1}(E)) > 0$. By the area formula, we have

$$\mathcal{H}^1(E) \geq \mathcal{H}^1(\Gamma_\tau \circ \Gamma_\tau^{-1}(E)) = \int_{\Gamma_\tau^{-1}(E)} (\mathcal{X}_s^2 + \mathcal{Y}_s^2)^{1/2}\, ds \geq \text{meas}(\Gamma_\tau^{-1}(E))$$

because $(\mathcal{X}_s^2 + \mathcal{Y}_s^2)^{1/2} \geq \frac{1}{2}(\mathcal{X}_s + \mathcal{Y}_s) = 1$. Hence, our claim is proved and it follows that $\mu_\tau(\{x \in \mathbb{R} \mid c'(u(\tau, x)) \neq 0\}) > 0$ for at most almost every $\tau \in \mathbb{R}$ and we have proved (7.45). □

Theorem 7.11 (Finite speed of propagation). *Given $\mathbf{t} \geq 0$ and $\mathbf{x} \in \mathbb{R}$, for any two initial datas $\zeta_0 = (u_0, R_0, S_0, \mu_0, \nu_0)$ and $\bar{\zeta}_0 = (\bar{u}_0, \bar{R}_0, \bar{S}_0, \bar{\mu}_0, \bar{\nu}_0)$ in $\mathcal{D}$, we consider the corresponding solutions $(u, R, S, \mu, \nu)(t)$ and $(u, R, S, \mu, \nu)(t)$ given by Definition 6.4. If the restrictions of ζ_0 and $\bar{\zeta}_0$ are equal on $[\mathbf{x} - \kappa\mathbf{t}, \mathbf{x} + \kappa\mathbf{t}]$, then the two solutions coincide at $(\mathbf{t}, \mathbf{x})$, that is, $u(\mathbf{t}, \mathbf{x}) = \bar{u}(\mathbf{t}, \mathbf{x})$.*

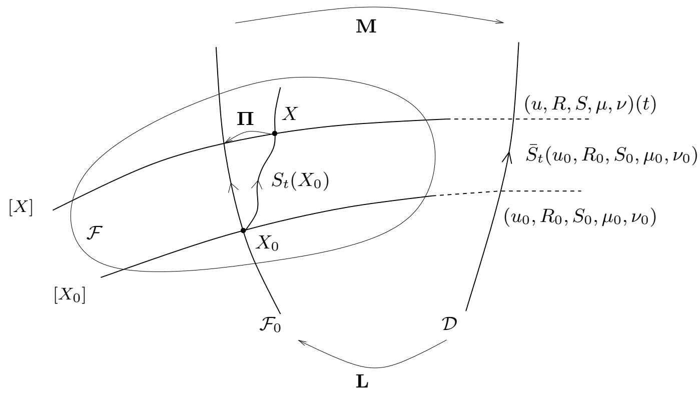

FIGURE 6. The semigroup $\bar{S}_t$.

Proof. For a given $\zeta_0 = (u_0, R_0, S_0, \mu_0, \nu_0)$, we define $\bar\zeta_0$ equal to ζ_0 on $[\mathbf{x} - \kappa\mathbf{t}, \mathbf{x} + \kappa\mathbf{t}]$ and zero otherwise, i.e.,

$$(\bar u_0, \bar R_0, \bar S_0)(x) = \begin{cases} (u_0, R_0, S_0)(x) & \text{if } x \in [\mathbf{x} - \kappa\mathbf{t}, \mathbf{x} + \kappa\mathbf{t}] \\ 0 & \text{otherwise} \end{cases}$$

and

$$\bar\mu_0(E) = \mu_0(E \cap [\mathbf{x} - \kappa\mathbf{t}, \mathbf{x} + \kappa\mathbf{t}]), \quad \bar\nu_0(E) = \nu_0(E \cap [\mathbf{x} - \kappa\mathbf{t}, \mathbf{x} + \kappa\mathbf{t}])$$

for any Borel set E. It is enough to prove that the theorem holds for this particular $\bar\zeta_0$. We have to compute the solutions for ζ_0 and $\bar\zeta_0$. Let us denote $x_l = \mathbf{x} - \kappa\mathbf{t}$, $x_r = \mathbf{x} + \kappa\mathbf{t}$. We set $\psi = \mathbf{C}(\zeta_0)$ and $\bar\psi = \mathbf{C}(\bar\zeta_0)$.

Step 1. We want to compute $\bar\psi$ as a function of ψ. We denote $X_l = x_l$, $Y_l = x_l$, $X_r = x_r + \mu_0([x_l, x_r])$, $Y_r = x_r + \nu_0([x_l, x_r])$ and $\Omega = [X_l, X_r] \times [Y_l, Y_r]$. Let us prove that

$$\bar x_1(X) = \begin{cases} X & \text{if } X \le X_l \\ x_1(X + \mu_0(-\infty, x_l)) & \text{if } X_l < X \le X_r \\ X - \mu_0([x_l, x_r]) & \text{if } X_r < X \end{cases} \tag{7.48}$$

and

$$\bar x_2(Y) = \begin{cases} Y & \text{if } Y \le Y_l \\ x_2(Y + \nu_0(-\infty, x_l)) & \text{if } Y_l < Y \le Y_r \\ Y - \nu_0([x_l, x_r]) & \text{if } Y_r < Y. \end{cases} \tag{7.49}$$

From the definition (3.34a), we have

$$\bar x_1(X) = \sup\{x' \in \mathbb{R} \mid x' + \bar\mu_0(-\infty, x') < X\} \tag{7.50}$$

First case: $X \le x_l$. For any x' such that $x' + \bar\mu_0(-\infty, x') < X$ we have $x' < X$. Hence, $x' < x_l$ and $\bar\mu_0(-\infty, x') = 0$. It follows that $\bar x_1(X) = X$. Second case: $X_l < X \le X_r$. For any x' such that $x' + \bar\mu_0(-\infty, x') < X$, we have $x' \le x_r$. Let us assume the opposite, i.e., $x' > x_r$, then $\bar\mu_0(-\infty, x') = \mu_0((-\infty, x') \cap [x_l, x_r]) = \mu_0([x_l, x_r])$ and therefore $x' + \mu_0([x_l, x_r]) < X \le x_r + \mu_0([x_l, x_r])$, which gives a contradiction. We can assume without loss of generality that $x' \geq x_l$ because for $x' = x_l$, $\bar{\mu}_0(-\infty, x') = 0$ and we have $x' + \bar{\mu}_0(-\infty, x') = x' = x_l < X$. Thus we have $x' \in [x_l, x_r]$ and (7.50) rewrites

$$\bar{x}_1(X) = \sup\{x' \in [x_l, x_r] \mid x' + \mu_0[x_l, x') < X\}. \tag{7.51}$$

We now want to prove that, for $X_l \leq X \leq X_r$,

$$x_1(X + \mu_0(-\infty, x_l)) = \sup\{x' \in [x_l, x_r] \mid x' + \mu_0[x_l, x') < X\}. \tag{7.52}$$

For any x' such that $x' + \mu_0(-\infty, x') < X + \mu_0(-\infty, x_l)$, we have $x' \leq x_r$. Let us assume the opposite, i.e., $x' > x_r$, then $x_r + \mu_0([x_l, x_r]) \leq x' + \mu_0([x_l, x')) < X$ implies a contradiction with the assumption that $X \leq X_r$. For $x' = x_l$, we have $x' + \mu_0(-\infty, x') < X + \mu_0(-\infty, x_l)$ so that we can assume without loss of generality that $x' \geq x_l$. Hence,

$$\begin{aligned} x_1(X + \mu_0(-\infty, x_l)) &= \sup\{x' \in \mathbb{R} \mid x' + \mu_0(-\infty, x') < X + \mu_0(-\infty, x_l)\} \\ &= \sup\{x' \in [x_l, x_r] \mid x' + \mu_0(-\infty, x') < X + \mu_0(-\infty, x_l)\} \end{aligned}$$

and (7.52) follows. By comparing (7.51) and (7.52), we get $\bar{x}_1(X) = x_1(X + \mu_0(-\infty, x_l))$ for $X_l < X < X_r$. Third case: $X_r < X$. For $x' = x_r$, we have $x' + \bar{\mu}_0(-\infty, x') \leq X_r = x_r + \mu_0[x_l, x_r] < X$. Hence, $\bar{x}_1(X) = \sup\{x' \in [x_r, \infty) \mid x' + \bar{\mu}_0(-\infty, x') < X\}$. Since, for $x' > x_r$, $\bar{\mu}_0(-\infty, x') = \mu_0([x_l, x_r])$, it follows that $\bar{x}_1(X) = X - \mu_0([x_l, x_r])$. This concludes the proof of proved (7.48). One proves in the same way (7.49). Let $\phi = (f, g) \in G^2$ where $f : X \mapsto X + \mu_0(-\infty, x_l)$ and $g : Y \mapsto Y + \nu_0(-\infty, x_r)$. We denote $\tilde{\psi} = \psi \cdot \phi$. We have proved that

$$\bar{x}_1(X) = \tilde{x}_1(X) \text{ for } X_l < X \leq X_r \tag{7.53}$$

and

$$\bar{x}_2(Y) = \tilde{x}_2(Y) \text{ for } X_l < Y \leq X_r. \tag{7.54}$$

We denote $\bar{\theta} = \mathbf{C}(\bar{\psi})$ and $\tilde{\theta} = \mathbf{C}(\tilde{\psi})$.

Step 2. We prove that

$$\bar{\mathcal{X}}(s) = \tilde{\mathcal{X}}(s) \quad \text{and} \quad \bar{\mathcal{Y}}(s) = \tilde{\mathcal{Y}}(s) \tag{7.55}$$

for $s \in [s_l, s_r]$ where $s_l = \frac{1}{2}(X_l + Y_l)$ and $s_r = \frac{1}{2}(X_r + Y_r)$. By using the definitions of x_1 and x_2, we obtain that, for any $x \in \mathbb{R}$,

$$x_1(x + \mu_0(-\infty, x)) = x_1(x + \mu_0(-\infty, x]) = x \tag{7.56}$$

and that the corresponding statement for x_2 holds. Let us now prove that

$$\tilde{\mathcal{X}}(s_l) = X_l \quad \text{and} \quad \tilde{\mathcal{Y}}(s_l) = Y_l. \tag{7.57}$$

It follows from (7.56) that

$$\tilde{x}_1(X_l) = \tilde{x}_2(Y_l) = x_l \tag{7.58}$$

as we have $\tilde{x}_1(X_l) = x_1(x_l + \mu_0(-\infty, x_l)) = x_l = x_2(x_l + \nu_0(-\infty, x_l)) = \tilde{x}_2(X_l)$. For any $X < X_l$, we have $\tilde{x}_1(X) \leq \tilde{x}_1(X_l) = x_l$. Let us prove that $\tilde{x}_1(X) < \tilde{x}_1(X_l)$. We assume the opposite, i.e., that $\tilde{x}_1(X) = \tilde{x}_1(X_l) = x_l$. Then, there exists an increasing sequence x_i such that $\lim_{i\to\infty} x_i = x_l$ and $x_i + \mu_0(-\infty, x_i) < X + \mu_0(-\infty, x_l)$. It implies that $x_l + \mu_0(-\infty, x_l) \leq X + \mu_0(-\infty, x_l)$ because of the lower semicontinuity of $x \mapsto \mu_0(-\infty, x)$. Hence, $x_l \leq X_l$, which is a contradiction. Thus we have proved that, for any $X < X_l$,

$\tilde{x}_1(X) < \tilde{x}_1(X_l) = \tilde{x}_2(Y_l) \leq \tilde{x}_2(2s - X)$. Hence, $\tilde{X}(s_l) = X_l$ and (7.57) holds. By using similar arguments, one also proves that

$$\tilde{\mathcal{X}}(s_r) = X_r, \quad \tilde{\mathcal{Y}}(s_r) = Y_r \quad \text{and} \quad \tilde{x}_1(X_r) = \tilde{x}_2(Y_r) = x_r. \tag{7.59}$$

and the corresponding results for $\bar{\mathcal{X}}$ and $\bar{\mathcal{Y}}$, that is,

$$\bar{\mathcal{X}}(s_l) = X_l, \quad \bar{\mathcal{Y}}(s_l) = Y_l, \quad \bar{x}_1(X_l) = \bar{x}_2(Y_l) = x_l \tag{7.60}$$

and

$$\bar{\mathcal{X}}(s_r) = X_r, \quad \bar{\mathcal{Y}}(s_r) = Y_r, \quad \bar{x}_1(X_r) = \bar{x}_2(Y_r) = x_r. \tag{7.61}$$

In particular, we have proved (7.55) for $s = s_l$ and $s = s_r$. For any $s \in (s_l, s_r)$, either $X_l < \bar{\mathcal{X}}(s) \leq X_r$ or $Y_l \leq \bar{\mathcal{Y}}(s) < Y_r$. We consider only the case where $X_l < \bar{\mathcal{X}}(s) \leq X_r$ as the other case can be treated similarly. By definition of $\bar{\mathcal{X}}$, there exists an increasing sequence X_i such that $\lim_{i\to\infty} X_i = \bar{\mathcal{X}}(s)$ and $\bar{x}_1(X_i) < \bar{x}_2(Y_i)$ where $Y_i = 2s - X_i$. For i large enough, we have $X_l < X_i \leq X_r$ and, by (7.53), we get

$$\bar{x}_1(X_i) = \tilde{x}_1(X_i) < \bar{x}_2(Y_i) \tag{7.62}$$

If $Y_i \leq Y_r$ then $\bar{x}_2(Y_i) = \tilde{x}_2(Y_i)$ and

$$\tilde{x}_1(X_i) < \tilde{x}_2(Y_i). \tag{7.63}$$

If $Y_i > Y_r$ then (7.63) holds also. Indeed, let us assume the opposite. By the monotonicity of $\tilde{x}_1$ and $\tilde{x}_2$, we get

$$\tilde{x}_1(X_r) \geq \tilde{x}_1(X_i) \geq \tilde{x}_2(Y_i) \geq \tilde{x}_2(Y_r). \tag{7.64}$$

By (7.59), we have $\tilde{x}_1(X_r) = \tilde{x}_1(\mathcal{X}(s_r)) = \tilde{x}_2(\mathcal{Y}(s_r)) = \tilde{x}_2(Y_r)$ and therefore (7.64) implies that $\tilde{x}_2(Y_i) = \tilde{x}_2(Y_r) = x_r$. From the definitions of x_2 and $\tilde{x}_2$, we know that there exists a decreasing sequence x_j such that $\lim_{j\to\infty} x_j = \tilde{x}_2(Y_i)$ and $x_j + \nu_0(-\infty, x_j) \geq Y_i + \nu_0(-\infty, x_l)$. Letting j tend to infinity, we get $\tilde{x}_2(Y_i) + \nu_0(-\infty, x_2(Y_i)] \geq Y_i + \nu_0(-\infty, x_l)$. Hence, as $\tilde{x}_2(Y_i) = \tilde{x}_2(Y_r) = x_r$,

$$Y_r = x_r + \nu_0[x_l, x_r] \geq Y_i$$

which is a contradiction and we have proved that (7.63) holds. If $Y_l < \bar{\mathcal{Y}}(s)$, we get

$$\tilde{x}_1(\bar{\mathcal{X}}(s)) = \bar{x}_1(\bar{\mathcal{X}}(s)) = \bar{x}_2(\bar{\mathcal{Y}}(s)) = \tilde{x}_2(\bar{\mathcal{Y}}(s)) \tag{7.65}$$

from (7.48) and (7.49). If $Y_l = \bar{\mathcal{Y}}(s)$, $\tilde{x}_2(\bar{\mathcal{Y}}(s)) = x_l = \bar{x}_2(\bar{\mathcal{Y}}(s))$, by (7.59) and (7.61) so that (7.65) also holds. Then, it follows from (7.63) and (7.65) that $\bar{\mathcal{X}}(s) = \tilde{\mathcal{X}}(s)$ and the proof of (7.55) is complete.

Step 3. Let $\tilde{Z} = \mathbf{S}\tilde{\Theta}$ and $\bar{Z} = \mathbf{S}\bar{\Theta}$. We prove that

$$\bar{t}(X,Y) = \tilde{t}(X,Y), \quad \bar{x}(X,Y) = \tilde{x}(X,Y), \quad \bar{U}(X,Y) = \tilde{U}(X,Y) \tag{7.66}$$

for all $(X,Y) \in \Omega$. Since $\bar{x}_1 = \tilde{x}_1$ on $[X_l, X_r]$ and $\bar{x}_2 = \tilde{x}_2$ on $[Y_l, Y_r]$, we get, from the definition of $\mathbf{L}$, that

$$\bar{U}_1 = \tilde{U}_1, \quad \bar{V}_1 = \tilde{V}_1, \quad \tilde{J}_1 = \bar{J}_1 + \tilde{J}_1(X_l), \quad \tilde{K}_1 = \bar{K}_1 + \tilde{K}_1(X_l)$$

on $[X_l, X_r]$ and

$$\bar{U}_2 = \tilde{U}_2, \quad \bar{V}_2 = \tilde{V}_2, \quad \tilde{J}_2 = \bar{J}_2 + \tilde{J}_2(Y_l), \quad \tilde{K}_2 = \bar{K}_2 + \tilde{K}_2(Y_l)$$

on $[Y_l, Y_r]$. Since, by (7.55), the two paths $(\bar{\mathcal{X}}, \bar{\mathcal{Y}})$ and $(\tilde{\mathcal{X}}, \tilde{\mathcal{Y}})$ in $\mathcal{C}(\Omega)$ are equal, one can check, by using the definition of the mapping $\mathbf{C}$, that it implies that

$$\bar{t}(s) = \tilde{t}(s), \quad \bar{x}(s) = \tilde{x}(s), \quad \bar{U}(s) = \tilde{U}(s), \quad \tilde{J}(s) = \bar{J}(s) + \tilde{J}(s_l), \quad \tilde{K}(s) = \bar{K}(s) + \tilde{K}(s_l)$$

for $s \in [s_l, s_r]$ and

$$\bar{\mathcal{V}} = \tilde{\mathcal{V}}, \quad \bar{\mathcal{W}} = \tilde{\mathcal{W}}$$

on $[X_l, X_r]$ and $[Y_l, Y_r]$, respectively. The elements $\tilde{\Theta}$ and $\bar{\Theta}$ are equal in Ω except that the energy potentials J and K differ up to a constant. However one can check that the governing equation (2.13) is invariant with respect to addition of a constant to the energy potentials. Hence, by the uniqueness result of Lemma 4.10 which holds on finite domains, we get (7.66).

Step 4. We prove that there exists $(X_0, Y_0) \in \Omega$ such that

$$\bar{t}(X_0, Y_0) = \mathbf{t} \quad \text{and} \quad \bar{x}(X_0, Y_0) = \mathbf{x}. \tag{7.67}$$

We have

$$\bar{x}_1(X_l) = \bar{x}_2(Y_l) = \mathbf{x} - \kappa\mathbf{t} \quad \text{and} \quad \bar{x}_1(X_r) = \bar{x}_2(Y_r) = \mathbf{x} + \kappa\mathbf{t}$$

so that

$$\bar{x}(X_l, Y_l) = x_l \quad \text{and} \quad \bar{x}(X_r, Y_r) = x_r.$$

Let $P = (X_r, Y_l)$ denote the right-corner of Ω. We have

$$\bar{x}(P) - x_l = \int_{X_l}^{X_r} \bar{x}_X(X, Y_l)\, dX = \int_{X_l}^{X_r} c(\bar{U})\bar{t}_X(X, Y_l)\, dX$$

and

$$\bar{t}(P) = \int_{X_l}^{X_r} \bar{t}_X(X, Y_l)\, dX.$$

Hence, using the positivity of $\bar{t}_X$ and the assumption that $\frac{1}{\kappa} < c < \kappa$, we get

$$\bar{x}(P) - x_l \leq \kappa\bar{t}(P). \tag{7.68}$$

Similarly, one proves that $x_r - \bar{x}(P) \leq \kappa\bar{t}(P)$, which added to (7.68), yields $x_l - x_r \leq 2\kappa\bar{t}(P)$ or, after plugging the definition of x_l and x_r,

$$\mathbf{t} \leq \bar{t}(P). \tag{7.69}$$

The mapping $(X, Y) \mapsto (\bar{t}(X, Y), \bar{x}(X, Y))$ is surjective from $\mathbb{R}^2$ to $\mathbb{R}^2$ and there exists $P_0 = (X_0, Y_0) \in \mathbb{R}^2$, which may not be unique, such that (7.67) is fulfilled. Let us assume that

$$(\bar{t}(X, Y), \bar{x}(X, Y)) \neq (\mathbf{t}, \mathbf{x}) \tag{7.70}$$

for all $(X, Y) \in \Omega$. By using (7.69) and the monotonicity of the function t and x in the X and Y directions, we infer that either $X_0 > X_r$ and $Y_l \leq Y$ or $Y < Y_l$ and $X_0 \leq X_r$. We treat only the first case as the other case can be treated similarly. We have $Y_0 \leq Y_r$ as, otherwise, $x(X_0, Y_0) \geq x(X_r, Y_r) = x_r$. We introduce the point $P_1 = (X_r, Y_0) \in \Omega$. Let us assume $x(P_1) \geq x(P_0) = \mathbf{x}$. By the monotonicity of x, we get that $x(P_1) = x(P_0)$ and $x_X(X, Y_0) = 0$ for $X \in [X_r, X_0]$. It implies that $t_X(X, Y_0) = 0$ for $x \in [X_l, X_r]$ and therefore $t(P_1) = t(P_0) = \mathbf{t}$. However this contradicts the original assumption (7.70) and we must have that $x(P_1) < \mathbf{x}$. By following the same type of computation that lead to (7.68), we now get

$$\mathbf{x} > x(P_1) = x_r + \int_{Y_r}^{Y_0} x_Y(X_r, Y)\, dY \geq x_r - \kappa t(P_1) \geq x_r - \kappa\mathbf{t} \geq \mathbf{x},$$

which is a contradiction. Hence, (7.70) cannot hold and we have proved (7.67).

Step 5. We now conclude the argument. By definition, we have $\bar{u}(\mathbf{t},\mathbf{x}) = \bar{U}(X_0, Y_0)$ for any (X_0, Y_0) such that (7.67) holds. By (7.66), it follows that $\tilde{U}(X_0, Y_0) = \bar{U}(X_0, Y_0) = \bar{u}(\mathbf{t},\mathbf{x})$ and $\tilde{t}(X_0, Y_0) = \mathbf{t}$ and $\tilde{x}(X_0, Y_0) = \mathbf{x}$. It gives $U(f(X_0), g(Y_0)) = \bar{u}(\mathbf{t},\mathbf{x})$ and $t(f(X_0), g(Y_0)) = \mathbf{t}$ and $x(f(X_0), g(Y_0)) = \mathbf{x}$, so that $u(\mathbf{t},\mathbf{x}) = \bar{u}(\mathbf{t},\mathbf{x})$, by (6.13).

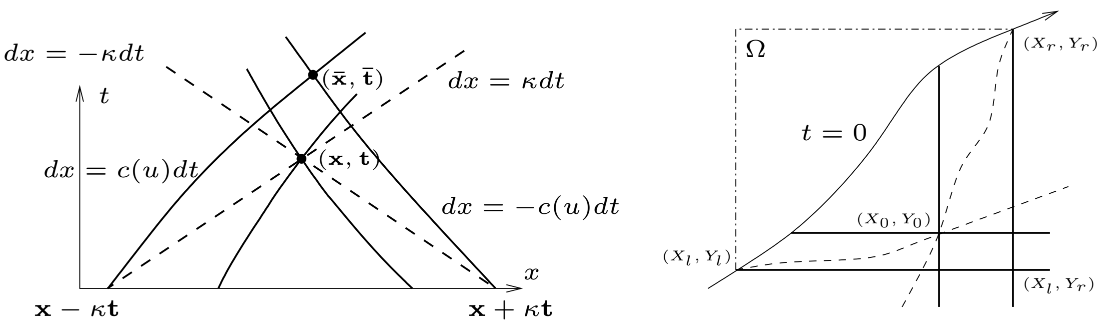

FIGURE 7. We have $\mathbf{t} = t(X_0, Y_0)$ and $\mathbf{x} = x(X_0, Y_0)$. In the new set of coordinates (X, Y), the domain of dependence is given by rectangles. We define the points (X_l, Y_l) and (X_r, Y_r) so that they correspond to the points $(\mathbf{x} - \kappa\mathbf{t}, 0)$ and $(\mathbf{x} + \kappa\mathbf{t}, 0)$. It then follows, from the boundedness of the function $c(u)$ that (X_0, Y_0) is contained in Ω.

□

8. Examples

There is a lack of explicit solutions for any choice of c except the trivial case of the linear wave equation for which c is constant. We here discuss two examples; first the linear case with general initial data, and second, a nonlinear case with very simple initial data.

8.1. **The linear wave equation.** In the case of the linear wave equation, the coefficient c is constant and the equivalent system (2.13) rewrites as

$$Z_{XY} = 0. \tag{8.1}$$

We consider general initial data $(u_0, R_0, S_0, \mu_0, \nu_0) \in \mathcal{D}$, let $(\psi_1, \psi_2) = \mathbf{L}(u_0, R_0, S_0, \mu_0, \nu_0)$ and $\Theta = (\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{V}, \mathcal{W}) = \mathbf{C}(\psi_1, \psi_2)$. From (8.1), we get that

$$Z_X(X, Y) = \mathcal{V}(X) \text{ and } Z_Y(X, Y) = \mathcal{W}(X).$$

Given a point $(X, Y) \in \mathbb{R}^2$, we denote $s_0 = \mathcal{Y}^{-1}(Y)$ and $s_1 = \mathcal{X}^{-1}(X)$ so that

$$\mathcal{Y}(s_0) = Y \text{ and } \mathcal{X}(s_1) = X. \tag{8.2}$$

We have

$$Z(X,Y) = \mathcal{Z}(s_0) + \int_{\mathcal{X}(Y)}^{X} Z_X(\bar{X},Y)\,d\bar{X} = \mathcal{Z}(s_0) + \int_{\mathcal{X}(Y)}^{X} \mathcal{V}(\bar{X})\,d\bar{X}$$

and

$$Z(X,Y) = \mathcal{Z}(s_1) + \int_{\mathcal{Y}(X)}^{Y} Z_Y(X,\bar{Y})\,d\bar{Y} = \mathcal{Z}(s_1) + \int_{\mathcal{Y}(X)}^{Y} \mathcal{W}(\bar{Y})\,d\bar{Y}.$$

By averaging these two equations, we get

$$Z(X,Y) = \frac{1}{2}(\mathcal{Z}(s_0) + \mathcal{Z}(s_1)) + \frac{1}{2}(\int_{\mathcal{X}(Y)}^{X} \mathcal{V}(\bar{X})\,d\bar{X} + \int_{\mathcal{Y}(X)}^{Y} \mathcal{W}(\bar{Y})\,d\bar{Y}).$$

After a change of variables, it yields

$$Z(X,Y) = \frac{1}{2}(\mathcal{Z}(s_0) + \mathcal{Z}(s_1)) + \frac{1}{2}\int_{s_0}^{s_1} (\mathcal{V}(\mathcal{X}(s))\dot{\mathcal{X}}(s) - \mathcal{W}(\mathcal{Y}(s))\dot{\mathcal{Y}}(s))\,ds. \tag{8.3}$$

We recall that for $\Theta \in \mathcal{G}$, we have $\mathcal{V}_2(\mathcal{X}(s))\dot{\mathcal{X}}(s) = \mathcal{W}_2(\mathcal{Y}(s))\dot{\mathcal{Y}}(s)$, see, for example, (3.32). For the first component $Z_1(X,Y)$ that we denote $t(X,Y)$, we have $\mathcal{Z}_1(s) = t(s) = 0$ for all $s \in \mathbb{R}$ and, after using (3.8b), we get

$$\begin{aligned} t(X,Y) &= \frac{1}{2c}\int_{s_0}^{s_1} (\mathcal{V}_2(\mathcal{X}(s))\dot{\mathcal{X}}(s) + \mathcal{W}_2(\mathcal{Y}(s))\dot{\mathcal{Y}}(s))\,ds \\ &= \frac{1}{2c}(\mathcal{Z}_2(s_1) - \mathcal{Z}_2(s_0)), && \text{by (3.7)} \\ &= \frac{1}{2c}(x_1(X) - x_2(Y)), && \text{by (3.24b)}. \end{aligned} \tag{8.4}$$

As far as the second component $Z_2(X,Y) = x(X,Y)$ is concerned, it follows directly from (8.3) and (3.24b) that

$$x(X,Y) = \frac{1}{2}(x_1(X) + x_2(Y)). \tag{8.5}$$

For the third component $Z_3(X,Y) = U(X,Y)$, we have $\mathcal{Z}_3(s_0) = u_0(x_1(X))$ and $\mathcal{Z}_3(s_1) = u_0(x_2(Y))$. After using (3.34e) and (3.31), we get, after a change of variables, that

$$\int_{s_0}^{s_1} (\mathcal{V}_3(\mathcal{X}(s))\dot{\mathcal{X}}(s) = \int_{s_0}^{s_1} (R_0(x_1(\mathcal{X}(s)))x_1'(\mathcal{X}(s))\dot{\mathcal{X}}(s)\,ds = \frac{1}{2c}\int_{x_2(Y)}^{x_1(X)} R_0(x)\,dx.$$

We use the fact that $x_1(\mathcal{X}(s_1)) = x_2(\mathcal{Y}(s_1)) = x_2(Y)$, which follows from (3.23) and (8.2). Similarly, we obtain that $\int_{s_0}^{s_1} W(\mathcal{Y}(s))\dot{\mathcal{Y}}(s)\,ds = -\frac{1}{2c}\int_{x_1(X)}^{x_2(Y)} S_0(x)\,dx$. Hence, (8.3) yields

$$U(X,Y) = \frac{1}{2}(u_0(x_1(X)) + u_0(x_2(Y))) + \frac{1}{4c}\int_{x_1(X)}^{x_2(Y)} (R_0 + S_0)\,dx. \tag{8.6}$$

From (8.4) and (8.5), it follows that $x_1(X) = x(X,Y) - ct(X,Y)$ and $x_2(Y) = x(X,Y) + ct(X,Y)$. Therefore, after using (6.13), we recover d'Alembert's formula from (8.6), i.e.,

$$u(t,x) = \frac{1}{2}(u_0(x-ct) + u_0(x+ct)) + \frac{1}{4c}\int_{x-ct}^{x+ct} (R_0 + S_0)\,dx$$

for the solution of the linear wave equation. Let us now look at the energy. We use the same notation as in the proof of Theorem 7.10. For a given time t, $(\mathcal{X}(t,s),\mathcal{Y}(t,s))$ denotes the curve corresponding to a given time, that is, $t(\mathcal{X}(t,s),\mathcal{Y}(t,s))=t$ (Beware of the notation, $t(\cdot,\cdot)$ denotes a function while t, without argument, denotes a constant). For any point x, we have

$$\mu(t)(-\infty,x)=\int_{x(\mathcal{X}(t,s),\mathcal{Y}(t,s))<x} J_X(\mathcal{X}(t,s),\mathcal{X}(t,s))\mathcal{X}_s(t,s)\,ds.$$

From (8.4) and (8.5), we get that $x(\mathcal{X}(t,s),\mathcal{Y}(t,s))<x$ if and only if $x_1(\mathcal{X}(t,s))<x+ct$. Since $J_X(X,Y)=\mathcal{V}_4(X)=J_1'(X)$, we get

$$\mu(t)(-\infty,x)=\int_{x_1(\mathcal{X}(t,s))<x+ct} J_1'(\mathcal{X}(t,s))\mathcal{X}_s(t,s)\,ds.$$

After a change of variables, it yields

$$\mu(t)(-\infty,x)=\int_{x_1(X)<x+ct} J_1'(X)\,dX=\mu_0(-\infty,x+ct).$$

Hence, for any Borel set B, we have

$$\mu(t)(B)=\mu_0(B+ct).$$

Similarly, we get

$$\nu(t)(B)=\nu_0(B-ct).$$

8.2. **An example with singular initial data.** Let

$$u_0(x)=1,\ R_0(x)=S_0(x)=0 \tag{8.7a}$$

for all $x\in\mathbb{R}$ and

$$\nu_0=2\mu_0=2\delta \tag{8.7b}$$

where δ denotes the Dirac delta function. Our intention is to consider initial data for which all the energy is concentrated in a set of zero measure (in this case the origin) and that is why we choose u_0 equal to a constant.[2] Since u_0 does not belong to $L^2(\mathbb{R})$, the theory we have developed does not apply directly. However, we can consider the sequence of solutions $(u_N,R_N,S_N,\mu_N,\nu_N)$ given by the semigroup $\bar{S}_t$ for the following initial data

$$u_0^N(x)=\begin{cases}1 & \text{for } x\in[-N,N]\\ 0 & \text{otherwise}\end{cases}$$

and $R_0^N(x)=S_0^N(x)=0$, $\nu_0=2\mu_0=2\delta$. Given a compact domain in time and space, we know that for N large enough the solutions will coincide on this compact domain due to the finite time of propagation, see Theorem 7.11. Thus we can define the solution of (1.1) for the initial data (8.7) as the limit of the solutions u^N when N tends to ∞. We see that, by using the same type of construction, we can actually construct solutions for any initial data such that u_0,R_0,S_0 belong to $L^2_{\text{loc}}(\mathbb{R})$ and μ, ν are (not necessarily finite) Radon measures (note that, by definition, a Radon measure is finite on compacts).

[2] If we choose $u_0=0$ (the only constant in $L^2(\mathbb{R})$) then, since $c'(0)=0$, one can check from the governing equations (2.13) that there is no evolution of the solution, and we have that $u(t,x)=0$, $2\mu(t)=\nu(t)=2\delta$ is the conservative solution.

For the initial data $(u_0, R_0, S_0, \mu_0, \nu_0)$ given by (8.7), let us denote $(\psi_1, \psi_2) = \mathbf{L}(u_0, R_0, S_0, \mu_0, \nu_0)$ as defined in Definition 3.8 and $\Theta = (\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{V}, \mathcal{W}) = \mathbf{C}(\psi_1, \psi_2)$ defined by Definition 3.5. We first find

$$x_1(X) = \begin{cases} X & \text{if } X < 0, \\ 0 & \text{if } 0 \le X \le 1, \\ X - 1 & \text{if } X > 1, \end{cases} \qquad x_2(Y) = \begin{cases} Y & \text{if } Y < 0, \\ 0 & \text{if } 0 \le Y \le 2, \\ Y - 2 & \text{if } Y > 2, \end{cases} \tag{8.8}$$

which yields

$$\begin{aligned} \Gamma_0 &= \{(X, Y) \mid x_1(X) = x_2(Y)\} \\ &= \{(X, X) \mid X \le 0\} \cup \big([0,1] \times [0,2]\big) \cup \{(X, X+1) \mid X \ge 1\}. \end{aligned}$$

Furthermore

$$\begin{aligned} J_1(X) &= \begin{cases} 0 & \text{if } X < 0, \\ X & \text{if } 0 \le X \le 1, \\ 1 & \text{if } X > 1, \end{cases} & J_2(Y) &= \begin{cases} 0 & \text{if } Y < 0, \\ Y & \text{if } 0 \le Y \le 2, \\ 2 & \text{if } Y > 2, \end{cases} \\ U_1(X) &= 1, & U_2(Y) &= 1, \\ V_1(X) &= 0, & V_2(Y) &= 0, \\ K_1(X) &= \begin{cases} 0 & \text{if } X < 0, \\ X/c(1) & \text{if } 0 \le X \le 1, \\ 1/c(1) & \text{if } X > 1, \end{cases} & K_2(Y) &= \begin{cases} 0 & \text{if } Y < 0, \\ -Y/c(1) & \text{if } 0 \le Y \le 2, \\ -2/c(1) & \text{if } Y > 2. \end{cases} \end{aligned}$$

Next, we obtain

$$\mathcal{X}(s) = \begin{cases} s & \text{if } s < 0, \\ 0 & \text{if } 0 \le s < 1, \\ 2s - 2 & \text{if } 1 \le s < 3/2, \\ s - 1/2 & \text{if } 3/2 \le s, \end{cases} \qquad \mathcal{Y}(s) = \begin{cases} s & \text{if } s < 0, \\ 2s & \text{if } 0 \le s < 1, \\ 2 & \text{if } 1 \le s < 3/2, \\ s + 1/2 & \text{if } 3/2 \le s, \end{cases} \tag{8.9a}$$

and

$$x(s) = \begin{cases} s & \text{if } s < 0, \\ 0 & \text{if } 0 \le s < 3/2, \\ s - 3/2 & \text{if } 3/2 \le s, \end{cases} \tag{8.9b}$$

and $U(s) = 1$ and

$$J(s) = \begin{cases} 0 & \text{if } s < 0, \\ 2s & \text{if } 0 \le s < 3/2, \\ 3 & \text{if } 3/2 \le s, \end{cases} \qquad K(s) = \begin{cases} 0 & \text{if } s < 0, \\ -\frac{2s}{c(1)} & \text{if } 0 \le s < 1, \\ \frac{2(s-2)}{c(1)} & \text{if } 1 \le s < \frac{3}{2}, \\ -\frac{1}{c(1)} & \text{if } 3/2 \le s, \end{cases} \tag{8.9c}$$

$$c(1)\mathcal{V}_1(X) = \mathcal{V}_2(X) = \begin{cases} 1/2 & \text{if } X < 0, \\ 0 & \text{if } 0 \le X < 1, \\ 1/2 & \text{if } 1 \le X, \end{cases} \tag{8.9d}$$

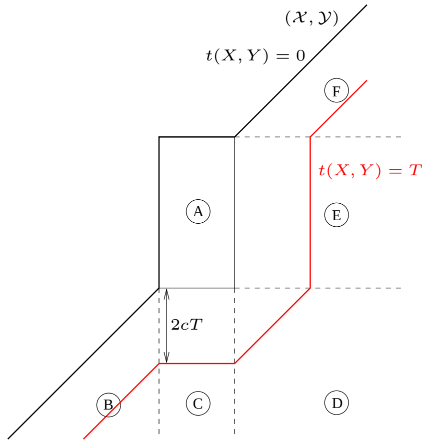

FIGURE 8. Plot of the initial data curve and the curve for a given time T.

$$c(1)\mathcal{W}_1(Y) = -\mathcal{W}_2(Y) = \begin{cases} 1/2 & \text{if } Y < 0, \\ 0 & \text{if } 0 \le Y < 2, \\ 1/2 & \text{if } 2 \le Y, \end{cases} \tag{8.9e}$$

and $\mathcal{V}_3 = \mathcal{W}_3 = 0$ and

$$c(1)\mathcal{V}_5(X) = \mathcal{V}_4(X) = \begin{cases} 0 & \text{if } X < 0, \\ 1 & \text{if } 0 \le X < 1, \\ 0 & \text{if } 1 \le X, \end{cases} \tag{8.9f}$$

$$c(1)\mathcal{W}_5(Y) = -\mathcal{W}_4(Y) = \begin{cases} 0 & \text{if } Y < 0, \\ 1 & \text{if } 0 \le Y < 2, \\ 0 & \text{if } 2 \le Y. \end{cases} \tag{8.9g}$$

In the case of the linear wave equation, the solution is explicit. In Figure 8, we plot the curve $(\mathcal{X},\mathcal{Y})$ and the curve $t(X,Y) = T$, for a given T. In Figure 8, the letters A to F denote the regions which are delimited by the neighboring solid or dashed black lines. The values of Z in these different regions are given in Table 1.

If we consider the choice

$$c(u)^2 = \beta\cos^2 u + \alpha\sin^2 u \tag{8.10}$$

where α and β are strictly positive constants, then no explicit solutions are available for the initial data (8.7). Due to the finite speed of propagation, we know that

$$U(X,Y) = 1 \text{ on } (-\infty,0]\times(-\infty,0]\bigcup[0,1]\times[0,2]\bigcup[1,\infty)\times[2,\infty). \tag{8.11}$$

The measures ν and μ will become regular measures for t nonzero. We take $\alpha = 0.2$ and $\beta = 0.1$. The solution is illustrated on Figs. 1, 9–12. Here we have used the numerical method described in Section 9.

	A	B	C	D	E	F
$t(X,Y)$	0	$\frac{X-Y}{2c}$	$-\frac{Y}{2c}$	$\frac{X-Y-1}{2c}$	$\frac{X-1}{2c}$	$\frac{X-Y+1}{2c}$
$x(X,Y)$	0	$\frac{X+Y}{2}$	$\frac{Y}{2}$	$\frac{X+Y-1}{2}$	$\frac{X-1}{2}$	$\frac{X+Y-3}{2}$
$U(X,Y)$	1	1	1	1	1	1
$J(X,Y)$	$X+Y$	0	X	1	$1+Y$	3
$K(X,Y)$	$\frac{X-Y}{c}$	0	$\frac{X}{c}$	$\frac{1}{c}$	$\frac{1-Y}{c}$	$-\frac{1}{c}$

TABLE 1. The values of the solution $Z=(t,x,U,J,K)$ of the linear wave equation for the initial data given by (8.7) in the different domains of the plane (see Figure 8, the regions A-F are delimited by the dashed and solid dark lines).

9. A numerical method for conservative solutions

Next we describe a general numerical approach to obtain conservative solutions of the NVW equation. Traditional finite difference methods will not yield conservative solutions, and we here use the full machinery of the analytical approach to derive an efficient numerical method for conservative solutions.

We discretize the problem as follows. Given N, $s_{\min}$ and $s_{\max}$, we set $h=(s_{\max}-s_{\min})/N$ and $s_i=s_{\min}+ih$ for $i=0,\dots,N$. Let

$$X_i=\mathcal{X}(s_i),\qquad Y_j=\mathcal{Y}(s_j),\qquad P_{i,j}=[X_i,X_j]$$

for $i=0,\dots,N$ and $j=0,\dots,N$. We compute the solution of (2.13) on the domain $\Omega=[X_0,X_N]\times[Y_0,Y_N]$. The algorithm follows the same type of iteration as in the proof of Lemma 4.10, and we use the same notation here. We approximate the form $Z_X(X,Y_j)\,dX$ on the interval $[X_{i-1},X_i]$ by the constant $V_{i,j}$ and the form $Z_Y(X_i,Y)\,dY$ on the interval $[Y_j,Y_{j+1}]$ by the constant $W_{i,j}$. We denote by $Z^h_{i,j}$ and $Z^v_{i,j}$ the approximation of Z on the segments $P_{i-1,j}-P_{i,j}$ and $P_{i,j}-P_{i,j+1}$, respectively. The initial curve is approximated on the piecewise horizontal and vertical line $\bigcup_{i=1}^{N-1}([P_{i,i},P_{i,i+1}]\cup[P_{i,i+1},P_{i+1,i+1}])$ and we set

$$Z^h_{i,i}=\mathcal{Z}(s_i),\quad\text{and}\quad V_{i,i}=\frac{1}{X_i-X_{i-1}}\int_{X_{i-1}}^{X_i}\mathcal{V}(X)\,dX\quad\text{for } i=1,\dots,N,$$

$$Z^v_{i,i}=\mathcal{Z}(s_i),\quad\text{and}\quad W_{i,i}=\frac{1}{Y_{i+1}-Y_i}\int_{Y_i}^{Y_{i+1}}\mathcal{W}(X)\,dX\quad\text{for } i=0,\dots,N-1,$$

where $\mathcal{Z}$, $\mathcal{V}$ and $\mathcal{W}$ are given by (8.9). If X_i-X_{i-1} (respectively $Y_{j+1}-Y_j$) is equal to zero, then we set $V_{i,i}$ (respectively $W_{i,i}$) to zero or an arbitrary value (this value will not have any impact on the computed solution). We compute the solution iteratively on vertical and horizontal strips: Given $n \in \{0, \dots, N\}$, we assume that the values of

$$Z^h_{i,j},\ V_{i,j} \quad \text{for } 1 \le i \le n, \quad 0 \le j \le n, \tag{9.1a}$$

$$Z^v_{i,j},\ W_{i,j} \quad \text{for } 0 \le i \le n, \quad 0 \le j \le n, \tag{9.1b}$$

have been computed. Then, we set iteratively, for $j = n+1, \dots, 1$,

$$\begin{aligned}
Z^h_{n+1,j-1} &= Z^h_{n+1,j} - (Y_j - Y_{j-1})W_{n,j-1},\\
V_{n+1,j-1} &= V_{n+1,j} - (Y_j - Y_{j-1})F(\frac{1}{2}(Z^h_{n+1,j} + Z^v_{n,j-1}))(V_{n+1,j}, W_{n,j-1}),\\
Z^v_{n+1,j-1} &= Z^v_{n,j-1} + (X_{n+1} - Y_n)V_{n+1,j},\\
W_{n+1,j-1} &= W_{n,j-1} + (X_{n+1} - X_n)F(\frac{1}{2}(Z^h_{n+1,j} + Z^v_{n,j-1}))(V_{n+1,j}, W_{n,j-1}),
\end{aligned}$$

and, for $i = n+1, \dots, 2$,

$$\begin{aligned}
Z^h_{i-1,n+1} &= Z^h_{i-1,n} + (Y_{n+1} - Y_n)W_{i-1,n},\\
V_{i-1,n+1} &= V_{i-1,n} + (Y_{n+1} - Y_n)F(\frac{1}{2}(Z^h_{i-1,n} + Z^v_{i-1,n}))(V_{i-1,n}, W_{i-1,n}),\\
Z^v_{i-1,n+1} &= Z^v_{i,n+1} - (X_i - X_{i-1})V_{i,n+1},\\
W_{i-1,n+1} &= W_{i,n+1} - (X_i - X_{i-1})F(\frac{1}{2}(Z^h_{i,n+1} + Z^v_{i,n+1}))(V_{i,n+1}, W_{i,n+1}),
\end{aligned}$$

and

$$\begin{aligned}
Z^v_{0,n+1} &= Z^v_{1,n+1} - (X_1 - X_0)V_{1,n+1},\\
W_{0,n+1} &= W_{1,n+1} - (X_1 - X_0)F(\frac{1}{2}(Z^h_{1,n+1} + Z^v_{1,n+1}))(V_{1,n+1}, W_{1,n+1}).
\end{aligned}$$

We have defined the quantities in (9.1) for n replaced by $n+1$. By induction we have computed the solution on the whole domain Ω. To compute the solution at a given time T, we have to extract a curve $(\mathcal{X}, \mathcal{Y}) \in \mathcal{C}$ such that $t(\mathcal{X}(s), \mathcal{Y}(s)) = T$ for all $s \in \mathbb{R}$. We proceed by iteration and compute a set of grid points that approximates well such a curve, for example, by taking

$$\begin{aligned}
i(k) &= \sup\{i \in \{0, \dots, N\} \mid t(X_{i,k}, Y_{i,k}) < T\},\\
j(k) &= k,
\end{aligned}$$

for $k = 0, \dots, N$. For a given T, the function $u(T,x)$ can be seen as the curve $(x, u(T,x))$ in $\mathbb{R}^2$ which is parametrized by $x \in \mathbb{R}$, and we approximate this curve by the points

$$(x(X_{i(k),j(k)}, Y_{i(k),j(k)}), U(X_{i(k),j(k)}, Y_{i(k),j(k)}))$$

for $k = 0, \dots, N$. This method has been used to produce the results presented in Figure 1.

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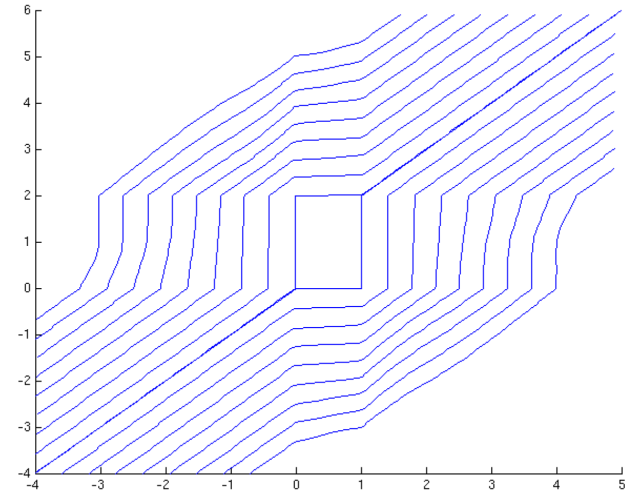

FIGURE 9. Plot of the isotimes, that is, the curves for which $t(X, Y)$ is a constant. Note the box in the middle in which t is constant and equal to zero.

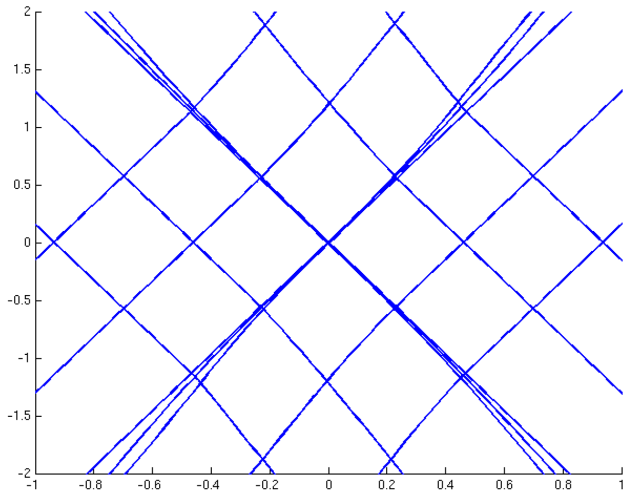

FIGURE 10. Forward and backward characteristics in the (x, t) plane. Both families have a point of intersection at zero. It corresponds to the point where the measures μ and ν become singular.

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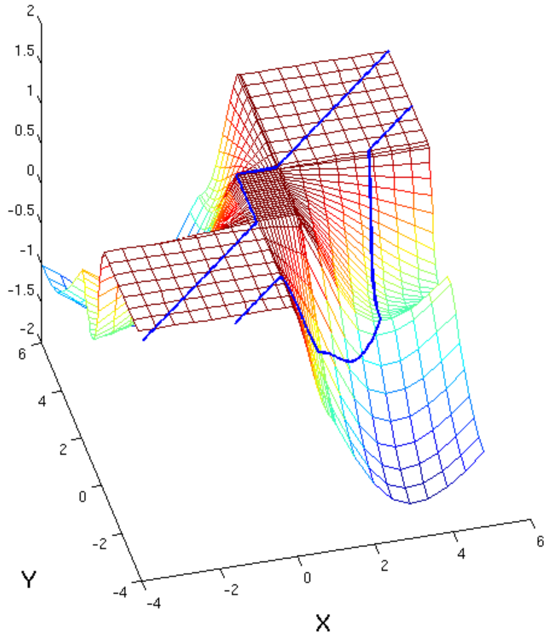

FIGURE 11. Plot of $U(X,Y)$. The blue curves single out the solution $U(X,Y)$ for times $t=0$ and $t=3$.

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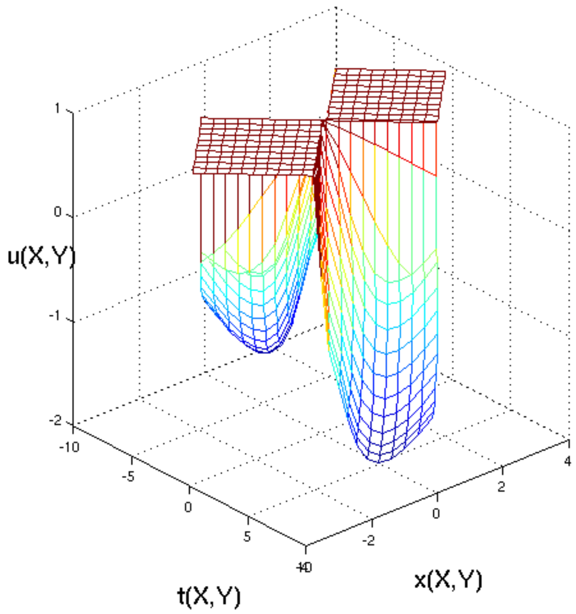

Figure 12. Plot of the surface $(t(X,Y), x(X,Y), U(X,Y))$ parametrized by (X,Y) and which is approximated by $(t(P_{i,j}), x(P_{i,j}), U(P_{i,j}))$ for $i,j = 0,\ldots,N$.

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