Mixed Systems: ODEs – Balance Laws

Raul Borsche[1] Rinaldo M. Colombo[2] Mauro Garavello[3]

May 18, 2011

Abstract

We prove the well posedness of mixed problems consisting of a system of ordinary differential equations coupled with systems of balance laws in domains with moving boundaries. The interfaces between the systems are provided by the boundary data and boundary positions. Various situations that fit into this framework are studied, both analytically and numerically. We consider a piston moving in a pipe full of fluid, a model for fluid–particle interaction and a traffic model. References to other examples in the literature are provided.

Keywords: Mixed PDE–ODE Problems, Conservation Laws, Ordinary Differential Equations

2010 MSC: 35L65, 35M33, 34A12

1 Introduction

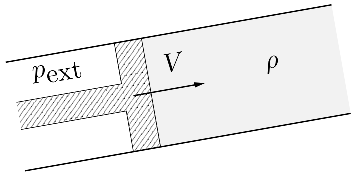
This paper deals with mixed problems consisting of 1D systems of hyperbolic balance laws coupled with ordinary differential equations. As a first example, consider the case in which the balance law is defined on a half–line and the coupling is provided by the boundary condition, i.e.

$$\begin{cases} \partial_t u + \partial_x f(u) = g(u) & x > \gamma(t) \\ b\left(u\left(t, \gamma(t)+\right)\right) = B\left(t, w(t)\right) \\ \dot{w} = F\left(t, u\left(t, \gamma(t)+\right), w(t)\right) \\ \dot{\gamma}(t) = \Pi\left(w(t)\right) . \end{cases} \tag{1.1}$$

Here, γ is a *free boundary* in the sense that its position is not known *a priori* but it is an unknown to be determined when solving (1.1). Below, we prove the well posedness of (1.1), extending the results in [6], where only existence was considered. Moreover, in the present framework the boundary may well move and we now also admit the presence of a (possibly nonlocal) source term g, extending the situation described in [6] where the boundary was fixed and no source term was considered.

The present construction comprehends, for instance, the Eulerian description of a fluid in a pipe with a piston at one end, which leads to the following system, studied in Section 3.3:

$$\begin{cases} \partial_t \rho + \partial_x q = 0 \\ \partial_t q + \partial_x \left(\frac{q^2}{\rho} + p(\rho)\right) = -\nu \frac{q|q|}{\rho} - g\,\rho \sin\alpha \\ V(t) = \dfrac{q\left(t, \gamma(t)+\right)}{\rho\left(t, \gamma(t)+\right)} \\ \dot{V} = \beta\left(p_{\text{ext}}(t) - p\left(\rho\left(t, \gamma(t)+\right)\right)\right) - g \sin\alpha \\ \dot{\gamma}(t) = V(t) . \end{cases} \tag{1.2}$$



[1]Fachbereich Mathematik, Technische Universität Kaiserslautern
[2]Dipartimento di Matematica, Università di Brescia
[3]Di.S.T.A., Università del Piemonte Orientale

Here, ρ is the gas density, q is its linear momentum density, $p = p(\rho)$ is a pressure law playing the role of the gas equation of state, V is the piston speed and γ its position. Friction is described by the term $-\nu q|q|/\rho$, with ν being a suitable constant. The slope of the pipe is α, while β is the ratio between the surface of the pipe and the mass of the piston; g is gravity, see the figure at (1.2).

The construction presented below applies also to the formally different situation of two balance laws defined on the two sides of two moving boundaries, say

$$\begin{cases} \partial_t u^- + \partial_x f^-(u^-) = g^-(u^-) & x < \gamma^-(t) \\ \partial_t u^+ + \partial_x f^+(u^+) = g^+(u^+) & x > \gamma^+(t) \\ b^-\Big(u^-\left(t,\gamma^-(t)-\right), u^+\left(t,\gamma^+(t)+\right)\Big) = B^-\left(t, w(t)\right) \\ b^+\Big(u^-\left(t,\gamma^-(t)-\right), u^+\left(t,\gamma^+(t)+\right)\Big) = B^+\left(t, w(t)\right) \\ \dot w = F\Big(t, u^-\left(t,\gamma^-(t)-\right), u^+\left(t,\gamma^+(t)+\right), w(t)\Big) \\ \dot\gamma^-(t) = \Pi^-\left(w(t)\right) \\ \dot\gamma^+(t) = \Pi^+\left(w(t)\right)\,. \end{cases} \tag{1.3}$$

In general, as long as the flow f in (1.1) is not explicitly dependent on time, the system (1.3) does not fall in the framework of (1.1). An example is the model for gas-particle interaction considered in Section 3.1, which fits in (1.3) but not in (1.1). This model reads

$$\begin{cases} \begin{cases} \partial_t \rho^- + \partial_x q^- = 0 \\ \partial_t q^- + \partial_x \left(\dfrac{{q^-}^2}{\rho^-} + p(\rho^-)\right) = -g\,\rho^- \end{cases} & x < \gamma^-(t) \\ \begin{cases} \partial_t \rho^+ + \partial_x q^+ = 0 \\ \partial_t q^+ + \partial_x \left(\dfrac{{q^+}^2}{\rho^+} + p(\rho^+)\right) = -g\,\rho^+ \end{cases} & x > \gamma^+(t) \\ \dfrac{q^-\left(t,\gamma^-(t)\right)}{\rho^-\left(t,\gamma^-(t)\right)} = \dfrac{q^+\left(t,\gamma^+(t)\right)}{\rho^+\left(t,\gamma^+(t)\right)} = V \\ \dot V = -g - \dfrac{p\Big(\rho^+\left(t,\gamma^+(t)\right)\Big) - p\Big(\rho^-\left(t,\gamma^-(t)\right)\Big)}{m} \\ \dot\gamma^-(t) = \dot\gamma^+(t) = V\,. \end{cases} \tag{1.4}$$

The space variable x is a vertical coordinate oriented upwards; $\rho^\pm$ and $q^\pm$ are the fluid mass and linear momentum density above $(+)$ and below $(-)$ the particle; $p = p(\rho)$ is the pressure law; V is the speed of the particle sited in $[\gamma^-(t), \gamma^+(t)]$ and m is its mass; g is gravity. A justification of the speed law for $\dot V$ in (1.4) is provided by the conservation of energy and is presented in Section 3.1, see also [12].

In Section 3.2 we present a new model describing the interaction between traffic flow and a large vehicle hindering the other vehicles. A similar model was presented in [18], where the existence of solutions to a system consisting of an ODE coupled with the Lighthill–Whitham and Richards model was proved. Below, we use the Aw–Rascle model [5] to describe traffic and, for the resulting system, we prove also the continuous dependence of the solutions from the initial data. Also this system fits into (1.3) but not into (1.1).

Other applications of (1.1) are collected in [6, Section 3]. They comprehend, for instance, a description of a sewer system with a manhole [6, § 3.2], the equations for a node of supply chains with queues [6, § 3.3], as well as a multiscale blood flow model, see [6, § 3.4] which summarizes [15, formulæ (2.3), (2.12), (2.14)], [1, Section 2] and [8]. These systems all fit in the present, more general, framework in the particular case $g \equiv 0$, $\gamma \equiv 0$.

The main result of this work is the local in time well posedness of (1.1) and of (1.3). In the spirit of the theory of conservation laws, by this we mean the existence of solutions and their

$\mathbf{L^1}$–Lipschitz dependence with respect to the initial data. In general, a global in time result is not feasible without major restrictions on (1.1) or (1.3). As it is well known, the presence of source terms may lead to nonexistence of solutions for large times. Moreover, also when the source term vanishes, the observation in [6, Remark 3.2] apply showing that in the present setting long time existence results are not possible.

To obtain the well posedness of (1.1) and (1.3), we needed to improve the analytical results in [10]. Therefore, as a byproduct, below we also prove new stability bounds on the variation of the trace of the solution at the boundary to a general initial boundary value problem for a balance law, see *(4)* in Proposition 2.2.

The analytical proof of the well posedness of (1.1) is very similar to that of (1.3). Below, the former is presented in detail, while the latter is only briefly sketched.

In the theory of conservation laws, results often refer either to the case of 1D systems, as the present work, or to scalar multiD equations. An analog of the present work in the scalar multiD case is provided by [11]. The well posedness proved therein refers to a Kružkov type conservation law coupled with an ordinary differential equation.

The next section is devoted to the main analytical results of this work: the well posedness of (1.1) and (1.3). Then, Section 3 is devoted to (1.2), to (1.4) and to the Aw–Rascle model with a moving obstacle. We first present the models, then prove that each of these examples fits into (1.1) or (1.3) so that Theorem 2.6 or Theorem 2.7 apply. Then, we provide sample numerical integrations. The final Section 4 presents all the technical proofs.

2 Analytical Results

Throughout, we denote $\mathbb{R}^+ = [0, +\infty[$ and $\mathring{\mathbb{R}}^+ =]0, +\infty[$. Let $\Omega \subseteq \mathbb{R}^n$ be an open set. With $\mathcal{B}_r(w)$ we denote the open ball centered at w with radius r. Fix the reference states $\hat u \in \Omega$, $\hat w \in \mathbb{R}^m$ and a point $\hat x \in \mathbb{R}$.

On system (1.1) we require the following conditions, where we refer to [7, 13] for the standard vocabulary about conservation laws.

(f) $f \in \mathbf{C^4}(\Omega; \mathbb{R}^n)$ is smooth and such that, for all $u \in \Omega$, $Df(u)$ is strictly hyperbolic and each characteristic field is either genuinely nonlinear or linearly degenerate.

For $u \in \Omega$ and $i = 1, \ldots, n$, call $\lambda_i(u)$ the i-th eigenvalue of $Df(u)$ and $r_i(u)$ the corresponding right eigenvector. By **(f)**, we may assume that $\lambda_{i-1}(u) < \lambda_i(u)$ for all $u \in \Omega$ and $i = 2, \ldots, n$. Define

$$\mathcal{U} = \left\{ u \in \hat u + (\mathbf{BV} \cap \mathbf{L^1})(\mathbb{R}; \mathbb{R}^n) \colon u(\mathbb{R}) \subset \Omega \right\} \quad \text{and} \quad \mathcal{U}_\delta = \left\{ u \in \mathcal{U} \colon \mathrm{TV}(u) \le \delta \right\} \tag{2.1}$$

for all positive δ. We add the following natural assumption on the source term of (1.1):

(g) For $\delta_o > 0$, $g \colon \mathcal{U}_{\delta_o} \to \mathbf{L^1}(\mathbb{R}; \mathbb{R}^n)$ is such that for suitable $L_1, L_2 > 0$, $\forall u, u' \in \mathcal{U}_{\delta_o}$

$$\left\| g(u) - g(u') \right\|_{\mathbf{L^1}} \le L_1 \left\| u - u' \right\|_{\mathbf{L^1}} \quad \text{and} \quad \mathrm{TV}\left(g(u)\right) \le L_2 \,.$$

The next hypothesis has a mainly technical role. It allows us to consider also higher order ordinary differential equations and in the applications below it is a linear map.

(Π) $\Pi \in \mathbf{C^{0,1}}(\mathbb{R}^m; \mathbb{R})$.

Concerning the boundary, we introduce the following conditions.

(NC) There exist $c > 0$ and $\ell \in \{1, 2, \ldots, n-1\}$ such that $\lambda_\ell(\hat u) < \Pi(\hat w) - c$ and $\lambda_{\ell+1}(\hat u) > \Pi(\hat w) + c$.

The above Non Characteristic condition on f is coordinated with the following assumption on b, which describes how the boundary data are assigned.

(b) $b \in \mathbf{C^1}(\Omega;\mathbb{R}^{n-\ell})$ is such that $\det\Big(D_u b(\hat u)\,\big[r_{\ell+1}(\hat u) \quad r_{\ell+2}(\hat u) \quad \cdots \quad r_n(\hat u)\big]\Big) \neq 0$.

Condition **(b)** above is the usual assumption on the assignment of boundary data in a noncharacteristic problem for a conservation law, see for instance [2, 3, 6, 10, 17, 20]. Besides, it imposes b to be *not* invertible. The case of an invertible b would formally correspond to $\ell = 0$ in **(b)** and would allow the decoupling of system (1.1) in a PDE and a separate ODE.

First, we consider the balance law with given boundary γ_* and boundary data B_*

$$\left\{\begin{array}{lrcl} \partial_t u + \partial_x f(u) = g(u) & (t,x) & \in & \mathbb{R}^+ \times \left]\gamma_*(t), +\infty\right[\\ b\Big(u\left(t,\gamma_*(t)+\right)\Big) = B_*(t) & t & \in & \mathbb{R}^+ \\ u(0,x) = u_o(x) & x & \in & \mathbb{R}^+. \end{array}\right. \tag{2.2}$$

The above problem will be related to (1.1) setting $B_*(t) = B\left(t, w(t)\right)$ and $\dot\gamma_*(t) = \Pi\left(w(t)\right)$. Following [10, Definition 3.1] and [6, Definition 2.2], we slightly modify the definition given in [17] of solution to (2.2) in the non characteristic case, see also [3]. Indeed, here we require the boundary condition to be satisfied by the solution only *almost everywhere.*

Definition 2.1. *[10, Definition 3.1] Let $T > 0$ and fix the state $\hat u$. A map $u = u(t,x)$ is a solution to (2.2) if*

1. *$u \in \mathbf{C^0}([0,T];\mathcal{U})$ with $u(t,x) \in \Omega$ for a.e. $t \in \mathbb{R}^+$, $x \in \left[\gamma_*(t), +\infty\right[$ and $u(t,x) = \hat u$ otherwise;*
2. *$u(0,x) = u_o(x)$ for a.e. $x \in \left[\gamma_*(0), +\infty\right[$ and $\lim_{x\to\gamma(t)+} b\left(u(t,x)\right) = B_*(t)$ a.e. $t \geq 0$;*
3. *for $x > \gamma(t)$, u is a weak entropy solution to $\partial_t u + \partial_x f(u) = g(u)$.*

We refer to [7, Chapter 4] for the entropy admissibility criterion in balance laws. Theorem 2.6 below shows the existence of solutions u to (1.1) in the class **BV**, more precisely $u(t) \in \mathbf{BV}(\mathbb{R}^+;\Omega)$ for all t. This, in turn, ensures the existence of the trace at 2.

First, we need to slightly extend [10, Theorem 3.2] to obtain a further estimate, namely (4), that will be used in Theorem 2.6.

Theorem 2.2. *Let system (2.2) satisfy **(f)**, **(g)** and **(b)**. Then, there exist positive δ, Δ, T and L such that for all $B_* \in \mathbf{BV}(\mathbb{R}^+;\mathbb{R}^{n-\ell})$ and $\gamma_* \in \mathbf{W^{1,\infty}}(\mathbb{R}^+;\mathbb{R})$ satisfying*

$$\begin{array}{c} \left\|b(\hat u) - B_*(0)\right\|_{\mathbb{R}^{n-\ell}} + \mathrm{TV}(B_*) < \delta \quad \text{and} \\ \lambda_\ell(\hat u) + c < \dot\gamma_*(t) < \lambda_{\ell+1}(\hat u) - c \text{ for a.e. } t \in [0,T], \end{array} \tag{2.3}$$

there exists a family of closed domains $\mathcal{D}_t$ with $\mathcal{U}_\delta \subseteq \mathcal{D}_t \subseteq \mathcal{U}_\Delta$ defined for all $t \in [0,T]$, and a process $P(t,t_o)\colon \mathcal{D}_{t_o} \to \mathcal{D}_{t_o+t}$, for all $t_o \in [0,T]$ and $t \in [0, T-t_o]$ such that

(1) for all $t_o \in [0,T]$ and $u \in \mathcal{D}_{t_o}$, $P(0,t_o)\,u = u$ while for all $t_o \in [0,T]$, $s \in [0,T-t_o]$, $t \in [0, T-t_o-s]$ and $u \in \mathcal{D}_{t_o}$, it holds that $P(t+s,t_o)\,u = P(t, t_o+s) \circ P(s,t_o)u$;

(2) for all $t_o \in [0,T]$, $t,t' \in [0,T-t_o]$ and for any $u \in \mathcal{D}_{t_o}$, we have the following Lipschitz estimate:

$$\left\|P(t,t_o)u - P(t',t_o)u\right\|_{\mathbf{L^1}} \leq L\left(1 + \|u\|_{\mathbf{L^1}}\right)\left|t-t'\right|;$$

(3) for all $u_o \in \mathcal{D}_0$, the map $u(t,x) = \left(P(t,0)u_o\right)(x)$ defined for $t \in [0,T]$ and $x \in \left[\gamma_*(t), +\infty\right[$, solves (2.2) in the sense of Definition 2.1;

(4) let P_1 and P_2 be the processes defined by (2.2) with (B^1_*, γ^1_*) and (B^2_*, γ^2_*). For any $t_o \in [0,T]$, $t \in [0, T-t_o]$ and for any $u_1, u_2 \in \mathcal{D}_{t_o}$, we have the following Lipschitz estimate:

$$\begin{aligned} &\left\|P_1(t,t_o)u_1 - P_2(t,t_o)u_2\right\|_{\mathbf{L^1}} + \int_{t_o}^{t_o+t} \left\|\left(P_1(\tau,t_o)u_1\right)\left(\gamma^1_*(\tau)+\right) - \left(P_2(\tau,t_o)u_2\right)\left(\gamma^2_*(\tau)+\right)\right\|_{\mathbb{R}^n} \mathrm{d}\tau \\ &\quad \leq L\left[\|u_1-u_2\|_{\mathbf{L^1}} + \int_{t_o}^{t_o+t} \left\|B^1_*(\tau) - B^2_*(\tau)\right\|_{\mathbb{R}^{n-\ell}} \mathrm{d}\tau + \sup_{\tau\in[t_o,t_o+t]} \left|\gamma^1_*(\tau) - \gamma^2_*(\tau)\right|\right]. \end{aligned}$$

The proofs of (1), (2) and (3) are as in [10, Theorem 3.2]. In Section 4 we provide the proof of the sharper estimate (4).

We impose to the ordinary differential equation in (1.1) to fit into the standard framework of Carathéodory equations, see [16, § 1], introducing the following conditions.

(F) The map $F\colon \mathbb{R}^+ \times \Omega \times \mathbb{R}^m \longrightarrow \mathbb{R}^m$ is such that

(F.1) For all $u \in \Omega$ and $w \in \mathbb{R}^m$, the function $\begin{array}{ccc} \mathbb{R}^+ & \longrightarrow & \mathbb{R}^m \\ t & \longmapsto & F(t,u,w) \end{array}$ is Lebesgue measurable.

(F.2) For all compact subset K of $\Omega \times \mathbb{R}^m$, there exists $C_K > 1$ such that for all $t \in \mathbb{R}^+$ and $(u_1,w_1),(u_2,w_2) \in K$

$$\left\|F(t,u_1,w_1) - F(t,u_2,w_2)\right\|_{\mathbb{R}^m} \leq C_K \, \left(\|u_1-u_2\|_{\mathbb{R}^n} + \|w_1-w_2\|_{\mathbb{R}^m}\right).$$

(F.3) There exists a function $C \in \mathbf{L^1_{loc}}(\mathbb{R}^+;\mathbb{R}^+)$ such that for all $t > 0$, $u \in \Omega$ and $w \in \mathbb{R}^m$

$$\left\|F(t,u,w)\right\|_{\mathbb{R}^m} \leq C(t)\, \left(1 + \|w\|_{\mathbb{R}^m}\right).$$

Above, we used the notation $C(t)$ and C_K to denote quantities whose precise value is not relevant in the sequel.

Consider now the problem

$$\begin{cases} \dot w = F_*(t,w) & t \in \mathbb{R}^+ \\ w(0) = w_o\,. \end{cases} \tag{2.4}$$

which is linked to (1.1) setting $F_*(t,w) = F\left(t, u\left(t,\gamma(t)+\right), w\right)$

Definition 2.3. *Let (2.4) be a Carathéodory equation in the sense of [16, § 1]. A function $w \in \mathbf{W^{1,1}}(\mathbb{R}^+;\mathbb{R}^m)$ is a solution to (2.4) if, for a.e. $t \in \mathbb{R}^+$, the integral equality $w(t) = w_o + \int_0^t F_*\left(\tau, w(\tau)\right)\,\mathrm{d}\tau$ holds.*

The following standard proposition ensures the well posedness of (2.4), see similar results in [16, Chapter 1].

Proposition 2.4. *Let $I \subseteq \mathbb{R}$ be an interval with $0 \in \mathring{I}$, and $F_*\colon I \times \mathbb{R}^m \to \mathbb{R}^m$ be a map measurable in $t \in I$ and such that there exist $A, B \in \mathbf{L^1_{loc}}(\mathbb{R}^+;\mathbb{R}^+)$ such that*

$$\left\|F_*(t,w)\right\|_{\mathbb{R}^m} \leq A(t) + B(t)\,\|w\|_{\mathbb{R}^m} \qquad \text{for all } t \in I \text{ and } w \in \mathbb{R}^m \tag{2.5}$$

and for any compact set $K \subset \mathbb{R}^m$ there exists a constant $C_K > 0$ satisfying

$$\left\|F_*(t,w_1) - F_*(t,w_2)\right\|_{\mathbb{R}^m} \leq C_K\,\|w_1 - w_2\|_{\mathbb{R}^m} \qquad \text{for all } t \in I \text{ and } w \in K\,. \tag{2.6}$$

Then, problem (2.4) admits a unique solution $w = w(t)$ in the sense of Definition 2.3. Moreover, given a sequence of vector fields $F_*^h\colon I \times \mathbb{R}^m \to \mathbb{R}^m$ all satisfying (2.5), (2.6) and converging a.e. on $I \times \mathbb{R}^m$ to F_*, call w_h the corresponding solutions to (2.4). Then, we have the convergence $\lim_{h\to+\infty} w_h = w$ uniformly on any compact time interval.

The proof is elementary and is sketched in Section 4.

Now we pass to the full problem (1.1), first providing a rigorous definition of solution to (1.1).

Definition 2.5. *Let $T > 0$ and the state $\hat u$ be fixed. A triple (u,w,γ) with*

$$u \in \mathbf{C^0}\left([0,T];\mathcal{U}\right) \qquad w \in \mathbf{W^{1,1}}\left([0,T];\mathbb{R}^m\right) \qquad \gamma \in \mathbf{W^{1,\infty}}\left([0,T];\mathbb{R}^m\right)$$

is a solution to (1.1) on $[0,T]$ with initial datum (u_o,w_o,x_o) such that $u_o \in \mathcal{U}$ with $u_o(x) = \hat u$ for $x < x_o$, $w_o \in \mathbb{R}^m$ and $x_o \in \mathbb{R}$, if

1. *u solves (2.2) on $[0,T]$ with $B_*(t) = B\left(t, w(t)\right)$, $\gamma_*(t) = \gamma(t)$ and initial datum u_o, in the sense of Definition 2.1;*
2. *w solves (2.4) on $[0,T]$ with $F_*(t) = F\left(t, u\left(t, \gamma(t)+\right), w\right)$ a.e. and initial datum w_o, in the sense of Definition 2.3;*
3. *$\gamma(t) = x_o + \int_0^t \Pi\left(w(\tau)\right) \mathrm{d}\tau$ for a.e. $t \in [0,T]$.*

We are now ready to state the main results of this paper.

(B) $B \in \mathbf{C^1}(\mathbb{R}^+ \times \mathbb{R}^m; \mathbb{R}^{n-\ell})$ is locally Lipschitz, i.e. for every compact subset K of $\mathbb{R}^m$, there exists a constant $\tilde{C}_K > 0$ such that, for every $t > 0$ and $w \in K$:

$$\left\| \frac{\partial}{\partial t} B(t,w) \right\|_{\mathbb{R}^{n-\ell}} + \left\| \frac{\partial}{\partial w} B(t,w) \right\|_{\mathbb{R}^{n-\ell}} \le \tilde{C}_K \,.$$

We now present the main result of this work, which extends [6, Theorem 2.8] allowing moving boundaries, comprising the source term, ensuring uniqueness and providing stability estimates.

Theorem 2.6. *Let **(f)**, **(g)**, **(Π)**, **(NC)**, **(b)**, **(F)** and **(B)** hold. Assume that $b(\hat{u}) = B(0, \hat{w})$. Then, there exist positive δ, Δ, L, T_δ, domains $\hat{\mathcal{D}}_t$ (for $t \in [0, T_\delta]$) and maps $\hat{P}(t, t_0) \colon \hat{\mathcal{D}}_{t_0} \to \hat{\mathcal{D}}_{t_0+t}$ ($t_0, t_0 + t \in [0, T_\delta]$) such that*

1. *$\left(\mathcal{U}_\delta \times \mathcal{B}_\delta(\hat{w}) \times]\hat{x} - \delta, \hat{x} + \delta[\right) \subseteq \hat{\mathcal{D}}_t \subseteq \left(\mathcal{U}_\Delta \times \mathcal{B}_\Delta(\hat{w}) \times]\hat{x} - \Delta, \hat{x} + \Delta[\right)$;*
2. *for all t_0, t_1, t_2 with $t_0 \in [0, T_\delta[$, $t_1 \in [0, T_\delta - t_0[$ and $t_2 \in [0, T - t_0 - t_1]$, then $\hat{P}(t_2, t_0 + t_1) \circ \hat{P}(t_1, t_0) = \hat{P}(t_1 + t_2, t_0)$ and $\hat{P}(0, t_0) = Id$;*
3. *for $t_0 \in [0, T_\delta[$, $t \in [0, T_\delta - t_0]$, and $(u, w, x), (\bar{u}, \bar{w}, \bar{x}) \in \hat{\mathcal{D}}_{t_0}$*
$$\left\| \hat{P}(t, t_0)(u, w, x) - \hat{P}(t, t_0)(\bar{u}, \bar{w}, \bar{x}) \right\|_{\mathbf{L^1} \times \mathbb{R}^m \times \mathbb{R}} \le L \left(\|u - \bar{u}\|_{\mathbf{L^1}} + \|w - \bar{w}\|_{\mathbb{R}^m} + |x - \bar{x}| \right);$$
4. *for all $(u_0, w_0, x_0) \in \hat{\mathcal{D}}_0$, the map $t \to \hat{P}(t, 0)(u_0, w_0, x_0)$, defined for $t \in [0, T_\delta]$, solves (1.1) in the sense of Definition 2.5.*

The proof is deferred to Section 4.

We consider now the well posedness of (1.3). To this aim, we have to slightly modify the various assumptions. The notation below is the obvious extension of that used above, for instance $\hat{u}^\pm$ are fixed reference states in $\Omega^\pm$ and the sets $\mathcal{U}^\pm_\delta$ are defined similarly to (2.1).

(f*) $f^\pm \in \mathbf{C^4}(\Omega^\pm; \mathbb{R}^{n^\pm})$ is smooth and such that, for all $u^\pm \in \Omega^\pm$, $Df^\pm(u^\pm)$ is strictly hyperbolic and each characteristic field is either genuinely nonlinear or linearly degenerate.

(g*) For $\delta_o > 0$, $g^\pm \colon \mathcal{U}^\pm_{\delta_o} \to \mathbf{L^1}(\mathbb{R}; \mathbb{R}^{n^\pm})$ is such that for suitable $L_1, L_2 > 0$, $\forall u, u' \in {\mathcal{U}^\pm}_{\delta_o}$

$$\left\| g^\pm(u) - g^\pm(u') \right\|_{\mathbf{L^1}} \le L_1 \left\| u - u' \right\|_{\mathbf{L^1}} \quad \text{and} \quad \mathrm{TV}\left(g^\pm(u)\right) \le L_2 \,.$$

(Π*) $\Pi^\pm \in \mathbf{C^{0,1}}(\mathbb{R}^m; \mathbb{R})$.

(NC*) There exist $c > 0$, $\ell^- \in \{2, \ldots, n^-\}$ and $\ell^+ \in \{1, 2, \ldots, n^+ - 1\}$ such that

$$\begin{array}{rclcrcl} \lambda_{\ell^- - 1}(\hat{u}^-) & < & \Pi^-(\hat{w}) - c & \text{and} & \lambda_{\ell^-}(\hat{u}^-) & > & \Pi^-(\hat{w}) + c \\ \lambda_{\ell^+}(\hat{u}^+) & < & \Pi^+(\hat{w}) - c & \text{and} & \lambda_{\ell^+ + 1}(\hat{u}^+) & > & \Pi^+(\hat{w}) + c \end{array}$$

(b*) $b^- \in \mathbf{C^1}(\Omega^- \times \Omega^+; \mathbb{R}^{\ell^-})$ and $b^+ \in \mathbf{C^1}(\Omega^- \times \Omega^+; \mathbb{R}^{n^+ - \ell^+})$ are such that

$$\begin{array}{rcl} \det\left(D_{u^-} b^-(\hat{u}^-, \hat{u}^+) \left[r^-_1(\hat{u}^-) \quad r^-_2(\hat{u}^-) \quad \cdots \quad r^-_{\ell^-}(\hat{u}^-) \right] \right) & \neq & 0 \\ \det\left(D_{u^+} b^+(\hat{u}^-, \hat{u}^+) \left[r^+_{\ell^+ + 1}(\hat{u}^+) \quad r^+_{\ell^+ + 2}(\hat{u}^+) \quad \cdots \quad r^+_{n^+}(\hat{u}^+) \right] \right) & \neq & 0 \end{array}$$

(F*) The map $F\colon \mathbb{R}^+ \times \Omega^- \times \Omega^+ \times \mathbb{R}^m \longrightarrow \mathbb{R}^m$ is such that

(F*.1) For all $u^\pm \in \Omega^\pm$ and $w \in \mathbb{R}^m$, the function $\begin{array}{ccc} \mathbb{R}^+ & \longrightarrow & \mathbb{R}^m \\ t & \longmapsto & F(t,u^-,u^+,w) \end{array}$ is Lebesgue measurable.

(F*.2) For all compact subset K of $\Omega^- \times \Omega^+ \times \mathbb{R}^m$, there exists $C_K > 1$ such that for all $t \in \mathbb{R}^+$ and $(u_1^-, u_1^+, w_1), (u_2^-, u_2^+, w_2) \in K$

$$\left\| F(t,u_1^-,u_1^+,w_1) - F(t,u_2^-,u_2^+,w_2) \right\|_{\mathbb{R}^m} \le C_K \left[\left\| u_1^- - u_2^- \right\|_{\mathbb{R}^{n^-}} + \left\| u_1^+ - u_2^+ \right\|_{\mathbb{R}^{n^+}} + \|w_1 - w_2\|_{\mathbb{R}^m} \right].$$

(F*.3) There exists a function $C \in \mathbf{L^1_{loc}}(\mathbb{R}^+;\mathbb{R}^+)$ such that for all $t > 0$, $u^\pm \in \Omega^\pm$ and $w \in \mathbb{R}^m$

$$\left\| F(t,u^-,u^+,w) \right\|_{\mathbb{R}^m} \le C(t)\, \left(1 + \|w\|_{\mathbb{R}^m}\right) .$$

(B*) $B^- \in \mathbf{C^1}(\mathbb{R}^+ \times \mathbb{R}^m;\mathbb{R}^{\ell^-})$ and $B^- \in \mathbf{C^1}(\mathbb{R}^+ \times \mathbb{R}^m;\mathbb{R}^{n^+-\ell^+})$ are locally Lipschitz, i.e. for every compact subset K of $\mathbb{R}^m$, there exists a constant $\tilde{C}_K > 0$ such that, for every $t > 0$ and $w \in K$:

$$\left\| \frac{\partial}{\partial t} B^\pm(t,w) \right\| + \left\| \frac{\partial}{\partial w} B^\pm(t,w) \right\| \le \tilde{C}_K .$$

The extension of Theorem 2.6 to the case of (1.3) is as follows.

Theorem 2.7. *Let **(f*)**, **(g*)**, **(Π*)**, **(NC*)**, **(b*)**, **(F*)** and **(B*)** hold. Assume moreover that* $b^\pm(\hat{u}^-,\hat{u}^+) = B^\pm(0,\hat{w})$. *Then, there exist positive* δ, Δ, L, T_δ, *domains* $\hat{\mathcal{D}}_t$ *(for* $t \in [0,T_\delta]$*) and maps* $\hat{P}(t,t_0)\colon \hat{\mathcal{D}}_{t_0} \to \hat{\mathcal{D}}_{t_0+t}$ $(t_0, t_0+t \in [0,T_\delta])$ *such that*

1. $\left(\mathcal{U}_\delta^- \times \mathcal{U}_\delta^+ \times \mathcal{B}_\delta(\hat{w}) \times]\hat{x}-\delta,\hat{x}+\delta[\right) \subseteq \hat{\mathcal{D}}_t \subseteq \left(\mathcal{U}_\Delta^- \times \mathcal{U}_\Delta^+ \times \mathcal{B}_\Delta(\hat{w}) \times]\hat{x}-\Delta,\hat{x}+\Delta[\right)$;
2. *for all* t_0,t_1,t_2 *with* $t_0 \in [0,T_\delta[$, $t_1 \in [0,T_\delta - t_0[$ *and* $t_2 \in [0,T-t_0-t_1]$, *then* $\hat{P}(t_2,t_0+t_1) \circ \hat{P}(t_1,t_0) = \hat{P}(t_1+t_2,t_0)$ *and* $\hat{P}(0,t_0) = Id$;
3. *for* $t_0 \in [0,T_\delta[$, $t \in [0,T_\delta - t_0]$, *and* $(u^-,u^+,w,x), (\bar{u}^-,\bar{u}^+,\bar{w},\bar{x}) \in \hat{\mathcal{D}}_{t_0}$

$$\begin{aligned} & \left\| \hat{P}(t,t_0)(u^-,u^+,w,x) - \hat{P}(t,t_0)(\bar{u}^-,\bar{u}^+,\bar{w},\bar{x}) \right\|_{\mathbf{L^1}\times\mathbf{L^1}\times\mathbb{R}^m\times\mathbb{R}} \\ \le \;& L \left(\left\| u^- - \bar{u}^- \right\|_{\mathbf{L^1}} + \left\| u^+ - \bar{u}^+ \right\|_{\mathbf{L^1}} + \|w - \bar{w}\|_{\mathbb{R}^m} + |x - \bar{x}| \right); \end{aligned}$$

4. *for all* $(u_0^-,u_0^+,w_0,x_0) \in \hat{\mathcal{D}}_0$, *the map* $t \to \hat{P}(t,0)(u_0^-,u_0^+,w_0,x_0)$, *defined for* $t \in [0,T_\delta]$, *solves (1.1) in the sense of Definition 2.5.*

The proof is a simple modification of that of Theorem 2.6 and is hence omitted.

3 Models and Numerical Integrations

Below, in the numerical integrations of the convective part of the PDE the moving mesh method of [14] is used. The ODE at the interface is solved using a two stage Runge–Kutta method. The ODE–PDE coupling, as well as the incorporation of the source terms, is realized by a Strang Splitting, see [19, Paragraph 17.4].

In all examples, the following parameters are chosen. The computational domain of the PDE $[0,1]$ is discretized with 1000 points. At the boundaries $x=0$ and $x=1$, free outflow conditions are imposed. The time steps are chosen adaptively corresponding to a CFL number 0.9.

3.1 Gas - Particle Interaction

We consider now the model (1.4) for the interaction of a gas with a particle. The gas is described by the classical p–system with a source term due to gravity and the pressure law satisfying

(p) $p \in \mathbf{C^4}(\mathring{\mathbb{R}}^+;\mathring{\mathbb{R}}^+)$, $p'(\rho) > 0$ and $p''(\rho) \geq 0$ for all $\rho \in \mathring{\mathbb{R}}^+$.

The particle fills the segment $[\gamma^-(t), \gamma^+(t)]$, interacts with the gas and is subject to gravity.

First, we observe that the smooth solutions to (1.4) conserve the total energy

$$\mathcal{E}(t) = \int_{\mathbb{R}\setminus[\gamma^-(t),\gamma^+(t)]} \Big(E\left(\rho(t,x), q(t,x)\right) + \rho(t,x)\, g\, x\Big)\, \mathrm{d}x + m\, g\, \frac{\gamma^-(t)+\gamma^+(t)}{2} + \frac{1}{2}\, m\, V^2(t)\,.$$

Above, the integral is the total energy of the gas while the latter terms are the gravity potential and the kinetic energy of the incompressible particle. Indeed

$$E(\rho, q) = \frac{q^2}{2\rho} + \rho \int_{\rho_o}^{\rho} \frac{p(r)}{r^2}\, \mathrm{d}r \qquad \text{and} \qquad F(\rho,q) = \frac{q}{\rho}\left(E(\rho,q) + p(\rho)\right)$$

are the gas energy density and flow, see [9, § 9.2]. Simple computations give:

$$\begin{aligned}
\frac{d}{dt}\mathcal{E}(t) &= \int_{\mathbb{R}\setminus[\gamma^-(t),\gamma^+(t)]} \partial_t \Big(E\left(\rho(t,x), q(t,x)\right) + \rho(t,x)\, g\, x\Big)\, \mathrm{d}x \\
&\quad + \Big(E(\rho^-, q^-)\, \dot\gamma^- - E(\rho^+, q^+)\, \dot\gamma^+\Big) + \Big(\rho^-\, g\, \gamma^-\, \dot\gamma^- - \rho^+\, g\, \gamma^+\, \dot\gamma^+\Big) \\
&\quad + m\, g\, \frac{\dot\gamma^-(t) + \dot\gamma^+(t)}{2} + m\, V\, \dot V
\end{aligned}$$

Recall that by (1.4), $V = \dot\gamma^- = \dot\gamma^+$. Moreover, along smooth solutions, the conservation of energy yields $\partial_t E + \partial_x F = -q\, g$, so that

$$\begin{aligned}
\frac{d}{dt}\mathcal{E}(t) &= \int_{\mathbb{R}\setminus[\gamma^-(t),\gamma^+(t)]} \Big(-\partial_x F\left(\rho(t,x), q(t,x)\right) - g\, \partial_x\left(q(t,x)\, x\right)\Big)\, \mathrm{d}x \\
&\quad + \Big(E(\rho^-, q^-) + \rho^-\, g\, \gamma^- - E(\rho^+, q^+) - \rho^+\, g\, \gamma^+ + m\,(g + \dot V)\Big)\, V \\
&= \Big(F(\rho^+, q^+) - F(\rho^-, q^-) - g(q^-\, \gamma^- - q^+ \gamma^+)\Big) \\
&\quad + \Big(E(\rho^-, q^-) + \rho^-\, g\, \gamma^- - E(\rho^+, q^+) - \rho^+\, g\, \gamma^+ + m\,(g + \dot V)\Big)\, V \\
&= \Big(p^+ - p^- + m\,(g + \dot V)\Big)\, V\,.
\end{aligned}$$

This shows that energy is conserved with the particle speed law as in (1.4). A further justification of model (1.4) is in [12]. There, a system consisting of 2 compressible fluids is considered. At the incompressible limit for one of the two fluids, system (1.4) is obtained.

Proposition 3.1. *System (1.4) is a particular case of (1.3), where*

$$\begin{array}{rclrcl}
u^-(t,x) &=& \begin{bmatrix} \rho^-(t,x) \\ q^-(t,x) \end{bmatrix} & f^\pm(u) &=& \begin{bmatrix} q \\ \frac{q^2}{\rho} + p(\rho) \end{bmatrix} \\
u^+(t,x) &=& \begin{bmatrix} \rho^+(t,x) \\ q^+(t,x) \end{bmatrix} & g^\pm(u) &=& \begin{bmatrix} 0 \\ -g\,\rho \end{bmatrix} \\
w &=& V & F(t,u^-,u^+,w) &=& -g - \left(p(\rho^+) - p(\rho^-)\right)/m \\
b^-(u^-,u^+) &=& q^-/\rho^- & B^\pm(w) &=& w \\
b^+(u^-,u^+) &=& q^+/\rho^+ & \Pi^\pm(w) &=& w\,.
\end{array}$$

Fix $\hat\rho^\pm \in \mathring{\mathbb{R}}^+$, $\hat V \in \mathbb{R}$ *and set* $\hat u^\pm = (\hat\rho^\pm, \hat\rho^\pm\, \hat V)$, $\hat w = \hat V$. *Let* p *satisfy* ***(p)****. Assume that* $p(\hat\rho^+) - p(\hat\rho^-) = -m\, g$. *Then, Theorem 2.7 applies, hence* (1.4) *is well posed.*

In the numerical integration of (1.4), we chose the following function, parameters and data:

$$\begin{array}{rclrclrcl}
p(\rho) &=& \rho^{1.4} & g &=& 9.81 & m &=& 0.025 \\
w(0) &=& 1 & \gamma^-(0) &=& 0.75 & \gamma^+(0) &=& 0.80 \\
u\left(0,[0,0.5]\right) &=& \begin{bmatrix} 2 \\ 2 \end{bmatrix} & u\left(0,[0.5,1]\right) &=& \begin{bmatrix} 0.25 \\ 0.25 \end{bmatrix}
\end{array} \tag{3.1}$$

On each side of the interface, 500 points are equally distributed. The time of integration is 0.4.

The Riemann Problem at $t = 0$ generates a 2-shock moving upward and a 1-rarefaction moving downward, see Figure 1, left and middle. The particle is first subject only to gravity, since the upper and lower gas pressure balance each other. The 2-shock is slightly bent by gravity. At time

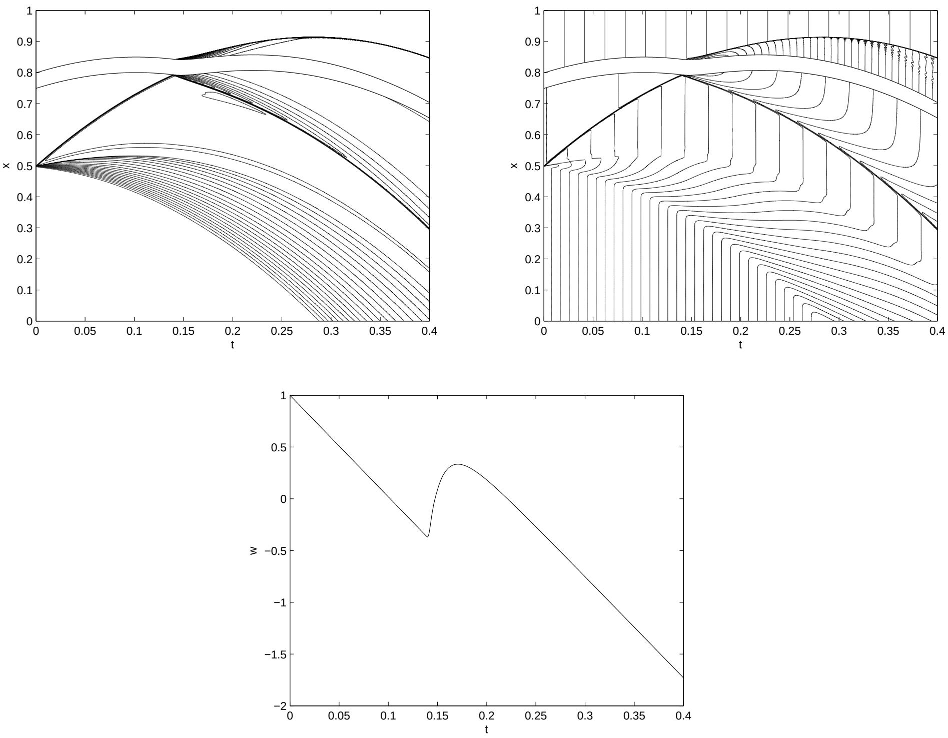


Figure 1: Numerical integration of (1.4)–(3.1). Above: the vertical axis is the x coordinate, the horizontal axis is time, the particle position is in the white strip. Left, the contour lines for ρ and, right, the ones for q. Below: the vertical axis is the particle's speed, the horizontal axis is time.

$t \simeq 0.14$, the 2-shock hits the particle. This interaction causes a sharp change in the particle's acceleration, see Figure 1, right. The shock is both reflected into a 1-shock and refracted into a 2-compression wave. Then, at $t \simeq 0.15$, the particle starts moving upward. The change in the particle speed also creates a 1-rarefaction that interacts with the 1-shock. Later, due to gravity, at $t \simeq 0.22$ the particle moves downward again.

3.2 A Moving Bottleneck

Consider a rectilinear road where traffic dynamics is described by the Aw–Rascle model [5]

$$\begin{cases} \partial_t \rho + \partial_x (\rho\, v) = 0 \\ \partial_t \Big(\rho\, \left(v + p(\rho)\right)\Big) + \partial_x \Big(\rho\, v\, \left(v + p(\rho)\right)\Big) = 0 \end{cases} \tag{3.2}$$

where $\rho = \rho(t,x)$, respectively $v = v(t,x)$, is the traffic density, respectively speed, at time t and position x. The *"pressure"* p can be chosen for instance $p(\rho) = k\rho^\gamma$ with $\gamma \geq 1$ and $k > 0$, see [5, formula (2.2)]. Below, we require the following general condition:

(P) $p \in \mathbf{C^4}(\mathring{\mathbb{R}}^+; \mathring{\mathbb{R}}^+)$ is such that $p'(\rho) > 0$ and $\frac{d^2}{d\rho^2}\left(\rho\, p(\rho)\right) \neq 0$ for all $\rho \in \mathring{\mathbb{R}}^+$.

A large vehicle at position $X = X(t)$ hinders the flow of traffic, so that next to it the maximal possible traffic flow is diminished, i.e.

$$\left\{\begin{array}{rcl}(\rho\, v)\left(t, X(t)-\right) & = & \left(R+\rho(t,X(t)-)\right)\dot X(t)\\ (\rho\, v)\left(t, X(t)+\right) & = & \left(R+\rho(t,X(t)+)\right)\dot X(t)\end{array}\right. \tag{3.3}$$

where R can be interpreted as the *"density"* of the large vehicle. The vehicle at X adjusts its speed to the traffic conditions in front of it as follows:

$$\ddot X = -\frac{1}{T_*}\left(\dot X - V_*\left(1-\frac{\rho\left(t,X(t)+\right)}{R_*}\right)\right)$$

where T_*, V_* and R_* are fixed positive constants. The whole model then reads

$$\left\{\begin{array}{ll}\left\{\begin{array}{l}\partial_t\rho^- + \partial_x(\rho^-\, v^-) = 0\\ \partial_t\left(\rho^-\,\left(v^- + p(\rho^-)\right)\right) + \partial_x\left(\rho^-\, v^-\,\left(v^- + p(\rho^-)\right)\right) = 0\end{array}\right. & x < X(t)\\ \left\{\begin{array}{l}\partial_t\rho^+ + \partial_x(\rho^+\, v^+) = 0\\ \partial_t\left(\rho^+\,\left(v^+ + p(\rho^+)\right)\right) + \partial_x\left(\rho^+\, v^+\,\left(v^+ + p(\rho^+)\right)\right) = 0\end{array}\right. & x > X(t)\\ (\rho^-\, v^-)\left(t,X(t)-\right) = \left(R+\rho^-(t,X(t)-)\right)\dot X(t) & \\ (\rho^+\, v^+)\left(t,X(t)+\right) = \left(R+\rho^+(t,X(t)+)\right)\dot X(t) & \\ \ddot X = -\frac{1}{T_*}\left(\dot X - V_*\left(1-\frac{\rho^+\left(t,X(t)+\right)}{R_*}\right)\right) & \end{array}\right. \tag{3.4}$$

and fits in the framework of Theorem 2.7.

Remark that, as is to be expected, the total mass of the solutions to (3.4) is conserved. Indeed, the usual Rankine–Hugoniot conditions [7, § 4.2] hold at any (t, x) with $x \neq X(t)$. Along the trajectory of the large vehicle, conditions (3.3) ensure that

$$(\rho^+\, v^+)\left(t,X(t)+\right) - (\rho^-\, v^-)\left(t,X(t)-\right) = \dot X(t)\,\left(\rho^+\left(t,X(t)+\right) - \rho^-\left(t,X(t)-\right)\right)$$

which is equivalent to the conservation of ρ.

Proposition 3.2. *Fix positive R, R_*, V_* and T_*. Then, system (3.4) fits into (1.3), where*

$$\begin{array}{rclrcl} u^\pm(t,x) & = & \left[\begin{array}{c}\rho^\pm\\ \rho^\pm\,\left(v^\pm + p(\rho^\pm)\right)\end{array}\right] & f^\pm(u) & = & \left[\begin{array}{c}\rho^\pm\, v^\pm\\ \rho^\pm\, v^\pm\,\left(v^\pm + p(\rho^\pm)\right)\end{array}\right]\\ & & & g^\pm(u) & = & 0\\ w & = & \dot X & F(t,u^-,u^+,w) & = & -\frac{1}{T_*}\left(\dot X - V_*\left(1-\frac{\rho^+}{R_*}\right)\right)\\ \gamma & = & X & \Pi^\pm(w) & = & \dot X\\ b^\pm(u^-,u^+) & = & \rho^\pm\, v^\pm/(R+\rho^\pm) & B^\pm(w) & = & \dot X\,. \end{array}$$

*Let now p satisfy **(P)**, fix $\hat u^\pm$ such that*

$$\hat\rho^-\,\hat v^-/(R+\hat\rho^-) \quad = \quad \hat\rho^+\,\hat v^+/(R+\hat\rho^+) \tag{3.5}$$

$$\hat v^\pm \neq 0 \quad \textit{and} \quad \hat v^\pm \neq \rho^\pm\left(1+\frac{\rho^\pm}{R}\right)p'(\rho^\pm) \tag{3.6}$$

$$v^\pm - {\rho^\pm}^2\, p'(\rho^\pm) < \quad \hat w \quad < v^\pm \tag{3.7}$$

where $\hat w = \hat\rho^\pm\,\hat v^\pm/(R+\hat\rho^\pm)$. Then, Theorem 2.7 applies, hence (3.4) *is well posed.*

In the numerical integration of (3.4), we chose the following functions, parameters and data:

$$
\begin{array}{rclrclrcl}
p(\rho) & = & 3\rho^2 & R & = & 0.2 & & & \\
T_* & = & 0.2 & V_* & = & 0.8 & R_* & = & 0.8 \\
w(0) & = & 0.4 & \gamma^-(0) & = & 0.2 & \gamma^+(0) & = & 0.2 \\
\rho\left(0,]-\infty, 1/6]\right) & = & 0.5 & \rho\left(0, [1/6, \infty[\right) & = & 0.4 & & & \\
v\left(0, \mathbb{R} \setminus [7/15, 11/15]\right) & = & 0.6 & v\left(0, [7/15, 11/15]\right) & = & 0.8 & & &
\end{array}
\qquad (3.8)
$$

On each side of the interface, 500 points are equally distributed. The time of integration is 1.4.

In the numerical integration of (3.4)–(3.8), at time $t = 0$, 2–waves arise from $x = 1/6$, $x = 7/15$ and $x = 11/15$. A 1-rarefaction arises from $x = 7/15$, while from $x = 11/15$ a 1-shock is born. The

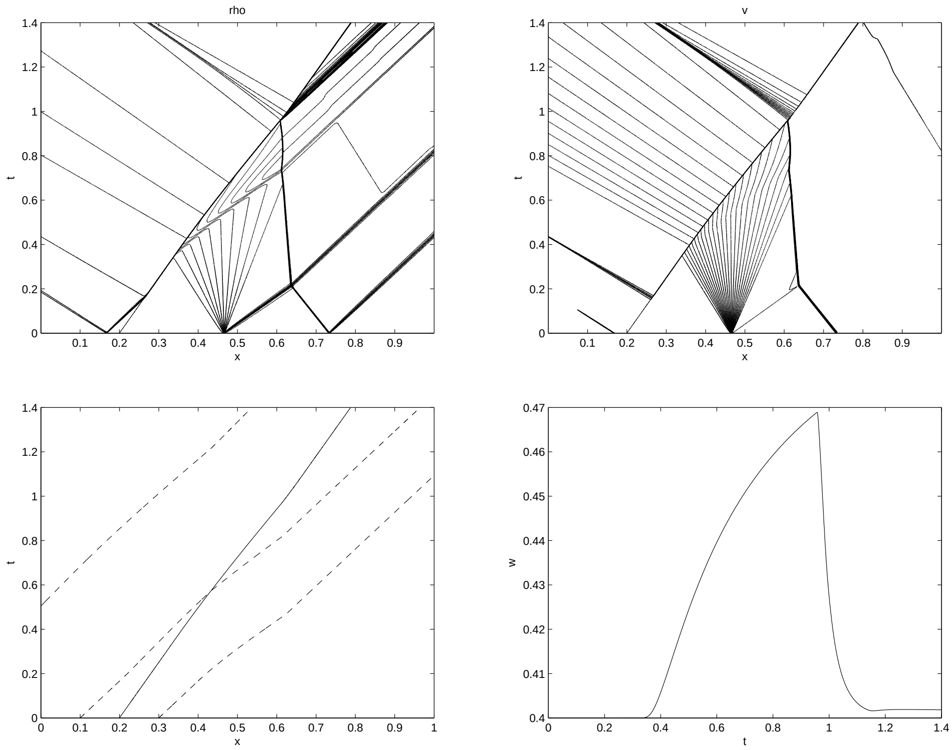


Figure 2: Numerical integration of (3.4)–(3.8). Above: the vertical axis is time, the horizontal axis is the space coordinate. Left, the contour lines for ρ and, right, the ones for v. Note that 2-waves are not seen in the right picture. Below: left, the 3 dashed lines represent the trajectories of 3 vehicles, while the solid line is trajectory of the truck, in the x (horizontal) t (vertical) plane; right, the vertical axis is the special vehicle's speed, the horizontal axis is time.

leftmost 2-wave reaches the truck at $t \simeq 0.15$ and is reflected into a 1-shock. Later, the truck enters the 1-rarefaction and, as it is physically reasonable, it accelerates, see Figure 2, bottom right. This interaction results in a refracted 1-rarefaction and in reflected 2-contact discontinuities, seen in the ρ-diagram but not in the v-diagram, see Figure 2, first line. At $t \simeq 0.96$, the truck hits a 1-shock and immediately slows down, see Figure 2, bottom right. This interaction results in a refracted 1-compression wave and in reflected 2-contact discontinuities. Note that the standard vehicles may well overtake the truck, see figure 2, bottom left.

3.3 The Piston

Now, we prove that the piston problem (1.2) in Eulerian coordinates, see the figure at (1.2), fits in the framework of Theorem 2.6.

Proposition 3.3. *System* (1.2) *is a particular case of* (1.1), *where*

$$
\begin{array}{rclrclrcl}
u & = & \begin{bmatrix}\rho\\ q\end{bmatrix} & f(u) & = & \begin{bmatrix} q \\ q^2/\rho + p(\rho)\end{bmatrix} & g(u) & = & \begin{bmatrix} 0 \\ -\nu\frac{q|q|}{\rho} - g\rho\sin\alpha\end{bmatrix}\\
w & = & V & F(t,u,w) & = & \beta\left(p_{\mathrm{ext}}(t) - p(\rho)\right) - g\sin\alpha & \Pi(w) & = & V\\
b(u) & = & q/\rho & B(t,w) & = & V\,. & & &
\end{array}
$$

Fix a state $(\hat\rho, \hat q) \in \mathring{\mathbb{R}}^+ \times \mathbb{R}$ *and call* $\hat V = \hat q/\hat\rho$. *Assume that:*

1. $p \in \mathbf{C^4}(\mathring{\mathbb{R}}^+; \mathbb{R}^+)$ *is such that* $p'(\rho) > 0$ *and* $p''(\rho) > 0$ *for all* $\rho \in \mathring{\mathbb{R}}^+$;
2. $p_{\mathrm{ext}} \in \mathbf{L^1_{loc}}(\mathbb{R}^+; \mathbb{R}^+)$;
3. $\left|\hat q/\hat\rho\right| < \sqrt{p'(\hat\rho)}$.

Then, the assumptions of Theorem 2.6 are satisfied, hence (1.2) *is well posed.*

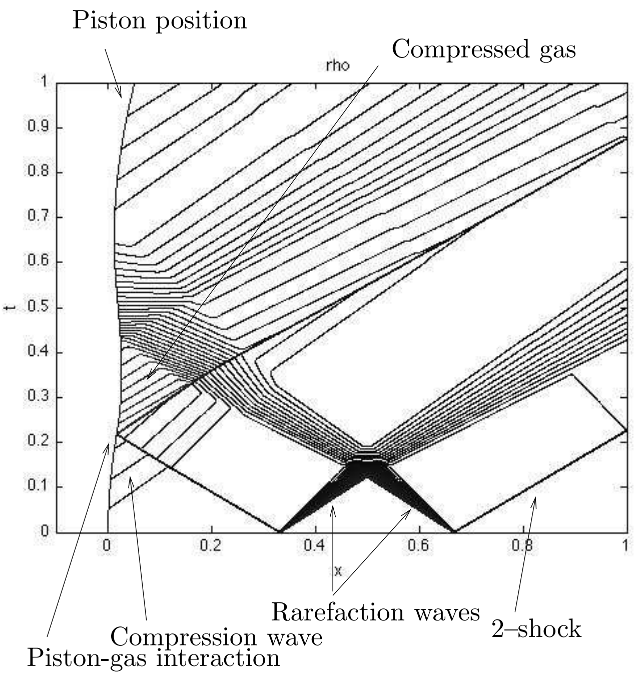


Figure 3: Contour lines of the numerical solution to (1.2)–(3.9) on the time interval $[0, 1]$. On the vertical axis is time, on the horizontal one is the space coordinate.

In the numerical integration below, we choose the following pressure functions, parameter values and initial data:

$$
\begin{array}{rclrcl}
p(\rho) & = & \rho^{1.4} & p_{ext}(t) & = & 1.5^{1.4}\\
\alpha & = & 0 & \nu & = & 0\\
V(0) & = & 0 & \gamma(0) & = & 0\\
u(0,[1/3,2/3]) & = & [2,0]^T & u(0,\mathbb{R}^+ \setminus [1/3,2/3]) & = & [1,0]^T
\end{array}
\tag{3.9}
$$

The choices $g = 0$ and $\nu = 0$ allow an easier identification of the various phenomena. The spatial grid consists of 1000 equally spaced points. The computation ends at $T = 1$.

At the beginning the piston is at rest in $x = 0$ and for $x \in [1/3, 2/3]$ the gas density is higher than outside it. The outer pressure pushes the piston to the right and a 1-compression wave in the gas is formed. At time $t \simeq 0.21$, the 1–shock in the gas hits the piston, see Figure 3. As a result of this interaction, a 2-shock is formed and interacts with the compression wave, so that the gas reaches densities higher than that in the initial data. The piston is slowed down by the high density of the gas until it starts moving to the left at $t \simeq 0.35$. The leftward movement of the piston causes a 2-rarefaction in the gas. The effect of the constant outer pressure causes the piston to move again to the right at $t \simeq 0.67$.

4 Technical Details

For later use, we state here without proof the Grönwall type lemma used in the sequel.

Lemma 4.1. *Fix $T > 0$. Let $\delta \in \mathbf{C^0}\left([0,T];\mathbb{R}^+\right)$, $\alpha \in \mathbf{L^\infty_{loc}}\left([0,T];\mathbb{R}^+\right)$ and $\beta \in \mathbf{L^1_{loc}}\left([0,T];\mathring{\mathbb{R}}^+\right)$. If*

$$\delta(t) \leq \alpha(t) + \int_0^t \beta(\tau)\,\delta(\tau)\,\mathrm{d}\tau \qquad \text{for a.e. } t \in [0,T]$$

then

$$\delta(t) \leq \alpha(t) + \int_0^t \alpha(\tau)\,\beta(\tau)\,e^{\int_\tau^t \beta(s)\,\mathrm{d}s}\,\mathrm{d}\tau \qquad \text{for a.e. } t \in [0,T]\,.$$

The proof is immediate and hence omitted.

Proof of Theorem 2.2. Thanks to [10, Theorem 3.2], we are left to prove only (4). Let δ_o be such that $\overline{B(\hat u, \delta_o)} \subseteq \Omega$. Consider first the case $g = 0$ and $\gamma_*^1 = \gamma_*^2$. We improve the construction in [10] as follows.

Let $\sigma \to R_j(\sigma)(u)$, respectively $\sigma \to S_j(\sigma)(u)$, be the j-rarefaction curve, respectively the j-shock curve, exiting u. If the j-th field is linearly degenerate, then the parameter σ above is the arc-length. In the genuinely nonlinear case, see [7, Definition 5.2], we choose σ so that (see [7, formula (5,37) and Remark 5.4])

$$\frac{\partial \lambda_j}{\partial \sigma}\left(R_j(\sigma)(u)\right) = 1 \qquad \text{and} \qquad \frac{\partial \lambda_j}{\partial \sigma}\left(S_j(\sigma)(u)\right) = 1\,.$$

Introduce the j-Lax curve $\sigma \to \psi_j(\sigma)(u) = \begin{cases} R_j(\sigma)(u) & \text{if} \quad \sigma \geq 0 \\ S_j(\sigma)(u) & \text{if} \quad \sigma < 0 \end{cases}$ and for $\boldsymbol{\sigma} \equiv (\sigma_1, \ldots, \sigma_n)$, define the map $\boldsymbol{\Psi}(\boldsymbol{\sigma}) = \psi_n(\sigma_n) \circ \ldots \circ \psi_1(\sigma_1)$. By **(f)**, see [7, Paragraph 5.3], given any two states $u^-, u^+ \in \Omega$ sufficiently close to $\hat u$, there exists a $\mathbf{C^2}$ map E such that

$$\boldsymbol{\sigma} = E(u^-, u^+) \quad \text{if and only if} \quad u^+ = \boldsymbol{\Psi}(\boldsymbol{\sigma})(u^-)\,. \tag{4.1}$$

Similarly, let the map $\boldsymbol{S}$ and the vector $\boldsymbol{q} = (q_1, \ldots, q_n)$ be defined by

$$u^+ = \boldsymbol{S}(\boldsymbol{q})(u^-) \quad \text{and} \quad \boldsymbol{S}(\boldsymbol{q}) = S_n(q_n) \circ \ldots \circ S_1(q_1)\,, \tag{4.2}$$

i.e. $\boldsymbol{S}$ is the gluing of the Rankine–Hugoniot curves.

We now use the usual ε-solutions to (2.2), defined by means of the classical wave front tracking technique, see [7] or [2, 3, 10] for the case with boundary. Let B_*^ε be a piecewise constant approximation of B_* such that $\left\|B_*^\varepsilon - B_*\right\|_{\mathbf{L^1}} < \varepsilon$. Recall the following definitions of the linear and quadratic potentials and the Glimm functional, given along an ε-solution $u = u(t,x)$, for suitable constants K, H_1, H_2 all greater than 1, see [7, 10]:

$$\begin{array}{rclrcl} V^\varepsilon_{B_*}(t) & = & \mathrm{TV}\left(B^\varepsilon_*; [t, +\infty[\right) & V^\varepsilon_u(t) & = & K \displaystyle\sum_{x \geq \gamma(t)} \sum_{i=1}^{\ell} \left|\sigma_{x,i}\right| + \sum_{x \geq \gamma(t)} \sum_{i=\ell+1}^{n+1} \left|\sigma_{x,i}\right| \\ Q^\varepsilon_u(t) & = & \displaystyle\sum_{(\sigma_{x,i}, \sigma_{y,j}) \in \mathcal{A}} \left|\sigma_{x,i}\sigma_{y,j}\right| & \Upsilon^\varepsilon_u(t) & = & V^\varepsilon_u(t) + H_1 V^\varepsilon_{B_*}(t) + H_2 Q^\varepsilon_u(t) \end{array} \tag{4.3}$$

where $\mathcal{A}$ is the usual set of approaching waves and $(\sigma_{x,1}, \ldots, \sigma_{x,n}) = E\left(u(t,x-), u(t,x-)\right)$ with E as in (4.1), see [7, 10, 13]. Recall that non-physical waves are assigned to the $(n+1)$-th family and all travel with the same speed $\widehat{\lambda} = \max_{i=1,n} \sup_{u \in \overline{B(\hat{u},\delta_o)}} \left|\lambda_i(u)\right|$.

Let u and v be ε-solutions corresponding to the two initial data u_o, v_o and the two boundary data B^1_* and B^2_*. Let ω be a piecewise constant function with the following properties: $\omega(t,\cdot)$ is an $\mathbf{L^1}$–function with small total variation, $\omega(t,x)$ has finitely many polygonal lines of discontinuity and the slope of any discontinuity line is bounded in absolute value by $\widehat{\lambda}$. The function ω does not need to have any relation with the conservation law.

Define the functions $u' = v + \omega$ and $\boldsymbol{q} \equiv (q_1, \ldots, q_n)$ implicitly by $u'(t,x) = \boldsymbol{S}\left(\boldsymbol{q}(t,x)\right)\left(u(t,x)\right)$ with $\boldsymbol{S}$ as in (4.2). We now consider the functional

$$\Phi(u,u')(t) = \bar{K} \sum_{i=1}^{\ell} \int_{\gamma(t)}^{\infty} \left|q_i(t)\right| W_i(t)\, \mathrm{d}x + \sum_{i=\ell+1}^{n} \int_{\gamma(t)}^{\infty} \left|q_i(t)\right| W_i(t)\, \mathrm{d}x \tag{4.4}$$

where $\bar{K}$ is a positive constant to be defined later. To define the W_i, recall that $J(u)$, respectively $J(v)$, denote the sets of all jumps in u, respectively in v, for $x > \gamma(t)$, while $\bar{J}(u)$, $\bar{J}(v)$ are the sets of the physical jumps only. If the i-th characteristic field is linearly degenerate, then we set

$$A_i(x) \doteq \sum \left\{ \left|\sigma_{y,\kappa}\right| : y \in \bar{J}(u) \cup \bar{J}(v) \text{ and } \begin{array}{l} y < x,\ i < \kappa \leq n, \text{ or} \\ y > x,\ 1 \leq \kappa < i \end{array} \right\}.$$

If the i-th field is genuinely nonlinear, the definition of A_i will contain an additional term, accounting for waves in u and in v of the same i-th family:

$$\begin{array}{rcl} A_i(x) & \doteq & \sum \left\{ \left|\sigma_{y,\kappa}\right| : y \in \bar{J}(u) \cup \bar{J}(v) \text{ and } \begin{array}{l} y < x,\ i < \kappa \leq n, \text{ or} \\ y > x,\ 1 \leq \kappa < i \end{array} \right\} \\ & & + \left\{ \begin{array}{ll} \sum \left\{ \left|\sigma_{y,i}\right| : \begin{array}{l} y \in \bar{J}(u),\ y < x \text{ or} \\ y \in \bar{J}(v),\ y > x \end{array} \right\} & \text{if } q_i(x) < 0, \\ \sum \left\{ \left|\sigma_{y,i}\right| : \begin{array}{l} y \in \bar{J}(v),\ y < x \text{ or} \\ y \in \bar{J}(u),\ y > x \end{array} \right\} & \text{if } q_i(x) \geq 0. \end{array} \right. \end{array} \tag{4.5}$$

Recall that non-physical fronts play no role in the definition of A_i. We remark that the function ω enters the definition of A_i only indirectly by influencing the sign of the scalar functions q_i.

Let now $W_i(t,x) = 1 + \kappa_1 A_i(t,x) + \kappa_2 \Big(\Upsilon^\varepsilon\left(u(t)\right) + \Upsilon^\varepsilon\left(v(t)\right)\Big)$. The constants κ_1, κ_2 are the same defined in [10], see also [7]. We also recall that, since δ_o is chosen small enough, the weights satisfy $1 \leq W_i(t,x) \leq 2$, hence for a suitable constant $C_3 > 1$,

$$\frac{1}{C_3} \left\|u'(t) - u(t)\right\|_{\mathbf{L^1}} \leq \Phi(u,u')(t) \leq C_3 \left\|u'(t) - u(t)\right\|_{\mathbf{L^1}}, \tag{4.6}$$

where the $\mathbf{L^1}$ norm is taken in the interval $\left]\gamma(t), +\infty\right[$.

We now want to prove that there exists a $\delta \in \left]0, \delta_o\right[$ and a $C > 0$ such that if u, v, ω, u' are the functions defined above satisfying $\Upsilon^\varepsilon_u(t)$, $\Upsilon^\varepsilon_v(t)$, $\Upsilon^\varepsilon_\omega(t)$, $\Upsilon^\varepsilon_{u'}(t) \leq \delta$, for any $t \geq 0$, then

$$\begin{array}{rl} & \Phi(u,u')(t_2) + \displaystyle\int_{t_1}^{t_2} \left\| u\left(s, \gamma^1_*(s)\right) - u'\left(s, \gamma^1_*(s)\right) \right\|_{\mathbb{R}^n} \mathrm{d}s \\ \leq & \Phi(u,u')(t_1) + C\varepsilon(t_2 - t_1) \\ & + C \displaystyle\int_{t_1}^{t_2} \left[\left\| b\left(u\left(s, \gamma^1_*(s)\right)\right) - b\left(v\left(s, \gamma^1_*(s)\right)\right) \right\|_{\mathbb{R}^{n-\ell}} + \mathrm{TV}\left(\omega(s,\cdot)\right) \right] \mathrm{d}s\,. \end{array}$$

To this aim, we use the main results obtained in [4, 7]. At each x define the intermediate states $U_0(x) = u(x)$, $U_1(x)$, $\ldots$, $U_n(x) = u'(x)$ by setting $U_i(x) \doteq S_i\left(q_i(x)\right) \circ S_{i-1}\left(q_{i-1}(x)\right) \circ \cdots \circ$

$S_1\left(q_1(x)\right)\left(u(x)\right)$. Moreover, call $\lambda_i(x) \doteq \lambda_i\left(U_{i-1}(x), U_i(x)\right)$ the speed of the i-shock connecting $U_{i-1}(x)$ with $U_i(x)$. For notational convenience, we write $q_i^{y+} \doteq q_i(y+)$, $q_i^{y-} \doteq q_i(y-)$ and similarly for $W_i^{y\pm}$, $\lambda_i^{y\pm}$. If $y < \tilde{y}$ are two consecutive points in $J = J(u) \cup J(v) \cup J(\omega)$, then $q_i^{y+} = q_i^{\tilde{y}-}$, $W_i^{y+} = W_i^{\tilde{y}-}$, $\lambda_i^{y+} = \lambda_i^{\tilde{y}-}$. Therefore, as in [7], outside the interaction times we have:

$$
\begin{aligned}
\frac{d}{dt}\Phi(u,u')(t) \;=\; & \bar{K}\sum_{y\in J}\sum_{i=1}^{\ell}\left(W_i^{y+}\left|q_i^{y+}\right|(\lambda_i^{y+}-\dot{x}_y) - W_i^{y-}\left|q_i^{y-}\right|(\lambda_i^{y-}-\dot{x}_y)\right) \\
& + \sum_{y\in J}\sum_{i=\ell+1}^{n}\left(W_i^{y+}\left|q_i^{y+}\right|(\lambda_i^{y+}-\dot{x}_y) - W_i^{y-}\left|q_i^{y-}\right|(\lambda_i^{y-}-\dot{x}_y)\right) \\
& + \bar{K}\sum_{i=1}^{\ell} W_i^{\gamma_*^1+}\left|q_i^{\gamma_*^1+}\right|(\lambda_i^{\gamma_*^1+}-\dot{\gamma}_*^1) + \sum_{i=\ell+1}^{n} W_i^{\gamma_*^1+}\left|q_i^{\gamma_*^1+}\right|(\lambda_i^{\gamma_*^1+}-\dot{\gamma}_*^1)
\end{aligned}
$$

where $\dot{x}_y$ is the velocity of the discontinuity at the point y. This is because the quantities q_i vanish outside a compact set. For each jump point $y \in J$ and every $i = 1, \ldots, n$, define

$$
\bar{q}_i^{y\pm} = \begin{cases} \bar{K} q_i^{y\pm} & \text{if } i \le \ell \\ q_i^{y\pm} & \text{if } i \ge \ell+1 \end{cases}
$$

and $E_{y,i} = W_i^{y+}\left|\bar{q}_i^{y+}\right|(\lambda_i^{y+}-\dot{x}_y) - W_i^{y-}\left|\bar{q}_i^{y-}\right|(\lambda_i^{y-}-\dot{x}_y)$, so that

$$
\frac{d\Phi}{dt}(u,u')(t) = \sum_{\substack{i=1,\ldots,n \\ y\in J}} E_{y,i} + \bar{K}\sum_{i=1}^{\ell} W_i^{\gamma_*^1+}\left|q_i^{\gamma_*^1+}\right|(\lambda_i^{\gamma_*^1+}-\dot{\gamma}_*^1) + \sum_{i=\ell+1}^{n} W_i^{\gamma_*^1+}\left|q_i^{\gamma_*^1+}\right|(\lambda_i^{\gamma_*^1+}-\dot{\gamma}_*^1)\,.
$$

Note that $\bar{q}_i^{y\pm}$ is a reparametrization of the shock curve equivalent to that provided by $q_i^{y\pm}$ and that satisfies the key property, see [7, Remark 5.4], $\left(S_i(\bar{q}_i) \circ S_i(-\bar{q}_i)\right)(u) = u$. Therefore, the computations in [4, Section 4] and [7, Chapter 8] apply. As in [4, formula (4.13)], we have $\sum_{y\in J}\sum_{i=1}^{n} E_{y,i} \le C \cdot \left(\varepsilon + \mathrm{TV}(\omega)\right)$ for a suitable positive constant C. Concerning the term on the boundary, **(NC)** implies that if $i \le \ell$, then $\lambda_i^{\gamma+} - \dot{\gamma} \le -c$. Moreover, $W_i^{\gamma+} \ge 1$. Hence, if $B_*^\varepsilon = b\left(u(t,\gamma_*^1(t)+\right)$, $\bar{B}_*^\varepsilon = b\left(v(t,\gamma_*^1(t)+\right)$, [10, Lemma 4.2] implies

$$
\begin{aligned}
& \bar{K}\sum_{i=1}^{\ell} W_i^{\gamma_*^1+}\left|q_i^{\gamma_*^1+}\right|(\lambda_i^{\gamma_*^1+}-\dot{\gamma}_*^1) + \sum_{i=\ell+1}^{n} W_i^{\gamma_*^1+}\left|q_i^{\gamma_*^1+}\right|(\lambda_i^{\gamma_*^1+}-\dot{\gamma}_*^1) \\
\le\;& -c\bar{K}\sum_{i=1}^{\ell}\left|q_i^{\gamma_*^1+}\right| + C\sum_{i=\ell+1}^{n}\left|q_i^{\gamma_*^1+}\right| \\
\le\;& -c\bar{K}\sum_{i=1}^{\ell}\left|q_i^{\gamma_*^1+}\right| + (C+1)\sum_{i=1}^{\ell}\left|q_i^{\gamma_*^1+}\right| - \sum_{i=\ell+1}^{n}\left|q_i^{\gamma_*^1+}\right| + (C+1)\left[\left\|B_*^\varepsilon - \bar{B}_*^\varepsilon\right\|_{\mathbb{R}^{n-\ell}} + \left\|\omega^{\gamma_*^1+}\right\|\right] \\
\le\;& (C+1)\left[\left\|B_*^\varepsilon - \bar{B}_*^\varepsilon\right\|_{\mathbb{R}^{n-\ell}} + \left\|\omega^{\gamma_*^1+}\right\|\right] - \sum_{i=1}^{n}\left|q_i^{\gamma_*^1+}\right|
\end{aligned}
$$

provided $\bar{K} > (2+C)/c$ is sufficiently large. Reinserting the t variable, we obtain

$$
\begin{aligned}
& \left(\frac{d}{dt}\Phi(u,u')(t)\right) + \left\|u\left(t,\gamma_*^1(t)\right) - u'\left(t,\gamma_*^1(t)\right)\right\|_{\mathbb{R}^n} \\
\le\;& (C+1)\left[\varepsilon + \mathrm{TV}\left(\omega(t,\cdot)\right) + \left\|b\left(u\left(s,\gamma_*^1(s)\right)\right) - b\left(v\left(s,\gamma_*^1(s)\right)\right)\right\|_{\mathbb{R}^{n-\ell}}\right].
\end{aligned}
$$

Then, standard computations (see [7, Theorem 8.2]) show that when an interaction occurs, the possible increase in $A_i(x)$ is compensated by a decrease in Υ^ε. Therefore, the functional Φ is not increasing at interaction times. Hence, integrating the previous inequality, we obtain

$$\begin{aligned}
& \Phi(u,u')(t_2) + \int_{t_1}^{t_2} \left\| u\left(t, \gamma_*^1(t)\right) - u'\left(t, \gamma_*^1(t)\right) \right\|_{\mathbb{R}^n} \mathrm{d}t \\
\leq\ & \Phi(u,u')(t_1) + (C+1)\varepsilon(t_2 - t_1) \\
& + (C+1) \int_{t_1}^{t_2} \left(\left\| B_*^{1,\varepsilon}(s) - B_*^{2,\varepsilon}(s) \right\|_{\mathbb{R}^{n-\ell}} + \mathrm{TV}\left(\omega(s,\cdot)\right) \right) \mathrm{d}s\,.
\end{aligned}$$

Hence, point (4) in Theorem 2.2 is proved in the case $\gamma_*^1 = \gamma_*^2$ and $g = 0$.

In the case $B_*^1 = B_*^2$ and $g = 0$, point (4) is proved by [10, Proposition 2.3].

Finally, the proof of point (4) in the general case $g \neq 0$ is obtained using exactly the same technique adopted in [10, Theorem 3.2], based on operator splitting. □

Proof of Proposition 2.4. Existence and uniqueness of a global solution to (2.4) follow from [16, § 1]. To prove continuous dependence from the vector field, use (2.5) to find the *a priori* estimate

$$\left\| w(t) \right\|_{\mathbb{R}^m} \leq \|w_o\|_{\mathbb{R}^m} + \int_0^t \left\| F\left(\tau, w(\tau)\right) \right\|_{\mathbb{R}^m} \mathrm{d}\tau \leq \|w_o\|_{\mathbb{R}^m} + \int_0^t \left(A(\tau) + B(\tau) \left\| w(\tau) \right\|_{\mathbb{R}^m} \right) \mathrm{d}\tau$$

so that by Lemma 4.1 with $\alpha(t) = \|w_o\|_{\mathbb{R}^m} + \int_0^t A(\tau)\,\mathrm{d}\tau$ and $\beta(t) = B(t)$,

$$\left\| w(t) \right\|_{\mathbb{R}^m} \leq \|w_o\|_{\mathbb{R}^m} + \int_0^t \left(A(\tau) + \left(\|w_o\|_{\mathbb{R}^m} + \int_0^\tau A(s)\,\mathrm{d}s \right) B(\tau) e^{\int_\tau^t B(s)\mathrm{d}s} \right) \mathrm{d}\tau\ .$$

Define

$$R_t = \|w_o\|_{\mathbb{R}^m} + \int_0^t \left(A(\tau) + \left(\|w_o\|_{\mathbb{R}^m} + \int_0^\tau A(s)\,\mathrm{d}s \right) B(\tau) e^{\int_\tau^t B(s)\mathrm{d}s} \right) \mathrm{d}\tau\ .$$

Now, following usual procedures based on Grönwall Lemma

$$\begin{aligned}
\left\| w_h(t) - w(t) \right\|_{\mathbb{R}^m} \leq\ & \int_0^t \left\| F_*^h\left(\tau, w_h(\tau)\right) - F_*\left(\tau, w(\tau)\right) \right\|_{\mathbb{R}^m} \mathrm{d}\tau \\
\leq\ & \int_0^t \left\| F_*\left(\tau, w_h(\tau)\right) - F_*\left(\tau, w_h(\tau)\right) \right\|_{\mathbb{R}^m} \mathrm{d}\tau \\
& + \int_0^t \left\| F_*^h\left(\tau, w_h(\tau)\right) - F_*\left(\tau, w(\tau)\right) \right\|_{\mathbb{R}^m} \mathrm{d}\tau\ .
\end{aligned}$$

Let $K_t = \left\{ w\colon \|w\|_{\mathbb{R}^m} \leq R_t \right\}$ and call C_{K_t} the corresponding constant in **(F.2)**. Call $A_h(t)$ the latter summand above, apply **(F.2)** and Lemma 4.1 with $\alpha = A_h$ and $\beta = C_{K_t}$ to obtain

$$\begin{aligned}
\sup_{t\in[0,T]} \left\| w_h(t) - w(t) \right\|_{\mathbb{R}^m} \leq\ & \sup_{t\in[0,T]} \left(A_h(t) + C_{K_t} \int_0^t A_h(\tau)\, e^{C_{K_t}(t-\tau)}\,\mathrm{d}\tau \right) \\
\leq\ & A_h(T) + C_{K_T} \int_0^T A_h(\tau)\, e^{C_{K_t}(T-\tau)}\,\mathrm{d}\tau\ .
\end{aligned}$$

At the limit $h \to 0$, by Lebesgue Dominated Convergence Theorem we have that $A_h(t) \to 0$ on any compact time interval and the proof is completed. □

Lemma 4.2. *Assume that the sequence* $h_n \in \mathbf{C^0}([0,T];\mathbb{R}^+)$ *satisfies*

$$h_n(t) \leq \alpha + \beta \int_0^t h_{n-2}(\tau)\,\mathrm{d}\tau \qquad \textit{with} \qquad h_0(t) \in [0,H] \qquad \textit{and} \qquad h_1(t) \in [0,H]$$

for positive numbers α, β *and* H. *Then, for all* $n \geq 1$,

$$\max\left\{h_{2n}(t), h_{2n+1}(t)\right\} \leq \alpha \sum_{i=0}^{n-1} \frac{\beta^i t^i}{i!} + H \frac{\beta^n t^n}{n!}\,.$$

The proof is elementary and obtained by induction.

Proof of Theorem 2.6. Without loss of generality, we assume throughout this proof that $\hat{u} = 0$, $\hat{w} = 0$ and $\hat{x} = 0$. The proof is obtained by an iterative method through several steps.

1. Definition of u_k, w_k and γ_k. Let δ_1 be the δ in Theorem 2.2. Let $\delta_2 > 0$ be such that

$$\sup_{u \in \mathcal{B}_{\delta_2}(0)} \lambda_\ell(u) < \lambda_\ell(0) + c/2 < \Pi(w) < \lambda_{\ell+1}(0) - c/2 < \inf_{u \in \mathcal{B}_{\delta_2}(0)} \lambda_{\ell+1}(u) \tag{4.7}$$

for every w such that $\|w\|_{\mathbb{R}^m} < \delta_2$. By **(B)** and by the fact that $b(0) = B(0,0)$, there exists $0 < \tilde{\delta} < \delta_2$ such that $\left\|B(0,w) - b(0)\right\|_{\mathbb{R}^m} < \delta_1/2$ for every w with $\|w\|_{\mathbb{R}^m} < \tilde{\delta}$. Define for $t > 0$

$$\begin{array}{rclrcl} \delta &=& \min\left\{\tilde{\delta}, \delta_1\right\} & H_t &=& \left[1 + \|C\|_{\mathbf{L^1}([0,t])} e^{\|C\|_{\mathbf{L^1}([0,t])}}\right]\left[\tilde{\delta} + \|C\|_{\mathbf{L^1}([0,t])}\right] \\ K &=& \left\{u \in \Omega :\ \|u\|_{\mathbb{R}^n} \leq \Delta\right\} & K_{1,t} &=& \left\{w \in \mathbb{R}^m :\ \|w\|_{\mathbb{R}^m} \leq H_t\right\} \end{array} \tag{4.8}$$

where Δ is defined in Theorem 2.2 and C in **(F)**. Let $\tilde{C}_{K_{1,t}}$ be as in **(B)** and let L and T be the constants defined in Theorem 2.2. Choose $T_\delta \in \,]0,T[$ such that

$$\left\{\begin{array}{l} H_{T_\delta} < \delta_2, \\ T_\delta < \delta_1 \Big/ \left(4\tilde{C}_{K_{1,T_\delta}}\right), \\ \|C\|_{\mathbf{L^1}(0,T_\delta)} < \delta_1 \Big/ \left(4\left(1 + H_{T_\delta}\right)\tilde{C}_{K_{1,T_\delta}}\right). \end{array}\right. \tag{4.9}$$

Note that it is possible to choose T_δ in this way, since $H_0 = \tilde{\delta}$ and $\tilde{C}_{K_{1,0}} > 0$. Denote

$$\begin{array}{rclrcl} H &=& H_{T_\delta} & L_\Pi &=& \text{a Lipschitz constant of } \Pi \\ K_1 &=& K_{1,T_\delta} & M &=& \max\left\{e^{C_{K\times K_1} T_\delta}, C_{K\times K_1} L\left(L_\Pi + \tilde{C}_{K_1}\right) e^{C_{K\times K_1} T_\delta}\right\} \end{array} \tag{4.10}$$

see **(Π)**. Fix $w_o \in \mathbb{R}^m$, $x_o \in \mathbb{R}$ and $u_o \in \left(\mathbf{L^1} \cap \mathbf{BV}\right)(\mathbb{R};\Omega)$ with $u_o(x) = 0$ for $x < x_o$, such that

$$\mathrm{TV}(u_o) + \|w_o\|_{\mathbb{R}^m} + |x_o| < \delta\,. \tag{4.11}$$

Define $u_0(t,x) = u_o$, $w_0(t) = w_o$ and $\gamma_0(t) = x_o$ for $t \in \mathbb{R}^+$ and $x \in \mathbb{R}^+$.

By (4.11), we easily get $w_o \in K_1$, $u_o \in \mathcal{D}_0$, where $\mathcal{D}_0$ is defined in Theorem 2.2. Since the function $t \mapsto B(t, w_o)$ is absolutely continuous, then

$$\begin{array}{rcl} \mathrm{TV}\left(B(\cdot,\, w_o(\cdot))_{|[0,T_\delta]}\right) + \left\|B(0,w_o) - b(0)\right\|_{\mathbb{R}^m} &<& \displaystyle\int_0^{T_\delta} \left\|\frac{\partial}{\partial s} B(s,\, w_o)\right\|_{\mathbb{R}^{n-\ell}} \mathrm{d}s + \frac{\delta_1}{2} \\ &\leq& \tilde{C}_{K_1} T_\delta + \dfrac{\delta_1}{2} < \delta_1 \end{array}$$

by (4.9). Define $\gamma_1(t) = x_o + \int_0^t \Pi\left(w_0(\tau)\right)\mathrm{d}\tau$. Note that γ_1 is non characteristic, by **(NC)** and **(Π)**. Use now Theorem 2.2, for every $t \in [0,T_\delta]$, there exists $u_1 = u_1(t,x)$ solving

$$\left\{\begin{array}{l} \partial_t u + \partial_x f(u) = g(u) \\ b\left(u\left(t, \gamma_1(t)\right)\right) = B\left(t, w_0(t)\right) \\ u(0,x) = u_o(x)\,. \end{array}\right.$$

Note that $\left\|u_1(t,x)\right\|_{\mathbb{R}^n} \le \Delta$ for a.e. $t>0$ and $x>\gamma_1(t)$. Hypothesis **(F)** implies that there exists a unique solution w_1 on $[0,T_\delta]$ to the Cauchy problem

$$\left\{\begin{array}{l} \dot w = F(t, u_0(t,x_o), w) \\ w(0) = w_o \,. \end{array}\right.$$

By **(F.3)**, we get that, for every $t \in [0,T_\delta]$,

$$\begin{aligned} \left\|w_1(t)\right\|_{\mathbb{R}^m} &\le \|w_o\|_{\mathbb{R}^m} + \int_0^t \left\|F(s, u_0(s,x_o), w_1(s))\right\|_{\mathbb{R}^m} \mathrm{d}s \\ &\le \|w_o\|_{\mathbb{R}^m} + \|C\|_{\mathbf{L^1}([0,t])} + \int_0^t C(s)\left\|w_1(s)\right\|_{\mathbb{R}^m} \mathrm{d}s \end{aligned}$$

and, by Lemma 4.1,

$$\begin{aligned} \left\|w_1(t)\right\|_{\mathbb{R}^m} &\le \left[\|w_o\|_{\mathbb{R}^m} + \|C\|_{\mathbf{L^1}([0,t])}\right]\left[1 + \int_0^t C(s) e^{\int_s^t C(r)\mathrm{d}r}\,\mathrm{d}s\right] \\ &\le \left[1 + \|C\|_{\mathbf{L^1}([0,t])} e^{\|C\|_{\mathbf{L^1}([0,t])}}\right]\left[\|w_o\|_{\mathbb{R}^m} + \|C\|_{\mathbf{L^1}([0,t])}\right] \le H\,. \end{aligned} \tag{4.12}$$

Introduce recursively, for $k \ge 2$, on the time interval $[0,T_\delta]$ the quantities

$$\gamma_k(t) = x_o + \int_0^t \Pi\left(w_{k-1}(\tau)\right)\mathrm{d}\tau$$

$$u_k \text{ as the solution to } \left\{\begin{array}{l} \partial_t u + \partial_x f(u) = g(u) \\ b\Big(u\left(t,\gamma_k(t)\right)\Big) = B\left(t, w_{k-1}(t)\right) \\ u(0,x) = u_o(x) \end{array}\right.$$

$$w_k \text{ as the solution to } \left\{\begin{array}{l} \dot w = F(t, u_{k-1}\left(t,\gamma_{k-1}(t)\right), w) \\ w(0) = w_o \end{array}\right. \qquad \text{by Proposition 2.4.}$$

By **(NC)**, (4.7) and (4.8), the non characteristic condition is satisfied by the initial boundary value problem defining u_k. The same estimate as (4.12) holds on $\left\|w_k(t)\right\|_{\mathbb{R}^m}$ for all $k \ge 2$. Moreover, since the function $t \mapsto B\left(t, w_{k-1}(t)\right)$ is absolutely continuous, by **(B)**, **(F.3)** and (4.10) we have

$$\begin{aligned} \mathrm{TV}\left(B\left(\cdot,\, w_{k-1}(\cdot)\right)_{|[0,T_\delta]}\right) &= \int_0^{T_\delta} \left\|\frac{d}{ds} B\left(s,\, w_{k-1}(s)\right)\right\|_{\mathbb{R}^{n-\ell}} \mathrm{d}s \\ &= \int_0^{T_\delta} \left\|\frac{\partial}{\partial s} B\left(s,\, w_{k-1}(s)\right) + \frac{\partial}{\partial w} B\left(s,\, w_{k-1}(s)\right) \circ w'_{k-1}(s)\right\|_{\mathbb{R}^{n-\ell}} \mathrm{d}s \\ &\le \tilde C_{K_1}\left[T_\delta + \int_0^{T_\delta} \left\|w'_{k-1}(s)\right\|_{\mathbb{R}^m} \mathrm{d}s\right] \\ &= \tilde C_{K_1}\left[T_\delta + \int_0^{T_\delta} \left\|F\left(s, u_{k-2}(s,\gamma_{k-2}(s)), w_{k-1}(s)\right)\right\|_{\mathbb{R}^m} \mathrm{d}s\right] \\ &= \tilde C_{K_1}\left[T_\delta + \int_0^{T_\delta} C(s)\left[1 + \left\|w_{k-1}(s)\right\|_{\mathbb{R}^m}\right] \mathrm{d}s\right] \\ &= \tilde C_{K_1}\left[T_\delta + (1+H)\,\|C\|_{\mathbf{L^1}(0,T_\delta)}\right]\,. \end{aligned}$$

Then, by (4.9), we deduce that $\mathrm{TV}\left(B\left(\cdot,\, w_{k-1}(\cdot)\right)_{|[0,T_\delta]}\right) + \left\|B(0,w_o) - b(0)\right\|_{\mathbb{R}^m} < \delta_1$ and so Theorem 2.2 applies and u_k exists in the time interval $[0,T_\delta]$.

2. The w_k is a Cauchy sequence in $\mathbf{C^0}\left([0,T_\delta];\mathbb{R}^m\right)$. For $t \in [0,T_\delta]$ and $k \in \mathbb{N}$ we have

$$\begin{aligned}
& \left\|w_k(t) - w_{k-1}(t)\right\|_{\mathbb{R}^m} \\
\leq\ & \int_0^t \left\|F(s,u_{k-1}(s,\gamma_{k-1}(s)),w_k(s)) - F(s,u_{k-2}(s,\gamma_{k-2}(s)),w_{k-1}(s))\right\|_{\mathbb{R}^m} \mathrm{d}s \\
\leq\ & C_{K\times K_1}\left[\int_0^t \left\|w_k(s)-w_{k-1}(s)\right\|_{\mathbb{R}^m}\mathrm{d}s + \int_0^t \left\|u_{k-1}(s,\gamma_{k-1}(s)) - u_{k-2}(s,\gamma_{k-2}(s))\right\|_{\mathbb{R}^n}\right] \\
\leq\ & C_{K\times K_1}\int_0^t \left\|w_k(s)-w_{k-1}(s)\right\|_{\mathbb{R}^m}\mathrm{d}s + C_{K\times K_1}L\int_0^t \left\|B(s,w_{k-2}(s)) - B(s,w_{k-3}(s))\right\|_{\mathbb{R}^{n-\ell}}\mathrm{d}s \\
& + C_{K\times K_1}L\sup_{s\in[0,t]}\left|\gamma_{k-1}(s)-\gamma_{k-2}(s)\right| \\
\leq\ & C_{K\times K_1}\int_0^t \left\|w_k(s)-w_{k-1}(s)\right\|_{\mathbb{R}^m}\mathrm{d}s + C_{K\times K_1}L\int_0^t \left\|B(s,w_{k-2}(s)) - B(s,w_{k-3}(s))\right\|_{\mathbb{R}^{n-\ell}}\mathrm{d}s \\
& + C_{K\times K_1}L\int_0^t \left|\Pi(w_{k-2}(s)) - \Pi(w_{k-3}(s))\right|\mathrm{d}s \\
\leq\ & C_{K\times K_1}\int_0^t \left\|w_k(s)-w_{k-1}(s)\right\|_{\mathbb{R}^m}\mathrm{d}s + C_{K\times K_1}L(\tilde{C}_{K_1}+L_\Pi)\int_0^t \left\|w_{k-2}(s)-w_{k-3}(s)\right\|_{\mathbb{R}^m}\mathrm{d}s\,,
\end{aligned}$$

where we used the definition of w_k, $(\boldsymbol{F.2})$, (4) of Theorem 2.2, the definition of γ_{k-1}, $(\boldsymbol{B})$ and $(\boldsymbol{\Pi})$. Using Lemma 4.1 with

$$\begin{aligned}
\alpha(t) &= C_{K\times K_1}L(\tilde{C}_{K_1}+L_\Pi)\int_0^t \left\|w_{k-2}(s)-w_{k-3}(s)\right\|_{\mathbb{R}^m}\mathrm{d}s \\
\beta(t) &= C_{K\times K_1} \\
\delta(t) &= \left\|w_k(t)-w_{k-1}(t)\right\|_{\mathbb{R}^m}
\end{aligned}$$

we deduce that

$$\left\|w_k(t)-w_{k-1}(t)\right\|_{\mathbb{R}^m} \leq C_{K\times K_1}L(\tilde{C}_{K_1}+L_\Pi)\,e^{C_{K\times K_1}t}\int_0^t \left\|w_{k-2}(s)-w_{k-3}(s)\right\|_{\mathbb{R}^m}\mathrm{d}s\ .$$

By Lemma 4.2, with

$$\alpha = 0 \qquad \beta = C_{K\times K_1}L(\tilde{C}_{K_1}+L_\Pi)\,e^{C_{K\times K_1}T_\delta} \qquad h_k(t) = \left\|w_k(t)-w_{k-1}(t)\right\|_{\mathbb{R}^m}\,,$$

both $\left\|w_{2k}(t)-w_{2k-1}(t)\right\|_{\mathbb{R}^m}$ and $\left\|w_{2k+1}(t)-w_{2k}(t)\right\|_{\mathbb{R}^m}$ are bounded by

$$\max\left\{\sup_{t\in[0,T_\delta]}\left\|w_1(t)-w_0(t)\right\|_{\mathbb{R}^m},\ \sup_{t\in[0,T_\delta]}\left\|w_2(t)-w_1(t)\right\|_{\mathbb{R}^m}\right\}\frac{\left(C_{K\times K_1}L(\tilde{C}_{K_1}+L_\Pi)e^{C_{K\times K_1}T_\delta}\right)^k {T_\delta}^k}{k!}.$$

Thus, we conclude that the sequence w_k is a Cauchy sequence in $\mathbf{C^0}\left([0,T_\delta];\mathbb{R}^m\right)$, since

$$\sum_{k=0}^{+\infty}\frac{\left(C_{K\times K_1}L(\tilde{C}_{K_1}+L_\Pi)\,e^{C_{K\times K_1}T_\delta}\right)^k {T_\delta}^k}{k!} < +\infty\,.$$

Therefore there exists a $w_* \in \mathbf{C^0}\left([0,T_\delta];\mathbb{R}^m\right)$ such that w_k converges to w_* in $\mathbf{C^0}\left([0,T_\delta];\mathbb{R}^m\right)$.

3. Definition of u_* and of γ_*. The Dominated Convergence Theorem implies that the sequence γ_k uniformly converges to the function

$$\gamma_*(t) = x_o + \int_0^t \Pi(w_*(\tau))\,\mathrm{d}\tau$$

on $[0,T_\delta]$. Moreover, for $t \in [0,T_\delta]$ and $h,k \in \mathbb{N}$, we have

$$\begin{aligned}
\left\|u_k(t) - u_h(t)\right\|_{\mathbf{L^1}} &\leq L\left[\int_0^t \left\|B(s,w_{k-1}(s)) - B(s,w_{h-1}(s))\right\|_{\mathbb{R}^{n-\ell}} \mathrm{d}s + \sup_{\tau\in[0,t]} \left|\gamma_k(\tau) - \gamma_h(\tau)\right|\right] \\
&\leq L\tilde{C}_{K_1} \int_0^t \left\|w_{k-1}(s) - w_{h-1}(s)\right\|_{\mathbb{R}^m} \mathrm{d}s + L \sup_{\tau\in[0,t]} \left|\gamma_k(\tau) - \gamma_h(\tau)\right| \\
&\leq L\tilde{C}_{K_1} T_\delta \sup_{t\in[0,T_\delta]} \left\|w_{k-1}(t) - w_{h-1}(t)\right\|_{\mathbb{R}^m} + L \sup_{\tau\in[0,T_\delta]} \left|\gamma_k(\tau) - \gamma_h(\tau)\right|,
\end{aligned}$$

where we used (4) of Theorem 2.2 and ($\boldsymbol{B}$). By the previous results, the sequence u_k is a Cauchy sequence in $\mathbf{C^0}\left([0,T_\delta];\mathbf{L^1}(\mathbb{R}^+;\mathbb{R}^n)\right)$. Let u_* be the corresponding limit.

4. The triple (u_*, w_*, γ_*) **solves (1.1) in the sense of Definition 2.5.** Let $\bar{w}$ solve (2.4) with $F_*(\tau,w) = F\left(\tau, u_*\left(\tau,\gamma_*(\tau)\right), w\right)$. We prove that $w_* = \bar{w}$. Let $F_*^k(t,w) = F\left(t, u_k\left(t,\gamma_k(t)\right), w\right)$ and apply the last part of Proposition 2.4. This is possible, since $u_k\left(t,\gamma_k(t)\right) \to u_*\left(t,\gamma_*(t)\right)$ for a.e. $t \in [0,T_\delta]$, which is shown as in the proof of [2, Theorem 1.2], thanks to **(NC)**.

It is sufficient to prove that u_* satisfies (2.2) with $B_*(t) = B\left(t, w_*(t)\right)$. As in Theorem 2.2,

$$\begin{aligned}
\left\|P_{B_*}(t,0)u_o - u_*(t)\right\|_{\mathbf{L^1}} &= \lim_{k\to+\infty} \left\|P_{B_*}(t,0)u_o - u_k(t)\right\|_{\mathbf{L^1}} \\
&\leq L \lim_{k\to+\infty} \left[\int_0^t \left\|B_*(t) - B\left(t, w_{k-1}(\tau)\right)\right\|_{\mathbb{R}^{n-\ell}} \mathrm{d}\tau + \sup_{\tau\in[0,t]} \left|\gamma_*(\tau) - \gamma_k(\tau)\right|\right] \\
&= 0
\end{aligned}$$

where we used **(B)** and the uniform convergence both of w_k to w_* and of γ_k to γ_*. Finally, 3. in Definition 2.5 is satisfied by construction.

5. Stability inequalities. Consider two triples $(u_{0,1}, w_{0,1}, x_{0,1})$ and $(u_{0,2}, w_{0,2}, x_{0,2})$ such that $u_{0,i} \in (\mathbf{L^1} \cap \mathbf{BV})(\mathbb{R};\Omega)$, $w_{0,i} \in \mathbb{R}^m$, $x_{0,i} \in \mathbb{R}$ and $\mathrm{TV}(u_{0,i}) + \left\|w_{0,i}\right\|_{\mathbb{R}^m} + \left|x_{o,i}\right| < \delta$ for $i = 1,2$. Denote with $u_{k,i}$, $w_{k,i}$, $\gamma_{k,i}$ and $u_{k,i}$ the sequences defined in point 1 starting from $(u_{0,i}, w_{0,i}, x_{0,i})$. By Theorem 2.2 and **(B)**, for every $k \geq 1$, we have

$$\begin{aligned}
\left\|u_{k,1}(t) - u_{k,2}(t)\right\|_{\mathbf{L^1}} &\leq L\left\|u_{0,1} - u_{0,2}\right\|_{\mathbf{L^1}} + L\int_0^t \left\|B(\tau, w_{k-1,1}(\tau)) - B(\tau, w_{k-1,2}(\tau))\right\|_{\mathbb{R}^{n-\ell}} \mathrm{d}\tau \\
&\quad + L \sup_{\tau\in[0,t]} \left|\gamma_{k,1}(\tau) - \gamma_{k,2}(\tau)\right| \\
&\leq L\left[\left\|u_{0,1} - u_{0,2}\right\|_{\mathbf{L^1}} + \tilde{C}_{K_1} \int_0^t \left\|w_{k-1,1}(\tau) - w_{k-1,2}(\tau)\right\|_{\mathbb{R}^m} \mathrm{d}\tau\right] \\
&\quad + L \sup_{\tau\in[0,t]} \left|\gamma_{k,1}(\tau) - \gamma_{k,2}(\tau)\right| \qquad (4.13)
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^t \left\|u_{k,1}\left(\tau, \gamma_{k,1}(\tau)\right) - u_{k,2}\left(\tau, \gamma_{k,2}(\tau)\right)\right\|_{\mathbb{R}^n} \mathrm{d}\tau \qquad (4.14) \\
&\leq L\left[\left\|u_{0,1} - u_{0,2}\right\|_{\mathbf{L^1}} + \tilde{C}_{K_1} \int_0^t \left\|w_{k-1,1}(\tau) - w_{k-1,2}(\tau)\right\|_{\mathbb{R}^m} \mathrm{d}\tau + \sup_{\tau\in[0,t]} \left|\gamma_{k,1}(\tau) - \gamma_{k,2}(\tau)\right|\right]
\end{aligned}$$

while, using (4.10) and ($\boldsymbol{\Pi}$), the distance between $\gamma_{k,1}$ and $\gamma_{k,2}$ is estimated by

$$\left|\gamma_{k,1}(t) - \gamma_{k,2}(t)\right| \leq \left|x_{0,1} - x_{0,2}\right| + \int_0^t \left|\Pi\left(w_{k-1,1}(\tau)\right) - \Pi\left(w_{k-1,2}(\tau)\right)\right| \mathrm{d}\tau$$

$$\leq \quad \left|x_{0,1} - x_{0,2}\right| + L_\Pi \int_0^t \left|w_{k-1,1}(\tau) - w_{k-1,2}(\tau)\right| \mathrm{d}\tau\,. \tag{4.15}$$

Moreover, by **(F.2)**, for every $k \geq 1$,

$$\begin{aligned}
& \left\|w_{k,1}(t) - w_{k,2}(t)\right\|_{\mathbb{R}^m} \\
\leq \; & \left\|w_{0,1} - w_{0,2}\right\|_{\mathbb{R}^m} \\
& + \int_0^t \left\| F\left(\tau, u_{k-1,1}\left(\tau, \gamma_{k-1,1}(\tau)\right), w_{k,1}(\tau)\right) - F\left(\tau, u_{k-1,2}\left(\tau, \gamma_{k-1,2}(\tau)\right), w_{k,2}(\tau)\right)\right\|_{\mathbb{R}^m} \mathrm{d}\tau \\
\leq \; & \left\|w_{0,1} - w_{0,2}\right\|_{\mathbb{R}^m} + C_{K\times K_1} \int_0^t \left\|w_{k,1}(\tau) - w_{k,2}(\tau)\right\|_{\mathbb{R}^m} \mathrm{d}\tau \\
& + C_{K\times K_1} \int_0^t \left\|u_{k-1,1}\left(\tau, \gamma_{k-1,1}(\tau)\right) - u_{k-1,2}\left(\tau, \gamma_{k-1,2}(\tau)\right)\right\|_{\mathbb{R}^n} \mathrm{d}\tau\,.
\end{aligned}$$

By Lemma 4.1, we deduce that

$$\left\|w_{k,1}(t) - w_{k,2}(t)\right\|_{\mathbb{R}^m} \leq A_k(t) + C_{K\times K_1} \int_0^t A_k(\tau) e^{C_{K\times K_1}(t-\tau)} \mathrm{d}\tau$$

where

$$A_k(t) = C_{K\times K_1} \int_0^t \left\|u_{k-1,1}\left(\tau, \gamma_{k-1,1}(\tau)\right) - u_{k-1,2}\left(\tau, \gamma_{k-1,2}(\tau)\right)\right\|_{\mathbb{R}^n} \mathrm{d}\tau + \left\|w_{0,1} - w_{0,2}\right\|_{\mathbb{R}^m}.$$

Since $A_k(t)$ is non decreasing w.r.t. t, we obtain that

$$\left\|w_{k,1}(t) - w_{k,2}(t)\right\|_{\mathbb{R}^m} \leq e^{C_{K\times K_1} t} A_k(t).$$

By (4.14) and (4.15), for $k \geq 2$,

$$\begin{aligned}
& A_k(t) \\
\leq \; & \left\|w_{0,1} - w_{0,2}\right\|_{\mathbb{R}^m} + C_{K\times K_1} L \left\|u_{0,1} - u_{0,2}\right\|_{\mathbf{L^1}} \\
& + C_{K\times K_1} L \tilde{C}_{K_1} \int_0^t \left\|w_{k-2,1}(\tau) - w_{k-2,2}(\tau)\right\|_{\mathbb{R}^m} \mathrm{d}\tau + C_{K\times K_1} L \sup_{\tau\in[0,t]} \left|\gamma_{k-1,1}(\tau) - \gamma_{k-1,2}(\tau)\right| \\
\leq \; & \left\|w_{0,1} - w_{0,2}\right\|_{\mathbb{R}^m} + C_{K\times K_1} L \left\|u_{0,1} - u_{0,2}\right\|_{\mathbf{L^1}} + C_{K\times K_1} L \left|x_{0,1} - x_{0,2}\right| \\
& + C_{K\times K_1} L (L_\Pi + \tilde{C}_{K_1}) \int_0^t \left\|w_{k-2,1}(\tau) - w_{k-2,2}(\tau)\right\|_{\mathbb{R}^m} \mathrm{d}\tau
\end{aligned}$$

and so, by (4.10),

$$\begin{aligned}
& \left\|w_{k,1}(t) - w_{k,2}(t)\right\|_{\mathbb{R}^m} \\
\leq \; & \left(\left\|w_{0,1} - w_{0,2}\right\|_{\mathbb{R}^m} + C_{K\times K_1} L \left\|u_{0,1} - u_{0,2}\right\|_{\mathbf{L^1}} + C_{K\times K_1} L \left|x_{0,1} - x_{0,2}\right|\right) e^{C_{K\times K_1} t} \\
& + C_{K\times K_1} L (L_\Pi + \tilde{C}_{K_1}) e^{C_{K\times K_1} t} \int_0^t \left\|w_{k-2,1}(\tau) - w_{k-2,2}(\tau)\right\|_{\mathbb{R}^m} \mathrm{d}\tau \\
\leq \; & M \left[\left\|w_{0,1} - w_{0,2}\right\|_{\mathbb{R}^m} + \left\|u_{0,1} - u_{0,2}\right\|_{\mathbf{L^1}} + \left|x_{0,1} - x_{0,2}\right|\right] \\
& + M \int_0^t \left\|w_{k-2,1}(s) - w_{k-2,2}(s)\right\|_{\mathbb{R}^m} \mathrm{d}\tau\,.
\end{aligned}$$

Clearly we have that

$$\left\|w_{k,1}(t) - w_{k,2}(t)\right\|_{\mathbb{R}^m} \leq M \left\|w_{0,1} - w_{0,2}\right\|_{\mathbb{R}^m}$$

for $k \in \{0, 1\}$. Hence we may apply Lemma 4.2 with

$$\begin{array}{rclrcl} H & = & M\left\|w_{0,1} - w_{0,2}\right\|_{\mathbb{R}^m} & \alpha & = & M\left[\left\|w_{0,1} - w_{0,2}\right\|_{\mathbb{R}^m} + \left\|u_{0,1} - u_{0,2}\right\|_{\mathbf{L}^1} + \left|x_{0,1} - x_{0,2}\right|\right] \\ \beta & = & M & h_k(t) & = & \left\|w_{k,1}(t) - w_{k,2}(t)\right\|_{\mathbb{R}^m} \end{array}$$

and obtain

$$\begin{aligned} \left\|w_{k,1}(t) - w_{k,2}(t)\right\|_{\mathbb{R}^m} &\leq M\left[\sum_{i=0}^{\lfloor k/2 \rfloor - 1} \frac{M^i {T_\delta}^i}{i!} + \frac{M^{\lfloor k/2 \rfloor - 1} {T_\delta}^{\lfloor k/2 \rfloor - 1}}{\left(\lfloor k/2 \rfloor - 1\right)!}\right] \\ &\quad \cdot \left[\left\|u_{0,1} - u_{0,2}\right\|_{\mathbf{L}^1} + \left\|w_{0,1} - w_{0,2}\right\|_{\mathbb{R}^m} + \left|x_{0,1} - x_{0,2}\right|\right] \\ &\leq \tilde{M}\left[\left\|u_{0,1} - u_{0,2}\right\|_{\mathbf{L}^1} + \left\|w_{0,1} - w_{0,2}\right\|_{\mathbb{R}^m} + \left|x_{0,1} - x_{0,2}\right|\right] \quad (4.16) \end{aligned}$$

for every $k \in \mathbb{N}$, where $\lfloor \cdot \rfloor$ denotes the integer part and $\tilde{M} = M \sum_{i=0}^{+\infty} M^i {T_\delta}^i / i!$, which is a convergent series. Inserting (4.16) into (4.13) and (4.15), we deduce that

$$\left\|u_{k,1} - u_{k,2}\right\|_{\mathbf{L}^1} + \left\|\gamma_{k,1} - \gamma_{k,2}\right\|_{\mathbf{C}^0([0,T_\delta];\mathbb{R})} \leq M_1\left[\left\|u_{0,1} - u_{0,2}\right\|_{\mathbf{L}^1} + \left\|w_{0,1} - w_{0,2}\right\|_{\mathbb{R}^m} + \left|x_{0,1} - x_{0,2}\right|\right]$$

for all $k \in \mathbb{N}$ and M_1 is a constant. This inequality, together with (4.16), concludes the proof. □

Proof of Proposition 3.1. In the present case, $n^\pm = 2$ and $m = 1$. We verify the various assumptions. **(f*)** Holds by **(p)**, see for instance [9, § 6.3]. **(g*)** holds since the source terms $g^\pm$ are linear. **(Π)** is immediate. **(NC*)** holds with $\ell^- = 2$ and $\ell^+ = 1$ by the choice of $\hat{u}^\pm$ and $\hat{w}$. **(b*)**: recall that the eigenvectors of the p-system have the expressions

$$r_1 = \left[\begin{array}{c} \rho \\ q - \rho\sqrt{p'(\rho)} \end{array}\right] \qquad \text{and} \qquad r_2 = \left[\begin{array}{c} \rho \\ q + \rho\sqrt{p'(\rho)} \end{array}\right]$$

so that the determinants attain the values $\pm\sqrt{p'(\hat{\rho}^\pm)}$, which do not vanish by **(p)**. **(F*)** holds by **(p)**. **(B*)** is immediate. □

Proof of Proposition 3.2. Recall first the basic quantities related to (3.2):

$$\begin{array}{llll} \text{Eigenvalues:} & \begin{array}{rcl} \lambda_1 & = & v - \rho p'(\rho) \\ \lambda_2 & = & v \end{array} & \text{Invariants:} & \begin{array}{rcl} z & = & v \\ w & = & v + p(\rho) \end{array} \\ \text{Eigenvectors:} & r_1 = \left[\begin{array}{c} -\rho \\ -\rho\left(v + p(\rho)\right) \end{array}\right] & & r_2 = \left[\begin{array}{c} \rho \\ \rho\left(v + p(\rho)\right) + \rho^2 p'(\rho) \end{array}\right] \\ & \begin{array}{rcl} \nabla\lambda_1 \cdot r_1 & = & \frac{d^2}{d\rho^2}\left(\rho\, p(\rho)\right) \\ \nabla\lambda_2 \cdot r_2 & = & 0 \end{array} & \text{Lax curves:} & \begin{array}{rcl} L_1(\rho; \rho_0, v_0) & = & v_0 + p(\rho_0) - p(\rho) \\ L_2(\rho; \rho_0, v_0) & = & v_0\,. \end{array} \end{array} \quad (4.17)$$

Consider now the various assumptions separately, with $n^\pm = 2$ and $m = 1$. **(f*)** holds by **(P)** and by (4.17). **(g*)**, **(B*)** and **(Π)** are immediate. **(NC*)** holds with $\ell^- = 2$ and $\ell^+ = 1$, by the choice of $\hat{u}^\pm$ and $\hat{w}$, thanks to (3.7). **(b*)** follows from (3.6). **(F*)** holds by (4.17) and **(P)** thanks to (3.6). The condition $b^\pm(\hat{u}^-, \hat{u}^+) = B^\pm(\hat{w})$ follows from (3.5). □

Proof of Proposition 3.3. Set $n = 2$, $m = 1$ and $\ell = 1$. Conditions **(Π)**, **(B)** and the equality $b(\hat{\rho}, \hat{q}) = B(0, \hat{V})$ are immediate. Condition **(f)** follows from 1., **(NC)** from 3. and **(F)** from 1. and 2. Assumption **(g)** holds since g is locally Lipschitz. To prove **(b)**, introduce the right eigenvectors of Df, i.e. $r_{1,2}(\rho, q) = \left[1 \quad \lambda_{1,2}(\rho, q)\right]^T$ where $\lambda_{1,2}(\rho, q) = q/\rho \pm \sqrt{p'(\rho)}$, and compute

$$\det\left(D_{(\hat{\rho},\hat{q})} b(\hat{\rho}, \hat{q})\; r_2(\hat{\rho}, \hat{q})\right) = \det\left[\begin{array}{cc} -\hat{q}/\hat{\rho}^2 & 1/\hat{\rho} \end{array}\right] \left[\begin{array}{c} 1 \\ \hat{q}/\hat{\rho} + \sqrt{p'(\hat{\rho})} \end{array}\right] = \sqrt{p'(\hat{\rho})}/\hat{\rho} > 0$$

completing the proof. □

Acknowledgment. The authors acknowledge the support of the GNAMPA project *Control for Conservation Laws* and of the Vigoni project *Non–Local Transport Processes Modeling, Analysis, Numerics and Optimal Control*.

References

[1] J. Alastruey, K. H. Parker, J. Peiró, and S. J. Sherwin. Lumped parameter outflow models for 1-D blood flow simulations: effect on pulse waves and parameter estimation. *Commun. Comput. Phys.*, 4(2):317–336, 2008.

[2] D. Amadori. Initial-boundary value problems for nonlinear systems of conservation laws. *NoDEA Nonlinear Differential Equations Appl.*, 4(1):1–42, 1997.

[3] D. Amadori and R. M. Colombo. Continuous dependence for 2×2 conservation laws with boundary. *J. Differential Equations*, 138(2):229–266, 1997.

[4] D. Amadori and G. Guerra. Uniqueness and continuous dependence for systems of balance laws with dissipation. *Nonlinear Anal.*, 49(7, Ser. A: Theory Methods):987–1014, 2002.

[5] A. Aw and M. Rascle. Resurrection of "second order" models of traffic flow. *SIAM J. Appl. Math.*, 60(3):916–938 (electronic), 2000.

[6] R. Borsche, R. M. Colombo, and M. Garavello. On the coupling of systems of hyperbolic conservation laws with ordinary differential equations. *Nonlinearity*, 23(11):2749–2770, 2010.

[7] A. Bressan. *Hyperbolic systems of conservation laws*, volume 20 of *Oxford Lecture Series in Mathematics and its Applications*. Oxford University Press, Oxford, 2000. The one-dimensional Cauchy problem.

[8] S. Čanić and E. H. Kim. Mathematical analysis of the quasilinear effects in a hyperbolic model blood flow through compliant axi-symmetric vessels. *Math. Methods Appl. Sci.*, 26(14):1161–1186, 2003.

[9] G.-Q. Chen and D. Wang. The Cauchy problem for the Euler equations for compressible fluids. In *Handbook of mathematical fluid dynamics, Vol. I*, pages 421–543. North-Holland, Amsterdam, 2002.

[10] R. M. Colombo and G. Guerra. On general balance laws with boundary. *Journal of Differential Equations*, 248(5):1017–1043, 2010.

[11] R. M. Colombo and M. Mercier. An analytical framework to describe the interactions between individuals and a continuum. Submitted, 2010.

[12] R. M. Colombo and V. Schleper. On the compressible to incompressible limit in two-phase flows. In preparation, 2011.

[13] C. M. Dafermos. *Hyperbolic conservation laws in continuum physics*, volume 325 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, Berlin, second edition, 2005.

[14] R. Fazio and R. J. LeVeque. Moving-mesh methods for one-dimensional hyperbolic problems using clawpack. *Computers & Mathematics with Applications*, 45(1-3):273 – 298, 2003.

[15] M. Á. Fernández, V. Milišić, and A. Quarteroni. Analysis of a geometrical multiscale blood flow model based on the coupling of ODEs and hyperbolic PDEs. *Multiscale Model. Simul.*, 4(1):215–236 (electronic), 2005.

[16] A. F. Filippov. *Differential equations with discontinuous righthand sides*. Kluwer Academic Publishers Group, Dordrecht, 1988. Translated from the Russian.

[17] J. Goodman. *Initial Boundary Value Problems for Hyperbolic Systems of Conservation Laws*. PhD thesis, California University, 1982.

[18] C. Lattanzio, A. Maurizi, and B. Piccoli. Moving bottlenecks in car traffic flow: a PDE-ODE coupled model. *SIAM J. Math. Anal.*, 43(1):50–67, 2011.

[19] R. J. LeVeque. *Finite volume methods for hyperbolic problems*. Cambridge Texts in Applied Mathematics. Cambridge University Press, Cambridge, 2002.

[20] M. Sablé-Tougeron. Méthode de Glimm et problème mixte. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 10(4):423–443, 1993.