Formation of singularity and smooth wave propagation for the non-isentropic compressible Euler equations

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Abstract

We define compressive and rarefactive waves and give the differential equations describing smooth wave steepening for the compressible Euler equations with a varying entropy profile and general pressure laws. Using these differential equations, we directly generalize P. Lax's singularity (shock) formation results in [9] for hyperbolic systems with two variables to the 3×3 compressible Euler equations for a polytropic ideal gas. Our results are valid globally without restriction on the size of the variation of initial data.

Key Words: Conservation laws, Compressible Euler equation, Gradient blowup, Large amplitude, Rarefactive and compressive waves.

1 Introduction

In this paper, we consider the initial value problem for the compressible Euler equations in Lagrangian coordinates in one space dimension,

$$\tau_t - u_x = 0, \tag{1.1}$$

$$u_t + p_x = 0, \tag{1.2}$$

$$(\frac{1}{2}u^2 + e)_t + (up)_x = 0, \tag{1.3}$$

where ρ is the density, $\tau = \rho^{-1}$ is the specific volume, p is the pressure, u is the velocity, and e is the internal energy. For C^1 solutions, (1.3) can be equivalently replaced by

$$S_t = 0, \tag{1.4}$$

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where S is the entropy. When the entropy is a constant (isentropic fluid), (1.1) and (1.2) become a complete system, known as the p-system [15].

The formation of shock waves from smooth initial data is one of the central problems of conservation laws. The formation of shock waves takes place when one or more gradients blows up. In this paper, we consider the formation of such kind of singularity from smooth initial data. In [9], Lax gave singularity formation results for hyperbolic conservation laws with two variables, including the p-system as an example. This says that the singularity forms from smooth initial data if and only if the initial data includes compressive waves. For the conservation laws with a coordinate system of Riemann invariants, similar singularity formation results hold, c.f. [4]. However, for the conservation laws with more than two variables and without a coordinate system of Riemann invariants, including (1.1)$\sim$(1.3), the singularity formation results are only for restrictive initial data, see [7], [12], [11]. In this paper, we directly generalize Lax's singularity formation results for the p-system to (1.1)$\sim$(1.3). The singularity formations for the compressible Euler equations in multiple space dimensions are considered by [14] and [13].

Comparing to the p-system, one of the main difficulties for studying (1.1) $\sim$ (1.3), even for the smooth solutions, is the smoothly varying entropy field. Wave propagation in a smoothly varying entropy field is still not well understood: even some basic questions, like what should be considered as the compressive and rarefactive waves have not previously been answered. Furthermore, the study of smoothly varying entropy field is also the basis for studying the formation of singularity, and interactions between shock waves and simple waves.

In an isentropic domain, people have long known how to define the compressive and rarefactive waves for smooth solutions by the changes of density, pressure, wave speed or Riemann invariants, c.f. [3], [8], [9], [16], [19]. However, in a smoothly varying entropy field, there are no Riemann invariants, and the forward and backward waves can't be geometrically divided. In order to define rarefactive and compressive waves, we need to find a variable which is not impacted by the variation of entropy but still can discriminate rarefactive and compressive waves. Pressure is an appropriate variable, since it is invariant inside the stationary entropy field.

We use superscripts $'$ and $\backprime$ to denote the directional derivatives on the forward and backward characteristic lines, respectively:

$$' = \partial_t + c\partial_x,\ \backprime = \partial_t - c\partial_x, \tag{1.5}$$

where c is the wave speed, c.f. [9]. We define the Rarefactive (R) and Compressive (C) waves (or R/C character) as follows:

Definition 1.1. Consider a solution of (1.1)$\sim$(1.3), which is smooth in an open set U in the (t, x)-plane and A is a point in U. We say that the solution is forward (backward) rarefactive at A, if and only if $p^\backprime < 0$ ($p' < 0$); it is forward (backward) compressive at A, if and only if $p^\backprime > 0$ ($p' > 0$).

Here, we use the derivative of pressure in the *opposite* direction to define the

R/C character, which helps us discount the disturbance from the waves in the opposite direction. This definition gives us appropriate physical explanation of R/C characters. This definition is not only for the simple waves [15], but also for waves in the wave interaction regions.

Then we restrict our consideration on the polytropic ideal gas dynamics, where

$$p = Ke^{\frac{S}{c_v}}\tau^{-\gamma}, \tag{1.6}$$

K and c_v are positive constants, and $\gamma > 1$ is the adiabatic gas constant. We introduce variables z and m:

$$z = \frac{2\sqrt{K\gamma}}{\gamma - 1}\tau^{-\frac{\gamma-1}{2}}, \quad m = e^{\frac{S}{2c_v}}. \tag{1.7}$$

In this paper, the smooth solution means that u, τ, S are C^2, which is equivalent to that u, z, m are C^2 and z, m are both positive, by (1.7). If the initial data are smooth, m is smooth and positive until either the τ or u profile has some singularities (gradient blowups), since the entropy is stationary in C^1 solutions by (1.4). Since we only consider smooth wave propagation and singularity formation from smooth initial data in this paper, the entropy profile is always smooth and stationary.

We introduce variables

$$\alpha = -\tfrac{p^{\backprime}}{c^2} = u_x + mz_x + \tfrac{\gamma-1}{\gamma}m_x z, \tag{1.8}$$

$$\beta = -\tfrac{p'}{c^2} = u_x - mz_x - \tfrac{\gamma-1}{\gamma}m_x z. \tag{1.9}$$

α and β are the generalization of s_x and r_x in a smoothly varying entropy field, where s, r are Riemann invariants in a constant entropy field.

Theorem 1.2. *The smooth solutions in (1.1)∼(1.3) satisfy*

$$\alpha' = k_1\{k_2(3\alpha + \beta) + \alpha\beta - \alpha^2\}, \tag{1.10}$$

and

$$\beta^{\backprime} = k_1\{-k_2(\alpha + 3\beta) + \alpha\beta - \beta^2\}, \tag{1.11}$$

where

$$k_1 = \tfrac{(\gamma+1)K_c}{2(\gamma-1)}z^{\frac{2}{\gamma-1}}, \quad k_2 = \tfrac{\gamma-1}{\gamma(\gamma+1)}zm_x, \quad K_c \text{ is a positive constant.} \tag{1.12}$$

The equations (1.10) and (1.11) are not pure ODEs because they aren't closed.

Then we transform (1.10) and (1.11) into "decoupled ODEs" by changing α and β into new variables y and q, where

$$y = m^{-\frac{3(3-\gamma)}{2(3\gamma-1)}}z^{\frac{\gamma+1}{2(\gamma-1)}}((u + mz)_x - \tfrac{2}{3\gamma-1}m_x z), \tag{1.13}$$

$$q = m^{-\frac{3(3-\gamma)}{2(3\gamma-1)}}z^{\frac{\gamma+1}{2(\gamma-1)}}((u - mz)_x + \tfrac{2}{3\gamma-1}m_x z). \tag{1.14}$$

Similarly, we define

$$\tilde{y} = z^{\frac{\gamma+1}{2(\gamma-1)}}((u+mz)_x - \tfrac{2}{3\gamma-1}m_x z), \tag{1.15}$$

$$\tilde{q} = z^{\frac{\gamma+1}{2(\gamma-1)}}((u-mz)_x + \tfrac{2}{3\gamma-1}m_x z). \tag{1.16}$$

Theorem 1.3. *The smooth solutions in (1.1)∼(1.3) satisfy*

$$y' = a_0 + a_2 y^2, \tag{1.17}$$

$$q^{\backprime} = a_0 + a_2 q^2, \tag{1.18}$$

where

$$a_0 = \tfrac{K_c}{\gamma} m^{-\frac{3(3-\gamma)}{2(3\gamma-1)}}[\tfrac{\gamma-1}{3\gamma-1}mm_{xx} - \tfrac{(3\gamma+1)(\gamma-1)}{(3\gamma-1)^2}m_x^2]z^{\frac{3(\gamma+1)}{2(\gamma-1)}+1}, \tag{1.19}$$

$$a_2 = -K_c\frac{\gamma+1}{2(\gamma-1)}m^{\frac{3(3-\gamma)}{2(3\gamma-1)}}z^{\frac{\gamma+1}{2(\gamma-1)}-1} < 0. \tag{1.20}$$

Clearly, the functions a_0 and a_2 only depend on the density and initial entropy profile. These "ODEs" generalize Lax's "ODEs" for p-system in [9]. In fact, when entropy is constant, $a_0 = 0$, and we recover Lax's equations. In this paper, "$\rightarrow$" denotes the limit when (t, x) approaches a fixed (finite) point (t_0, x_0).

For later reference, we state two assumptions that will be used in some of our results.

Assumption 1 : z is not equal to zero or infinity for any x and t.

Assumption 2 : There exist positive constants Z_L, Z_U, $M_1 \sim M_4$, such that

$$Z_L < z < Z_U, \ \textit{for all } x \textit{ and } t; \tag{1.21}$$

$$M_1 < m^0 < M_2, \quad |m_x^0| < M_3, \quad |m_{xx}^0| < M_4, \tag{1.22}$$

where $m^0(x) = m(0,x)$ is given by the prescribed entropy profile $s(0,x) \equiv s(t,x)$ by (1.4) and (1.7).

By (1.7), Assumption 1 means that the density is not zero or infinity, and (1.21) means that the density has positive upper and lower bounds. Furthermore, all realistic smooth (C^2) initial entropy profiles satisfy (1.22). We impose Z_L to avoid potential problems at vacuum, which are addressed in an upcoming paper [20]. Z_L can be arbitrarily small.

Using (1.17) and (1.18), we can now give some singularity formation results.

Theorem 1.4. *Assume the initial data are smooth, and Assumption 2 holds. The constants Z_U and $M_1 \sim M_4$ are given in Assumption 2. Then there exist positive constants N and $\tilde{N}$ depending only on Z_U and $M_1 \sim M_4$, such that, if y or q is less than $-N$ or $\tilde{y}$ or $\tilde{q}$ is less than $-\tilde{N}$ somewhere in the initial data, then $|u_x|$ and/or $|\tau_x|$ blow up in finite time. When $\gamma \geqslant 3$, all the results hold without the lower bound of density in Assumption 2. (Here y, q, $\tilde{y}$ and $\tilde{q}$ are defined in (1.13)∼(1.16).)*

Theorem 1.5. *Assume the initial data are smooth, and Assumption 2 holds. Furthermore, assume that there exists a point $A \in \mathbb{R}$, such that the initial entropy profile satisfies*

$$(m^{-\frac{2}{3\gamma-1}})_{xx} \geqslant 0, \ \ for\ x > A. \tag{1.23}$$

If, in the initial data, $y_0 = y(0, A^*) < 0$ at some point $x = A^*$ with $A^* > A$, then $|u_x|$ and/or $|\tau_x|$ blow up before some finite time T_*, where

$$0 < T_* \leqslant -\frac{1}{y_0 \min(-a_2)}, \tag{1.24}$$

and $\min(-a_2)$ *is a positive constant depending on Z_L (Z_U) and M_1 (M_2) when $1 < \gamma < 3$ ($\gamma > 3$). When $\gamma = 3$, all the results hold without Assumption 2. Symmetric results hold for q. (Here a_2 is defined in (1.20).)*

Theorem 1.4 implies that gradients of solutions blow up if the initial compressions are strong enough, which is a direct generalization of Lax's singularity formation results for p-system. When the variation of entropy is mild, N and $\tilde{N}$ are close to zero, so the shock free solutions are "almost rarefactive", which is consistent with Lax's singularity formation results. In [16] and a forthcoming paper [2], examples are given, of solutions containing compressive waves, but the gradients of the solutions don't blow up.

Our "ODEs" and the singularity formation results are for arbitrarily large smooth initial data, where there is no restriction on the amplitude of the waves.

We divide this paper into 7 sections. In sections 2 and 3, we review the background and definition of R/C character in the p-system. In section 4, we define rarefactive and compressive waves in a smoothly varying entropy field. In sections 5 and 6, we give "ODEs" for smooth solutions. In section 7, we prove the singularity formation results.

2 Equations and Coordinates

In this paper, we focus on the polytropic ideal gas dynamics, where the equation of state is given by,

$$p\tau = RT, \tag{2.1}$$

and

$$e = c_v T = \frac{1}{\gamma - 1} p\tau, \tag{2.2}$$

with

$$p = K e^{\frac{S}{c_v}} \tau^{-\gamma}. \tag{2.3}$$

Here S is the entropy, T is the temperature, R, K, c_v are all positive constants, and the adiabatic gas constant $\gamma > 1$, c.f. [3]. The (Lagrangian) wave (sound) speed is given by

$$c = \sqrt{-p_\tau} = \sqrt{K\gamma} \tau^{-\frac{\gamma+1}{2}} e^{\frac{S}{2c_v}}. \tag{2.4}$$

We use the coordinates provided by B. Temple and R. Young, in [16]. Define new variables m and z for S and τ, by

$$m = e^{\frac{S}{2c_v}}, \tag{2.5}$$

and

$$z = \int_{\tau}^{\infty} \frac{c}{m} d\tau = \frac{2\sqrt{K\gamma}}{\gamma - 1} \tau^{-\frac{\gamma-1}{2}}, \tag{2.6}$$

where we use (2.4) and (2.5). It follows that

$$\tau = K_\tau z^{-\frac{2}{\gamma-1}}, \tag{2.7}$$

$$p = K_p m^2 z^{\frac{2\gamma}{\gamma-1}}, \tag{2.8}$$

$$c = c(z,m) = K_c m z^{\frac{\gamma+1}{\gamma-1}}, \tag{2.9}$$

where K_τ, K_p and K_c are positive constants given by

$$K_\tau = \left(\frac{2\sqrt{K\gamma}}{\gamma - 1}\right)^{\frac{2}{\gamma-1}}, \tag{2.10}$$

$$K_p = K K_\tau^{-\gamma}, \tag{2.11}$$

$$K_c = \sqrt{K\gamma} K_\tau^{-\frac{\gamma+1}{2}}, \tag{2.12}$$

and

$$K_p = \frac{\gamma - 1}{2\gamma} K_c. \tag{2.13}$$

By (2.5) and (2.6), for C^1 solutions, the Lagrangian equations (1.1)$\sim$(1.3) are equivalent to

$$z_t + \frac{c}{m} u_x = 0, \tag{2.14}$$

$$u_t + mcz_x + 2\frac{p}{m} m_x = 0, \tag{2.15}$$

$$m_t = 0, \tag{2.16}$$

where the last equation is coming from (1.4) instead of (1.3), c.f. [15].

When the entropy is constant, $p = p(\tau)$, so (2.14)$\sim$(2.16) change to the p-system:

$$z_t + \frac{c}{m} u_x = 0, \tag{2.17}$$

$$u_t + mcz_x = 0. \tag{2.18}$$

So the corresponding Riemann invariants are

$$r = u - mz, \tag{2.19}$$

$$s = u + mz, \tag{2.20}$$

which satisfy, by (2.17) and (2.18),

$$r_t - cr_x = 0, \tag{2.21}$$

$$s_t + cs_x = 0. \tag{2.22}$$

3 Rarefactive and Compressive Waves in a Constant Entropy Domain

We first review the definition of the rarefactive and compressive waves for smooth solutions in a constant entropy domain, where the system is 2×2. At this time, the system has full set of Riemann coordinates (2.19) and (2.20). For a single simple wave, the wave is rarefactive (compressive) if pressure, density or wave speed, are decreasing (increasing) from ahead of to behind the wave, c.f. [8]. Here we consider not only the simple waves, but also the local rarefactive and compressive characters at some points or open sets in the wave interaction regions, c.f. [9], [16], [19]. Sometimes, we use the names "rarefactive wave" and "compressive waves" or even "R" and "C", instead of the rarefactive and compressive character at some points or open sets, without confusion.

In this section, we assume entropy is a constant, hence p is a function which only depends on τ, so that

$$p^{\backslash} = p_\tau \tau^{\backslash}, \quad c^{\backslash} = c_\tau \tau^{\backslash}, \quad \rho^{\backslash} = -\frac{1}{\tau^2}\tau^{\backslash}. \tag{3.1}$$

Recall subscripts "$/$" and "$\backslash$" denote the directional derivatives along forward and backward characteristics, respectively, which are defined in (1.5). Note $p_\tau < 0$, and $c_\tau < 0$ since $p_{\tau\tau} > 0$ and (2.4). Furthermore,

$$-\frac{s_t}{c} = s_x = u_x - c\tau_x = \tau^{\backslash}, \tag{3.2}$$

where we use (2.22), (2.20) and (1.1), respectively.

Hence

$$\tau^{\backslash} < 0 \Leftrightarrow c^{\backslash} > 0 \Leftrightarrow p^{\backslash} > 0 \Leftrightarrow s_x < 0 \Leftrightarrow s_t > 0 \Leftrightarrow \rho^{\backslash} > 0, \tag{3.3}$$

which means that the forward waves are compressive; and

$$\tau^{\backslash} > 0 \Leftrightarrow c^{\backslash} < 0 \Leftrightarrow p^{\backslash} < 0 \Leftrightarrow s_x > 0 \Leftrightarrow s_t < 0 \Leftrightarrow \rho^{\backslash} < 0, \tag{3.4}$$

which means that the forward waves are rarefactive.

In fact, (3.3) and (3.4) mean that pressure, density and wave speed, are locally increasing or decreasing from ahead of to behind the forward wave along backward characteristic line, respectively. We use the directional derivative along backward characteristic to define the forward R/C character, which discounts the disturbance from the backward waves. Symmetrically, we define the backward R/C character.

4 Rarefactive and Compressive Solutions in a Smoothly Varying Entropy Domain

In the previous Section, we describe the equivalent definitions of rarefactive and compressive waves in a constant entropy domain. However, when the entropy is smoothly varying, there are no Riemann invariants anymore, and the forward and backward waves can't be geometrically divided, which give difficulties for defining the R/C characters. In this Section, we present a definition of rarefactive and compressive waves in a smoothly varying entropy domain, which is new and fundamental for the smooth waves of the compressible Euler equations.

The conditions shown in the previous Section are not equivalent anymore in the domain with smoothly varying entropy profile. This is because pressure and wave speed become functions depending on both density and entropy. In order to pick up the right condition, we first consider the stationary solutions of (1.1)$\sim$(1.3), where u, τ, S are stationary. So u, p are constant by (1.1)$\sim$(1.3). In particular, the density need not be constant when the entropy profile is varying.

We go back to check the rarefactive and compressive conditions given in the previous Section. In the stationary solutions, $p' = p^{\backprime} = 0$ while the other quantities are nonzero if the entropy profile is smoothly varying. In the other words, pressure is the only thermodynamic variable which is not impacted by the stationary entropy field. Moreover, change of pressure contributes to the wave propagation on the (non-vertical) characteristic lines, i.e. rarefactive or compressive waves. So pressure is the only possible variable which still can distinguish rarefactive and compressive waves in a smoothly varying entropy field. So we can get the definition of rarefactive and compressive waves in Definition 1.1. Note that we use the directional derivative of pressure along the *opposite* characteristic to define the R/C character, which helps us discount the disturbance from waves in the opposite direction. This definition also applies for the general pressure law with $p_\tau < 0,\ p_{\tau\tau} > 0$.

We define
$$\alpha = -\frac{p^{\backprime}}{c^2}, \quad \beta = -\frac{p'}{c^2}. \tag{4.1}$$

Lemma 4.1.
$$\alpha = u_x + m z_x + \tfrac{\gamma-1}{\gamma} m_x z, \tag{4.2}$$

and
$$\beta = u_x - m z_x - \tfrac{\gamma-1}{\gamma} m_x z. \tag{4.3}$$

Proof.
$$\begin{aligned} -c^2\alpha &= p^{\backprime} \\ &= (K_p m^2 z^{\frac{2\gamma}{\gamma-1}})^{\backprime} \\ &= K_p m^2 \tfrac{2\gamma}{\gamma-1} z^{\frac{\gamma+1}{\gamma-1}} z_t - cK_p m^2 \tfrac{2\gamma}{\gamma-1} z^{\frac{\gamma+1}{\gamma-1}} z_x - 2cK_p m m_x z^{\frac{2\gamma}{\gamma-1}} \\ &= -K_p m c \tfrac{2\gamma}{\gamma-1} z^{\frac{\gamma+1}{\gamma-1}} (u_x + m z_x + \tfrac{\gamma-1}{\gamma} m_x z) \\ &= c^2 (u_x + m z_x + \tfrac{\gamma-1}{\gamma} m_x z), \end{aligned} \tag{4.4}$$

where we use (2.9) and (2.13). Similarly, we can prove (4.3). □

By the Definition 1.1 and (4.1), we can equivalently define the R/C character by α and β.

Lemma 4.2. *In the polytropic ideal gas, the local R/C character of the smooth solution is given by:*

$$\begin{array}{llll} \textit{Forward} & R & \textit{iff} & \alpha > 0, \\ \textit{Forward} & C & \textit{iff} & \alpha < 0, \\ \textit{Backward} & R & \textit{iff} & \beta > 0, \\ \textit{Backward} & C & \textit{iff} & \beta < 0. \end{array} \tag{4.5}$$

When Assumption 1 holds,

$$|\alpha| \ \textit{or} \ |\beta| \to \infty \ \textit{iff} \ |u_x| \ \textit{or} \ |\tau_x| \to \infty. \tag{4.6}$$

Proof. Clearly

$$p^{\backprime} \gtreqless 0 \Leftrightarrow \alpha \lesseqgtr 0, \tag{4.7}$$

and,

$$p' \gtreqless 0 \Leftrightarrow \beta \lesseqgtr 0. \tag{4.8}$$

By (4.2) and (4.3),

$$\alpha + \beta = 2u_x, \tag{4.9}$$

$$\alpha - \beta = 2(mz_x + \frac{\gamma - 1}{\gamma} m_x z). \tag{4.10}$$

By (4.2), (4.3), (4.9), (4.10), (2.6) and Assumption 1, we get (4.6), where we also use that m is stationary and positive in smooth solution by (2.5) and (2.16). □

Note: We give the definition of compressive and rarefactive waves in a smoothly varying entropy profile from a purely physical point of view, which also explains the physical meanings of Riemann invariants in the p-system. In fact, in the p-system, the physical meanings of the derivatives of Riemann invariants can be explained by the directional derivatives of pressure:

$$s_x = -\frac{p^{\backprime}}{c^2}, \quad r_x = -\frac{p'}{c^2}, \tag{4.11}$$

where we use that m is a constant, (2.19), (2.20), (4.1) and Lemma 4.1. So α, β can be considered as the generalization of s_x and r_x in a smoothly varying entropy field. When $m_x = 0$, α and β equal to s_x and r_x, respectively.

5 Smooth Wave Propagation

In this section, we consider the smooth wave propagation of (1.1)∼(1.3). By considering the directional derivatives of α and β, we construct "ordinary differential equations" (1.10) and (1.11) in Theorem 1.2, which together with the definition of rarefactive and compressive waves will give us a framework for the wave propagation of smooth solutions. These "ODEs" are not real ODEs since the derivatives are in different directions and (1.10) and (1.11) depend on the variable z, but they are much simpler than the original partial differential equations.

5.1 Proof of Theorem 1.2

Proof. By Lemma 4.1,

$$\begin{aligned}
\alpha' &= (u_x + mz_x + \tfrac{\gamma-1}{\gamma}m_x z)' \\
&= u_{xt} + mz_{xt} + \tfrac{\gamma-1}{\gamma}m_x z_t + c[u_{xx} + mz_{xx} + m_x z_x + \tfrac{\gamma-1}{\gamma}(m_x z_x + m_{xx}z)] \\
&= (u_{xt} + cmz_{xx}) + (cu_{xx} + mz_{xt}) + \tfrac{2\gamma-1}{\gamma}cm_x z_x + \tfrac{\gamma-1}{\gamma}m_x z_t + \tfrac{\gamma-1}{\gamma}cm_{xx}z.
\end{aligned} \tag{5.1}$$

By (2.14)$\sim$(2.16) and (2.8),

$$(u_t + cmz_x)_x = (-2\frac{p}{m}m_x)_x = -(2K_p m m_x z^{\frac{2\gamma}{\gamma-1}})_x, \tag{5.2}$$

and

$$(cu_x + mz_t)_x = 0. \tag{5.3}$$

So

$$\begin{aligned}
u_{xt} + cmz_{xx} \quad &= -2K_c m m_x z^{\frac{\gamma+1}{\gamma-1}} z_x - \tfrac{\gamma+1}{\gamma-1}K_c m^2 z^{\frac{2}{\gamma-1}}(z_x)^2 \\
&\quad -2K_p(m_x)^2 z^{\frac{2\gamma}{\gamma-1}} - 2K_p m m_{xx} z^{\frac{2\gamma}{\gamma-1}} - \tfrac{4\gamma}{\gamma-1}K_p m m_x z^{\frac{\gamma+1}{\gamma-1}} z_x,
\end{aligned} \tag{5.4}$$

and

$$cu_{xx} + mz_{xt} = -c_x u_x - m_x z_t = -\tfrac{\gamma+1}{\gamma-1}K_c m z^{\frac{2}{\gamma-1}} z_x u_x. \tag{5.5}$$

Furthermore, the sum of the second term in the right hand side of (5.4) and the right hand side of (5.5) is

$$\begin{aligned}
&-\tfrac{\gamma+1}{\gamma-1}K_c m z^{\frac{2}{\gamma-1}} z_x(mz_x + u_x) \\
= \;& -\tfrac{\gamma+1}{\gamma-1}K_c z^{\frac{2}{\gamma-1}}(\alpha - \tfrac{\gamma-1}{\gamma}m_x z - u_x)(\alpha - \tfrac{\gamma-1}{\gamma}m_x z) \\
= \;& -\tfrac{\gamma+1}{\gamma}K_c m_x z^{\frac{\gamma+1}{\gamma-1}} u_x - \tfrac{\gamma^2-1}{\gamma^2}K_c(m_x)^2 z^{\frac{2\gamma}{\gamma-1}} \\
&+\tfrac{\gamma+1}{\gamma-1}K_c z^{\frac{2}{\gamma-1}}[(\tfrac{2\gamma-2}{\gamma}m_x z + u_x)\alpha - \alpha^2].
\end{aligned} \tag{5.6}$$

By (5.4)$\sim$(5.6) and (2.9), the right hand side of (5.1) equals to,

$$\begin{aligned}
&-2K_c m m_x z^{\frac{\gamma+1}{\gamma-1}} z_x - 2K_p(m_x)^2 z^{\frac{2\gamma}{\gamma-1}} - 2K_p m m_{xx} z^{\frac{2\gamma}{\gamma-1}} - \tfrac{4\gamma}{\gamma-1}K_p m m_x z^{\frac{\gamma+1}{\gamma-1}} z_x \\
&-\tfrac{\gamma+1}{\gamma}K_c m_x z^{\frac{\gamma+1}{\gamma-1}} u_x - \tfrac{\gamma^2-1}{\gamma^2}K_c(m_x)^2 z^{\frac{2\gamma}{\gamma-1}} \\
&+\tfrac{2\gamma-1}{\gamma}K_c m m_x z^{\frac{\gamma+1}{\gamma-1}} z_x - \tfrac{\gamma-1}{\gamma}K_c m_x z^{\frac{\gamma+1}{\gamma-1}} u_x + \tfrac{\gamma-1}{\gamma}K_c m m_{xx} z^{\frac{2\gamma}{\gamma-1}} \\
&+\tfrac{\gamma+1}{\gamma-1}K_c z^{\frac{2}{\gamma-1}}[(\tfrac{2\gamma-2}{\gamma}m_x z + u_x)\alpha - \alpha^2],
\end{aligned} \tag{5.7}$$

where we use (2.14) to get rid of z_t. Plug (2.13) into (5.7), then (5.7) equals to,

$$\begin{aligned}
&-2K_c m_x z^{\frac{\gamma+1}{\gamma-1}} u_x - \tfrac{2\gamma^2-\gamma-1}{\gamma^2}K_c(m_x)^2 z^{\frac{2\gamma}{\gamma-1}} + \tfrac{-2\gamma-1}{\gamma}K_c m m_x z^{\frac{\gamma+1}{\gamma-1}} z_x \\
&+\tfrac{\gamma+1}{\gamma-1}K_c z^{\frac{2}{\gamma-1}}[(\tfrac{2\gamma-2}{\gamma}m_x z + u_x)\alpha - \alpha^2].
\end{aligned} \tag{5.8}$$

Using $mz_x = \alpha - u_x - \frac{\gamma-1}{\gamma}m_x z$, which is from (4.2), (5.8) can be simplified to

$$K_c z^{\frac{\gamma+1}{\gamma-1}}\{\tfrac{1}{\gamma}m_x u_x + (\tfrac{1}{\gamma}m_x + \tfrac{\gamma+1}{\gamma-1}\tfrac{u_x}{z})\alpha - \tfrac{\gamma+1}{\gamma-1}\tfrac{\alpha^2}{z}\}. \tag{5.9}$$

By (4.9),

$$u_x = \frac{\alpha + \beta}{2}. \tag{5.10}$$

Plugging (5.10) into (5.9), we get (1.10). The calculation of (1.11) is same. $\square$

Corollary 1. *In an open set U in the (t, x)-plane, assume the solutions of (1.1)∼(1.3) are smooth, $m_x \neq 0$, and Assumption 1 holds. If $\beta \equiv 0$ in U, then $\alpha \equiv 0$ in U; if $\alpha \equiv 0$ in U, then $\beta \equiv 0$ in U.*

Proof. If $\beta \equiv 0$ in U, $-k_1 k_2 \alpha \equiv 0$ by (1.11), hence $\alpha \equiv 0$ by Assumption 1. Another case can be proved similarly. $\square$

In this corollary, we show that there will be no "pure" forward or backward rarefactive or compressive waves inside any open set in (t, x)-plane with varying entropy, i.e. the forward and backward waves will exist or die out mutually. This is different from the isentropic domain, where we can find rarefactive or compressive simple waves.

6 "Decoupled" ODEs

Next, after introducing new variables y and q, we change (1.10) and (1.11) into "decoupled ODEs" (1.17) and (1.18) in Theorem 1.4, which are only coupled by z and x. The variables y and q are given in (1.13) and (1.14). These new "ODEs" will help us prove the singularity results. The equations (1.17) and (1.18) generalize Lax's "ODEs" for p-system in [9]. In fact, when the entropy is constant, $a_0 = 0$, so (1.17) and (1.18) change to

$$y' = a_2 y^2, \quad q` = a_2 q^2, \tag{6.1}$$

which is exactly the "ODEs" provided in [9] for the p-system.

6.1 Proof of Theorem 1.3

Proof. By (2.14), (2.9) and (4.3),

$$\begin{aligned} z' &= z_t + c z_x \\ &= -\frac{c}{m}(u_x - m z_x) \\ &= -K_c z^{\frac{\gamma+1}{\gamma-1}}\left(\beta + \frac{\gamma-1}{\gamma} m_x z\right). \end{aligned} \tag{6.2}$$

Hence

$$\beta = -\frac{z'}{K_c z^{\frac{\gamma+1}{\gamma-1}}} - \frac{\gamma-1}{\gamma} m_x z. \tag{6.3}$$

Plugging (6.3) into (1.10), we get

$$\alpha' = k_1\{k_2(3\alpha - \tfrac{z'}{K_c z^{\frac{\gamma+1}{\gamma-1}}} - \tfrac{\gamma-1}{\gamma} m_x z) + \alpha(-\tfrac{z'}{K_c z^{\frac{\gamma+1}{\gamma-1}}} - \tfrac{\gamma-1}{\gamma} m_x z) - \alpha^2\}. \tag{6.4}$$

We move the terms including z' to the left hand side, then we multiply by $z^{\frac{\gamma+1}{2(\gamma-1)}}$ on both sides. After simplification, we have

$$\begin{aligned} & z^{\frac{\gamma+1}{2(\gamma-1)}}\alpha' + \tfrac{m_x}{2\gamma} z^{\frac{\gamma+1}{2(\gamma-1)}} z' + \tfrac{\gamma+1}{2(\gamma-1)} \alpha z^{\frac{\gamma+1}{2(\gamma-1)}-1} z' \\ = \; & -\tfrac{\gamma-1}{2\gamma^2} K_c z^{\frac{3(\gamma+1)}{2(\gamma-1)}+1} m_x^2 + \tfrac{2-\gamma}{2\gamma} K_c m_x z^{\frac{3(\gamma+1)}{2(\gamma-1)}} \alpha - \tfrac{\gamma+1}{2(\gamma-1)} K_c z^{\frac{3(\gamma+1)}{2(\gamma-1)}-1} \alpha^2, \end{aligned} \tag{6.5}$$

where we use (1.12). The left hand side of (6.5) is equal to

$$(\alpha z^{\frac{\gamma+1}{2(\gamma-1)}} + \tfrac{\gamma-1}{\gamma(3\gamma-1)} z^{\frac{\gamma+1}{2(\gamma-1)}+1} m_x)' - \tfrac{\gamma-1}{\gamma(3\gamma-1)} z^{\frac{\gamma+1}{2(\gamma-1)}+1} (m_x)'. \tag{6.6}$$

We define a new variable $\tilde{y}$,

$$\tilde{y} = \alpha z^{\frac{\gamma+1}{2(\gamma-1)}} + \frac{\gamma-1}{\gamma(3\gamma-1)} z^{\frac{\gamma+1}{2(\gamma-1)}+1} m_x. \tag{6.7}$$

So

$$\alpha = \tilde{y} z^{-\frac{\gamma+1}{2(\gamma-1)}} - \frac{\gamma-1}{\gamma(3\gamma-1)} z m_x. \tag{6.8}$$

Hence, by (6.6), (6.8) and $(m_x)' = c m_{xx}$, (6.5) changes to

$$\begin{aligned} \tilde{y}' = \; & \tfrac{\gamma-1}{\gamma(3\gamma-1)} K_c z^{\frac{3(\gamma+1)}{2(\gamma-1)}+1} m m_{xx} - \tfrac{\gamma-1}{2\gamma^2} K_c z^{\frac{3(\gamma+1)}{2(\gamma-1)}+1} m_x^2 \\ & + \tfrac{2-\gamma}{2\gamma} K_c m_x z^{\frac{3(\gamma+1)}{2(\gamma-1)}} (\tilde{y} z^{-\frac{\gamma+1}{2(\gamma-1)}} - \tfrac{\gamma-1}{\gamma(3\gamma-1)} z m_x) \\ & - \tfrac{\gamma+1}{2(\gamma-1)} K_c z^{\frac{3(\gamma+1)}{2(\gamma-1)}-1} (\tilde{y} z^{-\frac{\gamma+1}{2(\gamma-1)}} - \tfrac{\gamma-1}{\gamma(3\gamma-1)} z m_x)^2. \end{aligned} \tag{6.9}$$

After simplification, we have

$$\tilde{y}' = \tilde{a}_0 + \tilde{a}_1 \tilde{y} + \tilde{a}_2 \tilde{y}^2, \tag{6.10}$$

where

$$\tilde{a}_0 = K_c \frac{1}{\gamma} [\frac{\gamma-1}{3\gamma-1} m m_{xx} - \frac{(3\gamma+1)(\gamma-1)}{(3\gamma-1)^2} m_x^2] z^{\frac{3(\gamma+1)}{2(\gamma-1)}+1}, \tag{6.11}$$

$$\tilde{a}_1 = K_c \frac{3(3-\gamma)}{2(3\gamma-1)} m_x z^{\frac{\gamma+1}{\gamma-1}}, \tag{6.12}$$

$$\tilde{a}_2 = -K_c \frac{\gamma+1}{2(\gamma-1)} z^{\frac{\gamma+1}{2(\gamma-1)}-1}. \tag{6.13}$$

Then we do one more simplification by multiplying

$$\bar{\mu} = m^{-\frac{3(3-\gamma)}{2(3\gamma-1)}}, \tag{6.14}$$

on (6.10). In fact, it is easy to check that

$$\bar{\mu}' = -\tilde{a}_1 \bar{\mu}, \tag{6.15}$$

since $m' = cm_x$. Then we denote

$$y = \bar{\mu}\tilde{y}. \tag{6.16}$$

Hence (6.10) changes to

$$y' = a_0 + a_2 y^2, \tag{6.17}$$

where

$$a_0 = \bar{\mu}\tilde{a}_0,\ a_2 = \tilde{a}_2/\bar{\mu}. \tag{6.18}$$

Similarly, we prove (1.18). By (2.14), (2.9) and (4.3),

$$\begin{aligned} z' &= z_t - cz_x \\ &= -\frac{c}{m}(u_x + mz_x) \\ &= -K_c z^{\frac{\gamma+1}{\gamma-1}}(\alpha - \frac{\gamma-1}{\gamma} m_x z). \end{aligned} \tag{6.19}$$

Hence

$$\alpha = -\frac{z'}{K_c z^{\frac{\gamma+1}{\gamma-1}}} + \frac{\gamma-1}{\gamma} m_x z. \tag{6.20}$$

Plugging (6.20) into (1.11), we get

$$\beta' = k_1\{-k_2(3\beta - \tfrac{z'}{K_c z^{\frac{\gamma+1}{\gamma-1}}} + \tfrac{\gamma-1}{\gamma} m_x z) + \beta(-\tfrac{z'}{K_c z^{\frac{\gamma+1}{\gamma-1}}} + \tfrac{\gamma-1}{\gamma} m_x z) - \beta^2\}. \tag{6.21}$$

We move the terms including z' to the left hand side, then we multiply by $z^{\frac{\gamma+1}{2(\gamma-1)}}$ on both sides. After simplification, we have

$$\begin{aligned} & z^{\frac{\gamma+1}{2(\gamma-1)}}\beta' - \tfrac{m_x}{2\gamma} z^{\frac{\gamma+1}{2(\gamma-1)}} z' + \tfrac{\gamma+1}{2(\gamma-1)}\beta z^{\frac{\gamma+1}{2(\gamma-1)}-1} z' \\ = \ & -\tfrac{\gamma-1}{2\gamma^2} K_c z^{\frac{3(\gamma+1)}{2(\gamma-1)}+1} m_x^2 - \tfrac{2-\gamma}{2\gamma} K_c m_x z^{\frac{3(\gamma+1)}{2(\gamma-1)}}\beta - \tfrac{\gamma+1}{2(\gamma-1)} K_c z^{\frac{3(\gamma+1)}{2(\gamma-1)}-1}\beta^2, \end{aligned} \tag{6.22}$$

where we use (1.12). The left hand side of (6.22) is equal to

$$(\beta z^{\frac{\gamma+1}{2(\gamma-1)}} - \tfrac{\gamma-1}{\gamma(3\gamma-1)} z^{\frac{\gamma+1}{2(\gamma-1)}+1} m_x)' + \tfrac{\gamma-1}{\gamma(3\gamma-1)} z^{\frac{\gamma+1}{2(\gamma-1)}+1}(m_x)'. \tag{6.23}$$

We define a new variable $\tilde{q}$,

$$\tilde{q} = \beta z^{\frac{\gamma+1}{2(\gamma-1)}} - \frac{\gamma-1}{\gamma(3\gamma-1)} z^{\frac{\gamma+1}{2(\gamma-1)}+1} m_x. \tag{6.24}$$

So

$$\beta = \tilde{q} z^{-\frac{\gamma+1}{2(\gamma-1)}} + \frac{\gamma-1}{\gamma(3\gamma-1)} z m_x. \tag{6.25}$$

Hence (6.22) changes to

$$\begin{aligned} \tilde{q}' = \ & \tfrac{\gamma-1}{\gamma(3\gamma-1)} K_c z^{\frac{3(\gamma+1)}{2(\gamma-1)}+1} m m_{xx} - \tfrac{\gamma-1}{2\gamma^2} K_c z^{\frac{3(\gamma+1)}{2(\gamma-1)}+1} m_x^2 \\ & -\tfrac{2-\gamma}{2\gamma} K_c m_x z^{\frac{3(\gamma+1)}{2(\gamma-1)}}(\tilde{q} z^{-\frac{\gamma+1}{2(\gamma-1)}} + \tfrac{\gamma-1}{\gamma(3\gamma-1)} z m_x) \\ & -\tfrac{\gamma+1}{2(\gamma-1)} K_c z^{\frac{3(\gamma+1)}{2(\gamma-1)}-1}(\tilde{q} z^{-\frac{\gamma+1}{2(\gamma-1)}} + \tfrac{\gamma-1}{\gamma(3\gamma-1)} z m_x)^2, \end{aligned} \tag{6.26}$$

where we use $(m_x)^{\backprime} = -cm_{xx}$. After simplification, we have

$$\tilde{q}^{\backprime} = \tilde{a}_0 - \tilde{a}_1\tilde{q} + \tilde{a}_2\tilde{q}^2, \tag{6.27}$$

where $\tilde{a}_i$ are in (6.11)$\sim$(6.13). Similarly, we denote

$$q = \bar{\mu}\tilde{q}, \tag{6.28}$$

where $\bar{\mu}$ is in (6.14). Since

$$\bar{\mu}^{\backprime} = \tilde{a}_1\bar{\mu}, \tag{6.29}$$

(6.27) changes to

$$q^{\backprime} = a_0 + a_2 q^2, \tag{6.30}$$

where a_0, a_2 are defined in (6.18). $\square$

It is easy to check that $a_2 < 0$,

$$a_0 \gtreqless 0 \Leftrightarrow (3\gamma - 1)mm_{xx} \gtreqless (3\gamma + 1)m_x^2 \Leftrightarrow (m^{-\frac{2}{3\gamma-1}})_{xx} \lesseqgtr 0. \tag{6.31}$$

So the sign of a_0 only depends on the entropy profile, for fixed γ. Note that the entropy profile of smooth solution is stationary because of (2.16). So the sign of a_0 only depends on the initial data.

Corollary 2. *When Assumption 1 holds,*

$$|y| \ \mathit{or}\ |q| \to \infty \ \mathit{iff}\ |u_x| \ \mathit{or}\ |\tau_x| \to \infty. \tag{6.32}$$

Proof. By (6.7), (6.16), (6.24) and (6.28),

$$y + q = 2m^{-\frac{3(3-\gamma)}{2(3\gamma-1)}} u_x z^{\frac{\gamma+1}{2(\gamma-1)}}, \tag{6.33}$$

$$y - q = 2m^{-\frac{3(3-\gamma)}{2(3\gamma-1)}} (m z^{\frac{\gamma+1}{2(\gamma-1)}} z_x + \frac{3\gamma-3}{3\gamma-1} z^{\frac{\gamma+1}{2(\gamma-1)}+1} m_x). \tag{6.34}$$

By the same argument as the proof of (4.6) in Lemma 4.2, (6.32) is right. We complete the proof. $\square$

Similarly, when Assumption 1 holds,

$$|\tilde{y}| \ \mathit{or}\ |\tilde{q}| \to \infty \ \mathit{iff}\ |u_x| \ \mathit{or}\ |\tau_x| \to \infty. \tag{6.35}$$

7 Singularity Formation

In this section, using the *quadratic* "ODEs" in the Theorem 1.3, we will give several singularity formation (gradient blowup) results for (1.1)$\sim$(1.3) with smooth initial data. By Corollary 2, we need to find conditions on the initial data, under which $|y|$ or $|q|$ blows up in finite time.

We consider the quadratic equation ($\xi = y$ or q),

$$0 = a_0 + a_2\xi^2, \tag{7.1}$$

where a_0 and a_2 are defined in (1.19) and (1.20). When $a_0 < 0$, there is no real root; when $a_0 = 0$, there is a unique root 0; when $a_0 > 0$, the two roots of (7.1) are

$$\pm\sqrt{-\frac{a_0}{a_2}} = \pm\sqrt{\frac{2(\gamma-1)^2}{\gamma(\gamma+1)(3\gamma-1)}(mm_{xx} - \frac{3\gamma+1}{3\gamma-1}m_x^2)}\ z^{\frac{\gamma+1}{2(\gamma-1)}+1}m^{-\frac{3(3-\gamma)}{2(3\gamma-1)}}. \tag{7.2}$$

By considering (1.17) and (1.18), we summarize the dynamic system properties for smooth solutions in Fig. 1. Moreover, $\{y > 0, q > 0\}$ is an invariant domain for the region with initial entropy profile satisfying $a_0 \geqslant 0$, by Fig. 1.

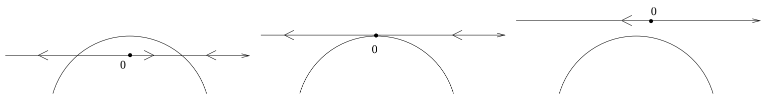

Figure 1: $a_0 > 0$, $a_0 = 0$, $a_0 < 0$ from left to right. The arrows indicate y or q increases or decreases.

For the p-system, m is a constant. Hence, $a_0 = 0$, and

$$y = s_x m^{-\frac{3(3-\gamma)}{2(3\gamma-1)}} z^{\frac{\gamma+1}{2(\gamma-1)}}, \quad q = r_x m^{-\frac{3(3-\gamma)}{2(3\gamma-1)}} z^{\frac{\gamma+1}{2(\gamma-1)}}, \tag{7.3}$$

so y and s_x (q and r_x) have same signs. Hence, the p-system follows the middle picture of Fig. 1. When y or q is negative somewhere, y or q will approach negative infinity in finite time. When y and q are nonnegative everywhere in the initial data, there is no such kind of singularity, since $\{y > 0, q > 0\}$ is an invariant domain. These results are given by Lax, c.f. [9].

We don't expect such "clean" results when the entropy profile is varying, because the dynamic properties in this case are much more complicated than those in a constant entropy domain, as we can see from Fig. 1. In [16] and a forthcoming paper [2], there are shock free examples with compressive waves and entropy jumps. We also expect the smooth examples including compressive waves.

Using the nonlinear *Riccati* type equations (1.17) and (1.18), we can prove Theorems 1.4 and 1.5 for breakdown of y and q, which directly generalize Lax's results in [9]. This kind of formation of singularity corresponds to the formation of a shock.

Theorem 1.4 says that the shock free solutions don't include "strong" compressive waves, since y, q, $\tilde{y}$, $\tilde{q}$ and α, β are larger than some negative constants in shock free solutions. When M_3, M_4 are close to zero, i.e. the variation of entropy profile is mild, N and $\tilde{N}$ are close to zero, where N and $\tilde{N}$ are given at (7.5) and (7.10). The shock free solutions should be "almost rarefactive". Theorem 1.4 is quite sharp at this time, since it is consistent with Lax's singularity results in [9]. In fact, when entropy is a constant, $N = \tilde{N} = 0$, Theorem 1.4 is the same as the singularity results for the p-system in [9]. The bounds N and $\tilde{N}$ only depend on Z_U, M_1, M_2, M_3, M_4. N and $\tilde{N}$ don't depend on Z_L, so Z_L can be arbitrarily small.

In Theorem 1.5, we consider the singularity formation for some special initial entropy profiles. Here, we give an exact example with entropy profile satisfying condition (1.23) and bounded away from infinity and negative infinity. For example,

$$m(x) = \begin{cases} f(x), & x \leqslant A, \\ (e^{-x}+1)^{-\frac{3\gamma-1}{2}}, & x > A, \end{cases} \tag{7.4}$$

satisfies (1.23), where $f(x)$ is a smooth function that makes $m(x)$ smooth and bounded away from zero and infinity. A large class of entropy functions satisfies (1.23).

7.1 Proof of Theorem 1.4

Proof. By (7.2) and Assumption 2, the roots of (7.1), if these exist, have uniform lower bound $-N$, where if $1 < \gamma \leqslant 3$,

$$N = (1+\varepsilon)\sqrt{\tfrac{2(\gamma-1)^2}{\gamma(\gamma+1)(3\gamma-1)}M_2M_4}\ Z_U^{\frac{\gamma+1}{2(\gamma-1)}+1} M_1^{-\frac{3(3-\gamma)}{2(3\gamma-1)}}; \tag{7.5}$$

if $\gamma > 3$, M_1 changes to M_2 in (7.5). Here, ε is a fixed positive constant, which can be arbitrarily small. So

$$a_0 + a_2\frac{N^2}{(1+\varepsilon)^2} < 0, \tag{7.6}$$

since $a_2 < 0$. Recall, m is stationary by (2.16).

If $y < -N$ somewhere in the initial data, then by (7.6) and $a_2 < 0$,

$$a_0 + a_2\frac{y^2}{(1+\varepsilon)^2} < 0. \tag{7.7}$$

So (1.17) gives that

$$y' = a_0 + a_2y^2 < (1 - \frac{1}{(1+\varepsilon)^2})a_2y^2. \tag{7.8}$$

Hence y is decreasing at this time by (1.20), so y is always less than $-N$ along forward characteristic line. Furthermore, by (7.8),

$$\frac{1}{y(t)} \geqslant \frac{1}{y(0)} - \int_0^t (1 - \frac{1}{(1+\varepsilon)^2})a_2 dt, \tag{7.9}$$

where the integral is along the characteristic. By (1.20) and Assumption 2, a_2 is negative and bounded below. So the right hand side of (7.9) approaches zero in finite time, which gives that $y(t)$ approaches $-\infty$ in finite time. By (6.32), $|\tau_x|$ and/or $|u_x|$ blow up.

We have another version of the singularity formation results if we start from (6.10) and (6.27) for $\tilde{y}$ and $\tilde{q}$. We assume that

$$\tilde{N} = (1+\varepsilon)\frac{(\gamma-1)[\ |9-3\gamma|M_3+\sqrt{|A_1|M_3^2+|A_2|M_2M_4}\ ]}{2(3\gamma-1)(\gamma+1)} Z_U{}^{\frac{\gamma+1}{2(\gamma-1)}+1}, \tag{7.10}$$

where ε is a fixed positive constant which can be arbitrarily small,

$$\begin{aligned} A_1 &= (9\gamma^2 - 54\gamma + 81) - \tfrac{1}{1+\varepsilon}\tfrac{1}{\gamma}(24\gamma^2 + 32\gamma + 8), \\ A_2 &= \tfrac{1}{1+\varepsilon}\tfrac{1}{\gamma}(24\gamma^2 + 16\gamma - 8). \end{aligned} \tag{7.11}$$

The solutions of

$$\tilde{a}_0 \pm \tilde{a}_1\tilde{y} + \frac{1}{1+\varepsilon}\tilde{a}_2\tilde{y}^2 = 0, \tag{7.12}$$

if existing, are

$$(1+\varepsilon)\frac{(\gamma-1)[\ \pm(9-3\gamma)m_x \pm \sqrt{A_1 m_x^2 + A_2 m m_{xx}}\ ]z^{\frac{\gamma+1}{2(\gamma-1)}+1}}{2(3\gamma-1)(\gamma+1)}, \tag{7.13}$$

which are always larger than $-\tilde{N}$. If $\tilde{y} < -\tilde{N}$ somewhere in the initial data

$$\tilde{y}' < (1 - \frac{1}{(1+\varepsilon)})\tilde{a}_2\tilde{y}^2. \tag{7.14}$$

Hence $\tilde{y}$ will go to negative infinity in finite time, by the same proof of the singularity result for y. By (6.35), the absolute values of τ_x and/or u_x blow up.

We have symmetric results for q and $\tilde{q}$. □

7.2 Proof of Theorem 1.5

Proof. We use $y(t)$ to denote the function y along the forward characteristic starting from $(0, A^*)$, and assume $y_0 = y(0) < 0$. By (6.31), $a_0 \leqslant 0$ when $x > A$. So (1.17) changes to

$$y' \geqslant a_2 y^2. \tag{7.15}$$

By solving (7.15), we get

$$\frac{1}{y} \leqslant \frac{1}{y_0} - \int a_2 dt, \tag{7.16}$$

where the integral is along the characteristic. Hence, when the right hand side of (7.16) goes to zero, y goes to $-\infty$. So, by $y_0 < 0$, (1.20) and Assumption 2, the blowup happens before $T_* \leqslant -\frac{1}{y_0 \min(-a_2)}$, where $\min(-a_2)$ is a positive constant depending on Z_L (Z_U) and M_1 (M_2) when $1 < \gamma < 3$ ($\gamma > 3$). When $\gamma = 3$, a_2 is a constant, so we don't need Assumption 2. Symmetric results apply for q. □

When $1 < \gamma < 3$, we don't need the upper bounds for z, $|m_x|$ and $|m_{xx}|$.

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