WAVE PROPAGATION IN MULTICOMPONENT FLOW MODELS

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ABSTRACT. We consider systems of hyperbolic balance laws governing flows of an arbitrary number of components equipped with general equations of state. The components are assumed to be immiscible.

We compare two such models; one in which thermal equilibrium is attained trough a relaxation procedure, and a fully relaxed model in which equal temperatures are instantaneously imposed. We describe how the relaxation procedure may be made consistent with the second law of thermodynamics.

Exact wave velocities for both models are obtained and compared. In particular, our formulation directly proves a general subcharacteristic condition: For an arbitrary number of components and thermodynamically stable equations of state, the mixture sonic velocity of the relaxed system can never exceed the sonic velocity of the relaxation system.

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1. INTRODUCTION

Dynamic simulations of multicomponent flows often involve non-equilibrium processes. Driving forces towards equilibrium occur in the equations as *relaxation source terms*, which may be extremely stiff if the relaxation time towards equilibrium is small. In this paper, we consider hyperbolic relaxation systems in a form similar to the description by Chen et al. [7]:

$$\frac{\partial \boldsymbol{U}}{\partial t} + \frac{\partial \boldsymbol{F}(\boldsymbol{U})}{\partial x} + \boldsymbol{A}(\boldsymbol{U})\frac{\partial \boldsymbol{W}(\boldsymbol{U})}{\partial x} + \frac{1}{\epsilon}\boldsymbol{R}(\boldsymbol{U}) = 0, \tag{1}$$

to be solved for the unknown M-vector $\boldsymbol{U}$. The system is endowed with a $m \times M$ constant-coefficient matrix $\mathcal{Q}$ with rank $m < M$ such that

$$\mathcal{Q}\boldsymbol{R} = 0 \qquad \forall \boldsymbol{U}. \tag{2}$$

Furthermore, we assume that $\mathcal{Q}\boldsymbol{A}\,\mathrm{d}\boldsymbol{W}$ is an exact differential:

$$\mathcal{Q}\boldsymbol{A}\,\mathrm{d}\boldsymbol{W} = \mathrm{d}\boldsymbol{G}(\boldsymbol{U}). \tag{3}$$

Multiplying (1) on the left by $\mathcal{Q}$ we obtain a conservation law for the reduced variable $\boldsymbol{V} = \mathcal{Q}\boldsymbol{U}$:

$$\frac{\partial \boldsymbol{V}}{\partial t} + \frac{\partial}{\partial x}\left(\mathcal{Q}\boldsymbol{F}(\boldsymbol{U}) + \boldsymbol{G}(\boldsymbol{U})\right) = 0. \tag{4}$$

We now assume that each $\boldsymbol{V}$ uniquely determines a local equilibrium value $\boldsymbol{U} = \mathcal{E}(\boldsymbol{V})$, satisfying $\boldsymbol{R}(\mathcal{E}(\boldsymbol{V})) = 0$ as well as

$$\mathcal{Q}\mathcal{E}(\boldsymbol{V}) = \boldsymbol{V} \qquad \forall \boldsymbol{V}. \tag{5}$$

Now (4) can be closed as a reduced system by imposing the local equilibrium condition for $\boldsymbol{U}$, namely

$$\boldsymbol{U} = \mathcal{E}(\boldsymbol{V}), \tag{6}$$

$$\frac{\partial \boldsymbol{V}}{\partial t} + \frac{\partial \mathcal{F}(\boldsymbol{V})}{\partial x} = 0, \tag{7}$$

where the reduced flux $\mathcal{F}$ is defined by

$$\mathcal{F}(\boldsymbol{V}) \equiv \mathcal{Q}\boldsymbol{F}\left(\mathcal{E}(\boldsymbol{V})\right) + \boldsymbol{G}(\mathcal{E}(\boldsymbol{V})). \tag{8}$$

Chen et al. [7] studied stability of solutions to such relaxation systems for the special case $\boldsymbol{A} = 0$, i.e. the hyperbolic part of (1) is conservative. In particular, they based their analysis on the requirement that the relaxation term should be *entropy dissipative*.

We remark that for the general case of non-conservative hyperbolic balance laws, where $\boldsymbol{A}\,\mathrm{d}\boldsymbol{W}$ in (1) is not an exact differential, the study of uniqueness of solutions requires an extension of the standard theory for conservative systems. This has been an active area of research in recent years, see for instance [6, 8].

1.1. The Subcharacteristic Condition.
Central to the question of stability of relaxation systems is the *subcharacteristic condition*, a concept introduced by Liu [13]. Within our formulation, this concept may be defined as follows:

Definition 1. *Let the M eigenvalues of the relaxing system* (1) *be given by*

$$\lambda_1 \leq \ldots \leq \lambda_k \leq \lambda_{k+1} \leq \ldots \leq \lambda_M \tag{9}$$

and the m eigenvalues of the relaxed system (6)–(7) *be given by*

$$\tilde{\lambda}_1 \leq \ldots \leq \tilde{\lambda}_j \leq \tilde{\lambda}_{j+1} \leq \ldots \leq \tilde{\lambda}_m. \tag{10}$$

Herein, the relaxation system (1) *is applied to a local equilibrium state $\boldsymbol{U} = \mathcal{E}(\boldsymbol{V})$ such that*

$$\lambda_k = \lambda_k(\mathcal{E}(\boldsymbol{V})), \qquad \tilde{\lambda}_j = \tilde{\lambda}_j(\boldsymbol{V}). \tag{11}$$

Now let the $\tilde{\lambda}_j$ be interlaced *with λ_k in the following sense: Each $\tilde{\lambda}_j$ lies in the closed interval $[\lambda_j, \lambda_{j+M-m}]$. Then the relaxed system* (6)–(7) *is said to satisfy the* **subcharacteristic condition** *with respect to* (1).

Chen et al. [7] were able to prove the following: If the relaxation system (1) may be equipped with a convex entropy function that is dissipated by the relaxation term, then the subcharacteristic condition holds. Furthermore, a converse holds for linear systems and general 2×2 systems.

Although the subcharacteristic condition is formally neither a necessary nor sufficient condition for stability *in general*, it is nevertheless an essential condition for linear stability and is in practice required for most physically meaningful relaxation processes. Hence the literature commonly puts a strong focus on this condition, see for instance Baudin et al. [4, 5] for an application to a two-phase flow model.

A main result of this paper is a proof that the subcharacteristic condition holds for the models we are studying. In particular, we present explicit expressions for the eigenvalues of the models by which it may easily be verified that the condition holds.

1.2. Applications to Multiphase Flows.
In addition to modelling actual physical processes, relaxation systems are significant also from the viewpoint of pure numerical analysis – the relaxation system (1) may be used as a starting point for devising numerical methods for the relaxed system (6)–(7). A classic paper in this respect is the work of Jin and Xin [11], who devised a general method in which a conservative system in the form (7) is recast as the limit $\epsilon \to 0$ of (1), where $M = 2m$ and the hyperbolic part of (1) is fully linear. By this, they were able to construct a numerical method where all nonlinearities are encoded in the source terms. Variations of this approach were applied to the drift-flux two-phase flow model by Evje and Fjelde [10] as well as Baudin et al. [4, 5].

Since the works of Saurel and Abgrall [2, 20], there has been considerable interest in applying various relaxation techniques to multiphase flow models. The starting point for many such investigations is the two-pressure two-fluid model [3, 19]:

- *Conservation of mass:*

$$\frac{\partial}{\partial t}(\rho_{\mathrm{g}}\alpha_{\mathrm{g}}) + \frac{\partial}{\partial x}(\rho_{\mathrm{g}}\alpha_{\mathrm{g}}v_{\mathrm{g}}) = 0, \tag{12}$$

$$\frac{\partial}{\partial t}(\rho_{\ell}\alpha_{\ell}) + \frac{\partial}{\partial x}(\rho_{\ell}\alpha_{\ell}v_{\ell}) = 0, \tag{13}$$

- *Balance of momentum:*

$$\frac{\partial}{\partial t}(\rho_{\mathrm{g}}\alpha_{\mathrm{g}}v_{\mathrm{g}}) + \frac{\partial}{\partial x}\left(\rho_{\mathrm{g}}\alpha_{\mathrm{g}}v_{\mathrm{g}}^2 + \alpha_{\mathrm{g}}p_{\mathrm{g}}\right) - p^{\mathrm{i}}\frac{\partial\alpha_{\mathrm{g}}}{\partial x} = \mu_v(v_{\ell} - v_{\mathrm{g}}), \tag{14}$$

$$\frac{\partial}{\partial t}(\rho_{\ell}\alpha_{\ell}v_{\ell}) + \frac{\partial}{\partial x}\left(\rho_{\ell}\alpha_{\ell}v_{\ell}^2 + \alpha_{\ell}p_{\ell}\right) - p^{\mathrm{i}}\frac{\partial\alpha_{\ell}}{\partial x} = \mu_v(v_{\mathrm{g}} - v_{\ell}), \tag{15}$$

- *Balance of energy:*

$$\frac{\partial}{\partial t}\left(\rho_{\mathrm{g}}\alpha_{\mathrm{g}}\left(\frac{1}{2}v_{\mathrm{g}}^2 + e_{\mathrm{g}}\right)\right) + \frac{\partial}{\partial x}\left(\rho_{\mathrm{g}}\alpha_{\mathrm{g}}v_{\mathrm{g}}\left(\frac{1}{2}v_{\mathrm{g}}^2 + e_{\mathrm{g}} + \frac{p_{\mathrm{g}}}{\rho_{\mathrm{g}}}\right)\right) + p^{\mathrm{i}}\frac{\partial\alpha_{\mathrm{g}}}{\partial t} = \mu_v v^{\mathrm{i}}(v_{\ell} - v_{\mathrm{g}}), \tag{16}$$

$$\frac{\partial}{\partial t}\left(\rho_{\ell}\alpha_{\ell}\left(\frac{1}{2}v_{\ell}^2 + e_{\ell}\right)\right) + \frac{\partial}{\partial x}\left(\rho_{\ell}\alpha_{\ell}v_{\ell}\left(\frac{1}{2}v_{\ell}^2 + e_{\ell} + \frac{p_{\ell}}{\rho_{\ell}}\right)\right) + p^{\mathrm{i}}\frac{\partial\alpha_{\ell}}{\partial t} = \mu_v v^{\mathrm{i}}(v_{\mathrm{g}} - v_{\ell}), \tag{17}$$

- *Evolution of volume fraction:*

$$\frac{\partial\alpha_{\mathrm{g}}}{\partial t} + v^{\mathrm{i}}\frac{\partial\alpha_{\mathrm{g}}}{\partial x} = \mu_p(p_{\mathrm{g}} - p_{\ell}). \tag{18}$$

Herein, we use the following nomenclature for phase $k \in \{\mathrm{g}, \ell\}$:

$$\begin{array}{lll}
\rho_k & - & \text{density of phase } k, \\
p_k & - & \text{pressure of phase } k, \\
v_k & - & \text{velocity of phase } k, \\
\alpha_k & - & \text{volume fraction of phase } k, \\
e_k & - & \text{specific internal energy of phase } k, \\
p^{\mathrm{i}} & - & \text{pressure at the gas-liquid interface}, \\
v^{\mathrm{i}} & - & \text{local velocity at the gas liquid-interface}.
\end{array}$$

Furthermore, μ_v and μ_p are relaxation coefficients and the following relation holds:

$$\alpha_{\mathrm{g}} + \alpha_{\ell} = 1. \tag{19}$$

Munkejord [15] fixed $\mu_v = 0$ and studied the resulting relaxation system for $\mu_p \to \infty$, with an emphasis on assessing a relaxation scheme based on the Roe Riemann solver, and performing computations with finite μ_p. Here the energy equations were neglected.

Several authors [9, 12, 16, 21, 22] have performed analytical and numerical studies of the full relaxation process where both $\mu_p \to \infty$ and $\mu_v \to \infty$, which results in a five-equation simplified system also briefly described by Stewart and Wendroff [24]. This system may be written in the following form [16]:

$$\frac{\partial}{\partial t}(\rho_{\mathrm{g}}\alpha_{\mathrm{g}}) + \frac{\partial}{\partial x}(\rho_{\mathrm{g}}\alpha_{\mathrm{g}}v) = 0, \tag{20}$$

$$\frac{\partial}{\partial t}(\rho_{\ell}\alpha_{\ell}) + \frac{\partial}{\partial x}(\rho_{\ell}\alpha_{\ell}v) = 0, \tag{21}$$

$$\frac{\partial}{\partial t}(\rho v) + \frac{\partial}{\partial x}(\rho v + p) = 0, \tag{22}$$

$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x}(v(E + p)) = 0, \tag{23}$$

$$\frac{\partial\alpha_{\mathrm{g}}}{\partial t} + v\frac{\partial\alpha_{\mathrm{g}}}{\partial x} = \frac{\alpha_{\mathrm{g}}\alpha_{\ell}(\rho_{\ell}c_{\ell}^2 - \rho_{\mathrm{g}}c_{\mathrm{g}}^2)}{\alpha_{\ell}\rho_{\mathrm{g}}c_{\mathrm{g}}^2 + \alpha_{\mathrm{g}}\rho_{\ell}c_{\ell}^2}\frac{\partial v}{\partial x}, \tag{24}$$

where the mixture density ρ is given by

$$\rho = \rho_{\mathrm{g}}\alpha_{\mathrm{g}} + \rho_\ell\alpha_\ell, \tag{25}$$

the mixture total energy E is given by

$$E = \rho_{\mathrm{g}}\alpha_{\mathrm{g}}e_{\mathrm{g}} + \rho_\ell\alpha_\ell e_\ell + \frac{1}{2}\rho v^2, \tag{26}$$

and v and p are the velocity and pressure common to both phases. In addition to p, v and α_{g}, the independent physical variables are here the temperatures T_{g} and T_ℓ.

1.3. **Outline of This Paper.** This paper is motivated by the observation that most existing works related to the model (20)–(24) assume that the number of independent phases is fixed to 2. We are interested in generalizing this model to apply to an arbitrary number of components, and then applying relaxation heat-transfer terms that will drive the model towards thermal equilibrium. In the fully relaxed limit, we then recover the *homogeneous equilibrium model*, studied for instance in [1, 18].

The usefulness of such an extension is twofold:

(1) Several immiscible fluids may coexist without being in thermal equilibrium, and modelling individual temperatures for each species may be required. For instance, this can occur for mixtures of hydrocarbons and water relevant for the petroleum industry.
(2) Direct equilibrium calculations for multicomponent mixtures are computationally expensive. Therefore, relaxation schemes based on non-equilibrium models may provide benefits in terms of efficiency compared to solving equilibrium models directly.

This paper is organized as follows: In Section 2, we detail the models we will be working with. In Section 2.1, we present the $(2N + 1)$-equation relaxation model for N components involving N individual temperatures. We derive necessary and sufficient restrictions on the relaxation terms imposed by the first and second law of thermodynamics. In Section 2.1.2, we explicitly state our model in the form (1). In Section 2.1.3, we show that our model reduces to the standard five-equation model for the special case $N = 2$. In Section 2.2, we explicitly perform the relaxation procedure to recover the reduced form (6)–(7).

In Sections 3.1–3.2, we obtain exact expressions for the wave velocities of the models. Our formulation allows for a direct proof that the subcharacteristic condition as stated in Definition 1 is satisfied. This is stated in Section 3.2.4.

For completeness, we derive an explicit quasilinear formulation of the relaxed system in Section 3.3. In Section 4, we summarize and comment on the results of our paper.

2. The Models

The foundation for the models we consider in this paper consists of one mass conservation equation for each component:

$$\frac{\partial}{\partial t}\left(\rho_i\alpha_i\right) + \frac{\partial}{\partial x}\left(\rho_i\alpha_i v\right) = 0 \quad \forall i \in \{1, \ldots, N\}, \tag{27}$$

as well as a conservation equation for the total momentum of the mixture:

$$\frac{\partial \rho v}{\partial t} + \frac{\partial}{\partial x}\left(\rho v^2 + p\right) = 0, \tag{28}$$

where for the purposes of this analysis we neglect any momentum source terms. Here

$$
\begin{array}{ll}
\rho_i & \text{-} \quad \text{density of component } i, \\
\rho & \text{-} \quad \text{density of the mixture}, \\
v & \text{-} \quad \text{velocity of the mixture}, \\
\alpha_i & \text{-} \quad \text{volume fraction of component } i, \\
p & \text{-} \quad \text{pressure common to all components},
\end{array}
$$

and the following relations hold:

$$\rho = \sum_{i=1}^{N} \rho_i \alpha_i, \tag{29}$$

$$\sum_{i=1}^{N} \alpha_i = 1. \tag{30}$$

We now state some observations that will prove useful later.

Lemma 1. *The mixture density evolution equation can be written as*

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) = 0. \tag{31}$$

Proof. Sum (27) over all i. □

Lemma 2. *The evolution equation for the mass fraction*

$$Y_i = \frac{\rho_i \alpha_i}{\rho} \tag{32}$$

can be written as

$$\frac{\partial Y_i}{\partial t} + v \frac{\partial Y_i}{\partial x} = 0. \tag{33}$$

Proof. Write

$$\rho_i \alpha_i = \rho Y_i \tag{34}$$

and use (27) and (31). □

Remark 1. *Note that since*

$$\sum_{i=1}^{N} Y_i = 1, \tag{35}$$

we have only $N-1$ independent mass fraction equations, expressible in vector form

$$\frac{\partial \boldsymbol{Y}}{\partial t} + v \frac{\partial \boldsymbol{Y}}{\partial x} = 0, \tag{36}$$

where

$$\boldsymbol{Y} = \begin{bmatrix} Y_1 \\ \vdots \\ Y_{N-1} \end{bmatrix}. \tag{37}$$

Lemma 3. *The following momentum evolution equation is valid for each component i:*

$$\frac{\partial}{\partial t} \left(\rho_i \alpha_i v \right) + \frac{\partial}{\partial x} \left(\rho_i \alpha_i v^2 \right) + \frac{\rho_i \alpha_i}{\rho} \frac{\partial p}{\partial x} = 0. \tag{38}$$

Proof. We have

$$d(\rho_i \alpha_i v) = \rho_i \alpha_i \, dv + v \, d(\rho_i \alpha_i) \tag{39}$$

and also

$$dv = \frac{1}{\rho} \left(d(\rho v) - v \, d\rho \right). \tag{40}$$

Substituting (40) into (39), and using (27)–(28) and (31), we obtain

$$\frac{\partial}{\partial t} \left(\rho_i \alpha_i v \right) + \frac{\rho_i \alpha_i}{\rho} \left(\frac{\partial}{\partial x}(\rho v^2 + p) - v \frac{\partial}{\partial x}(\rho v) \right) + v \frac{\partial}{\partial x}(\rho_i \alpha_i v) = 0, \tag{41}$$

which simplifies to

$$\frac{\partial}{\partial t} \left(\rho_i \alpha_i v \right) + \rho_i \alpha_i v \frac{\partial v}{\partial x} + \frac{\rho_i \alpha_i}{\rho} \frac{\partial p}{\partial x} + v \frac{\partial}{\partial x}(\rho_i \alpha_i v) = 0 \tag{42}$$

by expansion of derivatives. Lemma 3 now follows from the product rule for derivatives.

□

Lemma 4. *The velocity evolution equation can be formulated as follows:*

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0. \tag{43}$$

Proof. Expand derivatives in (38) and use (27).

$\square$

Lemma 5. *The following kinetic energy evolution equation is valid for each component i:*

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho_i\alpha_i v^2\right) + \frac{\partial}{\partial x}\left(\frac{1}{2}\rho_i\alpha_i v^3\right) + \frac{\rho_i\alpha_i v}{\rho}\frac{\partial p}{\partial x} = 0. \tag{44}$$

Proof. We expand the time derivative as

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho_i\alpha_i v^2\right) = \frac{1}{2}v\frac{\partial}{\partial t}\left(\rho_i\alpha_i v\right) + \frac{1}{2}\rho_i\alpha_i v\frac{\partial v}{\partial t}. \tag{45}$$

If now now substitute (38) and (43) into (45), we recover Lemma 5 after collecting derivatives.

$\square$

2.1. **Relaxation System.** In this section, we derive separate energy evolution equations for each component, where heat is transferred between the components at a rate proportional to their temperature difference. We start with the assumption that in Lagrangian coordinates, entropy change is due only to the heat-transfer terms:

$$\rho_i\alpha_i T_i\left(\frac{\partial s_i}{\partial t} + v\frac{\partial s_i}{\partial x}\right) = \sum_{j\neq i} H_{ij}(T_j - T_i), \tag{46}$$

where

$$s_i = s_i(p, T_i) \tag{47}$$

is the specific entropy of component i. We further assume that the relaxation coefficients H_{ij} are independent of the temperatures T_k. From (46), we may then derive energy evolution equations for each component, using the kinetic energy equation (44) and the fundamental thermodynamic differential

$$\mathrm{d}e_i = T_i\,\mathrm{d}s_i + \frac{p}{\rho_i^2}\,\mathrm{d}\rho_i. \tag{48}$$

Proposition 1. *To be consistent with the second law of thermodynamics, the relaxation coefficients* H_{ij} *must satisfy*

$$H_{ij} = H_{ji} \geq 0. \tag{49}$$

Proof. For the total cross-sectional entropy given by

$$\omega = \sum_{i=1}^{N} \rho_i\alpha_i s_i, \tag{50}$$

we obtain the evolution equation

$$\frac{\partial \omega}{\partial t} + \frac{\partial}{\partial x}(\omega v) = \sum_{i=1}^{N}\sum_{j\neq i} H_{ij}\frac{T_j - T_i}{T_i} \tag{51}$$

from (46). Now, inside a *closed* region R the global entropy Ω is given by:

$$\Omega(t) = \int_R \omega(x, t)\,\mathrm{d}x. \tag{52}$$

Hence the second law

$$\frac{\mathrm{d}\Omega}{\mathrm{d}t} \geq 0 \tag{53}$$

imposes

$$\sum_{i=1}^{N}\sum_{j\neq i} H_{ij}\frac{T_j - T_i}{T_i} \geq 0. \tag{54}$$

Now

$$\sum_{i=1}^{N}\sum_{j\neq i} H_{ij}\frac{T_j - T_i}{T_i} = \sum_{i,j>i}\left(\frac{H_{ij}}{T_i} - \frac{H_{ji}}{T_j}\right)(T_j - T_i)$$

$$= \sum_{i,j>i} H_{ij}\frac{(T_j - T_i)^2}{T_i T_j} + \sum_{i,j>i}(H_{ij} - H_{ji})\frac{T_j - T_i}{T_j}, \tag{55}$$

which remains unconditionally non-negative only if

$$H_{ij} \geq 0, \qquad H_{ij} - H_{ji} = 0 \qquad \forall i,j. \tag{56}$$

$\square$

Proposition 2. *The entropy evolution equations* (46) *with the condition* (49) *respect conservation of total energy.*

Proof. From (46) and the fundamental differential (48), we obtain

$$\rho_i \alpha_i \left(\frac{\partial e_i}{\partial t} + v\frac{\partial e_i}{\partial x}\right) - \frac{p\alpha_i}{\rho_i}\left(\frac{\partial \rho_i}{\partial t} + v\frac{\partial \rho_i}{\partial x}\right) = \sum_{j\neq i} H_{ij}(T_j - T_i), \tag{57}$$

where e_i is the specific internal energy of component i. Using (27), we can rewrite this as

$$\frac{\partial}{\partial t}(\rho_i \alpha_i e_i) + \frac{\partial}{\partial x}(\rho_i \alpha_i e_i v) + p\left(\frac{\partial \alpha_i}{\partial t} + \frac{\partial}{\partial x}(\alpha_i v)\right) = \sum_{j\neq i} H_{ij}(T_j - T_i). \tag{58}$$

Summing over all i and using (30) we obtain

$$\frac{\partial}{\partial t}\left(\sum_{i=1}^{N}\rho_i \alpha_i e_i\right) + \frac{\partial}{\partial x}\left(v\sum_{i=1}^{N}\rho_i \alpha_i e_i\right) + p\frac{\partial v}{\partial x} = \sum_{i=1}^{N}\sum_{j\neq i} H_{ij}(T_j - T_i), \tag{59}$$

which by (49) may be simplified to

$$\frac{\partial}{\partial t}\left(\sum_{i=1}^{N}\rho_i \alpha_i e_i\right) + \frac{\partial}{\partial x}\left(v\sum_{i=1}^{N}\rho_i \alpha_i e_i\right) + p\frac{\partial v}{\partial x} = 0. \tag{60}$$

We now define the total energy E as

$$E = \sum_{i=1}^{N}\rho_i \alpha_i \left(e_i + \frac{1}{2}v^2\right). \tag{61}$$

Summing (44) over all i and adding (60), we obtain an evolution equation for E in conservative form:

$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x}(v(E + p)) = 0. \tag{62}$$

$\square$

2.1.1. *Energy Evolution Equations.* In this section, we aim to transform (46) into evolution equations for the energy E_i of each component:

$$E_i = \rho_i \alpha_i \left(\frac{1}{2}v^2 + e_i\right). \tag{63}$$

We start by deriving some preliminary results.

Lemma 6. *The pressure evolution equation can be written as*

$$\frac{\partial p}{\partial t} + v\frac{\partial p}{\partial x} + \rho c^2\frac{\partial v}{\partial x} = \rho c^2 \sum_{i,j>i}\left(H_{ij}\left(\frac{\Gamma_i}{\rho_i c_i^2} - \frac{\Gamma_j}{\rho_j c_j^2}\right)(T_j - T_i)\right), \tag{64}$$

where

$$c^2 = \left(\rho\sum_{i=1}^{N}\frac{\alpha_i}{\rho_i c_i^2}\right)^{-1}. \tag{65}$$

Here

$$c_i^2 = \left(\frac{\partial p}{\partial \rho_i}\right)_{s_i} \tag{66}$$

represents the single-component velocity of sound, and Γ_i is the Grüneisen coefficient

$$\Gamma_i = \frac{1}{\rho_i}\left(\frac{\partial p}{\partial e_i}\right)_{\rho_i}. \tag{67}$$

Proof. The differential (48) may be rewritten as

$$\mathrm{d}p = c_i^2\,\mathrm{d}\rho_i + \Gamma_i\rho_i T_i\,\mathrm{d}s_i. \tag{68}$$

From (68) and (46) we obtain

$$\frac{\partial p}{\partial t} + v\frac{\partial p}{\partial x} = c_i^2\left(\frac{\partial \rho_i}{\partial t} + v\frac{\partial \rho_i}{\partial x}\right) + \frac{\Gamma_i}{\alpha_i}\sum_{j\neq i} H_{ij}(T_j - T_i), \tag{69}$$

which by (27) may be rewritten as

$$\frac{\alpha_i}{\rho_i c_i^2}\left(\frac{\partial p}{\partial t} + v\frac{\partial p}{\partial x}\right) + \frac{\partial \alpha_i}{\partial t} + \frac{\partial}{\partial x}(\alpha_i v) = \frac{\Gamma_i}{\rho_i c_i^2}\sum_{j\neq i} H_{ij}(T_j - T_i). \tag{70}$$

Lemma 6 follows from summing over all i. $\qquad\square$

Lemma 7. *The internal energy evolution equation for component i can be written as*

$$\frac{\partial}{\partial t}(\rho_i\alpha_i e_i) + \frac{\partial}{\partial x}(\rho_i\alpha_i e_i v) + \alpha_i p\frac{\rho c^2}{\rho_i c_i^2}\frac{\partial v}{\partial x}$$
$$= \theta_i\sum_{j\neq i} H_{ij}(T_j - T_i) + \frac{\rho c^2}{\rho_i c_i^2}\alpha_i\sum_{k,j>k}(H_{kj}(\theta_j - \theta_k)(T_j - T_k)), \quad (71)$$

where

$$\theta_i = 1 - \frac{\Gamma_i p}{\rho_i c_i^2} \equiv \frac{1}{T_i}\left(\frac{\partial e_i}{\partial s_i}\right)_p. \tag{72}$$

Proof. Substitute (70) into (58) to obtain

$$\frac{\partial}{\partial t}(\rho_i\alpha_i e_i) + \frac{\partial}{\partial x}(\rho_i\alpha_i e_i v) - \frac{\alpha_i p}{\rho_i c_i^2}\left(\frac{\partial p}{\partial t} + v\frac{\partial p}{\partial x}\right) = \left(1 - \frac{\Gamma_i p}{\rho_i c_i^2}\right)\sum_{j\neq i} H_{ij}(T_j - T_i). \tag{73}$$

Now (71) follows by substituting (64) into (73).

$\square$

Proposition 3. *The evolution equation for the total energy of component i can be written as*

$$\frac{\partial E_i}{\partial t} + \frac{\partial}{\partial x}(E_i v) + \frac{\rho_i\alpha_i v}{\rho}\frac{\partial p}{\partial x} + \alpha_i p\frac{\rho c^2}{\rho_i c_i^2}\frac{\partial v}{\partial x}$$
$$= \theta_i\sum_{j\neq i} H_{ij}(T_j - T_i) + \frac{\rho c^2}{\rho_i c_i^2}\alpha_i\sum_{k,j>k}(H_{kj}(\theta_j - \theta_k)(T_j - T_k)), \quad (74)$$

or equivalently

$$\frac{\partial E_i}{\partial t} + \frac{\partial}{\partial x}(E_i v) + \frac{\rho_i\alpha_i}{\rho}\frac{\partial}{\partial x}(pv) + \alpha_i p\left(\frac{\rho^2 c^2 - \rho_i^2 c_i^2}{\rho\rho_i c_i^2}\right)\frac{\partial v}{\partial x}$$
$$= \theta_i\sum_{j\neq i} H_{ij}(T_j - T_i) + \frac{\rho c^2}{\rho_i c_i^2}\alpha_i\sum_{k,j>k}(H_{kj}(\theta_j - \theta_k)(T_j - T_k)). \quad (75)$$

Proof. Add (44) and (71) to obtain (74). $\qquad\square$

2.1.2. *Canonical Relaxation Form.* In this section, we explicitly express the above model in the form (1). We emphasize that since the system is partially non-conservative, there is no obvious preferred choice of variables in which to express the balance equations; however, conservation of total energy must be respected.

For the ($N = 2$)-model previously investigated, the authors [9, 12, 16, 21, 22] commonly choose to express the equations in terms of total energy and volume fraction, as stated by (20)–(24). This formulation naturally follows from performing the relaxation procedure on the model (12)–(18).

However, to preserve the symmetry in the equations, we here choose to express our model in terms of the energy evolution equations for each component. Summing these equations then automatically yields conservation of total energy, as stated by Proposition 2. In the context of (1), we obtain:

$$
\boldsymbol{U} = \begin{bmatrix} \rho_1 \alpha_1 \\ \vdots \\ \rho_N \alpha_N \\ \rho v \\ E_1 \\ \vdots \\ E_N \end{bmatrix}, \qquad \boldsymbol{F}(\boldsymbol{U}) = \begin{bmatrix} \rho_1 \alpha_1 v \\ \vdots \\ \rho_N \alpha_N v \\ \rho v^2 + p \\ E_1 v \\ \vdots \\ E_N v \end{bmatrix}, \qquad \boldsymbol{W}(\boldsymbol{U}) = \begin{bmatrix} pv \\ v \end{bmatrix}. \tag{76}
$$

Furthermore, the $(2N + 1) \times 2$ matrix $\boldsymbol{A}$ is given by

$$
\boldsymbol{A}(\boldsymbol{U}) = \frac{1}{\rho} \begin{bmatrix} 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 0 & 0 \\ \rho_1 \alpha_1 & \alpha_1 p \left(\frac{\rho c^2 - \rho_1 c_1^2}{\rho_1 c_1^2} \right) \\ \vdots & \vdots \\ \rho_N \alpha_N & \alpha_N p \left(\frac{\rho c^2 - \rho_N c_N^2}{\rho_N c_N^2} \right) \end{bmatrix}. \tag{77}
$$

The relaxation source term is given by

$$
\boldsymbol{R}(\boldsymbol{U}) = - \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ \theta_1 \sum_{j \neq 1} h_{1j}(T_j - T_1) + \frac{\rho c^2}{\rho_1 c_1^2} \alpha_1 \sum_{k,j>k} \left(h_{kj}(\theta_j - \theta_k)(T_j - T_k) \right) \\ \vdots \\ \theta_N \sum_{j \neq N} h_{Nj}(T_j - T_N) + \frac{\rho c^2}{\rho_N c_N^2} \alpha_N \sum_{k,j>k} \left(h_{kj}(\theta_j - \theta_k)(T_j - T_k) \right) \end{bmatrix}, \tag{78}
$$

where

$$
h_{ij} = \epsilon H_{ij}. \tag{79}
$$

2.1.3. *Relation to Five-Equation Model.* In this section, we wish to illustrate that our model essentially reduces to the five-equation model [9, 12, 16, 21, 22] for the special case $N = 2$.

From our general model (76)–(78), we may derive an evolution equation for the volume fraction:

Lemma 8. *The evolution equation for the volume fraction of component i can be written as*

$$
\frac{\partial \alpha_i}{\partial t} + v \frac{\partial \alpha_i}{\partial x} + \alpha_i \frac{\rho_i c_i^2 - \rho c^2}{\rho_i c_i^2} \frac{\partial v}{\partial x}
$$
$$
= \frac{\Gamma_i}{\rho_i c_i^2} \sum_{j \neq i} H_{ij}(T_j - T_i) - \alpha_i \frac{\rho c^2}{\rho_i c_i^2} \sum_{k,j>k} \left(H_{kj} \left(\frac{\Gamma_k}{\rho_k c_k^2} - \frac{\Gamma_j}{\rho_j c_j^2} \right) (T_j - T_k) \right). \tag{80}
$$

Proof. Substitute (64) into (70) and expand derivatives. □

Now for $N = 2$, this may be written as

$$\frac{\partial \alpha_1}{\partial t} + v \frac{\partial \alpha_1}{\partial x} = \frac{\alpha_1 \alpha_2 (\rho_2 c_2^2 - \rho_1 c_1^2)}{\alpha_2 \rho_1 c_1^2 + \alpha_1 \rho_2 c_2^2} \frac{\partial v}{\partial x} + \frac{\alpha_1 \alpha_2}{\alpha_2 \rho_1 c_1^2 + \alpha_1 \rho_2 c_2^2} \left(\frac{\Gamma_1}{\alpha_1} + \frac{\Gamma_2}{\alpha_2} \right) H_{12}(T_2 - T_1). \tag{81}$$

Augmenting this with the mass, total momentum and total energy equations (20)–(23), we recover the formulation of the five-equation model stated in [21], Section 5.5.

2.2. **Relaxed System.** We now consider the system obtained by letting the relaxation coefficients H_{ij} tend to infinity, i.e. we achieve instantaneous thermal equilibrium. In addition to the mass and momentum conservation equations (27) and (28), we replace the componentwise energy evolution equations (74) with:

- *Equality of temperatures:*
$$T_i = T_j = T \quad \forall i, j; \tag{82}$$

- *Conservation of total energy:*
$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x}(v(E + p)) = 0. \tag{83}$$

In the context of Section 1, the $(N + 2) \times (2N + 1)$ matrix $\mathcal{Q}$ is given by

$$\mathcal{Q} = [Q_{ij}], \qquad Q_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 1 & \text{if } j > i \text{ and } i = N + 2, \\ 0 & \text{otherwise.} \end{cases} \tag{84}$$

We may then verify that (2) holds. Furthermore, we obtain:

$$\boldsymbol{V}(\boldsymbol{U}) = \begin{bmatrix} \rho_1 \alpha_1 \\ \vdots \\ \rho_N \alpha_N \\ \rho v \\ E \end{bmatrix}, \qquad \boldsymbol{G}(\boldsymbol{U}) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ pv \end{bmatrix}, \qquad \mathcal{F}(\boldsymbol{V}) = \begin{bmatrix} \rho_1 \alpha_1 v \\ \vdots \\ \rho_N \alpha_N v \\ \rho v^2 + p \\ Ev \end{bmatrix}. \tag{85}$$

and the local equilibrium value $\mathcal{E}(\boldsymbol{V})$ is determined by (82).

Remark 2. *Note that the matrix $\mathcal{Q}$ reduces to the identity matrix for the special case $N = 1$, where the equilibrium condition is already satisfied by the relaxation system. However, in Section 1, we explicitly assume that*

$$\text{rank}(\mathcal{Q}) < M = 2N + 1. \tag{86}$$

Throughout this paper, we will assume that $N \geq 2$ so that (86) holds.

3. Wave Structure

In this section, we derive the wave velocities associated with the relaxation and relaxed models, formally given by the eigenvalues of the coefficient matrix of the system in quasilinear form. Our derivation will rely heavily on the similarities between our systems and the well known Euler system for single-component gas dynamics.

3.1. **Relaxation System.** The system (76)–(78) may be expressed in an alternative form as a composition of 3 parts:

- An "isentropic Euler part", consisting of (28) and (31);
- A mass fraction part (36);
- An entropy part (46).

From (36) and (46) we immediately see that s_i and $\boldsymbol{Y}$ are characteristic variables. Hence v is an eigenvalue of the system with multiplicity $(2N-1)$, corresponding to N entropy waves and $N-1$ mass fraction waves.

From (28) and (31) we then obtain the remaining eigenvalues:

$$\lambda = v \pm \hat{c}, \tag{87}$$

where

$$\hat{c}^2 = \left(\frac{\partial p}{\partial \rho}\right)_{\boldsymbol{Y}, s_1, \dots, s_N}. \tag{88}$$

Proposition 4. *The mixture sonic velocity $\hat{c}$ is given by*

$$\hat{c}^2 = \left(\rho \sum_{i=1}^{N} \frac{\alpha_i}{\rho_i c_i^2}\right)^{-1}. \tag{89}$$

Proof. Consider the differential

$$\sum_{i=1}^{N} \frac{\mathrm{d}(\rho_i \alpha_i)}{\rho_i} = \sum_{i=1}^{N} \mathrm{d}\alpha_i + \sum_{i=1}^{N} \left(\frac{\alpha_i}{\rho_i}\mathrm{d}\rho_i\right) = \sum_{i=1}^{N}\left(\frac{\alpha_i}{\rho_i c_i^2}\mathrm{d}p + \mathcal{O}(\mathrm{d}s_i)\right), \tag{90}$$

which can also be written as

$$\sum_{i=1}^{N} \frac{\mathrm{d}(\rho_i \alpha_i)}{\rho_i} = \sum_{i=1}^{N} \frac{\mathrm{d}(\rho Y_i)}{\rho_i} = \rho \sum_{i=1}^{N} \frac{\mathrm{d}Y_i}{\rho_i} + \sum_{i=1}^{N} \left(\frac{Y_i}{\rho_i}\right)\mathrm{d}\rho. \tag{91}$$

We then have

$$\hat{c}^2 = \left(\frac{\partial p}{\partial \rho}\right)_{\boldsymbol{Y}, s_1, \dots, s_N} = \frac{\sum_{i=1}^{N} \frac{Y_i}{\rho_i}}{\sum_{i=1}^{N} \frac{\alpha_i}{\rho_i c_i^2}}, \tag{92}$$

and (89) follows. $\square$

Remark 3. *Note that when $N = 2$, (89) reduces to a classical expression for the two-phase sonic velocity, sometimes referred to as the "Wood speed of sound" [21]. This expression is also derived in [17] by considering one phase as an elastic wall for the other.*

3.2. **Relaxed System.** The relaxed system (85) may also be expressed in a convenient alternative form as:

- A mass fraction part (36);
- A "mixture Euler" part, consisting of:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) = 0, \tag{93}$$

$$\frac{\partial}{\partial t}(\rho v) + \frac{\partial}{\partial x}(\rho v^2 + p) = 0, \tag{94}$$

$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x}\left((E + p)v\right). \tag{95}$$

From (36) we see that there are $N-1$ characteristics with velocity v corresponding to mass fraction waves. The remaining 3 eigenvalues may now be found from the Euler system (93)–(95), by means of the following result:

Proposition 5. *The mixture entropy given by*

$$s = \sum_{i=1}^{N} Y_i s_i \tag{96}$$

satisfies the characteristic equation

$$\frac{\partial s}{\partial t} + v \frac{\partial s}{\partial x} = 0. \tag{97}$$

Proof. The assumption of immiscibility implies that the differential (48) holds individually for each component. Substituting (48) into (95) and using (61), we recover (97) by textbook simplifications made possible by (93) and (94).

$\square$

Hence, in addition to the $N-1$ mass fraction waves and the mixture entropy wave (97), we obtain two sonic waves with velocities $v \pm \tilde{c}$, calculated in a standard way from the reduced Euler system (93)–(94). Herein, the sonic velocity $\tilde{c}$ is given by

$$\tilde{c} = \left(\frac{\partial p}{\partial \rho} \right)_{\boldsymbol{Y},s}. \tag{98}$$

3.2.1. *Some Thermodynamic Derivatives.* In order to obtain an explicit expression for $\tilde{c}$, we will first need some intermediate results. In particular, the following parameter will prove useful:

$$\zeta_i = \left(\frac{\partial T}{\partial p} \right)_{s_i} = -\frac{1}{\rho_i^2} \left(\frac{\partial \rho_i}{\partial s_i} \right)_p. \tag{99}$$

A number of useful thermodynamic derivatives may now be expressed in terms of ζ.

Lemma 9.

$$\left(\frac{\partial s_i}{\partial p} \right)_T = \frac{1}{\rho_i^2} \left(\frac{\partial \rho_i}{\partial T} \right)_p = -\frac{\zeta_i c_{p,i}}{T}, \tag{100}$$

where the specific heat capacity $c_{p,i}$ is given by

$$c_{p,i} = T \left(\frac{\partial s_i}{\partial T} \right)_p. \tag{101}$$

Proof. The result follows directly from (99) and (101). $\square$

Lemma 10.

$$\left(\frac{\partial \rho_i}{\partial p} \right)_T = \frac{1}{c_i^2} + \frac{\rho_i^2 \zeta_i^2 c_{p,i}}{T}. \tag{102}$$

Proof. The result follows from (66), (99), (101) and the relation

$$\left(\frac{\partial \rho_i}{\partial p} \right)_T = \left(\frac{\partial \rho_i}{\partial p} \right)_{s_i} + \left(\frac{\partial \rho_i}{\partial s_i} \right)_p \left(\frac{\partial s_i}{\partial p} \right)_T. \tag{103}$$

$\square$

Lemma 11.

$$\left(\frac{\partial e_i}{\partial T} \right)_p = c_{p,i} \left(1 - \zeta_i \frac{p}{T} \right). \tag{104}$$

Proof. The result follows from

$$\left(\frac{\partial e_i}{\partial T} \right)_p = T \left(\frac{\partial s_i}{\partial T} \right)_p + \frac{p}{\rho_i^2} \left(\frac{\partial \rho_i}{\partial T} \right)_p \tag{105}$$

and (100). $\square$

Lemma 12.

$$\left(\frac{\partial e_i}{\partial p} \right)_T = \frac{p}{(\rho_i c_i)^2} - \zeta_i c_{p,i} \left(1 - \zeta_i \frac{p}{T} \right). \tag{106}$$

Proof. The result follows from

$$\left(\frac{\partial e_i}{\partial p} \right)_T = T \left(\frac{\partial s_i}{\partial p} \right)_T + \frac{p}{\rho_i^2} \left(\frac{\partial \rho_i}{\partial p} \right)_T \tag{107}$$

as well as (100) and (102). $\square$

3.2.2. *The Relaxed Sound Velocity $\tilde{c}$.* Armed with these results, we are now able to obtain an explicit expression for $\tilde{c}$ as given by (98). To this end, we first state the following Lemma:

Lemma 13. *The differential* (90) *can be written as*

$$\sum_{i=1}^{N} \frac{\mathrm{d}(\rho_i \alpha_i)}{\rho_i} = \sum_{i=1}^{N} \left(\frac{\alpha_i}{\rho_i c_i^2} \right) \mathrm{d}p - \rho \frac{\sum_{i=1}^{N} \zeta_i C_{p,i}}{\sum_{i=1}^{N} C_{p,i}} \mathrm{d}s$$
$$+ \left(T \sum_{i=1}^{N} C_{p,i} \right)^{-1} \left(\sum_{i=1}^{N} \zeta_i^2 C_{p,i} \cdot \sum_{i=1}^{N} C_{p,i} - \left(\sum_{i=1}^{N} \zeta_i C_{p,i} \right)^2 \right) \mathrm{d}p + \mathcal{O}(\mathrm{d}\boldsymbol{Y}), \quad (108)$$

where the extensive heat capacity $C_{p,i}$ is given by

$$C_{p,i} = \rho_i \alpha_i c_{p,i}. \tag{109}$$

Proof. Use (100) and (102) to obtain

$$\sum_{i=1}^{N} \frac{\mathrm{d}(\rho_i \alpha_i)}{\rho_i} = \sum_{i=1}^{N} \left(\frac{\alpha_i}{\rho_i} \mathrm{d}\rho_i \right) = \sum_{i=1}^{N} \left(\frac{\alpha_i}{\rho_i c_i^2} + \zeta_i^2 \frac{C_{p,i}}{T} \right) \mathrm{d}p - \sum_{i=1}^{N} \left(\zeta_i \frac{C_{p,i}}{T} \right) \mathrm{d}T. \tag{110}$$

Furthermore, use (100) and (101) when differentiating (96) to obtain

$$\mathrm{d}s = \sum_{i=1}^{N} \left(Y_i \frac{c_{p,i}}{T} \right) \mathrm{d}T - \sum_{i=1}^{N} \left(Y_i \zeta_i \frac{c_{p,i}}{T} \right) \mathrm{d}p + \mathcal{O}(\mathrm{d}\boldsymbol{Y}). \tag{111}$$

Substitute (111) for $\mathrm{d}T$ in (110), and (108) follows. $\square$

To achieve further simplification, we will find use for a general summation lemma:

Lemma 14.

$$\sum_i (x_i^2 y_i) \cdot \sum_i y_i - \left(\sum_i (x_i y_i) \right)^2 = \sum_{j>i} y_i y_j (x_j - x_i)^2. \tag{112}$$

Proof.

$$\sum_i (x_i^2 y_i) \cdot \sum_i y_i - \left(\sum_i (x_i y_i) \right)^2$$
$$= \sum_i (x_i^2 y_i^2) + \sum_{i \neq j} x_i^2 y_i y_j - \sum_i (x_i^2 y_i^2) - \sum_{i \neq j} x_i x_j y_i y_j$$
$$= \sum_{i \neq j} \left(x_i^2 y_i y_j - x_i x_j y_i y_j \right) = \sum_{j>i} \left((x_i^2 + x_j^2) y_i y_j - 2 x_i x_j y_i y_j \right)$$
$$= \sum_{j>i} y_i y_j (x_j - x_i)^2. \tag{113}$$

$\square$

Proposition 6. *The relaxed mixture sonic velocity* (98) *may be written as*

$$\tilde{c}^{-2} = \hat{c}^{-2} + \rho \left(T \sum_{i=1}^{N} C_{p,i} \right)^{-1} \sum_{j>i} C_{p,i} C_{p,j} (\zeta_j - \zeta_i)^2, \tag{114}$$

where $\hat{c}$, given by (89), *is the mixture sonic velocity of the relaxation system of Section 2.1.*

Proof. Lemma 14 allows us to write (108) as

$$\sum_{i=1}^{N} \frac{\mathrm{d}(\rho_i \alpha_i)}{\rho_i} = \left(\sum_{i=1}^{N} \frac{\alpha_i}{\rho_i c_i^2} + \left(T \sum_{i=1}^{N} C_{p,i} \right)^{-1} \sum_{j>i} C_{p,i} C_{p,j} (\zeta_j - \zeta_i)^2 \right) \mathrm{d}p - \rho \frac{\sum_{i=1}^{N} \zeta_i C_{p,i}}{\sum_{i=1}^{N} C_{p,j}} \mathrm{d}s + \mathcal{O}(\mathrm{d}\boldsymbol{Y}). \tag{115}$$

Using (91), we may then express the mixture sound velocity as

$$\tilde{c}^{-2} = \left(\frac{\partial \rho}{\partial p}\right)_{\boldsymbol{Y},s} = \left(\sum_{i=1}^{N} \frac{Y_i}{\rho_i}\right)^{-1} \left(\sum_{i=1}^{N} \frac{\alpha_i}{\rho_i c_i^2} + \left(T\sum_{i=1}^{N} C_{p,i}\right)^{-1} \sum_{j>i} C_{p,i} C_{p,j} (\zeta_j - \zeta_i)^2\right), \quad (116)$$

and (114) follows.

$\square$

3.2.3. *Alternative Formulations.* Several equivalent formulations of the equilibrium mixture sound velocity $\tilde{c}$ are known from the literature. For equilibrium flow of two immiscible components, Städtke [23] obtained the following result:

$$\tilde{c}^{-2} = \rho \left(\alpha_1 \gamma_1 + \alpha_2 \gamma_2 - \frac{T}{C_{p,1} + C_{p,2}} (\alpha_1 \beta_1 + \alpha_2 \beta_2)^2\right), \quad (117)$$

where

$$\beta_i = -\frac{1}{\rho_i}\left(\frac{\partial \rho_i}{\partial T}\right)_p, \quad (118)$$

$$\gamma_i = \frac{1}{\rho_i}\left(\frac{\partial \rho_i}{\partial p}\right)_T. \quad (119)$$

Proposition 7. *The expression* (114) *is equivalent to* (117) *when* $N = 2$.

Proof. Using (100) and (102) to substitute for β_i and γ_i in (117), we recover

$$\tilde{c}^{-2} = \hat{c}^{-2} + \frac{\rho}{T}\frac{C_{p,1} C_{p,2} (\zeta_2 - \zeta_1)^2}{C_{p,1} + C_{p,2}} \quad (120)$$

which corresponds to (114) for $N = 2$.

$\square$

Furthermore, Abgrall [1] derived the general result

$$\tilde{c}^2 = \sum_{i=1}^{N} Y_i \mathcal{P}_i + \frac{\rho e + p}{\rho}\mathcal{P}_\epsilon, \quad (121)$$

where the parameters $\mathcal{P}$ are defined through

$$\mathrm{d}p = \sum_{i=1}^{N} \mathcal{P}_i \, \mathrm{d}(\rho_i \alpha_i) + \mathcal{P}_\epsilon \, \mathrm{d}(\rho e) \quad (122)$$

with

$$e = \sum_{i=1}^{N} Y_i e_i. \quad (123)$$

In Section 3.3, we will show that the expression (114) can be written in the form (121) for our model.

3.2.4. *The Subcharacteristic Condition.* Although related formulations of the mixture sound velocity $\tilde{c}$ already exist in the literature, the particular formulation (114) we have obtained in this paper will now prove useful. In particular, it straightforwardly leads to the following result:

Proposition 8. *Assume that the relaxation sonic velocity $\hat{c}$ given by* (89) *is real and non-zero, i.e. $\hat{c}^2 > 0$. Then the relaxed system of Section 2.2 satisfies the subcharacteristic condition given by Definition 1, with respect to the relaxation system of Section 2.1, subject only to the condition*

$$C_{p,i} > 0 \qquad \forall i, \quad (124)$$

which is assured by thermodynamic stability theory.

Proof. We observe that the difference

$$\tilde{c}^{-2} - \hat{c}^{-2} = \rho \left(T \sum_{i=1}^{N} C_{p,i} \right)^{-1} \sum_{j>i} C_{p,i} C_{p,j} (\zeta_j - \zeta_i)^2 \tag{125}$$

is strictly non-negative under the condition (124). Hence

$$\tilde{c} \le \hat{c}, \tag{126}$$

and the equality holds only if all ζ_i are equal. Furthermore, in the context of Definition 1, we have that $M = 2N + 1$ and $m = N + 2$, and we assume that $N \ge 2$ as stated in Remark 2. The eigenvalues are given by

$$\lambda_1 = v - \hat{c}, \tag{127}$$
$$\lambda_2, \ldots, \lambda_{2N} = v, \tag{128}$$
$$\lambda_{2N+1} = v + \hat{c} \tag{129}$$

and

$$\tilde{\lambda}_1 = v - \tilde{c}, \tag{130}$$
$$\tilde{\lambda}_2, \ldots, \tilde{\lambda}_{N+1} = v, \tag{131}$$
$$\tilde{\lambda}_{N+2} = v + \tilde{c}. \tag{132}$$

The interlacing condition of Definition 1 becomes

$$\tilde{\lambda}_j \in [\lambda_j, \lambda_{j+N-1}] \quad \forall j, \tag{133}$$

which by inspection of (127)–(132) yields the following conditions:

$$v - \tilde{c} \in [v - \hat{c}, v], \tag{134}$$
$$v \in [v, v], \tag{135}$$
$$v + \tilde{c} \in [v, v + \hat{c}], \tag{136}$$

which by (126) are all satisfied. $\qquad \square$

3.3. **Quasilinear Formulation.** In this section, we derive an explicit quasilinear formulation of the relaxed system described in Section 2.2. More precisely, we express the system in the form

$$\frac{\partial \boldsymbol{V}}{\partial t} + \mathcal{A}(\boldsymbol{V}) \frac{\partial \boldsymbol{V}}{\partial x} = 0, \tag{137}$$

where

$$\mathcal{A}(\boldsymbol{V}) = \frac{\partial \mathcal{F}(\boldsymbol{V})}{\partial \boldsymbol{V}}. \tag{138}$$

In addition to facilitating further analysis, such a formulation provides advantages when devising numerical methods for the model. An application of this has already been presented in [14].

3.3.1. *Some Intermediate Results.* We will start by deriving some intermediate differentials that will prove useful for our further analysis.

Lemma 15. *The internal-energy differentials satisfy*

$$\sum_{i=1}^{N} V_i \, \mathrm{d}e_i = \left(p - T \frac{\sum_{i=1}^{N} C_{p,i}}{\sum_{i=1}^{N} \zeta_i C_{p,i}} \right) \sum_{i=1}^{N} \frac{\mathrm{d}V_i}{\rho_i} + \left(\frac{T}{\rho \hat{c}^2} \frac{\sum_{i=1}^{N} C_{p,i}}{\sum_{i=1}^{N} \zeta_i C_{p,i}} + \frac{\sum_{j>i} C_{p,i} C_{p,j} (\zeta_j - \zeta_i)^2}{\sum_{i=1}^{N} \zeta_i C_{p,i}} \right) \mathrm{d}p. \tag{139}$$

Proof. Use Lemmas 11 and 12, as well as the definition (89), to obtain

$$\sum_{i=1}^{N} V_i \, \mathrm{d}e_i = \sum_{i=1}^{N} \left(C_{p,i} \left(1 - \zeta_i \frac{p}{T} \right) \right) \mathrm{d}T + \left(\frac{p}{\rho \hat{c}^2} - \sum_{i=1}^{N} \zeta_i C_{p,i} \left(1 - \zeta_i \frac{p}{T} \right) \right) \mathrm{d}p. \tag{140}$$

Use (110) to eliminate $\mathrm{d}T$ from (140), and simplify by use of Lemma 14. $\qquad \square$

Lemma 16. *The total specific internal-energy differential may be expressed as*

$$\mathrm{d}(\rho e) = \sum_{i=1}^{N} \left(e_i + \frac{p}{\rho_i} \right) \mathrm{d}V_i - T \frac{\sum_{i=1}^{N} C_{p,i}}{\sum_{i=1}^{N} \zeta_i C_{p,i}} \sum_{i=1}^{N} \frac{\mathrm{d}V_i}{\rho_i} + \frac{T}{\rho \tilde{c}^2} \frac{\sum_{i=1}^{N} C_{p,i}}{\sum_{i=1}^{N} \zeta_i C_{p,i}} \mathrm{d}p. \tag{141}$$

Proof. Use

$$\sum_{i=1}^{N} V_i \, \mathrm{d}e_i = \mathrm{d}(\rho e) - \sum_{i=1}^{N} e_i \, \mathrm{d}V_i \tag{142}$$

in (139) and simplify using (114). $\square$

Lemma 17. *The pressure differential may be expressed as*

$$\mathrm{d}p = \rho \tilde{c}^2 \sum_{i=1}^{N} \frac{\mathrm{d}V_i}{\rho_i} - \frac{\rho \tilde{c}^2}{T} \frac{\sum_{i=1}^{N} \zeta_i C_{p,i}}{\sum_{i=1}^{N} C_{p,i}} \left(\sum_{i=1}^{N} \left(e_i + \frac{p}{\rho_i} - \frac{1}{2} v^2 \right) \mathrm{d}V_i + v \, \mathrm{d}V_{N+1} - \mathrm{d}V_{N+2} \right). \tag{143}$$

Proof. Use

$$\mathrm{d}(\rho e) = \frac{1}{2} v^2 \sum_{i=1}^{N} \mathrm{d}V_i - v \, \mathrm{d}V_{N+1} + \mathrm{d}V_{N+2} \tag{144}$$

in (141) and solve for $\mathrm{d}p$. $\square$

Lemma 18. *The pressure-transport differential may be expressed as*

$$\begin{aligned}
\mathrm{d}(pv) = {}& v\rho \tilde{c}^2 \sum_{i=1}^{N} \frac{\mathrm{d}V_i}{\rho_i} - v \frac{\rho \tilde{c}^2}{T} \frac{\sum_{i=1}^{N} \zeta_i C_{p,i}}{\sum_{i=1}^{N} C_{p,i}} \sum_{i=1}^{N} \left(e_i + \frac{p}{\rho_i} - \frac{1}{2} v^2 \right) \mathrm{d}V_i - \frac{pv}{\rho} \sum_{i=1}^{N} \mathrm{d}V_i \\
& \left(\frac{p}{\rho} - v^2 \frac{\rho \tilde{c}^2}{T} \frac{\sum_{i=1}^{N} \zeta_i C_{p,i}}{\sum_{i=1}^{N} C_{p,i}} \right) \mathrm{d}V_{N+1} + v \frac{\rho \tilde{c}^2}{T} \frac{\sum_{i=1}^{N} \zeta_i C_{p,i}}{\sum_{i=1}^{N} C_{p,i}} \mathrm{d}V_{N+2}.
\end{aligned} \tag{145}$$

Proof. use

$$\mathrm{d}(pv) = v \, \mathrm{d}p + p \, \mathrm{d}v \tag{146}$$

together with (29) and (40) in (143). $\square$

3.3.2. *The Jacobi Matrix.* We will find it convenient to split the flux vector into convective and pressure terms as follows:

$$\mathcal{F}(\boldsymbol{V}) = \mathcal{F}_\mathrm{c}(\boldsymbol{V}) + \mathcal{F}_p(\boldsymbol{V}), \tag{147}$$

where

$$\mathcal{F}_\mathrm{c}(\boldsymbol{V}) = v\boldsymbol{V} \tag{148}$$

and

$$\mathcal{F}_p(\boldsymbol{V}) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ p \\ pv \end{bmatrix}, \qquad \boldsymbol{V} = \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_N \\ V_{N+1} \\ V_{N+2} \end{bmatrix} = \begin{bmatrix} \rho_1 \alpha_1 \\ \rho_2 \alpha_2 \\ \vdots \\ \rho_N \alpha_N \\ \rho v \\ \rho e + \frac{1}{2} v^2 \end{bmatrix}. \tag{149}$$

Then we may write

$$\mathcal{A}(\boldsymbol{V}) = \mathcal{A}_\mathrm{c}(\boldsymbol{V}) + \mathcal{A}_p(\boldsymbol{V}), \tag{150}$$

where

$$\mathcal{A}_\mathrm{c}(\boldsymbol{V}) = \frac{\partial \mathcal{F}_\mathrm{c}(\boldsymbol{V})}{\partial \boldsymbol{V}} \qquad \text{and} \qquad \mathcal{A}_p(\boldsymbol{V}) = \frac{\partial \mathcal{F}_p(\boldsymbol{V})}{\partial \boldsymbol{V}}. \tag{151}$$

Proposition 9. *The convective Jacobian matrix $\mathcal{A}_c$ can be written as*

$$
\mathcal{A}_c(\boldsymbol{V}) =
\begin{bmatrix}
(1 - Y_1)v & -Y_1 v & \dots & -Y_1 v & Y_1 & 0 \\
-Y_2 v & (1 - Y_2)v & \dots & -Y_2 v & Y_2 & 0 \\
\vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\
-Y_N v & -Y_N v & \dots & (1 - Y_N)v & Y_N & 0 \\
-v^2 & -v^2 & \dots & -v^2 & 2v & 0 \\
-\left(e + \frac{1}{2}v^2\right)v & -\left(e + \frac{1}{2}v^2\right)v & \dots & -\left(e + \frac{1}{2}v^2\right)v & e + \frac{1}{2}v^2 & v
\end{bmatrix},
\tag{152}
$$

where

$$
e = \sum_{i=1}^{N} Y_i e_i.
\tag{153}
$$

Proof. From (148) we obtain

$$
\mathrm{d}\mathcal{F}_c = v\,\mathrm{d}\boldsymbol{V} + \boldsymbol{V}\,\mathrm{d}v,
\tag{154}
$$

which together with (29) and (40) yields the result.

□

Proposition 10. *The pressure Jacobian $\mathcal{A}_p$ can be written as*

$$
\mathcal{A}_p = \mathcal{A}_1 + \frac{\rho \tilde{c}^2}{T} \frac{\sum_{i=1}^{N} \zeta_i C_{p,i}}{\sum_{i=1}^{N} C_{p,i}} \mathcal{A}_2,
\tag{155}
$$

where

$$
\mathcal{A}_1(\boldsymbol{V}) =
\begin{bmatrix}
0 & 0 & \dots & 0 & 0 & 0 \\
0 & 0 & \dots & 0 & 0 & 0 \\
\vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & \dots & 0 & 0 & 0 \\
\frac{\rho \tilde{c}^2}{\rho_1} & \frac{\rho \tilde{c}^2}{\rho_2} & \dots & \frac{\rho \tilde{c}^2}{\rho_N} & 0 & 0 \\
v\left(\frac{\rho \tilde{c}^2}{\rho_1} - \frac{p}{\rho}\right) & v\left(\frac{\rho \tilde{c}^2}{\rho_2} - \frac{p}{\rho}\right) & \dots & v\left(\frac{\rho \tilde{c}^2}{\rho_N} - \frac{p}{\rho}\right) & \frac{p}{\rho} & 0
\end{bmatrix}
\tag{156}
$$

and

$$
\mathcal{A}_2(\boldsymbol{V}) =
\begin{bmatrix}
0 & 0 & \dots & 0 & 0 & 0 \\
0 & 0 & \dots & 0 & 0 & 0 \\
\vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & \dots & 0 & 0 & 0 \\
\frac{1}{2}v^2 - e_1 - \frac{p}{\rho_1} & \frac{1}{2}v^2 - e_2 - \frac{p}{\rho_2} & \dots & \frac{1}{2}v^2 - e_N - \frac{p}{\rho_N} & -v & 1 \\
v\left(\frac{1}{2}v^2 - e_1 - \frac{p}{\rho_1}\right) & v\left(\frac{1}{2}v^2 - e_2 - \frac{p}{\rho_2}\right) & \dots & v\left(\frac{1}{2}v^2 - e_N - \frac{p}{\rho_N}\right) & -v^2 & v
\end{bmatrix}.
\tag{157}
$$

Proof. The result follows directly from Lemmas 17 and 18 applied to (149). □

By the above calculations, it follows that the relaxed system of Section 2.2 can be written in the form (137), with

$$
\mathcal{A} = \mathcal{A}_c + \mathcal{A}_1 + \frac{\rho \tilde{c}^2}{T} \frac{\sum_{i=1}^{N} \zeta_i C_{p,i}}{\sum_{i=1}^{N} C_{p,i}} \mathcal{A}_2,
\tag{158}
$$

where $\mathcal{A}_c$, $\mathcal{A}_1$ and $\mathcal{A}_2$ are given by (152), (156) and (157).

We are now in position to prove the following:

Proposition 11. *The mixture sound velocity $\tilde{c}$, given by (114), satisfies Abgrall's formula (121).*

Proof. From Lemma 17 it follows that

$$
\mathcal{P}_i = \frac{\rho \tilde{c}^2}{\rho_i} - \frac{\rho \tilde{c}^2}{T} \frac{\sum_{i=1}^{N} \zeta_i C_{p,i}}{\sum_{i=1}^{N} C_{p,i}} \left(e_i + \frac{p}{\rho_i}\right)
\tag{159}
$$

and

$$\mathcal{P}_\epsilon = \frac{\rho \tilde{c}^2}{T} \frac{\sum_{i=1}^{N} \zeta_i C_{p,i}}{\sum_{i=1}^{N} C_{p,i}} \tag{160}$$

in the context of (122). By this, (121) simplifies to the trivial identity

$$\tilde{c}^2 = \sum_{i=1}^{N} Y_i \frac{\rho \tilde{c}^2}{\rho_i} = \sum_{i=1}^{N} \alpha_i \tilde{c}^2 = \tilde{c}^2. \tag{161}$$

$\square$

4. Summary

We have studied a *relaxation system* modelling the flow of an arbitrary number of immiscible fluids. The fluids are assumed to flow with the same velocities and to be in mechanical equilibrium, i.e., to have the same pressure. Thermal equilibrium is not assumed, instead heat transfer has been modelled by a relaxation procedure. The relaxation procedure has been carefully chosen to respect the first and second law of thermodynamics. In this respect, we have extended upon previous works [12, 21], that considered the special case of two separate fluids.

Furthermore, we have studied the *relaxed* limit where thermal equilibrium is instantaneously imposed. This relaxed limit is sometimes referred to as the *homogeneous equilibrium model*. We have derived a formulation of the mixture sound velocity of this relaxed model, from which it is straightforward to see that the relaxed system unconditionally satisfies the *subcharacteristic condition*. The physical interpretation of this result is that the instantaneous equilibrium condition imposes a slower mixture sound velocity compared to the non-equilibrium case. Although this result may be obtained by other means, the proof presented in this paper seems original, and provides insights into the effects of relevant thermodynamic parameters on sonic propagation.

Acknowledgements

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