WELL-POSEDNESS FOR MULTIDIMENSIONAL SCALAR CONSERVATION LAWS WITH DISCONTINUOUS FLUX

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Abstract. We obtain a well-posedness result of an entropy solution to a multidimensional scalar conservation law with discontinuous (quasi-homogeneous) flux satisfying crossing conditions, but with no genuine nonlinearity assumptions. The proof is based on the kinetic formulation of the equation under consideration and it does not involve any transformation of the original equation or existence of strong traces. Also, we propose Brenier's transport-collapse type operator corresponding to the problem under consideration.

1. Introduction

In the paper, we are looking for the existence and uniqueness to the following Cauchy problem:

$$\partial_t u + \mathrm{Div}_x\, f(x,u) = 0, \quad u = u(t,x), \quad t \geq 0,\ x \in \mathbb{R}^d. \tag{1}$$

$$u|_{t=0} = u_0(x) \in L^\infty(\mathbb{R}^d), \quad a \leq u_0 \leq b, \tag{2}$$

where the flux vector $f(x,\lambda) = (f_1(x,\lambda), \dots, f_d(x,\lambda))$, $\lambda \in \mathbb{R}$, is assumed to be continuously differentiable with respect to $u \in \mathbb{R}$ and discontinuous with respect to $x \in \mathbb{R}^d$ so that, for every $\lambda \in \mathbb{R}$, the discontinuity is placed on the manifold $\Gamma \subset \mathbb{R}^d$ of co-dimension one which divides the space $\mathbb{R}^d$ into two domains. This assumption is not substantial, and considerations can be easily repeated for more complicated situations of $(d-1)$-dimensional manifolds.

More precisely, we assume that there exist two domains Ω_L and Ω_R such that:

$$\mathbb{R}^d = \Omega_L \cup \Gamma \cup \Omega_R, \quad \overline{\Omega}_L \cap \overline{\Omega}_R = \Gamma, \tag{3}$$

and that, by denoting

$$\kappa_L(x) = \begin{cases} 1, & x \in \Omega_L \\ 0, & x \notin \Omega_L \end{cases}, \quad \kappa_R(x) = \begin{cases} 1, & x \in \Omega_R \\ 0, & x \notin \Omega_R \end{cases},$$

we can rewrite (1) in the form:

$$\partial_t u + \mathrm{Div}_x\, (g_L(x,u)\kappa_L(x) + g_R(x,u)\kappa_R(x)) = 0. \tag{4}$$

In the sequel, by $A(\hat{x_i})$ we imply that the quantity A does not depend on x_i but only on $x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_d$. With such a convention, we assume that the functions $g_L, g_R \in C^1(\mathbb{R}^{d+1}; \mathbb{R}^d)$ are of the form:

$$g_L(x,u) = (g_{1L}(\hat{x_1}, u), \dots, g_{dL}(\hat{x_d}, u)),$$
$$g_R(x,u) = (g_{1R}(\hat{x_1}, u), \dots, g_{dR}(\hat{x_d}, u)).$$

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The substantial demand on the functions g_L and g_R is so called "crossing condition" introduced in [8].

Definition 1. (The crossing conditions) We say that the functions g_L and g_R satisfy the crossing conditions if for every $i = 1, \dots, d$

$$g_{iR}(\hat{x}_i, u) - g_{iL}(\hat{x}_i, u) < 0 < g_{iR}(\hat{x}_i, v) - g_{iL}(\hat{x}_i, v) \Rightarrow u < v. \tag{5}$$

Conservation laws like (1) have a number of important applications, and pose several analytical challenges not present in the now classical situation where the flux is continuous. We shall avoid listing of numerous references and remind readers on [18], where the problem was opened, and address them to [2] and references therein where one can find thorough description, analysis, but also a kind of unification of previous works on the problem. We stress that most of papers on the subject were addressed on the one-dimensional situation of the problem, while in the multidimensional case there are very few results. We are listing them right now without getting into details of the papers [1, 11, 14, 15].

Concerning existence, the main difficulty is the lack of a spatial variation bound. The uniqueness question is perhaps more difficult. First there is the fact that most equations like (1) have more than one reasonable notion of solution [5]. Once a notion of solution is selected, there is the problem of characterizing it in a way that is useful for analysis.

In order to explain our ideas more concisely, let us consider one-dimensional variant of the problem:

$$\begin{aligned} &\partial_t u + \partial_x (f(u)H(x) + g(u)H(-x)) = 0 \\ &u|_{t=0} = u_0(x), \quad 0 \le u_0 \le 1, \end{aligned} \tag{6}$$

where $f(0) = g(0) = f(1) = g(1) = 0$. If it holds $f \equiv g$, we can apply the Kruzhkov entropy admissibility concept [9] which provides existence and uniqueness of a weak solution to (6). Probably the first successful attempt to adapt Kruzhkov's concept on the case of scalar conservation law with discontinuous flux was made in [8]. There, the following definition is used:

Definition 2. [8] Let u be a weak solution to problem (6).

We say that u is an entropy admissible weak solution to (6) if the following entropy condition is satisfied for every fixed $\xi \in \mathbf{R}$:

$$\begin{aligned} \partial_t |u - \xi| + \partial_x \Big\{ \mathrm{sgn}(u - \xi) \Big[H(x)(f(u) - f(\xi)) + H(-x)(g(u) - g(\xi)) \Big] \Big\} \\ - |f(\xi) - g(\xi)| \delta(x) \le 0 \ \text{ in } \ \mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}). \end{aligned} \tag{7}$$

This definition, together with the "crossing condition" ($f(u) - g(u) < 0 < f(v) - g(v) \Rightarrow u < v$) provided uniqueness to the considered Cauchy problem. In [10], by introducing the change of the unknown function $u = \alpha(v)H(x) + \beta(v)H(-x)$ for appropriate bijections

$$\alpha, \beta : [0, 1] \to [a', b'] \subset \mathbb{R} \tag{8}$$

we transformed the equation in (6) so that the crossing conditions were satisfied. This provided uniqueness in a rather general situation. Existence is obtained by considering separately subintervals of $[0, 1]$ on which functions f and g are genuinely nonlinear, and the subintervals where the genuine nonlinearity is lost. This enabled us to apply results from [14, 13] to complete the paper.

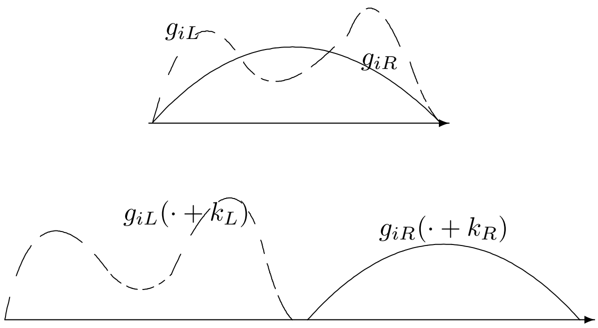

FIGURE 1. Situation from [11] for $i = 1, \dots, d$.

We were not able to extend latter techniques on the multidimensional case. It appears that a proper way out lies in using semi-entropies

$$|z|^+ = \begin{cases} z, & z > 0 \\ 0, & z \leq 0 \end{cases}, \quad |z|^- = \begin{cases} 0, & z > 0 \\ -z, & z \leq 0 \end{cases}$$

rather than standard entropies. First substantial application of the semi-entropies was given in [3]. There, the function u is said to be an entropy solutions if for every $\xi \in (0,1)$:

$$\begin{split} \partial_t |u - \xi|^{\pm} + \partial_x \Big\{ \mathrm{sgn}_{\pm}(u-\xi) \Big[H(x)(f(u) - f(\xi)) + H(-x)(g(u) - g(\xi)) \Big] \Big\} \\ - |g(\xi) - f(\xi)|^{\pm} \delta(x) \leq 0 \ \text{ in } \ \mathcal{D}'(I\!R^+ \times I\!R), \end{split} \tag{9}$$

where $\mathrm{sgn}_{\pm}(z) = (|z|^{\pm})'$. It is clear that by putting separately $+$ and $-$ in (9), and then adding the resulting expressions, we reach to (7). However, it can be (fairly easily) shown that (7) implies (9) in the case when a solution to (4), (2) satisfying (7) admits traces at $x = 0$. This means that the problem of non-uniqueness also remains when one uses semi-entropies instead of entropies. Continuing in this direction, we remark that, if the crossing condition is violated, as shown in [2], uniqueness of the entropy admissible solution to (6) does not hold in general even in the one-dimensional case.

We shall show in this paper that if semi-entropies are applied and the crossing condition is satisfied, then we have uniqueness to (6) no matter whether the traces exist. However, mere application of semi-entropies and the classical Kruzhkov's techniques was not enough to prove existence and uniqueness. It was necessary to introduce original kinetic type formulation of problem (6) (compare [3, Section 3.3.1] with [12]), and to apply original techniques in the proof (compare [3, Section 3.4] with [17]). In [11], we have adapted techniques from [3] to prove existence and uniqueness to the multidimensional problem (4), (2), but the constructed stable semigroup was rather special. It was tacitly assumed that the functions f and g are compactly supported so that the transformation $u = (k_R + v)H(x) + (k_L + v)H(-x)$ for appropriate k_L, k_R rendered the flux from (4) in a position in which it was possible to apply techniques from [3] (see Figure 1). Also, we had certain structural conditions on the interface.

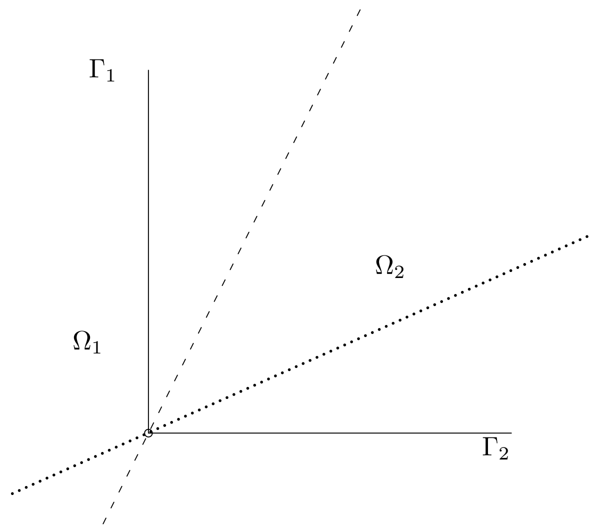

FIGURE 2. Dotted line is the boundary of Ω_1. Dashed line is the boundary of Ω_2. Both lines contain the edge of the interface (hollow point). $\Gamma = \Gamma_1 \cup \Gamma_2$ is the interface.

Here, we further develop techniques from [3, 11] which enables us to prove well-posedness to (4), (2) assuming that the flux is such that the crossing conditions and the maximum principle is satisfied. Finally, we remark that by using the transformation of type (8) which render fluxes not satisfying the crossing conditions to the ones satisfying the crossing conditions (as in [10]), we obtain existence and uniqueness for different semigroups (see [10, Remark 2] and [2, Section 6.2]) corresponding to different physical situations (see [5]).

The paper is organized as follows.

In the first part of the paper (Section 2), we shall consider the case when the discontinuity manifold is piecewise parallel to the coordinate hyper-planes (Figure 2). We shall prove that the Cauchy problem corresponding to such situation is well-posed.

In the second part (Section 3), we extend the results on the case of an arbitrary manifold which can be approximated by a manifold which is piecewise parallel to coordinate hyper-planes.

In the Appendix, we give a remark on a possible extension of Brenier's transport-collapse scheme [4] on scalar conservation laws with discontinuous flux.

2. Admissibility conditions for the piecewise parallel interface

In this section, we shall introduce admissibility conditions first for the situation when the interface is piecewise parallel to coordinate hyper-planes.

Assume that the interface (hyper-polyhedron) Γ contains n parts parallel to the coordinate hyper-planes. We can split the space $\mathbb{R}^d$ (excluding the edges of the interface) on open sets Ω_j, $j = 1, \dots, n$, such that each Ω_j, $j = 1, \dots, n$, contains a single part of the interface which is parallel to a coordinate hyper-plane (excluding edges of the hyper-polyhedron forming the interface; see Figure 2). In other words:

$$\begin{aligned}
&\Gamma = Cl\left(\bigcup_{j=1}^{n}\{x\in I\!R^d:\ x_{k_j}=c_j\}\cap\Omega_j\right);\quad I\!R^d=\Gamma\cup\left(\bigcup_{j=1}^{n}\bar\Omega_j\right);\\
&\Gamma_j = Cl\left(\Gamma\cap\Omega_j\right);\quad \Gamma_j\cap\Omega_s=\begin{cases}\emptyset, & s\neq j,\\ \Gamma_j, & s=j\end{cases},\ s,j=1,\dots,n,
\end{aligned} \tag{10}$$

where for $j\in\{1,\dots,n\}$, we have appropriate k_j-th coordinate (i.e. $k_j\in\{1,\dots,d\}$), and c_j, $j=1,\dots,n$, are appropriate constants.

Definition 3. Let $u_0\in L^\infty(I\!R^d)$, $a\le u_0\le b$ a.e. on $I\!R^d$. Let $u\in L^\infty(I\!R^+\times I\!R^d)$.

1. The function u is an entropy sub-solution (respectively entropy super-solution) of problem (4), (2) if for any $\xi\in I\!R$ and any $\varphi\in C^1_0(I\!R\times\Omega_j)$, $j\in 1,\dots,n$:

$$\begin{aligned}
&\int_{I\!R^+\times I\!R^d}(u-\xi)^\pm\partial_t\varphi dtdx \\
&+\int_{I\!R^+\times I\!R^d}\sum_{i=1}^d \operatorname{sgn}_\pm(u-\xi)\Big(\left(g_{iL}(\hat x_i,u)-g_{iL}(\hat x_i,\xi)\right)H(c_j-x_{k_j})\\
&\qquad\qquad+\left(g_{iR}(\hat x_i,u)-g_{iR}(\hat x_i,\xi)\right)H(x_{k_j}-c_j)\Big)\partial_{x_i}\varphi dtdx\\
&+\int_{I\!R^d}(u_0-\xi)^\pm\varphi(0,x)dx\\
&-\int_{I\!R^+\times I\!R^{d-1}}\left(g_{k_jL}(\hat x_{k_j},\xi)-g_{k_jR}(\hat x_{k_j},\xi)\right)^\pm\varphi|_{x_{k_j}=c_j}d\hat x_{k_j}dt\ge 0.
\end{aligned} \tag{11}$$

2. The function u is an entropy solution if it is a weak entropy process sub- and super-solution at the same time.

We shall also need notions of nonlinear weak-$\star$ convergence and entropy process sub and super solution.

Definition 4. Let Ω be an open subset of $I\!R^d$ and $(u_n)\subset L^\infty(\Omega)$ and $u\in L^\infty(\Omega\times(0,1))$. The sequence (u_n) converges towards u in the nonlinear weak-$\star$ sense if

$$\int_\Omega g(u_n(x))\psi(x)dx\to\int_0^1\int_\Omega g(u(x,\lambda))\psi(x)dxd\lambda\quad as\quad n\to\infty,$$
$$\forall\psi\in L^1(\Omega),\quad\forall g\in C(I\!R).$$

Any bounded sequence of $L^\infty(\Omega)$ has a subsequence converging in the nonlinear weak-$\star$ sense.

Theorem 5. [7] *Let Ω be an open subset of $I\!R^d$ and (u_n) be a bounded sequence of $L^\infty(\Omega)$. Then (u_n) admits a subsequence converging in the nonlinear weak-$\star$ sense.*

Using the nonlinear weak-$\star$ convergence concept, we introduce the notion of entropy process super and sub solutions.

Definition 6. Let $u_0\in L^\infty(I\!R^d)$, $a\le u_0\le b$ a.e. on $I\!R^d$. Let $u\in L^\infty([0,\infty)\times I\!R^d\times(0,1))$.

1. The function u is an entropy process sub-solution (respectively entropy process super-solution) of problem (4), (2) if for any $\xi\in I\!R$ and any $\varphi\in C^1_0(I\!R^+\times\Omega_j)$,

$j \in 1, \dots, n$:

$$\int_0^1 d\lambda \int_{I\!R^+ \times I\!R^d} (u-\xi)^{\pm} \partial_t \varphi dt dx \tag{12}$$

$$+ \int_0^1 d\lambda \int_{I\!R^+ \times I\!R^d} \sum_{i=1}^d \mathrm{sgn}_{\pm}(u-\xi) \Big(\left(g_{iL}(\hat{x_i}, u) - g_{iL}(\hat{x_i}, \xi) \right) H(c_j - x_{k_j})$$

$$+ \left(g_{iR}(\hat{x_i}, u) - g_{iR}(\hat{x_i}, \xi) \right) H(x_{k_j} - c_j) \Big) \partial_{x_i} \varphi dt dx$$

$$+ \int_{I\!R^d} (u_0 - \xi)^{\pm} \varphi(0,x) dx$$

$$- \int_{I\!R^+ \times I\!R^{d-1}} \left(g_{k_j L}(\hat{x}_{k_j}, \xi) - g_{k_j R}(\hat{x}_{k_j}, \xi) \right)^{\pm} \varphi|_{x_{k_j} = c_j} d\hat{x}_{k_j} dt \geq 0.$$

2. The function u is an entropy solution if it is a weak entropy process sub- and super-solution at the same time.

It is not difficult to prove the existence of an entropy process solution to (4), (2). The proof is application of standard arguments given in e.g. [11, Theorem 2.8].

Theorem 7. *There exists an entropy process solution to* (4), (2).

Proof: We shall briefly recall arguments proving the theorem. First, we regularize the characteristic functions appearing in (4). Then, since $g_L(x,a) = g_L(x,b) = g_R(x,a) = g_R(x,b) = 0$, we conclude that family of entropy admissible solutions $(u_\varepsilon)_{\varepsilon \in (0,1)}$ to the regularized equation satisfy $a \leq u_\varepsilon \leq b$. According to Theorem 5, (u_ε) admits the nonlinear weak-$\star$ limit along a subsequence which we denote by $u \in L^\infty(I\!R^+ \times I\!R^d \times (0,1))$. Since (u_ε) are entropy admissible, by letting $\varepsilon \to 0$ along the subsequence defining the function u, we reach to (11). □

We shall prove the following comparison principle which establishes the uniqueness and existence of entropy admissible solutions to (4), (2) in the case when the interface is piecewise parallel to coordinate axis.

Theorem 8. *Any two entropy process solutions u and v to* (4), *where g_L and g_R satisfy the crossing conditions* (5), *with initial conditions u_0 and v_0, respectively, satisfy the following relation for any $T > 0$ and any ball $B(0,R) \subset I\!R^d$:*

$$\int_0^1 d\eta \int_0^1 d\lambda \int_0^T \int_{B(0,R)} (u(t,x,\lambda) - v(t,x,\eta))^{\pm} dx dt \leq T \int_{B(0,R+CT)} (u_0(x) - v_0(x))^{\pm} dx, \tag{13}$$

for a constant $C > 0$ independent of $T, R > 0$.

The proof of the theorem is based on the kinetic formulation of (11). Before we introduce it, we need some auxiliary notions. For functions $u, v \in L^\infty(I\!R^+ \times I\!R^d \times (0,1))$, $u_0, v_0 \in L^\infty(I\!R^d; [a,b])$, we denote:

$$h_{\pm}(t,x,\lambda,\xi) = \mathrm{sgn}_{\pm}(u(t,x,\lambda) - \xi), \quad j_{\pm}(t,x,\eta,\xi) = \mathrm{sgn}_{\pm}(v(t,x,\eta) - \xi)$$

$$h^0_{\pm}(x,\xi) = \mathrm{sgn}_{\pm}(u_0(x) - \xi), \quad j^0_{\pm}(x,\xi) = \mathrm{sgn}_{\pm}(u_0(x) - \xi).$$

The functions $h_\pm$ and $j_\pm$ we call equilibrium functions.

In the sequel, we shall imply $\int_{t,x,\xi} \cdot = \int_{I\!R^+ \times I\!R^d \times I\!R} \cdot \, dt dx d\xi$.

Definition 9. Denote

$$G_{iL}(x,\xi)=\partial_\xi g_{iL}(x,\xi), \quad G_{iR}(x,\xi)=\partial_\xi g_{iR}(x,\xi).$$

Let $u_0\in L^\infty(I\!R^d;[a,b])$ and $u\in L^\infty(I\!R^+\times I\!R^d\times(0,1))$.

The function u is a kinetic process super-solution (respectively kinetic process sub-solution) to (4), (2) if there exists $m_\pm\in C(I\!R_\xi;w-\star\mathcal{M}_+(I\!R^+\times I\!R^d))$ such that $m_+(\cdot,\xi)$ vanishes for large ξ (respectively, $m_-(\cdot,\xi)$ vanishes for large $-\xi$), and such that for any $j=1,\dots,n$, and every $\varphi\in C^1(I\!R^+\times\Omega_j\times(0,1))$,

$$\begin{aligned}
&\int_0^1 d\lambda\int_{t,x,\xi} h_\pm\times \qquad (14)\\
&\qquad\times\Big(\partial_t+\sum_{i=1}^d\big(G_{iL}(\hat{x_i},\xi)H(c_j-x_{k_j})+G_{iR}(\hat{x_i},\xi)H(x_{k_j}-c_j)\big)\,\partial_{x_i}\Big)\varphi\\
&+\int_{x,\xi} h^0_{\pm,k}\varphi|_{t=0}-\int_{t,\hat{x}_{k_j},\xi}\partial_\xi\left(g_{k_jL}(\hat{x_{k_j}},\xi)-g_{iR}(\hat{x_{k_j}},\xi)\right)^\pm\varphi|_{x_{k_j}=c_j}\\
&=\int_{t,x,\xi}\partial_\xi\varphi dm_\pm.
\end{aligned}$$

As in [11, Proposition 1] we can prove the following proposition. It is basically obtained by appealing on the Schwarz lemma for non-negative distributions and (then) differentiating (12) with respect to $\xi\in I\!R$.

Proposition 10. *The entropy process admissible solution is at the same time the kinetic process solution.*

In the sequel, we shall denote by $h_\pm$ and $j_\pm$ equilibrium functions corresponding to the entropy process solutions u and v to (4) with initial conditions $u_0\in L^\infty(I\!R^d;(a,b))$ and $v_0\in L^\infty(I\!R^d;(a,b))$, respectively.

We shall also need the following known formula. It holds for a $\theta\in C^1_0(I\!R^+\times I\!R^d)$, and arbitrary functions $\alpha_i: I\!R^{d-1}\to I\!R$:

$$\begin{aligned}
&\int_0^1 d\lambda\int_0^1 d\eta\int_{t,x,\xi}(-h_+j_-)\Big(\partial_t\theta \qquad (15)\\
&\qquad+\sum_{i=1}^d\left(G_{iL}(\hat{x_i},\xi)H(\alpha_i(\hat{x_i})-x_i)+G_{iR}(\hat{x_i},\xi)H(x_i-\alpha_i(\hat{x_i}))\right)\partial_{x_i}\theta\Big)\\
&=\int_0^1 d\lambda\int_0^1 d\eta\int_{t,x}\Big(|u(t,x,\lambda)-v(t,x,\eta)|^+\partial_t\theta\\
&\qquad+\sum_{i=1}^d \mathrm{sgn}_+(u(t,x,\lambda)-v(t,x,\eta))\times\\
&\qquad\times\Big(\big(g_{iL}(\hat{x_i},u(t,x,\lambda))-g_{iL}(\hat{x_i},v(t,x,\eta))\big)H(\alpha_i(\hat{x_i})-x_i)\\
&\qquad\quad+\big(g_{iR}(\hat{x_i},u(t,x,\lambda))-g_{iR}(\hat{x_i},v(t,x,\eta))\big)H(x_i-\alpha_i(\hat{x_i}))\Big)\partial_{x_i}\theta
\end{aligned}$$

Finally, we need a lemma concerning the traces of the entropy process solutions along the line $t=0$.

Lemma 11. *Assume that the bounded functions $u = u(t,x,\lambda)$ and $v = v(t,x,\eta)$ are two entropy process solutions to (4) corresponding to the initial condition $u_0 \in L^\infty(I\!R^d;[a,b])$ and $v_0 \in L^\infty(I\!R^d;[a,b])$, respectively.*

Introduce the cut-off function

$$\omega_\varepsilon(s) = \int_0^{|s|} \rho_\varepsilon(r)dr, \quad \rho_\varepsilon(r) = \varepsilon^{-1}\rho(\varepsilon^{-1}r), \quad s \in I\!R^d,\ r \in I\!R, \tag{16}$$

where $\rho \in C_c^\infty((0,1))$ is a non-negative function with total mass one. The proof is the same as the proof of [11, Lemma 3.2.].

It holds for every $\varphi \in C_0^1(I\!R \times I\!R^d)$:

$$\begin{aligned} \lim_{n\to\infty} \int_0^1 d\lambda \int_0^1 d\eta \int_{I\!R^+\times I\!R^d} &|u(t,x,\lambda) - v(t,x,\eta)|^{\pm}\omega'_{1/n}(t)\varphi(t,x)dtdx \\ &\le \int_{I\!R^d} |u_0(x) - v_0(x)|^{\pm}\varphi(0,x)dtdx \end{aligned} \tag{17}$$

Finally, we shall need a lemma concerning properties of convolution operators. Notations are taken from the previous lemma.

Lemma 12. [6, Lemma II.1.] *Suppose that $a \in C^1(I\!R^d)$ and $u \in L^p_{loc}(I\!R^d)$, $1 \le p < \infty$. Then $(au) \star \rho_\varepsilon - a(u \star \rho_\varepsilon) \to 0$ as $\varepsilon \to 0$ in the Sobolev space $W^{1,p}_{loc}(I\!R^d)$.*

2.1. **Proof of Theorem 8.** Assume that for every $i = 1,\dots,d$ the functions g_{iL} and g_{iR} satisfy the crossing conditions, i.e. they are such that there exists a unique point $p_i \in (a,b)$ such that for every $x \in I\!R^d$

$$g_{iL}(\hat{x_i},\xi) \le g_{iR}(\hat{x_i},\xi),\ \xi > p_i; \quad \text{and} \quad g_{iL}(\hat{x_i},\xi) \ge g_{iR}(\hat{x_i},\xi),\ \xi < p_i. \tag{18}$$

Let $\psi^L_{p,\varepsilon}, \psi^R_{p,\varepsilon} \in C^1(I\!R)$ be non-negative monotonic functions such that

$$\begin{aligned} &\psi^L_{p,\varepsilon}(\xi) + \psi^R_{p,\varepsilon}(\xi) \equiv 1, \quad \xi \in I\!R, \\ &\psi^L_{p,\varepsilon}(\xi) \equiv 0, \quad \xi \le p + \varepsilon, \\ &\psi^R_{p,\varepsilon}(\xi) \equiv 0, \quad \xi \ge p - \varepsilon. \end{aligned} \tag{19}$$

Next, take the functions:

$$I\!R^+ \times I\!R^d \times I\!R \ni (t,x,\xi) \mapsto \rho_{\varepsilon,\sigma,\zeta}(t,x,\xi) = \sum_{i=1}^d \rho^i_{\varepsilon,\sigma,\zeta}(t,x,\xi) = \sum_{i=1}^d \rho_\varepsilon(t)\rho_\zeta(\xi)\rho_\sigma(x_i),$$

$$I\!R^+ \times I\!R^d \ni (t,x) \mapsto \rho_{\varepsilon,\sigma}(t,x) = \sum_{i=1}^d \rho^i_{\varepsilon,\sigma}(t,x) = \sum_{i=1}^d \rho_\varepsilon(t)\rho_\sigma(x_i),$$

where ρ_ε is defined in (16), and let

$$\begin{aligned} j_{\pm,\varepsilon_j,\sigma_j,\zeta_j}(t,x,\xi,\eta) &= j_\pm \star \rho_{\varepsilon_j,\sigma_j,\zeta_j}(t,x,\xi,\eta) \\ h_{\pm,\varepsilon_h,\sigma_h,\zeta_h}(t,x,\xi,\lambda) &= h_\pm \star \rho_{\varepsilon_h,\sigma_h,\zeta_h}(t,x,\xi,\lambda). \end{aligned}$$

Notice that

$$\begin{aligned} j_{\pm,\varepsilon_j,\sigma_j} &= \lim_{\zeta_j\to 0} j_{\pm,\varepsilon_j,\sigma_j,\zeta_j} = j_\pm \star \rho_{\varepsilon_j,\sigma_j} \\ h_{\pm,\varepsilon_h,\sigma_h} &= \lim_{\zeta_h\to 0} h_{\pm,\varepsilon_h,\sigma_h,\zeta_h} = h_\pm \star \rho_{\varepsilon_h,\sigma_h}, \end{aligned}$$

where the limit is understood in the strong L^1_{loc} sense.

Now, fix $s \in \{1, \dots, n\}$ and choose the following test function

$$\varphi(t, x, \xi) = \theta \star \rho_{\varepsilon,\sigma,\zeta},$$

where $\text{supp}\theta \subset I\!R^+ \times (\Omega_s \backslash \Gamma) \times I\!R$, in the place of the function φ from (14).

For $\varepsilon, \sigma, \zeta$ small enough, the following also holds:

$$\text{supp}\theta \star \rho_{\varepsilon,\sigma,\zeta} \subset I\!R^+ \times (\Omega_s \backslash \Gamma) \times I\!R.$$

Therefore, for the equilibrium functions $h_\pm$, (14) becomes:

$$\begin{aligned}
&\int_0^1 d\lambda \int_{t,x,\xi} h_\pm \star \rho_{\varepsilon_h,\sigma_h,\zeta_h} \partial_t \theta \\
&\quad + \sum_{i=1}^d \Big(h_\pm \big(G_{iL}(\hat{x_i}, \xi) H(c_s - x_{k_s}) + G_{iR}(\hat{x_i}, \xi) H(x_{k_s} - c_s) \big) \Big) \star \rho_{\varepsilon_h,\sigma_h,\zeta_h} \partial_{x_i} \theta \\
&\quad = \int_{t,x,\xi} \partial_\xi \theta m_\pm^{\varepsilon_h,\sigma_h,\zeta_h}.
\end{aligned} \tag{20}$$

where $m_\pm^{\varepsilon_h,\sigma_h,\zeta_h} = m_\pm \star \rho_{\varepsilon_h,\sigma_h,\zeta_h}$, while for the equilibrium functions $j_\pm$

$$\begin{aligned}
&\int_0^1 d\eta \int_{t,x,\xi} j_\pm \star \rho_{\varepsilon_j,\sigma_j,\zeta_j} \partial_t \theta \\
&\quad + \sum_{i=1}^d \Big(j_\pm \big(G_{iL}(\hat{x_i}, \xi) H(c_s - x_{k_s}) + G_{iR}(\hat{x_i}, \xi) H(x_{k_s} - c_s) \big) \Big) \star \rho_{\varepsilon_j,\sigma_j,\zeta_j} \partial_{x_i} \theta \\
&\quad = \int_{t,x,\xi} \partial_\xi \theta q_\pm^{\varepsilon_j,\sigma_j,\zeta_j},
\end{aligned} \tag{21}$$

where $q_\pm^{\varepsilon_j,\sigma_j,\zeta_j} = q_\pm \star \rho_{\varepsilon_j,\sigma_j,\zeta_j}$.

Next, in (20) take instead of $\pm$ the sign $+$ and $\theta(t, x, \xi) = -\psi^L_{\epsilon,p_{k_s}}(\xi)\varphi(t, x) j_{-,\varepsilon_j,\sigma_j,\zeta_j}$ where $\varphi \in C^1_0(I\!R^+ \times (\Omega_s \backslash \Gamma))$, and integrate over $\eta \in (0, 1)$. Similarly, for the same function φ, in (21) take instead of $\pm$ the sign $-$ and $\theta(t, x, \xi) = -\psi^L_{\epsilon,p_{k_s}}(\xi)\varphi(t, x) h_{+,\varepsilon_h,\sigma_h,\zeta_h}$, and integrate over $\lambda \in (0, 1)$.

By adding the resulting expressions, we obtain:

$$\begin{aligned}
&\int_0^1 d\lambda \int_0^1 d\eta \int_{t,x,\xi} (-h_{+,\varepsilon_h,\sigma_h,\zeta_h} j_{-,\varepsilon_j,\sigma_j,\zeta_j}) \psi^L_{\epsilon,p} \Big(\partial_t \\
&\qquad + \sum_{i=1}^d (G_{iL}(\hat{x_i}, \xi) H(c_s - x_{k_s}) + G_{iR}(\hat{x_i}, \xi) H(x_{k_s} - c_s)) \, \partial_{x_i} \Big) \varphi \\
&= \int_0^1 d\eta \int_{t,x,\xi} \varphi \partial_\xi (-\psi^L_{\epsilon,p_{k_s}} j_{-,\varepsilon_j,\sigma_j,\zeta_j}) m_+^{\varepsilon_h,\sigma_h,\zeta_h} \\
&\quad + \int_0^1 d\lambda \int_{t,x,\xi} \varphi \partial_\xi (-\psi^L_{\epsilon,p_{k_s}} h_{+,\varepsilon_h,\sigma_h,\zeta_h}) q_-^{\varepsilon_j,\sigma_j,\zeta_j} \\
&\quad + R^{k_s}_{\varepsilon_h,\sigma_h,\zeta_h}(\varphi j_{-,\varepsilon_j,\sigma_j,\zeta_j}) + Q^{k_s}_{\varepsilon_j,\sigma_j,\zeta_j}(\varphi h_{+,\varepsilon_h,\sigma_h,\zeta_h}) + o_{\varepsilon_j+\sigma_j+\zeta_j}(1) + o_{\varepsilon_h+\sigma_h+\zeta_h}(1),
\end{aligned} \tag{22}$$

where the estimates $o_{\varepsilon_j+\sigma_j+\zeta_j}(1) \to 0$ as $\varepsilon_j + \sigma_j + \zeta_j \to 0$, and $o_{\varepsilon_h+\sigma_h+\zeta_h}(1) \to 0$ as $\varepsilon_h + \sigma_h + \zeta_h \to 0$ follow from Lemma 12 and we shall omit them in the sequel.

The terms $R^{k_s}_{\varepsilon_h,\sigma_h,\zeta_h}$ and $Q^{k_s}_{\varepsilon_j,\sigma_j,\zeta_j}$ are defined by

$$R^{k_s}_{\varepsilon_h,\sigma_h,\zeta_h}(\varphi)=\int_{t,x,\xi} h_{+,\varepsilon_h,\sigma_h,\zeta_h}\Big(G_{k_s L}(\hat{x}_{k_s},\xi)H(c_s-x_{k_s})+G_{k_s R}(\hat{x}_{k_s},\xi)H(x_{k_s}-c_s)\Big)\partial_{x_{k_s}}\varphi$$
$$-\Big(h_+\Big(G_{k_s L}(\hat{x}_{k_s},\xi)H(c_s-x_{k_s})+G_{k_s R}(\hat{x}_{k_s},\xi)H(x_{k_s}-c_s)\Big)\Big)\star\rho_{\varepsilon_h,\sigma_h,\zeta_h}\partial_{x_{k_s}}\varphi,$$

$$Q^{k_s}_{\varepsilon_j,\sigma_j,\zeta_j}(\varphi)=\int_{t,x,\xi} j_{+,\varepsilon_j,\sigma_j,\zeta_j}\Big(G_{k_s L}(\hat{x}_{k_s},\xi)H(c_s-x_{k_s})+G_{k_s R}(\hat{x}_{k_s},\xi)H(x_{k_s}-c_s)\Big)\partial_{x_{k_s}}\varphi$$
$$-\Big(j_-\Big(G_{k_s L}(\hat{x}_{k_s},\xi)H(c_s-x_{k_s})+G_{k_s R}(\hat{x}_{k_s},\xi)H(x_{k_s}-c_s)\Big)\Big)\star\rho_{\varepsilon_j,\sigma_j,\zeta_j}\partial_{x_{k_s}}\varphi,$$

and, according to the Friedrichs lemma:

$$R^{k_s}_{\varepsilon_h,\sigma_h,\zeta_h}(\varphi j_{-,\varepsilon_j,\sigma_j,\zeta_j})=\mathcal{O}(\frac{\zeta_h}{\sigma_j}),\quad Q^{k_s}_{\varepsilon_j,\sigma_j,\zeta_j}(\varphi h_{+,\varepsilon_h,\sigma_h,\zeta_h})=\mathcal{O}(\frac{\zeta_j}{\sigma_h}).$$

Finding the derivative in ξ on the right-hand of (22), and bearing in mind that $\partial_\xi(-j_{-,\varepsilon_j,\sigma_j,\zeta_j})>0$ and $\partial_\xi(-h_{+,\varepsilon_h,\sigma_h,\zeta_h})>0$, we conclude from (22) after letting $\zeta_h,\zeta_j\to 0$:

$$\int_0^1 d\lambda\int_0^1 d\eta\int_{t,x,\xi}(-h_{+,\varepsilon_h,\sigma_h}j_{-,\varepsilon_j,\sigma_j})\psi^L_{\epsilon,p_{k_s}}\Big(\partial_t \tag{23}$$
$$+\sum_{i=1}^d(G_{iL}(\hat{x}_i,\xi)H(c_s-x_{k_s})+G_{iR}(\hat{x}_i,\xi)H(x_{k_s}-c_s))\,\partial_{x_i}\Big)\varphi$$
$$\geq-\int_0^1 d\eta\int_{t,x,\xi}\varphi j_{-,\varepsilon_j,\sigma_j}\partial_\xi\psi^L_{\epsilon,p_{k_s}}dm_+^{\varepsilon_h,\sigma_h}-\int_0^1 d\lambda\int_{t,x,\xi}\varphi h^k_{+,\varepsilon_h,\sigma_h}\partial_\xi\psi^L_{\epsilon,p_{k_s}}dq_-^{\varepsilon_j,\sigma_j}$$

Let us now remove the conditions imposed on the support of function φ. In (23), for an arbitrary function $\theta\in C^1_0(I\!R^+\times\Omega_s)$, put:

$$\varphi(t,x)=\theta(t,x)\omega_{1/n}(x_{k_s}-c_s),$$

where ω is given by (16). We get:

$$\int_0^1 d\lambda\int_0^1 d\eta\int_{t,x,\xi}(-h_{+,\varepsilon_h,\sigma_h}j_{-,\varepsilon_j,\sigma_j})\psi^L_{\epsilon,p}\omega_{1/n}(x_{k_s}-c_s)\times \tag{24}$$
$$\times\Big(\partial_t+\sum_{i=1}^d(G_{iL}(\hat{x}_i,\xi)H(c_s-x_{k_s})+G_{iR}(\hat{x}_i,\xi)H(x_{k_s}-c_s))\,\partial_{x_i}\Big)\theta$$
$$+\int_0^1 d\lambda\int_0^1 d\eta\int_{t,x,\xi}(-h_{+,\varepsilon_h,\sigma_h}j_{-,\varepsilon_j,\sigma_j})\psi^L_{\epsilon,p_{k_s}}\theta\times$$
$$\times\Big(\partial_t+\sum_{i=1}^d\big(G_{iL}(\hat{x}_i,\xi)H(c_s-x_{k_s})+G_{iR}(\hat{x}_i,\xi)H(x_{k_s}-c_s)\big)\partial_{x_i}\Big)\omega_{1/n}(x_{k_s}-c_s)$$
$$\geq-\int_0^1 d\eta\int_{t,x,\xi}\theta j_{-,\varepsilon_j,\sigma_j}\partial_\xi\psi^L_{\epsilon,p_{k_s}}\omega_{1/n}(x_{k_s}-c_s)dm_+^{\varepsilon_h,\sigma_h}$$
$$-\int_0^1 d\lambda\int_{t,x,\xi}\theta h_{+,\varepsilon_h,\sigma_h}\omega_{1/n}(x_{k_s}-c_s)\partial_\xi\psi^L_{\epsilon,p_{k_s}}dq_-^{\varepsilon_j,\sigma_j}.$$

In order to cope with the problematic term in the previous expression containing derivatives of $\omega_{1/n}$, choose in (14) the sign $+$ instead of $\pm$, and for an arbitrary $\theta \in C_0^1(I\!R^+ \times \Omega_j \times I\!R)$:

$$\varphi(t,x,\xi) = (-(1-\omega_{1/n}(x_{k_s}-c_s))\psi^L_{\epsilon,p_{k_s}}\theta j_{-,\varepsilon_j,\sigma_j,\zeta_j}) \star \rho_{\varepsilon_h,\sigma_h,\zeta_h}.$$

Noticing that $\partial_\xi(-j_{-,\varepsilon_j,\sigma_j,\zeta_j}) > 0$, we get after letting $\zeta_j, \zeta_h \to 0$ (see the transition from (22) to (23)):

$$\begin{aligned}
&\int_0^1 d\lambda \int_0^1 d\eta \int_{t,x,\xi} (-h_{+,\varepsilon_h,\sigma_h} j_{-,\varepsilon_j,\sigma_j})\theta\psi^L_{\epsilon,p}\Big(\partial_t \qquad (25)\\
&\qquad + \sum_{i=1}^d (G_{iL}(\hat{x}_i,\xi)H(c_s - x_{k_s}) + G_{iR}(\hat{x}_i,\xi)H(x_{k_s}-c_s))\,\partial_{x_i}\Big)\omega_{1/n}(x_{k_s}-c_s)\\
&\quad + o_n(1) + \mathcal{O}(\epsilon)\\
&\leq \int_0^1 d\eta \int_{t,x,\xi} (1-\omega_{1/n}(x_{k_s}-c_s))\theta j_{-,\varepsilon_j,\sigma_j}\partial_\xi \psi^L_{\epsilon,p_{k_s}} dm_+^{\varepsilon_h,\sigma_h},
\end{aligned}$$

where $o_n(1)$ ($\to 0$ as $n \to \infty$) and $\mathcal{O}(\epsilon)$ are standard Landau symbols which depend only on θ, $\nabla_x\theta$ and $\partial_t\theta$. The term $\mathcal{O}(\epsilon)$ comes from the following relation:

$$\int_{t,x,\xi} \partial_\xi \left(g_{k_s L}(\hat{x}_{k_s},\xi) - g_{k_s R}(\hat{x}_{k_s},\xi)\right)^+ \psi^L_{\epsilon,p_{k_s}}\theta = \mathcal{O}(\epsilon),$$

where $\mathcal{O}(\epsilon)$ obviously depends on $\|\psi^L_{\epsilon,p_{k_s}}\theta\|_\infty$ and $\operatorname{supp}(\psi^L_{\epsilon,p_{k_s}}\theta)$. The term $o_n(1)$ is a consequence of the fact $1-\omega_{1/n}(x_{k_s}-c_s) \to 0$, $n\to\infty$, almost everywhere.

Taking into account (25), we get from (24):

$$\begin{aligned}
&\int_0^1 d\lambda \int_0^1 d\eta \int_{t,x,\xi} (-h_{+,\varepsilon_h,\sigma_h} j_{-,\varepsilon_j,\sigma_j})\psi^L_{\epsilon,p_{k_s}} \times \qquad (26)\\
&\quad \times \Big(\partial_t + \sum_{i=1}^d (G_{iL}(\hat{x}_i,\xi)H(c_s - x_{k_s}) + G_{iR}(\hat{x}_i,\xi)H(x_{k_s}-c_s))\,\partial_{x_i}\Big)\theta + o_n(1) + \mathcal{O}(\epsilon)\\
&\geq -\int_0^1 d\eta \int_{t,x,\xi} \theta j_{-,\varepsilon_j,\sigma_j}\partial_\xi \psi^L_{\epsilon,p} dm_+^{\varepsilon_h,\sigma_h}\\
&\quad - \int_0^1 d\lambda \int_{t,x,\xi} \omega_{1/n}(x_{k_s}-c_s)\theta h_{+,\varepsilon_h,\sigma_h}\partial_\xi\psi^L_{\epsilon,p} dq_-^{\varepsilon_j,\sigma_j}.
\end{aligned}$$

Now, in (20), take instead of $\pm$ the sign $+$ and $-\psi^R_{\epsilon,p}(\xi)\theta(t,x)j_{-,\varepsilon_j,\sigma_j,\zeta_j}$ in place of the test function, where $\theta \in C_0^1(I\!R^+\times\Omega_s)$ disappears in the neighborhood of the discontinuity manifold Γ, and integrate over $\eta \in (0,1)$. Similarly, for the same function φ, in (21), take instead of $\pm$ the sign $-$ and $-\psi^R_{\epsilon,p}(\xi)\theta(t,x)h_{+,\varepsilon_h,\sigma_h,\zeta_h}$ in place of the test function, and integrate over $\lambda \in (0,1)$. Summing the resulting expressions, letting $\zeta_h,\zeta_j \to 0$, and applying the procedure which led from (23) to

(26) with changed roles of h_+ and j_-, we get for an arbitrary $\theta \in C^1_0(I\!R^+ \times \Omega_s)$:

$$\int_0^1 d\lambda \int_0^1 d\eta \int_{t,x,\xi} (-h_{+,\varepsilon_h,\sigma_h} j_{-,\varepsilon_j,\sigma_j}) \psi^R_{\epsilon,p} \Big(\partial_t \tag{27}$$

$$+ \sum_{i=1}^d \left(G_{iL}(\hat{x}_i,\xi) H(c_s - x_{k_s}) + G_{iR}(\hat{x}_i,\xi) H(x_{k_s} - c_s) \right) \partial_{x_i} \Big) \theta + \mathcal{O}(\epsilon) + o_n(1)$$

$$\geq - \int_0^1 d\eta \int_{t,x,\xi} \omega_{1/n}(x_{k_s} - c_s) \theta j_{-,\varepsilon_j,\sigma_j} \partial_\xi \psi^R_{\epsilon,p} dm_{+,\varepsilon_h,\sigma_h}$$

$$- \int_0^1 d\lambda \int_{t,x,\xi} \theta h_{+,\varepsilon_h,\sigma_h} \partial_\xi \psi^R_{\epsilon,p} dq_{-,\varepsilon_j,\sigma_j}.$$

Now, add (26) and (27). We get after taking into account $\psi^L_{\epsilon,p} + \psi^R_{\epsilon,p} \equiv 1$, $\partial_\xi \psi^L_{\varepsilon,p} = -\partial_\xi \psi^L_{\varepsilon,p} \geq 0$ (due to the crossing conditions), $-j_- > 0$, and $h_+ \geq 0$:

$$\int_0^1 d\eta \int_0^1 d\lambda \int_{t,x,\xi} (-h_{+,\varepsilon_h,\sigma_h} j_{-,\varepsilon_j,\sigma_j}) \Big(\partial_t \theta$$

$$+ \sum_{i=1}^d (G_{iL}(\hat{x}_i,\xi) H(\alpha_i(c_s - x_{k_s}) + G_{iR}(\hat{x}_i,\xi) H(x_{k_s} - c_s)) \partial_{x_i} \theta \Big) + o_n(1)$$

$$\geq - \int_0^1 d\eta \int_{t,x,\xi} (1 - \omega_{1/n}(x_{k_s} - c_s)) \theta j_{-,\varepsilon_j,\sigma_j} \partial_\xi \psi^R_{\epsilon,p} dm_{+,\varepsilon_h,\sigma_h}$$

$$- \int_0^1 d\lambda \int_{t,x,\xi} (1 - \omega_{1/n}(x_{k_s} - c_s)) \theta h_{+,\varepsilon_h,\sigma_h} \partial_\xi \psi^R_{\epsilon,p} dq_{-,\varepsilon_j,\sigma_j} \geq 0,$$

and from here, letting $\varepsilon_h, \varepsilon_j, \sigma_h, \sigma_j \to 0$, and $n \to \infty$, and appealing to (15), we conclude:

$$\int_0^1 d\lambda \int_0^1 d\eta \int_{t,x} \Big(|u(t,x,\lambda) - v(t,x,\eta)|^+ \partial_t \theta + \sum_{i=1}^d \operatorname{sgn}_+(u(t,x,\lambda) - v(t,x,\eta)) \times \tag{28}$$

$$\times \Big(g_{iL}(\hat{x}_i, u(t,x,\lambda)) - g_{iL}(\hat{x}_i, v(t,x,\eta)) \Big) H(c_s - x_{k_s})$$

$$+ \Big(g_{iR}(\hat{x}_i, u(t,x,\lambda)) - g_{iR}(\hat{x}_i, v(t,x,\eta)) \Big) H(x_{k_s} - c_s) \Big) \partial_{x_i} \theta \geq 0.$$

To proceed, denote by $\tilde{\Gamma}$ set of edges of the interface Γ, and notice that any test function $\varphi \in C^1_0(I\!R \times (I\!R^d \backslash \tilde{\Gamma}))$ can be written as a sum

$$\varphi = \sum_{j=1}^n \varphi_j,$$

where $\operatorname{supp}(\varphi_j) \subset I\!R \times \Omega_j$, $j = 1, \dots, n$.

Therefore, from (28), we conclude that the following holds for every $\varphi \in C_0^1(I\!R\times (I\!R^d\backslash\tilde\Gamma))$:

$$
\begin{aligned}
&\int_0^1 d\lambda \int_{I\!R^+\times I\!R^d} (v-u)^{\pm}\partial_t\varphi dtdx \qquad (29)\\
&+\int_0^1 d\lambda \int_{I\!R^+\times I\!R^d} \sum_{i=1}^d \operatorname{sgn}_{\pm}(v-u)\times\\
&\quad\times\Big(\left(g_{iL}(\hat{x}_i,v)-g_{iL}(\hat{x}_i,u)\right)\kappa_L(x)+\left(g_{iR}(\hat{x}_i,v)-g_{iR}(\hat{x}_i,u)\right)\kappa_R(x)\Big)\partial_{x_i}\varphi\\
&+\int_{I\!R^d}(v_0-u_0)^{\pm}\varphi(0,x)dx\geq 0.
\end{aligned}
$$

Now, denote by $\tilde\Gamma_\varepsilon$ an ε-neighborhood of the set $\tilde\Gamma$. Let $\omega_\varepsilon\in C^1(I\!R^d)$ be such that

$$
\tilde\omega_\varepsilon(x)=\begin{cases}1, & x\notin\tilde\Gamma_{2\varepsilon}\\ 0, & x\in\tilde\Gamma_\varepsilon.\end{cases}
$$

Notice that

$$
\begin{aligned}
&|\partial_{x_i}\tilde\omega_\varepsilon|\leq\frac{C}{\varepsilon}\\
&\operatorname{meas}(\operatorname{supp}(\partial_{x_i}\tilde\omega_\varepsilon))\leq\tilde C\varepsilon^2,
\end{aligned}
\qquad (30)
$$

for some constants C and $\tilde C$, since $codim(\tilde\Gamma)\geq 2$.

Then, take an arbitrary $\varphi\in C_0^1(I\!R\times I\!R^d)$ and put in (29) $\varphi\tilde\omega_\varepsilon$. We conclude from (30):

$$
\begin{aligned}
&\int_0^1 d\lambda \int_{I\!R^+\times I\!R^d} (v-u)^{\pm}\partial_t\varphi dtdx\\
&+\int_0^1 d\lambda \int_{I\!R^+\times I\!R^d} \sum_{i=1}^d \operatorname{sgn}_{\pm}(v-u)\times\\
&\quad\times\Big(\left(g_{iL}(\hat{x}_i,v)-g_{iL}(\hat{x}_i,u)\right)\kappa_L(x)+\left(g_{iR}(\hat{x}_i,v)-g_{iR}(\hat{x}_i,u)\right)\kappa_R(x)\Big)\partial_{x_i}\varphi\\
&+\int_{I\!R^d}(v_0-u_0)^{\pm}\varphi(0,x)dx\geq \mathcal{O}(\varepsilon).
\end{aligned}
$$

From here, using the standard procedure (e.g. [9]) and (17), we arrive at (13). This completes the proof.

A simple corollary of Theorem 7 and Theorem 8 is (see e.g. [3, Page 377]):

Corollary 13. *There exists a unique entropy weak solution to* (4), (2).

3. Admissibility conditions for the general interface

In order to formulate the admissibility conditions in the general case, we shall assume that for every $i\in\{1,\dots,d\}$ the interface Γ can be represented in the form

$$
\Gamma=\{x\in I\!R^d:\ x_i=\alpha(\hat{x}_i)\}, \qquad (31)
$$

for Lipschitz continuous functions α_i, $i=1,\dots,d$. We can safely assume that the latter representation holds only locally, but to avoid unnecessary complications, we shall assume exactly (31).

If not stated differently, we will use notions and notations from the previous section. We are going to use the following admissibility conditions.

Definition 14. Let $u_0 \in L^\infty(\mathbb{R}^d)$, $a \leq u_0 \leq b$ a.e. on $\mathbb{R}^d$. Let $u \in L^\infty([0,\infty) \times \mathbb{R}^d)$.

1. The function u is an entropy sub-solution (respectively entropy super-solution) of problem (4), (2) if for any $\xi \in \mathbb{R}$ and any $\varphi \in C_0^1(\mathbb{R} \times \mathbb{R}^d)$:

$$\begin{aligned}
&\int_{\mathbb{R}^+ \times \mathbb{R}^d} (u-\xi)^{\pm} \partial_t \varphi dt dx \\
&+ \int_{\mathbb{R}^+ \times \mathbb{R}^d} \sum_{i=1}^d \operatorname{sgn}_{\pm}(u-\xi) \times \\
&\quad \times \Big(\left(g_{iL}(\hat{x_i}, u) - g_{iL}(\hat{x_i}, \xi) \right) H(\alpha_i(\hat{x_i}) - x_i) \\
&\quad\quad + \left(g_{iR}(\hat{x_i}, u) - g_{iR}(\hat{x_i}, \xi) \right) H(x_i - \alpha_i(\hat{x_i})) \Big) \partial_{x_i} \varphi dt dx \\
&+ \int_{\mathbb{R}^d} (u_0 - \xi)^{\pm} \varphi(0,x) dx \\
&- \sum_{i=1}^d \int_{\mathbb{R}^+ \times \mathbb{R}^{d-1}} (g_{iL}(\hat{x_i}, \xi) - g_{iR}(\hat{x_i}, \xi))^{\pm} \varphi|_{x_i = \alpha_i(\hat{x_i})} dt d\hat{x_i} \geq 0.
\end{aligned} \tag{32}$$

2. The function u is an entropy solution if it is a weak k-entropy process sub- and super-solution at the same time.

If we assume that $u \in L^\infty([0,\infty) \times \mathbb{R}^d \times (0,1))$ and integrate (32) over $\lambda \in (0,1)$ then the function u is an entropy process sub (super) solution to (4), (2). Existence of the entropy process sub and super solutions exist for the general case as well and the proof is completely the same as the proof of [11, Theorem 2.8] (in other words, Theorem 7 holds).

3.1. **Proof of Theorem 8 in the case of general interface.** We shall prove that every entropy admissible solution for the Cauchy problem with the interface in the general form can be approximated by a sequence of entropy solutions to the Cauchy problem with the interface in the special form. Since solutions to Cauchy problems with special interface are stable, this will imply stability of the solution to the Cauchy problem in the general situation that we are considering here.

So, fix an $\delta > 0$ and approximate the given interface Γ by a manifold Γ_δ which is piecewise parallel to the coordinate hyper-planes and satisfies $\operatorname{dist}(\Gamma, \Gamma_\delta) \leq \delta$, and it divides the space $\mathbb{R}^d$ into two parts $\Omega_{L,\delta}$ and $\Omega_{R,\delta}$. We assume that for every relatively compact $K \subset\subset \mathbb{R}^d$ it holds

$$\int_K \left(|(\kappa_{L,\delta} - \kappa_L)(x)| + |(\kappa_{R,\delta} - \kappa_R)(x)| \right) dx = \mathcal{O}(\delta), \tag{33}$$

where $\kappa_{L,\delta}$ and $\kappa_{R,\delta}$ are characteristic functions to $\Omega_{L,\delta}$ and $\Omega_{R,\delta}$, respectively, and $\mathcal{O}(\delta)$ depends only on K.

Denote by u_δ a unique entropy admissible solution in the sense of Definition 3 to the following Cauchy problem

$$
\begin{aligned}
\partial_t u_\delta + \mathrm{Div}_x \left(g_L(x,u_\delta)\kappa_{L,\delta}(x) + g_R(x,u_\delta)\kappa_{R,\delta}(x) \right) = 0, \\
u_\delta \Big|_{t=0} = u_0(x).
\end{aligned} \tag{34}
$$

Denote by Ω_j^δ, $j=1,\dots,r_\delta$, sets such that (10) is satisfied with Γ replaced by Γ_δ, and n replaced by r_δ. Since u_δ is entropy admissible, it satisfies for every $\varphi \in C^1_0(I\!R \times \Omega_j^\delta \times I\!R)$, $j=1,\dots,r_\delta$, the following kinetic relation:

$$
\begin{aligned}
&\int_0^1 d\lambda \int_{t,x,\xi} h^\delta_\pm \times \\
&\qquad \times \Big(\partial_t + \sum_{i=1}^d \left(G_{iL}(\hat{x}_i,\xi) H(c_j^\delta - x_{k_j}) + G_{iR}(\hat{x}_i,\xi) H(x_{k_j} - c_j^\delta) \right) \partial_{x_i} \Big) \varphi \\
&+ \int_{x,\xi} h^0_{\pm,k} \varphi|_{t=0} - \int_{t,\hat{x}_{k_j},\xi} \partial_\xi \left(g_{k_j L}(\hat{x_{k_j}},\xi) - g_{iR}(\hat{x_{k_j}},\xi) \right)^{\pm} \varphi|_{x_{k_j}=c_j^\delta} \\
&= \int_{t,x,\xi} \partial_\xi \varphi \, dm^\delta_\pm,
\end{aligned} \tag{35}
$$

where $h^\delta_\pm$ are equilibrium functions corresponding to u_δ, and $m^\delta_\pm$ are non-negative measures satisfying the corresponding conditions from Definition 9.

Next, take the entropy process solution $u \in L^\infty(I\!R^+ \times I\!R^d \times (0,1))$ to (4), (2) (the Cauchy problem that corresponds to the interface Γ). It satisfies the following kinetic relation for every $\varphi \in C^1_0(I\!R \times I\!R^d \times I\!R)$

$$
\begin{aligned}
&\int_0^1 d\lambda \int_{t,x,\xi} h_\pm \times \\
&\qquad \times \Big(\partial_t + \sum_{i=1}^d \left(G_{iL}(\hat{x}_i,\xi) H(\alpha_i(\hat{x}_i) - x_i) + G_{iR}(\hat{x}_i,\xi) H(x_i - \alpha_i(\hat{x}_i)) \right) \partial_{x_i} \Big) \varphi \\
&+ \int_{x,\xi} h^0_{\pm,k} \varphi|_{t=0} dx d\xi - \sum_{i=1}^d \int_{t,\hat{x}_i,\xi} \partial_\xi \left(g_{iL}(\hat{x}_i,\xi) - g_{iR}(\hat{x}_i,\xi) \right)^{\pm} \varphi|_{x_i=\alpha(\hat{x}_i)} d\hat{x}_i dt d\xi \\
&= \int_{t,x,\xi} \partial_\xi \varphi \, dm_\pm d\xi,
\end{aligned}
$$

for non-negative measures $m_\pm$ (satisfying the appropriate conditions from Definition 9), and equilibrium functions $h_\pm$ corresponding to the function u. We can rewrite the latter relation for $\varphi \in C^1_0(I\!R \times \Omega^\delta_j \times I\!R)$ in the form

$$\begin{aligned}
&\int_0^1 d\lambda \int_{t,x,\xi} h_\pm \times \\
&\quad\times\Big(\partial_t + \sum_{i=1}^d \big(G_{iL}(\hat{x_i},\xi)H(c^\delta_j - x_{k_j}) + G_{iR}(\hat{x_i},\xi)H(x_{k_j} - c^\delta_j)\big)\,\partial_{x_i}\Big)\varphi \\
&+ \int_{x,\xi} h^0_\pm \varphi|_{t=0} - \int_{t,\hat{x_i},\xi} \partial_\xi\left(g_{iL}(\hat{x_i},\xi) - g_{iR}(\hat{x_i},\xi)\right)^\pm \varphi|_{x_i=\alpha(\hat{x_i})} \\
&= \int_{t,x,\xi} \partial_\xi \varphi dm_\pm \\
&+ \int_0^1 d\lambda \int_{t,x,\xi} h_\pm \Big(\sum_{i=1}^d \Big(G_{iL}(\hat{x_i},\xi)\left(H(\alpha_i(\hat{x_i}) - x_i) - H(x_{k_j} - c^\delta_j)\right) \\
&\qquad\qquad + G_{iR}(\hat{x_i},\xi)\left(H(x_i - \alpha_i(\hat{x_i})) - H(x_{k_j} - c^\delta_j)\right)\Big)\partial_{x_i}\Big)\varphi,
\end{aligned} \tag{36}$$

Since the left-hand sides of (35) and (36) are the same, and the measures $m_\pm$ and $m^\delta_\pm$ are non-negative, we can repeat the procedure from the proof of Theorem 8 to obtain for $\theta \in C^1_0(I\!R \times \Omega^\delta_s)$, $s \in \{1,\dots,r_\delta\}$ fixed:

$$\begin{aligned}
&\int_0^1 d\lambda \int_{t,x,\xi} (-h_{+,\varepsilon_h,\sigma_h} h^\delta_{-,\varepsilon_j,\sigma_j}) \times \\
&\times\Big(\partial_t + \sum_{i=1}^d \left(G_{iL}(\hat{x_i},\xi)H(c_s - x_{k_s}) + G_{iR}(\hat{x_i},\xi)H(x_{k_s} - c_s)\right)\partial_{x_i}\Big)\theta \\
&+ \mathcal{O}(\epsilon) + o_n(1) \\
&\geq \int_0^1 d\lambda \int_{t,x,\xi} h_+ \Big(\sum_{i=1}^d \Big(G_{iL}(\hat{x_i},\xi)\left(H(\alpha_i(\hat{x_i}) - x_i) - H(x_{k_s} - c^\delta_s)\right) \\
&\qquad\qquad + G_{iR}(\hat{x_i},\xi)\left(H(x_i - \alpha_i(\hat{x_i})) - H(x_{k_s} - c^\delta_s)\right)\Big)\partial_{x_i}\Big)(\theta \psi^L_{\epsilon,p_{k_s}} h^\delta_{-,\varepsilon_j,\sigma_j}) \\
&+ \int_0^1 d\lambda \int_{t,x,\xi} h_{+,\varepsilon_h,\sigma_h} \Big(\sum_{i=1}^d \Big(G_{iL}(\hat{x_i},\xi)\left(H(\alpha_i(\hat{x_i}) - x_i) - H(x_{k_s} - c^\delta_s)\right) \\
&\qquad\qquad + G_{iR}(\hat{x_i},\xi)\left(H(x_i - \alpha_i(\hat{x_i})) - H(x_{k_s} - c^\delta_s)\right)\Big)\partial_{x_i}\Big) \times \\
&\qquad\qquad\qquad \times \left((\omega_{1/n}(x_{k_s} - c_s) - 1)\psi^L_{\epsilon,p_{k_s}} \theta h^\delta_{-,\varepsilon_j,\sigma_j}\right)
\end{aligned} \tag{37}$$

Now, take an arbitrary $\theta \in C^1_0(I\!R^+ \times I\!R^d)$. Denote by $\tilde{\Gamma}_\delta$ set of edges of the hyper-polyhedron Γ_δ. Denote by $\tilde{\omega}_m$ the cut off function such that $\tilde{\omega}_m(x) \equiv 0$ in the $1/m$-neighborhood of $\tilde{\Gamma}$, and $\tilde{\omega}_m(x) \equiv 1$ out of $2/m$-neighborhood of $\tilde{\Gamma}$. Since $codim \tilde{\Gamma}_\delta \geq 2$, by inserting $\tilde{\omega}_m \theta$ into (37) and letting $m \to \infty$, we conclude that it holds:

$$\int_0^1 d\lambda \int_{t,x,\xi} (-h_{+,\varepsilon_h,\sigma_h} h^\delta_{-,\varepsilon_j,\sigma_j})\times \tag{38}$$
$$\times\Big(\partial_t + \sum_{i=1}^d \left(G_{iL}(\hat{x_i},\xi)H(c_s - x_{k_s}) + G_{iR}(\hat{x_i},\xi)H(x_{k_s}-c_s)\right)\partial_{x_i}\Big)\theta$$
$$+\mathcal{O}(\epsilon) + o_n(1)$$
$$\geq \int_0^1 d\lambda \int_{t,x,\xi} h_+ \Big(\sum_{i=1}^d \Big(G_{iL}(\hat{x_i},\xi)\sum_{s=1}^r \kappa_{\Omega^\delta_s}\left(H(\alpha_i(\hat{x_i})-x_i) - H(x_{k_s}-c^\delta_s)\right)$$
$$+ G_{iR}(\hat{x_i},\xi)\sum_{s=1}^r \kappa_{\Omega^\delta_s}\left(H(x_i-\alpha_i(\hat{x_i})) - H(x_{k_s}-c^\delta_s)\right)\Big)\partial_{x_i}\Big)(\theta\psi^L_{\epsilon,p_{k_s}} h^\delta_{-,\varepsilon_j,\sigma_j})$$
$$+\int_0^1 d\lambda \int_{t,x,\xi} h_{+,\varepsilon_h,\sigma_h} \Big(\sum_{i=1}^d \Big(G_{iL}(\hat{x_i},\xi)\sum_{s=1}^r \kappa_{\Omega^\delta_s}\left(H(\alpha_i(\hat{x_i})-x_i) - H(x_{k_s}-c^\delta_s)\right)$$
$$+G_{iR}(\hat{x_i},\xi)\sum_{s=1}^r \kappa_{\Omega^\delta_s}\left(H(x_i-\alpha_i(\hat{x_i})) - H(x_{k_s}-c^\delta_s)\right)\Big)\partial_{x_i}\Big)\times$$
$$\times\left((\omega_{1/n}(x_{k_s}-c_s)-1)\psi^L_{\epsilon,p_{k_s}}\theta h^\epsilon_{-,\varepsilon_j,\sigma_j}\right),$$

where $\kappa_{\Omega^\delta_s}$ is the characteristic function of the set Ω^δ_s.

Next, denote by $v \in L^\infty(I\!R^+ \times I\!R^d \times (0,1))$ the nonlinear weak-$\star$ limit along a subsequence to the family (u_δ). Denote $j_-(t,x,\xi) = \mathrm{sgn}_-(\xi - v(t,x,\eta))$. It holds for any $\theta \in C^1_0(I\!R^+ \times I\!R^d)$

$$\lim_{\delta\to 0}\int_{t,x} h^\delta_{-,\varepsilon_j,\sigma_j}\theta = \int_0^1 d\eta \int_{t,x} j_- \star \rho_{\varepsilon_j,\sigma_j}\theta.$$

Having this in mind, we get from (38) after letting $\delta \to 0$ and taking (33) into account:

$$\int_0^1 d\eta \int_0^1 d\lambda \int_{t,x,\xi} (-h_{+,\varepsilon_h,\sigma_h} j_{-,\varepsilon_j,\sigma_j})\times \tag{39}$$
$$\times\Big(\partial_t + \sum_{i=1}^d \left(G_{iL}(\hat{x_i},\xi)H(c_s - x_{k_s}) + G_{iR}(\hat{x_i},\xi)H(x_{k_s}-c_s)\right)\partial_{x_i}\Big)\theta$$
$$+\mathcal{O}(\epsilon) + o_n(1) \geq 0.$$

Finally, letting here $n \to \infty$, and $\epsilon, \varepsilon_h, \sigma_h, \varepsilon_j, \sigma_j \to 0$, we reach to (28). The rest of the proof is standard and relies on (17) and the procedure from [9].

The proof is over.

4. Appendix

In this section, we shall propose a possible extension of the transport-collapse operator of Y.Brenier [4]. It seems to be rather nontrivial to rigorously prove appropriate convergence results, and we will keep our considerations at the level of informal arguing.

Original transport-collapse scheme [4] was actually based on the kinetic formulation which was explicitly formulated more than ten years later [16, 12]. The kinetic formulation for the Cauchy problem for a homogeneous scalar conservation law

$$\begin{aligned} &\partial_t u + \partial_x f(u) = 0, \quad (t,x) \in I\!R^+ \times I\!R^d, \\ &u|_{t=0} = u_0(x), \end{aligned} \tag{40}$$

has the form

$$\begin{aligned} &\partial_t h + f'(\xi)\partial_x h = -\partial_\xi m, \\ &h|_{t=0} = h(u_0,\xi) = \begin{cases} 1, & 0 \leq \xi \leq u_0(x) \\ -1, & u_0(x) \leq \xi \leq 0, \\ 0, & else \end{cases}, \end{aligned} \tag{41}$$

where m is a non-negative measure, and

$$h(u,\xi) = \begin{cases} 1, & 0 \leq \xi \leq u(t,x) \\ -1, & u(t,x) \leq \xi \leq 0, \\ 0, & else \end{cases}$$

where u is the entropy solution to (40). Then, instead of (41), Brenier (implicitly) considers

$$\partial_t h + f'(\xi)\partial_x h = 0, \quad h|_{t=0} = h(u_0,\xi), \tag{42}$$

which he solves using the standard method of characteristics. The solution is $h(u,\xi) = h(u_0(x - f'(\xi)t), \xi)$. Then, relying on the fact that

$$\int h(u,\xi)d\xi = u(t,x), \tag{43}$$

the transport-collapse operator $T(t)$ is introduced

$$(T(t)u_0)(x) = \int_\xi h(u_0(x - f'(\xi)t), \xi). \tag{44}$$

The main statement in the transport-collapse framework is the following theorem.

Theorem 15. *Denote by u the entropy solution to Cauchy problem* (40). *It holds*

$$u(t,x) = \lim_{n\to\infty} \left(T\left(\frac{t}{n}\right) \right)^n u_0(x).$$

We shall explain in one-dimensional case how one could extend the transport-collapse framework on the discontinuous flux case. The transport-collapse principle in the multidimensional case is bit more involved but the extension is rather straightforward. We shall spend couple of lines on this issue at the end of the section.

Accordingly, consider Cauchy problem (6). We have proved in Section 2 that the latter Cauchy problem admits a unique weak solution satisfying the following kinetic relation

$$\partial_t h_\pm + \partial_x \left(g'(\xi)H(-x) + f'(\xi)H(x)\right) - \partial_\xi (g(\xi) - f(\xi))^\pm \delta(x) = \partial_\xi m_\pm(t,x,\xi), \tag{45}$$

where, as before, $h_\pm(u,\xi) = \mathrm{sgn}_\pm(u(t,x) - \xi)$, and $m_\pm$ are positive measures.

Now, assume that the functions g and f have intersection at the point $0 < p < 1$, and are such that

$$f(\xi) - g(\xi) \geq 0, \quad \xi \in (p,1),$$
$$f(\xi) - g(\xi) \leq 0, \quad \xi \in (0,p).$$

Following the transport-collapse principle, we shall replace (45) by the following linear equations:

$$\partial_t h_+ + \partial_x \left(g'(\xi)H(-x) + f'(\xi)H(x)\right) h_+ = 0, \quad \xi \in (0,p), \tag{46}$$
$$\partial_t h_- + \partial_x \left(g'(\xi)H(-x) + f'(\xi)H(x)\right) h_- = 0, \quad \xi \in (p,1). \tag{47}$$

Remark that we have removed not only the measures $m_\pm$ from (45) but also term containing δ-distributions. It is necessary to be done since the terms containing δ distributions disappear for ξ belonging to the intervals where equations (46) and (47) are defined.

Consider only (46) since (47) is considered analogically. Appropriate system of characteristics has the form:

$$\dot{x} = g'(\xi)H(-x) + f'(\xi)H(x), \quad x(0) = x_0,$$
$$\dot{h}_+ = (g'(\xi) - f'(\xi))\delta(x), \quad h_+(0) = h_+(u_0(x_0),\xi), \quad \xi \in (0,p).$$

The functions on the right-hand side of the latter equations are too singular to provide existence of solutions. Therefore, following the construction of the entropy solution to (6) (see Theorem 7), we regularize the functions H and δ to obtain the following, globally solvable, system of characteristics:

$$\begin{aligned} &\dot{x} = g'(\xi)H_\varepsilon(-x) + f'(\xi)H_\varepsilon(x), \quad x(0) = x_0, \\ &\dot{h}_+ = (g'(\xi) - f'(\xi))\delta_\varepsilon(x), \quad h_+(0) = h_+(u_0(x_0),\xi), \quad \xi \in (0,p). \end{aligned} \tag{48}$$

Denote by $h^{TC}_{+,\varepsilon}(t,x,\xi)$ the solution to the latter system. Applying the similar procedure on (47), we reach to the function $h^{TC}_{-,\varepsilon}(t,x_0,\xi)$ representing the solution to (48) with h_+ replaced by h_- for $\xi \in (p,1)$.

Next, notice that

$$h_+(u,\xi) = h(u,\xi) - \mathrm{sgn}_-(\xi),$$
$$h_-(u,\xi) = h(u,\xi) - \mathrm{sgn}_+(\xi),$$

and thus

$$\int_0^p h_+(u,\xi)d\xi = \int_0^p \big(h(u,\xi)dx - \mathrm{sgn}_-(\xi)\big)\,d\xi,$$
$$\int_p^1 h_-(u,\xi)d\xi = \int_p^0 \big(h(u,\xi)dx - \mathrm{sgn}_+(\xi)\big)\,d\xi,$$

Summing the latter two equations, we conclude:

$$\int_0^p h_+(u,\xi)d\xi + \int_p^1 h_-(u,\xi)d\xi = u - p.$$

Having this in mind, analogically with (42), (43), and (44), we introduce the approximate transport-collapse operator:

$$T_\varepsilon(t)u_0(x) = \int_0^p h^{TC}_{+,\varepsilon}(u,\xi)d\xi + \int_p^1 h^{TC}_{-,\varepsilon}(u,\xi)d\xi - p. \tag{49}$$

We conjecture the following:

Conjecture 16. *Denote by u the entropy solution to Cauchy problem* (40)*. It holds*

$$u(t,x)=\lim_{\varepsilon\to 0}\lim_{n\to\infty}\left(T_\varepsilon\left(\frac{t}{n}\right)\right)^n u_0(x). \tag{50}$$

In the multidimensional case, we need to approximate the interface by a manifold which is piecewise parallel to the coordinate hyper-planes. With the notation from Section 3 (more precisely (33)), we introduce the approximate transport-collapse operator for every Ω_j^δ, $j=1,\dots,n_\delta$:

$$T^j_{\varepsilon,\delta}(t)u_0(x)=\int_0^{p^j}h^{j,TC}_{+,\varepsilon,\delta}(u,\xi)d\xi+\int_{p^j}^1 h^{j,TC}_{-,\varepsilon,\delta}(u,\xi)d\xi-p^j,$$

where p^j, $j=1,\dots,d$, are the intersection point of the functions g_{i_jL} and g_{i_jR} for $i_j\in\{1,\dots,d\}$ such that $\Gamma\cap\Omega_j\subset\{x_{i_j}=c_j\}$. The functions $h^{j,TC}_{\pm,\varepsilon,\delta}$, $j=1,\dots,n_\delta$, are solutions to the multi-dimensional analogue of the system of characteristics given by (48).

Assume for simplicity that Ω_j, $j=1,\dots,n_\delta$, are disjoint, and that instead (10) they satisfy $I\!R^d=Cl\left(\bigcup_{j=1}^{n_\delta}\Omega_j\right)$. We conjecture the following:

Conjecture 17. *Denote by u the entropy solution to Cauchy problem* (4)*,* (2) *with the interface Γ in the general form. It holds*

$$u(t,x)=\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\lim_{n\to\infty}\sum_{k=1}^{n_\delta}\kappa_{\Omega_k}(x)\left(T^k_{\varepsilon,\delta}\left(\frac{t}{n}\right)\right)^n u_0(x), \tag{51}$$

where κ_{Ω_k} is the characteristic function of the set Ω_k, $k=1,\dots,n_\delta$.

The transport-collapse procedure is of importance since it provides an explicit formula for the solution to the considered Cauchy problem. For instance, by fixing large $n=N^k$, $N\in I\!N$, $k>1$, and putting $\varepsilon=1/N$ (we probably need a larger order of convergence for n than for ε since we first let $n\to\infty$ and then $\varepsilon\to 0$) we obtain a numerical scheme that should converge toward the entropy solution to (6). More precisely, we define:

$$u(t+\Delta t,x)=\left(T_{1/N}\left(\frac{\Delta t}{N}\right)\right)^{N^k}u(t,x).$$

We believe that this approximation depends only on the rate of n/ε, and that it is independent on the choice of subsequences since we have a unique entropy solution to (4), (2). In the multidimensional case, we have three parameters and we should probably choose them so that rate of convergence of n is larger than the rate of convergence of ε, and the rate of convergence of ε is larger that the rate of convergence of δ.

Still, in order to justify the latter procedures it is necessary first to prove Conjecture 16 and Conjecture 17, and then to obtain error estimates for the transport-collapse procedures (50) and (51) in order to understand relations between parameters n, ε, and δ. At the moment, we are far from being able to settle these questions.

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