A GENERALIZATION OF H-MEASURES AND APPLICATION ON PURELY FRACTIONAL SCALAR CONSERVATION LAWS

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ABSTRACT. We extend the notion of H-measures on test functions defined on $I\!\!R^d \times P$, where $P \subset I\!\!R^d$ is an arbitrary compact simply connected Lipschitz manifold such that there exists a family of regular nonintersecting curves issuing from the manifold and fibrating $I\!\!R^d$. We introduce a concept of quasi-solutions to purely fractional scalar conservation laws and apply our extension of the H-measures to prove strong L^1_{loc} precompactness of such quasi-solutions.

1. INTRODUCTION

Suppose that we wish to solve a nonlinear PDE which we write symbolically as $A[u] = f$, where A denotes a given nonlinear operator. One of usual ways to do it is to approximate the PDE by a collection of "nicer" problems $A_k[u_k] = f_k$, where (A_k) is a sequence of operators which is somehow close to A. Then, we try to prove that the sequence (u_k) converges toward a solution to the original problem $A[u] = f$. The overall impediment is of course nonlinearity which prevents us from obtaining necessary uniform estimates on the sequence (u_k). The typical situation is the following.

Let Ω be an open set in $I\!\!R^d$, and let (u_k) be a bounded sequence in $L^2(\Omega)$ converging in the sense of distributions to $u \in L^2(\Omega)$. In order to prove that u is a solution to $A[u] = f$, we need to prove that (u_k) converges strongly to u, say, in $L^1_{loc}(\Omega)$ (often situation in conservation laws; see e.g. [1, 5, 16]). One of the ways is to consider the sequence $\nu_k = |u_k - u|^2$ bounded in the space of Radon measures $\mathcal{M}(I\!\!R^d)$. Since it is bounded, there exists a measure ν such that $\nu_k \rightharpoonup \nu$ along a subsequence in $\mathcal{M}(I\!\!R^d)$. The support of ν is the set of points in Ω near which (u_k) does not converge to u for the strong topology of $L^2(I\!\!R^d)$. The measure ν is called a defect measure and it was systematically studied by P.L.Lions. For instance, if we are able to prove that ν is equal to zero out of a negligible set, then (u_k) will L^2-strongly converge toward u on a set large enough to state that u is a solution to $A[u] = f$. Such method is called the concentrated compactness method [10, 11].

A shortcoming of the latter defect measure is that they are not sensitive to oscillation corresponding to different frequencies. For instance, consider the sequence $(u_k(x))_{k \in I\!\!N} = (exp(ikx\xi))_{k \in I\!\!N}$, where i is the imaginary unit, $\xi \in I\!\!R^d$ is a fixed vector, and $x \in I\!\!R^d$ is a variable. The sequence is bounded which implies that it is bounded in $L^2(\Omega)$ for any bounded $\Omega \subset I\!\!R^d$. Furthermore, it is well known that $u_k \rightharpoonup 0$ in the sense of distributions but (u_k) does not converge strongly in L^p_{loc} for any $p > 0$. On the other hand, the defect measure ν corresponding to the sequence

(u_k) is the Lebesgue measure for any $\xi \in \mathbb{R}^d$ (and ξ determines the frequency of the rapidly oscillating sequence (u_k)).

Step forward in this direction was made at the beginning of 90's when L.Tartar [18] and P.Gerard [7] independently introduced the H-measures (microlocal defect measures). They are given by the following theorem:

Theorem 1. [18] *If* $(u_n) = ((u_n^1, \ldots, u_n^r))$ *is a sequence in* $L^2(\mathbb{R}^d; \mathbb{R}^r)$ *such that* $u_n \rightharpoonup 0$ *in* $L^2(\mathbb{R}^d; \mathbb{R}^r)$, *then there exists its subsequence* $(u_{n'})$ *and a positive definite matrix of complex Radon measures* $\mu = \{\mu^{ij}\}_{i,j=1,\ldots,r}$ *on* $\mathbb{R}^d \times S^{d-1}$ *such that for all* $\varphi_1, \varphi_2 \in C_0(\mathbb{R}^d)$ *and* $\psi \in C(S^{d-1})$:

$$
\begin{aligned}
\lim_{n'\to\infty} \int_{\mathbb{R}^d} (\varphi_1 u_{n'}^i)(x)\overline{\mathcal{A}_\psi(\varphi_2 u_{n'}^j)(x)}dx &= \langle \mu^{ij}, \varphi_1\overline{\varphi_2}\psi \rangle \\
&= \int_{\mathbb{R}^d \times S^{d-1}} \varphi_1(x)\overline{\varphi_2(x)}\psi(\xi)d\mu^{ij}(x,\xi), \quad i,j = 1,\ldots,r,
\end{aligned}
\tag{1}
$$

where $\mathcal{A}_\psi$ *is a multiplier operator with the symbol* $\psi \in C(S^{d-1})$.

The complex matrix Radon measure $\{\mu^{ij}\}_{i,j=1,\ldots,r}$ defined in the previous theorem we call the H-*measure* corresponding to the subsequence $(u_{n'}) \in L^2(\mathbb{R}^d; \mathbb{R}^r)$.

The H-measures describe a loss of strong L^2 compactness for the corresponding sequence $(u_n) \in L^2(\mathbb{R}^d; \mathbb{R}^r)$. In order to clarify the latter, assume that we are dealing with one dimensional sequence (u_n) (this means that $r = 1$). Then, notice that, by applying the Plancherel theorem, the term under the limit sign in Theorem 1 takes the form

$$
\int_{\mathbb{R}^d} \widehat{\varphi_1 u_{n'}}\overline{\widehat{\psi\varphi_2 u_{n'}}}\, d\xi ,
\tag{2}
$$

where by $\hat{u}(\xi) = (\mathcal{F}u)(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i x\cdot\xi}u(x)\, dx$ we denote the Fourier transform on $\mathbb{R}^d$ (with the inverse $(\bar{\mathcal{F}}v)(x) := \int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi}v(\xi)\, d\xi$). Now, it is not difficult to see that if (u_n) is strongly convergent in L^2, then the corresponding H-measure is trivial. Conversely, if the H-measure is trivial, then $u_n \longrightarrow 0$ in $L^2_{loc}(\mathbb{R}^d)$ (see [3]).

One of constraints in using the H-measures concept is that the symbols of the defining multipliers appearing in (1) are defined on the unit sphere. This makes the concepts adapted for usage basically only on hyperbolic problems (see e.g. [1, 7, 15] and exceptions [17, 14]). The reason for the mentioned confinement lies in the lemma which provides linearity of the integral on the right-hand side of (1). This is so called first commutation lemma and is stated as follows:

Lemma 2. [18, Lemma 1.7] *(First commutation lemma) Let* $a \in C(S^{d-1})$ *and* $b \in C_0(\mathbb{R}^d)$. *Let* $\mathcal{A}$ *be a multiplier operator with the symbol* a, *and* B *be an operator of multiplication given by the formulae:*

$$
\mathcal{F}(\mathcal{A}u)(\xi) = a\big(\frac{\xi}{|\xi|}\big)\mathcal{F}(u)(\xi) \quad a.e. \quad \xi \in \mathbb{R}^d,
$$

$$
Bu(x) = b(x)u(x) \quad a.e. \quad x \in \mathbb{R}^d,
$$

where $\mathcal{F}$ *is the Fourier transform. Then* $C = \mathcal{A}B - B\mathcal{A}$ *is a compact operator from* $L^2(\mathbb{R}^d)$ *into* $L^2(\mathbb{R}^d)$.

As we can see, the symbol a given above is defined on the unit sphere. Recently, in [4] the first commutation lemma was extended for symbol a defined on the parabolic manifold $P = \{(\tau, \xi) \in \mathbb{R} \times \mathbb{R}^d : \tau^2 + |\xi|^4 = 1\}$, and then, in an analog

fashion, in [14] on the ultra-parabolic manifold $UP = \{(\tilde{\xi}, \bar{\xi}) \in {I\!\!R}^k \times {I\!\!R}^{(d-k)} : |\tilde{\xi}|^2 + |\hat{\xi}|^4 = 1\}$. This enabled the authors of [4] and [14] to replace in Theorem 1 the sphere S^{d-1} by P and UP, respectively.

We have noticed that the proof of the first commutation lemma relies only on the fact that if we project any compact set K on the sphere along the rays issuing from the origin, the projection will be smaller as the distance of K from the origin is larger. Furthermore, it is clear that we do not need to project the set $K \subset {I\!\!R}^d$ along the rays – the projection curves can be arbitrary smooth nonintersecting curves fibrating the space (see Figure 1). We will use this observation in Section 2 to replace the sphere S^{d-1} in Theorem 1 by an arbitrary compact simply connected Lipschitz manifold such that there exists a family of regular nonintersecting curves issuing from the manifold and fibrating ${I\!\!R}^d$.

In Section 3, we consider the fractional scalar conservation law:

$$\sum_{k=1}^{d} \partial_{x_k}^{\alpha_k} f_k(x, u) = 0, \tag{3}$$

where $\alpha_k \in (0, 1]$, $f_k \in BV({I\!\!R}^d; C^1({I\!\!R}))$, $k = 1, \ldots, d$. We start by introducing a notion of quasi-solutions to (3) which are basically functions $u \in L^\infty({I\!\!R}^d)$ such that for every $\lambda \in {I\!\!R}$, the operator $\sum_{k=1}^{d} \partial_{x_k}^{\alpha_k} \mathrm{sgn}(u-\lambda)(f_k(x,u) - f_k(x,\lambda))$ is compact as mapping from $W^{1,\infty}({I\!\!R}^d)$ to $L^1_{loc}({I\!\!R}^d)$ (for a more precise definition see Definition 8). In the case of the classical scalar conservation law, the latter operator is nothing but the entropy defect measure. The main result of the section is the fact that under a genuine nonlinearity conditions (see Definition 9), any bounded sequence of quasi-solutions to (3) is strongly L^1_{loc}-precompact.
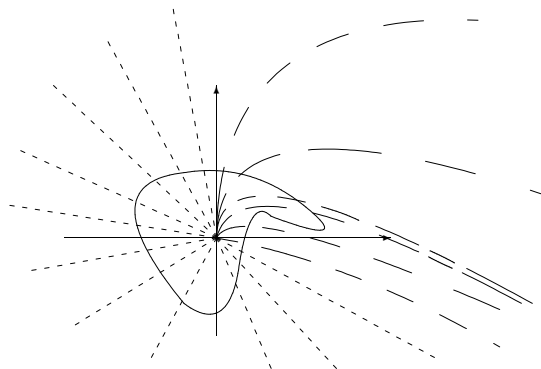

FIGURE 1. The manifold P is represented by normal line. Fibres are dashed. Notice that a fibre must not intersect P twice.

2. THE H-MEASURES REVISITED

In order to improve Theorem 1, we need a new variant of the first commutation lemma. To introduce it, we need the following operators. Let $\mathcal{A}$ be a multiplier operator with a symbol $a \in C({I\!\!R}^d)$, and B be an operator of multiplication by a

function $b \in C_0(I\!R^d)$, given by the formulae:

$$\mathcal{F}(\mathcal{A}u)(\xi) = a(\xi)\mathcal{F}(u)(\xi) \quad a.e. \ \ \xi \in I\!R^d, \tag{4}$$

$$Bu(x) = b(x)u(x) \quad a.e. \ \ x \in I\!R^d, \tag{5}$$

where $\mathcal{F}$ is the Fourier transform.

Following the proof from [18, Lemma 1.7], we shall see in Lemma 4 that the commutator $C = \mathcal{A}B - B\mathcal{A}$ is a compact operator from $L^2(I\!R^d)$ into $L^2(I\!R^d)$ if $a \in L^\infty(I\!R^d)$ satisfies the following condition (see [19, Lemma 28.2]):

$$(\forall R > 0)(\forall \varepsilon > 0)(\exists r > 0) \ |\xi|, |\eta| > r \ \wedge \ \xi - \eta \in B(0, R) \Rightarrow |a(\xi) - a(\eta)| < \varepsilon, \tag{6}$$

where $B(0, R) \subset I\!R^d$ is the ball centered in zero with the radius R.

Here, we want conditions that are more intuitive than (6). They are given by the following definition.

Definition 3. Let $\Omega \subset I\!R^d$ be an arbitrary open subset of the Euclidean space $I\!R^d$. We say that the set Ω admits a complete fibration along the family of curves (below, I denotes a set of indices)

$$\mathcal{C} = \{\varphi_\lambda : I\!R^+ \to \Omega : \ \lambda \in I\}$$

if for every $x \in \Omega$ there exist a unique $t \in I\!R^+$ and unique $\lambda \in I$ such that $x = \varphi_\lambda(t)$.

Assume that we have a family of curves

$$\mathcal{C} = \{\varphi_\lambda : I\!R^+ \to I\!R^d : \ \varphi_\lambda(t) = t\psi_\lambda(t); \ \lambda, \psi_\lambda(t) \in S^{d-1}; \ \psi_\lambda(1) = \lambda\}, \tag{7}$$

parameterized by the distance of the origin, which completely fibrates $I\!R^d \setminus \{0\}$. We have chosen the unit sphere S^{d-1} intentionally since we would like $\lambda \in S^{d-1}$ to determine the "direction" of the curve φ_λ.

Furthermore, assume that there exist a constant $c > 0$ and an increasing real function f satisfying $f(z) \to \infty$ as $z \to \infty$ such that, for any $\lambda_1, \lambda_2 \in S^{d-1}$ and any $t_1, t_2 \in I\!R^+$, it holds:

$$|t_1 \psi_{\lambda_1}(t_1) - t_2 \psi_{\lambda_2}(t_2)| \geq cf(\min\{t_1, t_2\})|\lambda_1 - \lambda_2|, \tag{8}$$

where ψ_λ are defined in (7).

Finally, let $a \in L^\infty(I\!R^d)$ and $a_\infty \in C(S^{d-1})$ be functions such that:

$$\lim_{t \to \infty} a(\varphi_\lambda(t)) = a_\infty(\lambda), \quad \text{uniformly in } \lambda \in S^{d-1}, \tag{9}$$

and let $b : I\!R^d \to I\!R$ be a continuous function converging to zero at infinity. We associate to a and b operators $\mathcal{A}$ and B, respectively, as defined in (4) and (5). The following commutation lemma holds.

Lemma 4. *The operator $C = \mathcal{A}B - B\mathcal{A}$ is a compact operator from $L^2(I\!R^d)$ into $L^2(I\!R^d)$.*

Proof: The proof initially follows steps from the proof of Tartar's First commutation lemma. On the first step notice that we can assume $b \in C_0^1(I\!R^d)$. Indeed, if we assume merely $b \in C_0(I\!R^d)$ then we can uniformly approach the function b by a sequence $(b_n) \in C_0^1(I\!R^d)$ such that for every $n \in I\!N$ the function $\mathcal{F}(b_n)$ has a compact support. The corresponding sequence of commutators $C_n = \mathcal{A}B_n - B_n\mathcal{A}$, where $B_n(u) = b_n u$, converges in norm toward C. So, if we prove that C_n are

compact for each n, the same will hold for C as well. Then, consider the Fourier transform of the operator $\mathcal{C}$. It holds:

$$\mathcal{F}(\mathcal{C}u)(\xi) = \int_{\mathbb{R}^d} \mathcal{F}b(\xi - \eta)\,(a(\xi) - a(\eta))\,\mathcal{F}u(\eta)d\eta.$$

So, following the proof of [18, Lemma 1.7] (or directly from [19, Lemma 28.2]), to complete the proof of our lemma, it is enough to prove (6).

First, notice that for all $\xi, \eta \in \mathbb{R}^d \setminus \{0\}$ such that $\xi = \varphi_{\lambda_1}(t_1)$, $\eta = \varphi_{\lambda_2}(t_2)$, we have according to (8)

$$|\lambda_1 - \lambda_2| \leq \frac{|\xi - \eta|}{cf(\min\{|\xi|, |\eta|\})}. \tag{10}$$

Now, let $M > 0$ and $\varepsilon > 0$ be arbitrary, and let $\xi, \eta \in \mathbb{R}^d \setminus \{0\}$ be such that $\xi - \eta \in B(0, M)$.

According to our assumptions from Definition 4, there are unique $\lambda_1, \lambda_2 \in S^{d-1}$ and $t_1, t_2 \in \mathbb{R}^+$ such that $\xi = \varphi_{\lambda_1}(t_1)$, $\eta = \varphi_{\lambda_2}(t_2)$. Second, S^{d-1} is compact, and so a_∞ is uniformly continuous:

$$(\exists \delta > 0)\ |\lambda_1 - \lambda_2| < \delta \Rightarrow |a_\infty(\lambda_1) - a_\infty(\lambda_2)| < \frac{\varepsilon}{3}.$$

Third, according to (9) there is $R_1 > 0$ such that

$$t_1, t_2 > R_1 \Rightarrow |a(\xi) - a_\infty(\lambda_1)| < \frac{\varepsilon}{3} \text{ and } |a(\eta) - a_\infty(\lambda_2)| < \frac{\varepsilon}{3}.$$

Finally, (10) imply

$$(\exists R_2 > 0)\ t_1, t_2 > R_2 \Rightarrow |\lambda_1 - \lambda_2| \leq \frac{|\xi - \eta|}{cf(R_2)} \leq \frac{\text{diam}K}{cf(R_2)} < \delta,$$

and so for $R = \max\{R_1, R_2\}$, $|\xi|, |\eta| \geq R$ and $\xi - \eta \in B(0, M)$ we have

$$|a(\xi) - a(\eta)| \leq |a(\xi) - a_\infty(\lambda_1)| + |a_\infty(\lambda_1) - a_\infty(\lambda_2)| + |a_\infty(\lambda_2) - a(\eta)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

The proof is over. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ ❑

Definition 5. We say that a manifold $P \subset \mathbb{R}^d$ is admissible if there exists a fibration of the space $\mathbb{R}^d \setminus \{0\}$ along a family of curves $\mathcal{C}$ of form (7) such that for every $y \in P$ there exists a unique $\varphi_{\lambda(y)} \in \mathcal{C}$ such that $y \in \{\varphi_{\lambda(y)}(t) : t \in \mathbb{R}^+\}$ and $\mathbb{R}^d \setminus \{0\} = \dot{\bigcup}_{y \in P} \{\varphi_{\lambda(y)}(t) : t \in \mathbb{R}^+\}$, where $\dot{\bigcup}$ denotes the disjoint union.

We say that the function $\tilde{\psi} \in C(\mathbb{R}^d)$ is an admissible symbol if for every $\varphi_\lambda \in \mathcal{C}$ it holds $\lim_{t \to \infty} \tilde{\psi}(\varphi_\lambda(t)) = \psi(y)$, where $y \in P$ is such that $y \in \{\varphi_\lambda(t) : t \in \mathbb{R}^+\}$, $\lambda \in S^{d-1}$, and $\psi \in C(P)$.

We shall also write

$$\lim_{\xi \to \infty} (\tilde{\psi} - (\psi \circ \pi_P))(\xi) = 0, \tag{11}$$

where π_P is the projection of the point ξ on the manifold P along the fibres $\mathcal{C}$.

We shall define an extension of the H-measures on the set $\mathbb{R}^d \times P$, where P is a manifold admissible in the sense of Definition 5. The following theorem holds:

Theorem 6. *Denote by P a manifold admissible in the sense of Definition 5. If $(u_n) = ((u_n^1, \ldots, u_n^r))$ is a sequence in $L^2(\mathbb{R}^d; \mathbb{R}^r)$ such that $u_n \rightharpoonup 0$ in $L^2(\mathbb{R}^d; \mathbb{R}^r)$, then there exists its subsequence $(u_{n'})$ and a positive definite matrix of complex*

Radon measures $\mu = \{\mu^{ij}\}_{i,j=1,\dots,r}$ on $\mathbb{R}^d \times P$ such that for all $\varphi_1, \varphi_2 \in C_0(\mathbb{R}^d)$ and an admissible symbol $\tilde{\psi} \in C(\mathbb{R}^d)$:

$$\lim_{n' \to \infty} \int_{\mathbb{R}^d} (\varphi_1 u_{n'}^i)(x)\overline{\mathcal{A}_{\tilde{\psi}}(\varphi_2 u_{n'}^j)(x)}dx = \langle \mu^{ij}, \varphi_1\overline{\varphi_2}\psi \rangle$$

$$= \int_{\mathbb{R}^d \times P} \varphi_1(x)\overline{\varphi_2(x)}\psi(\xi)d\mu^{ij}(x,\xi), \quad \xi \in P, \tag{12}$$

where $\mathcal{A}_{\tilde{\psi}}$ is a multiplier operator with the (admissible) symbol $\tilde{\psi} \in C(\mathbb{R}^d)$, and $\psi \in C(P)$ is such that (11) is satisfied.

Proof: First, notice that

$$\int_{\mathbb{R}^d} (\varphi_1 u_{n'}^i)(x)\overline{\mathcal{A}_{\tilde{\psi}}(\varphi_2 u_{n'}^j)(x)}dx \tag{13}$$

$$= \int_{\mathbb{R}^d} \mathcal{F}(\varphi_1 u_{n'}^i)(\xi)\overline{\mathcal{F}(\varphi_2 u_{n'}^j)(\xi)}\tilde{\psi}(\xi)d\xi,$$

according to the Plancherel theorem. Then, denote by $\pi_P(x)$ the projection of the point $x \in \mathbb{R}^d$ on the manifold P along the corresponding fibres. It holds

$$\int_{\mathbb{R}^d} \mathcal{F}(\varphi_1 u_{n'}^i)(\xi)\overline{\mathcal{F}(\varphi_2 u_{n'}^j)(\xi)}\tilde{\psi}(\xi)d\xi \tag{14}$$

$$= \int_{\mathbb{R}^d} \mathcal{F}(\varphi_1 u_{n'}^i)(\xi)\overline{\mathcal{F}(\varphi_2 u_{n'}^j)(\xi)}(\psi \circ \pi_P)(\xi))d\xi$$

$$+ \int_{\mathbb{R}^d} \mathcal{F}(\varphi_1 u_{n'}^i)(\xi)\overline{\mathcal{F}(\varphi_2 u_{n'}^j)(\xi)}\left(\tilde{\psi}(\xi) - (\psi \circ \pi_P)(\xi)\right)d\xi.$$

From the fact that the symbol $\tilde{\psi}$ is admissible in the sense of Definition 5 and the Lebesgue dominated converges theorem, it follows

$$\lim_{n' \to \infty} \int_{\mathbb{R}^d} \mathcal{F}(\varphi_1 u_{n'}^i)(\xi)\overline{\mathcal{F}(\varphi_2 u_{n'}^j)(\xi)}\left(\tilde{\psi}(\xi) - (\psi \circ \pi_P)(\xi)\right)d\xi = 0.$$

From here, (13) and (14), we conclude

$$\lim_{n' \to \infty} \int_{\mathbb{R}^d} (\varphi_1 u_{n'}^i)(x)\overline{\mathcal{A}_{\tilde{\psi}}(\varphi_2 u_{n'}^j)(x)}dx$$

$$= \lim_{n' \to \infty} \int_{\mathbb{R}^d} (\varphi_1 u_{n'}^i)(x)\overline{\mathcal{A}_{\psi \circ \pi_P}(\varphi_2 u_{n'}^j)(x)}dx,$$

implying that, in order to prove (12), it is enough to prove it for the multipliers with symbols defined on P. Now, the proof completely follows the one of [18, Theorem 1.1]. Let us briefly recall it.

Notice that, according to the first commutation lemma (Lemma 4), the mapping

$$(\varphi_1\overline{\varphi_2}, \psi) \mapsto \int_{\mathbb{R}^d} (\varphi_1 u_{n'}^i)(x)\overline{\mathcal{A}_{\psi}(\varphi_2 u_{n'}^j)(x)}dx$$

is a positive bilinear functional on $C_0(\mathbb{R}^d) \times C(P)$. According to the Schwartz kernel theorem, the functional can be extended to a continuous linear functional on $\mathcal{D}(\mathbb{R}^d \times P)$. Since it is positive, due to the Schwartz lemma on non-negative distributions, it follows that the mentioned extension is a Radon measure. ❑

Remark 7. If we assume that the sequence (u_n) defining the H-measure is bounded in $L^p(\mathbb{R}^d)$ for $p > 2$, then we can take the test functions φ_1, φ_2 from Theorem 1 such that $\varphi_1 \in L^q(\mathbb{R}^d)$ where $1/q + 2/p \le 1$, and $\varphi_2 \in C_0(\mathbb{R}^d)$ (see [18, Corollary 1.4] and [16, Remark 2, a)]).

3. Strong precompactness property of a sequence of quasisolutions to a fractional scalar conservation law

Differential equations involving fractional derivatives have received considerable amount of attention recently (see e.g. [2, 6] and references therein). Here, we shall consider a sequence of quasi-solutions to a (purely) fractional scalar conservation law. The definition of a quasi-solution for a classical conservation law can be found in [15, Definition 1.2]. It actually represents a slightly relaxed version of Kružkov's admissibility conditions [8]. Among other facts, the mentioned conditions are obtained relying on the Leibnitz rule for the derivatives of product. This rule does not hold for the fractional derivatives. Therefore, we need to modify slightly Panov's definition of quasisolutions. The motivation for the modification lies in the procedure from [17] (see also [1]) where the existence of solution to an ultra-parabolic equation is proved relying on the H-measures and compactness of appropriate operators.

Definition 8. We say that a function $u \in L^\infty(\mathbb{R}^d)$ is a quasisolution to equation (3) if for every $\lambda \in \mathbb{R}$, $\varphi_1 \in C_c^\infty(\mathbb{R}^d)$ and $\varphi_2 \in L^\infty(\mathbb{R}^d)$, it holds

$$\int_{\mathbb{R}^d} \sum_{k=1}^d \operatorname{sgn}(u-\lambda)(f_k(x,u)-f_k(x,\lambda))\varphi_1(x)\overline{\mathcal{A}_{\frac{(i\xi_k)^{\alpha_k}}{|\xi_1|^{\alpha_1}+|\xi_2|^{\alpha_2}+\ldots+|\xi_d|^{\alpha_d}}}\varphi_2(x)}dx$$

$$= \int_{\mathbb{R}^d} L_{\lambda,\varphi_1}[\varphi_2]dx, \tag{15}$$

where

- $\mathcal{A}_{\frac{(i\xi_k)^{\alpha_k}}{|\xi_1|^{\alpha_1}+|\xi_2|^{\alpha_2}+\cdots+|\xi_d|^{\alpha_d}}}$ is a multiplier operator with the symbol $\frac{(i\xi_k)^{\alpha_k}}{|\xi_1|^{\alpha_1}+|\xi_2|^{\alpha_2}+\cdots+|\xi_d|^{\alpha_d}}$;
- the linear operator $L_{\lambda,\varphi_1} : L^\infty(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$ is compact.

The operator L_{λ,φ_1} we call an **entropy defect operator**. In the case of classical scalar conservation laws, the operators L_{λ,φ_1}, $\lambda \in \mathbb{R}$, will correspond to the appropriate entropy defect measures weighted by $\varphi_1 \mathcal{A}_{\frac{1}{|\xi|}}(\cdot)$, where $\mathcal{A}_{\frac{1}{|\xi|}}$ is the multiplier operator with the symbol $\frac{1}{|\xi|}$.

An interesting question might be how to define a weak solution to (3) analog to the standard weak solution for a PDE of an integer order. Let us recall how one can (formally) reach to a definition of weak solution for a first order partial differential equation.

So, for a function $f(x,\lambda) = (f_1(x,\lambda),\ldots,f_d(x,\lambda)) \in BV(\mathbb{R}^d; C(\mathbb{R}))$, $(x,\lambda) \in \mathbb{R}^d \times \mathbb{R}$, consider

$$\operatorname{div} f(x,u) = 0, \quad u \in L^\infty(\mathbb{R}^d).$$

Finding the Fourier transform of the last expression, we obtain

$$\sum_{k=1}^d i\xi_k \mathcal{F}(f_k(\cdot,u))(\xi) = 0, \quad \xi \in \mathbb{R}^d. \tag{16}$$

Then, take an arbitrary function $\varphi \in C_c^1(I\!\!R^d)$ and multiply (16) by $\overline{\mathcal{F}(\varphi)}(\xi)$ (inverse Fourier transform of φ). We obtain

$$\sum_{k=1}^{d} i\xi_k \mathcal{F}(f_k(\cdot, u))(\xi) \overline{\mathcal{F}(\varphi)}(\xi) = -\sum_{k=1}^{d} \mathcal{F}(f_k(\cdot, u))(\xi) \overline{i\xi_k \mathcal{F}(\varphi)}(\xi)$$

$$= -\sum_{k=1}^{d} \mathcal{F}(f_k(\cdot, u))(\xi) \overline{\mathcal{F}(\partial_{x_k}\varphi)}(\xi) = 0.$$

Integrating this over $\xi \in I\!\!R^d$ and applying the Plancherel formula, we get

$$-\int_{I\!\!R^d} \sum_{k=1}^{d} \mathcal{F}(f_k(\cdot, u))(\xi) \overline{\mathcal{F}(\partial_{x_k}\varphi)}(\xi) d\xi = -\int_{I\!\!R^d} \sum_{k=1}^{d} f_k(x, u)\partial_{x_k}\varphi(x)dx = 0,$$

which is the classical definition of a weak solution.

From the latter considerations, it is natural to define an integrable function u to be a weak solution to (3) if for every $\varphi \in C_c^\infty(I\!\!R^d)$, it holds

$$\int_{I\!\!R^d} \sum_{k=1}^{d} f_k(x, u(x)) \overline{\partial_{x_k}^{\alpha_k}\varphi(x)}dx = 0,$$

where $\partial_{x_k}^{\alpha_k}$ is the multiplier operator with the symbol $(i\xi_k)^{\alpha_k}$, $k = 1, \ldots, d$.

Existence of a sequence of quasisolutions to (3) is an open question which we will deal with in a future. Existence of the sequence of quasisolutions together with the strong precompactness result (Theorem 11) would immediately give existence of a weak solution to (3).

The latter notion of quasisolution can be rewritten in the so called kinetic formulation which appeared to be very powerful in the field of conservation laws [9]. It reduces equation (3) to a linear equation with the right-hand side in the form of a distribution of order one.

It is enough to find derivative in λ to (15). Thus, in the sense of distributions, we have

$$-\int_{I\!\!R^d} \sum_{k=1}^{d} h(x, \lambda)\partial_\lambda f_k(x, \lambda)\varphi_1(x) \overline{\mathcal{A}_{\frac{(i\xi_k)^{\alpha_k}}{|\xi_1|^{\alpha_1}+|\xi_2|^{\alpha_2}+\cdots+|\xi_d|^{\alpha_d}}}\varphi_2(x)}dx \qquad (17)$$

$$= \int_{I\!\!R^d} \partial_\lambda L_{\lambda,\varphi_1}[\varphi_2]dx,$$

where $h(x, \lambda) = \mathrm{sgn}(u(x) - \lambda)$, or equivalently, for any $\rho \in C_0^1(I\!\!R)$

$$\int_{I\!\!R} \int_{I\!\!R^d} \sum_{k=1}^{d} h(x, \lambda)\partial_\lambda f(x, \lambda)\rho(\lambda)\varphi_1(x) \overline{\mathcal{A}_{\frac{(i\xi_k)^{\alpha_k}}{|\xi_1|^{\alpha_1}+|\xi_2|^{\alpha_2}+\cdots+|\xi_d|^{\alpha_d}}}\varphi_2(x)}dxd\lambda \qquad (18)$$

$$= \int_{I\!\!R} \int_{I\!\!R^d} L_{\lambda,\varphi_1}[\varphi_2]\rho'(\lambda)dxd\lambda.$$

We shall prove that under a genuine nonlinearity condition for the flux function $f(x, \lambda) = (f_1(x, \lambda), \ldots, f_d(x, \lambda))$ from the previous definition, a sequence of quasisolutions to (3) is strongly precompact in $L_{loc}^1(I\!\!R^d)$.

Definition 9. We say that equation (3) is genuinely nonlinear if for almost every $x \in \mathbf{R}^d$ the mapping

$$\lambda \mapsto \sum_{k=1}^{d} (i\xi_k)^{\alpha_k} f_k(x, \lambda), \tag{19}$$

where i is the imaginary unit, is not identically equal to zero on any set of positive measure $X \subset I\!\!R$.

To continue, denote by $P = \{\xi \in I\!\!R^d : \sum_{k=1}^{d} |\xi_k|^{\alpha_k} = 1\}$ where α_k, $k = 1, \ldots, d$, are given in (3). Notice that the manifold P is admissible manifold in the sense of Definition 5. For the family $\mathcal{C}$ from Definition 5 corresponding to the manifold P, we will take the family of curves defined by

$$\xi_k(t) = \eta_k t^{1/\alpha_k}, \quad t \geq 0, \quad k = 1, \ldots, d, \quad (\eta_1, \ldots, \eta_d) \in P \tag{20}$$

Therefore, there exists an H-measure μ defined on $I\!\!R^d \times P$ as given in Theorem 6.

Remark 10. Remark that there can be several manifolds (compare [3] and [4] in the parabolic case) as well as several fibrations that we could use. If we need a smoother manifold, we could take $\tilde{P} = \{\xi \in I\!\!R^d : \left(\sum_{k=1}^{d} |\xi_k|^{2\alpha_k} \right)^{1/2} = 1\}$. Also, we can take several fibrations, but the one that should be used here is exactly (20) since in that case the symbols $\frac{(i\xi_k)^{\alpha_k}}{|\xi_1|^{\alpha_1} + |\xi_2|^{\alpha_2} + \cdots + |\xi_d|^{\alpha_d}}$, $k = 1, \ldots, d$, are admissible test functions in (23) and we can pass to the limit as $n' \to \infty$ in (24). We would like to thank to E.Yu.Panov for helping us to clear up this situation.

To proceed, denote by (u_n) a family of quasi-solutions to (3) satisfying the non-degeneracy condition in the sense of Definition 9. The following theorem holds:

Theorem 11. *Let (u_n) be a bounded sequence of quasi-solutions to (3). Assume that there exists a subsequence (not relabeled) (u_n) of the given sequence such that, for every $\lambda \in I\!\!R$ and $\varphi_1 \in C_c^\infty(I\!\!R^d)$, the corresponding sequence of entropy defects operators $(L_{\lambda,\varphi_1}^n)$ admits a limit in the sense that there exists a compact operator $L_{\lambda,\varphi_1} : L^\infty(I\!\!R^d) \to L^1(I\!\!R^d)$ such that for any $\rho \in C_0^1(I\!\!R)$ and any sequence (φ_n) weakly-$\star$ converging to zero in $L^\infty(I\!\!R^d)$, it holds*

$$\lim_{n \to \infty} \int_{I\!\!R} \int_{I\!\!R^d} \left(L_{\lambda,\varphi_1}^n[\varphi_n] - L_{\lambda,\varphi_1}[\varphi_n] \right) \rho(\lambda) dx d\lambda = 0.$$

Then, the sequence (u_n) is strongly precompact in $L_{loc}^1(I\!\!R^d)$.

Notice that we have the situation from the latter theorem in the case of a classical scalar conservation law (see e.g. [1, 12] and the comments after Definition 8).

Denote

$$h_n(x, \lambda) = \text{sgn}(u_n(x) - \lambda) \tag{21}$$

and assume that for a function $h \in L^\infty(I\!\!R^d \times I\!\!R)$, it holds

$$h_n(x, \lambda) \xrightarrow{\quad * \quad} h(x, \lambda) \text{ in } L^\infty(I\!\!R^d) \tag{22}$$

along a subsequence of the sequence (h_n). Taking Remark 7 into account, the following extension of Theorem 6 can be proved in the exactly same way as [13, Theorem 3] (see also [16, Remark 2, a)], [17, Theorem N], [14, Proposition 2]):

Theorem 12. *1.   For the sequence (h_n) and the function h defined by (21) and (22), respectively, there exists a set $E \subset \mathbb{R}$ of a full measure such that there exists a family of complex Radon measures $\mu = \{\mu^{pq}\}_{p,q \in E}$ on $\mathbb{R}^d \times P$ such that there exists a subsequence $(h_{n'} - h)$ of the sequence $(h_n - h)$ such that for all $\varphi_1 \in L^2(\mathbb{R}^d)$, $\varphi_2 \in C_c(\mathbb{R}^d)$ and a symbol $\psi \in C(\mathbb{R}^d)$ admissible in the sense of Definition 5:*

$$\lim_{n' \to \infty} \int_{\mathbb{R}^d} \varphi_1(x)(h_{n'} - h)(x, p)\overline{\mathcal{A}_\psi(\varphi_2(\cdot)(h_{n'} - h)(\cdot, q))(x)}dx$$
$$= \langle \mu^{pq}, \varphi_1\overline{\varphi_2}\psi \circ \pi_P \rangle = \int_{\mathbb{R}^d \times P} \varphi_1(x)\overline{\varphi_2(x)}\psi \circ \pi_P(\xi)d\mu^{pq}(x, \xi), \tag{23}$$

where $(x, \xi) \in \mathbb{R}^d \times P$, and $\mathcal{A}_\psi$ is a multiplier operator with the (admissible) symbol $\psi \in C(\mathbb{R}^d)$.

2. The mapping $(p, q) \mapsto \mu^{pq}$ as the mapping from $E \times E$ to the space $\mathcal{M}(\mathbb{R}^d \times P)$ of complex Radon measures is continuous with the topology generated by the semi-norms $||\mu||_K = Var(\mu)(K)$, K-compact in $\mathbb{R}^d \times P$.

Now, we can prove Theorem 11.

Proof of Theorem 11: The proof uses the kinetic formulation (18) of (15).

First, take the functions h_n and h defined by (21) and (22), respectively. Then, notice that according to (18), a subsequence $(h_{n'} - h)$ of the sequence $(h_n - h)$ given in Theorem 12 satisfies

$$\int_{\mathbb{R}} \int_{\mathbb{R}^d} \sum_{k=1}^{d}(h_{n'} - h)(x, \lambda)\partial_\lambda f_k(x, \lambda)\rho(\lambda)\varphi_1(x)\overline{\mathcal{A}_{\frac{(i\xi_k)^{\alpha_k}}{|\xi_1|^{\alpha_1} + |\xi_2|^{\alpha_2} + \ldots + |\xi_d|^{\alpha_d}}}\varphi_2(x)}dx$$
$$= \int_{\mathbb{R}} \int_{\mathbb{R}^d}(L_{\lambda, \varphi_1}^{n'}[\varphi_2] - L_{\lambda, \varphi_1}[\varphi_2])\rho'(\lambda)dxd\lambda, \tag{24}$$

where $\rho \in C_c^1(\mathbb{R})$, $\varphi_1 \in C_c^\infty(\mathbb{R}^d)$, and $\varphi_2 \in L^\infty(\mathbb{R}^d)$ are arbitrary. Then, for a fixed $p \in \mathbb{R}$, put

$$\varphi_2(x) = \varphi_2^{n'}(x, p) = (h_{n'} - h)(x, p)\phi_2(x, p), \quad \phi_2 \in C_c^\infty(\mathbb{R}^d \times \mathbb{R}).$$

After letting $n' \to \infty$ in (24), from Theorem 12 and conditions on L_{λ, φ_1}^n and L_{λ, φ_1} given in Theorem 11, we conclude that for almost every $p \in \mathbb{R}$:

$$\int_{\mathbb{R}} \int_{\mathbb{R}^d \times P} \sum_{i=1}^{d} \xi_i^{\alpha_k} f_i(x, \lambda)\varphi_1(x)\rho(\lambda)\phi_2(x, p)d\mu^{\lambda p}(x, \xi)dp = 0,$$

where μ is an H-measure corresponding to the sequence $(h_n - h)$, as given in Theorem 12. For a fixed q, put here $\rho(\lambda) = \frac{1}{\varepsilon}\tilde{\rho}(\frac{\lambda - q}{\varepsilon})\bar{\rho}(\frac{q + \lambda}{2})$ and $\phi_2(x, p) = \frac{1}{\varepsilon}\tilde{\rho}(\frac{p - q}{\varepsilon})\bar{\rho}(\frac{q + p}{2})\varphi_1(x)$, where $\tilde{\rho}$ is a non-negative compactly supported real function with total mass one, and $\bar{\rho} \in C_0^1(\mathbb{R})$ is arbitrary. Integrating over $p, q \in \mathbb{R}$, and letting $\varepsilon \to 0$, we obtain:

$$\int_{\mathbb{R}} \int_{\mathbb{R}^d \times P} \sum_{k=1}^{d}(i\xi_k)^{\alpha_k} f_k(x, q)\varphi_1^2(x)\bar{\rho}(q)d\mu^{qq}(x, \xi)dq = 0.$$

From the genuine nonlinearity condition, we conclude $\mu^{\lambda\lambda} \equiv 0$ for almost every $\lambda \in E$ (see e.g. [13, Theorem 5]). This actually means that $h_{n'} \to h$ strongly in $L_{loc}^2(\mathbb{R}^d \times \mathbb{R})$, and that $h(x, \lambda) = \text{sgn}(u(x) - \lambda)$ for some $u \in L^\infty(\mathbb{R}^d)$. From here, it is not difficult to conclude that $u_{n'} \to u$ strongly in $L_{loc}^1(\mathbb{R}^d)$. This concludes the proof. $\square$

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