A blow-up criterion in terms of the density for compressible viscous flows [*]

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Abstract

We study an initial boundary value problem for the two-dimensional Navier-Stokes equations of compressible fluids in the unit square domain. We establish a blow-up criterion for the local strong solutions in terms of the density only. Namely, if the density is away from vacuum ($\rho = 0$) and the concentration of mass ($\rho = \infty$), then a local strong solution can be continued globally in time.

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1 Introduction

This paper is concerned with a blow-up criterion for the two-dimensional Navier-Stokes equations of compressible isentropic flows which describe the conservation of mass and momentum, and can be written in the following form:

$$\rho_t + \mathrm{div}\,(\rho\mathbf{u}) = 0, \tag{1.1}$$

$$\rho(\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u}) = \mu\Delta\mathbf{u} + (\lambda+\mu)\nabla\mathrm{div}\,\mathbf{u} - \nabla P, \tag{1.2}$$

where $\mathbf{u} = (u, v) \in \mathbb{R}^2$ and $\rho \in \mathbb{R}_+$ denote the velocity and density, respectively. The physical constants μ, λ are the viscosity coefficients satisfying $\mu > 0$, $\lambda + \mu \geq 0$, P is the pressure having the following form in the isentropic flow case:

$$P = a\rho^\gamma, \tag{1.3}$$

where $\gamma \geq 1$ is the specific heat ratio and $a > 0$ is a gas constant.

For the sake of simplicity, we will consider an initial boundary value problem for (1.1), (1.2) in the domain $\Omega := \{(x, y) \in \mathbb{R}^2;\ 0 < x, y < 1\}$ with initial data

$$(\rho, \mathbf{u})|_{t=0} = (\rho_0, \mathbf{u}_0) \ \text{ in } \ \Omega \tag{1.4}$$

and boundary conditions

$$\begin{aligned} &u|_{x=0} = u|_{x=1} = 0, \quad u_y|_{y=0} = u_y|_{y=1} = 0, \\ &v|_{y=0} = v|_{y=1} = 0, \quad v_x|_{x=0} = v_x|_{x=1} = 0. \end{aligned} \tag{1.5}$$

In the last decades mathematical aspects for the compressible Navier-Stokes equations have been extensively studied and significant progress has been made in the study of global in time existence for the system (1.1)–(1.3). Assuming that the initial data are sufficiently small, Matsumura and Nishida [15, 16] first proved the global existence of smooth solutions to initial boundary value problems and the Cauchy problem for (1.1)–(1.3), and the existence of global weak solutions was shown by Hoff [9]. For large data, however, it is still an open question whether a global smooth solution to (1.1)–(1.3) exists or not, except certain special cases, such as the spherically symmetric case in domains without the origin, see [11] for example. Concerning weak solutions to the multidimensional compressible Navier-Stokes equations, the existence of global weak solutions in the isentropic flow case was first shown by Lions [14], and his result was then improved and generalized in [6, 12, 13], and among others. We also mention that for non-isentropic flows, Feireisl [7, 8] recently obtained the global existence of the so-called "variational solution" in the case of real gases in the sense that the energy equation is replaced by an energy inequality. However, this result excludes the case of ideal gases unfortunately.

Xin [20], Rozanova [17] showed the non-existence of global smooth solutions when the initial density is compactly supported, or the initial mass is bounded and solutions decay to zero sufficiently fast. Since the system (1.1)–(1.3) is a model of non-dilute fluids, these non-existence results are natural to expect when vacuum regions are present initially. Thus, it is very interesting to investigate whether a strong or smooth solution will still blow up in finite time, when there is no vacuum initially.

Fan and Jiang [4] proved the following blow-up criterion for the local strong solutions to (1.1)–(1.5) in the case of two dimensions:

$$\lim_{T\to T_*}\Big(\sup_{0\le t\le T}\|\rho\|_{L^\infty}+\int_0^T\big(\|\rho\|_{W^{1,q_0}}+\|\nabla\rho\|_{L^2}^4\big)\Big)=\infty,\quad \text{provided}\ \ 7\mu>9\lambda. \tag{1.6}$$

The condition (1.6) shows that if ρ is regular, the smoothness of u can then be guaranteed, and therefore, a strong solution $(\rho,\mathbf{u})$ can exist for all time.

Recently, Huang and Xin [10] established the following blow-up criterion, similar to the Beale-Kato-Majda criterion for ideal incompressible flows [1], for the isentropic compressible Navier-Stokes equations:

$$\lim_{t\to T_*}\int_0^T\|\nabla\mathbf{u}\|_{L^\infty}dt=\infty, \tag{1.7}$$

provided

$$7\mu>\lambda. \tag{1.8}$$

Very recently, Huang and Xin's result [10] for isentropic flows has been extended to non-isentropic flows [5].

In view of the non-existence results of global smooth solutions due to Xin [20] and Rozanova [17], we think that presence of vacuum could be the reason that prevents a local smooth solution from extending to any time. In other words, if the density is away from vacuum ($\rho=0$) and the concentration of mass ($\rho=\infty$), then a strong solution should exist globally in time. The aim of the current paper is to prove this assertion. More precisely, a global strong solution to (1.1)–(1.5) exists provided that the density is pointwise bounded from below and above.

Before giving our main result, we state the following local existence of the strong solutions with initial vacuum, the proof of which can be found in [3].

Proposition 1.1 (Local Existence) *Assume that the initial data ρ_0,u_0,θ_0 satisfy*

$$\rho_0\ge 0,\quad \rho_0\in W^{1,q}(\Omega)\ \textit{for some}\ 2<q\le 6,\quad \mathbf{u}_0\in H_0^1(\Omega)\cap H^2(\Omega), \tag{1.9}$$

and the compatibility condition

$$-\mu\Delta\mathbf{u}_0-(\mu+\lambda)\nabla\mathrm{div}\mathbf{u}_0+\nabla P(\rho_0)=\rho_0^{1/2}g\quad \textit{for some}\ \ g\in L^2(\Omega). \tag{1.10}$$

Then, there exist a $T_*>0$ and a unique strong solution $(\rho,\mathbf{u})$ to (1.1)–(1.5), such that

$$\begin{aligned}&\rho\ge 0,\ \ \rho\in C([0,T_*],W^{1,q}),\ \ \rho_t\in C([0,T_*],L^q),\\ &\mathbf{u}\in C([0,T_*],H_0^1\cap H^2)\cap L^2(0,T_*;W^{2,q}),\ \ \mathbf{u}_t\in L^\infty(0,T_*;L^2)\cap L^2(0,T_*;H_0^1).\end{aligned} \tag{1.11}$$

Remark 1.1 *The local existence of strong solutions is shown for the three-dimensional non-isentropic case in [3], where $q > 3$ is required (because of $W^{1,q} \hookrightarrow L^\infty$ in $\mathbb{R}^3$). Here for our two-dimensional isentropic case, $q > 2$ is sufficient.*

Now, we are in a position to state the main result of this paper.

Theorem 1.1 (Blow-up criterion) *Let $\rho_0 \in W^{1,q}$ $(q > 2)$, $\mathbf{u}_0 \in H^2$, and $\bar{m} \le \rho_0 \le \bar{M}$ for some positive constants $\bar{m}, \bar{M}$. Also assume that the initial data $\rho_0, \mathbf{u}_0$ are compatible with the boundary conditions (1.5). If $T^* < \infty$ is the maximal time of existence, then*

$$\lim_{T \to T^*} \Big(\|\rho\|_{L^\infty((0,T)\times\Omega)} + \|\rho^{-1}\|_{L^\infty((0,T)\times\Omega)} \Big) = \infty.$$

We will prove Theorem 1.1 by contradiction in the next section. In fact, the proof of the theorem is based on a priori estimates under the assumption that $\|\rho\|_{L^\infty((0,T)\times\Omega)} + \|\rho^{-1}\|_{L^\infty((0,T)\times\Omega)}$ is bounded for any $T \in [0, T^*)$. The a priori estimates are then sufficient for us to apply the local existence theorem to extend a local solution beyond the maximal time of existence T^*, consequently, contradicting to the assumption of boundedness of $\|\rho\|_{L^\infty((0,T)\times\Omega)} + \|\rho^{-1}\|_{L^\infty((0,T)\times\Omega)}$.

The key step in getting the a priori estimates is to bound higher order derivatives of $(\rho, \mathbf{u})$. The boundary conditions (1.5) could induce some difficulties, if we try to bound the higher order derivatives of $(\rho, \mathbf{u})$ directly. Instead, we will adapt and modify the arguments in [18] to first bound the vorticity $\mathrm{curl}\,\mathbf{u}$ and the viscous flux $(2\mu + \lambda)\mathrm{div}\,\mathbf{u} - P$ (cf. [14]) as well as their first-order derivative, since the vorticity and the viscous flux satisfy the evolution equations (2.3) and (2.4) with the classical Dirichlet and Neumann boundary conditions (2.7) that are easier to handle (than (1.5)). Then, with the help of the estimates for the vorticity and the viscous flux, we can bound the higher order derivatives of $(\rho, \mathbf{u})$.

Throughout this paper, we will use the following abbreviations:

$$L^p \equiv L^p(\Omega), \quad H^m \equiv H^m(\Omega), \quad H_0^m \equiv H_0^m(\Omega).$$

2 Proof of Theorem 1.1

Let $0 < T < T^*$ be arbitrary but fixed. Throughout this section we denote by C (or $C(X, \cdots)$ to emphasize the dependence of C on $X, \cdots$) a general positive constant which may depend continuously on T. Let $(\rho, \mathbf{u})$ be a strong solution to the problem (1.1)–(1.5) in the function space given in (1.11) on the time interval $[0, T]$.

We will prove Theorem 1.1 by a contradiction argument. To this end, we suppose that

$$\|\rho\|_{L^\infty((0,T)\times\Omega)} + \|\rho^{-1}\|_{L^\infty((0,T)\times\Omega)} \le C \quad \text{for any } T < T^*, \tag{2.1}$$

we will deduce a contradiction to the maximality of T^*.

First, we show the standard energy estimate. In fact, denoting

$$h(\rho) := \begin{cases} a\Big(\dfrac{\rho^\gamma - \rho}{\gamma - 1} - \rho + 1\Big), & \gamma > 1, \\ a(\rho \log \rho - \rho + 1), & \gamma = 1, \end{cases}$$

we multiply (1.2) by $\mathbf{u}$ in $L^2(\Omega)$, integrate by parts, use (1.5) and (1.1) to deduce

$$\frac{d}{dt}\int_\Omega \Big[\frac{\rho|\mathbf{u}|^2}{2} + h(\rho)\Big]dx + \int_\Omega \Big(\mu|\nabla \mathbf{u}|^2 + (\mu+\lambda)|\mathrm{div}\,\mathbf{u}|^2\Big)dx = 0.$$

Integration of the above identity with respect to t and use of Poincaré's inequality give

$$\|\mathbf{u}\|_{L^\infty(0,T;L^2)} + \|\mathbf{u}\|_{L^2(0,T;H^1)} \le C. \tag{2.2}$$

Without loss of generality, we may assume $\mu = 1$ in the calculations that follow. Next, we will adapt and modify the arguments due to Vagaint and Kazhikhov [18] to derive a priori estimates on the vorticity and the viscous flux under the assumption (2.1). Introducing

$$A := u_y - v_x = -\mathrm{curl}\,\mathbf{u}, \quad B := (2+\lambda)\mathrm{div}\,\mathbf{u} - P,$$
$$L := \frac{1}{\rho}(A_y + B_x), \quad H := \frac{1}{\rho}(-A_x + B_y),$$

and recalling $\mu = 1$ as well as the equations (1.1) and (1.2), we obtain, after a straightforward calculation, the following equations (see [18, (14)–(15)]):

$$A_t + \mathbf{u}\cdot\nabla A + A\mathrm{div}\,\mathbf{u} = L_y - H_x, \tag{2.3}$$

$$B_t + \mathbf{u}\cdot\nabla B - \rho P'(\rho)\mathrm{div}\,\mathbf{u} + (2+\lambda)(u_x^2 + 2v_x u_y + v_y^2) = (2+\lambda)(L_x + H_y), \tag{2.4}$$

$$\begin{aligned}&\rho(L_t + \mathbf{u}\cdot\nabla L) - \rho L\,\mathrm{div}\,\mathbf{u} + \mathbf{u}_y\cdot\nabla A + \mathbf{u}_x\cdot\nabla B + \big[A\,\mathrm{div}\,\mathbf{u}\big]_y - \big[\rho P'(\rho)\mathrm{div}\,\mathbf{u}\big]_x \\ &\quad +(2+\lambda)(u_x^2 + v_y^2 + 2v_x u_y)_x = (L_y - H_x)_y + (2+\lambda)(L_x + H_y)_x,\end{aligned} \tag{2.5}$$

$$\begin{aligned}&\rho(H_t + \mathbf{u}\nabla H) - \rho H\,\mathrm{div}\,\mathbf{u} - \mathbf{u}_x\cdot\nabla A + \mathbf{u}_y\cdot\nabla B - \big[A\,\mathrm{div}\,\mathbf{u}\big]_x - \big[\rho P'(\rho)\mathrm{div}\,\mathbf{u}\big]_y \\ &\quad +(2+\lambda)(u_x^2 + v_y^2 + 2v_x u_y)_y = -(L_y - H_x)_x + (2+\lambda)(L_x + H_y)_y,\end{aligned} \tag{2.6}$$

with the following boundary conditions:

$$\begin{aligned}&A|_{\partial\Omega} = 0, \quad B_x|_{x=0} = B_x|_{x=1} = B_y|_{y=0} = B_y|_{y=1} = 0, \\ &L|_{x=0} = L|_{x=1} = 0, \quad L_y|_{y=0} = L_y|_{y=1} = 0, \\ &H|_{y=0} = H|_{y=1} = 0, \quad H_x|_{x=0} = H_x|_{x=1} = 0.\end{aligned} \tag{2.7}$$

If we multiply (2.3) and (2.4) by A and $(2+\lambda)^{-1}B$ in $L^2(\Omega)$ respectively, integrate by parts and make use of (2.7), we get

$$\begin{aligned}
&\int_\Omega A\big(A_t + \mathbf{u}\cdot\nabla A + A\,\mathrm{div}\,\mathbf{u}\big)dxdy + \int_\Omega (LA_y - HA_x)dxdy\\
&+\frac{1}{2+\lambda}\int_\Omega B(B_t + \mathbf{u}\cdot\nabla B)dxdy - \frac{1}{2+\lambda}\int_\Omega \rho BP'(\rho)\mathrm{div}\,\mathbf{u}\,dxdy\\
&+\int_\Omega B(u_x^2 + 2v_xu_y + v_y^2)dxdy + \int_\Omega (LB_x + HB_y)dxdy = 0,
\end{aligned}$$

which, by recalling the definition of (L,H) and employing the boundary conditions for (u,v), yields

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega \Big(A^2 + \frac{B^2}{2+\lambda}\Big)dxdy + \int_\Omega \frac{(A_y+B_x)^2 + (-A_x+B_y)^2}{\rho}dxdy\\
&+\frac{1}{2}\int_\Omega A^2\mathrm{div}\,\mathbf{u}\,dxdy - \frac{1}{2(2+\lambda)}\int_\Omega B^2\,\mathrm{div}\,\mathbf{u}\,dxdy - \frac{1}{2+\lambda}\int_\Omega \rho BP'(\rho)\mathrm{div}\,\mathbf{u}\,dxdy\\
&+\int_\Omega B\,|\mathrm{div}\,\mathbf{u}|^2dxdy + 2\int_\Omega B(u_yv_x - u_xv_y)dxdy = 0. \qquad (2.8)
\end{aligned}$$

Recalling that $B := (2+\lambda)\mathrm{div}\,\mathbf{u} - P$, we easily find

$$|\mathrm{div}\,\mathbf{u}|^2 = \frac{1}{2+\lambda}\mathrm{div}\,\mathbf{u}\,(B+P).$$

Hence, the identity (2.8) turns to

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega \Big(A^2 + \frac{B^2}{2+\lambda}\Big)dxdy + \int_\Omega \frac{(A_y+B_x)^2 + (-A_x+B_y)^2}{\rho}dxdy\\
&= -\frac{1}{2}\int_\Omega A^2\,\mathrm{div}\,\mathbf{u}\,dxdy + \frac{1}{2(2+\lambda)}\int_\Omega B^2\mathrm{div}\,\mathbf{u}\,dxdy\\
&\quad+\frac{1}{2+\lambda}\int_\Omega \big(\rho P' - P(\rho)\big)B\,\mathrm{div}\,\mathbf{u}\,dxdy - 2\int_\Omega B(u_yv_x - u_xv_y)dxdy\\
&\equiv I_1 + I_2 + I_3 + I_4. \qquad (2.9)
\end{aligned}$$

We have to estimate each term in (2.9). First, the second term on the left-hand side of (2.9) can be bounded as follows, using (2.1), integration by parts and the boundary conditions (2.7).

$$\begin{aligned}
I_5 :&= \int_\Omega \frac{(A_y+B_x)^2 + (-A_x+B_y)^2}{\rho}dxdy\\
&\geq C\int_\Omega \Big((A_y+B_x)^2 + (-A_x+B_y)^2\Big)dxdy\\
&= C\int_\Omega (A_y^2 + A_x^2 + B_x^2 + B_y^2)dxdy. \qquad (2.10)
\end{aligned}$$

Keeping in mind that

$$\operatorname{div}\mathbf{u} = \frac{1}{2+\lambda}(B+P), \qquad \operatorname{curl}\mathbf{u} = -A,$$

we have (see [2, 19])

$$\begin{aligned}\|\nabla u\|_{L^p} + \|\nabla v\|_{L^p} &\le C\|\operatorname{div}\mathbf{u}\|_{L^p} + \|\operatorname{curl}\mathbf{u}\|_{L^p}\\ &\le C(\|A\|_{L^p} + \|B\|_{L^p} + \|P\|_{L^p}).\end{aligned} \tag{2.11}$$

Obviously,

$$\|A\|_{L^2}^2 + \|B\|_{L^2}^2 \le C(\|\nabla u\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 + \|P\|_{L^2}^2). \tag{2.12}$$

Thus, we use Cauchy-Schwarz's inequality, (2.1), and (2.11) with $p = 2$, (2.12) and Gagliardo-Nirenberg's inequality in two dimensions ($\|\cdot\|_{L^4}^2 \le C\|\cdot\|_{L^2}\|\nabla\cdot\|_{L^2}$) to bound I_1 as follows.

$$\begin{aligned}I_1 &\le C\|\operatorname{div}\mathbf{u}\|_{L^2}\|A\|_{L^4}^2\\ &\le C(1 + \|A\|_{L^2} + \|B\|_{L^2})\|A\|_{L^2}\|\nabla A\|_{L^2}\\ &\le C\epsilon^{-1}(1 + \|A\|_{L^2}^4 + \|B\|_{L^2}^4) + \epsilon\|\nabla A\|_{L^2}^2, \qquad 0 < \epsilon < 1.\end{aligned} \tag{2.13}$$

In the same manner, we can obtain

$$\begin{aligned}I_2 &\le C\|\operatorname{div}\mathbf{u}\|_{L^2}\|B\|_{L^4}^2\\ &\le C\epsilon^{-1}(1 + \|A\|_{L^2}^4 + \|B\|_{L^2}^4) + \epsilon\|\nabla B\|_{L^2}^2\end{aligned} \tag{2.14}$$

and

$$I_3 \le C\|\operatorname{div}\mathbf{u}\|_{L^2}\|B\|_{L^2}^2 \le C(1 + \|A\|_{L^2}^4 + \|B\|_{L^2}^4). \tag{2.15}$$

The estimate of I_4 is more involved but in a similar way, we have

$$\begin{aligned}I_4 &= -2\int_\Omega B(v_x u_y - v_y u_x)\,dxdy\\ &\le 2\|B\|_{L^2}\|v_x u_y - v_y u_x\|_{L^2}\\ &\le C\|B\|_{L^2}\|\nabla\mathbf{u}\|_{L^4}^2\\ &\le C\|B\|_{L^2}\big(\|\operatorname{div}\mathbf{u}\|_{L^4}^2 + \|\operatorname{curl}\mathbf{u}\|_{L^4}^2\big)\\ &\le C\|B\|_{L^2}\big(\|B+P\|_{L^4}^2 + \|A\|_{L^4}^2\big)\\ &\le C\|B\|_{L^2}\big(1 + \|A\|_{L^4}^2 + \|B\|_{L^4}^2\big)\\ &\le C\|B\|_{L^2}\big(1 + \|A\|_{L^2}\|\nabla A\|_{L^2} + \|B\|_{L^2}^2 + \|B\|_{L^2}\|\nabla B\|_{L^2}\big)\\ &\le C\epsilon^{-1}(1 + \|A\|_{L^2}^4 + \|B\|_{L^2}^4) + \epsilon\|\nabla A\|_{L^2}^2 + \epsilon\|\nabla B\|_{L^2}^2, \qquad 0 < \epsilon < 1.\end{aligned} \tag{2.16}$$

Notice that from (2.1) and (2.2) it follows that

$$\int_0^t (\|A\|_{L^2}^2 + \|B\|_{L^2}^2)(t)\,dt \le C. \tag{2.17}$$

Therefore, substituting (2.10)–(2.16) into (2.9), taking ϵ appropriately small, applying Gronwall's inequality and using (2.17), we conclude

Lemma 2.1

$$\begin{aligned}&\|(A,B)\|_{L^\infty(0,T;L^2)}+\|(A,B)\|_{L^2(0,T;H^1)}\le C,\\&\|\mathbf{u}\|_{L^\infty(0,T;H^1)}\le C.\end{aligned}$$

Next, we derive bounds on L and H by careful calculations. Multiplying the equations (2.5) and (2.6) by L and H in $L^2(\Omega)$, respectively, integrating by parts, using (2.7) and (1.1), we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega \rho(L^2+H^2)dxdy+\int_\Omega\Big((L_y-H_x)^2+(2+\lambda)(L_x+H_y)^2\Big)dxdy\\
&=\int_\Omega \rho(L^2+H^2)\mathrm{div}\,\mathbf{u}\,dxdy-\int_\Omega A(H_x-L_y)\mathrm{div}\,\mathbf{u}\,dxdy\\
&\quad-\int_\Omega \rho P'(\rho)(L_x+H_y)\mathrm{div}\,\mathbf{u}\,dxdy-\int_\Omega L(\mathbf{u}_y\cdot\nabla A+\mathbf{u}_x\cdot\nabla B)dxdy\\
&\quad-\int_\Omega H(-\mathbf{u}_x\cdot\nabla A+\mathbf{u}_y\nabla B)dxdy-(2+\lambda)\int_\Omega(L_x+H_y)(u_x^2+2u_yv_x+v_y^2)dxdy\\
&\le C\|\mathrm{div}\,\mathbf{u}\|_{L^2}\|(L,H)\|_{L^4}^2+C\|A\|_{L^4}\|\mathrm{div}\,\mathbf{u}\|_{L^4}\|(H_x,L_y)\|_{L^2}+C\|\mathrm{div}\,\mathbf{u}\|_{L^2}\|L_x+H_y\|_{L^2}\\
&\quad+C\|L\|_{L^4}(\|u_y\|_{L^4}\|A_x\|_{L^2}+\|v_y\|_{L^4}\|A_y\|_{L^2}+\|u_x\|_{L^4}\|B_x\|_{L^2}+\|v_x\|_{L^4}\|B_y\|_{L^2})\\
&\quad+C\|H\|_{L^4}(\|u_x\|_{L^4}\|A_x\|_{L^2}+\|v_x\|_{L^4}\|A_y\|_{L^2}+\|u_y\|_{L^4}\|B_x\|_{L^2}+\|v_y\|_{L^4}\|B_y\|_{L^2})\\
&\quad+C\|L_x+H_y\|_{L^2}(\|u_x\|_{L^4}^2+\|v_x\|_{L^4}\|u_y\|_{L^4}+\|v_y\|_{L^4}^2)\\
&\le C(1+\|B\|_{L^2})\|(L,H)\|_{L^4}^2+C(1+\|B\|_{L^4})\|A\|_{L^4}\|(\nabla L,\nabla H)\|_{L^2}\\
&\quad+C(1+\|B\|_{L^2}+\|\nabla\mathbf{u}\|_{L^4}^2)\|(\nabla L,\nabla H)\|_{L^2}+C\|(L,H)\|_{L^4}\|\nabla\mathbf{u}\|_{L^4}\|(\nabla A,\nabla B)\|_{L^2}\\
&\le C\|(L,H)\|_{L^4}^2+C\|(A,B)\|_{L^4}^2\|(\nabla L,\nabla H)\|_{L^2}+\frac{1}{16}\|(\nabla L,\nabla H)\|_{L^2}^2\\
&\quad+C\|(L,H)\|_{L^4}\|(A,B)\|_{L^4}\|(\nabla A,\nabla B)\|_{L^2}+C\|(A,B)\|_{L^4}^2\|(\nabla L,\nabla H)\|_{L^2}\\
&\le C\|(L,H)\|_{L^2}^2+\frac{1}{8}\|(\nabla L,\nabla H)\|_{L^2}^2+C\|(A,B)\|_{L^4}^4+C\|(\nabla A,\nabla B)\|_{L^2}^2,
\end{aligned}$$

which, by applying Gronwall's inequality, yields

Lemma 2.2

$$\|(L,H)\|_{L^\infty(0,T;L^2)}+\|(L,H)\|_{L^2(0,T;H^1)}\le C.$$

Finally, we estimate the spatial and temporal derivatives of the density in $L^q(\Omega)$ space for $q>2$. More precisely, we have

Lemma 2.3

$$\|\nabla\rho\|_{L^\infty(0,T;L^q)}\le C,$$

where $q>2$ is the same as in Theorem 1.1.

Proof. We differentiate (2.1) with ∂_i, multiply the resulting equation by $|\partial_i\rho|^{q-2}\partial\rho$ $(q>2)$ in $L^2(\Omega)$, and make use of Hölder's inequality to obtain

$$\frac{1}{q}\frac{d}{dt}\int_\Omega |\nabla\rho|^q dxdy \le C\|\nabla\mathbf{u}\|_{L^\infty}\|\nabla\rho\|_{L^q}^q + C\|\Delta\mathbf{u}\|_{L^q}\|\nabla\rho\|_{L^q}^{q-1},$$

whence,

$$\frac{d}{dt}\|\nabla\rho\|_{L^q} \le C\|\nabla\mathbf{u}\|_{L^\infty}\|\nabla\rho\|_{L^q} + C\|\Delta\mathbf{u}\|_{L^q}. \tag{2.18}$$

To control the term ΔU on the right-hand of (2.18), we first note that A and B satisfy

$$\begin{cases} A_y + B_x = \rho L, \\ -A_x + B_y = \rho H \end{cases} \tag{2.19}$$

with boundary conditions $(2.7)_1$. Thus, we can apply the regularity result in [18] to the problem (2.19) to obtain

$$\|\nabla A\|_{L^q} + \|\nabla B\|_{L^q} \le C(\|L\|_{L^q} + \|H\|_{L^q}). \tag{2.20}$$

Recalling the definition of (A, B, L, H) and the inequality $(2.11)_1$, employing (2.1), we find

$$\|\Delta\mathbf{u}\|_{L^q} \le C(\|\nabla A\|_{L^q} + \|\nabla B\|_{L^q} + \|\nabla\rho\|_{L^q}). \tag{2.21}$$

Now, if we define

$$I(t) := \sup_\Omega |\nabla\mathbf{u}(\cdot,t)|, \qquad M(t) := 1 + \sup_\Omega |\operatorname{div}\mathbf{u}(\cdot,t)| + \sup_\Omega |\operatorname{curl}\mathbf{u}(\cdot,t)|,$$

then we have by an inequality from [18, 21] that

$$\begin{aligned} I(t) &\le CM(t)\log(e + \|D^2\mathbf{u}\|_{L^q}) \\ &\le CM(t)\log(e + \|\nabla A\|_{L^q} + \|\nabla B\|_{L^q} + \|\nabla\rho\|_{L^q}) \\ &\le CM(t)(\|\nabla A\|_{L^q} + \|\nabla B\|_{L^q})\log(e + \|\nabla\rho\|_{L^q}) \\ &\le C(\|A\|_{L^\infty} + \|B\|_{L^\infty} + 1)(\|\nabla A\|_{L^q} + \|\nabla B\|_{L^q})\log(e + \|\nabla\rho\|_{L^q}) \\ &\le C(\|\nabla A\|_{L^q} + \|\nabla B\|_{L^q})^2\log(e + \|\nabla\rho\|_{L^q}) \\ &\le C\|(L,H)\|_{L^q}^2\log(e + \|\nabla\rho\|_{L^q}) \\ &\le C\|(\nabla L,\nabla H)\|_{L^2}^2\log(e + \|\nabla\rho\|_{L^q}). \end{aligned} \tag{2.22}$$

So, inserting (2.21) and (2.22) into (2.18), applying Gronwall's inequality and Lemma 2.2, we conclude

$$\|\nabla\rho\|_{L^\infty(0,T;L^q)} \le C, \qquad q > 2,$$

which proves the lemma. □

Lemma 2.4

$$\|\mathbf{u}_t\|_{L^2(0,T;L^2)} + \|\mathbf{u}\|_{L^2(0,T;H^2)} \le C, \tag{2.23}$$
$$\|\mathbf{u}_t\|_{L^\infty(0,T;L^2)} + \|\mathbf{u}_t\|_{L^2(0,T;H^1)} + \|\mathbf{u}\|_{L^\infty(0,T;H^2)} \le C. \tag{2.24}$$

Proof. Multiplying the momentum equation (1.2) by $\mathbf{u}_t$ and integrating with respect to x, we find that

$$\int_\Omega \rho|\mathbf{u}_t|^2 dx + \frac{\mu}{2}\int_\Omega |\nabla \mathbf{u}|_t^2 dx = -\int_\Omega \rho\mathbf{u}\cdot\nabla\mathbf{u}\mathbf{u}_t dx + \int_\Omega \nabla P\cdot \mathbf{u}_t dx. \tag{2.25}$$

Using Lemma 2.1 and (2.1), the terms on the right-hand side of (2.25) can be bounded as follows.

$$\int_0^T\int_\Omega \nabla P\cdot\mathbf{u}_t dxdt \le C\epsilon^{-1}\|P\|_{L^2(0,T;H^1)} + \epsilon\|\mathbf{u}_t\|_{L^2(0,T;L^2)},$$

$$\begin{aligned}\int_0^T\int_\Omega \rho\mathbf{u}\cdot\nabla\mathbf{u}\mathbf{u}_t dxdt &\le C\epsilon^{-1}\|\mathbf{u}\cdot\nabla\mathbf{u}\|_{L^2(0,T;L^2)} + \epsilon\|\mathbf{u}_t\|_{L^2(0,T;L^2)}\\ &\le C\epsilon^{-1}(\|\mathbf{u}\|_{L^2(0,T;L^p)} + \|\nabla\mathbf{u}\|_{L^2(0,T;L^q)}) + \epsilon\|\mathbf{u}_t\|_{L^2(0,T;L^2)}\\ &\le C\epsilon^{-1} + \epsilon\|\mathbf{u}_t\|_{L^2(0,T;L^2)},\end{aligned}$$

here $1/p+1/q=1$. Inserting the above two estimates into (2.25) and taking ϵ small, we get

$$\|\mathbf{u}_t\|_{L^2(0,T;L^2)} + \|\nabla\mathbf{u}\|_{L^\infty(0,T;L^2)} \le C. \tag{2.26}$$

On the other hand, we may apply the regularity theory of elliptic systems to the momentum equation (1.2):

$$\mu\Delta\mathbf{u} + (\lambda+\mu)\nabla\mathrm{div}\,\mathbf{u} = \rho(\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u}) + \nabla P,$$

to obtain

$$\|\mathbf{u}\|_{H^2} \le C(\|\mathbf{u}_t\|_{L^2} + \|\mathbf{u}\cdot\nabla\mathbf{u}\|_{L^2} + \|\nabla P\|_{L^2}),$$

which, together with (2.26) and (2.1), gives

$$\|\mathbf{u}\|_{L^2(0,T;H^2)} \le C. \tag{2.27}$$

Now, to show (2.24), we differentiate (1.2) with respect to t, multiply the resulting equation by $\mathbf{u}_t$ in $L^2(\Omega)$, make use of (1.1) and integrate by parts to infer, after a straightforward calculation, that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega \rho|\mathbf{u}_t|^2 dx + \mu\int_\Omega |\nabla\mathbf{u}_t|^2 + (\lambda+\mu)(\mathrm{div}\,\mathbf{u}_t)^2 dx = \int_\Omega P_t\,\mathrm{div}\,\mathbf{u}_t dx\\ &\quad -2\int_\Omega \rho\mathbf{u}\cdot\nabla\mathbf{u}_t\cdot\mathbf{u}_t dx - \int_\Omega \rho\mathbf{u}\cdot\nabla(\mathbf{u}\cdot\nabla\mathbf{u}\cdot\mathbf{u}_t)dx - \int_\Omega \rho\mathbf{u}_t\cdot\nabla\mathbf{u}\cdot\mathbf{u}_t dx.\end{aligned} \tag{2.28}$$

By the equation of state, P satisfies

$$P_t + \mathrm{div}\,(P\mathbf{u}) + (\gamma-1)P\mathrm{div}\,\mathbf{u} = 0,$$

which combined with (2.28) results in

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega (\rho|\mathbf{u}_t|^2 + \gamma P(\mathrm{div}\,\mathbf{u})^2)dx + \int_\Omega \mu|\nabla\mathbf{u}_t|^2 dx \\
&\le \int_\Omega \Big(2\rho|\mathbf{u}|\,|\mathbf{u}_t|\,|\nabla\mathbf{u}_t| + \rho|\mathbf{u}|\,|\mathbf{u}_t|\,|\nabla\mathbf{u}|^2 + \rho|\mathbf{u}|^2|\mathbf{u}_t|\,|\nabla^2\mathbf{u}| + \rho|\mathbf{u}|^2|\nabla\mathbf{u}|\,|\nabla\mathbf{u}_t| \\
&\quad + \rho|\mathbf{u}_t|^2|\nabla\mathbf{u}| + |\nabla P|\,|\mathbf{u}|\,|\nabla\mathbf{u}_t| + \gamma P|\mathbf{u}|\,|\nabla\mathbf{u}|\,|\nabla^2\mathbf{u}| + \frac{\gamma(\gamma-1)}{2}P|\nabla\mathbf{u}|^3\Big)dx \\
&= \sum_{i=1}^{8} J_i. \qquad (2.29)
\end{aligned}$$

Next, we have to estimate $J_i, (i = 1, \cdots, 8)$. Utilizing (2.1), Hölder's inequality, and Sobolev's imbedding theorem, Lemma 2.1, and the interpolation inequality, we can bound J_i $(i = 1, 2)$ as follows.

$$\begin{aligned}
J_1 &= \int_\Omega 2\rho|\mathbf{u}|\,|\mathbf{u}_t|\,|\nabla\mathbf{u}_t| dx \\
&\le C\|\mathbf{u}\|_{L^6}\|\mathbf{u}_t\|_{L^3}\|\nabla\mathbf{u}_t\|_{L^2} \\
&\le C\|\nabla\mathbf{u}\|_{L^2}\|\mathbf{u}_t\|_{L^2}^{1/2}\|\mathbf{u}_t\|_{L^6}^{1/2}\|\nabla\mathbf{u}_t\|_{L^2} \\
&\le C\epsilon^{-1}\|\mathbf{u}_t\|_{L^2}\|\nabla\mathbf{u}_t\|_{L^2} + \epsilon\|\nabla\mathbf{u}_t\|_{L^2}^2 \\
&\le C\epsilon^{-3}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2 + \epsilon\|\nabla\mathbf{u}_t\|_{L^2}^2,
\end{aligned}$$

and

$$\begin{aligned}
J_2 &= \int_\Omega \rho|\mathbf{u}|\,|\mathbf{u}_t|\,|\nabla\mathbf{u}|^2 dx \\
&\le C\|\mathbf{u}\|_{L^4}\|\mathbf{u}_t\|_{L^2}\|\nabla\mathbf{u}\|_{L^8}^2 \\
&\le C\|\nabla\mathbf{u}\|_{L^2}\|\mathbf{u}_t\|_{L^2}\|\mathbf{u}\|_{H^2}^2 \\
&\le C\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}\|\mathbf{u}\|_{H^2}^2 \\
&\le \|\mathbf{u}\|_{H^2}^2\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2 + C\|\mathbf{u}\|_{H^2}^2.
\end{aligned}$$

In the same manner, the terms J_i $(i = 3, \cdots, 8)$ can be bounded, and we have

$$\begin{aligned}
J_3 &= \int_\Omega \rho|\mathbf{u}|^2|\mathbf{u}_t|\,|\nabla^2\mathbf{u}| dx \\
&\le C\|\mathbf{u}\|_{L^{12}}^2\|\mathbf{u}_t\|_{L^3}\|\nabla^2\mathbf{u}\|_{L^2} \\
&\le C\|\nabla\mathbf{u}\|_{L^2}^2\|\nabla\mathbf{u}_t\|_{L^2}\|\nabla^2\mathbf{u}\|_{L^2} \\
&\le \epsilon\|\nabla\mathbf{u}_t\|_{L^2}^2 + C\epsilon^{-1}\|\mathbf{u}\|_{H^2}^2,
\end{aligned}$$

$$
\begin{aligned}
J_4 &= \int_\Omega \rho|\mathbf{u}|^2|\nabla\mathbf{u}|\,|\nabla\mathbf{u}_t|dx\\
&\le C\|\mathbf{u}\|_{L^{12}}^2\|\nabla\mathbf{u}\|_{L^3}\|\nabla\mathbf{u}_t\|_{L^2}\\
&\le C\epsilon^{-1}\|\mathbf{u}\|_{H^2}^2+\epsilon\|\nabla\mathbf{u}_t\|_{L^2}^2,
\end{aligned}
$$

and

$$
\begin{aligned}
J_5 &= \int_\Omega \rho|\mathbf{u}_t|^2|\nabla\mathbf{u}|dx\\
&\le C\|\mathbf{u}_t\|_{L^4}^2\|\nabla\mathbf{u}\|_{L^2}\\
&\le C\|\mathbf{u}_t\|_{L^2}\|\nabla\mathbf{u}_t\|_{L^2}\\
&\le C\epsilon^{-1}\|\mathbf{u}_t\|_{L^2}^2+\epsilon\|\nabla\mathbf{u}_t\|_{L^2}^2,
\end{aligned}
$$

where Gagliardo-Nirenberg's inequality has also been used,

$$
J_6 = \int_\Omega |\nabla P|\,|\mathbf{u}|\,|\nabla\mathbf{u}_t|dx \le C\epsilon^{-1}\|\nabla\rho\|_{L^2}^2\|\mathbf{u}\|_{L^\infty}^2+\epsilon\|\nabla\mathbf{u}_t\|_{L^2} \le C\epsilon^{-1}\|\mathbf{u}\|_{H^2}^2+\epsilon\|\nabla\mathbf{u}_t\|_{L^2},
$$

where we have also applied Lemma 2.3,

$$
J_7 = \gamma\int_\Omega P\,|\mathbf{u}|\,|\nabla\mathbf{u}|\,|\nabla^2\mathbf{u}|dx \le C\|\mathbf{u}\|_{L^6}\|\nabla\mathbf{u}\|_{L^3}\|\nabla^2\mathbf{u}\|_{L^2}\le C\|\mathbf{u}\|_{H^2}^2,
$$

and

$$
J_8 = \frac{\gamma(\gamma-1)}{2}\int_\Omega P|\nabla\mathbf{u}|^3dx \le C\|\nabla\mathbf{u}\|_{L^4}^2\|\nabla\mathbf{u}\|_{L^2}\le C\|\mathbf{u}\|_{H^2}^2.
$$

Substituting the above estimates for J_i into (2.29) and taking ϵ small, we obtain by applying Gronwall's inequality and using (2.27) that

$$
\|\mathbf{u}_t\|_{L^\infty(0,T;L^2)}^2+\|\mathbf{u}_t\|_{L^2(0,T;H^1)}^2\le C. \tag{2.30}
$$

With the help of (2.30), an application of the regularity theory of elliptic systems to (1.2) implies

$$
\|\mathbf{u}\|_{L^\infty(0,T;H^2)}\le C. \tag{2.31}
$$

Thus, from (2.26), (2.27), (2.30) and (2.31) we get the lemma.

□

Remark 2.1 *We should point out here that in fact, the estimate of* $\mathbf{u}$ *in* $L^\infty(0,T;H^2)$ *can simply follow from Lemmas 2.2 and 2.3. Indeed, recalling the definition of* L, H*, and the fact that* $L, H, \nabla\rho\in L^\infty(0,T;L^2)$*, one can apply the elliptic regularity to get* $\|\mathbf{u}\|_{L^\infty(0,T;H^2)}\le C$ *immediately. With the help of this boundedness, the proof of Lemma 2.4 can be much simplified.*

Lemma 2.5
$$\|\rho_t\|_{L^\infty(0,T;L^q)} + \|\mathbf{u}\|_{L^2(0,T;W^{2,q})} \le C.$$

Proof. By virtue of Sobolev's imbedding theorem and (2.24), one easily obtains

$$\|\mathbf{u}\|_{L^\infty((0,T)\times\Omega)} \le C. \tag{2.32}$$

Since $\rho_t = -\mathbf{u}\cdot\nabla\rho - \rho\,\mathrm{div}\,\mathbf{u}$, we have by (2.1), (2.32) and Lemma 2.3 that

$$\|\rho_t(t)\|_{L^q} \le \|\mathbf{u}\|_{L^\infty}\|\nabla\rho\|_{L^q} + \|\rho\|_{L^\infty}\|\mathrm{div}\,\mathbf{u}\|_{L^q} \le C, \quad t\in[0,T],$$

while applying the regularity theory of elliptic equations to (1.2) and using Sobolev's imbedding theorem, we obtain

$$\begin{aligned}\|\mathbf{u}(t)\|_{W^{2,q}} &\le C(\|\mathbf{u}_t\|_{L^q} + \|\mathbf{u}\cdot\nabla\mathbf{u}\|_{L^q} + \|\nabla\rho\|_{L^q})\\ &\le C(\|\nabla\mathbf{u}_t\|_{L^2} + \|\mathbf{u}\|_{L^\infty}\|\nabla\mathbf{u}\|_{L^q} + \|\nabla\rho\|_{L^q})\\ &\le C(\|\nabla\mathbf{u}_t\|_{L^2} + \|\mathbf{u}\|_{H^2} + \|\nabla\rho\|_{L^q}).\end{aligned}$$

If we integrate the above inequality over t, we immediately see that $\|\mathbf{u}\|_{L^2(0,T;W^{2,q})}$ is bounded from above. The proof of the lemma is complete. $\square$

By virtue of Lemmas 2.1, 2.3–2.5, we see that at time $t = T^*$, the function $(\rho,\mathbf{u})|_{t=T^*} = \lim_{t\to T^*}(\rho,\mathbf{u})$ satisfy the conditions imposed on the initial data in the local existence theorem given in Proposition 1.1. Hence we can take $(\rho,\mathbf{u})|_{t=T^*}$ as the initial data at $t = T^*$ and apply Proposition 1.1 to extend our local solution beyond T^* in time. This contradicts the maximality of T^*, and therefore the assumption (2.1) does not hold. This completes the proof of Theorem 1.1.

References

[1] J.T. Beale, T. Kato and A. Majda, Remarks on the breakdown of smooth solutions for the 3-D Euler equations. *Comm. Math. Phys.* **94** (1984), 61-66.

[2] J. Bolik and W. von Wahl, Estimating ∇u in terms of $\mathrm{div}\,u$, $\mathrm{curl}\,u$, either $(\nu\cdot u)$ or $(\nu\times u)$ and the topology. *Math. Meth. Appl. Sci.* **20** (1997), 737-744.

[3] Y. Cho and H. Kim, Existence results for viscous polytropic fluids with vacuum. *J. Diff. Eqns.* **228** (2006), 377-411.

[4] J. Fan and S. Jiang, Blow-up criteria for the Navier-Stokes equations of compressible fluids. *J. Hyper. Diff. Eqns.* **5** (2008), 167-185

[5] J. Fan, S. Jiang, Y. Ou, A blow-up criterion for compressible viscous heat-conductive flows. Preprint, 2009.

[6] E. Feireisl, A. Novotný and H.Petzeltová, On the existence of globally defined weak solutions to the Navier-Stokes equations of isentropic compressible fluids. *J. Math. Fluid Mech.* **3** (2001), 358-392.

[7] E. Feireisl, Dynamics of Viscous Compressible Fluids. *Oxford Univ. Press*, Oxford, 2004.

[8] E. Feireisl, On the motion of a viscous, compressible and heat conducting fluid. *Indiana Univ. Math. J.* **53** (2004), 1705-1738.

[9] D. Hoff, Discontinuous solutions of the Navier-Stokes equations for multidimensional flows of heat-conducting fluids. *Arch. Rat. Mech. Anal.* **139** (1997), 303-354.

[10] X. Huang and Z. Xin, A blow-up criterion for classical solutions to the compressible Navier-Stokes equations. arXiv: 0903.3090 v2 [math-ph]. 19 March, 2009.

[11] S. Jiang, Global spherically symmetric solutions to the equations of a viscous polytropic ideal gas in an exterior domain. *Comm. Math. Phys.* **178** (1996), 339-374.

[12] S. Jiang and P. Zhang, On spherically symmetric solutions of the compressible isentropic Navier-Stokes equations, *Comm. Math. Phys.* **215** (2001), 559-581.

[13] S. Jiang and P. Zhang, Axisymmetric solutions of the 3-D Navier-Stokes equations for compressible isentropic fluids. *J. Math. Pure Appl.* **82** (2003), 949-973.

[14] P.L. Lions, Mathematical Topics in Fluid Mechanics, Vol. 2, *Oxford Lecture Series in Math. and Its Appl.* 10, Clarendon Press, Oxford, 1998.

[15] A. Matsumura and T. Nishida, The initial value problem for the equations of motion of viscous and heat-conductive gases. *J. Math. Kyoto Univ.* **20** (1980), 67-104.

[16] A. Matsumura and T. Nishida, The initial boundary value problems for the equations of motion of compressible and heat-conductive fluids. *Comm. Math. Phys.* **89** (1983), 445-464.

[17] O. Rozanova, Blow up of smooth solutions to the compressible Navier-Stokes equations with the data highly decreasing at infinity. *J. Diff. Eqns.* **245** (2008), 1762-1774.

[18] V.A. Vagaint and A.V. Kazhikhov, On existence of global solutions to the two-dimensional Navier-Stokes equations for a compressible viscous fluid. *Siberian Math. J.* **36** (1995), 1108-1141.

[19] W. von Wahl, Estimating ∇u by $\operatorname{div} u$ and $\operatorname{curl} u$. *Math. Meth. Appl. Sci.* **12** (1992), 123-143.

[20] Z.P. Xin, Blow up of smooth solutions to the compressible Navier-Stokes equation with compact density. *Comm. Pure Appl. Math.* **51** (1998), 229-240.

[21] V.I. Yudovich, A two-dimensional nonstationary problem of the flowing of an ideal incompressible fluid through a fixed domain. *Math. Sb.* **64** (1964), 562-588.