STRONG TRACES FOR ENTROPY SOLUTIONS OF HETEROGENEOUS ULTRA-PARABOLIC EQUATIONS

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Abstract. We prove that entropy solutions of heterogeneous ultra-parabolic equations satisfying a traceability condition admit strong traces at $t = 0$. In particular, this property is satisfied by entropy solutions to heterogeneous scalar conservation laws. The tools that we are using are (Panov's extension of) H-measures and the kinetic approach.

1. Introduction

The aim of this paper is to prove the existence of strong traces at $t = 0$ for entropy solutions of an ultra-parabolic equation,

$$\partial_t u + \mathrm{div}_x\, f(x,u) = \sum_{i,j=1}^{k} \partial_{x_i}(b_{ij}(x,u)\partial_{x_j} u), \tag{1}$$

where $u = u(t,x) \in L^\infty(I\!R^+ \times I\!R^d)$ is the unknown function, $(t,x) \in I\!R^d_+ := I\!R^+ \times I\!R^d \equiv (0,\infty) \times I\!R^d$, $k \le d$, $k, d \in I\!N$.

We assume the following:

- The function $f \in C^1(I\!R^d \times I\!R; I\!R^d)$;
- The matrix $b(x,\lambda) = [b_{ij}(x,\lambda)]_{i,j=1,\dots,k} \in \big(C(I\!R^d \times I\!R)\big)^{k\times k}$, is nonnegative definite in the sense that for almost all $x \in I\!R^d$,
$$\langle b(x,\lambda)\xi, \xi\rangle \ge c(x,\lambda)|\xi|^2, \quad \xi \in I\!R^k,\ \lambda \in I\!R, \tag{2}$$
where $\langle \cdot,\cdot \rangle$ denotes the scalar product in $I\!R^k$. The nonnegative function c fulfills the following: there exists a partition $\{\Omega_m\}_{m\in I\!N}$, Ω_m are open, of $I\!R^d$ (i.e. $I\!R^d = Cl(\bigcup_{m=1}^\infty \Omega_m)$) such that for every $m \in I\!N$ there exist increasing sequence of real numbers $\{\lambda_i^m\}_{i\in I\!N}$ such that for almost every $x \in \Omega_m$,
$$c(x,\lambda) > 0 \ \text{ for } \ \lambda \in \bigcup_{i=1}^{\infty} (\lambda_i^m, \lambda_{i+1}^m); \tag{3}$$
- The elements of the matrix b are of the following form,
$$b_{ij}(x,\lambda) = \sum_{l=1}^{k} \sigma^b_{il}(x,\lambda)\sigma^b_{lj}(x,\lambda), \quad i,j = 1,..,k, \tag{4}$$
where $\sigma^b_{il}(x,\lambda) = \sigma^b_{li}(x,\lambda)$.

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The latter equation describes transport processes in heterogeneous media in which the diffusion (which is represented by the second order terms) can be neglected in certain directions [12]. Such equations were firstly considered by Graetz [7], and Nusselt [16] in their investigations concerning the heat transfer. Besides the heat transfer, equations of type (1) describe processes in porous media (cf. [21]) such as oil extraction or CO2 sequestration which typically occur in highly heterogeneous surroundings. One can also find applications in sedimentation processes, traffic flow, radar shape-from-shading problems, blood flow, gas flow in a variable duct and so on.

Before we continue, let us recall the definition of the strong traces.

Definition 1. The strong trace of a solution u to (1) at $t = 0$ is a function $u_0 \in L^\infty(I\!R^d)$ such that for any relatively compact $K \subset\subset I\!R^d$,

$$\lim_{t\to 0} \int_K |u(t,x) - u_0(x)| dx = 0.$$

The question of existence of traces was firstly raised in the context of limit of hyperbolic relaxation toward a scalar conservation laws . The notion of traces was also used to formulate the solution concept for boundary value problems for scalar conservation laws ([2] and references therein). The traces played an essential role in proving uniqueness for scalar conservation laws with discontinuous flux, cf. [9, 14].

However, all previous results were given for the scalar conservation laws in homogeneous media (see e.g. [11, 19] and references therein). Here, we shall extend the results on the case of ultra-parabolic equations which govern processes in heterogeneous media.

The basic tools that we are going to use are the H-measures [5, 22], more precisely its variant introduced in [1] and generalized in [8, 20]. We combine this technique with the classical blow up techniques, [6, 18, 24], induction with respect to the space dimension [18], and the kinetic formulation of (1), [4] (see also the classical work [13]).

The extension of the mentioned techniques (concretely the ones from [18]) from the homogeneous scalar conservation laws to (substantially) homogeneous ultra-parabolic equations does not demand significant changes in the strategy. However, the situation is much more complicated if we assume that the flux depends on the space variable. It seems necessary to pass to the kinetic formulation to the considered problem, not in order to use the averaging lemmas as in [11, 24], but in order to introduce appropriate change of variables which, locally, reduces the non-homogeneous equation to the homogeneous one.

Therefore, we shall split the proof in two parts. First, in Section 3, we consider the function $f = (f_1, ..., f_d)$ with components $f_{k+1}, ..., f_d$ depending only on the state variable u, i.e. without explicit dependence on the space variable x,

$$\partial_t u + \sum_{i=1}^{k} \partial_{x_i} f_i(x,u) + \sum_{i=k+1}^{d} \partial_{x_i} f_i(u) = \sum_{i,j=1}^{k} \partial_{x_i}(b_{ij}(x,u)\partial_{x_j} u). \tag{5}$$

Then, in Section 4, we shall reformulate (1) in the kinetic framework, and use appropriate change of variables to reduce the equation on an equation of type (5) in a neighborhood of every point where the existence of traces could be lost.

In Section 2 we give notions and auxiliary results, as well as formulation of the main result.

In the final section, we discuss the traceability condition and give several examples of conservation laws satisfying the condition.

2. Notions, notations, and auxiliary results

We shall first introduce the entropy admissibility concept for (1). It comes as an extension of the Kruzhkov admissibility concept for scalar conservation laws [10], cf. [3, 4, 20].

Definition 2. Denote by $B(x,u)=\big(B_{ij}(x,u)\big)_{i,j=0,\dots,d}$ the matrix such that

$$\begin{aligned}&B_{ij}(x,u)=0, \text{ for } i\in\{0,k+1,...,k+d\} \text{ or } j\in\{0,k+1,...,k+d\},\\&\partial_u B_{ij}(x,u)=b_{ij}(x,u), \text{ for } i,j\in\{1,...,k\}.\end{aligned}$$

A function $u\in L^\infty(I\!\!R^d_+)$ is an entropy solution to (1) if it satisfies the condition

$$\sum_{i=1}^k \sigma^b_{li}(u)\partial_{x_i}u\in L^2(I\!\!R^d_+),\quad l=1,...,k, \tag{6}$$

and the following entropy inequality: for any $c\in I\!\!R$,

$$\begin{aligned}&\partial_t|u-c|+\sum_{i=1}^d\partial_{x_i}\left(\operatorname{sgn}(u-c)(f_i(x,u)-f_i(x,c))\right)+\operatorname{sgn}(u-c)\sum_{i=1}^d D_{x_i}f_i(x,c)\\&-\sum_{i,j=1}^k\partial^2_{x_ix_j}\left(\operatorname{sgn}(u-c)(B_{ij}(x,u)-B_{ij}(x,c))\right)-\operatorname{sgn}(u-c)\sum_{i,j=1}^k D^2_{x_ix_j}B_{ij}(x,c)\\&+\operatorname{sgn}(u-c)\sum_{i,j=1}^k\Big(D^2_{x_ix_j}B_{ij}(x,u)+D_{x_j}b_{ij}(x,u)u_{x_i}\Big)\leq 0,\end{aligned} \tag{7}$$

in $\mathcal{D}'(I\!\!R^d_+)$, cf. [3].

In order to prove that an entropy solution to (1) admits strong traces at $t=0$, we shall prove more, that any quasi-solution to (1) admits the strong traces at the initial hyperspace $t=0$. The concept of quasi-solutions was introduced in [18]. Having in mind that

$$\sum_{i=1}^d D_{x_i}f_i(x,c)-\sum_{i,j=1}^k D^2_{x_i,x_j}B_{ij}(x,c)=\gamma_c\in\mathcal{M}(I\!\!R^d),\quad c\in I\!\!R,$$

we have the following definition of the quasi solutions to (1), cf. [20].

Definition 3. We say that $u\in L^\infty(I\!\!R^d_+)$ is a quasi-solution to (1) if for every $c\in I\!\!R$, and almost every $(t,x)\in I\!\!R^d_+$, there exists a Radon measure $\mu_c\in\mathcal{M}(I\!\!R^d_+)$, such that

$$\begin{aligned}L^c_u(t,x)&\equiv\partial_t|u-c|+\operatorname{div}\left[\operatorname{sgn}(u-c)(f(x,u)-f(x,c))\right]\\&-\sum_{i,j=1}^k\partial^2_{x_ix_j}\left(\operatorname{sgn}(u-c)\int_c^u b_{ij}(x,v)dv\right)=-\mu_c(t,x),\end{aligned} \tag{8}$$

in $\mathcal{D}'(I\!\!R^d_+)$. The Radon measures μ_c, $c\in I\!\!R$, are called the defect measure corresponding to u.

From (8), with $c > \|u\|_\infty$, it follows that there exists $\mu \in \mathcal{M}(I\!R^d_+)$, such that

$$\partial_t u + \mathrm{div}_x f(x,u) - \sum_{i,j=1}^k \partial_{x_i}(b_{ij}(x,u)\partial_{x_j} u) = -\mu, \tag{9}$$

in $\mathcal{D}'(I\!R^d_+)$.

To proceed, we need the truncation function. For $a,b \in I\!R$, $a<b$, denote

$$s_{a,b}(u)(t,x) = \max\{a, \min\{u(t,x), b\}\}.$$

A simple consequence of (2) and (6) is the following lemma.

Lemma 4. *Let u be an entropy solution to* (1). *Then for every $i \in I\!N$ and any $a<b$ such that $(a,b) \subset (\lambda_i^m, \lambda_{i+1}^m)$, $m \in I\!N$,*

$$\partial_{x_i} s_{a,b}(u)(t,x) \in L^2(I\!R^+ \times \Omega_m), \quad i = 1,\dots,k.$$

Proof: First, notice that

$$s'_{a,b}(\lambda) = \begin{cases} 1, & a<\lambda<b \\ 0, & \lambda < a \text{ or } \lambda > b \end{cases}.$$

From here and (2), we conclude,

$$\begin{aligned}\sum_{i=1}^k |\partial_{x_i} s_{a,b}(u)|^2 &\le \frac{s'_{a,b}(u)}{c(x,u)} \sum_{i,j=1}^k b_{ij}(x,u) u_{x_i} u_{x_j} \\ &\le \max_{a\le\lambda\le b} (c(x,\lambda))^{-1} \sum_{j=1}^k \left(\sum_{i=1}^k \sigma_{ij} \partial_{x_i} s_{a,b}(u) \right)^2 \in L^1(I\!R^+ \times \Omega_m),\end{aligned}$$

where the last relation follows from (6). This concludes the proof. ❑

The main result of the paper is the following theorem.

Theorem 5. *If u is a quasi-solution to* (1) *then there exists a function $u_0 \in L^\infty(I\!R^d)$ such that*

$$L^1_{\mathrm{loc}}(I\!R^d) - \lim_{t\to 0} u(t,\cdot) = u_0.$$

Important part in the proof of Theorem 5 plays the notion of nondegeneracy for equation (1). It is given in [20, Definition 2]. Denote by $X \subset I\!R^{d+1}$ the linear subspace of $I\!R^{d+1}$ such that

$$X := \left\{ \tilde\xi = (\xi_0, ..., \xi_d) \in I\!R^{d+1} \,\middle|\, \xi_1 = ... = \xi_k = 0 \right\}. \tag{10}$$

We see that for all $\tilde\xi \in X$, $\langle B(x,\lambda)\tilde\xi, \tilde\xi\rangle = 0$, which in terms of [20, Definition 2], means that equation (1) is nondegenerate if for almost all $(t,x) \in I\!R^d_+$ and for all nonzero $\tilde\xi = (\xi_0, 0, ..., 0, \xi_{k+1}, ..., \xi_d) \in X$ and $\bar\xi = (0, \xi_1, ..., \xi_k, 0, ..., 0,) \in X^\perp$, the mappings

$$\lambda \mapsto \xi_0\lambda + \sum_{i=k+1}^d \xi_i f_i(x,\lambda) \qquad \text{and} \qquad \lambda \mapsto \sum_{i,j=1}^k B_{ij}(x,\lambda)\xi_i\xi_j \tag{11}$$

are not simultaneously constant on nondegenerate intervals. From conditions (2)-(3), we see that the second statement from (11) is fulfilled, which gives us the following definition.

Definition 6. We say that equation (1) is *nondegenerate* if for almost all $(t,x) \in \mathbb{R}^d_+$ and for all $\tilde{\xi} \in X$, $\tilde{\xi} \neq 0$, the mapping

$$\lambda \mapsto \xi_0 \lambda + \sum_{i=k+1}^{d} \xi_i f_i(x,\lambda) \tag{12}$$

is not constant on nondegenerate intervals.

We shall also need the following statements whose proofs, with negligible adaptations, can be found in [18].

Proposition 7 (Existence of a weak trace)**.** [18, Proposition 1 and Corrolary 1] *If u is a quasi solution to* (1)*, then there exists weak trace in the sense that there exists $u_0 \in L^\infty(\mathbb{R}^d)$ such that*

$$u(t,\cdot) \rightharpoonup u_0, \ \ast\text{-weakly in } L^\infty(\mathbb{R}^d), \text{as } t \to 0, \ t \in E,$$

where $E := \{t > 0 \,|\, (t,x) \text{ is a Lebesque point to } u(t,x) \text{ for a.e. } x \in \mathbb{R}^d\}$.

The following proposition concerns sufficient condition for existence of the strong trace.

Proposition 8. [18, Proposition 3] *Let $u \in L^\infty(\mathbb{R}^d_+)$ be a quasi solution to* (1) *and there is a sequence $t_m \in \mathcal{E}$ such that $t_m \to 0$, $m \to \infty$, and $u(t_m,\cdot) \to u_0$, as $m \to \infty$ in $L^1_{\rm loc}(\mathbb{R}^d)$. Then $u(t,\cdot) \to u_0$, in $L^1_{\rm loc}(\mathbb{R}^d)$, as $t \to 0$, $t \in \mathcal{E}$.*

2.1. ***H*-measures.** We use here the concept of the parabolic H-measures introduced in [1] and generalized in [20]. The parabolic H-measures are modifications of the H-measures introduced in [5, 22].

Recall that a measure valued function on $\mathbb{R}^d_+$ is a weakly measurable mapping $(t,x) \mapsto \nu_{t,x}$, where $\nu_{t,x}$ are Borel probability measures with compact support in $\mathbb{R}$. If ${\rm supp}\nu_{t,x} \subset [-M,M]$, we say that $\nu_{t,x}$ is *bounded* and define $\|\nu_{t,x}\|_\infty = \inf M$. If a measure valued function has the form $\nu_{t,x}(\lambda) = \delta(\lambda - u(t,x))$, where δ is the Dirac measure, then we say that $\nu_{t,x}$ is *regular*. Regular measure valued function $\nu_{t,x}(\lambda) = \delta(\lambda - u(t,x))$ is identified with the function u, so we can embed the space $L^\infty(\mathbb{R}^d_+)$ into the space of bounded measure valued function, $MV(\mathbb{R}^d_+)$.

A sequence of bounded measure valued function, $\nu^m_{t,x} \in MV(\mathbb{R}^d_+)$, $n \in \mathbb{N}$,

- weakly converges to $\nu_{t,x} \in MV(\mathbb{R}^d_+)$, $\nu^m_{t,x} \rightharpoonup \nu_{t,x}$, if for every $\phi \in C(\mathbb{R})$,
$$\int \phi(\lambda) d\nu^m_{t,x}(\lambda) \rightharpoonup \int \phi(\lambda) d\nu_{t,x}(\lambda) \text{ weakly}-\ast \text{ in } L^\infty(\mathbb{R}^d_+), \text{ as } m \to \infty;$$
- strongly converges to $\nu_{t,x} \in MV(\mathbb{R}^d_+)$, $\nu^m_{t,x} \to \nu_{t,x}$, if for every $\phi \in C(\mathbb{R})$,
$$\int \phi(\lambda) d\nu^m_{t,x}(\lambda) \to \int \phi(\lambda) d\nu_{t,x}(\lambda), \text{ in } L^1_{\rm loc}(\mathbb{R}^d_+), \text{ as } m \to \infty;$$
- is bounded if $\sup_{m \in \mathbb{N}} \|\nu^m_{t,x}\|_\infty < \infty$.

Every bounded sequence $\nu^m_{t,x} \in MV(\mathbb{R}^d_+)$ is weakly precompact, cf. [17, 20].

Let $\nu^m_{t,x} \rightharpoonup \nu^0_{t,x}$, as $m \to \infty$. Denote,

$$u_m(t,x,\lambda) = \nu^m_{t,x}((\lambda,+\infty)), \qquad u_0(t,x,\lambda) = \nu^0_{t,x}((\lambda,+\infty)),$$

for $(t,x)\in I\!R^+_d,\ \lambda\in I\!R$. Recall [20], the distribution functions $u_m(t,x,\lambda), u_0(t,x,\lambda)$ are measurable in $(t,x)\in I\!R^d_+$, and

$$U_m^\lambda(t,x):=u_m(t,x,\lambda)-u_0(t,x,\lambda)\rightharpoonup 0,\ \text{weakly}-*\ \text{in}\ L^\infty(I\!R^d_+),\ \text{as}\ m\to\infty.$$

for all $\lambda\in\mathcal{E}:=\{\lambda_0\in I\!R|\, u_0(t,x,\lambda)\to u_0(t,x,\lambda_0),\ \text{as}\ \lambda\to\lambda_0,\ \text{in}\ L^1_{\rm loc}(I\!R^d_+)\}$. The complement $I\!R\backslash\mathcal{E}$ is at most countable.

For X given by (10), we define $S_X:=\{\xi\in I\!R^{d+1}:\ \xi_0^2+\xi_1^4+...+\xi_k^4+\xi_{k+1}^2+..+\xi_d^2=1\}$ and $p(\xi):=(\xi_0^2+\xi_1^4+...+\xi_k^4+\xi_{k+1}^2+..+\xi_d^2)^{1/4}$. Notice that for $\xi\in I\!R^{d+1}$, $\xi=\tilde\xi+\bar\xi$, $\tilde\xi\in X$ and $\bar\xi\in X^\perp$,

$$\pi_X(\xi):=\frac{\tilde\xi}{p(\xi)^2}+\frac{\bar\xi}{p(\xi)}\in S_X.$$

Now, we introduce the ultra-parabolic H-measure, $\{\mu^{pq}\}_{p,q\in\mathcal{E}}$, associated to the bounded (sub)sequence of measure valued functions $\{\nu^m_{t,x}\}_m$ as well as notions connecting the equation under consideration with the H-measures.

Proposition 9. [20, Proposition 2] *There exists a family of locally finite Borel measures $\{\mu^{pq}\}_{p,q\in\mathcal{E}}$ in $I\!R^d_+\times S_X$ and a subsequence $U_m(t,x)=\{U_m^\lambda(t,x)\}_{\lambda\in\mathcal{E}}$ such that for all $\phi_1,\phi_2\in C_0(I\!R^d_+)$ and $\psi\in C(S_X)$*

$$\langle\mu^{pq},\phi_1(t,x)\phi_2(t,x)\psi(\xi)\rangle=\lim_{m\to\infty}\int_{I\!R^d_+}\mathcal{F}[\phi_1U_p^m](\xi)\mathcal{F}[\phi_2U_q^m](\xi)\psi(\pi_X(\xi))d\xi.$$

Definition 10. We say that the bounded sequence of measure valued functions $\{\nu^m_{t,x}\}_m$ fulfills the condition (C) if the sequence of distributions

$$\partial_t\int_p^{+\infty}(\lambda-p)d\nu^m_{t,x}(\lambda)+{\rm div}_x\int_p^{+\infty}(f(x,\lambda)-f(x,p))d\nu^m_{t,x}(\lambda)$$
$$-\sum_{i,j=1}^k\partial^2_{x_ix_j}\int_p^{+\infty}(B_{ij}(x,\lambda)-B_{ij}(x,p))d\nu^m_{t,x}(\lambda)$$

is precompact in $H^{-1,-2}_{\rm loc}(I\!R^d_+)$.

Here, $H^{-1,-2}_{\rm loc}(I\!R^d_+)$ stands for the locally convex Sobolev space $H^{-1,-2}_{\rm loc}(I\!R^d_+)=\{u\in\mathcal{D}'(I\!R^d_+)|\,(\forall\phi\in C_0^\infty(I\!R^d_+))u\phi\in H^{-1,-2}(I\!R^d_+)\}$, where $H^{-1,-2}(I\!R^d_+)$ is the anisotropic Sobolev space

$$H^{-1,-2}(I\!R^d_+)=\left\{u\in\mathcal{D}'(I\!R^d_+)\Big|\,(\exists w\in L^2(I\!R^d_+))\frac{\mathcal{F}[u](\xi)}{(1+|\tilde\xi|^2+|\bar\xi|^4)1/2}=\mathcal{F}[w](\xi)\right\}.$$

Recall [20], $H^{-1}(I\!R^d_+)\subset H^{-1,-2}(I\!R^d_+)\subset H^{-2}(I\!R^d_+)$, and also

$$H^{-1}_{\rm loc}(I\!R^d_+)\subset H^{-1,-2}_{\rm loc}(I\!R^d_+)\subset H^{-2}_{\rm loc}(I\!R^d_+).\tag{13}$$

The condition (C) is important because of the following proposition. The proof is the same as the proof of [18, Theorem 5].

Proposition 11. *If the H-measure $\{\mu^{pq}\}_{p,q\in P}$ associated to the sequence $\{\nu^m_{t,x}\}_m$ is not trivial, and condition (C) is fulfilled, then there exists an interval $I=[p_0,p_0+\delta]$, $\delta>0$, and $\xi_0,\tilde\xi\neq 0$, such that $\xi_0\lambda+\xi_{k+1}f_{k+1}(\lambda)+...+\xi_df_d(\lambda)=const$, for $\lambda\in I$, i.e. the genuine nonlinearity condition is not fulfilled.*

2.2. **Scaling.** This part represents a more significant modification of the standard techniques [18, 24]. Namely, in the hyperbolic case, one of the crucial steps in the proof of traces existence was the change of variables $(t,x) \mapsto (\varepsilon t, y+\varepsilon x)$, $y \in I\!R^d$ is fixed. Since we have the ultra-parabolic terms, we need a different scaling.

Accordingly, denote

$$\bar{x} = (x_1, ...x_k, 0, ...0) \in I\!R^d, \quad \tilde{x} = (0, ..., 0, x_{k-1}, ..., x_d) \in I\!R^d, \quad \bar{x} + \tilde{x} = x \in I\!R^d,$$

and change the variables in the following way,

$$(t,x) \mapsto (\varepsilon_m t, \sqrt{\varepsilon_m}\bar{x} + \varepsilon\tilde{x} + y), \quad y \in I\!R^d,$$

where $(\varepsilon_m)_{m\in I\!N}$ is a sequence of positive numbers converging to zero.

With the new variables, for the weak trace u_0 from Remark 7, i.e. for $u_{0_m}(x) = u_0(\sqrt{\varepsilon_m}\bar{x} + \varepsilon\tilde{x} + y)$, it is easy to prove, cf. [18, 24], that, up to a subsequence, as $m \to \infty$,

$$u_{0_m} \to u_0(y), \text{ in } L^1_{\rm loc}(I\!R^d), \text{ for a.e. } y \in I\!R^d.$$

Moreover, for

$$u^m := u(\varepsilon_m t, \sqrt{\varepsilon_m}\bar{x} + \varepsilon_m\tilde{x} + y) \text{ and } u^m_{x_i} = u_{x_i}(\varepsilon_m t, \sqrt{\varepsilon_m}\bar{x} + \varepsilon\tilde{x} + y), \tag{14}$$

we obtain

$$\begin{aligned} &L^c_u(\varepsilon_m t, \sqrt{\varepsilon_m}\bar{x} + \varepsilon\tilde{x} + y) = -\varepsilon_m \mu_c(\varepsilon_m t, \sqrt{\varepsilon_m}\bar{x} + \varepsilon_m\tilde{x} + y) \\ &\quad + (\sqrt{\varepsilon_m} - \varepsilon_m)\mathrm{sgn}(u^m - c)\sum_{i=1}^k \Big[(\partial_u f_i)(\sqrt{\varepsilon_m}\bar{x} + \varepsilon_m\tilde{x} + y, u^m) u^m_{x_i} \\ &\quad + D_{x_i} f_i(\sqrt{\varepsilon_m}\bar{x} + \varepsilon_m\tilde{x} + y, u^m) - D_{x_i} f_i(\sqrt{\varepsilon_m}\bar{x} + \varepsilon_m\tilde{x} + y, c)\Big] =: -\mu^m_c, \end{aligned} \tag{15}$$

in $\mathcal{D}'(I\!R^d_+)$. Since according to Lemma 4, for $i = 1, \dots, k$ it holds $\partial_{x_i} u \in L^2(I\!R^d) \subset \mathcal{M}(I\!R^d_+)$, we can rely on the proof of [24, Lemma 2] to state:

Lemma 12. *If $\mu_c \in \mathcal{M}(I\!R^d_+)$ then, up to a subsequence,*

$$L^c_{u_m} \to 0 \text{ in } \mathcal{M}(I\!R^d_+),$$

for almost every $y \in I\!R^d$ and almost every $c \in I\!R$.

The topology in the space of locally bounded Borel measures $\mathcal{M}_{loc}(I\!R^d_+)$ is generated by the semi-norms $\|\gamma\|_K = Var(\mu_c)(K)$ for compact subsets $K \subset\subset I\!R^d_+$.

Applying this property to $\{\mu_c\}_{c\in C}$, C is a dense countable subset of $I\!R$, and using the standard diagonal extraction, we can choose a subsequence of $\{\varepsilon_m\}$ being common for all $c \in C$, such that

$$\mu^m_c \to 0 \text{ in } \mathcal{M}(I\!R^d_+), \text{ as } m \to \infty, \tag{16}$$

for a.e. $y \in I\!R^d$.

From (16), by slightly modifying the proof of [18, Theorem 2], one has the following theorem.

Theorem 13. *Existence of the strong trace $\lim_{t\to 0} u(t,\cdot) = u_0$ in $L^1_{\rm loc}(I\!R^d)$ is equivalent to the condition that, for a.e. $y \in I\!R^d$, the sequence u_m converges, up to a subsequence, in $L^1_{\rm loc}(I\!R^d_+)$.*

3. Existence of traces for equation (5)

In this section we consider equation (5). We modify the assumptions from the previous section in the following manner:

- $f_i \in C^1(I\!R^d \times I\!R)$, $i = 1, ..., k$ and $f_j \in C^1(I\!R)$, $j = k+1, ..., d$;
- The function c from (2) and (3) can be simplified as $c = c(\lambda) \geq 0$ is such that there exist points $-\infty < \lambda_i < \lambda_{i+1} < +\infty$, $i \in I\!N$ such that
$$c(\lambda) > 0 \ \text{ for } \ \lambda \in \bigcup_{i=1}^{\infty} (\lambda_i, \lambda_{i+1}). \tag{17}$$
In other words, the splitting Ω_m, $m \in I\!N$, implied in (3) reduces to a single set which is equal to the space $I\!R^d$ itself. This does not affect the generality of our consideration since it is of the local nature. Therefore, we can always split our analysis on several subdomains of $I\!R^d$.

The proof of existence of traces at $t = 0$ for (5) will be given via the method of mathematical induction with respect to the space dimension. Therefore we need the following theorem whose proof is the same as the proof of [18, Theorem 3].

Theorem 14. *Suppose that, in* (5)*, the component f_d of the flux vector f is absent, i.e. equation* (5) *has form*
$$\partial_t u + \sum_{i=1}^{k} \partial_{x_i} f_i(x, u) + \sum_{i=k+1}^{d-1} \partial_{x_i} f_i(u) = \sum_{i,j=1}^{k} \partial_{x_i} (b_{ij}(x, u) \partial_{x_j} u). \tag{18}$$
Then, if $u = u(t, x)$ is a quasi-solution to (5)*, then for a.e. $x_d \in I\!R$, $\tilde{u}(t, x') := u(t, x', x_d)$ is a quasi-solution to reduced equation* (18)*, where $x' = (x_1, ..., x_{d-1})$ and x_d is treated like a parameter.*

The following theorem is the main result of this section.

Theorem 15. *If u is a quasi-solution to* (5) *then it admits the strong trace at $t = 0$, i.e. there exists a function $u_0 \in L^\infty(I\!R^d)$ such that*
$$L^1_{\text{loc}}(I\!R^d) - \lim_{t \to 0} u(t, \cdot) = u_0.$$

Proof: We use the method of mathematical induction with respect to $d - k$.

Step 1. Assume that $d - k = 0$. In this case, from Lemma 4, we conclude that for almost every $(a, b) \in I\!R$, the function $s_{a,b}(u) \in L^\infty(I\!R^+; BV(I\!R^d))$. From [25], it follows that $s_{a,b}(u)$ admits strong traces at $t = 0$ which, since a, b belong to the set of full measure, implies that u admits the strong traces as well.

Step 2. Assume that if $u \in L^\infty(I\!R^{d-1}_+)$, $u = u(t, x_1, ..., x_{d-1})$, is a quasi-solution to
$$\partial_t u + \sum_{i=1}^{k} \partial_{x_i} f_i(x, u) + \sum_{i=k+1}^{d-1} \partial_{x_i} f_i(u) = \sum_{i,j=1}^{k} \partial_{x_i} (b_{ij}(x, u) \partial_{x_j} u), \tag{19}$$
then there exists a function $u_0 \in L^\infty(I\!R^{d-1})$ such that $L^1_{\text{loc}}(I\!R^{d-1}) - \lim u(t, \cdot) = u_0$.

Step 3. Let $u \in L^\infty(I\!R^d_+)$ be a quasi-solution to (5). Assume that the genuine nonlinearity is lost in an interval (a, b). To restrict our considerations on a case when the quasi-solution u takes values in the interval (a, b), we use the truncation function $v = v(t, x) = s_{a,b}(u(t, x))$. The point is that for any continuous function $F = F(x, \lambda)$, one can verify that

$$\begin{aligned}\operatorname{sgn}(v-c)(F(x,v)-F(x,c)) &= \operatorname{sgn}(u-c')(F(x,u)-F(x,c'))-\\ &-\frac{1}{2}\Big(\operatorname{sgn}(u-a)(F(x,u)-F(x,a))+\operatorname{sgn}(u-b)(F(x,u)-F(x,b))\Big)\\ &+\frac{1}{2}(F(x,b)-F(x,a)),\end{aligned}$$

where $c' = \max\{a, \min\{c, b\}\}$. This enables us to conclude that

$$L_v^c = -\mu_{c'} + \frac{1}{2}(\mu_a + \mu_b),$$

which proves that v is a quasi-solution to (5).

Since the genuine nonlinearity is lost in (a, b), there exists nonzero vector $(\xi_0, \xi_{k+1}, ..., \xi_d) \in I\!R^{d-k+1}$ such that

$$\xi_0\lambda + \xi_{k+1}f_{k+1}(\lambda) + ... + \xi_d f_d(\lambda) = const, \quad \lambda \in (a, b). \tag{20}$$

We will use this fact to reduce spatial dimension in the following sense. Introduce the change of spatial variables $(x_{k+1}, ...x_d) \in I\!R^{d-k} \mapsto (z_{k+1}, ...z_d) \in I\!R^{d-k}$ as $\tilde{z} = ct + A\tilde{x}$, where $c = (c_{k+1}, ..., c_d)^\top$ and $A = [a_{ij}]_{i,j=k+1,...,d} \in I\!R^{d-k\times d-k}$, $a_{ij} = a_{ji}$. Other spatial variables will remain unchanged, i.e. $z_1 = x_1$, ..., $z_k = x_k$. With this change, for $u = u(t, z)$, equation (5) becomes

$$u_t + \sum_{i=1}^{k}\partial_{z_i} f_i(z,u) + \sum_{l=k+1}^{d}\partial_{z_l}\left(c_l u + \sum_{i=k+1}^{d} a_{li} f_i(u)\right) = \sum_{i,j=1}^{k}\partial_{z_i}\left(b_{ij}(z,u)\partial_{z_j} u\right).$$

Denote $\tilde{f}_l(u) := c_l u + \sum_{i=k+1}^{d} a_{li} f_i(u)$, $l = k+1, ..., d$ and $\tilde{f}_i(z, u) := f_i(z, u)$, $i = 1, ..., k$. According to (20), we choose $c_d := \xi_0$, $a_{d,k+1} := \xi_{k+1}$, ..., $a_{d,d} := \xi_d$ and obtain $\tilde{f}_d(u) := c_d u + \sum_{i=k+1}^{d} a_{di} f_i(u) = const$, for $u \in (a, b)$. This means that $\partial_{z_d}\tilde{f}_d(u(t, z)) = 0$, and the equation takes the following form,

$$\partial_t u + \sum_{i=1}^{k}\partial_{z_i} f_i(z,u) + \sum_{i=k+1}^{d-1}\partial_{z_i}\tilde{f}_i(u) = \sum_{i,j=1}^{k}\partial_{z_j}\left(b_{ij}(z,u)\partial_{z_j} u\right). \tag{21}$$

According to Theorem 14, for a fixed (parameter) z_d, the function $v = v(t, z', z_d)$, $z' \in I\!R^{d-1}$ is a quasi solution to (21). Applying Theorem 14 we conclude that $v(t, z', z_d)$ is a quasi-solution to (21), for a.e. $z_d \in I\!R$.

According to inductive hypothesis, for a.e. $z_d \in I\!R$, there exists $v_0(\cdot, z_d) \in L^\infty(I\!R^{d-1})$ such that

$$L^1_{\rm loc}(I\!R^{d-1}) - \lim_{t\to 0} v(t, \cdot, z_d) = v_0(\cdot, z_d).$$

We need a special choice of (t, z_d) to obtain the analogical assertion in $I\!R^d$. Thus, as in [18], we use the following construction. Let

$$\begin{aligned}E :=&\{t > 0 \,|\, (t, x) \text{ is a Lebesque point to } u(t, x) \text{ for a.e. } x \in I\!R^d\}\\ \mathcal{M} :=&\{(t, z) \equiv (t, z', z_d) \,|\, (t, z) \text{ is a Lebesque point to } u \text{ and}\\ &\qquad\qquad (t, z') \text{ is a Lebesque point to } u(\cdot, z_d)\}\\ E_1 :=&\{t > 0 \,|\, \mathcal{M}_t \text{ is of full measure}\}, \text{ where } \mathcal{M}_t := \{z \,|\, (t, z) \in \mathcal{M}\}\end{aligned}$$

From E_1, which is of full measure, we choose a sequence $\{t_r\}_{r\in\mathbb{N}}$ such that $t_r \to 0$, $r\to\infty$. Then, we take z_d from

$$\mathcal{Z}=\bigcap_r \mathcal{Z}_r, \text{ where } \mathcal{Z}_r := \{s\in\mathbb{R}\,|\,(z',s)\in\mathcal{M}_{t_r}\}.$$

Applying the inductional hypothesis to $v(t_r,z',z_d)$ we obtain that there exists $v_0(\cdot,z_d)\in L^\infty(\mathbb{R}^{d-1})$ such that

$$L^1_{\rm loc}(\mathbb{R}^{d-1})-\lim_{r\to\infty} v(t_r,\cdot,z_d)=v_0(\cdot,z_d).$$

With this choice of (t,z_d) we have that $v_0(\cdot,z_d)\in L^\infty(\mathbb{R}^d)$ and then apply the Lebesgue dominated convergence theorem to conclude that

$$L^1_{\rm loc}(\mathbb{R}^{d})-\lim_{r\to\infty} v(t_r,z)=v_0(z).$$

From here, the same limit relation follows for the original variable x, i.e.

$$L^1_{\rm loc}(\mathbb{R}^{d})-\lim_{r\to\infty} v(t_r,x)=v_0(x).$$

Then, from Proposition 8, we have that

$$L^1_{\rm loc}(\mathbb{R}^{d})-\lim_{t\in E,\,t\to 0} v(t,\cdot)=v_0.$$

Applying Theorem 13, we have that there is a sequence of positive numbers $\varepsilon_m\to 0$, as $m\to\infty$, such that the sequence of functions $v_m(t,x)=v(\varepsilon_m t,\sqrt{\varepsilon_m}\bar{x}+\varepsilon_m\tilde{x}+y)$ converges strongly in $L^1_{\rm loc}(\mathbb{R}^d_+)$, for a.e. $y\in\mathbb{R}^d$.

The analysis in this proof is done on the interval $I=(a,b)$. Now we want to collect all intervals where the genuine nonlinearity is lost. To accomplish countable many intervals we will restrict our attention to the numbers $a,b\in C$, where C is countable dense subset of $\mathbb{R}$ used in the previous section. Than $\mathcal{I}:=\{I=(a,b):a,b\in C\}$ is countable set of intervals. By the diagonal extraction we can choose $\varepsilon_m\to 0$, such that for all $I=(a_I,b_I)\in\mathcal{I}$, and a.e. $y\in\mathbb{R}^d$, $v_m^I:=\max\{a_I,\min\{u_m,b_I\}\}$ converges strongly in $L^1_{\rm loc}(\mathbb{R}^d_+)$, as $m\to\infty$.

Since, $\{u_m\}_m$ is bounded sequence in $L^\infty(\mathbb{R}^d_+)$, $\delta(\cdot-u_m(t,x)) =: \nu^m_{t,x}(\cdot)\in MV(\mathbb{R}^d_+)$ presents a sequence of bounded regular measure-valued functions, weakly convergent (up to a sequence) to a measure-valued function $\nu_{t,x}\in MV(\mathbb{R}^d_+)$. We keep the same notation for a subsequence. Furthermore, there is a H-measure $\{\mu^{pq}\}_{p,q\in P}$ associated to the sequence $\{\nu^m_{t,x}\}_m$, defied in Proposition 9.

Now, we prove that the sequence $\nu^m_{t,x}(\cdot):=\delta(\cdot-u_m(t,x))$ fulfills the condition (C) from Definition 10. Denote $|u-p|^+=\max\{u-p,0\}$ and ${\rm sgn}_+(u-p)={\rm sgn}\,(|u-p|^+)$. Since for any continuous function ϕ,

$${\rm sgn}_+(u-p)(\phi(u)-\phi(p))=\frac{1}{2}\left({\rm sgn}(u-p)(\phi(u)-\phi(p))+\phi(u)-\phi(p)\right),$$

and $\nu^m_{t,x}((p,+\infty)) = \mathrm{sgn}_+(u_m(t,x)-p)$, we compute

$$\begin{aligned}\mathcal{L}^p_m =&\partial_t\left(\frac{1}{2}(|u_m-p|+u_m-p)\right)\\ &+\sum_{i=1}^k \partial_{x_i}\left(\frac{1}{2}(\mathrm{sgn}(u_m-p)(f_i(x,u_m)-f_i(x,p))+f_i(x,u_m)-f_i(x,p))\right)\\ &+\sum_{i=k+1}^d \partial_{x_i}\left(\frac{1}{2}(\mathrm{sgn}(u_m-p)(f_i(u_m)-f_i(p))+f_i(u_m)-f_i(p))\right)\\ &-\sum_{i,j=1}^k \partial^2_{x_i,x_j}\left(\frac{1}{2}(\mathrm{sgn}(u_m-p)(B_{i,j}(x,u_m)-B_{i,j}(x,p))+B_{i,j}(x,u_m)-B_{i,j}(x,p))\right)\\ =&-\frac{1}{2}(\mu^m_c+\mu^m),\end{aligned}$$

where μ^m_c and μ^m are measures from (8) and (9), associated to u_m. We see in (16) that the right hand side tends to zero, as $m\to\infty$, in $\mathcal{M}(\bar{\mathbb{R}}^d_+)$. From Murat's lemma we conclude that $\mathcal{L}^p_m \to 0$ in H^{-1}_{loc}, and from (13) we conclude that $\mathcal{L}^p_m\to 0$ in $H^{-1,-2}_{\mathrm{loc}}$ as well.

To continue, notice that there exists a constant $M>0$ such that

$$|\langle \mathcal{L}^p_m - \mathcal{L}^q_m,\, \phi\rangle| \leq M|p-q|\|\phi\|_{H^{1,2}}$$

where $\phi\in H^{1,2}_c(\mathbb{R}^d)$ is a test function $\phi(t,x)$ with a compact support such that $\phi_t, \phi_{x_{k+1}}, ...\phi_{x_d} \in L^2(\mathbb{R}^d_+)$ and $\phi_{x_ix_j}\in L^2(\mathbb{R}^d_+)$, for $i,j=1,...,k$. The proven equicontinuity of the function $\mathcal{L}^p_m$ with respect to $p\in\mathbb{R}$ implies that the condition (C) is fulfilled for every $p\in\mathbb{R}$, which in turn implies that the H-measure $\mu^{pq}\equiv 0$. Thus, for almost every $y\in\mathbb{R}^d$

$$u(\varepsilon_m t, \varepsilon_m\bar{x}+\sqrt{\varepsilon_m}\tilde{x}+y)\to u_0(x),\quad t\to 0,$$

in $L^1_{\mathrm{loc}}(\mathbb{R}^d)$. Then, we apply Theorem 13 to conclude about existence of the strong traces on $t=0$ to (5). Details of the procedure can be found in the final steps of the proof of [18, Theorem 1]. ❑

4. The heterogeneous case; proof of Theorem 5

In this section, we shall prove the main result of the paper – Theorem 5. In addition to conditions (1)-(4), existence of the strong traces at $t=0$ to (1) will be proved under the following traceability assumptions.

Definition 16. We say that the flux f is *traceable* if for almost every $x_0 = (x^0_1,\dots,x^0_d)\in\mathbb{R}^d$ there exists its neighborhood $U(x_0)\subset\mathbb{R}^d$ and a partition $\{\lambda^l_{x_0}\}_{l\in\mathbb{N}}$ of the real line such that for every $l\in\mathbb{N}$, either of the following two assumptions hold on $(\lambda^l_{x_0},\lambda^{l+1}_{x_0})$

- there exists a transformation $a:\mathbb{R}^{d-k}\to\mathbb{R}^{d-k}$, $a\in C^2(\mathbb{R}^{d-k+1};\mathbb{R}^{d-k})$, defined by

$$\begin{aligned}\hat{x}_{k+1} &= a_{k+1}(x_{k+1},\dots,x_d,\lambda)\\ &\cdots\\ \hat{x}_d &= a_d(x_{k+1},\dots,x_d,\lambda)\end{aligned}\tag{22}$$

which is regular with respect to $\tilde{x} \equiv (x_{k+1}, \dots, x_d)$ in a neighborhood $U(x_0)$, i.e.

$$J(\lambda, x) = \Big|\frac{\partial a}{\partial \tilde{x}}\Big| > 0, \qquad \lambda \in (\lambda^l_{x_0}, \lambda^{l+1}_{x_0}),\ x \in U(x_0), \tag{23}$$

such that there exist functions $p_i : I\!R \to I\!R$, $i = k+1, \dots, d$, satisfying

$$\sum_{j=k+1}^{d} \frac{\partial a_i}{\partial x_j} \partial_\lambda f_j(x, \lambda) = p_i(\lambda), \qquad \lambda \in (\lambda^l_{x_0}, \lambda^{l+1}_{x_0}),\ x \in U(x_0). \tag{24}$$

or

- the flux f is nondegenerate on $(\lambda^l_{x_0}, \lambda^{l+1}_{x_0}) \times U(x_0)$ in the sense that for almost every $x \in U(x_0)$ the mapping

$$\lambda \mapsto \xi_0 \lambda + \sum_{i=k+1}^{d} \xi_i f_i(x, \lambda) \tag{25}$$

is not constant on nondegenerate subintervals of $(\lambda^l_{x_0}, \lambda^{l+1}_{x_0})$.

In order to make use of the traceability conditions, we need to rewrite (8) in the kinetic formulation. The appropriate procedure can be found in [4] in the homogeneous case, and it can be easily adapted to the situation that we have here. The following proposition holds.

Proposition 17. *The function u represents a quasi-solution to* (1) *if and only if the kinetic function*

$$h(t, x, \lambda) = \begin{cases} 1, & 0 \le \lambda \le u(t, x) \\ -1, & u(t, x) \le \lambda \le 0 \\ 0, & else \end{cases}$$

satisfies the following linear equation:

$$\begin{aligned} &\partial_t h(t, x, \lambda) + \operatorname{div}\left[\partial_\lambda f(x, \lambda) h(t, x, \lambda)\right] \\ &- \sum_{i,j=1}^{k} \partial^2_{x_i x_j} b_{ij}(x, \lambda) h(t, x, \lambda) = -\partial_\lambda \mu_\lambda(t, x), \qquad in\ \mathcal{D}'(I\!R \times I\!R^d_+), \end{aligned} \tag{26}$$

where $\mu = \mu_\lambda(t, x) \in \mathcal{M}(I\!R \times I\!R^d_+)$.

Remark 18. It can be proved that the functional $\mu = \mu_\lambda(t, x)$ has more regularity (see e.g [4]), but since it is not necessary here, we shall not get into that issue.

Proof: As we have already said, the latter proposition is basically proved in [4]. Here, we shall briefly propose another proof.

Accordingly, assume first that (26) is satisfied. Integrating it over $\int_{-M}^{c} d\lambda$, where M is such that $-M \le u \le M$, we immediately reach to (8).

In order to prove the inverse implication, we shall assume that (8) is satisfied for the semi-entropies $|u - c|^+ = \begin{cases} u - c, & u - c \ge 0 \\ 0, & u - c \le 0 \end{cases}$. It is well known that entropy formulations via entropies and semi-entropies are equivalent. Then, notice that

$$\partial_c |u - c|^+ = -\mathrm{sgn}_+(u - c) = h(t, x, c) - \mathrm{sgn}_+(c), \tag{27}$$

and differentiate (8) (with the latter semi-entropies) with respect to c. We get

$$\begin{aligned}&\partial_t \mathrm{sgn}_+(u-c) + \mathrm{div}\left[\partial_\lambda f(x,c)\mathrm{sgn}_+(u-c)\right]\\&-\sum_{i,j=1}^{k} \partial^2_{x_i x_j} b_{ij}(x,\lambda)\mathrm{sgn}_+(u-c) = -\partial_c \mu_\lambda(t,x), \quad \text{ in } \mathcal{D}'(I\!R\times I\!R^d_+).\end{aligned}$$

Taking (27) into account, we immediately reach to (26).

❑

Proof of Theorem 5. Fix $x_0 \in I\!R^d$ and its neighborhood $U(x_0)$ so that the traceability condition is satisfied.

Case 1. If in the neighborhood $U(x_0)$ and an interval $(\lambda^l_{x_0}, \lambda^{l+1}_{x_0})$ nondegeneracy condition (25) is satisfied, then, according to Proposition 11, the H-measures corresponding to the sequence $\nu^m_{t,x} = \delta(\cdot - s_{\lambda^l_{x_0},\lambda^{l+1}_{x_0}}(u_m(t,x)))$, where u_m is defined in (14), is identically equal to zero. Therefore, $s_{\lambda^l_{x_0},\lambda^{l+1}_{x_0}}(u_m(t,x)) \to v_{l,l+1}$ in $L^1_{\mathrm{loc}}(I\!R^d_+)$ along a subsequence. According to Theorem 8, we conclude that the latter convergence holds for the entire sequence.

Case 2. Assume that the first item of the traceability condition is satisfied, i.e. that there exists the transformation a from (22) satisfying (24) for the neighborhood $U(x_0)$ and the interval $(\lambda^l_{x_0}, \lambda^{l+1}_{x_0})$. Recall that the function $s_{\lambda^l_{x_0},\lambda^{l+1}_{x_0}}(u) := u_{l,l+1}$ also represent a quasi-solution to (1). Therefore, it satisfies the kinetic relation (26) for the function $h(t,x,\lambda) = \begin{cases} 1, & 0\leq \lambda \leq u_{l,l+1}(t,x)\\ -1, & u_{l,l+1}(t,x)\leq\lambda\leq 0\,.\\ 0, & else\end{cases}$

To proceed, notice that

$$\begin{aligned}&\sum_{i=k+1}^{d} \partial_{x_i}\left(f_i(x,\lambda)h\right)\\&=\sum_{i=k+1}^{d}\left(\sum_{j=k+1}^{d}\partial_{\hat{x}_j}(f_i(x,\lambda)h)\frac{\partial a_j}{\partial x_i}\right)\\&=\sum_{j=k+1}^{d}\partial_{\hat{x}_j}\left(\sum_{i=k+1}^{d} f_i(x,\lambda)\frac{\partial a_j}{\partial x_i}h\right)-\sum_{i=k+1}^{d}\sum_{i=k+1}^{d} f_i(x,\lambda)h\frac{\partial^2 a_j}{\partial \hat{x}_j\partial x_i}\\&=\sum_{j=k+1}^{d}\partial_{\hat{x}_j}\left(p_j(\lambda)h\right)+\hat{\gamma}(t,x,\lambda),\end{aligned}$$

where $\hat{\gamma} \in \mathcal{M}(I\!R^+ \times U(x_0))$. Having this in mind, we substitute the change of variables given by (22) in (26). We obtain:

$$\begin{aligned} &\partial_t h(t,x,\lambda) + \sum_{j=1}^{k} \partial_{x_j}(\partial_\lambda f_j(x,\lambda) h(t,x,\lambda)) + \sum_{j=k+1}^{d} \partial_{\hat{x}_j}(\partial_\lambda p_j(\lambda) h(t,x,\lambda)) \\ &- \sum_{i,j=1}^{k} \partial^2_{x_i x_j} b_{ij}(x,\lambda) h(t,x,\lambda) = -\partial_\lambda \mu_\lambda(t,x) + \hat{\gamma}(t,x,\lambda), \quad \text{ in } \mathcal{D}'(I\!R \times I\!R^d_+). \end{aligned} \tag{28}$$

From here and Proposition 17, we conclude that the function $u = u(t,\bar{x},\hat{x})$ given in the new coordinates $(x_1,\dots,x_k,\hat{x_{k+1}},\dots,\hat{x}_d)$ satisfies in $U(x_0)$:

$$\begin{aligned} &\partial_t |u-\lambda| + \sum_{j=1}^{k} \partial_{x_j}(\operatorname{sgn}(u-\lambda)(f_j(x,u) - f_j(x,\lambda)) h(t,x,\lambda)) \\ &+ \sum_{j=k+1}^{d} \partial_{\hat{x}_j} \operatorname{sgn}(u-\lambda)(p_j(u) - p_j(\lambda)) \\ &- \sum_{i,j=1}^{k} \partial^2_{x_i x_j}\left(\operatorname{sgn}(u-c)\int_c^u b_{ij}(v)dv\right) = \gamma_\lambda(t,x) + \int_{-M}^{\lambda} \hat{\gamma}(t,x,\lambda')d\lambda. \end{aligned} \tag{29}$$

From here, we see that in $I\!R^+ \times U(x_0)$ the function u represents a quasi-solution to an equation of type (5) for which we have proved existence of traces in Section 3. By choosing countably many intervals $U(x_0)$ for point $x_0 \in I\!R^d$ in which we have the traceability assumptions fulfilled, we can cover entire $I\!R^d$ (excluding the set of measure zero). Since the traces at $t = 0$ exist in each of the latter neighborhoods, they exist globally as well. The proof is over. □

5. Conclusion and examples

The question that naturally arises is to find conditions on the flux of (1) under which the traceability conditions hold. We guess that it is enough to assume merely $f \in C^1(I\!R^d \times I\!R; I\!R^d)$. At the moment, we are not able to provide any rigorous statement or to provide an equation which does not satisfy the traceability conditions. However, we find necessary to include several examples.

We start with the simplest one dimensional scalar conservation law:

$$\partial_t + \partial_x(xu) = 0.$$

In this case, we simply take $a(x,\lambda) = \ln(x)$ on the segments $(-\infty, 0)$ and $(0,\infty)$.

Less trivial example is the two dimensional scalar conservation law which is linear in the direction of the first space-variable (i.e. it is not non-degenerate).

$$\partial_t + \partial_{x_1}(x_1 u) + \partial_{x_2}(x_1 u^2) = 0.$$

In this case, we can choose $a_1(x_1,x_2,\lambda) = \ln(x_1)$ and $a_2(x_1,x_2,\lambda) = x_2 - 2\lambda x_1 + p_2(\lambda)\ln(x_1)$ for an arbitrary continuous function p_2 on appropriate subdomains of $I\!R^2 \times I\!R$.

In the latter two examples, we could locally reduce the equations on the homogeneous ones by introducing appropriate change of variables without passing to the kinetic formulation (although not quite obvious in the case of the second equation). This is not so for the following conservation law.

$$\partial_t + \partial_{x_1}(x_1 u) + \partial_{x_2}(\sqrt{x_2^2 + u}) = 0.$$

In this case, wanted transformations are $a_1(x_1, x_2, \lambda) = \ln(x_1)$ and $a_2(x_1, x_2, \lambda) = (x_2^2 + \lambda)^{3/2}$.

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