A CONVERGENT MIXED METHOD FOR THE STOKES APPROXIMATION OF VISCOUS COMPRESSIBLE FLOW

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Abstract. We propose a mixed finite element method for the motion of a strongly viscous, ideal, and isentropic gas. At the boundary we impose a Navier–slip condition such that the velocity equation can be posed in mixed form with the vorticity as an auxiliary variable. In this formulation we design a finite element method, where the velocity and vorticity is approximated with the div- and curl- conforming Nédélec elements, respectively, of the first order and first kind. The mixed scheme is coupled to a standard piecewise constant upwind discontinuous Galerkin discretization of the continuity equation. For the time discretization, implicit Euler time stepping is used. Our main result is that the numerical solution converges to a weak solution as the discretization parameters go to zero. The convergence analysis is inspired by the continuous analysis of Feireisl and Lions for the compressible Navier–Stokes equations. Tools used in the analysis include an equation for the effective viscous flux and various renormalizations of the density scheme.

Contents

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1. Introduction

Let $\Omega \subset \mathbb{R}^N$, $N = 2$, 3, be an open, convex, polygonal domain with Lipschitz boundary $\partial\Omega$ and let $T > 0$ be a final time. We consider the flow of an ideal isentropic viscous gas governed by the *Stokes approximation equations*

$$\partial_t \varrho + \operatorname{div}_x(\varrho \boldsymbol{u}) = 0, \quad \text{in } (0,T) \times \Omega, \tag{1.1}$$

$$\partial_t \boldsymbol{u} - \mu\Delta \boldsymbol{u} - \lambda \nabla_x \operatorname{div}_x \boldsymbol{u} + \nabla_x p(\varrho) = 0, \quad \text{in } (0,T) \times \Omega. \tag{1.2}$$

Here, the unknowns are the density $\varrho = \varrho(t,x) > 0$ and velocity $\boldsymbol{u} = \boldsymbol{u}(t,x) \in \mathbb{R}^N$. The operators ∇_x and div_x are respectively the spatial gradient and divergence operators, and $\Delta = \operatorname{div}_x \nabla_x$ is the Laplace operator. The viscosity coefficients μ, λ are assumed to be constant and to satisfy $\mu > 0$, $N\lambda + 2\mu \geq 0$.

The pressure is given by *Boyle's law* which in the isentropic regime takes the form $p(\varrho) = a\varrho^\gamma$, where $a > 0$ is constant. In real applications the value of γ ranges from a maximum of $\frac{5}{3}$ for monoatomic gases, to values close to one for polyatomic gases at high temperatures. In this paper, we will for purely technical reasons be forced to require $\gamma > \frac{N}{2}$.

From the point of view of applications, the model (1.1)–(1.2) can be justified for flows at very low Reynolds numbers so that the effects of convection may be neglected. It is also on the same form as various shallow water models [11]. From a mathematical perspective, the system (1.1)–(1.2) is a model problem containing some, but not all, of the difficulties associated with compressible fluid dynamics.

Mathematical analysis concerning the well-posedness of the system (1.1)–(1.2) seems to originate with the papers [12, 14] by Kazhikov and collaborators. Several other contributions on the existence and long term stability exist, also in the context of similar shallow water models. However, for our purpose here, the most relevant study is that of Lions [11] in which the global existence of (weak) solutions and some higher regularity results are established.

In this paper we impose the following boundary conditions:

$$\boldsymbol{u} \cdot \nu = 0, \quad \text{on } (0,T) \times \partial\Omega, \tag{1.3}$$

$$\operatorname{curl}_x \boldsymbol{u} \times \nu = 0, \quad \text{on } (0,T) \times \partial\Omega, \tag{1.4}$$

where ν is the unit outward normal on $\partial\Omega$ and curl_x is the curl operator. Here, in 2D, curl_x denotes the rotation operator taking vectors into scalars. The first condition is a natural condition of impermeability type on the normal velocity. The second condition is in the literature commonly referred to as the Navier–slip condition. While these boundary conditions are not motivated by physics, they are widely used in numerical methods. In particular, in the context of geophysical flows they are often preferred over classical Dirichlet conditions since the latter necessitates expensive calculations of boundary layers. Of more importance to this paper, the boundary conditions (1.3)– (1.4) allow us to pose the system (1.1)–(1.2) in mixed form with $\operatorname{curl}_x \boldsymbol{u}$ as an auxiliary variable. This fact will play a crucial role in the upcoming analysis.

While many numerical methods appropriate for the Stokes approximation and Navier–Stokes equations have been proposed, the convergence properties of these methods are mostly unsettled. In fact, it is not clear whether or not any of these methods, in more than one dimension, converge to a (weak) solution as discretization parameters tend to zero. In one dimension, there are some available results due to D. Hoff and his collaborators. However, these results apply to an ideal gas in Lagrangian coordinates and with initial data of bounded variation. In more than one dimension, there are some recent results for simplified models. In the papers [6, 7], a convergent finite element method for a stationary compressible Stokes system is proposed and analyzed. The system considered there are similar to (1.1)–(1.2) but without temporal dependence. In [9], we established convergence of a finite element method for a semi–stationary version of (1.1)–(1.2) ($\partial_t \boldsymbol{u} = 0$) and homogenous Dirichlet boundary conditions. This paper can be seen as a continuation of the recent study [8] in which a convergent numerical method for the same semi–stationary system ((1.1)–(1.2) with $\partial_t \boldsymbol{u} = 0$) with boundary conditions (1.3)–(1.4) was established. The main novelty of this paper is consequently the addition of the time derivative term $\partial_t \boldsymbol{u}$ in the velocity equation (1.2).

Let us now discuss our choice of numerical method for the Stokes approximation equations. For the time discretization, we will use implicit time stepping in both equations. To approximate the continuity equation (1.1) we will use a standard piecewise constant upwind discontinuous Galerkin method. To approximate the velocity, we will use a mixed finite element method with the Nédélec's spaces of the first order and first kind. The mixed formulation is motivated by introducing the vorticity $\boldsymbol{w} = \operatorname{curl}_x \boldsymbol{u}$ as an auxiliary unknown and recasting the velocity equation (1.2) in the form:

$$\partial_t \boldsymbol{u} + \mu \operatorname{curl}_x \boldsymbol{w} - (\lambda+\mu)\nabla_x \operatorname{div}_x \boldsymbol{u} + \nabla_x p(\varrho) = 0, \tag{1.5}$$

where the identity $-\Delta = \operatorname{curl}_x \operatorname{curl}_x - \nabla_x \operatorname{div}_x$ is used. This leads to a natural mixed formulation in which the requirement $\boldsymbol{w} = \operatorname{curl}_x \boldsymbol{u}$ plays the role of a lagrangian multiplier.

Denote by $\boldsymbol{W}_0^{\operatorname{div},2}(\Omega)$ the vector fields $\boldsymbol{u}$ on Ω for which $\operatorname{div}_x \boldsymbol{u} \in L^2$ and $\boldsymbol{u} \cdot \nu|_{\partial\Omega} = 0$, and by $\boldsymbol{W}_0^{\operatorname{curl},2}(\Omega)$ the vector fields $\boldsymbol{w}$ on Ω for which $\operatorname{curl}_x \boldsymbol{w} \in L^2$ and $\boldsymbol{w} \times \nu|_{\partial\Omega} = 0$. We choose corresponding finite element spaces $\boldsymbol{V}_h \subset \boldsymbol{W}_0^{\operatorname{div},2}(\Omega)$ and $\boldsymbol{W}_h \subset \boldsymbol{W}_0^{\operatorname{curl},2}(\Omega)$ based on Nédélec's elements of the first order and first kind [13]. The mixed finite element methods seeks, for each time step $k = 1, \dots, M$, functions $(\boldsymbol{w}_h^k, \boldsymbol{u}_h^k) \in \boldsymbol{W}_h \times \boldsymbol{V}_h$ such that

$$\begin{aligned} &\int_\Omega \partial_t^h\left(\boldsymbol{u}_h^k\right) \boldsymbol{v}_h + \mu \operatorname{curl}_x \boldsymbol{w}_h^k \boldsymbol{v}_h + \left[(\lambda+\mu)\operatorname{div}_x \boldsymbol{u}_h^k - p(\varrho_h^k)\operatorname{div}_x \boldsymbol{v}_h\right]\, dx = 0, \\ &\int_\Omega \boldsymbol{w}_h^k \boldsymbol{\eta}_h - \boldsymbol{u}_h^k \operatorname{curl}_x \boldsymbol{\eta}_h \, dx = 0, \end{aligned} \tag{1.6}$$

for all $(\boldsymbol{v}_h, \boldsymbol{\eta}_h) \in \boldsymbol{W}_h \times \boldsymbol{V}_h$, where ϱ_h^k is given and $\partial_t^h\left(\boldsymbol{u}_h^k\right) = (\Delta t)^{-1}[\boldsymbol{u}_h^k - \boldsymbol{u}_h^{k-1}]$ denotes implicit time stepping. Note that the boundary conditions (1.3)–(1.4) are mandatory to obtain this formulation.

Our main result is that $\{(\boldsymbol{w}_h, \boldsymbol{u}_h, \varrho_h)\}_{h>0}$ converges to a weak solution of the *Stokes approximation equations*, at least along a subsequence. The major difficulty is to obtain strong compactness of the density approximation $\{\varrho_h\}_{h>0}$ which is needed in order to pass to the limit in the nonlinear pressure function. Since the density approximations are only bounded in $L^\infty(0,T;L^\gamma(\Omega))$ this is intricate. At the heart of the convergence analysis lies the *effective viscous flux* $P_{\text{eff}}(\varrho_h, \boldsymbol{u}_h) = p(\varrho_h) - (\lambda+\mu)\operatorname{div}_x \boldsymbol{u}_h$. In particular, strong convergence of the density approximation follows from the property:

$$\lim_{h\to 0} \int\int \psi P_{\text{eff}}(\varrho_h, \boldsymbol{u}_h)\varrho_h \, dxdt = \int\int \psi \overline{P_{\text{eff}}(\varrho, \boldsymbol{u})}\varrho \, dxdt, \tag{1.7}$$

for all $\psi \in C_c^\infty(0,T)$. It is in the process of obtaining (1.7) that the carefully selected finite element spaces and mixed form prove useful. Specifically, we obtain (1.7) by setting $\boldsymbol{v}_h = \Pi_h^V \nabla_x \Delta^{-1} \varrho_h$ in (1.6), where Π_h^V is the canonical interpolation operator into $\boldsymbol{V}_h$. This test function satisfies $\operatorname{div}_x \boldsymbol{v}_h = \varrho_h$ and is almost orthogonal to curls. The main difficult in obtaining (1.7) is to treat the time derivative term, which, with $\boldsymbol{v}_h$ as described above, is of the form

$$\begin{aligned}\int\int \partial_t^h(\boldsymbol{u}_h)\,\Pi_h^V\nabla_x\Delta^{-1}[\varrho_h]\;dxdt &= \int\int \partial_t^h(\boldsymbol{u}_h)\,\nabla_x\Delta^{-1}[\varrho_h]\;dxdt + O(h)\\ &= \int\int \Delta^{-1}\left[\operatorname{div}_x\boldsymbol{u}_h\right]\partial_t^h(\varrho_h)\;dxdt + O(h).\end{aligned}$$

Using the continuity scheme, the last term the last term can be shown to converge. The property (1.7) then follows. Our analysis resembles that of Lions and Feireisl for the compressible Navier–Stokes equations.

As part of the analysis, we will need that the discrete velocity $\boldsymbol{u}_h$ converges strongly to a function $\boldsymbol{u}$. This is not immediate since the approximation space $\boldsymbol{V}_h$ is only div_x conforming. To obtain strong convergence, we utilize the discrete Hodge decomposition $\boldsymbol{V}_h = \operatorname{curl}_x\boldsymbol{W}_h + \boldsymbol{V}_h^{0,\perp}$ satisfied by the chosen Nédélec spaces. When writing $\boldsymbol{u}_h = \operatorname{curl}_x\boldsymbol{\zeta}_h + \boldsymbol{z}_h$, it can be seen that $\boldsymbol{\zeta}_h$ does not depend on the density ϱ_h and as a consequence converges strongly. The remaining term $\boldsymbol{z}_h$ is then weakly discrete curl free with bounded divergence and an estimate from the previous paper [8] yields

$$\|\boldsymbol{z}_h(t,x) - \boldsymbol{z}_h(t,x-\xi)\|_{L^2(0,T;\boldsymbol{L}^2(\Omega))} \to 0, \quad \text{as } |\xi|\to 0, \tag{1.8}$$

uniformly in h. From the velocity scheme, we deduce a weak time-continuity estimate of the form

$$\partial_t^h(\boldsymbol{z}_h) \in L^1(0,T;W^{-1,1}(\Omega)), \tag{1.9}$$

independently of h. The two estimates (1.8) and (1.9) tells us that $\boldsymbol{z}_h$ satisfies the hypotheses of an Aubin–Lions type lemma (see Lemma 2.3 below for details). Strong convergence of $\boldsymbol{z}_h$ follows from this lemma.

The paper is organized as follows: In Section 2, we introduce notation and list some basic results needed for the later analysis. Moreover, we recall the usual notion of weak solution and introduce a mixed weak formulation of the velocity equation. Finally, we introduce the finite element spaces and review some of their basic properties. In Section 3, we present the numerical method and state our main convergence result. Section 4 is devoted to deriving basic estimates. In Section 5, we establish higher integrability of the density. Finally, in Section 6, we prove the main convergence result stated in Section 3. The proof is divided into several steps (subsections), including convergence of the continuity scheme, weak continuity of the discrete viscous flux, strong convergence of the density approximations, and convergence of the velocity scheme.

2. Preliminary material

We will write $W^{m,p}(\Omega)$ for the Sobolev space of functions with derivatives of all orders up to m belonging to the space $L^p(\Omega)$. To distinguish between scalar and vector functions, we will write vector functions with a bold face. Similarly, a functions space written in bold face denotes the vector analog of the corresponding scalar space.

We make frequent use of the divergence and curl operators and denote these by div_x and curl_x, respectively. In the 2D case, we will denote both the rotation operator taking scalars into vectors and the curl operator taking vectors into scalars by curl_x.

We will make use of the spaces

$$\begin{aligned} L_0^2(\Omega) &= \left\{\phi\in L^2(\Omega) : \int_\Omega \phi\;dx = 0\right\},\\ \boldsymbol{W}^{\operatorname{div},2}(\Omega) &= \left\{\boldsymbol{v}\in\boldsymbol{L}^2(\Omega) : \operatorname{div}_x\boldsymbol{v}\in L^2(\Omega)\right\},\end{aligned}$$

$$\boldsymbol{W}^{\mathrm{curl},2}(\Omega) = \left\{ \boldsymbol{v} \in \boldsymbol{L}^2(\Omega) : \mathrm{curl}_x \, \boldsymbol{v} \in \boldsymbol{L}^2(\Omega) \right\},$$

where ν denotes the outward pointing unit normal vector on $\partial\Omega$. If $\boldsymbol{v} \in \boldsymbol{W}^{\mathrm{div},2}(\Omega)$ satisfies $\boldsymbol{v} \cdot \nu|_{\partial\Omega} = 0$, we write $\boldsymbol{v} \in \boldsymbol{W}_0^{\mathrm{div},2}(\Omega)$. Similarly, $\boldsymbol{v} \in \boldsymbol{W}_0^{\mathrm{curl},2}(\Omega)$ means $\boldsymbol{v} \in \boldsymbol{W}^{\mathrm{curl},2}(\Omega)$ and $\boldsymbol{v} \times \nu|_{\partial\Omega} = 0$. In two dimensions, $\boldsymbol{w}$ is a scalar function and the space $\boldsymbol{W}_0^{\mathrm{curl},2}(\Omega)$ is to be understood as $W_0^{1,2}(\Omega)$. To define weak solutions, we shall use the space

$$\mathcal{W}(\Omega) = \left\{ \boldsymbol{v} \in \boldsymbol{L}^2(\Omega) : \mathrm{div}_x \, \boldsymbol{v} \in L^2(\Omega), \mathrm{curl}_x \, \boldsymbol{v} \in \boldsymbol{L}^2(\Omega), \boldsymbol{v} \cdot \nu|_{\partial\Omega} = 0 \right\},$$

which coincides with $\boldsymbol{W}_0^{\mathrm{div},2}(\Omega) \cap \boldsymbol{W}_0^{\mathrm{curl},2}(\Omega)$. The space $\mathcal{W}(\Omega)$ is equipped with the norm $\|\boldsymbol{v}\|_{\mathcal{W}}^2 = \|\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)}^2 + \|\mathrm{div}_x \, \boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)}^2 + \|\mathrm{curl}_x \, \boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)}^2$. It is known that $\|\cdot\|_{\mathcal{W}}$ is equivalent to the $\boldsymbol{W}^{1,2}$ norm on the space $\left\{ v \in \boldsymbol{W}^{1,2}(\Omega) : \boldsymbol{v} \cdot \nu|_{\partial\Omega} = 0 \right\}$, see, e.g., [11].

For the convenience of the reader we list some basic functional analysis results to be utilized (often without mentioning) in the subsequent arguments (for proofs, see, e.g., [5]). Throughout the paper we use overbars to denote weak limits.

Lemma 2.1. *Let O be a bounded open subset of $\mathbb{R}^M$ with $M \geq 1$. Suppose $g \colon \mathbb{R} \to (-\infty, \infty]$ is a lower semicontinuous convex function and $\{v_n\}_{n\geq 1}$ is a sequence of functions on O for which $v_n \rightharpoonup v$ in $L^1(O)$, $g(v_n) \in L^1(O)$ for each n, $g(v_n) \rightharpoonup \overline{g(v)}$ in $L^1(O)$. Then $g(v) \leq \overline{g(v)}$ a.e. on O, $g(v) \in L^1(O)$, and $\int_O g(v) \, dy \leq \liminf_{n\to\infty} \int_O g(v_n) \, dy$. If, in addition, g is strictly convex on an open interval $(a, b) \subset \mathbb{R}$ and $g(v) = \overline{g(v)}$ a.e. on O, then, passing to a subsequence if necessary, $v_n(y) \to v(y)$ for a.e. $y \in \{y \in O \mid v(y) \in (a, b)\}$.*

Let X be a Banach space and denote by $X^\star$ its dual. The space $X^\star$ equipped with the weak-$\star$ topology is denoted by $X^\star_{\mathrm{weak}}$, while X equipped with the weak topology is denoted by X_{weak}. By the Banach-Alaoglu theorem, a bounded ball in $X^\star$ is $\sigma(X^\star, X)$-compact. If X separable, then the weak-$\star$ topology is metrizable on bounded sets in $X^\star$, and thus one can consider the metric space $C\left([0, T]; X^\star_{\mathrm{weak}}\right)$ of functions $v : [0, T] \to X^\star$ that are continuous with respect to the weak topology. We have $v_n \to v$ in $C\left([0, T]; X^\star_{\mathrm{weak}}\right)$ if $\langle v_n(t), \phi \rangle_{X^\star, X} \to \langle v(t), \phi \rangle_{X^\star, X}$ uniformly with respect to t, for any $\phi \in X$. The following lemma is a consequence of the Arzelà-Ascoli theorem:

Lemma 2.2. *Let X be a separable Banach space, and suppose $v_n \colon [0, T] \to X^\star$, $n = 1, 2, \dots,$ is a sequence for which $\|v_n\|_{L^\infty([0,T];X^\star)} \leq C$, for some constant C independent of n. Suppose the sequence $[0, T] \ni t \mapsto \langle v_n(t), \Phi \rangle_{X^\star, X}$, $n = 1, 2, \dots,$ is equi-continuous for every Φ that belongs to a dense subset of X. Then v_n belongs to $C\left([0, T]; X^\star_{\mathrm{weak}}\right)$ for every n, and there exists a function $v \in C\left([0, T]; X^\star_{\mathrm{weak}}\right)$ such that along a subsequence as $n \to \infty$ there holds $v_n \to v$ in $C\left([0, T]; X^\star_{\mathrm{weak}}\right)$.*

In what follows, we will often obtain a priori estimates for a sequence $\{v_n\}_{n\geq 1}$ that we write as "$v_n \in_{\mathrm{b}} X$" for some functional space X. What this really means is that we have a bound on $\|v_n\|_X$ that is independent of n.

The following discrete version of a lemma due to Lions [11, Lemma 5.1] will prove useful in the convergence analysis. A proof of this lemma can be found in [9].

Lemma 2.3. *Given $T > 0$ and a small number $h > 0$, write $(0, T] = \cup_{k=1}^M (t_{k-1}, t_k]$ with $t_k = hk$ and $Mh = T$. Let $\{f_h\}_{h>0}^\infty$, $\{g_h\}_{h>0}^\infty$ be two sequences such that:*

(1) *the mappings $t \to g_h(t, x)$ and $t \to f_h(t, x)$ are constant on each interval $(t_{k-1}, t_k]$, $k = 1, \dots, M$.*

(2) *$\{f_h\}_{h>0}$ and $\{g_h\}_{h>0}$ converge weakly to f and g in $L^{p_1}(0,T;L^{q_1}(\Omega))$ and $L^{p_2}(0,T;L^{q_2}(\Omega))$, respectively, where $1<p_1,q_1<\infty$ and*

$$\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{q_1}+\frac{1}{q_2}=1.$$

(3) *the discrete time derivative satisfies*

$$\frac{g_h(t,x)-g_h(t-h,x)}{h}\in_b L^1(0,T;W^{-1,1}(\Omega))$$

(4) *$\|f_h(t,x)-f_h(t,x-\xi)\|_{L^{p_2}(0,T;L^{q_2}(\Omega))}\to 0$ as $|\xi|\to 0$, uniformly in h.*

Then $g_hf_h\rightharpoonup gf$ in the sense of distributions on $(0,T)\times\Omega$.

2.1. **Weak and renormalized solutions.**

Definition 2.4 (Weak solutions)**.** We say that a pair $(\varrho,\boldsymbol{u})$ of functions constitutes a weak solution of the Stokes approximation equations (1.1)–(1.2) with initial data

$$(\varrho^0,\boldsymbol{u}^0)\in L^\gamma(\Omega)\times\boldsymbol{L}^2(\Omega),\quad \gamma>\frac{N}{2},$$

and Navier-slip type boundary conditions (1.3)–(1.4), provided the following conditions hold:

(1) $(\varrho,\boldsymbol{u})\in L^\infty(0,T;L^\gamma(\Omega))\times L^2(0,T;\mathcal{W})\cap L^\infty(0,T;\boldsymbol{L}^2(\Omega))$;

(2) $\varrho_t+\operatorname{div}_x(\varrho\boldsymbol{u})=0$ in the weak sense, i.e, $\forall\phi\in C^\infty([0,T)\times\overline{\Omega})$,

$$\int_0^T\int_\Omega \varrho\left(\phi_t+\boldsymbol{u}\nabla_x\phi\right)\,dxdt+\int_\Omega\varrho^0\phi|_{t=0}\,dx=0; \tag{2.1}$$

(3) $\boldsymbol{u}_t-\mu\Delta\boldsymbol{u}-\lambda\nabla_x\operatorname{div}_x\boldsymbol{u}+\nabla_x p(\varrho)=0$ weakly, i.e, $\forall\boldsymbol{\phi}\in\boldsymbol{C}^\infty([0,T)\times\overline{\Omega})$ for which $\boldsymbol{\phi}\cdot\nu=0$ on $(0,T)\times\partial\Omega$,

$$\begin{aligned}\int_0^T\int_\Omega &-\boldsymbol{u}\boldsymbol{\phi_t}+\mu\operatorname{curl}_x\boldsymbol{u}\operatorname{curl}_x\boldsymbol{\phi}\\ &+\left[(\mu+\lambda)\operatorname{div}_x\boldsymbol{u}-p(\varrho)\right]\operatorname{div}_x\boldsymbol{\phi}\,dxdt=\int_\Omega\boldsymbol{u}^0\boldsymbol{\phi}|_{t=0}\,dx.\end{aligned} \tag{2.2}$$

For the convergence analysis we shall also need the DiPerna-Lions concept of renormalized solutions of the continuity equation.

Definition 2.5 (Renormalized solutions)**.** Given $\boldsymbol{u}\in L^2(0,T;\mathcal{W}(\Omega))$, we say that $\varrho\in L^\infty(0,T;L^\gamma(\Omega))$ is a renormalized solution of (1.1) provided

$$B(\varrho)_t+\operatorname{div}_x\left(B(\varrho)\boldsymbol{u}\right)+b(\varrho)\operatorname{div}_x\boldsymbol{u}=0\quad\text{in the weak sense on } [0,T)\times\overline{\Omega},$$

for any $B\in C[0,\infty)\cap C^1(0,\infty)$ with $B(0)=0$ and $b(\varrho):=\varrho B'(\varrho)-B(\varrho)$.

We shall need the following lemma. A proof can be found in [8].

Lemma 2.6. *Suppose $(\varrho,\boldsymbol{u})$ is a weak solution according to Definition 2.4. If $\varrho\in L^2((0,T)\times\Omega))$, then ϱ is a renormalized solution according to Definition 2.5.*

2.2. **A mixed formulation.**

In view of the Navier–slip boundary condition (1.4) the velocity equation (1.2) admits the following mixed weak formulation, which we will use to design a mixed finite element method: Determine functions

$$(\boldsymbol{w},\boldsymbol{u})\in L^2(0,T;\boldsymbol{W}_0^{\mathrm{curl},2}(\Omega))\times L^2(0,T;\boldsymbol{W}_0^{\mathrm{div},2}(\Omega))$$

such that

$$\begin{aligned}&\int_0^T\int_\Omega -\boldsymbol{u}\boldsymbol{v}_t+\mu\operatorname{curl}_x\boldsymbol{w}\boldsymbol{v}+\left[(\mu+\lambda)\operatorname{div}_x\boldsymbol{u}-p(\varrho)\right]\operatorname{div}_x\boldsymbol{v}\,dxdt=\int_\Omega\boldsymbol{u}^0\boldsymbol{v}|_{t=0}\,dx,\\ &\int_0^T\int_\Omega\boldsymbol{w}\boldsymbol{\eta}-\operatorname{curl}_x\boldsymbol{\eta}\boldsymbol{u}\,dxdt=0,\end{aligned} \tag{2.3}$$

for all $(\boldsymbol{\eta}, \boldsymbol{v}) \in L^2(0,T;\boldsymbol{W}_0^{\text{curl},2}(\Omega)) \times L^2(0,T;\boldsymbol{W}_0^{\text{div},2}(\Omega)) \cap W^{1,2}(0,T;\boldsymbol{L}^2(\Omega))$.

Note that if $(\boldsymbol{w}, \boldsymbol{u}, \varrho)$ is a triple satisfying the mixed formulation (2.3) then the pair $(\boldsymbol{u}, \varrho)$ satisfies the weak formulation (2.4) [8].

2.3. **Finite Element spaces and basic results.** Throughout this paper, $\{E_h\}_h$ denotes a shape regular family of tetrahedral meshes of Ω, where h is the maximal diameter. By shape regular we mean that there exists a constant $\kappa > 0$ such that every $E \in E_h$ contains a ball of radius $\lambda_E \geq \frac{h_E}{\kappa}$, where h_E is the diameter of E. For each fixed $h > 0$, we let Γ_h denote the set of faces in E_h and $\mathcal{V}_h$ the set of edges. In two dimensions, Γ_h is the set of edges and $\mathcal{V}_h$ the set of vertices. We will use $\mathbb{P}_j^k(E)$ to denote the space of vector polynomials on E with l components and maximal order k.

To approximate the vorticity $\boldsymbol{w}$, we will use the curl-conforming Nédélec space of the first order and kind [13]:

$$\begin{aligned}\boldsymbol{W}_h(\Omega) = \Big\{ \boldsymbol{w} \in \boldsymbol{W}_0^{\text{curl},2}(\Omega) :\ \boldsymbol{w}|_E \in \boldsymbol{W}(E),\ \forall E \in E_h,& \\ \int_e [\![\boldsymbol{w} \cdot t]\!]_e \ dS(x) = 0,\ \forall e \in \mathcal{V}_h &\Big\},\end{aligned} \tag{2.4}$$

where t is the unit tangential along the edge e, $[\![\cdot]\!]_e$ is the jump over the edge e, and

$$\boldsymbol{W}(E) = \begin{cases} \mathbb{P}_1^1(E), & N = 2, \\ \left\{ \boldsymbol{w} \in \mathbb{P}_1^3(E) :\ \nabla_x \boldsymbol{w} + \nabla_x \boldsymbol{w}^T = 0 \right\}, & N = 3. \end{cases}$$

In two dimensions, the continuity requirement $\int_e [\![\boldsymbol{w} \cdot t]\!] \ dS(x) = 0$ in (2.4) is to be understood as continuity at vertices. For the velocity $\boldsymbol{u}$, we will use the div-conforming Nédélec space of the first order and kind [13]:

$$\boldsymbol{V}_h(\Omega) = \left\{ \boldsymbol{v} \in \boldsymbol{W}_0^{\text{div},2} :\ \boldsymbol{v}|_E \in \boldsymbol{V}(E),\ \forall E \in E_h,\ \int_\Gamma [\![\boldsymbol{v} \cdot \nu]\!] \ dS(x) = 0,\ \forall \Gamma \in \Gamma_h \right\},$$

where $\boldsymbol{V}(E) = \mathbb{P}_0^N \oplus \mathbb{P}_0^1 \boldsymbol{x}$, and $[\![\cdot]\!]_\Gamma$ is the jump over Γ. The density ϱ will be approximated in the space of piecewise constants on E_h:

$$Q_h(\Omega) = \left\{ q \in L^2(\Omega) :\ q|_E \in \mathbb{P}_0^1(E),\ \forall E \in E_h \right\}.$$

Next, we introduce the canonical interpolation operators:

$$\begin{aligned} &\Pi_h^S : W_0^{1,2} \cap\ W^{2,2} \to S_h, \quad \Pi_h^W : \boldsymbol{W}_0^{\text{curl},2} \cap\ \boldsymbol{W}^{2,2} \to \boldsymbol{W}_h, \\ &\Pi_h^V : \boldsymbol{W}_0^{\text{div},2} \cap \boldsymbol{W}^{1,2} \to \boldsymbol{V}_h, \quad \Pi_h^Q : L_0^2 \to Q_h, \end{aligned}$$

using the available degrees of freedom of the involved spaces. That is, the operators (in three dimensions) are defined by [2, 13]

$$\begin{aligned} \left(\Pi_h^S s\right)(x_i) &= s(x_i), \quad \forall x_i \in \mathcal{N}_h; \\ \int_e \left(\Pi_h^W \boldsymbol{w}\right) \times \nu \ dS(x) &= \int_e \boldsymbol{w} \times \nu \ dS(x), \quad \forall e \in \mathcal{V}_h; \\ \int_\Gamma \left(\Pi_h^V \boldsymbol{v}\right) \cdot \nu \ dS(x) &= \int_\Gamma \boldsymbol{v} \cdot \nu \ dS(x), \quad \forall \Gamma \in \Gamma_h; \\ \int_E \Pi_h^Q q \ dx &= \int_E q \ dx, \quad \forall E \in E_h, \end{aligned}$$

where $\mathcal{N}_h$ it the set of vertices of E_h. It is well known that the following diagram commutes ([2,3]):

$$\begin{array}{ccccccc}
W_0^{1,2}\cap W^{2,2} & \xrightarrow{\text{grad}} & \boldsymbol{W}_0^{\text{curl},2}\cap\ \boldsymbol{W}^{2,2} & \xrightarrow{\text{curl}} & \boldsymbol{W}_0^{\text{div},p}\cap\boldsymbol{W}^{1,2} & \xrightarrow{\text{div}} & \boldsymbol{L}_0^2 \\
\Pi_h^S\Big\downarrow & & \Pi_h^W\Big\downarrow & & \Pi_h^V\Big\downarrow & & \Pi_h^Q\Big\downarrow \\
S_h & \xrightarrow{\text{grad}} & \boldsymbol{W}_h & \xrightarrow{\text{curl}} & \boldsymbol{V}_h & \xrightarrow{\text{div}} & Q_h.
\end{array}$$

Remark 2.7. The interpolation operators Π_h^S, Π_h^W, and Π_h^V, are defined on function spaces with enough regularity to ensure that the corresponding degrees of freedom are functionals on these spaces. This is reflected in writing $\boldsymbol{W}^{\text{curl},2}\cap\boldsymbol{W}^{2,2}$ instead of merely $\boldsymbol{W}^{\text{curl},2}$ and so on.

In view of the above commuting diagram, we can define the spaces orthogonal to the range of the previous operator, i.e.,

$$\boldsymbol{W}_h^{0,\perp} := \{\boldsymbol{w}_h\in\boldsymbol{W}_h;\operatorname{curl}_x\boldsymbol{w}_h=0\}^\perp\cap\boldsymbol{W}_h,$$
$$\boldsymbol{V}_h^{0,\perp} := \{\boldsymbol{v}_h\in\boldsymbol{V}_h;\operatorname{div}_x\boldsymbol{v}_h=0\}^\perp\cap\boldsymbol{V}_h,$$

to obtain decompositions (cf. [2])

$$\boldsymbol{W}_h = \nabla_x S_h + \boldsymbol{W}_h^{0,\perp}, \tag{2.5}$$

$$\boldsymbol{V}_h = \operatorname{curl}_x\boldsymbol{W}_h + \boldsymbol{V}_h^{0,\perp}. \tag{2.6}$$

The following discrete Poincaré inequalities hold [3]

$$\|\boldsymbol{v}_h\|_{\boldsymbol{L}^2(\Omega)} \le C\,\|\operatorname{div}_x\boldsymbol{v}_h\|_{L^2(\Omega)}\,,\quad \forall\boldsymbol{v}\in\boldsymbol{V}_h^{0,\perp}, \tag{2.7}$$

$$\|\boldsymbol{w}_h\|_{\boldsymbol{L}^2(\Omega)} \le C\,\|\operatorname{curl}_x\boldsymbol{w}_h\|_{L^2(\Omega)}\,,\quad \forall\boldsymbol{w}\in\boldsymbol{W}_h^{0,\perp}, \tag{2.8}$$

where the constant C is independent of h.

In the subsequent convergence analysis, we make frequent use of the canonical projection operators. To bound these we shall need the following ([4,13])

Lemma 2.8. *There exists a constant $C>0$, depending only on the shape regularity of E_h and the size of Ω, such that for any $1\le p\le\infty$,*

$$\left\|\phi-\Pi_h^Q\phi\right\|_{L^p(\Omega)} \le Ch\,\|\nabla_x\phi\|_{\boldsymbol{L}^p(\Omega)}\,,$$
$$\left\|\boldsymbol{v}-\Pi_h^V\boldsymbol{v}\right\|_{\boldsymbol{L}^p(\Omega)} + h\left\|\operatorname{div}_x(\boldsymbol{v}-\Pi_h^V\boldsymbol{v})\right\|_{L^p(\Omega)} \le Ch^s\,\|\nabla_x^s\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}\,,\quad r=1,2,$$
$$\left\|\boldsymbol{w}-\Pi_h^W\boldsymbol{w}\right\|_{\boldsymbol{L}^p(\Omega)} + h\left\|\operatorname{curl}_x(\boldsymbol{w}-\Pi_h^W\boldsymbol{w})\right\|_{\boldsymbol{L}^p(\Omega)} \le Ch^s\|\nabla_x^s\boldsymbol{w}\|_{\boldsymbol{L}^p(\Omega)},\quad s=1,2,$$

for all $\phi\in W^{1,p}(\Omega),\boldsymbol{v}\in W^{s,p}(\Omega)$, and $\boldsymbol{w}\in W^{2,p}(\Omega)$.

We will also need the following lemma. It follows from scaling arguments and the equivalence of finite dimensional norms [4].

Lemma 2.9. *There exists a constant $C>0$, such that for $1\le q,p\le\infty$, and $r=0,1$,*

$$\|\phi_h\|_{W^{r,p}(E)} \le Ch^{-r+\frac{N}{p}-\frac{N}{q}}\,\|\phi_h\|_{L^q(E)}\,,$$

for any $E\in E_h$ and all polynomial functions $\phi_h\in\mathbb{P}_k(E)$, $k=0,1,..$ The constant C depends only on the shape regularity of E_h and polynomial degree k.

The next result follows from scaling arguments and the trace theorem [1]

Lemma 2.10. *Fix any $E\in E_h$ and let $\phi\in W^{1,2}(E)$ be arbitrary. There exists a constant $C>0$, depending only on the shape regularity of E_h such that,*

$$\|\phi\|_{L^2(\Gamma)} \le Ch^{-\frac{1}{2}}\left(\|\phi\|_{L^2(E)} + h\|\nabla_x\phi\|_{\boldsymbol{L}^2(E)}\right),\quad \forall\Gamma\in\Gamma_h\cap\partial E.$$

We now establish a Sobolev embedding estimate for the discrete decompositions (2.6) and (2.5).

Lemma 2.11. *The finite element spaces $\boldsymbol{V}_h^{0,\perp}(\Omega)$ and $\boldsymbol{W}_h^{0,\perp}(\Omega)$ satisfies the following embedding results independent of h:*

(1) *The space $\boldsymbol{V}_h^{0,\perp}(\Omega) \cap \boldsymbol{W}_0^{\mathrm{div},2}(\Omega)$ is embedded in $\boldsymbol{L}^{2^*}(\Omega)$,*
(2) *The space $\boldsymbol{W}_h^{0,\perp}(\Omega) \cap \boldsymbol{W}_0^{\mathrm{curl},2}(\Omega)$ is embedded in $\boldsymbol{L}^{2^*}(\Omega)$,*

where $2^* = 6$ if $N = 3$, and 2^* is any large finite number if $N = 2$.

Proof. We first prove (1). By virtue of the decomposition (2.6) we can for any $\boldsymbol{v}_h \in_{\mathrm{b}} \boldsymbol{V}_h^{0,\perp}(\Omega) \cap \boldsymbol{W}_0^{\mathrm{div},2}(\Omega)$ find functions $\boldsymbol{\zeta}_h \in \boldsymbol{W}_h(\Omega)$ and $\boldsymbol{z}_h \in \boldsymbol{V}_h^{0,\perp}(\Omega)$ such that

$$\Pi_h^V \left(\nabla_x \Delta^{-1} \left[\mathrm{div}_x\, \boldsymbol{v}_h\right]\right) = \mathrm{curl}_x\, \boldsymbol{\xi}_h + \boldsymbol{z}_h.$$

Using the commutative diagram and the definition of $\Pi_h^V \left(\nabla_x \Delta^{-1} [\cdot]\right)$ we easily verify that

$$\mathrm{div}_x\, \Pi_h^V \left(\nabla_x \Delta^{-1} \left[\mathrm{div}_x\, \boldsymbol{v}_h\right]\right) = \mathrm{div}_x\, \boldsymbol{v}_h.$$

Hence, since $(\boldsymbol{z}_h - \boldsymbol{v}_h) \in \boldsymbol{V}_h^{0,\perp}(\Omega)$ we can use the discrete Poincaré inequality (2.7) to conclude that

$$\|\boldsymbol{v}_h - \boldsymbol{z}_h\|_{\boldsymbol{L}^2(\Omega)} \leq C \|\, \mathrm{div}_x(\boldsymbol{v}_h - \boldsymbol{z}_h)\|_{L^2(\Omega)} = 0.$$

Thus, $\boldsymbol{z}_h = \boldsymbol{v}_h$ a.e in Ω and we easily calculate

$$\begin{aligned} \|\boldsymbol{v}_h\|_{\boldsymbol{L}^{2^*}(\Omega)} = \|\boldsymbol{z}_h\|_{\boldsymbol{L}^{2^*}(\Omega)} &\leq \|\Pi_h^V \left(\nabla_x \Delta^{-1} \left[\mathrm{div}_x\, \boldsymbol{v}_h\right]\right)\|_{\boldsymbol{L}^{2^*}(\Omega)} \\ &\leq C_1 \|\nabla_x \Delta^{-1} \left[\mathrm{div}_x\, \boldsymbol{v}_h\right]\|_{\boldsymbol{L}^{2^*}(\Omega)} \leq C_2 \|\, \mathrm{div}_x\, \boldsymbol{v}_h\|_{L^2(\Omega)}, \end{aligned}$$

where the last inequality is the standard Sobolev embedding $\boldsymbol{W}^{1,2}(\Omega) \subset \boldsymbol{L}^{2^*}(\Omega)$.

In two spatial dimensions, (2) follows directly from the standard Sobolev embedding $W^{1,2}(\Omega) \subset L^{2^*}(\Omega)$. To prove (2) in three spatial dimensions, fix any $\boldsymbol{w}_h \in_{\mathrm{b}} \boldsymbol{W}_h^{0,\perp}(\Omega) \cap \boldsymbol{W}_0^{\mathrm{curl},2}(\Omega)$ and let $\boldsymbol{\eta} \in \boldsymbol{W}_0^{\mathrm{curl},2}(\Omega) \cap \boldsymbol{W}^{\mathrm{div},2}(\Omega) \subset \boldsymbol{W}^{1,2}(\Omega)$ solve (cf. [10])

$$\begin{aligned} \mathrm{curl}_x\, \boldsymbol{\eta} &= \mathrm{curl}_x\, \boldsymbol{w}_h, \text{ in } \Omega, \\ \mathrm{div}_x\, \boldsymbol{\eta} &= 0, \text{ in } \Omega, \\ \boldsymbol{\eta} \times \nu &= 0, \text{ on } \partial\Omega. \end{aligned}$$

Using the decomposition (2.5) of the space $\boldsymbol{W}_h(\Omega)$, we can find functions $s_h \in S_h(\Omega)$ and $\boldsymbol{\zeta}_h \in \boldsymbol{W}_h^{0,\perp}(\Omega)$ such that

$$\Pi_h^W \boldsymbol{\eta} = \nabla_x s_h + \boldsymbol{\zeta}_h.$$

Hence, from the commuting diagram property, we deduce

$$\mathrm{curl}_x\, \boldsymbol{\zeta}_h = \mathrm{curl}_x\, \Pi_h^W \boldsymbol{\eta} = \Pi_h^V\, \mathrm{curl}_x\, \boldsymbol{\eta} = \Pi_h^V\, \mathrm{curl}_x\, \boldsymbol{w}_h = \mathrm{curl}_x\, \boldsymbol{w}_h.$$

Thus, since $(\boldsymbol{w}_h - \boldsymbol{\zeta}_h) \in \boldsymbol{W}_h^{0,\perp}(\Omega)$ we can use the Poincaré inequality (2.8) to conclude that

$$\|\boldsymbol{w}_h - \boldsymbol{\zeta}_h\|_{\boldsymbol{L}^2(\Omega)} \leq C \|\, \mathrm{curl}_x(\boldsymbol{w}_h - \boldsymbol{\zeta}_h)\|_{\boldsymbol{L}^2(\Omega)} = 0,$$

and hence that $\boldsymbol{w}_h = \boldsymbol{\zeta}_h$. Moreover, we easily calculate

$$\|\boldsymbol{w}_h\|_{\boldsymbol{L}^{2^*}(\Omega)} = \|\boldsymbol{\zeta}_h\|_{\boldsymbol{L}^{2^*}(\Omega)} \leq \|\Pi_h^W \boldsymbol{\eta}\|_{\boldsymbol{L}^{2^*}(\Omega)} \leq C_1 \|\boldsymbol{\eta}\|_{\boldsymbol{L}^{2^*}(\Omega)} \leq C_2 \|\nabla_x \boldsymbol{\eta}\|_{\boldsymbol{L}^2(\Omega)},$$

where the last inequality is the standard Sobolev embedding $\boldsymbol{W}^{1,2}(\Omega) \subset \boldsymbol{L}^2(\Omega)$. This concludes the proof. $\square$

We end this section by recalling a compactness result from [8, Theorem A.1].

Lemma 2.12. *Let $\{\boldsymbol{v}_h\}_{h>0}$ be a sequence of functions in $\boldsymbol{V}_h^{0,\perp}$ with $\operatorname{div}_x \boldsymbol{v}_h \in_b L^2(\Omega)$. For any $\xi \in \mathbb{R}^N$,*

$$\|\boldsymbol{v}_h(x) - \boldsymbol{v}_h(x-\xi)\|_{\boldsymbol{L}^2(\Omega)} \leq C(|\xi|^{\frac{4-N}{2}} + |\xi|^2)^{\frac{1}{2}} \|\operatorname{div}_x \boldsymbol{v}_h\|_{L^2(\Omega)},$$

where the constant $C > 0$ is independent of both h and ξ.

3. Numerical method and main result

In this section we define the numerical method for the Stokes approximation equations and the state the main convergence theorem. The proof of the main theorem is deferred to subsequent sections.

Given a time step $\Delta t > 0$, we discretize the time interval $[0,T]$ in terms of the points $t^m = m\Delta t$, $m = 0,\dots,M$, where we assume that $M\Delta t = T$. Regarding the spatial discretization, we let $\{E_h\}_h$ be a shape regular family of tetrahedral meshes of Ω, where h is the maximal diameter. It will be a standing assumption that h and Δt are related such that $\Delta t = ch$, for some constant c. Furthermore, for each h, let Γ_h denote the set of faces in E_h.

For each fixed $h > 0$, we let $\boldsymbol{W}_h(\Omega)$ and $\boldsymbol{V}_h(\Omega)$ denote the Nédélec spaces of the first order and kind on E_h (cf. Section 2.3) and $Q_h(\Omega)$ the space of piecewise constants on E_h. To incorporate boundary conditions, we let the degrees of freedom of $\boldsymbol{W}_h(\Omega)$ and $\boldsymbol{V}_h(\Omega)$ located at the boundary $\partial\Omega$ vanish.

Before defining our numerical scheme, we shall need to introduce some additional notation related to the discontinuous Galerkin scheme. Concerning the boundary ∂E of an element E, we write f_+ for the trace of the function f achieved from within the element E and f_- for the trace of f achieved from outside E. Concerning an edge Γ that is shared between two elements E_- and E_+, we will write f_+ for the trace of f achieved from within E_+ and f_- for the trace of f achieved from within E_-. Here E_- and E_+ are defined such that ν points from E_- to E_+, where ν is fixed (throughout) as one of the two possible normal components on each edge Γ throughout the discretization. We also write $[\![f]\!]_\Gamma = f_+ - f_-$ for the jump of f across the edge Γ, while forward time-differencing of f is denoted by $[\![f^m]\!] = f^{m+1} - f^m$. Discrete implicit time discretization of a function f is denoted by the operator $\partial_t^h(f^m) = \frac{1}{\Delta t}[\![f^{m-1}]\!]$.

Let us now define our numerical scheme.

Definition 3.1 (Numerical scheme)**.** Let $\left\{\varrho_h^0(x)\right\}_{h>0}$ be a sequence (of piecewise constant functions) in $Q_h(\Omega)$ that satisfies $\varrho_h^0 > 0$ for each fixed $h > 0$ and $\varrho_h^0 \to \varrho^0$ a.e. in Ω and in $L^1(\Omega)$ as $h \to 0$. Let the sequence $\{\boldsymbol{u}_h^0\}_{h>0}$ be such that for each fixed $h > 0$, $\boldsymbol{u}_h^0 \in \boldsymbol{V}_h(\Omega)$ and satisfies

$$\int_\Omega \boldsymbol{u}_h^0 \boldsymbol{v}_h \, dx = \int_\Omega \boldsymbol{u}^0 \boldsymbol{v}_h \, dx, \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h(\Omega). \tag{3.1}$$

Now, determine functions

$$(\varrho_h^m, \boldsymbol{w}_h^m, \boldsymbol{u}_h^m) \in Q_h(\Omega) \times \boldsymbol{W}_h(\Omega) \times \boldsymbol{V}_h(\Omega), \quad m = 1,\dots,M,$$

such that for all $\phi_h \in Q_h(\Omega)$,

$$\int_\Omega \partial_t^h(\varrho_h^m)\,\phi_h \, dx = \Delta t \sum_{\Gamma \in \Gamma_h^I} \int_\Gamma \left(\varrho_-^m(\boldsymbol{u}_h^m \cdot \nu)^+ + \varrho_+^m(\boldsymbol{u}_h^m \cdot \nu)^-\right) [\![\phi_h]\!]_\Gamma \, dS(x), \tag{3.2}$$

and for all $(\boldsymbol{\eta}_h, \boldsymbol{v}_h) \in \boldsymbol{W}_h(\Omega) \times \boldsymbol{V}_h(\Omega)$,

$$\begin{aligned}
&\int_\Omega \partial_t^h(\boldsymbol{u}_h^m)\,\boldsymbol{v}_h + \mu \operatorname{curl}_x \boldsymbol{w}_h^m \boldsymbol{v}_h + [(\mu+\lambda)\operatorname{div}_x \boldsymbol{u}_h^m - p(\varrho_h^m)] \operatorname{div}_x \boldsymbol{v}_h \, dx = 0,\\
&\int_\Omega \boldsymbol{w}_h^m \boldsymbol{\eta}_h - \boldsymbol{u}_h^m \operatorname{curl}_x \boldsymbol{\eta}_h \, dx = 0,
\end{aligned} \tag{3.3}$$

for $m = 1, \dots, M$.

In (3.2), $(\boldsymbol{u}_h \cdot \nu)^+ = \max\{\boldsymbol{u}_h \cdot \nu, 0\}$ and $(\boldsymbol{u}_h \cdot \nu)^+ = \min\{\boldsymbol{u}_h \cdot \nu, 0\}$, so that $\boldsymbol{u}_h \cdot \nu = (\boldsymbol{u}_h \cdot \nu)^+ + (\boldsymbol{u}_h \cdot \nu)^-$, i.e., in the evaluation of $\varrho(\boldsymbol{u} \cdot \nu)$ at the edge Γ the trace of ϱ is taken in the upwind direction.

Remark 3.2. Recall that $\varrho_\pm$ and $(\boldsymbol{u}_h \cdot \nu)^\pm$ related to a face Γ has a different meaning than $\varrho_\pm$ and $(\boldsymbol{u}_h \cdot \nu)^\pm$ related to the boundary of an element ∂E. By direct calculation, one can verify the identity

$$\begin{aligned}\Delta t \sum_{E \in E_h} &\int_{\partial E \setminus \partial \Omega} \left(\varrho_+^m (\boldsymbol{u}_h^m \cdot \nu)^+ + \varrho_-^m (\boldsymbol{u}_h^m \cdot \nu)^- \right) \phi_h \, dS(x) \\ &= -\Delta t \sum_{\Gamma \in \Gamma_h^I} \int_\Gamma \left(\varrho_-^m (\boldsymbol{u}_h^m \cdot \nu)^+ + \varrho_+^m (\boldsymbol{u}_h^m \cdot \nu)^- \right) [\phi_h]_\Gamma \, dS(x).\end{aligned}$$

Using this identity, we can state (3.2) on the following form:

$$\begin{aligned}\int_\Omega \varrho_h^m \phi_h \, dx + \Delta t \sum_{E \in E_h} &\int_{\partial E \setminus \partial \Omega} \left(\varrho_+^m (\boldsymbol{u}_h^m \cdot \nu)^+ + \varrho_-^m (\boldsymbol{u}_h^m \cdot \nu)^- \right) \phi_h \, dS(x) \\ = \int_\Omega \varrho_h^{m-1} \phi_h \, dx.\end{aligned} \tag{3.4}$$

The form (3.4) will be used frequently in the subsequent analysis.

For each fixed $h > 0$, the numerical solution $\{(\varrho_h^m, \boldsymbol{w}_h^m, \boldsymbol{u}_h^m)\}_{m=0}^M$ is extended to the whole of $(0, T) \times \Omega$ by setting

$$(\varrho_h, \boldsymbol{w}_h, \boldsymbol{u}_h)(t) = (\varrho_h^m, \boldsymbol{w}_h^m, \boldsymbol{u}_h^m), \qquad t \in (t_{m-1}, t_m), \quad m = 1, \dots, M. \tag{3.5}$$

In addition, we set $\varrho_h(0) = \varrho_h^0$ and $\boldsymbol{u}_h(0) = \boldsymbol{u}_h^0$.

The continuity scheme (3.2) clearly preserves the total mass. The following lemma from [8, Lemma 4.1] states that the density is strictly positive whenever the initial density is strictly positive.

Lemma 3.3. *Fix any $m = 1, \dots, M$ and suppose $\varrho_h^{m-1} \in Q_h(\Omega)$, $\boldsymbol{u}_h^m \in \boldsymbol{V}_h(\Omega)$ are given bounded functions. Then the solution $\varrho_h^m \in Q_h(\Omega)$ of the discontinuous Galerkin scheme* (3.2) *satisfies*

$$\min_{x \in \Omega} \varrho_h^m \geq \min_{x \in \Omega} \varrho_h^{m-1} \left(\frac{1}{1 + \Delta t \| \operatorname{div}_x \boldsymbol{u}_h^m \|_{L^\infty(\Omega)}} \right).$$

Consequently, if $\varrho_h^{m-1}(\cdot) > 0$, then $\varrho_h^m(\cdot) > 0$.

Existence of a solution to the nonlinear–implicit discrete scheme follows from a topological degree argument. This argument is essentially identical to that of [8, Lemma 4.2] with a minor modification to accommodate the discrete time derivative $\partial_t^h(\boldsymbol{u}_h)$.

Lemma 3.4. *For each fixed $h > 0$, there exists a solution*

$$(\varrho_h^m, \boldsymbol{w}_h^m, \boldsymbol{u}_h^m) \in Q_h(\Omega) \times \boldsymbol{W}_h(\Omega) \times \boldsymbol{V}_h(\Omega), \quad \varrho_h^m(\cdot) > 0, \quad m = 1, \dots, M,$$

to the nonlinear–implicit discrete problem posed in Definition 3.1.

Our main result is that, passing if necessary to a subsequence, $\{(\varrho_h, \boldsymbol{w}_h, \boldsymbol{u}_h)\}_{h>0}$ converges to a weak solution. More precisely, there holds

Theorem 3.5 (Convergence)**.** *Suppose $(\varrho^0, \boldsymbol{u}^0) \in L^\gamma(\Omega) \cap \boldsymbol{L}^2(\Omega)$, $\gamma > \frac{N}{2}$. Let $\{(\varrho_h, \boldsymbol{w}_h, \boldsymbol{u}_h)\}_{h>0}$ be a sequence of numerical solutions constructed according to* (3.5) *and Definition 3.1. Then, passing if necessary to a subsequence as $h \to 0$, $\boldsymbol{u}_h \to \boldsymbol{u}$, a.e in $(0, T) \times \Omega$, $\varrho_h \boldsymbol{u}_h \rightharpoonup \varrho \boldsymbol{u}$ in the sense of distributions on $(0, T) \times \Omega$, and $\varrho_h \to \varrho$ a.e. in $(0, T) \times \Omega$, where the limit triplet $(\varrho, \boldsymbol{w}, \boldsymbol{u})$ satisfies the mixed formulation* (2.3)*, and thus $(\varrho, \boldsymbol{u})$ is a weak solution according to Definition 2.4.*

4. Basic estimates

In this section we gather some basic estimates for our numerical method. The results include stability and weak time-continuity of both the density and velocity. We however commence by recalling (from [8]) the following renormalized version of the continuity scheme.

Lemma 4.1 (Renormalized continuity scheme)**.** *Fix any $m = 1, \dots, M$ and let the pair $(\varrho_h^m, \boldsymbol{u}_h^m) \in Q_h \times \boldsymbol{V}_h$ satisfy the continuity scheme* (3.2)*. Then $(\varrho_h^m, \boldsymbol{u}_h^m)$ also satisfies the renormalized continuity scheme*

$$
\begin{aligned}
&\int_\Omega B(\varrho_h^m)\phi_h \, dx \\
&\qquad - \Delta t \sum_{\Gamma \in \Gamma_h^I} \int_\Gamma \left(B(\varrho_-^m)(\boldsymbol{u}_h^m \cdot \nu)^+ + B(\varrho_+^m)(\boldsymbol{u}_h^m \cdot \nu)^- \right) [\![\phi_h]\!]_\Gamma \, dx \\
&\qquad + \Delta t \int_\Omega b(\varrho_h^m) \operatorname{div}_x \boldsymbol{u}_h^m \phi_h \, dx + \int_\Omega B''(\xi(\varrho_h^m, \varrho_h^{m-1})) \left[\!\left[\varrho_h^{m-1}\right]\!\right]^2 \phi_h \, dx \\
&\qquad + \Delta t \sum_{\Gamma \in \Gamma_h^I} \int_\Gamma B''(\xi^\Gamma(\varrho_+^m, \varrho_-^m)) [\![\varrho_h^m]\!]_\Gamma^2 (\phi_h)_- (\boldsymbol{u}_h^m \cdot \nu)^+ \\
&\qquad\qquad\qquad - B''(\xi^\Gamma(\varrho_-^m, \varrho_+^m)) [\![\varrho_h^m]\!]_\Gamma^2 (\phi_h)_+ (\boldsymbol{u}_h^m \cdot \nu)^- \, dS(x) \\
&= \int_\Omega B(\varrho_h^{m-1})\phi_h \, dx, \qquad \forall \phi_h \in Q_h(\Omega),
\end{aligned}
\tag{4.1}
$$

for any $B \in C[0,\infty) \cap C^2(0,\infty)$ with $B(0) = 0$ and $b(\varrho) := \varrho B'(\varrho) - B(\varrho)$. Given two positive real numbers a_1 and a_2, we denote by $\xi(a_1, a_2)$ and $\xi^\Gamma(a_1, a_2)$ two numbers between a_1 and a_2 (See [8] *for a precise definition).*

In what follows we will need the following discrete Hodge decomposition.

Lemma 4.2. *Let $\{(\varrho_h, \boldsymbol{w}_h, \boldsymbol{u}_h)\}_{h>0}$ be a sequence of numerical solutions constructed according to* (3.5) *and Definition 3.1. For each fixed $h > 0$, there exist unique functions $\boldsymbol{\zeta}_h^m \in \boldsymbol{W}_h^{0,\perp}$ and $\boldsymbol{z}_h^m \in \boldsymbol{V}_h^{0,\perp}$ such that*

$$
\boldsymbol{u}_h^m = \operatorname{curl}_x \boldsymbol{\zeta}_h^m + \boldsymbol{z}_h^m, \qquad m = 0, \dots, M. \tag{4.2}
$$

Moreover, if we let $\boldsymbol{\zeta}_h(t,x)$, $\boldsymbol{z}_h(t,x)$ denote the functions obtained by extending, as in (3.5)*, $\{\boldsymbol{\zeta}_h^m\}_{m=1}^M$, $\{\boldsymbol{z}_h^m\}_{m=1}^M$ to the whole of $(0,T] \times \Omega$, then*

$$
\boldsymbol{u}_h(t,\cdot) = \operatorname{curl}_x \boldsymbol{\zeta}_h(\cdot,t) + \boldsymbol{z}_h(\cdot,t), \qquad t \in (0,T).
$$

Finally, let $\operatorname{curl}_x \boldsymbol{\zeta}^0 \in \boldsymbol{L}^2(\Omega)$ and $\nabla_x s^0 \in \boldsymbol{L}^2(\Omega)$ satisfy the standard continuous Hodge decomposition $\boldsymbol{u}^0 = \operatorname{curl}_x \boldsymbol{\zeta}^0 + \nabla_x s^0$. Then,

$$
\operatorname{curl}_x \boldsymbol{\zeta}_h^0 \to \operatorname{curl}_x \boldsymbol{\zeta}^0, \quad \boldsymbol{z}_h^0 \to \nabla_x s^0, \quad \text{in } \boldsymbol{L}^2(\Omega),
$$

where $\boldsymbol{\zeta}_h^0$ and $\boldsymbol{z}_h^0$ are given by (4.2)*.*

Proof. The first two statements are consequences of (2.6).

To prove the last statement, fix any $\boldsymbol{\phi} \in \boldsymbol{C}_c^\infty(\Omega)$ and set $\boldsymbol{v}_h = \Pi_h^W \boldsymbol{\phi}$ in (3.1) to obtain

$$
\int_\Omega \operatorname{curl}_x \boldsymbol{\zeta}_h^0 \operatorname{curl}_x(\Pi_h^W \boldsymbol{\phi}) \, dx = \int_\Omega \operatorname{curl}_x \boldsymbol{\zeta}^0 \operatorname{curl}_x(\Pi_h^W \boldsymbol{\phi}) \, dx, \tag{4.3}
$$

where we have used that $\boldsymbol{u}_h^0 = \operatorname{curl}_x \boldsymbol{\zeta}_h^0 + \boldsymbol{z}_h^0$ and $\int_\Omega \boldsymbol{z}_h^0 \operatorname{curl}_x \Pi_h^W \boldsymbol{\phi} \, dx = 0$. Now, since $\|\operatorname{curl}_x \boldsymbol{\zeta}_h^0\|_{\boldsymbol{L}^2(\Omega)} \le C \|\boldsymbol{u}_h^0\|_{\boldsymbol{L}^2(\Omega)} \le C \|\boldsymbol{u}^0\|_{\boldsymbol{L}^2(\Omega)}$, there exists a function $\overline{\operatorname{curl}_x \boldsymbol{\zeta}^0}$ such that $\operatorname{curl}_x \boldsymbol{\zeta}_h^0 \rightharpoonup \overline{\operatorname{curl}_x \boldsymbol{\zeta}^0}$ in $\boldsymbol{L}^2(\Omega)$. Sending $h \to 0$ in (4.3) yields

$$
\int_\Omega (\overline{\operatorname{curl}_x \boldsymbol{\zeta}^0} - \operatorname{curl}_x \boldsymbol{\zeta}^0) \operatorname{curl}_x \boldsymbol{\phi} \, dx = 0, \quad \forall \boldsymbol{\phi} \in \boldsymbol{C}_c^\infty(\Omega).
$$

Hence, $\overline{\operatorname{curl}_x \boldsymbol{\zeta}^0} = \operatorname{curl}_x \boldsymbol{\zeta}^0$ a.e in Ω.

Next, let $\boldsymbol{v}_h = \operatorname{curl}_x \boldsymbol{\zeta}_h$ in (3.1) to discover

$$\|\operatorname{curl}_x \boldsymbol{\zeta}_h^0\|_{\boldsymbol{L}^2(\Omega)}^2 = \int_\Omega \operatorname{curl}_x \boldsymbol{\zeta}^0 \operatorname{curl}_x \boldsymbol{\zeta}_h^0 \, dx \to \|\operatorname{curl}_x \boldsymbol{\zeta}^0\|_{\boldsymbol{L}^2(\Omega)}^2,$$

as $h \to 0$. Then, $\operatorname{curl}_x \boldsymbol{\zeta}_h^0 \to \operatorname{curl}_x \boldsymbol{\zeta}^0$ in $\boldsymbol{L}^2(\Omega)$.

By setting $\boldsymbol{v}_h = \boldsymbol{u}_h^0$ in (3.1) we deduce

$$\|\boldsymbol{u}_h^0\|_{\boldsymbol{L}^2(\Omega)}^2 = \int_\Omega \boldsymbol{u}^0 \boldsymbol{u}_h^0 \, dx \to \|\boldsymbol{u}^0\|_{\boldsymbol{L}^2(\Omega)}^2,$$

as $h \to 0$. Hence, $\boldsymbol{u}_h^0 \to \boldsymbol{u}^0$ in $\boldsymbol{L}^2(\Omega)$.

Finally, a direct calculation shows that

$$\begin{aligned} 0 = \lim_{h\to 0} \|\boldsymbol{u}_h^0 - \boldsymbol{u}^0\|_{\boldsymbol{L}^2(\Omega)}^2 &= \lim_{h\to 0} \left[\|\operatorname{curl}_x \boldsymbol{\zeta}_h^0 - \operatorname{curl}_x \boldsymbol{\zeta}^0\|_{\boldsymbol{L}^2(\Omega)}^2 + \|\boldsymbol{z}_h^0 - \nabla_x s^0\|_{\boldsymbol{L}^2(\Omega)}^2\right] \\ &\quad - 2 \lim_{h\to 0} \left[\int_\Omega (\operatorname{curl}_x \boldsymbol{\zeta}_h^0 - \operatorname{curl}_x \boldsymbol{\zeta}^0)\,(\boldsymbol{z}_h^0 - \nabla_x s^0) \, dx\right], \end{aligned}$$

where the last term converges to zero since $\operatorname{curl}_x \boldsymbol{\zeta}_h^0 \to \operatorname{curl}_x \boldsymbol{\zeta}^0$ in $\boldsymbol{L}^2(\Omega)$. Thus, $\boldsymbol{z}_h^0 \to \nabla_x s^0$ in $\boldsymbol{L}^2(\Omega)$ and the proof is complete. □

We now derive a basic stability estimate satisfied by the numerical scheme.

Lemma 4.3 (Stability)**.** *Let $\{(\varrho_h, \boldsymbol{w}_h, \boldsymbol{u}_h)\}_{h>0}$ be a sequence of numerical solutions constructed according to* (3.5) *and Definition 3.1. For $\varrho(\cdot) > 0$, let*

$$\mathcal{E}(\varrho, \boldsymbol{u}) = \frac{a}{\gamma - 1}\varrho^\gamma + \frac{1}{2}|\boldsymbol{u}|^2.$$

For any $m = 1, \dots, M$, there holds

$$\begin{aligned} \int_\Omega \mathcal{E}(\varrho_h^m, \boldsymbol{u}_h^m) \, dx + \sum_{k=1}^m \Delta t \|\boldsymbol{u}_h^k\|_{\boldsymbol{W}^{\mathrm{div},2}(\Omega)}^2 + \sum_{k=1}^m \Delta t \|\boldsymbol{w}_h^k\|_{\boldsymbol{W}^{\mathrm{curl},2}(\Omega)}^2 + \mathcal{N}_{diffusion}^m \\ \leq \int_\Omega \mathcal{E}(\varrho^0, \boldsymbol{u}^0) \, dx, \end{aligned}$$

where the numerical diffusion term is given by

$$\begin{aligned} \mathcal{N}_{diffusion}^m = \frac{1}{2}\sum_{k=1}^m \|\, [\![\boldsymbol{u}_h^{m-1}]\!]\, \|_{\boldsymbol{L}^2(\Omega)}^2 + \sum_{k=1}^m \int_\Omega P''(\xi^{k-\frac{1}{2}}(\varrho_h^k, \varrho_h^{k-1}))\, [\![\varrho_h^{k-1}]\!]^2 \, dx \\ + \sum_{k=1}^m \sum_{\Gamma \in \Gamma_h^I} \Delta t \int_\Gamma P''(\varrho_\dagger^k)\, [\![\varrho_h^k]\!]_\Gamma^2 \left|\boldsymbol{u}_h^k \cdot \nu\right| \, dx. \end{aligned}$$

Proof. The proof is almost identical to that of Lemma 5.3 in [8] and follows directly from standard arguments. We omit the details. □

Since the finite element spaces are not conforming in $W^{1,2}(\Omega)$ it is not clear that the velocity and vorticity are embedded in $L^{2^*}(\Omega)$. Knowing this is essential for the later convergence analysis.

Lemma 4.4. *Let $\{(\varrho_h, \boldsymbol{w}_h, \boldsymbol{u}_h)\}_{h>0}$ be a sequence of numerical solutions constructed according to* (3.5) *and Definition 3.1. Then*

$$\boldsymbol{w}_h \in_b L^2(0,T; \boldsymbol{L}^{2^*}(\Omega)), \qquad \boldsymbol{u}_h \in_b L^2(0,T; \boldsymbol{L}^{2^*}(\Omega)),$$

where $2^* = 6$ if $N = 3$ and 2^* is any large finite number if $N = 2$.

Proof. The second equation in (3.3) with test function $\boldsymbol{\eta}_h = \nabla_x s_h$ reads:

$$\int_\Omega \boldsymbol{w}_h^m \nabla_x s_h \, dx = 0, \quad \forall s_h \in S_h(\Omega), \quad m = 1, \dots, M,$$

where the space $S_h(\Omega)$ is defined in Section 2.3. By definition, this means that $\boldsymbol{w}_h^m \in \boldsymbol{W}_h^{0,\perp}(\Omega)$ and hence Lemmma 2.11 is applicable and yields the desired estimate:

$$\boldsymbol{w}_h \in_{\mathrm{b}} L^2(0,T;\boldsymbol{L}^{2^*}(\Omega)). \tag{4.4}$$

Next, we make use of Lemma 4.2 and let $\{\boldsymbol{\zeta}_h\}_{h>0}$, $\{\boldsymbol{z}_h\}_{h>0}$ satisfy

$$\begin{aligned} &\boldsymbol{u}_h(\cdot,t) = \operatorname{curl}_x \boldsymbol{\zeta}_h(\cdot,t) + \boldsymbol{z}_h(\cdot,t), \\ &\boldsymbol{\zeta}_h(\cdot,t) \in \boldsymbol{W}_h^{0,\perp}(\Omega), \quad \boldsymbol{z}_h(\cdot,t) \in \boldsymbol{V}_h^{0,\perp}(\Omega), \end{aligned}$$

for all $t \in (0,T)$.

Another application of Lemma 2.11 yields

$$\boldsymbol{z}_h \in_{\mathrm{b}} L^2(0,T;\boldsymbol{L}^{2^*}(\Omega)). \tag{4.5}$$

Fix $\boldsymbol{\eta} \in \boldsymbol{W}_0^{\operatorname{curl}_x,(2^*)'}(\Omega)$ and let $\boldsymbol{\eta}_h \in \boldsymbol{W}_h^{0,\perp}(\Omega)$ satisfy

$$\int_\Omega \operatorname{curl}_x \boldsymbol{\eta}_h \operatorname{curl}_x \boldsymbol{\phi}_h \, dxdt = \int_\Omega \operatorname{curl}_x \boldsymbol{\eta} \operatorname{curl}_x \boldsymbol{\phi}_h \, dxdt, \quad \forall \boldsymbol{\phi}_h \in \boldsymbol{W}_h(\Omega).$$

Then, by utilizing the second equation in (3.3) with $\boldsymbol{\eta}_h$ as test function (the second equality below), we calculate

$$\begin{aligned} \sum_{m=1}^M \Delta t \left| \int_\Omega \operatorname{curl}_x \boldsymbol{\eta} \operatorname{curl}_x \boldsymbol{\zeta}_h^m \, dx \right|^2 &= \sum_{m=1}^M \Delta t \left| \int_\Omega \operatorname{curl}_x \boldsymbol{\eta}_h \operatorname{curl}_x \boldsymbol{\zeta}_h^m \, dx \right|^2 \\ &= \sum_{m=1}^M \Delta t \left| \int_\Omega \boldsymbol{w}_h^m \boldsymbol{\eta}_h^m dxdt \right|^2 \leq \|\boldsymbol{w}_h\|^2_{L^2(0,T;\boldsymbol{L}^{2^*}(\Omega))} \|\boldsymbol{\eta}_h\|^2_{\boldsymbol{L}^{(2^*)'}(\Omega))} \\ &\leq \|\boldsymbol{w}_h\|^2_{L^2(0,T;\boldsymbol{L}^{2^*}(\Omega))} \|\operatorname{curl}_x \boldsymbol{\eta}\|^2_{\boldsymbol{L}^{(2^*)'}(\Omega)}, \end{aligned} \tag{4.6}$$

where the last inequality follows from the discrete Poincaré inequality (2.8).

Now, for an arbitrary $\boldsymbol{\phi} \in L^{(2^*)'}$ let $\operatorname{curl}_x \boldsymbol{\eta}$ be given through the Hodge decomposition $\boldsymbol{\phi} = \operatorname{curl}_x \boldsymbol{\eta} + \nabla_x \lambda$. Then, we can use (4.6) to deduce

$$\begin{aligned} &\int_0^T \left(\sup_{\boldsymbol{\phi} \in L^{(2^*)'}(\Omega)} \frac{\left| \int_\Omega \boldsymbol{\phi} \operatorname{curl}_x \boldsymbol{\zeta}_h dx \right|}{\|\boldsymbol{\phi}\|_{\boldsymbol{L}^{(2^*)'}(\Omega)}} \right)^2 dt \\ &\quad = \int_0^T \left(\sup_{\boldsymbol{\phi} \in L^{(2^*)'}(\Omega)} \frac{\left| \int_\Omega \operatorname{curl}_x \boldsymbol{\eta} \operatorname{curl}_x \boldsymbol{\zeta}_h dx \right|}{\|\boldsymbol{\phi}\|_{\boldsymbol{L}^{(2^*)'}(\Omega)}} \right)^2 dt \\ &\quad = \int_0^T \left(\sup_{\boldsymbol{\phi} \in L^{(2^*)'}(\Omega)} \frac{\left| \int_\Omega \boldsymbol{\eta}_h \boldsymbol{w}_h dx \right|}{\|\boldsymbol{\phi}\|_{\boldsymbol{L}^{(2^*)'}(\Omega)}} \right)^2 dt \leq \|\boldsymbol{w}_h\|^2_{L^2(0,T;\boldsymbol{L}^{2^*}(\Omega))}, \end{aligned}$$

where the last term is bounded from (4.4). Hence, $\operatorname{curl}_x \boldsymbol{\zeta}_h \in_b L^2(0,T;\boldsymbol{L}^{2^*}(\Omega))$ and, keeping in mind (4.5), $\boldsymbol{u}_h \in_b L^2(0,T;\boldsymbol{L}^{2^*}(\Omega))$. □

In the upcoming convergence analysis and in order to establish weak time-continuity of the density we shall need to control the artificial diffusion introduced by the upwind discretization of the continuity equation. The following lemma provides the required bound.

Lemma 4.5. *Let $\{(\varrho_h, \boldsymbol{w}_h, \boldsymbol{u}_h)\}_{h>0}$ be a sequence of numerical solutions constructed according to* (3.5) *and Definition 3.1. Then there exists a constant $C > 0$*

depending only on the initial energy $\mathcal{E}(\varrho^0, \boldsymbol{u}^0)$, the shape–regularity of E_h, T, and $|\Omega|$, such that

$$\begin{aligned}&\sum_{E\in E_h}\int_0^T\int_{\partial E}[\![\varrho_h]\!]\,(\boldsymbol{u}_h\cdot\nu)^-(\Pi_h^Q\phi-\phi)\;dS(x)dt\\&\le h^{\theta(\gamma)}C\|\nabla_x\phi\|_{L^2(0,T;L^{2^*}(\Omega))},\quad \forall\phi\in L^2(0,T;W^{1,2^*}(\Omega)),\end{aligned}$$

where $2^*=6$, if $N=3$, and 2^* is a sufficiently large number, if $N=2$. Here, $\theta(\gamma)>0$ is given by (4.14) *below.*

Proof. Let $\phi\in L^2(0,T;W^{1,2^*}(\Omega))$ be arbitrary and set

$$\phi^m=\frac{1}{\Delta t}\int_{t^{m-1}}^{t^m}\phi(s,x)\;ds,\quad \phi_h^m=\Pi_h^Q\phi^m,\quad m=1,\dots,M.$$

We will need the auxilary function $B(z)=z^\alpha$. where

$$\alpha=\frac{\gamma}{i+1}\text{ and } i\in\mathbb{N}\text{ is chosen such that }\gamma\in(i+1,i+2].$$

Using $B''(z)>0$ and the Hölder inequality, we obtain

$$\begin{aligned}I^2&:=\left|\sum_{m=1}^M\sum_{E\in E_h}\Delta t\int_{\partial E\setminus\partial\Omega}[\![\varrho_h^m]\!]_{\partial E}\,(\boldsymbol{u}_h^m\cdot\nu)^-(\phi_h^m-\phi^m)dS(x)\right|^2\\&\le\left(\sum_{m=1}^M\sum_{E\in E_h}\Delta t\int_{\partial E\setminus\partial\Omega}B''(\varrho_\dagger^m)\,[\![\varrho_h^m]\!]^2\,|\boldsymbol{u}_h^m\cdot\nu|dS(x)\right)\\&\quad\times\left(\sum_{m=1}^M\sum_{E\in E_h}\Delta t\int_{\partial E\setminus\partial\Omega}\left(B''(\varrho_\dagger^m)\right)^{-1}|\boldsymbol{u}_h^m\cdot\nu||\Pi_h^Q\phi^m-\phi^m|^2dS(x)\right)\\&=:I_1\times I_2.\end{aligned}$$

In the case $\frac{N}{2}<\gamma\le2$, $\alpha=\gamma$ and Lemma 4.3 yields

$$I_1\le C\int_\Omega B(\varrho_0)\;dx=C\int_\Omega(\varrho^0)^\gamma\;dx.$$

Conversely, if $\gamma>2$ then $2\alpha\le\gamma$ and the renormalized scheme (4.1) with $\phi_h:=1$ yields

$$\begin{aligned}I_1&\le(\alpha-1)\left|\sum_{m=1}^M\Delta t\int_\Omega(\varrho_h^m)^\alpha\operatorname{div}_x\boldsymbol{u}_h^m dx\right|+\int_\Omega(\varrho^0)^\alpha dx\\&\le C\left(\|\varrho_h\|^\alpha_{L^\infty(0,T;L^\gamma(\Omega))}\|\operatorname{div}_x\boldsymbol{u}_h\|_{L^2(0,T;L^2(\Omega))}+\int_\Omega(\varrho^0)^\gamma dx\right),\end{aligned}$$

which is bounded by Lemma 4.3. Consequently, in both cases, we conclude that

$$I_1\le C. \tag{4.7}$$

To bound the I_2 term, we utilize the Hölder inequality:

$$\begin{aligned}&\int_0^T\int_\Omega|fgh^2|\;dxdt\\&\quad\le\int_0^T\left(\int_\Omega|fg|^{\frac{2^*}{2^*-2}}\;dx\right)^{\frac{2^*-2}{2^*}}\left(\int_\Omega|h|^{2^*}\;dx\right)^{\frac{2}{2^*}}dt\\&\quad\le\int_0^T\left(\int_\Omega|f|^{m_1}\;dx\right)^{\frac{1}{m_1}}\|g\|_{L^{2^*}(\Omega)}\|h\|^2_{L^{2^*}(\Omega)}\;dt\\&\quad\le\|f\|_{L^\infty(0,T;L^{m_1}(\Omega))}\|g\|_{L^\infty(0,T;L^{2^*}(\Omega))}\|h\|^2_{L^2(0,T;L^{2^*}(\Omega))},\end{aligned}\tag{4.8}$$

where $1 < m_1 = \frac{2^*}{2^*-3} \leq 2$ and $\frac{1}{m_1} + \frac{1}{2^*} + \frac{2}{2^*} = 1$.

Now, using (4.8), we deduce

$$
\begin{aligned}
I_2 \leq \alpha(\alpha-1) \max_{m=1,\dots,M} & \left(\sum_{E\in E_h} \int_{\partial E\setminus\partial\Omega} |\boldsymbol{u}_h^m \cdot \nu|^2 \, dS(x) \right)^{\frac{1}{2}} \\
& \times \sum_{m=1}^{M} \Delta t \left(\sum_{E\in E_h} \int_{\partial E\setminus\partial\Omega} \left| \Pi_h^Q \phi^m - \phi^m \right|^{2^*} \, dS(x) \right)^{\frac{2}{2^*}} \\
& \times \max_{m=1,\dots,M} \left(\sum_{E\in E_h} \int_{\partial E\setminus\partial\Omega} \left| \left(B''(\varrho_\dagger^m)\right)^{-1} \right|^{m_1} \, dS(x) \right)^{\frac{1}{m_1}}.
\end{aligned}
\tag{4.9}
$$

Next, we apply Lemma 2.10 to deduce

$$
\max_{m=1,\dots,M} \left(\sum_{E\in E_h} \int_{\partial E\setminus\partial\Omega} |\boldsymbol{u}_h^m \cdot \nu|^{2^*} \, dS(x) \right)^{\frac{1}{2^*}} \leq Ch^{-\frac{1}{2^*}} \|\boldsymbol{u}_h\|_{L^\infty(0,T;\boldsymbol{L}^{2^*}(\Omega))}. \tag{4.10}
$$

Similarly, we find that

$$
\begin{aligned}
& \sum_{m=1}^{M} \Delta t \left(\sum_{E\in E_h} \int_{\partial E\setminus\partial\Omega} \left| \Pi_h^Q \phi^m - \phi^m \right|^{2^*} \, dS(x) \right)^{\frac{2}{2^*}} \\
& \qquad \leq Ch^{-\frac{2}{2^*}} \|\Pi_h^Q \phi - \phi\|^2_{L^2(0,T;L^{2^*}(\Omega))} \leq Ch^{2-\frac{2}{2^*}} \|\nabla_x \phi\|^2_{L^2(0,T;L^{2^*}(\Omega))},
\end{aligned}
$$

where the last inequality is an application of Lemma 2.8.

To derive a similar bound for the B'' term in (4.9), we first note that, since ϱ_h^m is everywhere positive and $2-\alpha < 1$,

$$
\left| \left(B''(\varrho_\dagger^m)\right)^{-1} \right|^{m_1} \leq \left| \varrho_+^m + \varrho_-^m \right|^{(2-\alpha)m_1} \leq C(1 + \left|\varrho_+^m\right|^{m_1} + \left|\varrho_-^m\right|^{m_1}),
$$

on every $\Gamma \cap \partial E \setminus \partial\Omega$. From this, we conclude that

$$
\int_{\partial E} \left| \left(B''(\varrho_\dagger^m)\right)^{-1} \right|^{m_1} \, dS(x) \leq h^{-1} C \left(|E| + \int_{\mathcal{N}(E)\cup E} |\varrho_h|^{m_1} \, dx \right),
$$

where $\mathcal{N}(E)$ denotes the union of the neighboring elements of E. Applying this together with Lemma 2.10, we obtain

$$
\begin{aligned}
& \max_{m=1,\dots,M} \left(\sum_{E\in E_h} \int_{\partial E\setminus\partial\Omega} \left| \left(B''(\varrho_\dagger^m)\right)^{-1} \right|^{m_1} \, dS(x) \right)^{m_1} \\
& \qquad \leq Ch^{-\frac{1}{m_1}} \left(|\Omega|^{\frac{1}{m_1}} + \|\varrho_h\|_{L^\infty(0,T;L^{m_1}(\Omega))} \right) \\
& \qquad \leq Ch^{-\frac{1}{m_1}} \left(1 + h^{\min\{0,N(\frac{1}{m_1}-\frac{1}{\gamma})\}} \|\varrho_h\|_{L^\infty(0,T;L^\gamma(\Omega))} \right),
\end{aligned}
\tag{4.11}
$$

where the last inequality is a standard inverse estimate (Lemma 2.9). Setting (4.10)–(4.11) into (4.9) leads to the bound

$$
\begin{aligned}
I_2 & \leq Ch \|\boldsymbol{u}_h\|_{L^\infty(0,T;\boldsymbol{L}^{2^*}(\Omega))} \|\nabla_x \phi\|^2_{L^2(0,T;L^{2^*}(\Omega))} \\
& \qquad \times \left(1 + h^{\min\{0,N(\frac{1}{m_1}-\frac{1}{\gamma}\}} \|\varrho_h\|_{L^\infty(0,T;L^\gamma(\Omega))} \right) \\
& \leq Ch^{\frac{1}{2}} \|\boldsymbol{u}_h\|_{L^2(0,T;\boldsymbol{L}^{2^*}(\Omega))} \|\nabla_x \phi\|^2_{L^2(0,T;L^{2^*}(\Omega))} \\
& \qquad \times \left(1 + h^{\min\{0,N(\frac{1}{m_1}-\frac{1}{\gamma}\}} \|\varrho_h\|_{L^\infty(0,T;L^\gamma(\Omega))} \right),
\end{aligned}
\tag{4.12}
$$

where the last inequality is an application Lemma 2.9 in time (keeping in mind $\Delta t = \kappa h$). We have also used that

$$h^{-\left(\frac{1}{m_1}+\frac{1}{2^*}+\frac{2}{2^*}\right)} = h^{-1}.$$

In 2D, 2^* is any large finite number. Consequently, we can always make sure that $m_1 \leq \gamma$. Using this, it is straight forward to check that

$$h^{\frac{1}{2}} h^{\min\{0, N(\frac{1}{m_1}-\frac{1}{\gamma}\}} = h^{2\theta(\gamma)}, \tag{4.13}$$

where

$$0 < \theta(\gamma) := \begin{cases} \frac{1}{4}, & N = 2, \\ \frac{1}{2} + \min\{0, 3(\frac{1}{2} - \frac{1}{\gamma})\}, & N = 3, \end{cases} \tag{4.14}$$

By setting (4.13) into (4.12) and applying Lemma 4.3, we obtain

$$I_2 \leq h^{2\theta(\gamma)} C \|\nabla_x \phi\|^2_{L^2(0,T;L^{2^*}(\Omega))}.$$

This and (4.7) gives

$$I^2 = I_1 \times I_2 \leq C \|\nabla_x \phi\|_{L^2(0,T;L^{2^*}(\Omega))} h^{2\theta(\gamma)},$$

which brings the proof to an end. □

4.1. **Weak time-continuity estimates.** We end this section by establishing weak time-continuity of the density and velocity.

Lemma 4.6. *Let $\{(\varrho_h, \boldsymbol{w}_h, \boldsymbol{u}_h)\}_{h>0}$ be a sequence of numerical solutions constructed according to* (3.5) *and Definition 3.1. Then*

$$\partial_t^h(\varrho_h) \in_b L^2(0,T; W^{-1,(2^*)'}(\Omega)),$$

where $(2^*)' = \frac{2^*}{2^*-1}$ *and* 2^* *is as in the previous lemma.*

Proof. The proof is almost identical to the proof of Lemma 5.6 in [8] and is only included for the sake of completeness.

Fix $\phi \in L^2(0,T; W^{1,2^*}(\Omega))$, and introduce the piecewise constant approximations $\phi_h := \Pi_h^Q \phi$, $\phi_h^m := \Pi_h^Q \phi^m$, and $\phi^m := \frac{1}{\Delta t}\int_{t^{m-1}}^{t^m} \phi(t,\cdot)\, dt$.

The continuity scheme (3.2) with ϕ_h^m as test function reads

$$\begin{aligned} \Delta t \int_\Omega \partial_t^h(\varrho_h^m)\, \phi^m \, dxdt \\ = \Delta t \sum_{\Gamma \in \Gamma_h^I} \int_\Gamma \left(\varrho_-^m (\boldsymbol{u}_h^m \cdot \nu)^+ + \varrho_+^m (\boldsymbol{u}_h^m \cdot \nu)^-\right) [\![\phi_h^m]\!]_\Gamma \, dS(x). \end{aligned} \tag{4.15}$$

Since the traces of ϕ^m taken from either side of a face are equal, we can write

$$
\begin{aligned}
&\sum_{\Gamma\in\Gamma_h^I}\int_\Gamma \left(\varrho_-^m(\boldsymbol{u}_h^m\cdot\nu)^+ + \varrho_+^m(\boldsymbol{u}_h^m\cdot\nu)^-\right)[\![\phi_h^m]\!]_\Gamma \; dx\\
&= \sum_{\Gamma\in\Gamma_h^I}\int_\Gamma \left(\varrho_+^m(\boldsymbol{u}_h^m\cdot\nu)^- + \varrho_-^m(\boldsymbol{u}_h^m\cdot\nu)^+\right)[\![\phi_h^m-\phi^m]\!] \; dS(x),\\
&= -\sum_{E\in E_h}\int_{\partial E\setminus\partial\Omega} \left(\varrho_+^m(\boldsymbol{u}_h^m\cdot\nu)^+ + \varrho_-^m(\boldsymbol{u}_h^m\cdot\nu)^-\right)(\phi_h^m-\phi^m) \; dS(x),\\
&= \sum_{E\in E_h}\int_E -\operatorname{div}_x(\varrho_h^m\boldsymbol{u}_h^m(\phi_h^m-\phi^m))\; dx\\
&\qquad\qquad + \sum_{E\in E_h}\int_{\partial E\setminus\partial\Omega}[\![\varrho_h^m]\!]_{\partial E}\,(\boldsymbol{u}_h^m\cdot\nu)^-(\phi_h^m-\phi^m)\; dS(x)\\
&= \int_\Omega \varrho_h^m\boldsymbol{u}_h^m\cdot\nabla_x\phi^m\; dx + \sum_{E\in E_h}\int_{\partial E\setminus\partial\Omega}[\![\varrho_h^m]\!]_{\partial E}\,(\boldsymbol{u}_h^m\cdot\nu)^-(\phi_h^m-\phi^m)\; dS(x).
\end{aligned}
\tag{4.16}
$$

To conclude the last equality, we have used

$$
\int_E \varrho_h^m \operatorname{div}_x \boldsymbol{u}_h^m(\phi_h^m-\phi^m)\; dx = (\varrho_h^m \operatorname{div}_x \boldsymbol{u}_h^m)|_E\int_E \Pi_h^Q\phi^m-\phi^m\; dx = 0, \quad \forall E\in E_h,
$$

since both ϱ_h^m and $\operatorname{div}_x \boldsymbol{u}_h^m$ are piecewise constant.

By summing (4.15) over m, taking absolute values, and using the above identity, we find

$$
\begin{aligned}
&\left|\sum_{m=1}^M \Delta t\int_\Omega \partial_t^h\left(\varrho_h^m\right)\phi^m\; dxdt\right|\\
&\qquad\qquad \le \left|\sum_{m=1}^M \Delta t\int_\Omega \varrho_h^m\boldsymbol{u}_h^m\nabla_x\phi^m\; dx\right|\\
&\qquad\qquad\quad + \left|\sum_{m=1}^M\sum_{E\in E_h}\Delta t\int_{\partial E\setminus\partial\Omega}[\![\varrho_h^m]\!]_{\partial E}\,(\boldsymbol{u}_h^m\cdot\nu)^-(\phi_h^m-\phi^m)\; dS(x)\right|.
\end{aligned}
$$

Using Lemma 4.5, together with an application of Hölder's inequality, we deduce

$$
\begin{aligned}
&\left|\sum_{m=1}^M \Delta t\int_\Omega \partial_t^h\left(\varrho_h^m\right)\phi^m\; dx\right|\\
&\le \sum_{m=1}^M \Delta t\|\varrho_h^m\|_{L^\alpha(\Omega)}\|\boldsymbol{u}_h^m\|_{\boldsymbol{L}^{2^*}(\Omega)}\|\nabla_x\phi^m\|_{\boldsymbol{L}^{2^*}(\Omega)} + Ch^{\theta(\gamma)}\|\nabla_x\phi\|_{L^2(0,T;\boldsymbol{L}^{2^*}(\Omega))}\\
&\le \|\varrho_h\|_{L^\infty(0,T;L^\alpha(\Omega))}\|\boldsymbol{u}_h\|_{L^\infty(0,T;\boldsymbol{L}^{2^*}(\Omega))}\|\nabla_x\phi\|_{L^2(0,T;\boldsymbol{L}^{2^*}(\Omega))}\\
&\qquad + C\,h^{\theta(\gamma)}\,\|\nabla_x\phi\|_{L^2(0,T;\boldsymbol{L}^{2^*}(\Omega))},
\end{aligned}
$$

where $\alpha = \frac{2^*(2^*)'}{2^*-(2^*)'} < \gamma$ since $\gamma > \frac{N}{2}$ and $\frac{1}{\alpha}+\frac{1}{2^*}+\frac{1}{2^*} = 1$. By Lemma 4.3, the right–hand side is bounded, so we conclude that

$$
\begin{aligned}
&\left|\int_{\Delta t}^T\int_\Omega \partial_t^h\left(\varrho_h\right)\phi\; dxdt\right|\\
&= \left|\sum_{m=1}^M \Delta t\int_\Omega \partial_t^h\left(\varrho_h^m\right)\phi^m\; dx\right| \le C\left(1+h^{\theta(\gamma)}\right)\|\nabla_x\phi\|_{L^2(0,T;\boldsymbol{L}^{2^*}(\Omega))}.
\end{aligned}
$$

This brings the proof to an end. □

Lemma 4.7. *Let $\{(\varrho_h, \boldsymbol{w}_h, \boldsymbol{u}_h)\}_{h>0}$ be a sequence of numerical solutions constructed according to* (3.5) *and Definition 3.1. Then*

$$\partial_t^h(\boldsymbol{u}_h) \in_b L^2(0,T;W^{-1,1}(\Omega)),$$

Proof. By adding and subtracting, we see that for any $\boldsymbol{\phi} \in L^2(0,T;\boldsymbol{W}_0^{1,\infty}(\Omega))$,

$$\begin{aligned}
&\int_0^T \int_\Omega \partial_t^h(\boldsymbol{u}_h)\,\boldsymbol{\phi}\,dxdt \\
&= \int_0^T \int_\Omega \partial_t^h(\boldsymbol{u}_h)\,\Pi_h^V\boldsymbol{\phi}\,dxdt + \int_0^T\int_\Omega \partial_t^h(\boldsymbol{u}_h)\left(\boldsymbol{\phi}-\Pi_h^V\boldsymbol{\phi}\right)\,dxdt.
\end{aligned}$$

From the first equation of the momentum scheme (3.3) with $\boldsymbol{v}_h = \Pi_h^V\boldsymbol{\phi}$, we have that

$$\begin{aligned}
\int_0^T\int_\Omega \partial_t^h(\boldsymbol{u}_h)\,\Pi_h^V\boldsymbol{\phi}dxdt &= -\int_0^T\int_\Omega \mu\operatorname{curl}_x\boldsymbol{w}_h\left(\Pi_h^V\boldsymbol{\phi}\right) + (\mu+\lambda)\operatorname{div}_x\boldsymbol{u}_h\operatorname{div}_x\Pi_h^V\boldsymbol{\phi}\,dxdt \\
&\quad + \int_0^T\int_\Omega a\varrho_h^\gamma\operatorname{div}_x\Pi_h^V\boldsymbol{\phi}\,dxdt \\
&\le C\Big(\|\operatorname{curl}_x\boldsymbol{w}_h\|_{L^2(0,T;\boldsymbol{L}^2(\Omega)}\|\boldsymbol{\phi}\|_{L^2(0,T;\boldsymbol{L}^2(\Omega)} \\
&\quad + \|\operatorname{div}_x\boldsymbol{u}_h\|_{L^2(0,T;L^2(\Omega))}\|\operatorname{div}_x\boldsymbol{\phi}\|_{L^2(0,T;L^2(\Omega))} \\
&\quad + \|\varrho_h\|_{L^\infty(0,T;L^\gamma(\Omega))}\|\operatorname{div}_x\boldsymbol{\phi}\|_{L^1(0,T;L^\infty(\Omega))}\Big) \\
&\le C\|\boldsymbol{\phi}\|_{L^2(0,T;\boldsymbol{W}^{1,\infty}(\Omega))},
\end{aligned}$$

where the last inequality follows from Lemma 4.3.

From Lemma 4.3, we also have the estimate

$$\|\partial_t^h(\boldsymbol{u}_h)\|_{L^2(0,T;\boldsymbol{L}^2(\Omega))} = (\Delta t)^{-\frac12}\left(\sum_{m=1}^M\int_\Omega [\![\boldsymbol{u}_h^{m-1}]\!]^2\,dx\right)^{\frac12} \le h^{-\frac12}C. \tag{4.17}$$

Using (4.17), we estimate

$$\begin{aligned}
\int_{\Delta t}^T\int_\Omega \frac{d\left(\Pi_{\mathcal{L}}\boldsymbol{u}_h\right)}{dt}\left(\boldsymbol{\phi}-\Pi_h^V\boldsymbol{\phi}\right)dxdt &\le C\|\partial_t^h(\boldsymbol{u}_h)\|_{L^2(0,T;\boldsymbol{L}^2(\Omega))}\|\boldsymbol{\phi}-\Pi_h^V\boldsymbol{\phi}\|_{L^2(0,T;\boldsymbol{L}^2(\Omega)} \\
&\le C\frac{h}{\sqrt{\Delta t}}\|\nabla_x\phi\|_{L^2(0,T;\boldsymbol{L}^2(\Omega))} \le Ch^{\frac12},
\end{aligned}$$

where we in the last inequality have used the relation $\Delta t = \kappa h$. Combining the previous estimates concludes the proof. □

Recall our notation for the Hodge decomposition of the solution $\boldsymbol{u}$,

$$\boldsymbol{u} = \operatorname{curl}_x\boldsymbol{\zeta} + \nabla_x s.$$

In the next result, we prove that $\partial_t(\operatorname{curl}_x\boldsymbol{\zeta}) \in L^2(0,T;\boldsymbol{L}^2(\Omega))$. To see why such a bound is reasonable, apply the curl_x operator to the velocity equation (1.5)

$$\operatorname{curl}_x(\operatorname{curl}_x\boldsymbol{\zeta})_t + \mu\operatorname{curl}_x\operatorname{curl}_x\boldsymbol{w} = 0.$$

Multiplying with $\boldsymbol{\zeta}_t$, integrating by parts in space, and applying Hölder's inequality,

$$\|\operatorname{curl}_x\boldsymbol{\zeta}_t\|_{L^2(\Omega)}^2 \le \epsilon\|\operatorname{curl}_x\boldsymbol{\zeta}_t\|_{L^2(\Omega)}^2 + \frac{C}{\epsilon}\|\operatorname{curl}_x\boldsymbol{w}\|_{\boldsymbol{L}^2(\Omega)}^2.$$

Fixing ϵ small, and integrating in time

$$\int_0^T\|\operatorname{curl}_x\boldsymbol{\zeta}_t\|_{L^2(\Omega)}^2\,dt \le C\int_0^T\|\operatorname{curl}_x\boldsymbol{w}\|_{\boldsymbol{L}^2(\Omega)}^2\,dt,$$

where the right-hand side is bounded. Consequently, it is the higher regularity on $\boldsymbol{w} = \operatorname{curl}_x \boldsymbol{u}$ that enable us to obtain the bound.

Lemma 4.8. *Let $\{(\varrho_h, \boldsymbol{w}_h, \boldsymbol{u}_h)\}_{h>0}$ be a sequence of numerical solutions constructed according to (3.5) and Definition 3.1. Let $\{(\boldsymbol{\zeta}_h, \boldsymbol{z}_h)\}_{h>0}$ be the sequence given by the decomposition $\boldsymbol{u}_h(\cdot,t) = \operatorname{curl}_x \boldsymbol{\zeta}_h(\cdot,t) + \boldsymbol{z}_h(\cdot,t)$ and $\boldsymbol{\zeta}_h(\cdot,t) \in \boldsymbol{W}_h^{0,\perp}(\Omega)$, $\boldsymbol{z}_h(\cdot,t) \in \boldsymbol{V}_h^{0,\perp}(\Omega)$, for $t \in (0,T)$. Then*

$$\partial_t^h \left(\operatorname{curl}_x \boldsymbol{\zeta}_h\right) \in_b L^2(0,T;\boldsymbol{L}^2(\Omega)).$$

Proof. For any $m = 1, \dots, M$, let $\boldsymbol{v}_h^m = \partial_t^h \left(\operatorname{curl}_x \boldsymbol{\zeta}_h^m\right) \in \boldsymbol{V}_h$. Observe that by the orthogonality of the Hodge decomposition,

$$\int_\Omega \partial_t^h \left(\boldsymbol{u}_h^m\right) \partial_t^h \left(\operatorname{curl}_x \boldsymbol{\zeta}_h^m\right) \, dx = \int_\Omega \left|\partial_t^h \left(\operatorname{curl}_x \boldsymbol{\zeta}_h^m\right)\right|^2 \, dx.$$

Hence, by setting $\boldsymbol{v}_h^m$ as test function in the first equation of the momentum scheme (3.3), multiplying with Δt, and summing over all $m = 1, \dots, M$, we obtain

$$\begin{aligned}
&\sum_{m=1}^M \Delta t \int_\Omega \left|\partial_t^h \left(\operatorname{curl}_x \boldsymbol{\zeta}_h^m\right)\right|^2 \, dxdt \\
&\qquad = -\sum_{m=1}^M \Delta t \int_\Omega \mu \operatorname{curl}_x \boldsymbol{w}_h^m \partial_t^h \left(\operatorname{curl}_x \boldsymbol{\zeta}_h^m\right) \, dxdt \\
&\qquad \leq \mu \left(\sum_{m=1}^M \Delta t \|\partial_t^h \left(\operatorname{curl}_x \boldsymbol{\zeta}_h^m\right)\|^2\right)^{\frac{1}{2}} \left(\sum_{m=1}^M \Delta t \|\operatorname{curl}_x \boldsymbol{w}_h^m\|_{\boldsymbol{L}^2(\Omega)}^2\right)^{\frac{1}{2}}.
\end{aligned}$$

An application of the Cauchy inequality with ϵ to the above estimate yields

$$\|\partial_t^h \left(\operatorname{curl}_x \boldsymbol{\zeta}_h\right)\|_{L^2(0,T;\boldsymbol{L}^2(\Omega))} \leq \frac{\mu}{2} \|\operatorname{curl}_x \boldsymbol{w}_h\|_{L^2(0,T;\boldsymbol{L}^2(\Omega))}.$$

Lemma 4.3 provides a bound on the right–hand side and hence the proof is complete. □

5. Higher intergrability on the density

The stability estimate only provides the bound $p(\varrho_h) \in_{\mathrm{b}} L^\infty(0,T;L^1(\Omega))$. Hence, it is not clear that $p(\varrho_h)$ converges weakly to an integrable function. Moreover, the subsequent analysis relies heavily on the pressure having higher integrability. In this section we establish that the density is in fact bounded in $L^{\gamma+1}(0,T;L^{\gamma+1}(\Omega))$, independently of h. The main technical tool used to achieve this is an equation for the effective viscous flux:

$$P_{\text{eff}}(\varrho, \boldsymbol{u}) = p(\varrho) - (\lambda + \mu) \operatorname{div}_x \boldsymbol{u}.$$

We start by deriving this equation. For this purpose, fix any $\phi \in L^2(0,T;L_0^2(\Omega))$ and, for each fixed $h > 0$, let

$$\boldsymbol{v}_h(t,\cdot) = \Pi_h^V \left(\nabla_x \Delta^{-1} [\phi]\right)(t,\cdot), \quad t \in (0,T),$$

and

$$\boldsymbol{v}_h^m = \frac{1}{\Delta t} \int_{t^{m-1}}^{t^m} \boldsymbol{v}_h(s,\cdot) \, ds, \quad m = 1, \dots, M.$$

Observe that $\boldsymbol{v}_h$ is constructed such that

$$\operatorname{div}_x \boldsymbol{v}_h^m = \frac{1}{\Delta t} \int_{t^{m-1}}^{t^m} \phi \, dt, \quad m = 1, \dots, M.$$

By inserting $\boldsymbol{v}_h^m$ as test function in the momentum scheme (3.3), multiplying with Δt, and summing over all $m = 1, \dots, M$, we are led to the identity

$$\int_0^T \int_\Omega P_{\text{eff}}(\varrho_h, \boldsymbol{u}_h)\phi \, dxdt = \int_0^T \int_\Omega \left(\partial_t^h (\boldsymbol{u}_h) + \mu \operatorname{curl}_x \boldsymbol{w}_h\right) \Pi_h^V \left(\nabla_x \Delta^{-1}[\phi]\right) \, dxdt.$$

Since $\int_\Omega (\operatorname{curl}_x \boldsymbol{w}_h^m) \nabla_x \Delta^{-1}[\phi] \, dx = 0$, for all $m = 1, \dots, M$, we can further write

$$\begin{aligned}
&\int_0^T \int_\Omega P_{\text{eff}}(\varrho_h, \boldsymbol{u}_h)\phi \, dxdt \\
&= \int_0^T \int_\Omega \partial_t^h (\boldsymbol{u}_h) \nabla_x \Delta^{-1}[\phi] \, dxdt \\
&\qquad + \int_0^T \int_\Omega \left(\partial_t^h (\boldsymbol{u}_h) + \mu \operatorname{curl}_x \boldsymbol{w}_h\right) \left(\Pi_h^V \left(\nabla_x \Delta^{-1}[\phi]\right) - \nabla_x \Delta^{-1}[\phi]\right) \, dxdt.
\end{aligned} \tag{5.1}$$

As ϕ was fixed arbitrary, we can conclude that (5.1) holds for all $\phi \in L^2(0,T; L_0^2(\Omega))$.

The following lemma ensures that the last term of (5.1) converges to zero.

Lemma 5.1. *Let $\{(\varrho_h, \boldsymbol{w}_h, \boldsymbol{u}_h)\}_{h>0}$ be a sequence of numerical solutions constructed according to* (3.5) *and Definition 3.1. Then there exists a constant $C > 0$, depending only on the initial data and the shape regularity of E_h, such that*

$$\begin{aligned}
&\left| \int_0^T \int_\Omega \left(\partial_t^h (\boldsymbol{u}_h) + \mu \operatorname{curl}_x \boldsymbol{w}_h\right) \left(\Pi_h^V \left(\nabla_x \Delta^{-1}[\phi]\right) - \nabla_x \Delta^{-1}[\phi]\right) \, dxdt \right| \\
&\leq C(h^{\frac{1}{2}} + h)\|\phi\|_{L^2(0,T;L^2(\Omega))}, \quad \forall \phi \in L^2(0,T; L_0^2(\Omega)).
\end{aligned}$$

Proof. By this, the Hölder inequality, and Lemma 2.8, we deduce

$$\begin{aligned}
&\left| \int_0^T \int_\Omega \left(\partial_t^h (\boldsymbol{u}_h) + \mu \operatorname{curl}_x \boldsymbol{w}_h\right) \left(\Pi_h^V \left(\nabla_x \Delta^{-1}[\phi]\right) - \nabla_x \Delta^{-1}[\phi]\right) dxdt \right| \\
&\qquad \leq ch \|\nabla_x \nabla_x \Delta^{-1}[\phi]\|_{L^2(0,T;\boldsymbol{L}^2(\Omega))} \\
&\qquad\qquad \times \left(\|\partial_t^h (\boldsymbol{u}_h)\|_{L^2(0,T;\boldsymbol{L}^2(\Omega))} + \|\operatorname{curl}_x \boldsymbol{w}_h\|_{L^2(0,T;\boldsymbol{L}^2(\Omega))}\right) \\
&\leq C(h^{\frac{1}{2}} + h)\|\phi\|_{L^2(0,T;L^2(\Omega))},
\end{aligned}$$

where we in the last inequality have used Lemma 4.3 and (4.17). □

We are now in a position to prove higher integrability of the density. To increase readability of the proof, we introduce the notation

$$\langle \phi \rangle_\Omega = \frac{1}{|\Omega|} \int_\Omega \phi \, dx,$$

for the spatial average value of a function.

Lemma 5.2 (Higher integrability on the density)**.** *Let $\{(\varrho_h, \boldsymbol{w}_h, \boldsymbol{u}_h)\}_{h>0}$ be a sequence of numerical solutions constructed according to* (3.5) *and Definition 3.1. Then*

$$\varrho_h \in_b L^{\gamma+1}((0,T) \times \Omega).$$

Proof. Setting $\phi = \varrho_h - \langle \varrho_h^0 \rangle_\Omega$ in (5.1) yields the identity

$$\begin{aligned}
&\int_0^T \int_\Omega p(\varrho_h)\varrho_h \, dxdt \\
&\quad = \int_0^T \int_\Omega p(\varrho_h)\langle \varrho_h^0 \rangle_\Omega + (\lambda + \mu) \operatorname{div}_x \boldsymbol{u}_h \varrho_h + \partial_t^h (\boldsymbol{u}_h) \nabla_x \Delta^{-1} \left[(\varrho_h - \langle \varrho_h^0 \rangle_\Omega) \right] \, dxdt \\
&\qquad + \int_0^T \int_\Omega \left(\partial_t^h (\boldsymbol{u}_h) + \mu \operatorname{curl}_x \boldsymbol{w}_h \right) \\
&\qquad\qquad \times \left(\Pi_h^V \left(\nabla_x \Delta^{-1} \left[\varrho_h - \langle \varrho_h^0 \rangle_\Omega \right] \right) - \nabla_x \Delta^{-1} \left[\varrho_h - \langle \varrho_h^0 \rangle_\Omega \right] \right) \, dxdt,
\end{aligned}$$

Applying the Hölder inequality and Lemmas 4.3 and 5.1 yields

$$\begin{aligned}
\left| \int_0^T \int_\Omega p(\varrho_h)\varrho_h \, dxdt \right| &\leq \left| \int_0^T \int_\Omega \partial_t^h (\boldsymbol{u}_h) \nabla_x \Delta^{-1} \left[\varrho_h - \langle \varrho_h^0 \rangle_\Omega \right] \, dxdt \right| \\
&\quad + C \left(1 + h^{\frac{1}{2}} + h \right) \|\varrho_h\|_{L^2(0,T;L^2(\Omega))}.
\end{aligned} \tag{5.2}$$

To bound the first term on the right–hand side, we first note that

$$\begin{aligned}
&\int_0^T \int_\Omega \partial_t^h (\boldsymbol{u}_h) \nabla_x \Delta^{-1} \left[\varrho_h - \langle \varrho_h^0 \rangle_\Omega \right] \, dxdt \\
&\qquad = \sum_{m=1}^M \Delta t \int_\Omega \frac{\boldsymbol{u}_h^m - \boldsymbol{u}_h^{m-1}}{\Delta t} \nabla_x \Delta^{-1} \left[\varrho_h^m - \langle \varrho_h^0 \rangle_\Omega \right] \, dx.
\end{aligned} \tag{5.3}$$

Then, we apply summation by parts to (5.3) and make use of the Hölder inequality to obtain

$$\begin{aligned}
&\left| \int_0^T \int_\Omega \partial_t^h (\boldsymbol{u}_h) \nabla_x \Delta^{-1} \left[\varrho_h - \langle \varrho_h^0 \rangle_\Omega \right] \, dxdt \right| \\
&\qquad = \left| - \sum_{k=1}^M \Delta t \int_\Omega \boldsymbol{u}_h^{m-1} \nabla_x \Delta^{-1} \left[\partial_t^h (\varrho_h^m) \right] \, dx \right. \\
&\qquad\qquad \left. - \frac{1}{\Delta t} \int_0^{\Delta t} \int_\Omega \boldsymbol{u}_h^0 \nabla_x \Delta^{-1} \left[\varrho_h - \langle \varrho_h^0 \rangle_\Omega \right] dxdt \right| \\
&\qquad \leq \left| \sum_{m=1}^M \Delta t \int_\Omega \boldsymbol{u}_h^{m-1} \nabla_x \Delta^{-1} \left[\partial_t^h (\varrho_h^m) \right] \, dx \right| \\
&\qquad\qquad + C \|\boldsymbol{u}^0\|_{\boldsymbol{L}^2(\Omega)} \|\varrho_h - \langle \varrho_h^0 \rangle_\Omega\|_{L^\infty(0,T;L^\gamma(\Omega))},
\end{aligned} \tag{5.4}$$

where we in the last inequality have used elliptic theory (and $\gamma^* > 2$ since $\gamma > \frac{N}{2}$) to conclude that

$$\left\| \nabla_x \Delta^{-1} \left[\varrho_h - \langle \varrho_h^0 \rangle_\Omega \right] \right\|_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))} \leq C \|\varrho_h - \langle \varrho_h^0 \rangle_\Omega\|_{L^\infty(0,T;L^\gamma(\Omega))}.$$

Next, using integration by parts,

$$
\begin{aligned}
&\sum_{m=1}^{M} \Delta t \int_\Omega \boldsymbol{u}_h^{m-1} \nabla_x \Delta^{-1}\left[\partial_t^h\left(\varrho_h^m\right)\right]\, dx \\
&= \sum_{m=1}^{M} \Delta t \int_\Omega \Delta^{-1}\left[\operatorname{div}_x \boldsymbol{u}_h^{m-1}\right]\left(\frac{\varrho_h^m - \varrho_h^{m-1}}{\Delta t}\right)\, dx \\
&= \sum_{m=1}^{M} \Delta t \int_\Omega \boldsymbol{u}_h^m \varrho_h^m \nabla_x \Delta^{-1}\left[\operatorname{div}_x \boldsymbol{u}_h^{m-1}\right]\, dx \\
&\qquad + \sum_{m=1}^{M} \Delta t \sum_{E\in E_h} \int_{\partial E} [\![\varrho_h^m]\!]_{\partial E} \left(\boldsymbol{u}_h^m \cdot \nu\right)^- (\Pi_h^V - \mathbb{I})\Delta^{-1}\left[\operatorname{div}_x \boldsymbol{u}_h^{m-1}\right]\, dS(x),
\end{aligned}
\tag{5.5}
$$

where the last equality is deduced as follows: Set $\phi_h^m = \Pi_h^Q \Delta^{-1}\left[\operatorname{div}_x \boldsymbol{u}_h^{m-1}\right]$ in the continuity scheme (3.2) and perform the calculation (4.16).

By setting (5.5) into (5.4), and applying Lemma 4.5, we obtain

$$
\begin{aligned}
&\left|\int_0^T \int_\Omega \partial_t^h\left(\boldsymbol{u}_h\right) \nabla_x \Delta^{-1}\left[\varrho_h - \langle \varrho_h^0\rangle_\Omega\right]\, dxdt\right| \\
&\qquad \le C\left(1 + \|\boldsymbol{u}_h\|_{L^2(0,T;\boldsymbol{L}^{\frac{2\gamma}{\gamma-1}}(\Omega))} \|\varrho_h\|_{L^\infty(0,T;L^\gamma(\Omega))}\right) \\
&\qquad\quad + h^{\theta(\gamma)} C \|\nabla_x \Delta^{-1}\left[\operatorname{div}_x \boldsymbol{u}_h\right]\|_{L^2(0,T;\boldsymbol{L}^{2^*}(\Omega))},
\end{aligned}
\tag{5.6}
$$

where $\theta(\gamma)$ is given by (4.14).

Finally, inserting (5.6) into (5.2) and recalling that $\frac{2\gamma}{\gamma-1} < 2^*$, since $\gamma > \frac{N}{2}$, gives

$$
\begin{aligned}
&\left|\int_0^T \int_\Omega a\varrho_h^{\gamma+1}\, dxdt\right| \\
&\le C\left(1 + \|\boldsymbol{u}_h\|_{L^2(0,T;\boldsymbol{L}^{2^*}(\Omega))} \|\varrho_h\|_{L^\infty(0,T;L^\gamma(\Omega))}\right) + h^{\theta(\gamma)} C \|\operatorname{div}_x \boldsymbol{u}_h\|_{L^2(0,T;L^2(\Omega))} \\
&\quad + \left(1 + h^{\frac{1}{2}} + h\right) \|\varrho_h\|_{L^2(0,T;L^2(\Omega))}.
\end{aligned}
$$

The proof then follows from the Hölder and Cauchy (with epsilon) inequalities. □

6. Convergence

Let $\{(\varrho_h, \boldsymbol{w}_h, \boldsymbol{u}_h)\}_{h>0}$ be a sequence of numerical solutions constructed according to (3.5) and Definition 3.1. In this section we establish that a subsequence of $\{(\varrho_h, \boldsymbol{w}_h, \boldsymbol{u}_h)\}_{h>0}$ converges to a weak solution of the semi–stationary Stokes system, thereby proving Theorem 3.5. The proof is divided into several steps:

(1) Strong convergence of the velocity.
(2) Convergence of the continuity scheme.
(3) Weak sequential continuity of the discrete viscous flux.
(4) Strong convergence of the density.
(5) Convergence of the velocity scheme.

Our starting point is that the results of the previous sections ensure us that the approximate solutions $(\boldsymbol{w}_h, \boldsymbol{u}_h, \varrho_h)$ satisfy the following h–independent bounds:

$$
\begin{aligned}
\varrho_h &\in_{\mathrm{b}} L^\infty(0,T;L^\gamma(\Omega)) \cap L^{\gamma+1}((0,T)\times\Omega), \\
\boldsymbol{w}_h &\in_{\mathrm{b}} L^\infty(0,T;\boldsymbol{L}^2(\Omega)) \cap L^2(0,T;\boldsymbol{W}_0^{\mathrm{curl},2}(\Omega)), \\
\boldsymbol{u}_h &\in_{\mathrm{b}} L^\infty(0,T;\boldsymbol{L}^2(\Omega)) \cap L^2(0,T;\boldsymbol{W}_0^{\mathrm{div},2}(\Omega)).
\end{aligned}
$$

Moreover, in view of Lemma 4.2, there exists sequences $\{\boldsymbol{\zeta}_h\}_{h>0}$, $\{\boldsymbol{z}_h\}_{h>0}$ such that

$$\begin{aligned}&\boldsymbol{u}_h(\cdot,t) = \operatorname{curl}_x \boldsymbol{\zeta}_h(\cdot,t) + \boldsymbol{z}_h(\cdot,t),\\ &\boldsymbol{\zeta}_h(\cdot,t) \in \boldsymbol{W}_h^{0,\perp}(\Omega), \quad \boldsymbol{z}_h(\cdot,t) \in \boldsymbol{V}_h^{0,\perp}(\Omega),\end{aligned} \tag{6.1}$$

for all $t \in (0,T)$ where

$$\boldsymbol{z}_h \in_{\mathrm{b}} L^\infty(0,T;\boldsymbol{L}^2(\Omega)) \cap L^2(0,T;\boldsymbol{W}_0^{\mathrm{div},2}(\Omega)).$$

$$\operatorname{curl}_x \boldsymbol{\zeta}_h \in_{\mathrm{b}} L^\infty(0,T;\boldsymbol{L}^2(\Omega)),$$

and

$$\partial_t^h(\operatorname{curl}_x \boldsymbol{\zeta}_h) \in_{\mathrm{b}} L^2(0,T;\boldsymbol{L}^2(\Omega)).$$

Consequently, we may assume that there exist functions $\varrho, \boldsymbol{w}, \boldsymbol{u}$ such that

$$\begin{aligned}&\varrho_h \overset{h\to 0}{\rightharpoonup} \varrho, \quad \text{in } L^\infty(0,T;L^\gamma(\Omega)) \cap L^{2\gamma}((0,T)\times\Omega),\\ &\boldsymbol{w}_h \overset{h\to 0}{\rightharpoonup} \boldsymbol{w}, \quad \text{in } L^\infty(0,T;\boldsymbol{L}^2(\Omega)) \cap L^2(0,T;\boldsymbol{W}_0^{\mathrm{curl},2}(\Omega)),\\ &\boldsymbol{u}_h \overset{h\to 0}{\rightharpoonup} \boldsymbol{u}, \quad \text{in } L^\infty(0,T;\boldsymbol{L}^2(\Omega)) \cap L^2(0,T;\boldsymbol{W}_0^{\mathrm{div},2}(\Omega)).\end{aligned} \tag{6.2}$$

Furthermore, using the standard Hodge decomposition $\boldsymbol{u} = \operatorname{curl}_x \boldsymbol{\zeta} + \nabla_x s$ and orthogonality,

$$\begin{aligned}\boldsymbol{z}_h &\overset{h\to 0}{\rightharpoonup} \nabla_x s, \quad \text{in } L^\infty(0,T;\boldsymbol{L}^2(\Omega)) \cap L^2(0,T;\boldsymbol{W}_0^{\mathrm{div},2}(\Omega)),\\ \operatorname{curl}_x \boldsymbol{\zeta}_h &\overset{h\to 0}{\rightharpoonup} \operatorname{curl}_x \boldsymbol{\zeta}, \quad \text{in } C(0,T;\boldsymbol{L}^2(\Omega)) \cap W^{1,2}(0,T;\boldsymbol{L}^2(\Omega)).\end{aligned} \tag{6.3}$$

In addition,

$$\varrho_h^\gamma \overset{h\to 0}{\rightharpoonup} \overline{\varrho^\gamma}, \quad \varrho_h^{\gamma+1} \overset{h\to 0}{\rightharpoonup} \overline{\varrho^{\gamma+1}}, \quad \varrho_h \log \varrho_h \overset{h\to 0}{\rightharpoonup} \overline{\varrho \log \varrho},$$

where each $\overset{h\to 0}{\rightharpoonup}$ signifies weak convergence in a suitable L^p space with $p > 1$.

Finally, ϱ_h, $\varrho_h \log \varrho_h$ converge respectively to ϱ, $\overline{\varrho \log \varrho}$ in $C([0,T];L^p_{\mathrm{weak}}(\Omega))$ for some $1 < p < \gamma$, cf. Lemma 2.2 and also [5,11]. In particular, ϱ, $\varrho \log \varrho$, and $\overline{\varrho \log \varrho}$ belong to $C([0,T];L^p_{\mathrm{weak}}(\Omega))$.

6.1. **Strong convergence of the velocity.**

Lemma 6.1. *Let $\{(\varrho_h, \boldsymbol{w}_h, \boldsymbol{u}_h)\}_{h>0}$ be a sequence of numerical solutions constructed according to* (3.5) *and Definition 3.1. Then*

$$\boldsymbol{u}_h \to \boldsymbol{u}, \quad \textit{in } L^2(0,T;\boldsymbol{L}^2(\Omega)).$$

Proof. By virtue of (6.1) we can consider each component of the decomposition $\boldsymbol{u}_h = \operatorname{curl}_x \boldsymbol{\zeta}_h + \boldsymbol{z}_h$. In Lemma 6.2 below we prove that

$$\operatorname{curl}_x \boldsymbol{\zeta}_h \to \operatorname{curl}_x \boldsymbol{\zeta}, \quad \text{in } L^2(0,T;\boldsymbol{L}^2(\Omega)),$$

and hence it only remains to prove that $\boldsymbol{z}_h \to \boldsymbol{z}$ in the sense of distributions.

From Lemma 4.7, we have the the weak time-continuity estimate:

$$\partial_t^h(\boldsymbol{z}_h) \in_{\mathrm{b}} L^2(0,T;W^{-1,1}(\Omega)).$$

Lemma 2.12 provides the spatial translation estimate:

$$\|\boldsymbol{z}_h(t,x) - \boldsymbol{z}_h(t,x-\xi)\|_{L^2(0,T;\boldsymbol{L}^2(\Omega))} \le C(|\xi|^2 + |\xi|^{\frac{4-N}{2}})^{\frac{1}{2}} \|\operatorname{div}_x \boldsymbol{z}_h\|_{L^2(0,T;L^2(\Omega))},$$

where the constant $C > 0$ is independent of h and ξ. Lemma 2.3 can then be applied (recalling (6.3)) to obtain the desired result;

$$\boldsymbol{z}_h \to \nabla_x s, \quad \text{in } L^2(0,T;\boldsymbol{L}^2(\Omega)).$$

□

Lemma 6.2. *Given* (6.2) *and* (6.3),

$$\boldsymbol{w}_h \overset{h\to 0}{\to} \boldsymbol{w}, \quad \operatorname{curl}_x \boldsymbol{\zeta}_h \overset{h\to 0}{\to} \operatorname{curl}_x \boldsymbol{\zeta} \quad \text{in } L^2(0,T;\boldsymbol{L}^2(\Omega)).$$

Proof. Fix any $t \in (0,T)$ and m such that $t \in (t^{m-1}, t^m]$, where $t^m = m\Delta t$. Subtract the first equation of (3.3) with $\boldsymbol{v}_h = \operatorname{curl}_x \boldsymbol{\xi}_h^m$ from μ times the second equation of (3.3). Multiplying the result with Δt and summing over all $k = 1, \dots, m$ yields

$$\begin{aligned}\int_0^t \int_\Omega \mu \operatorname{curl}_x \boldsymbol{\eta}_h \operatorname{curl}_x \boldsymbol{\zeta}_h - \mu \operatorname{curl}_x \boldsymbol{w}_h \operatorname{curl}_x \boldsymbol{\xi}_h \, dxdt& \\ = \int_0^t \int_\Omega \mu \boldsymbol{w}_h \boldsymbol{\eta}_h + \partial_t^h (\operatorname{curl}_x \boldsymbol{\zeta}_h) \operatorname{curl}_x \boldsymbol{\xi}_h \, dxdt,&\end{aligned} \tag{6.4}$$

for all $\boldsymbol{\eta}_h, \boldsymbol{\xi}_h$ that are piecewise constant in time with values in $\boldsymbol{W}_h(\Omega)$. Fixing $\boldsymbol{\eta}, \boldsymbol{\xi} \in C_c^\infty((0,T)\times\Omega)$, we use in (6.4) the test functions

$$\boldsymbol{\eta}_h(t,\cdot) = \boldsymbol{\eta}_h^m(\cdot) := \frac{1}{\Delta t}\int_{t^{m-1}}^{t^m} \Pi_h^W \boldsymbol{\eta}(\cdot, s)\, ds, \quad t \in (t_{m-1}, t_m),\ m = 1, \dots, M.$$
$$\boldsymbol{\xi}_h(t,\cdot) = \boldsymbol{\xi}_h^m(\cdot) := \frac{1}{\Delta t}\int_{t^{m-1}}^{t^m} \Pi_h^W \boldsymbol{\xi}(\cdot, s)\, ds, \quad t \in (t_{m-1}, t_m),\ m = 1, \dots, M.$$

Due to Lemma 2.8, $\operatorname{curl}_x \boldsymbol{\xi}_h \to \operatorname{curl}_x \boldsymbol{\xi}$ and $\operatorname{curl}_x \boldsymbol{\eta}_h \to \operatorname{curl}_x \boldsymbol{\eta}$ in $L^2(0,T;\boldsymbol{L}^2(\Omega))$. As a consequence, keeping in mind (6.2) and (6.3), we let $h \to 0$ in (6.4) to obtain

$$\begin{aligned}\int_0^t \int_\Omega \mu \operatorname{curl}_x \boldsymbol{\eta} \operatorname{curl}_x \boldsymbol{\zeta} - \mu \operatorname{curl}_x \boldsymbol{w} \operatorname{curl}_x \boldsymbol{\xi} \, dxdt& \\ = \int_0^t \int_\Omega \mu \boldsymbol{w} \boldsymbol{\eta} + \partial_t (\operatorname{curl}_x \boldsymbol{\zeta}) \operatorname{curl}_x \boldsymbol{\xi} \, dxdt, \quad \forall \boldsymbol{\eta}, \boldsymbol{\xi} \in \boldsymbol{C}_c^\infty((0,T)\times\Omega).&\end{aligned} \tag{6.5}$$

Since $\boldsymbol{C}_c^\infty((0,T)\times\Omega)$ is dense in $L^2(0,T;\boldsymbol{W}_0^{\operatorname{curl}_x,2}(\Omega))$ ([10]), we see that (6.5) holds for all $\boldsymbol{\eta}, \boldsymbol{\xi} \in L^2(0,T;\boldsymbol{W}_0^{\operatorname{curl}_x,2}(\Omega))$. Hence, taking $\boldsymbol{\eta} = \boldsymbol{w}$, $\boldsymbol{\xi} = \boldsymbol{\zeta}$ in (6.5),

$$\frac{1}{2}\int_\Omega |\operatorname{curl}_x \boldsymbol{\zeta}^0|^2 \, dx = \int_0^t \int_\Omega \mu |\boldsymbol{w}|^2 \, dxdt + \frac{1}{2}\left(\int_\Omega |\operatorname{curl}_x \boldsymbol{\zeta}|^2 \, dx\right)(t), \tag{6.6}$$

where $\operatorname{curl}_x \boldsymbol{\zeta}^0$ is given by the Hodge decomposition $\boldsymbol{u}^0 = \operatorname{curl}_x \boldsymbol{\zeta}^0 + \nabla_x s^0$.

Next, setting $\boldsymbol{\eta}_h = \boldsymbol{w}_h$ and $\boldsymbol{\xi}_h = \boldsymbol{\zeta}_h$ in (6.4), we observe that

$$\begin{aligned}\frac{1}{2}\int_\Omega |\operatorname{curl}_x \boldsymbol{\zeta}_h^0|^2 \, dx &= \int_0^t \int_\Omega \mu |\boldsymbol{w}_h|^2 \, dxdt \\ &\quad + \frac{1}{2}\left(\int_\Omega |\operatorname{curl}_x \boldsymbol{\zeta}_h|^2 \, dx\right)(t) + \frac{1}{2}\sum_{m=1}^M \int_\Omega [\![\operatorname{curl}_x \boldsymbol{\zeta}_h^m]\!]^2 \, dx.\end{aligned} \tag{6.7}$$

Subtracting (6.6) from (6.7)

$$\begin{aligned}&\lim_{h\to 0}\left[\int_0^t \int_\Omega \mu(|\boldsymbol{w}_h|^2 - |\boldsymbol{w}|^2)\, dxdt + \frac{1}{2}\left(\int_\Omega |\operatorname{curl}_x \boldsymbol{\zeta}_h|^2 - |\operatorname{curl}_x \boldsymbol{\zeta}|^2 \, dx\right)(t)\right] \\ &\quad \le \lim_{h\to 0}\left[\frac{1}{2}\int_\Omega |\operatorname{curl}_x \boldsymbol{\zeta}_h^0|^2 - |\operatorname{curl}_x \boldsymbol{\zeta}^0|^2 \, dx\right] = 0,\end{aligned}$$

where the last equality is an application of Lemma 4.2. Consequently, for any $t \in (0,T)$,

$$
\begin{aligned}
&\lim_{h\to 0}\left[\mu\|\boldsymbol{w}_h - \boldsymbol{w}\|^2_{L^2(0,t;\boldsymbol{L}^2(\Omega))} + \frac{1}{2}\|\operatorname{curl}_x \boldsymbol{\zeta}_h(t,\cdot) - \operatorname{curl}_x \boldsymbol{\zeta}(t,\cdot)\|^2_{\boldsymbol{L}^2(\Omega)}\right] \\
&\qquad = \lim_{h\to 0}\left[\int_0^t \int_\Omega \mu\left(|\boldsymbol{w}|^2 - \boldsymbol{w}_h \boldsymbol{w}\right)\, dxdt\right] \\
&\qquad\quad + \lim_{h\to 0}\left(\int_\Omega |\operatorname{curl}_x \boldsymbol{\zeta}|^2 - (\operatorname{curl}_x \boldsymbol{\zeta}_h)(\operatorname{curl}_x \boldsymbol{\zeta})\, dx\right)(t) = 0,
\end{aligned}
$$

where the last term converges to zero due to the weak convergences (6.3). □

In the subsequent analysis, we will need the following technical lemma. For notational convenience, we define the linear time interpolant $\Pi_{\mathcal{L}}$:

$$
(\Pi_{\mathcal{L}} f)(t) = f^{m-1} + \frac{t - t^{m-1}}{\Delta t}(f^m - f^{m-1}), \quad t \in (t^{m-1}, t^m). \tag{6.8}
$$

Lemma 6.3. *Given* (6.2) *and* (6.3),

$$
\begin{aligned}
\Delta^{-1}[\operatorname{div}_x \boldsymbol{u}_h] &\overset{h\to 0}{\rightarrow} \Delta^{-1}[\operatorname{div}_x \boldsymbol{u}], \quad \textit{in } L^2(0,T;\boldsymbol{W}^{1,2}(\Omega)), \\
\Delta^{-1}[\operatorname{div}_x \Pi_{\mathcal{L}} \boldsymbol{u}_h] &\overset{h\to 0}{\rightarrow} \Delta^{-1}[\operatorname{div}_x \boldsymbol{u}], \quad \textit{in } L^2(0,T;\boldsymbol{W}^{1,2}(\Omega)).
\end{aligned}
$$

Proof. Recall the continuous Hodge decomposition

$$
\boldsymbol{u} = \operatorname{curl}_x \boldsymbol{\zeta} + \nabla_x s.
$$

As in the proof of Lemma 2.11, we have that $\boldsymbol{z}_h = \Pi_h^V\left(\nabla_x \Delta^{-1}\left[\operatorname{div}_x \boldsymbol{u}_h\right]\right)$. Hence,

$$
\begin{aligned}
&\|\Pi_h^V\left(\nabla_x \Delta^{-1}\left[\operatorname{div}_x \boldsymbol{u}_h\right]\right) - \nabla_x \Delta^{-1}\left[\operatorname{div}_x \boldsymbol{u}\right]\|_{L^2(0,T;\boldsymbol{L}^2(\Omega))} \\
&\qquad = \|\boldsymbol{z}_h - \nabla_x s\|_{L^2(0,T;\boldsymbol{L}^2(\Omega))}.
\end{aligned}
$$

From Lemma 6.1, we have that the right-hand side converges to zero. Hence, we conclude that $\Pi_h^V\left(\nabla_x \Delta^{-1}\left[\operatorname{div}_x \boldsymbol{u}_h\right]\right) \to \nabla_x \Delta^{-1}\left[\operatorname{div}_x \boldsymbol{u}\right]$ in $L^2(0,T;\boldsymbol{L}^2(\Omega))$. Next, we write

$$
\begin{aligned}
&\|\nabla_x \Delta^{-1}\left[\operatorname{div}_x \boldsymbol{u}_h\right] - \nabla_x \Delta^{-1}\left[\operatorname{div}_x \boldsymbol{u}\right]\|_{L^2(0,T;\boldsymbol{L}^2(\Omega))} \\
&\qquad \le \|\nabla_x \Delta^{-1}\left[\operatorname{div}_x \boldsymbol{u}_h\right] - \Pi_h^V\left(\nabla_x \Delta^{-1}\left[\operatorname{div}_x \boldsymbol{u}_h\right]\right)\|_{L^2(0,T;\boldsymbol{L}^2(\Omega))} \\
&\qquad\quad + \|\Pi_h^V\left(\nabla_x \Delta^{-1}\left[\operatorname{div}_x \boldsymbol{u}_h\right]\right) - \nabla_x \Delta^{-1}\left[\operatorname{div}_x \boldsymbol{u}\right]\|_{L^2(0,T;\boldsymbol{L}^2(\Omega))} \\
&\qquad \le Ch\|\operatorname{div}_x \boldsymbol{u}_h\|_{L^2(0,T;L^2(\Omega))} \\
&\qquad\quad + \|\Pi_h^V\left(\nabla_x \Delta^{-1}\left[\operatorname{div}_x \boldsymbol{u}_h\right]\right) - \nabla_x \Delta^{-1}\left[\operatorname{div}_x \boldsymbol{u}\right]\|_{L^2(0,T;\boldsymbol{L}^2(\Omega))}.
\end{aligned}
$$

By sending $h \to 0$, we discover

$$
\nabla_x \Delta^{-1}\left[\operatorname{div}_x \boldsymbol{u}_h\right] \to \nabla_x \Delta^{-1}\left[\operatorname{div}_x \boldsymbol{u}\right] \quad \text{in } L^2(0,T;\boldsymbol{L}^2(\Omega)),
$$

which proves the first part of the lemma.

A direct calculation gives

$$
|\Pi_{\mathcal{L}} \boldsymbol{u}_h(t,\cdot) - \boldsymbol{u}_h(t,\cdot)|^2 \le \left|[\![\boldsymbol{u}_h^{k-1}(\cdot)]\!]^2\right|, \quad t \in (t^{k-1}, t^k).
$$

Hence, integrating over $(0,T) \times \Omega$ yields

$$
\|\Pi_{\mathcal{L}} \boldsymbol{u}_h - \boldsymbol{u}_h\|^2_{L^2(0,T;\boldsymbol{L}^2(\Omega))} \le \Delta t \sum_{k=1}^{M-1} \|[\![\boldsymbol{u}_h^k]\!]\|^2_{\boldsymbol{L}^2(\Omega)} \le C\Delta t, \tag{6.9}
$$

where the last inequality follows from Lemma 4.3.

Using elliptic theory and (6.9), we conclude

$$
\left\|\Delta^{-1}\left[\Pi_{\mathcal{L}} \boldsymbol{u}_h - \boldsymbol{u}_h\right]\right\|_{L^2(0,T;\boldsymbol{W}^{1,2}(\Omega))} \le C(\Delta t)^{\frac{1}{2}}.
$$

Hence, the limits are equal and consequently the second part of the lemma now follows from the first.

□

6.2. **Density scheme.** Having established strong convergence of the velocity we now prove that the numerical solutions converge to a weak solution of the continuity equation (1.1).

Lemma 6.4 (Convergence of the continuity approximation)**.** *The limit pair* $(\varrho, \boldsymbol{u})$ *constructed in* (6.2) *is a weak solution of the continuity equation* (1.1) *in the sense of Definition 2.4.*

Proof. The proof is essentially identical to the proof of Lemma 6.4 in [8] and is only included for the sake of completeness.

Fix a test function $\phi \in C_c^\infty([0,T) \times \overline{\Omega})$, and introduce the piecewise constant approximations $\phi_h := \Pi_h^Q \phi$, $\phi_h^m := \Pi_h^Q \phi^m$, and $\phi^m := \frac{1}{\Delta t}\int_{t^{m-1}}^{t^m} \phi(t,\cdot)\, dt$.

Let us employ ϕ_h^m as test function in the continuity scheme (3.2) and sum over $m = 1, \dots, M$. The resulting equation reads

$$\begin{aligned}
&\sum_{m=1}^M \Delta t \int_\Omega \partial_t^h \left(\varrho_h^m\right) \phi_h^m \; dxdt \\
&\qquad = \sum_{\Gamma \in \Gamma_h^I} \sum_{m=1}^M \Delta t \int_\Gamma \left(\varrho_-^m (\boldsymbol{u}_h^m \cdot \nu)^+ + \varrho_+^m (\boldsymbol{u}_h^m \cdot \nu)^-\right) [\![\phi_h^m]\!]_\Gamma \; dS(x).
\end{aligned}$$

As in the proof of Lemma 4.6 we can rewrite this as

$$\begin{aligned}
&\sum_{m=1}^M \Delta t \int_\Omega \partial_t^h \left(\varrho_h^m\right) \phi_h^m \; dxdt \\
&= \sum_{m=1}^M \Delta t \int_\Omega \varrho_h^m \boldsymbol{u}_h^m \nabla_x \phi^m \; dx \\
&\qquad + \sum_{E \in E_h} \sum_{m=1}^M \Delta t \int_{\partial E \setminus \partial\Omega} [\![\varrho_h^m]\!]_{\partial E} \, (\boldsymbol{u}_h^m \cdot \nu)^- (\phi_h^m - \phi^m) \; dS(x) \\
&= \int_0^T \int_\Omega \varrho_h \boldsymbol{u}_h \nabla_x \phi \; dxdt \\
&\qquad + \sum_{E \in E_h} \int_0^T \int_{\partial E \setminus \partial\Omega} [\![\varrho_h]\!]_{\partial E} \, (\boldsymbol{u}_h \cdot \nu)^- (\phi_h - \phi) \; dS(x)dt.
\end{aligned} \tag{6.10}$$

Lemma 4.5 tells us that

$$\left| \sum_{E \in E_h} \int_0^T \int_{\partial E \setminus \partial\Omega} [\![\varrho_h]\!]_{\partial E} \, (\boldsymbol{u}_h \cdot \nu)^- (\phi_h - \phi) \; dS(x)dt \right| \le C\, h^{\frac{1}{4}} \left\| \nabla_x \phi \right\|_{L^2(0,T;\boldsymbol{L}^{2^*}(\Omega))} .$$

In view of Lemma 6.1,

$$\lim_{h \to 0} \int_0^T \int_\Omega \varrho_h \boldsymbol{u}_h \nabla_x \phi \; dxdt = \int_0^T \int_\Omega \varrho \boldsymbol{u} \nabla_x \phi \; dxdt.$$

Summation by parts gives

$$\begin{aligned}
&\sum_{m=1}^{M} \Delta t \int_\Omega \partial_t^h \left(\varrho_h^m\right) \phi_h^m \; dxdt \\
&= - \int_{\Delta t}^T \int_\Omega \varrho_h(t-\Delta t, x) \frac{\partial}{\partial t} \left(\Pi_{\mathcal{L}} \phi_h\right) \; dxdt - \int_\Omega \varrho_h^0 \phi_h^1 \; dx \\
&\overset{h\to 0}{\to} - \int_0^T \int_\Omega \varrho \phi_t \; dxdt - \int_\Omega \varrho_0 \phi(0,x) \; dx.
\end{aligned}$$

where (6.2), together with the strong convergence $\varrho_h^0 \overset{h\to 0}{\to} \varrho_0$, was used to pass to the limit. Summarizing, letting $h \to 0$ in (6.10) delivers the desired result (2.1). □

6.3. **Strong convergence of the density approximation.** To obtain strong convergence of the density approximations ϱ_h, the main ingredient is a weak continuity property of the quantity $P_{\text{eff}}(\varrho_h, \boldsymbol{u}_h)$. To derive this property we use (5.1) and a corresponding equation for the weak limit $\overline{P_{\text{eff}}(\varrho, \boldsymbol{u})}$. We start by deriving the latter.

Let $\psi \in C_c^\infty(0,T)$ be arbitrary, set $\phi = \psi(\varrho - \langle \varrho^0 \rangle_\Omega)$ in (5.1), take the limit $h \to 0$, and apply Lemmas 5.1 and 5.2 to obtain

$$\begin{aligned}
&\lim_{h\to 0} \int_0^T \int_\Omega P_{\text{eff}}(\varrho_h, \boldsymbol{u}_h) \psi(\varrho - \langle \varrho^0 \rangle_\Omega) \; dxdt \\
&\qquad = \lim_{h\to 0} \int_0^T \int_\Omega \partial_t^h \left(\boldsymbol{u}_h\right) \psi \nabla_x \Delta^{-1} \left[\varrho - \langle \varrho^0 \rangle_\Omega\right] \; dxdt.
\end{aligned} \tag{6.11}$$

Since the operator Δ^{-1} is self-adjoint, we can integrate by parts to obtain

$$\begin{aligned}
&\int_0^T \int_\Omega \partial_t^h \left(\boldsymbol{u}_h\right) \psi \nabla_x \Delta^{-1} \left[\varrho - \langle \varrho^0 \rangle_\Omega\right] \; dxdt \\
&= - \int_0^T \int_\Omega \partial_t^h \left(\Delta^{-1}[\operatorname{div}_x \boldsymbol{u}_h]\right) \psi(\varrho - \langle \varrho^0 \rangle_\Omega) \; dxdt \\
&= - \int_0^T \int_\Omega \frac{\partial}{\partial t} \left(\Delta^{-1}[\operatorname{div}_x \Pi_{\mathcal{L}} \boldsymbol{u}_h]\right) \psi(\varrho - \langle \varrho^0 \rangle_\Omega) \; dxdt,
\end{aligned}$$

where the last equality follows by definition of $\Pi_{\mathcal{L}}$ (6.8).

Next, we move ψ inside the time integration and use that $\langle \varrho^0 \rangle_\Omega$ is independent of time. This gives

$$\begin{aligned}
&\int_0^T \int_\Omega \partial_t^h \left(\boldsymbol{u}_h\right) \psi \nabla_x \Delta^{-1} \left[\varrho - \langle \varrho^0 \rangle_\Omega\right] \; dxdt \\
&= - \int_0^T \int_\Omega \frac{\partial}{\partial t} \left(\psi \Delta^{-1}[\operatorname{div}_x \Pi_{\mathcal{L}} \boldsymbol{u}_h]\right) \varrho \; dxdt \\
&\quad + \int_0^T \int_\Omega \psi'(t) \left(\Delta^{-1}[\operatorname{div}_x \Pi_{\mathcal{L}} \boldsymbol{u}_h]\right) (\varrho - \langle \varrho^0 \rangle_\Omega) \; dxdt.
\end{aligned} \tag{6.12}$$

At this point, we recall that $(\varrho, \boldsymbol{u})$ is a weak solution to the continuity equation (Lemma 6.4). Inserting $\phi = \psi \Delta^{-1}[\operatorname{div}_x \Pi_{\mathcal{L}} \boldsymbol{u}_h]$ as test function in the weak form of continuity equation gives

$$\int_0^T \int_\Omega \frac{\partial}{\partial t} \left(\psi \Delta^{-1}[\operatorname{div}_x \Pi_{\mathcal{L}} \boldsymbol{u}_h]\right) \varrho \; dxdt = - \int_0^T \int_\Omega \psi \varrho \boldsymbol{u} \nabla_x \Delta^{-1} \left[\Pi_{\mathcal{L}} \boldsymbol{u}_h\right] \; dxdt$$

Setting this into (6.12) gives

$$\begin{aligned}
&\int_0^T \int_\Omega \partial_t^h(\boldsymbol{u}_h)\,\psi\nabla_x\Delta^{-1}\left[\varrho-\langle\varrho^0\rangle_\Omega\right]\,dxdt\\
&\quad=\int_0^T \int_\Omega \psi\varrho\boldsymbol{u}\left(\nabla_x\Delta^{-1}\left[\operatorname{div}_x\Pi_{\mathcal{L}}\boldsymbol{u}_h\right]\right)+\psi'(t)\Delta^{-1}[\operatorname{div}_x\Pi_{\mathcal{L}}\boldsymbol{u}_h](\varrho-\langle\varrho^0\rangle_\Omega)\,dxdt.
\end{aligned}$$

Sending $h\to 0$ and applying Lemma 6.3

$$\begin{aligned}
&\lim_{h\to 0}\int_0^t \int_\Omega \partial_t^h(\boldsymbol{u}_h)\,\psi\nabla_x\Delta^{-1}\left[\varrho-\langle\varrho^0\rangle_\Omega\right]\,dxdt\\
&\quad=\int_0^t \int_\Omega \psi\varrho\boldsymbol{u}\left(\nabla_x\Delta^{-1}\left[\operatorname{div}_x\boldsymbol{u}\right]\right)+\psi'(t)\Delta^{-1}[\operatorname{div}_x\boldsymbol{u}](\varrho-\langle\varrho^0\rangle_\Omega)\,dxdt.
\end{aligned}$$

Finally, we insert this expression in (6.11) and obtain

$$\begin{aligned}
&\int_0^t \int_\Omega \overline{P_{\text{eff}}(\varrho,\boldsymbol{u})\varrho}\psi\,dxdt\\
&\qquad=\int_0^t \int_\Omega \psi\varrho\boldsymbol{u}\left(\nabla_x\Delta^{-1}\left[\operatorname{div}_x\boldsymbol{u}\right]\right)+\psi'(t)\Delta^{-1}[\operatorname{div}_x\boldsymbol{u}](\varrho-\langle\varrho^0\rangle_\Omega)\,dxdt.
\end{aligned}\tag{6.13}$$

Lemma 6.5 (Effective viscous flux)**.** *Given the convergences in* (6.2),

$$\lim_{h\to 0}\int_0^T \int_\Omega \psi P_{\mathit{eff}}(\varrho_h,\boldsymbol{u}_h)\varrho_h\,dxdt=\int_0^T \int_\Omega \psi\overline{P_{\mathit{eff}}(\varrho,\boldsymbol{u})\varrho}\,dxdt,$$

for all $\psi\in C_c^1(0,T)$.

Proof. Let $\psi\in C_c^1(0,T)$ be arbitrary and set $\phi=\psi(\varrho_h-\langle\varrho^0\rangle_\Omega)$ in (5.1) to obtain

$$\lim_{h\to 0}\int_0^T \int_\Omega \psi P_f(\varrho_h,\boldsymbol{u}_h)\varrho_h\,dxdt=\lim_{h\to 0}\int_0^T \int_\Omega \partial_t^h(\boldsymbol{u}_h)\,\psi\nabla_x\Delta^{-1}\left[\varrho_h-\langle\varrho^0\rangle_\Omega\right]dxdt,\tag{6.14}$$

where we have also used Lemmas 5.2 and 5.1. As in (5.4) and (5.5) we can use summation by parts and the continuity scheme (3.2) to deduce the following equality for the the right–hand side:

$$\begin{aligned}
&\int_0^T \int_\Omega \partial_t^h(\boldsymbol{u}_h)\,\psi\nabla_x\Delta^{-1}\left[\varrho_h-\langle\varrho_h^0\rangle_\Omega\right]dxdt\\
&=-\sum_{m=1}^M\Delta t\int_\Omega \boldsymbol{u}_h^{m-1}\partial_t^h(\psi^m)\,\nabla_x\Delta^{-1}\left[\varrho_h^m-\langle\varrho_h^0\rangle_\Omega\right]+\boldsymbol{u}_h^{m-1}\psi^{m-1}\nabla_x\Delta^{-1}\left[\partial_t^h(\varrho_h^m)\right]\,dx\\
&\qquad\qquad-\frac{1}{\Delta t}\int_0^{\Delta t}\int_\Omega \psi\boldsymbol{u}_h^0\nabla_x\Delta^{-1}\left[(\varrho_h-\langle\varrho_h^0\rangle_\Omega)\right]dxdt\\
&=-\sum_{m=1}^M\Delta t\int_\Omega \boldsymbol{u}_h^{m-1}\partial_t^h(\psi^m)\,\nabla_x\Delta^{-1}\left[\varrho_h^m-\langle\varrho_h^0\rangle_\Omega\right]\,dx\\
&\qquad+\sum_{m=1}^M\Delta t\int_\Omega \psi^{m-1}\boldsymbol{u}_h^m\varrho_h^m\nabla_x\Delta^{-1}\left[\operatorname{div}_x\boldsymbol{u}_h^{m-1}\right]\,dx\\
&\qquad\quad+\sum_{m=1}^M\Delta t\sum_{E\in E_h}\int_{\partial E}\psi^m\,[\![\varrho_h^m]\!]_{\partial E}\,(\boldsymbol{u}_h^m\cdot\nu)^-(\Pi_h^V-\mathbb{I})\Delta^{-1}(\operatorname{div}_x\boldsymbol{u}_h^{m-1})\,dS(x)\\
&\qquad-\frac{1}{\Delta t}\int_0^{\Delta t}\int_\Omega \psi\boldsymbol{u}_h^0\nabla_x\Delta^{-1}\left[(\varrho_h-\langle\varrho_h^0\rangle_\Omega)\right]\,dxdt.
\end{aligned}$$

Taking the limit $h \to 0$ and applying Lemma 4.5 gives

$$
\begin{aligned}
&\lim_{h\to 0}\int_0^T \int_\Omega \partial_t^h (\boldsymbol{u}_h)\, \psi \nabla_x \Delta^{-1}\left[\varrho_h - \langle \varrho_h^0\rangle_\Omega\right] dxdt \\
&= \lim_{h\to 0}\sum_{m=1}^{M} \Delta t \int_\Omega \psi^{m-1}\boldsymbol{u}_h^m \varrho_h^m \nabla_x \Delta^{-1}\left[\operatorname{div}_x \boldsymbol{u}_h^{m-1}\right]\, dx \\
&\qquad - \int_0^T \int_\Omega \boldsymbol{u}\psi'(t)\nabla_x\Delta^{-1}\left[\varrho - \langle\varrho^0\rangle_\Omega\right]\, dxdt.
\end{aligned}
\tag{6.15}
$$

We will now pass to the limit in the first term on the right-hand side.

From Lemmas 5.2, 6.1, and 6.3, we have that

$$
\varrho_h \rightharpoonup \varrho \quad \text{in } L^\infty(0,T;L^\gamma(\Omega)) \cap L^{\gamma+1}(0,T;L^{\gamma+1}(\Omega)),
$$

$$
\boldsymbol{u}_h \to \boldsymbol{u} \quad \text{in } L^2(0,T;\boldsymbol{L}^2(\Omega)),
\tag{6.16}
$$

$$
\nabla_x\Delta^{-1}\left[\operatorname{div}_x \boldsymbol{u}_h\right] \to \nabla_x\Delta^{-1}\left[\operatorname{div}_x \boldsymbol{u}\right] \quad \text{in } L^2(0,T;\boldsymbol{L}^2(\Omega)).
\tag{6.17}
$$

This is insufficient to pass to the limit in the desired term. However, since $\boldsymbol{u}_h \in_b L^\infty(0,T;\boldsymbol{L}^2(\Omega)) \cap L^2(0,T;\boldsymbol{L}^{2^*}(\Omega))$, we can by similar arguments as in the proof of Lemma 6.3 deduce that $\nabla_x\Delta^{-1}\left[\operatorname{div}_x \boldsymbol{u}_h\right] \in_b L^\infty(0,T;\boldsymbol{L}^2(\Omega)) \cap L^2(0,T;\boldsymbol{L}^{2^*}(\Omega))$. Let β be given by

$$
\frac{2}{\beta} = \frac{1}{2} + \frac{1}{2^*}.
$$

Then, $\beta \geq \frac{2N}{N-1} - \epsilon$, for any $\epsilon > 0$. Since $2 \leq \beta$, the standard interpolation inequality can be applied and yields

$$
\begin{aligned}
\int_0^T \|f\|_{L^\beta(\Omega)}^4\, dt &\leq \int_0^T \|f\|_{L^2(\Omega)}^2 \|f\|_{L^{2^*}(\Omega)}^2\, dt \\
&\leq \|f\|_{L^\infty(0,T;L^2(\Omega))}^2 \|f\|_{L^2(0,T;L^{2^*}(\Omega))}^2.
\end{aligned}
$$

From this inequality, we conclude

$$
\nabla_x\Delta^{-1}\left[\operatorname{div}_x \boldsymbol{u}_h\right], \boldsymbol{u}_h \in_{\mathrm{b}} L^4(0,T;L^\beta(\Omega)),
\tag{6.18}
$$

For notational convenience, we introduce the function g_h

$$
g_h(t,\cdot) = \boldsymbol{u}_h(t,\cdot)\cdot\nabla_x\Delta^{-1}\left[\operatorname{div}_x \boldsymbol{u}_h(t-\Delta t,\cdot)\right].
$$

Note that g_h is precisely the scalar product in (6.15). From the Hölder inequality and (6.18), we have in particular that

$$
g_h \in_{\mathrm{b}} L^2(0,T;L^2(\Omega))
$$

This, together with (6.16) and (6.17), tells us that

$$
g_h \to g := \boldsymbol{u}\nabla_x\Delta^{-1}\left[\operatorname{div}_x \boldsymbol{u}\right], \quad \text{in } L^p(0,T;L^p(\Omega)), \text{ for any } p<2, \text{ as } h\to 0.
$$

Hence, $g_h\varrho_h \rightharpoonup g\varrho$ in the sense of distributions on $(0,T)\times\Omega$. This is sufficient to pass to the limit in the first term on the right-hand side of (6.15). By sending $h \to 0$ in (6.15), we obtain the identity

$$
\begin{aligned}
&\lim_{h\to 0}\int_0^T \int_\Omega \partial_t^h (\boldsymbol{u}_h)\, \psi \nabla_x \Delta^{-1}\left[\varrho_h - \langle \varrho_h^0\rangle_\Omega\right] dxdt \\
&\quad = \int_0^T \int_\Omega \psi\varrho\boldsymbol{u}\nabla_x\Delta^{-1}\left[\operatorname{div}_x \boldsymbol{u}\right] - \boldsymbol{u}\psi'(t)\nabla_x\Delta^{-1}\left[\varrho - \langle\varrho^0\rangle_\Omega\right]\, dxdt.
\end{aligned}
\tag{6.19}
$$

Then, (6.19) in (6.14) yields

$$
\begin{aligned}
&\lim_{h\to 0}\int_0^T\int_\Omega P_{\text{eff}}(\varrho_h,\boldsymbol{u}_h)\psi\varrho_h\,dxdt\\
&\quad=\int_0^T\int_\Omega \psi\varrho\boldsymbol{u}\nabla_x\Delta^{-1}\left[\operatorname{div}_x\boldsymbol{u}\right]-\boldsymbol{u}\psi'(t)\nabla_x\Delta^{-1}\left[\varrho-\langle\varrho^0\rangle_\Omega\right]\,dxdt\\
&\quad=\int_0^T\int_\Omega \psi\varrho\boldsymbol{u}\left(\nabla_x\Delta^{-1}\left[\operatorname{div}_x\boldsymbol{u}\right]\right)+\psi'(t)(\varrho-\langle\varrho^0\rangle_\Omega)\left(\Delta^{-1}\operatorname{div}_x\boldsymbol{u}\right)\,dxdt\\
&=\int_0^T\int_\Omega \overline{P_{\text{eff}}(\varrho,\boldsymbol{u})}\psi\varrho\,dxdt,
\end{aligned}
$$

where the last equality is (6.13). This concludes the proof. □

We are now in a position to prove strong convergence of the density approximations.

Lemma 6.6 (Strong convergence of the density). *Suppose that* (6.2) *holds. Then, passing to a subsequence if necessary,*

$$\varrho_h\to\varrho\quad a.e.\ in\ (0,T)\times\Omega.$$

Proof. The proof is identical to that of Lemma 6.6 in [8] and is included for the sake of completeness.

In view of Lemma 6.4, the limit $(\varrho,\boldsymbol{u})$ is a weak solution of the continuity equation and hence, by Lemma 2.6, also a renormalized solution. In particular,

$$(\varrho\log\varrho)_t+\operatorname{div}_x\left((\varrho\log\varrho)\,\boldsymbol{u}\right)=\varrho\operatorname{div}_x\boldsymbol{u}\quad\text{in the weak sense on } [0,T)\times\overline{\Omega}.$$

Since $t\mapsto\varrho\log\varrho$ is continuous with values in some Lebesgue space equipped with the weak topology, we can use this equation to obtain for any $t>0$

$$\int_\Omega(\varrho\log\varrho)(t)\,dx-\int_\Omega\varrho_0\log\varrho_0\,dx=-\int_0^t\int_\Omega\varrho\operatorname{div}_x\boldsymbol{u}\,dxds \tag{6.20}$$

Next, we specify $\phi_h\equiv 1$ as test function in the renormalized scheme (4.1), multiply by Δt, and sum the result over m. Making use of the convexity of $z\log z$, we infer for any $m=1,\dots,M$

$$\int_\Omega\varrho_h^m\log\varrho_h^m\,dx-\int_\Omega\varrho_h^0\log\varrho_h^0\,dx\le-\sum_{k=1}^m\Delta t\int_\Omega\varrho_h^m\operatorname{div}_x\boldsymbol{u}_h^m\,dxdt. \tag{6.21}$$

In view of the convergences stated at the beginning of this section and strong convergence of the initial data, we can send $h\to 0$ in (6.21) to obtain

$$\int_\Omega\left(\overline{\varrho\log\varrho}\right)(t)\,dx-\int_\Omega\varrho_0\log\varrho_0\,dx\le-\int_0^t\int_\Omega\overline{\varrho\operatorname{div}_x\boldsymbol{u}}\,dxds. \tag{6.22}$$

Subtracting (6.20) from (6.22) gives

$$\int_\Omega\left(\overline{\varrho\log\varrho}-\varrho\log\varrho\right)(t)\,dx\le-\int_0^t\int_\Omega\overline{\varrho\operatorname{div}_x\boldsymbol{u}}-\varrho\operatorname{div}_x\boldsymbol{u}\,dxds,$$

for any $t\in(0,T)$. Lemma 6.5 tells us that

$$\int_0^t\int_\Omega\overline{\varrho\operatorname{div}_x\boldsymbol{u}}-\varrho\operatorname{div}_x\boldsymbol{u}\,dxds=\frac{a}{\mu+\lambda}\int_0^t\int_\Omega\overline{\varrho^{\gamma+1}}-\overline{\varrho^\gamma}\varrho\,dxds\ge 0,$$

where the last inequality follows as in [5, 11], so the following relation holds:

$$\overline{\varrho\log\varrho}=\varrho\log\varrho\quad\text{a.e. in }(0,T)\times\Omega.$$

Now an application of Lemma 2.1 brings the proof to an end. □

6.4. **Velocity scheme.**

Lemma 6.7 (Convergence of the momentum approximation)**.** *The limit triple* $(\boldsymbol{w}, \boldsymbol{u}, \varrho)$ *constructed in* (6.2) *is a weak solution of the velocity equation* (1.2) *in the sense of* (2.3).

Proof. Fix $(\boldsymbol{v}, \boldsymbol{\eta}) \in \boldsymbol{C}_c^\infty((0,T)\times\Omega)$, and introduce the projections $\boldsymbol{v}_h = \Pi_h^V \boldsymbol{v}$, $\boldsymbol{\eta}_h = \Pi_h^W \boldsymbol{\eta}$ and $\boldsymbol{v}_h^m = \frac{1}{\Delta t}\int_{t^{m-1}}^{t^m} \boldsymbol{v}_h\, dt$, $\boldsymbol{\eta}_h^m = \frac{1}{\Delta t}\int_{t^{m-1}}^{t^m} \boldsymbol{\eta}_h\, dt$.

Utilizing $\boldsymbol{v}_h^m$ and $\boldsymbol{\eta}_h^m$ as test functions in the velocity scheme (3.3), multiplying by Δt, summing the result over m, and applying summation by parts, we gather

$$
\begin{aligned}
-\int_{\Delta t}^T \int_\Omega & \boldsymbol{u}_h(t-\Delta t, x)\partial_t^h(\boldsymbol{v}_h)\ dxdt \\
&+ \mu \operatorname{curl}_x \boldsymbol{w}_h \boldsymbol{v}_h + [(\mu+\lambda)\operatorname{div}_x \boldsymbol{u}_h - p(\varrho_h)]\operatorname{div}_x \boldsymbol{v}_h\ dxdt = \int_\Omega \boldsymbol{u}_h^0 \boldsymbol{v}_h^1\ dx, \\
\int_0^T \int_\Omega & \boldsymbol{w}_h \boldsymbol{\eta}_h - \boldsymbol{u}_h \operatorname{curl}_x \boldsymbol{\eta}_h\ dxdt = 0.
\end{aligned}
\tag{6.23}
$$

In view of Lemma 2.8, $\boldsymbol{v}_h \to \boldsymbol{v}$ in $L^\infty(0,T;\boldsymbol{W}^{\mathrm{div},p})$ for any finite p and $\boldsymbol{\eta}_h \to \boldsymbol{\eta}$ in $L^\infty(0,T;\boldsymbol{W}^{\mathrm{curl},p})$. Furthermore, by Lemmas 5.2 and 6.6 $p(\varrho_h) \to p(\varrho)$ in $L^\alpha((0,T)\times\Omega)$ for any $\alpha < \gamma+1$. Hence, we can send $h \to 0$ in (6.23) to obtain that the limit constructed in (6.2) satisfies (2.2) for all test functions $(\boldsymbol{v}, \boldsymbol{\eta}) \in \boldsymbol{C}_c^\infty((0,T)\times\Omega)$. Since $\boldsymbol{C}_c^\infty((0,T)\times\Omega)$ is dense in both $L^2(0,T;\boldsymbol{W}_0^{\mathrm{curl},2}(\Omega))$ and $W^{1,2}(0,T;L^2(\Omega)) \cap L^2(0,T;\boldsymbol{W}_0^{\mathrm{div},2}(\Omega))$ [10] this concludes the proof. □

References

[1] Shmuel Agmon, *Lectures on elliptic boundary value problems*, Prepared for publication by B. Frank Jones, Jr. with the assistance of George W. Batten, Jr. Van Nostrand Mathematical Studies, No. 2, D. Van Nostrand Co., Inc., Princeton, N.J.-Toronto-London, 1965. ↑2.3

[2] Douglas N. Arnold, Richard S. Falk, and Ragnar Winther, *Multigrid in* $H(\mathrm{div})$ *and* $H(\mathrm{curl})$, Numer. Math. **85** (2000), no. 2, 197–217. ↑2.3, 2.3

[3] Douglas N. Arnold, Richard S. Falk, and Ragnar Winther, *Finite element exterior calculus, homological techniques, and applications*, Acta Numer. **15** (2006), 1–155. ↑2.3, 2.3

[4] Susanne C. Brenner and L. Ridgway Scott, *The mathematical theory of finite element methods*, 2nd ed., Texts in Applied Mathematics, vol. 15, Springer-Verlag, New York, 2002. ↑2.3, 2.3

[5] Eduard Feireisl, *Dynamics of viscous compressible fluids*, Oxford Lecture Series in Mathematics and its Applications, vol. 26, Oxford University Press, Oxford, 2004. ↑2, 6, 6.3

[6] Thierry Gallouët, Raphaele Herbin, and Jean-Claude Latché, *A convergent finite element-finite volume scheme for the compressible Stokes problem. Part I: The isothermal case*, Math. Comp., online (2009). ↑1

[7] Thierry Gallouët, Raphaele Herbin, and Jean-Claude Latché, *A convergent finite element-finite volume scheme for the compressible Stokes problem. Part II: The isentropic case*, Preprint (2009). ↑1

[8] Kenneth Karlsen and Trygve K. Karper, *Convergence of a mixed method for a semi–stationary compressible Stokes system*, Preprint (2009). ↑1, 1, 2.1, 2.2, 2.3, 3, 3, 4, 4.1, 4, 4.1, 6.2, 6.3

[9] Kenneth Karlsen and Trygve K. Karper, *A convergent nonconforming finite element method for compressible Stokes flow*, Preprint (2009). ↑1, 2

[10] Vivette Girault and Pierre-Arnaud Raviart, *Finite element methods for Navier-Stokes equations*, Springer Series in Computational Mathematics, vol. 5, Springer-Verlag, Berlin, 1986. Theory and algorithms. ↑2.3, 6.1, 6.4

[11] Pierre-Louis Lions, *Mathematical topics in fluid mechanics. Vol. 2*, Oxford Lecture Series in Mathematics and its Applications, vol. 10, The Clarendon Press Oxford University Press, New York, 1998. Compressible models; Oxford Science Publications. ↑1, 2, 2, 6, 6.3

[12] Lu Min, Alexandre V. Kazhikhov, and Seiji Ukai, *Global solutions to the Cauchy problem of the Stokes approximation equations for two-dimensional compressible flows*, Comm. Partial Differential Equations **23** (1998), no. 5-6, 985–1006. ↑1

[13] J.-C. Nédélec, *Mixed finite elements in* $\mathbf{R}^3$, Numer. Math. **35** (1980), no. 3, 315–341. ↑1, 2.3, 2.3, 2.3

[14] V. A. Vaĭgant and A. V. Kazhikhov, *Global solutions of equations of potential flows of a compressible viscous fluid for small Reynolds numbers*, Differentsial′nye Uravneniya **30** (1994), no. 6, 1010–1022, 1102 (Russian, with Russian summary); English transl., Differential Equations **30** (1994), no. 6, 935–947. ↑1

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