Front Tracing Approximations for Slow Erosion

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Abstract

In this paper we study an integro-differential equation that arises in modeling slow erosion of granular flow. We construct piecewise constant approximate solutions, using a front tracing technique. Convergence of the approximate solutions is established through proper a priori estimates, which in turn gives global existence of BV solutions. Furthermore, continuous dependence on initial data and on the erosion function is derived, achieving well-posedness of the solutions.

1 Introduction and preliminaries

Consider the scalar integro-differential equation

$$u(t,x)_t + \left(f(u(t,x)) \, \exp\left\{ \int_x^{\infty} f(u(t,s))\,ds \right\} \right)_x = 0\,, \tag{1.1}$$

associated with the initial data

$$u(0,x) = \bar{u}(x)\,, \qquad x \in \mathbb{R}\,. \tag{1.2}$$

This equation was first derived in [2] as the slow erosion limit for a granular flow model proposed in [15], with a specific function f. A more general model is later derived in [20] and [3] for more general functions f. Here, the unknown variable u describes the slope of the standing profile of granular matter, where small avalanches are passing over. The function f is called the *erosion function*, which denotes the erosion rate per unit length in space covered by the avalanche. See [20] for a more detailed derivation of the model.

Existence of global BV solutions and continuous dependence on initial data for a initial-boundary value problem of (1.1) are studied in [3], where we use an iteration technique with a frozen global term at every time step. In this paper, we propose a different approximation technique where we trace directly the wave fronts and follow their interactions. In more details, we construct piecewise constant approximate solutions, and design an algorithm in the style of front tracing. A somewhat similar algorithm is used in [20] where a Hamiltonian type integro-differential equation for the height of the profile is treated, and piecewise affine approximate solutions that allows discontinuities are constructed. Such front tracing algorithms give better intuition and control over wave interactions, and result in straight a-priori estimates. Convergence of solutions follows by compactness, yielding global existence of BV solutions. Furthermore, by directly comparing the $\mathbf{L}^1$ distance between two piecewise constant approximate solutions, we achieve the continuous dependence on both the initial data and the erosion function f. This paper is self-contained.

To simplify notation, we let F denote the integral term, i.e.,

$$F(x;u) \doteq \exp\left\{\int_x^\infty f(u(t,s))\,ds\right\}, \tag{1.3}$$

and we write (1.1) as

$$u_t + (fF)_x = 0\,. \tag{1.4}$$

The erosion function $f \in \mathcal{C}^2\{(0,+\infty)\}$ satisfies the assumptions **(F)**:

$$f(1) = 0\,, \qquad f' > 0\,, \qquad f'' < 0\,, \qquad \lim_{s\to 0^+} f(s) = -\infty\,, \qquad \lim_{s\to+\infty} \frac{f(s)}{s} = 0\,. \tag{1.5}$$

The physical meanings of these assumptions are as follows. (i) At the critical slope $u = 1$ there is no erosion or deposition, so $f(1) = 0$; (ii) When the slope approaches 0, there is infinite large deposition; (iii) When the slope is very large, the erosion function f grows slower than any linear functions. Examples of such functions could include the logarithm function, or $f(s) \approx s^a$ with $0 < a < 1$ for large values of s.

We remark that the assumptions in (1.5) are sharp to prevent blowup of u. In [20] it is proved that the slope u blows up to $+\infty$ if $f(s)$ approaches a linear asymptote as $s \to +\infty$.

Throughout the paper we will use $\|\cdot\|_{\mathbf{L}^1}$, $\|\cdot\|_{\mathbf{L}^\infty}$ and $\mathrm{TV}\{\cdot\}$ to denote the $\mathbf{L}^1$ norm, the $\mathbf{L}^\infty$ norm and the total variation, respectively, all in the space variable. We use $\mathrm{sign}(\cdot)$ to denote the sign function, and C to denote a generic bounded constant that does not depend on the critical variables.

Solutions of the Cauchy problem will be obtained within the class $\mathcal{W}$ consisting of all functions $w : \mathbb{R} \mapsto \mathbb{R}$ satisfying the property **(W)**:

(W) There exist positive constants κ_0, m_0, M_0 and a bounded interval $I = [a,b]$, such that

$$w(x) \geq \kappa_0\,, \qquad \|w(\cdot) - 1\|_{\mathbf{L}^1} \leq m_0\,, \qquad \mathrm{TV}\{w(\cdot)\} \leq M_0\,, \qquad \mathrm{supp}\{w(\cdot) - 1\} \in I\,. \tag{1.6}$$

A definition of weak solutions for (1.1)-(1.2) is now given.

Definition 1.1 *A function $u : [0,T] \times \mathbb{R} \mapsto \mathbb{R}$ is called an* entropy weak solution *for (1.1) with initial data $\bar{u}(x) \in \mathcal{W}$ if $u(0,x) = \bar{u}(x)$, and*

- *$u(t,\cdot) \in \mathcal{W}$ for every $t \in [0,T]$.*
- *For every test function $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^2)$, one has the integral identity*

$$\int_0^T \int_{\mathbb{R}} \left(u\phi_t + f(u)F(x;u)\phi_x\right)\,dx\,dt \;=\; \int_{\mathbb{R}} \Big(u(T,x)\phi(T,x) - u(0,x)\phi(0,x)\Big)\,dx\,. \tag{1.7}$$

- *For any $x < y$, there exists some constant C (that does not depend on x, y), such that*

$$u(t,x) - u(t,y) \leq C \max\{1/t,\ 1\}\,(y-x)\,. \tag{1.8}$$

The existence of entropy weak solutions is stated in next Theorem.

Theorem 1.1 *Let $T > 0$ and an initial data $\bar{u} \in \mathcal{W}$ be given. Then the Cauchy problem (1.1)-(1.2) admits an entropy weak solution $u = u(t,x)$ defined for all $t \in [0,T]$, that moreover satisfies* $\inf_x u(t,x) \geq \inf_x \bar{u}(x)$.

We also study continuous dependence of the solutions, on both initial data and the erosion function. Let $v = v(t,x)$ be an entropy weak solution for

$$v_t + \left(g(v)\exp\left\{\int_x^\infty g(v(t,s))\,ds\right\}\right)_x = 0\,, \qquad v(0,x) = \bar{v}(x) \in \mathcal{W} \tag{1.9}$$

with a different erosion function g that satisfies the assumptions **(F)**, i.e,

$$g(1) = 0\,, \qquad g' > 0\,, \qquad g'' < 0\,, \qquad \lim_{s\to 0} g(s) = -\infty\,, \qquad \lim_{s\to+\infty}\frac{g(s)}{s} = 0\,. \tag{1.10}$$

Note that it is important to have the same critical slope for both erosion functions, otherwise $\|\bar{u}-\bar{v}\|_{\mathbf{L}^1}$ would already be unbounded. Existence of weak solutions for (1.9) follows from Theorem 1.1. Let κ_0 and M be the lower and upper bounds (resp.) for both u and v, we define the norms

$$\|f-g\|_{\mathbf{L}^\infty} \doteq \max_{\kappa_0\leq s\leq M}|f(s)-g(s)|\,, \qquad \|f-g\|_{Lip} \doteq \max_{\kappa_0\leq s\leq M}\left|f'(s)-g'(s)\right|\,. \tag{1.11}$$

We have the following well-posedness Theorem.

Theorem 1.2 *Let u and v be entropy weak solutions for the Cauchy problems (1.1)-(1.2) and (1.9), respectively. Then, we have*

$$\begin{aligned}\|u(t,\cdot)-v(t,\cdot)\|_{\mathbf{L}^1} \;\leq\; & \|\bar{u}-\bar{v}\|_{\mathbf{L}^1} \;+\; C\int_0^t \|u(s,\cdot)-v(s,\cdot)\|_{\mathbf{L}^1}\,ds \\ & + Ct\left[\|f-g\|_{\mathbf{L}^\infty} + \|f-g\|_{Lip}\right]\,.\end{aligned} \tag{1.12}$$

where C depends only on the bounds of the initial data.

By Gronwall's Lemma, (1.12) gives continuous dependence.

Other PDE models for granular flow can be found in [13, 18, 4, 10]. For mathematical properties of the steady state solutions we refer to [8, 9]. A numerical study can be found in [14]. For time-dependent solutions, see the recent results [19, 1, 2, 3, 20]. Other well-known examples of conservation law involving integral terms include the Camassa-Holm equation [7, 5] and a variational wave equation [6]. For some related results on stability for general scalar balance law, we refer to [16, 11].

The rest of the paper is structured in the following way. In Section 2 we give the basic analysis and some formal arguments. In Section 3 we prove Theorem 1.1 by front tracing approximation. Finally, Theorem 1.2 is proved in Section 4.

2 Basic analysis

By the method of characteristics, for smooth solutions one has

$$\dot{x} = f'(u)F\,, \qquad \dot{u} = u_t + \dot{x}u_x = f^2(u)F\,. \tag{2.1}$$

Due to the non-linearity of the erosion function f, characteristics will merge, which leads to discontinuities in solutions. We call them *shocks* or *shock waves*. To see how these shocks form, let $z = u_x$, and consider its evolution along the characteristic,

$$\dot{z} = z_t + f'Fz_x = -f''Fz^2 + 3ff'Fz - f^3F\,. \tag{2.2}$$

Assuming u, f, f' bounded, the first term $(-f''Fz^2)$ dominates as $|z|$ is large. Since $-f''F > 0$, then z blows up to $+\infty$ in finite time, leading to an upward jump in u.

The traveling speed of the shock waves satisfies the Rankine-Hugoniot jump condition. Let u has a jump at x_0, with $u(x_0^-) = u^-$ and $u(x_0^+) = u^+$. The Rankine-Hugoniot condition gives

$$\lambda_s = F(x_0; u)\frac{f(u^-) - f(u^+)}{u^- - u^+}\,. \tag{2.3}$$

Since f is concave, only upward jumps are admissible. Initial downward jumps will open up into rarefaction waves. This is confirmed by (2.2), where z blows up only to $+\infty$, i.e.,

$$z \geq -C\max\{1/t,\, 1\}\,. \tag{2.4}$$

Therefore, an Oleinik-type one-sided entropy inequality (see [17]) holds: for any $t > 0$, and $x < y$, one has

$$u(t,x) - u(t,y) \leq (y - x)C\max\{1/t,\, 1\}\,. \tag{2.5}$$

Wave interactions are determined by the local behavior of the flux, i.e., the erosion function f, which is a concave function. The interactions are similar to those of a scalar conservation law. When two (or more) admissible shocks interact, they will simply merge into a bigger admissible shock, causing cancellation of waves. No new waves would be formed at interactions.

Next is a technical Lemma connecting properties of u with the global term F.

Lemma 2.1 *Let $u : \mathbb{R} \mapsto \mathbb{R}$ satisfy*

$$u(x) \geq \kappa_0 > 0, \qquad \|u(\cdot) - 1\|_{\mathbf{L}^1} \leq m_0\,.$$

Then, the function $f(u(x))$ is absolutely integrable, i.e.,

$$\|f(u(\cdot))\|_{\mathbf{L}^1} \doteq \int_{\mathbb{R}} |f(u(x))|\; dx = C < \infty\,. \tag{2.6}$$

Furthermore, the integral function $F(x;u)$ as defined in (1.3) satisfies

$$e^{-C} \leq F \leq e^{C}, \qquad TV\{F\} \leq Ce^{C}\,. \tag{2.7}$$

Proof. Since $\|u - 1\|_{\mathbf{L}^1} \leq m_0$, then the function $x \mapsto f(u(x))$ is absolutely integrable, because $u \mapsto f(u)$ is uniformly Lipschitz on $[\kappa_0, \infty]$ and $f(1) = 0$. This gives (2.6). The upper and lower bound on F is obvious by its definition and (2.6). Finally, since $x \mapsto F$ is Lipschitz continuous, we have

$$\mathrm{TV}\{F\} = \|F_x\|_{\mathbf{L}^1} = \|f(u)F\|_{\mathbf{L}^1} \leq \|F\|_{\mathbf{L}^\infty}\,\|f(u)\|_{\mathbf{L}^1} \leq Ce^{C}\,. \tag{2.8}$$

$\square$

Below we give some formal arguments, which serves as guideline for the a priori estimates for the approximate solutions.

(1). Lower bound on u. By (2.1), u is non-decreasing along characteristics, therefore the lower bound follows.

(2). Bound on total mass. The trivial solution is $u \equiv 1$. Equation (1.1) can be written as

$$(u-1)_t + (f(u)F)_x = 0\,. \tag{2.9}$$

By the assumptions (1.5) we have $\mathrm{sign}(u-1) = \mathrm{sign}(f(u))$. Since $F > 0$, we conclude that the $\mathbf{L}^1$ norm of $u-1$ is non-increasing in time.

By Lemma 2.1, F is uniformly bounded from below and above, and has bounded variation.

(3). Bounded support for $u-1$. By the lower bound on u, the characteristic speed $f'(u)F$ is now bounded. Therefore, for $t \le T$, the support for $u-1$ is bounded.

(4). Upper bound on u. Integrate the conservation law (2.9) over the region (t,y) with $0 \le t \le T$ and $y \le x(t)$ where $t \to x(t)$ is a characteristic, we get

$$\left| \int_0^T \big[(u-1)f'(u) - f(u)\big]\, F\, dt \right| = \left| \int_{-\infty}^{x(T)} (u(T,x)-1)dx - \int_{-\infty}^{x(0)} (u(0,x)-1)\, dx \right| \;\le\; 2m_0\,, \tag{2.10}$$

thanks to the bound on $\|u-1\|_{\mathbf{L}^1}$. Define an auxiliary function

$$\alpha(u) \doteq \frac{u-1}{f(u)} \quad \text{if } u \neq 1\,, \qquad \alpha(1) = 1/f'(1)\,. \tag{2.11}$$

This function is well-defined for all $u > 0$. At $u = 0$ we can set $\alpha(0) = 0$ by continuity. The function is nonnegative, $\alpha(u) > 0$ for $u > 0$, and is increasing in u, i.e.,

$$\alpha'(u) \;=\; \frac{f(u) - (u-1)f'(u)}{f^2(u)} \;>\; 0 \qquad \text{for} \;\; u > 0\,,\; u \neq 1\,.$$

By the last assumption on f in (1.5), $\alpha(u)$ grows to $+\infty$ as $u \to +\infty$,

$$\lim_{u \to +\infty} \alpha(u) = +\infty\,. \tag{2.12}$$

The evolution of $\alpha(u)$ along a characteristic is

$$\frac{d}{dt}\alpha(u(t,x(t))) = \alpha'(u)\dot{u} = \big[f(u) - (u-1)f'(u)\big]\, F\,. \tag{2.13}$$

By (2.10), we have, for all T,

$$\alpha(u(T,x(T))) \;\le\; \alpha(u(0,x(0))) + 2m_0\,. \tag{2.14}$$

By (2.12) we conclude that $u(t,x)$ remains bounded for all t,x.

(5). BV bound on u. Let $z = u_x$. Differentiating (1.1) in x, one gets

$$z_t + (f'(u)F(x;u)z)_x = (f^2(u)F(x;u))_x\,.$$

Recall κ_0 and M as the lower and upper bound for u. We define

$$\|f\|_{\mathbf{L}^\infty} \doteq \max_{\kappa_0 \le s \le M} |f(s)| \,. \tag{2.15}$$

Formally we have

$$\begin{aligned} \frac{d}{dt}\mathrm{TV}\{u\} &\le \mathrm{TV}\{f^2(u)F\} \le \|f\|^2_{\mathbf{L}^\infty} \mathrm{TV}\{F\} + 2\,\|f\|_{\mathbf{L}^\infty} f'(\kappa_0)\mathrm{TV}\{u\}\,\|F\|_{\mathbf{L}^\infty} \\ &\le C\,(1+\mathrm{TV}\{u\})\,. \end{aligned}$$

Therefore, $\mathrm{TV}\{u\}$ can grow exponentially, but remains bounded for finite time.

3 Front tracing approximate solutions

In this section we prove Theorem 1.1. The algorithm for the piecewise constant front tracing approximation is described in Section 3.1. Then we establish a priori estimates in Section 3.2. All estimates are used in Section 3.3 to achieve convergence, proving Theorem 1.1.

3.1 The algorithm

Let ε be the approximation parameter, and u^ε be the piecewise constant approximation for u that we now construct. For a given initial data $\bar{u} \in \mathcal{W}$, one can construct a piecewise constant approximation, call it $\bar{u}^\varepsilon$, such that $\bar{u}^\varepsilon \to \bar{u}$ in $\mathbf{L}^1_{loc}$, and $\bar{u}^\varepsilon \in \mathcal{W}$. The approximation could be achieved by a suitable sampling in $\bar{u}$. This will be the discrete initial data for the algorithm, i.e., $u^\varepsilon(0,x) = \bar{u}^\varepsilon(x)$. Let x_i $(i = 0, \cdots, N)$ be the points where u^ε has jumps, and write

$$u_{i+\frac{1}{2}}(t) = u^\varepsilon(t,x) \qquad \text{for } x \in [x_i, x_{i+1})\,.$$

The algorithm will result in a set of ODEs that govern the evolution of x_i and $u_{i+\frac{1}{2}}$ in t.

The approximation to the initial data must satisfy the following requirements.

- The downward jumps should be small because they are not admissible. Introduce the quantities
$$\eta_i(t) \doteq u_{i-\frac{1}{2}}(t) - u_{i+\frac{1}{2}}(t)\,, \qquad \eta(t) \doteq \max_i \eta_i(t)\,. \tag{3.1}$$
Note that $\eta(t)$ measures the size of the downward jumps at t. We require that
$$\eta(0) \le \varepsilon\,. \tag{3.2}$$
This ensures that possible initial (big) downward jumps will open up into a fan of downward jumps, each one of size $\le \varepsilon$.

- Whenever $\bar{u}$ crosses 1 with negative gradient, we will make sure that $u = 1$ is sampled. This will lead to a clean a priori $\mathbf{L}^1$ estimate.

- Denote the discrete version of the global term F^ε as
$$F^\varepsilon(t,x) \doteq F(x; u^\varepsilon) = \exp\left\{\int_x^\infty f(u^\varepsilon(t,y))\,dy\right\}\,. \tag{3.3}$$

For accuracy and convergence of F^ε, we define the quantities

$$\zeta_{i+\frac{1}{2}}(t) \doteq (x_{i+1}(t) - x_i(t)) \cdot |f(u_{i+\frac{1}{2}}(t))| \,, \qquad \zeta(t) \doteq \max_i \zeta_{i+\frac{1}{2}}(t)\,, \tag{3.4}$$

and we require

$$\zeta(0) \leq \varepsilon\,. \tag{3.5}$$

Then at $t=0$, F^ε satisfies

$$e^{-\varepsilon} \;\leq\; \frac{F^\varepsilon(0,x_{i+1})}{F^\varepsilon(0,x_i)} \;\leq\; e^{\varepsilon}\,. \tag{3.6}$$

Therefore, as $\varepsilon \to 0^+$, we have the convergences of F^ε at initial time $t=0$

$$F^\varepsilon \to F\,, \qquad F^\varepsilon_x \to F_x\,, \qquad \text{in } \mathbf{L}^1_{loc}\,. \tag{3.7}$$

Now we describe the algorithm. The jumps, either upward or downward, will all travel with Rankine-Hugoniot speed

$$\dot{x}_i = F^\varepsilon(x_i)\frac{f(u_{i+\frac{1}{2}}) - f(u_{i-\frac{1}{2}})}{u_{i+\frac{1}{2}} - u_{i-\frac{1}{2}}}\,, \tag{3.8}$$

and $u_{i+\frac{1}{2}}(t)$ evolves as

$$\dot{u}_{i+\frac{1}{2}} = -f(u_{i+\frac{1}{2}})\frac{F^\varepsilon(x_{i+1}) - F^\varepsilon(x_i)}{x_{i+1}-x_i}\,. \tag{3.9}$$

The logistics of the choice of $\dot{u}_{i+\frac{1}{2}}$ in (3.9) is as follows. In order to keep u^ε constant on the interval $[x_i, x_{i+1})$, u^ε_t must be piecewise constant. This leads to a piecewise constant approximation for F^ε_x, by a finite difference of the form

$$F^\varepsilon_x(x) \;\approx\; \frac{F^\varepsilon(x_{i+1}) - F^\varepsilon(x_i)}{x_{i+1}-x_i}\,, \qquad x \in [x_i, x_{i+1})\,. \tag{3.10}$$

Since F^ε is smooth on the interval, by the Mean Value Theorem we have

$$\frac{F^\varepsilon(x_{i+1}) - F^\varepsilon(x_i)}{x_{i+1}-x_i} = -f(u_{i+\frac{1}{2}})F^\varepsilon(\tilde{x}_{i+\frac{1}{2}})\,, \qquad \text{for some} \quad \tilde{x}_{i+\frac{1}{2}} \in (x_i, x_{i+1})\,. \tag{3.11}$$

This leads to

$$\dot{u}_{i+\frac{1}{2}} = f^2(u_{i+\frac{1}{2}})F^\varepsilon(\tilde{x}_{i+\frac{1}{2}})\,, \qquad \tilde{x}_{i+\frac{1}{2}} \in (x_i, x_{i+1})\,. \tag{3.12}$$

In the end, the piecewise constant approximate solution u^ε satisfies the approximate equation

$$u^\varepsilon_t + (f(u^\varepsilon)\bar{F}^\varepsilon)_x = 0\,, \tag{3.13}$$

where for every given t, $\bar{F}^\varepsilon$ is a linear interpolation of F^ε in x through nodal points, i.e.,

$$\bar{F}^\varepsilon(x) \doteq F^\varepsilon(x_i)\frac{x_{i+1}-x}{x_{i+1}-x_i} + F^\varepsilon(x_{i+1})\frac{x-x_i}{x_{i+1}-x_i}\,, \qquad \text{for } x \in [x_i, x_{i+1}]\,. \tag{3.14}$$

Merging of nodal points. Since the fronts travel with different speeds, nearby fronts could approach each other as they travel. If this happens, we will let all the approaching nodal points merge into one, and rearrange the indices. The new front will then travel with new Rankine-Hugoniot speed defined in (3.8). Total number of fronts will decrease in time.

We observe that two non-admissible (downward) jumps would never approach each other. If two nearby jumps approach, say $x_i(t) = x_{i+1}(t)$, then one of them must be an upward jump, and we must have

$$\dot{x}_i(t) \geq \dot{x}_{i+1}(t)\,, \qquad \Rightarrow \quad u_{i-\frac{1}{2}}(t) \leq u_{i+1+\frac{1}{2}}(t)\,,$$

so the out-coming jump must be admissible. If more than two jumps merge, say $x_i(t) = \cdots = x_j(t)$ with $i < j-1$, then between each two non-admissible jumps there must be at least one admissible jump. We can pair each non-admissible jump with a neighboring admissible jump, possibly leaving the last jump at x_j unpaired. By the discussion above, each pair must result in an upward jump, reducing the size of the possible non-admissible jump at x_j. As a result, the maximum size of downward jumps $\eta(t)$ would not increase at merging time.

3.2 A priori estimates

We define a discrete version of the maximal backward characteristic, in the same spirit as [12].

Definition 3.1 *For every point $(\bar{t}, \bar{x})$, the <u>discrete maximal backward characteristics</u> $[0, \bar{t}] \ni t \mapsto x(t)$ is a continuous curve that satisfies $x(\bar{t}) = \bar{x}$ and the following.*

- *(c1) If $u^\varepsilon(t,x)$ is continuous at (t,x), and $x \in (x_i, x_{i+1})$ then $\dot{x} = f'(u_{i+\frac{1}{2}})\bar{F}^\varepsilon(x)$, where $\bar{F}^\varepsilon(x)$ is defined as (3.14).*
- *(c2) If $x = x_i$, and u^ε has an admissible (upward) jump at x_i, then $\dot{x} = f'(u_{i+\frac{1}{2}})\bar{F}^\varepsilon(x)$.*
- *(c3) If $x = x_i$ and u^ε has a non-admissible (downward) jump at x_i, then $\dot{x} = \dot{x}_i$. This means, the backward characteristic will follow the nodal point as it goes backward.*
- *(c4) If two or several nodal points merge at (t,x), say $(x_{i-k}, \cdots, x_i)$ merge, then it depends only on the jump at x_i: If it is admissible, we follow (c2); If it is not admissible, then we follow (c3).*

Remark. Since nodal points can only merge in the algorithm, non-admissible jumps can be traced back to $t = 0$. Therefore, such backward characteristic is well defined, and it never crosses any nodal points (though it can join a non-admissible jump).

All the a priori estimates are summarized in the next Lemma.

Lemma 3.1 *Let u^ε be the piecewise constant function generated by the algorithm with initial data $\bar{u}^\varepsilon \in \mathcal{W}$ that satisfies (3.2) and (3.5). Then, for any $t \in [0,T]$, we have $x \to u^\varepsilon(t,x) \in \mathcal{W}$. For ε sufficiently small we have*

$$\eta(t) \leq C\varepsilon\,, \qquad \zeta(t) \leq C\varepsilon\,, \tag{3.15}$$

for some constant C indenpendent of ε.

Proof: *(1). Lower bound for u^ε.* By (3.12) we clearly have $\dot u_{i+\frac12} \ge 0$. The lower bound follows.

(2). Bound on $\|u^\varepsilon - 1\|_{\mathbf{L}^1}$. This follows from the facts that all jumps travel with Rankine-Hugoniot speed and $u = 1$ is always sampled when u^ε crosses 1 with negative slope. In more detail, since u^ε is piecewise constant, we have

$$\|u^\varepsilon - 1\|_{\mathbf{L}^1} = \sum_i \left| u_{i+\frac12} - 1 \right| (x_{i+1} - x_i)\,.$$

A direct computation gives (by using summation-by-parts)

$$\frac{d}{dt}\|u^\varepsilon - 1\|_{\mathbf{L}^1} = \sum_i \mathrm{sign}(u_{i+\frac12} - 1)\dot u_{i+\frac12}(x_{i+1} - x_i) + \left| u_{i+\frac12} - 1\right| (\dot x_{i+1} - \dot x_i) = \sum_i F^\varepsilon(x_i) I_i\,,$$

where

$$I_i = \left| f(u_{i+\frac12})\right| - \left| f(u_{i-\frac12})\right| + \frac{f(u_{i-\frac12}) - f(u_{i+\frac12})}{u_{i-\frac12} - u_{i+\frac12}} \left(\left| u_{i-\frac12} - 1\right| - \left| u_{i+\frac12} - 1 \right| \right).$$

There are several situations.

- If $\mathrm{sign}(u_{i-\frac12} - 1) = \mathrm{sign}(u_{i+\frac12} - 1)$, then $I_i = 0$;
- If $\mathrm{sign}(u_{i-\frac12} - 1) \ne \mathrm{sign}(u_{i+\frac12} - 1)$ and $u_{i-\frac12} \le 1 \le u_{i+\frac12}$, then by concavity of f we have

$$\left| f(u_{i+\frac12})\right| \le \frac{f(u_{i-\frac12}) - f(u_{i+\frac12})}{u_{i-\frac12} - u_{i+\frac12}} \left| u_{i+\frac12} - 1\right|, \qquad \left| f(u_{i-\frac12})\right| \ge \frac{f(u_{i-\frac12}) - f(u_{i+\frac12})}{u_{i-\frac12} - u_{i+\frac12}} \left| u_{i-\frac12} - 1\right|.$$

  Therefore $I_i \le 0$.
- If $\mathrm{sign}(u_{i-\frac12} - 1) \ne \mathrm{sign}(u_{i+\frac12} - 1)$ and $u_{i-\frac12} \ge 1 \ge u_{i+\frac12}$, then by construction we must have either $u_{i+\frac12} = 1$ or $u_{i-\frac12} = 1$. In either case we have $I_i = 0$.

In conclusion, we have for all $t \ge 0$,

$$\frac{d}{dt}\|u^\varepsilon(t,\cdot) - 1\|_{\mathbf{L}^1} \le 0\,, \qquad \Rightarrow \qquad \|u^\varepsilon(t,\cdot) - 1\|_{\mathbf{L}^1} \le \|u^\varepsilon(0,\cdot) - 1\|_{\mathbf{L}^1} \le m_0\,. \tag{3.16}$$

Now, by Lemma 2.1, $x \to f(u^\varepsilon)$ is absolutely integrable, and the global function F^ε satisfies

$$e^{-C} \le F^\varepsilon \le e^C\,, \qquad \mathrm{TV}\{F^\varepsilon\} \le Ce^C\,, \qquad \text{where} \quad C = \|f(u^\varepsilon(t,\cdot))\|_{\mathbf{L}^1}\,. \tag{3.17}$$

(3). Bound on the support for $u^\varepsilon - 1$. This is obvious since the nodal speeds for the first and last points are bounded, thanks to the lower bound on u^ε.

(4). Upper bound for u^ε and bounds on η and ζ. These bounds will be established together. First, we consider the upper bound for u^ε. Rewrite (3.13) as

$$(u^\varepsilon - 1)_t + (f(u^\varepsilon)\bar F^\varepsilon)_x = 0\,. \tag{3.18}$$

Consider a point (t,x) and let $t \to x(t)$ be the discrete maximal backward characteristic through it; let i be the index for the interval $[x_i, x_{i+1}]$, possibly depending on t, where the characteristic remains. Integrate the conservation law (3.18) over the region in (t,y) where $0 \le t \le T$, $y \le x(t)$ and get the estimate

$$
\begin{aligned}
&\left| \int_0^T (u_{i+\frac{1}{2}}(t) - 1)\dot{x}(t) - f(u_{i+\frac{1}{2}}(t))\bar{F}^\varepsilon(t, x(t))\, dt \right| \\
&\quad = \left| \int_{-\infty}^{x(T)} (u^\varepsilon(T,y) - 1)\, dy - \int_{-\infty}^{x(0)} (u^\varepsilon(0,y) - 1)\, dy \right| \ \le\ 2m_0\,, \qquad (3.19)
\end{aligned}
$$

thanks to the bound on $\|u^\varepsilon - 1\|_{\mathbf{L}^1}$.

If the characteristic does not join a downward jump on some interval $[t_1, t_2]$, i.e., if $x_i < x < x_{i+1}$, then $\dot{x} = f'(u_{i+\frac{1}{2}})\bar{F}^\varepsilon$ and by (3.19) we have

$$
\left| \int_{t_1}^{t_2} \left[(u_{i+\frac{1}{2}}(t) - 1) f'(u_{i+\frac{1}{2}}(t)) - f(u_{i+\frac{1}{2}}(t)) \right] \bar{F}^\varepsilon(t, x(t))\, dt \right| \ \le\ 2m_0\,, \qquad (3.20)
$$

uniformly in $0 \le t_1 \le t_2 \le T$. Recalling the auxiliary function $\alpha(u)$ in (2.11), we have

$$
\frac{d}{dt}\alpha(u_{i+\frac{1}{2}}(t)) = \alpha'(u_{i+\frac{1}{2}})\dot{u}_{i+\frac{1}{2}} = \left[(u_{i+\frac{1}{2}} - 1) f'(u_{i+\frac{1}{2}}) - f(u_{i+\frac{1}{2}}) \right] F^\varepsilon(\tilde{x}_{i+\frac{1}{2}})\,, \qquad (3.21)
$$

where $\tilde{x}_{i+\frac{1}{2}}$ is defined in (3.11). Thanks to (3.20) and the upper and lower bounds on F^ε, $\frac{d}{dt}\alpha(u_{i+\frac{1}{2}}(t))$ is integrable along $x(t)$.

If $x(t)$ joins a non-admissible jump, the situation is slightly different. Consider the case that $x(t)$ joins x_i for t in some $[t_1, t_2]$ (the case it joins x_{i+1} is completely similar). By (3.19) we have

$$
\left| \int_{t_1}^{t_2} \left[(u_{i+\frac{1}{2}}(t) - 1) \frac{f(u_{i+\frac{1}{2}}(t)) - f(u_{i-\frac{1}{2}}(t))}{u_{i+\frac{1}{2}}(t) - u_{i-\frac{1}{2}}(t)} - f(u_{i+\frac{1}{2}}(t)) \right] F^\varepsilon(x_i(t))\, dt \right| \ \le\ 2m_0\,. \qquad (3.22)
$$

The evolution of $\alpha(u_{i+\frac{1}{2}})$ along $x(t)$ is

$$
\frac{d}{dt}\alpha(u_{i+\frac{1}{2}}(t)) \ = \ \left[(u_{i+\frac{1}{2}} - 1) f'(u_{i+\frac{1}{2}}) - f(u_{i+\frac{1}{2}}) \right] F^\varepsilon(\tilde{x}_{i+\frac{1}{2}}) \ = \ I_1 + I_2
$$

where

$$
I_1(t) \ = \ \left[(u_{i+\frac{1}{2}} - 1) \frac{f(u_{i+\frac{1}{2}}) - f(u_{i-\frac{1}{2}})}{u_{i+\frac{1}{2}} - u_{i-\frac{1}{2}}} - f(u_{i+\frac{1}{2}}) \right] F^\varepsilon(\tilde{x}_{i+\frac{1}{2}})\,, \qquad (3.23)
$$

$$
I_2(t) \ \doteq \ (u_{i+\frac{1}{2}} - 1) \left[f'(u_{i+\frac{1}{2}}) - \frac{f(u_{i+\frac{1}{2}}) - f(u_{i-\frac{1}{2}})}{u_{i+\frac{1}{2}} - u_{i-\frac{1}{2}}} \right] F^\varepsilon(\tilde{x}_{i+\frac{1}{2}})\,. \qquad (3.24)
$$

The term I_1 is integrable along $x(t)$ thanks to (3.22) and the bounds on F^ε. For I_2, we have

$$
|I_2(t)| \le \left| u_{i+\frac{1}{2}} - 1 \right| \cdot \|F^\varepsilon\|_{\mathbf{L}^\infty}\, \eta(t) \sup_{u_{i+\frac{1}{2}} \le s \le u_{i-\frac{1}{2}}} \left| f''(s) \right|\,. \qquad (3.25)
$$

To control the possible growth in $\alpha(u_{i+\frac{1}{2}})$ caused by I_2, let M_1 be the upper bound for u^ε where we only consider the growth in α caused by I_1 or by (3.21). Notice that M_1 depends only on the initial data and properties of f.

Let $\tilde{t}$ be the first time in $[0,T]$ that $\alpha(u_{i+\frac{1}{2}}) = \alpha(M_1+1)$. Hence we have that $\tilde{t}$ belongs to an interval $[t_1,t_2]$ where $x(t)=x_i$ and $u_{i+\frac{1}{2}} < M_1+1$ for $t<\tilde{t}$. Moreover we have

$$\alpha(M_1) \;\geq\; \alpha(u_{i+\frac{1}{2}}(t_1)) + \int_{t_1}^{\tilde{t}} I_1(t)\,dt\,. \tag{3.26}$$

Before we proceed, we need to establish the estimates for η and ζ for $t<\tilde{t}$. We have

$$\begin{aligned}
\dot{\eta}_i(t) &= \dot{u}_{i-\frac{1}{2}}(t) - \dot{u}_{i+\frac{1}{2}}(t) \;=\; f^2(u_{i-\frac{1}{2}})F^\varepsilon(\tilde{x}_{i-\frac{1}{2}}) - f^2(u_{i+\frac{1}{2}})F^\varepsilon(\tilde{x}_{i+\frac{1}{2}}) \\
&= \left(f^2(u_{i-\frac{1}{2}}) - f^2(u_{i+\frac{1}{2}})\right) F^\varepsilon(\tilde{x}_{i-\frac{1}{2}}) + f^2(u_{i+\frac{1}{2}}) \left(F^\varepsilon(\tilde{x}_{i-\frac{1}{2}}) - F^\varepsilon(\tilde{x}_{i+\frac{1}{2}})\right) \\
&\leq C_1(M_1+1)\eta + C_2(M_1+1)^2\zeta\,.
\end{aligned} \tag{3.27}$$

For $\zeta(t)$, we have

$$\begin{aligned}
\dot{\zeta}_{i+\frac{1}{2}}(t) &= (\dot{x}_{i+1}-\dot{x}_i)\left|f(u_{i+\frac{1}{2}})\right| + (x_{i+1}-x_i)\mathrm{sign}(f(u_{i+\frac{1}{2}}))f'(u_{i+\frac{1}{2}})\dot{u}_{i+\frac{1}{2}} \\
&= (\dot{x}_{i+1}-\dot{x}_i)\left|f(u_{i+\frac{1}{2}})\right| - f'(u_{i+\frac{1}{2}})\left|f(u_{i+\frac{1}{2}})\right|(F^\varepsilon(t,x_{i+1}) - F^\varepsilon(t,x_i)) \\
&= \left|f(u_{i+\frac{1}{2}})\right| \cdot \left\{\left[\dot{x}_{i+1} - f'(u_{i+\frac{1}{2}})F^\varepsilon(t,x_{i+1})\right] - \left[\dot{x}_i - f'(u_{i+\frac{1}{2}})F^\varepsilon(t,x_i)\right]\right\}\,.
\end{aligned}$$

These two terms are negative if the jumps at x_i, x_{i+1} are upward (admissible). If one of the jumps is downward, say $u_{i+\frac{1}{2}} \leq u_{i-\frac{1}{2}}$, then we have

$$f'(u_{i+\frac{1}{2}})F^\varepsilon(t,x_i) - \dot{x}_i \leq \|F^\varepsilon\|_{\mathbf{L}^\infty} \sup_{1\leq s\leq M_1+1}\left|f''(s)\right|(u_{i-\frac{1}{2}} - u_{i+\frac{1}{2}}) \leq C(M_1)\eta\,.$$

The case of a downward jump at x_{i+1} is completely similar. In conclusion, we have

$$\dot{\zeta}_{i+\frac{1}{2}} \leq C(M_1)(M_1+1)\eta\,. \tag{3.28}$$

Taking supreme over all i in (3.27) and (3.28), we get

$$\dot{\eta} \leq C_1(M_1+1)\eta + C_2(M_1+1)^2\zeta, \qquad \dot{\zeta} \leq C(M_1)(M_1+1)\eta\,. \tag{3.29}$$

Notice that ζ is continuous when nodal points merge, while η may be discontinuous but it does not increase. Hence by standard comparison argument we arrive at

$$\eta(t) \leq C_3\varepsilon, \qquad \zeta(t) \leq C_3\varepsilon\,, \qquad \text{for all} \quad t\in[0,\tilde{t}\,] \tag{3.30}$$

for some $C_3 = C_3(M_1,T)$.

We now go back to the estimate on α. Using (3.26), we have

$$\alpha(M_1+1) \;=\; \alpha(t_1) + \int_{t_1}^{\tilde{t}} I_1(t)\,dt + \int_{t_1}^{\tilde{t}} I_2(t)\,dt \;\leq\; \alpha(M_1) + \int_{t_1}^{\tilde{t}} I_2(t)\,dt\,.$$

By the using (3.30), we have an estimate for the growth for $\alpha(u_{i+\frac{1}{2}})$ caused by I_2:

$$\int_{t_1}^{\tilde{t}} I_2(\tau)\,d\tau \;\; \leq \;\; (\tilde{t}-t_1)M_1 \left\|F^\varepsilon\right\|_{\mathbf{L}^\infty} \sup_{1\leq s\leq M_1+1} \left|f''(s)\right| \eta \; \leq \; (\tilde{t}-t_1)\left\|F^\varepsilon\right\|_{\mathbf{L}^\infty} C_4\,\varepsilon$$

for some $C_4 = C_4(M_1, T)$. Therefore

$$\alpha(M_1+1)-\alpha(M_1) \;\leq\; (\tilde{t}-t_1)\left\|F^\varepsilon\right\|_{\mathbf{L}^\infty} C_4\,\varepsilon\,,$$

which gives

$$\tilde{t}-t_1 \;\geq\; \frac{\alpha(M_1+1)-\alpha(M_1)}{C_4\left\|F^\varepsilon\right\|_{\mathbf{L}^\infty}}\cdot\frac{1}{\varepsilon} \;=\; \frac{\tilde{C}}{\varepsilon}\,.$$

By choosing ε small, $\tilde{t}$ can be arbitrarily large, leading to the upper bound for u^ε for any finite T. In turn, this gives the uniform bounds in (3.15) on η and ζ.

(5). BV bound for u^ε. For piecewise constant function u^ε we have

$$\begin{aligned}
\frac{d}{dt}\mathrm{TV}\{u^\varepsilon\} &= \frac{d}{dt}\sum_i\left|u_{i+\frac{1}{2}}-u_{i-\frac{1}{2}}\right| \;=\; \sum_i \mathrm{sign}(u_{i+\frac{1}{2}}-v_{i+\frac{1}{2}})\left[\dot{u}_{i+\frac{1}{2}}-\dot{u}_{i-\frac{1}{2}}\right]\\
&= \sum_i \mathrm{sign}(u_{i+\frac{1}{2}}-v_{i+\frac{1}{2}})\left[f^2(u_{i+\frac{1}{2}})F^\varepsilon(\tilde{x}_{i+\frac{1}{2}})-f^2(u_{i-\frac{1}{2}})F^\varepsilon(\tilde{x}_{i-\frac{1}{2}})\right]\\
&\leq \sum_i\left|f^2(u_{i+\frac{1}{2}})-f^2(u_{i-\frac{1}{2}})\right|F^\varepsilon(\tilde{x}_{i+\frac{1}{2}})+f^2(u_{i-\frac{1}{2}})\left|F^\varepsilon(\tilde{x}_{i+\frac{1}{2}})-F^\varepsilon(\tilde{x}_{i-\frac{1}{2}})\right|\\
&\leq 2\left\|f\right\|_{\mathbf{L}^\infty} f'(\kappa_0)\|F^\varepsilon\|_{\mathbf{L}^\infty}\mathrm{TV}\{u^\varepsilon\}+\|f\|^2_{\mathbf{L}^\infty}\mathrm{TV}\{F^\varepsilon\}\\
&\leq C\cdot\mathrm{TV}\{u^\varepsilon\}+C\,.
\end{aligned}$$

Therefore, total variation of u^ε can grow exponentially in time, but remains bounded for finite time $t \leq T$, completing the proof. □

Remark 1. By Lemma 3.1 and the fact that nodal points can only be cancelled, the set of ODEs for $x_i(t)$ in (3.8) and for $u_{i+\frac{1}{2}}(t)$ in (3.9) are well-posed, generating unique approximate solutions.

Remark 2. The $\mathbf{L}^1$ Continuity in time for u^ε and F^ε follows by a standard argument, as a consequence of the a priori bounds in Lemma 3.1. We omit the details.

In next Lemma we establish the discrete version of the entropy inequality.

Lemma 3.2 *A discrete version of a one-sided entropy inequality holds for u^ε,*

$$u^\varepsilon(t,x)-u^\varepsilon(t,y)\leq C\max\{1/t,\,1\}(y-x)+C\varepsilon\,,\qquad (x<y)\,. \tag{3.31}$$

Proof. For a given $t > 0$, consider two points $x < y$. Let $t \to x(t)$ and $t \to y(t)$ be the discrete maximal backward characteristics through them (resp.), and let i and j be the indices of the interval where the characteristics remain, respectively. If x and y are very close to each other, say $j - i \leq 10$, then by (3.15) we have

$$u^\varepsilon(t,x)-u^\varepsilon(t,y)\leq C\varepsilon\,. \tag{3.32}$$

Now consider $j - i >> 10$. Define an auxiliary function

$$H(t) \doteq \frac{u(x(t)) - u(y(t))}{x(t) - y(t)} = \frac{u_{i+\frac{1}{2}}(t) - u_{j+\frac{1}{2}}(t)}{x(t) - y(t)}. \tag{3.33}$$

The evolution of H as x and y move along the maximal backward characteristics is

$$H'(t) = \frac{\dot{u}_{i+\frac{1}{2}} - \dot{u}_{j+\frac{1}{2}}}{x - y} - H \cdot \frac{\dot{x} - \dot{y}}{x - y}. \tag{3.34}$$

Let's estimate each term. By using (3.12) and the a priori bounds in Lemma 3.1 we have

$$\begin{aligned}
\frac{\dot{u}_{i+\frac{1}{2}} - \dot{u}_{j+\frac{1}{2}}}{x - y} &= F^\varepsilon(\tilde{x}_{i+\frac{1}{2}}) \cdot \frac{f^2(u_{i+\frac{1}{2}}) - f^2(u_{j+\frac{1}{2}})}{x - y} + f^2(u_{j+\frac{1}{2}}) \cdot \frac{F^\varepsilon(\tilde{x}_{i+\frac{1}{2}}) - F^\varepsilon(\tilde{x}_{j+\frac{1}{2}})}{x - y} \\
&= \mathcal{O}(1) H + \mathcal{O}(1).
\end{aligned}$$

Here and in the rest of this proof, the notation $\mathcal{O}(1)$ denotes some uniformly bounded value that can be both positive or negative. For the second term we have

$$\begin{aligned}
\frac{\dot{x} - \dot{y}}{x - y} &= \bar{F}^\varepsilon(x) \cdot \frac{f'(u_{i+\frac{1}{2}}) - f'(u_{j+\frac{1}{2}})}{x - y} + f'(u_{j+\frac{1}{2}}) \cdot \frac{\bar{F}^\varepsilon(x) - \bar{F}^\varepsilon(y)}{x - y} + \frac{\mathcal{O}(1)\varepsilon}{x - y} \\
&= -\tilde{c} H + \mathcal{O}(1),
\end{aligned} \tag{3.35}$$

with

$$\tilde{c} = -\bar{F}^\varepsilon(x) f''(\tilde{u}) > 0, \qquad \text{for some bounded } \tilde{u}.$$

Note that the term $\varepsilon/(x - y)$ is of $\mathcal{O}(1)$ for $t \geq \varepsilon$. Putting these back into (3.34), we get

$$H'(t) = \tilde{c} H^2 + \mathcal{O}(1) H + \mathcal{O}(1). \tag{3.36}$$

For large values of H, the first term dominates, and H can blow up to $+\infty$ in finite time. By a standard comparison argument, we have

$$H(t) \geq -C \max\{1/t, 1\}, \quad \Rightarrow \quad u^\varepsilon(t, x) - u^\varepsilon(t, y) \leq C \max\{1/t, 1\}(y - x). \tag{3.37}$$

Combining (3.37) with (3.32), we achieve (3.31), completing the proof. □

3.3 Convergence of the approximate solutions and existence of entropy weak solutions

Since all nodal points x_i travels with the Rankine-Hugoniot speed, our piecewise constant function u^ε provides weak solutions to the modified conservation law (3.13). Rewrite it as

$$u_t^\varepsilon + (f(u^\varepsilon) F^\varepsilon)_x = E^\varepsilon, \qquad \text{where} \quad E^\varepsilon(t, x) \doteq \left[f(u^\varepsilon)(F^\varepsilon - \bar{F}^\varepsilon)\right]_x. \tag{3.38}$$

The following discrete weak formulation holds for all test functions $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^2)$,

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}} \left(u^\varepsilon \phi_t + f(u^\varepsilon) F^\varepsilon \phi_x\right) dx\, dt \\
&= \int_{\mathbb{R}} \Big(u^\varepsilon(T, x)\phi(T, x) - u^\varepsilon(0, x)\phi(0, x)\Big)\, dx - \int_0^T \int_{\mathbb{R}} E^\varepsilon \phi \, dx\, dt.
\end{aligned} \tag{3.39}$$

To achieve existence of weak solutions, we observe that, thanks to the a priori estimates in Lemma 3.1, there exist some limit functions $u(t, x)$ and $F(t, x)$ such that, by extracting a subsequence $\varepsilon \to 0$, one has

(i) $u^\varepsilon(0,\cdot)\to u(0,\cdot)$ and $u^\varepsilon(T,\cdot)\to u(T,\cdot)$ in $\mathbf{L}^1_{loc}(\mathbb{R})$;

(ii) $u^\varepsilon \to u$ and $F^\varepsilon \to F$ in $\mathbf{L}^1_{loc}([0,T]\times\mathbb{R})$;

(iii) For any given $a<b$, one has

$$\int_a^b |E^\varepsilon(t,x)|\,dx \le \mathrm{TV}\{f(u^\varepsilon)\}\left\|F^\varepsilon - \bar F^\varepsilon\right\|_{\mathbf{L}^\infty} + \|f\|_{\mathbf{L}^\infty}\mathrm{TV}\{F^\varepsilon - \bar F^\varepsilon\}\,.$$

Since $\bar F^\varepsilon$ is a linear interpolation of F^ε through nodal points, by using the estimates in Lemma 3.1 one has

$$\left\|F^\varepsilon - \bar F^\varepsilon\right\|_{\mathbf{L}^\infty} \le C\varepsilon\,,$$

and

$$\begin{aligned}\mathrm{TV}\{F^\varepsilon - \bar F^\varepsilon\} \;&=\; \left\|(F^\varepsilon - \bar F^\varepsilon)_x\right\|_{\mathbf{L}^1} \;=\; \sum_i \int_{x_i}^{x_{i+1}} \left|f(u_{i+\frac12})\right|\left|F^\varepsilon(x) - F^\varepsilon(\tilde x_{i+\frac12})\right|\,dx \\ &\le\; \sum_i \left|f(u_{i+\frac12})\right|(x_{i+1}-x_i)\mathrm{TV}\{F^\varepsilon;(x_i,x_{i+1})\} \;\le\; \zeta(t)\mathrm{TV}\{F^\varepsilon(t,\cdot)\} \;\le\; C\varepsilon\,.\end{aligned}$$

Therefore $\int_a^b |E^\varepsilon(t,x)|\,dx \to 0$ uniformly for $t\in[0,T]$.

(iv) Since u^ε and $f(u^\varepsilon)$ are uniformly bounded, the identity (3.3) holds in the limit, i.e.,

$$F(t,x) = \exp\left\{\int_x^\infty f(u(t,y))\,dy\right\} \qquad \text{a.e. } (t,x)\in[0,T]\times\mathbb{R}\,.$$

Furthermore, by taking the limit $\varepsilon\to 0$ in (3.31), the entropy inequality holds. The existence of entropy weak solutions follows, proving Theorem 1.1.

4 Continuous dependence on initial data and erosion function

In this section we prove Theorem 1.2. Introducing the notation

$$G(v;x) \doteq \exp\left\{\int_x^\infty g(v(t,y))\,dy\right\}\,, \tag{4.1}$$

we can write

$$v_t + (gG)_x = 0\,. \tag{4.2}$$

Let $u^\varepsilon, v^\varepsilon$ be the piecewise constant approximations to u, v, respectively, generated by our algorithm. Let x_i $(i=0,\cdots N)$ be the points where either u^ε or v^ε has a jump. We have

$$\|u^\varepsilon(t,\cdot) - v^\varepsilon(t,\cdot)\|_{\mathbf{L}^1} = \sum \left|u_{i+\frac12}(t) - v_{i+\frac12}(t)\right|(x_{i+1}(t) - x_i(t))\,. \tag{4.3}$$

Here and in the rest the summation $\sum$ is always over i. Differentiating (4.3) in t, we have

$$\frac{d}{dt}\|u^\varepsilon(t,\cdot) - v^\varepsilon(t,\cdot)\|_{\mathbf{L}^1} = A + B \tag{4.4}$$

where

$$A \doteq \sum \operatorname{sign}(u_{i+\frac{1}{2}} - v_{i+\frac{1}{2}})(\dot{u}_{i+\frac{1}{2}} - \dot{v}_{i+\frac{1}{2}})(x_{i+1} - x_i)\,, \tag{4.5}$$

$$B \doteq \sum \left|u_{i+\frac{1}{2}} - v_{i+\frac{1}{2}}\right| (\dot{x}_{i+1} - \dot{x}_i)\,. \tag{4.6}$$

Consider term A. Recalling $\dot{u}_{i+\frac{1}{2}}$ in (3.9), namely

$$\dot{u}_{i+\frac{1}{2}} = -f(u_{i+\frac{1}{2}})\Phi_{i+\frac{1}{2}}\,, \qquad \Phi_{i+\frac{1}{2}} \doteq \frac{F^\varepsilon(x_{i+1+m}) - F^\varepsilon(x_{i-n})}{x_{i+1+m} - x_{i-n}}\,, \tag{4.7}$$

where x_{i-n}, x_{i+1+m} are the two nearby points where u^ε has jumps, and for $\dot{v}_{i+\frac{1}{2}}$,

$$\dot{v}_{i+\frac{1}{2}} = -g(v_{i+\frac{1}{2}})\Psi_{i+\frac{1}{2}}\,, \qquad \Psi_{i+\frac{1}{2}} \doteq \frac{G^\varepsilon(x_{i+1+l}) - G^\varepsilon(x_{i-k})}{x_{i+1+l} - x_{i-k}}\,, \tag{4.8}$$

where x_{i-k}, x_{i+1+l} are the two nearby points where v^ε has jumps. Then

$$\dot{u}_{i+\frac{1}{2}} - \dot{v}_{i+\frac{1}{2}} = -(f(u_{i+\frac{1}{2}}) - f(v_{i+\frac{1}{2}}))\Phi_{i+\frac{1}{2}} - (f(v_{i+\frac{1}{2}}) - g(v_{i+\frac{1}{2}}))\Phi_{i+\frac{1}{2}} - g(v_{i+\frac{1}{2}})(\Phi_{i+\frac{1}{2}} - \Psi_{i+\frac{1}{2}})\,.$$

We can write

$$A = A_1 + A_2 + A_3$$

where

$$A_1 \doteq -\sum \left|f(u_{i+\frac{1}{2}}) - f(v_{i+\frac{1}{2}})\right| \Phi_{i+\frac{1}{2}}(x_{i+1} - x_i)\,, \tag{4.9}$$

$$A_2 \doteq -\sum \operatorname{sign}(u_{i+\frac{1}{2}} - v_{i+\frac{1}{2}})(f(v_{i+\frac{1}{2}}) - g(v_{i+\frac{1}{2}}))\Phi_{i+\frac{1}{2}}(x_{i+1} - x_i)\,, \tag{4.10}$$

$$A_3 \doteq -\sum \operatorname{sign}(u_{i+\frac{1}{2}} - v_{i+\frac{1}{2}})g(v_{i+\frac{1}{2}})(\Phi_{i+\frac{1}{2}} - \Psi_{i+\frac{1}{2}})(x_{i+1} - x_i)\,. \tag{4.11}$$

Note that $\Phi_{i+\frac{1}{2}}$ and $\Psi_{i+\frac{1}{2}}$ are approximations to F^ε_x and G^ε_x respectively on the interval $[x_i, x_{i+1})$. By our construction we have

$$\left|F^\varepsilon_x(x) - \Phi_{i+\frac{1}{2}}\right| \le C\varepsilon\,, \qquad \left|G^\varepsilon_x(x) - \Psi_{i+\frac{1}{2}}\right| \le C\varepsilon\,, \qquad \forall x \in [x_i, x_{i+1})\,. \tag{4.12}$$

We immediately have

$$A_2 \le \mathrm{TV}\{F^\varepsilon\} \cdot \|f - g\|_{\mathbf{L}^\infty} + C\varepsilon\,, \tag{4.13}$$

$$A_3 \le \|g\|_{\mathbf{L}^\infty} \mathrm{TV}\{F^\varepsilon - G^\varepsilon\} + C\varepsilon\,. \tag{4.14}$$

Now, consider the term B. By summation-by-parts, we have

$$B = \sum \left(\left|u_{i-\frac{1}{2}} - v_{i-\frac{1}{2}}\right| - \left|u_{i+\frac{1}{2}} - v_{i+\frac{1}{2}}\right|\right) \dot{x}_i\,. \tag{4.15}$$

At every x_i, we define the artificial speeds $s_i, \bar{s}_i$ as follows. If u^ε has a jumps at x_i, we let $s_i = \bar{s}_i = \dot{x}_i$. Otherwise, if v^ε has a jump at x_i, we let

$$s_i \doteq F^\varepsilon(x_i) \cdot \frac{f(v_{i+\frac{1}{2}}) - f(v_{i-\frac{1}{2}})}{v_{i+\frac{1}{2}} - v_{i-\frac{1}{2}}}\,, \qquad \bar{s}_i \doteq F^\varepsilon(x_i) \cdot \frac{g(v_{i+\frac{1}{2}}) - g(v_{i-\frac{1}{2}})}{v_{i+\frac{1}{2}} - v_{i-\frac{1}{2}}}\,. \tag{4.16}$$

Note that we use the F^ε for the global term in all these speeds. We now have

$$B = B_1 + B_2 + B_3$$

where

$$B_1 \doteq \sum s_i\left(\left|u_{i-\frac12} - v_{i-\frac12}\right| - \left|u_{i+\frac12} - v_{i+\frac12}\right|\right), \tag{4.17}$$

$$B_2 \doteq \sum(\bar{s}_i - s_i)\left(\left|u_{i-\frac12} - v_{i-\frac12}\right| - \left|u_{i+\frac12} - v_{i+\frac12}\right|\right), \tag{4.18}$$

$$B_3 \doteq \sum(\dot{x}_i - \bar{s}_i)\left(\left|u_{i-\frac12} - v_{i-\frac12}\right| - \left|u_{i+\frac12} - v_{i+\frac12}\right|\right). \tag{4.19}$$

Here in B_2, B_3 we only need to sum over all jumps in v^ε.

Now consider B_1. At every point x_i, we define λ_i^- and λ_i^+ as

$$\lambda_i^- \doteq \frac{f(u_{i-\frac12}) - f(v_{i-\frac12})}{u_{i-\frac12} - v_{i-\frac12}} F^\varepsilon(x_i), \qquad \lambda_i^+ \doteq \frac{f(u_{i+\frac12}) - f(v_{i+\frac12})}{u_{i+\frac12} - v_{i+\frac12}} F^\varepsilon(x_i). \tag{4.20}$$

The term B_1 can be written as

$$B_1 = B_{1,a} + B_{1,b},$$

where

$$B_{1,a} \doteq \sum \left|u_{i+\frac12} - v_{i+\frac12}\right| (\lambda_i^+ - s_i) - \left|u_{i-\frac12} - v_{i-\frac12}\right| (\lambda_i^- - s_i), \tag{4.21}$$

$$B_{1,b} \doteq \sum \left|u_{i-\frac12} - v_{i-\frac12}\right| \lambda_i^- - \left|u_{i+\frac12} - v_{i+\frac12}\right| \lambda_i^+. \tag{4.22}$$

Consider $B_{1,a}$ and write $B_{1,a} = \sum B_{1,a,i}$. There are various situations. Let's first consider if u^ε has a jump at x_i so $v_{i-\frac12} = v_{i+\frac12}$. There are several cases.

- If $u_{i-\frac12} \geq v_{i-\frac12}, u_{i+\frac12} \geq v_{i+\frac12}$, we have

$$B_{1,a,i} = (u_{i+\frac12} - v_{i+\frac12})\lambda_i^+ - (u_{i-\frac12} - v_{i-\frac12})\lambda_i^- - (u_{i+\frac12} - u_{i-\frac12})s_i = 0.$$

- If $u_{i-\frac12} \leq v_{i-\frac12}, u_{i+\frac12} \leq v_{i+\frac12}$, it is completely similar. We have $B_{1,a,i} = 0$.
- If $u_{i-\frac12} \leq v_{i-\frac12} = v_{i+\frac12} \leq u_{i+\frac12}$, then the jump is admissible. We have

$$\lambda_i^+ \leq s_i \leq \lambda_i^-, \qquad \text{therefore} \quad B_{1,a,i} \leq 0.$$

- If $u_{i-\frac12} \geq v_{i-\frac12} = v_{i+\frac12} \geq u_{i+\frac12}$, the jump is not admissible, therefore it is small. We have

$$\lambda_i^+ - s_i \leq C\varepsilon, \qquad s_i - \lambda_i^- \leq C\varepsilon, \qquad \text{therefore} \quad B_{1,a,i} \leq C\left|u_{i+\frac12} - u_{i-\frac12}\right|\varepsilon.$$

The cases where v^ε has a jump at x_i are completely similar. In summary, we have

$$B_{1,a} \leq C\varepsilon \sum \left|u_{i+\frac12} - u_{i-\frac12}\right| + \left|v_{i+\frac12} - v_{i-\frac12}\right| \leq C\varepsilon\left[\mathrm{TV}\{u^\varepsilon\} + \mathrm{TV}\{v^\varepsilon\}\right]. \tag{4.23}$$

For $B_{1,b}$, summation-by-parts again gives

$$B_{1,b} = \sum \left| f(u_{i+\frac{1}{2}}) - f(v_{i+\frac{1}{2}}) \right| \left(F^\varepsilon(x_{i+1}) - F^\varepsilon(x_i) \right).$$

Now, compare this with the term A_1 in (4.9). They are very close to each other in values but with opposite signs. By construction we have (similar to (4.12))

$$\left| \frac{F^\varepsilon(x_{i+1}) - F^\varepsilon(x_i)}{x_{i+1} - x_i} - \Phi_{i+\frac{1}{2}} \right| \le C\varepsilon . \tag{4.24}$$

This gives us

$$A_1 + B_{1,b} \le C\varepsilon \cdot \sum \left| f(u_{i+\frac{1}{2}}) - f(v_{i+\frac{1}{2}}) \right| (x_{i+1} - x_i) \le C\varepsilon \left[\|f(u^\varepsilon)\|_{\mathbf{L}^1} + \|f(v^\varepsilon)\|_{\mathbf{L}^1} \right] . \tag{4.25}$$

Here, $\|f(u^\varepsilon(\cdot))\|_{\mathbf{L}^1}$ and $\|f(v^\varepsilon(\cdot))\|_{\mathbf{L}^1}$ are both bounded. Now, consider the term B_2. Since we sum over all i where v^ε has a jump at x_i, we have $u_{i+\frac{1}{2}} = u_{i-\frac{1}{2}}$, therefore

$$\left| u_{i-\frac{1}{2}} - v_{i-\frac{1}{2}} \right| - \left| u_{i+\frac{1}{2}} - v_{i+\frac{1}{2}} \right| \le \left| v_{i+\frac{1}{2}} - v_{i-\frac{1}{2}} \right| . \tag{4.26}$$

And,

$$|\bar{s}_i - s_i| = F^\varepsilon(x_i) \left| \frac{(f(v_{i+\frac{1}{2}}) - g(v_{i+\frac{1}{2}})) - (f(v_{i-\frac{1}{2}}) - g(v_{i-\frac{1}{2}}))}{v_{i+\frac{1}{2}} - v_{i-\frac{1}{2}}} \right| \le \|F^\varepsilon\|_{\mathbf{L}^\infty} \|f - g\|_{Lip} .$$

Therefore,

$$B_2 \le \|F^\varepsilon\|_{\mathbf{L}^\infty} \|f - g\|_{Lip} \sum \left| v_{i+\frac{1}{2}} - v_{i-\frac{1}{2}} \right| \le \|F^\varepsilon\|_{\mathbf{L}^\infty} \|f - g\|_{Lip} \mathrm{TV}\{v^\varepsilon\} . \tag{4.27}$$

Finally, consider the term B_3. We have

$$|\dot{x}_i - \bar{s}_i| = \frac{g(v_{i+\frac{1}{2}}) - g(v_{i-\frac{1}{2}})}{v_{i+\frac{1}{2}} - v_{i-\frac{1}{2}}} \left| F^\varepsilon(x_i) - G^\varepsilon(x_i) \right| .$$

Combining this with (4.26), we get

$$B_3 \le \sum \left| g(v_{i+\frac{1}{2}}) - g(v_{i-\frac{1}{2}}) \right| \cdot |F^\varepsilon(x_i) - G^\varepsilon(x_i)| \le \|F^\varepsilon - G^\varepsilon\|_{\mathbf{L}^\infty} g'(\kappa_0) \mathrm{TV}\{v^\varepsilon\} . \tag{4.28}$$

Putting the estimates (4.13), (4.14), (4.23), (4.25), (4.27) and (4.28) back into (4.4), we get

$$\begin{aligned} \frac{d}{dt} \|u^\varepsilon - v^\varepsilon\|_{\mathbf{L}^1} \;\; \le \;\; & \|F^\varepsilon\|_{\mathbf{L}^\infty} \|f - g\|_{Lip} \mathrm{TV}\{v^\varepsilon\} \; + \; \|F^\varepsilon - G^\varepsilon\|_{\mathbf{L}^\infty} g'(\kappa_0) \mathrm{TV}\{v^\varepsilon\} \\ & + \mathrm{TV}\{F^\varepsilon\} \|f - g\|_{\mathbf{L}^\infty} \; + \; \|g\|_{\mathbf{L}^\infty} \mathrm{TV}\{F^\varepsilon - G^\varepsilon\} \; + \; C\varepsilon . \end{aligned} \tag{4.29}$$

By symmetry we also have

$$\begin{aligned} \frac{d}{dt} \|u^\varepsilon - v^\varepsilon\|_{\mathbf{L}^1} \;\; \le \;\; & \|G^\varepsilon\|_{\mathbf{L}^\infty} \|f - g\|_{Lip} \mathrm{TV}\{u^\varepsilon\} \; + \; \|F^\varepsilon - G^\varepsilon\|_{\mathbf{L}^\infty} f'(\kappa_0) \mathrm{TV}\{u^\varepsilon\} \\ & + \mathrm{TV}\{G^\varepsilon\} \|f - g\|_{\mathbf{L}^\infty} \; + \; \|f\|_{\mathbf{L}^\infty} \mathrm{TV}\{F^\varepsilon - G^\varepsilon\} \; + \; C\varepsilon . \end{aligned} \tag{4.30}$$

Now let's estimate the terms in (4.29)-(4.30). We have

$$\|f(v^\varepsilon) - g(v^\varepsilon)\|_{\mathbf{L}^1} = \|[f(v^\varepsilon) - f(1)] - [g(v^\varepsilon) - g(1)]\|_{\mathbf{L}^1} = \|v^\varepsilon - 1\|_{\mathbf{L}^1} \|f - g\|_{Lip} . \quad (4.31)$$

Note that it is important to have $f(1) = g(1) = 0$ to obtain (4.31). For $\|F^\varepsilon - G^\varepsilon\|_{\mathbf{L}^\infty}$ we have

$$\begin{aligned} \|F^\varepsilon - G^\varepsilon\|_{\mathbf{L}^\infty} &\leq \max\{\|F^\varepsilon\|_{\mathbf{L}^\infty}, \|G^\varepsilon\|_{\mathbf{L}^\infty}\} \int_{-\infty}^{\infty} |f(u^\varepsilon) - g(v^\varepsilon)| \, dy \\ &\leq \max\{\|F^\varepsilon\|_{\mathbf{L}^\infty}, \|G^\varepsilon\|_{\mathbf{L}^\infty}\} \int_{-\infty}^{\infty} |f(u^\varepsilon) - f(v^\varepsilon)| + |f(v^\varepsilon) - g(v^\varepsilon)| \, dy \\ &\leq \max\{\|F^\varepsilon\|_{\mathbf{L}^\infty}, \|G^\varepsilon\|_{\mathbf{L}^\infty}\} \left[f'(\kappa_0) \|u^\varepsilon - v^\varepsilon\|_{\mathbf{L}^1} + \|f(v^\varepsilon) - g(v^\varepsilon)\|_{\mathbf{L}^1}\right] \end{aligned} \quad (4.32)$$

and for $\mathrm{TV}\{F^\varepsilon - G^\varepsilon\}$ we have

$$\begin{aligned} \mathrm{TV}\{F^\varepsilon - G^\varepsilon\} = \|F_x^\varepsilon - G_x^\varepsilon\|_{\mathbf{L}^1} &= \int |f(u^\varepsilon(t,x))F^\varepsilon - g(v^\varepsilon(t,x))G^\varepsilon| \, dx \\ &\leq \int |f(u^\varepsilon) - g(v^\varepsilon)| \, F^\varepsilon + |g(v^\varepsilon)| \, |F^\varepsilon - G^\varepsilon| \, dx \\ &\leq \int |f(u^\varepsilon) - f(v^\varepsilon)| \, F^\varepsilon + |f(v^\varepsilon) - g(v^\varepsilon)| \, F^\varepsilon + |g(v^\varepsilon)| \, |F^\varepsilon - G^\varepsilon| \, dx \\ &\leq \|F^\varepsilon\|_{\mathbf{L}^\infty} \left[f'(\kappa_0) \|u^\varepsilon - v^\varepsilon\|_{\mathbf{L}^1} + \|f(v^\varepsilon) - g(v^\varepsilon)\|_{\mathbf{L}^1}\right] + \|g(v^\varepsilon)\|_{\mathbf{L}^1} \|F^\varepsilon - G^\varepsilon\|_{\mathbf{L}^\infty} . \end{aligned} \quad (4.33)$$

By using (4.31), (4.32) and (4.33), the estimates (4.29) and (4.30) become

$$\frac{d}{dt} \|u^\varepsilon(t,\cdot) - v^\varepsilon(t,\cdot)\|_{\mathbf{L}^1} \leq C \left[\|u^\varepsilon(t,\cdot) - v^\varepsilon(t,\cdot)\|_{\mathbf{L}^1} + \|f - g\|_{\mathbf{L}^\infty} + \|f - g\|_{Lip} + \varepsilon\right] , \quad (4.34)$$

for some bounded constant C that does not depend on ε. This gives the integral estimate

$$\begin{aligned} \|u^\varepsilon(t,\cdot) - v^\varepsilon(t,\cdot)\|_{\mathbf{L}^1} \leq\; & \|u^\varepsilon(0,\cdot) - v^\varepsilon(0,\cdot)\|_{\mathbf{L}^1} + \int_0^t \|u^\varepsilon(s,\cdot) - v^\varepsilon(s,\cdot)\|_{\mathbf{L}^1} \, ds \\ & + Ct\Big[\, \|f - g\|_{\mathbf{L}^\infty} + \|f - g\|_{Lip} + \varepsilon\Big] . \end{aligned} \quad (4.35)$$

Finally, by taking the limit $\varepsilon \to 0^+$ in (4.35), and using the fact that $u^\varepsilon \to u$ and $v^\varepsilon \to v$ in $\mathbf{L}^1_{loc}$ for a.e. t, we get (1.12), proving Theorem 1.2.

Remark. The estimates (4.29) and (4.30) are very similar to the ones in [16], Theorem 1.3, where the authors study a scalar conservation law with variable coefficients in multi space dimension

$$u_t + \nabla \cdot (k(x)f(u)) = \Delta A(u) .$$

Continuous dependence on initial data, on the coefficient k and on the flux function f is established with very similar results, by using Kruzkov inequality and a variable doubling technique. However, their coefficient $k(x)$ is local and does not depend on t.

On the other hand, the front tracing algorithm proposed here can be easily extended to conservation laws with variable coefficient in one space dimension

$$u_t + (k(t,x)f(u))_x = 0 ,$$

for k under suitable assumptions, such as in [3], Theorem 2. Existence and continuous dependence on initial data, on the coefficient k and on the function f would follow in a similar way.

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