Singular Solutions for Nonlinear Hyperbolic Systems

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Abstract

An extension of the method of weak asymptotics is presented which allows the construction of singular solutions of Riemann problems for systems of hyperbolic conservation laws. The method is based on using complex-valued approximations which become real-valued in the distributional limit. It is shown how this approach can be used to construct solutions containing combinations of classical hyperbolic shock waves and Dirac delta distributions. The method is applied to two particular systems of conservation laws in one spatial variable. First, existence of solutions for the shallow-water system is obtained for a class of initial data which includes delta distributions. Uniqueness is obtained in a smaller class of distributions which satisfy a condition of Oleinik type. As a second example, a hyperbolic system appearing in the study of magnetohydrodynamics is studied. This system was investigated in [22], and it was noticed that there exists no classical Lax-admissible solution for a particular Riemann problem. It was conjectured that this initial configuration might lead to singular solutions featuring combinations of Dirac delta distributions and shock waves. By introducing the concept of vanishing complex-valued correction in the weak asymptotic method, we are able to settle this question.

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1 Introduction

Consideration is given to existence and uniqueness of weak singular solutions of systems of hyperbolic conservation laws. In particular, initial-value problems of Riemann type in spaces of distributions including the Dirac delta distribution are under investigation. The study of such singular solutions of systems of conservation laws was initiated by Korchinski [33] and Keyfitz and Kranzer [31, 32]. In the last few years, interest in the topic has grown, and a sample of results may be found in [5, 13, 17, 22, 25, 26, 27, 34, 36, 40, 45, 46, 49]. One convenient tool for the construction of singular solutions for such systems is the method of weak asymptotics. For instance, the weak asymptotics method has been used recently to understand the evolution of nonlinear waves in scalar conservation laws as well as interaction and formation of δ-shock waves in the case of a triangular system of conservation laws [16, 13, 14]. Singular δ'-shock waves were also found as a new type of singular solution of hyperbolic systems of conservation laws [42, 44].

In the current work, an extension of the weak asymptotics method to the case where complex-valued corrections are considered for the approximate solutions is introduced and applied to systems of two conservation laws of the form

$$\left.\begin{array}{rcl} \partial_t u + \partial_x f(u,v) &=& 0, \\ \partial_t v + \partial_x g(u,v) &=& 0. \end{array}\right\} \tag{1.1}$$

Even though the imaginary parts of the solutions so constructed vanish in an appropriate limit, it appears that considering complex-valued weak asymptotic solutions significantly extends the range of possible singular solutions. This is also borne out by considering the recent definition of weak singular solutions of hyperbolic conservation laws given by Danilov and Shelkovich in [16]. Indeed, it is shown here how the complex-valued method of weak asymptotic solutions can be used to construct examples of solutions that fit in the framework of the definition given in [16].

To illustrate the power of the extension to complex-valued distributions, we apply it to two physically important systems of conservation laws. The first system to be studied is the classical shallow-water system

$$\left.\begin{array}{rcl} \partial_t u + \partial_x \left(v + \frac{u^2}{2}\right) &=& 0, \\ \partial_t v + \partial_x \left(u + uv\right) &=& 0. \end{array}\right\} \tag{1.2}$$

The physical motivation for studying this system comes from the realm of water-wave theory. The system (1.2) is a standard model in the field of hydraulics, and has been used in different forms for the study of surface waves and storm surges in rivers and channels [48]. The other important approximation in the study of water waves is the linear regime. In the context of linear models, such as the full linear water-wave problem, or the linear KdV equation, the solution emanating from Dirac delta data gives the linear propagator [43, 48]. In the nonlinear case, the KdV equation and some Boussinesq systems have been studied in the context of low-regularity solutions, in negative Sobolev classes, and in some cases allowing data singular enough to encompass the Dirac delta in one dimension [1, 6, 7, 21, 29]. On the other hand, the development of such singular initial data has not been studied for the shallow-water system.

The second system to be considered will be called the *Brio* system. It appears in the study of plasmas, and has the form

$$\begin{aligned} \partial_t u + \partial_x\big(\tfrac{u^2+v^2}{2}\big) &= 0, \\ \partial_t v + \partial_x(v(u-1)) &= 0. \end{aligned} \tag{1.3}$$

The system is derived in [4] from the classical MHD system, and it was thoroughly considered in [22]. There, it was noticed that for certain initial data no solution consisting of the Lax-admissible elementary waves (shock and rarefaction waves) exists. In [22], Riemann problems for system (1.3) are compared to Riemann problems for the system

$$\begin{aligned} \partial_t u + \partial_x\big(\tfrac{u^2}{2}\big) &= 0 \\ \partial_t v + \partial_x(v(u-1)) &= 0. \end{aligned} \tag{1.4}$$

Numerical computations of appropriate viscous profiles for (1.3) and (1.4) demonstrated surprising similarities. On the other hand, it was shown in [33], and later confirmed in numerous papers (see e.g. [14, 16, 17, 22, 24, 27, 39, 46]) that certain Riemann problems for (1.4) admit δ-shock wave solutions. The same fact could not be established for any Riemann problem corresponding to (1.3). Here, we bring our new method to bear on this problem, proving in effect that the numerical explorations in [22] provided the right intuition, and that (1.3) admits δ-shock wave solutions both in the weak asymptotic sense, and in the weak sense defined in [16]. Note that the flux functions $f(u,v)$ and $g(u,v)$ associated with the system (1.3) are nonlinear in both u and v, and it appears that the use of complex-valued approximations is essential for the construction of solutions of delta-shock type for systems nonlinear in both dependent variables.

Let us next define what we mean by complex-valued weak asymptotic solution, and highlight some methods to restrict the notion of solution with the goal of obtaining uniqueness. First we define a vanishing family of distributions.

Definition 1.1. *Let $f_\varepsilon(x) \in \mathcal{D}'(\mathbb{R})$ be a family of distributions depending on $\varepsilon \in (0,1)$, We say that $f_\varepsilon = o_{\mathcal{D}'}(1)$ if for any test function $\phi(x) \in \mathcal{D}(\mathbb{R})$, the estimate*

$$\langle f_\varepsilon, \phi \rangle = o(1), \quad \text{as} \quad \varepsilon \to 0$$

holds.

Note that the estimate on the right-hand side is understood in the usual Landau sense. Thus we may say that a family of distributions approach zero in the sense defined above if for a given test function ϕ, the pairing $\langle f_\varepsilon, \phi \rangle$ converges to zero as ε approaches zero. For families of distributions $f_\varepsilon(x,t)$, we write $f_\varepsilon = o_{\mathcal{D}'}(1) \subset \mathcal{D}'(\mathbb{R})$ if the estimate above holds uniformly in t. More succinctly, we require that

$$\langle f_\varepsilon(\cdot, t), \varphi \rangle \leq C_T g(\varepsilon) \text{ for } t \in [0,T],$$

where the function g depends on the test function $\varphi(x,t)$ and tends to zero as $\varepsilon \to 0$, and where C_T is a constant depending only on T. We define weak asymptotic solutions to a general system of two conservation laws (1.1) as follows.

Definition 1.2. *We say that the families of smooth complex-valued distributions (u_ε) and (v_ε) represent a weak asymptotic solution to* (1.1) *if there exist real-valued distributions $u, v \in C(\mathbb{R}_+; \mathcal{D}'(\mathbb{R}))$, such that for every fixed $t \in \mathbb{R}_+$*

$$u_\varepsilon \rightharpoonup u, \quad v_\varepsilon \rightharpoonup v \quad \text{as} \quad \varepsilon \to 0,$$

in the sense of distributions in $\mathcal{D}'(\mathbb{R})$, and

$$\left.\begin{array}{rcl} \partial_t u_\varepsilon + \partial_x f(u_\varepsilon, v_\varepsilon) & = & o_{\mathcal{D}'}(1), \\ \partial_t v_\varepsilon + \partial_x g(u_\varepsilon, v_\varepsilon) & = & o_{\mathcal{D}'}(1). \end{array}\right\} \tag{1.5}$$

Unlike previous definitions of the weak asymptotic solution [16, 42], Definition 1.2 explicitly allows the approximating distributions to be complex-valued. Although the imaginary parts of (u_ε) and (v_ε) will disappear in limit as $\varepsilon \to 0$, this definition effectively broadens the class of possible singular solutions of (1.1).

The weak asymptotic method is a convenient tool for constructing possible singular weak solutions to the Cauchy problem for conservation laws, but one problematic issue is the question of uniqueness. Indeed, by adding a constant term of order $\mathcal{O}(\varepsilon)$ to any weak asymptotic solution, one immediately obtains two different weak asymptotic solutions which correspond to the same solution if a more restrictive concept is used.

One way to narrow the class of solution candidates is to require distributional solutions to satisfy the equations in a stronger sense than the one defined in Definition 1.2. This strategy involves the problem of multiplication of singular distributions. It is well known that it is not possible to define such multiplication in an appropriate manner (the Schwartz impossibility result). The problem of taking products of singular distributions was overcome by Danilov and Shelkovich in [16] in a rather natural way. In their work, the weak asymptotic solution is constructed in such a way that the terms that do not have a distributional limit cancel in the limit as ε approaches zero. As a result, it is not necessary to include singular terms in the definition of the weak solution. Thus, the problem of multiplication of distributions is automatically eliminated, and the class of possible solution is significantly reduced. In Section 2, the weak solution concept of [16] is reviewed, and extended to flux functions $f(u, v)$ and $g(u, v)$ nonlinear in both u and v. As will be revealed in the examples provided in Sections 3 and 4, this extension is shown to be non-vacuous by the use of complex-valued correction terms in the weak asymptotic solutions. However, as also shown in Section 4, there is still strong uniqueness if the definition is used as a stand-alone concept of weak solution.

There are also several other reasonable ways to multiply Heaviside and Dirac distributions. In [8, 10, 24, 47], a number of definitions of weak solutions of (1.1) are introduced. Among the latter approaches, we emphasize the measure-type solution concept introduced by Huang in [24]. Indeed, this framework yields uniqueness of solutions if an additional condition of Oleinik-type is required, and [24] is probably the only work so far which obtains a uniqueness result for arbitrary initial data in a class of distributional solutions weak enough to allow delta-distributions. However, uniqueness has also been obtained for special classes of initial data by LeFloch in [35], and by Nedeljkov [40].

We remark that a systematic study of multiplication of distributions problem is investigated in the Colombeau algebra framework [8, 20, 38]. In these works, problems of the type considered here are also investigated. Actually, Definition 1.2 can be understood as a variant of appropriate definitions in [9, 39, 41]. The main difference is that in the present case, a solution is found pointwise with respect to $t \in \mathbb{R}_+$, and it is required that the distributional limit of the weak asymptotic solution be a distribution. The latter is not necessary in the framework of the Colombeau algebra though it may be tacitly assumed.

The plan of the present paper is as follows. We will provide a review of the definition of weak singular solutions in [16] in Section 2. It turns out that a somewhat more general statement is appropriate here. Moreover, it will be proved that any 2×2 system of hyperbolic conservation admits singular solutions of this type. In Section 3, Definition 1.2 is used to find weak asymptotic solutions of the shallow-water system. In Section 4, the definition of weak singular solutions of [16] is applied to the examples of Section 3, and various examples showing non-uniqueness are given. Finally, an attempt at defining a solution concept which will yield a unique singular solutions is made. In Section 5, weak asymptotic solutions of the Brio system are found, and it is shown that they also satisfy the the equation in the sense of Definition 2.1. In the conclusion, the pertinence of the δ-shock wave solutions is discussed.

2 Generalized weak solutions

In this section, the definition of weak singular solutions of a 2×2 system of conservation laws provided in [16] is reviewed. While the definition in [16] is given only for solutions singular in the second variable, while assuming that the flux functions f and g are linear in the second variable, it appears that the definition can actually be made more general. Suppose $\Gamma = \{\gamma_i \mid i \in I\}$ is a graph in the closed upper half plane, containing Lipschitz continuous arcs γ_i, $i \in I$, where I is a finite index set. Let I_0 be the subset of I containing all indices of arcs that connect to the x-axis, and let $\Gamma_0 = \{x_k^0 \mid k \in I_0\}$ be the set of initial points of the arcs γ_k with $k \in I_0$. Define the singular part by $\alpha(x,t)\delta(\Gamma) = \sum_{i \in I} \alpha_i(x,t)\delta(\gamma_i)$. Let (u, v) be a pair of distributions, where v is represented in the form

$$v(x,t) = V(x,t) + \alpha(x,t)\delta(\Gamma),$$

and where $u, V \in L^\infty(\mathbb{R} \times \mathbb{R}_+)$. Finally, the expression $\frac{\partial \varphi(x,t)}{\partial \mathbf{l}}$ denotes the tangential derivative of a function φ on the graph γ_i, and $\int_{\gamma_i}$ connotes the line integral over the arc γ_i.

Definition 2.1. a) *The pair of distributions u and $v = V + \alpha(x,t)\delta(\Gamma)$ are called a generalized δ-shock wave solution of system* (1.1) *with the initial data $u_0(x)$ and $V_0(x) + \sum_{I_0} \alpha_k(x_0^k, 0)\delta\big(x - x_k^0\big)$ if the integral identities*

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}} \left(u\partial_t\varphi + f(u,V)\partial_x\varphi\right)\, dxdt + \int_{\mathbb{R}} u_0(x)\varphi(x,0)\, dx = 0, \tag{2.1}$$

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}} \left(V\partial_t\varphi + g(u,V)\partial_x\varphi\right)\, dxdt \tag{2.2}$$

$$+ \sum_{i \in I} \int_{\gamma_i} \alpha_i(x,t)\frac{\partial \varphi(x,t)}{\partial \mathbf{l}} + \int_{\mathbb{R}} V^0(x)\varphi(x,0)\, dx + \sum_{k \in I_0} \alpha_k(x_k^0, 0)\varphi(x_k^0, 0) = 0,$$

hold for all test functions $\varphi \in \mathcal{D}(\mathbb{R} \times \mathbb{R}_+)$.

The next definition concerns the similar situation where the singular solution is contained in u, and v is a regular distribution. Thus we assume the representation

$$u(x,t) = U(x,t) + \alpha(x,t)\delta(\Gamma),$$

where now $U, v \in L^\infty(\mathbb{R} \times \mathbb{R}_+)$, and $\alpha(x,t)\delta(\Gamma)$ is defined as before.

Definition 2.1. b) *The pair of distributions* $u = U + \alpha(x,t)\delta(\Gamma)$ *and* v *is a generalized* δ*-shock wave solution of system* (1.1) *with the initial data* $u_0(x) + \sum_{I_0} \alpha_k(x_0^k, 0)\delta\big(x - x_k^0\big)$ *and* $v_0(x)$) *if the integral identities*

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}} \left(U\partial_t\varphi + f(U,v)\partial_x\varphi\right)\, dxdt \tag{2.3}$$

$$+\sum_{i\in I}\int_{\gamma_i} \alpha_i(x,t)\tfrac{\partial\varphi(x,t)}{\partial \mathbf{l}} + \int_{\mathbb{R}} u_0(x)\varphi(x,0)dx + \sum_{k\in I_0}\alpha_k(x_k^0,0)\varphi(x_k^0,0) = 0,$$

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}} \left(v\partial_t\varphi + g(U,v)\partial_x\varphi\right)\, dxdt + \int_{\mathbb{R}} v_0(x)\varphi(x,0)\, dx = 0, \tag{2.4}$$

hold for all test functions $\varphi \in \mathcal{D}(\mathbb{R}\times\mathbb{R}_+)$.

This definition is quite general, allowing a combination of initial steps and delta distributions; but its effectiveness is is already demonstrated by considering the Riemann problem with a single jump. Indeed, for this configuration it can be shown that a δ-shock wave solution exists for any 2×2 system of conservation laws. Consider the Riemann problem for (1.1) with initial data $u(x,0) = U_0(x)$ and $v(x,0) = V_0(x)$, where

$$U_0(x) = \begin{cases} u_1, & x < 0, \\ u_2, & x > 0, \end{cases}$$

$$V_0(x) = \begin{cases} v_1, & x < 0, \\ v_2, & x > 0. \end{cases}$$

Then, the following theorem holds.

Theorem 2.1. a) *If* $u_1 \neq u_2$ *then the pair of distributions*

$$u(x,t) = U_0(x-ct), \tag{2.5}$$

$$v(x,t) = V_0(x-ct) + \alpha(t)\delta(x-ct), \tag{2.6}$$

where

$$c = \frac{[f(u,V)]}{[u]} = \frac{f(u_2,v_2) - f(u_1,v_1)}{u_2 - u_1}, \quad \textit{and} \quad \alpha(t) = (c[V] - [g(u,V)])t,$$

represents the δ*-shock wave solution of* (1.1) *with initial data* $U_0(x)$ *and* $V_0(x)$ *in the sense of Definition 2.1(a).*

Theorem 2.1. b) *If* $v_1 \neq v_2$ *then the pair of distributions*

$$u(x,t) = U_0(x-ct) + \alpha(t)\delta(x-ct), \tag{2.7}$$

$$v(x,t) = V_0(x-ct), \tag{2.8}$$

where

$$c = \frac{[g(U,v)]}{[v]} = \frac{g(u_2,v_2) - g(u_1,v_1)}{v_2 - v_1}, \quad \alpha(t) = (c[U] - [f(U,v)])t$$

represents the δ-shock wave solution of (1.1) *with initial data $U_0(x)$ and $V_0(x)$ in the sense of Definition 2.1(b).*

Proof. We will prove only the first part of the theorem as the second part can be proved analogously. We immediately see that u and v given by (2.5) and (2.6) satisfy (2.1) since c is given exactly by the Rankine-Hugoniot condition derived from that system. By substituting u and v into (2.2), we get after standard transformations:

$$\int_{\mathbb{R}_+} \left(-c[V] + [g(u,V)]\right) \varphi(ct,t)\, dt - \int_{\mathbb{R}_+} \alpha'(t)\varphi(ct,t)\, dt = 0.$$

From here and since $\alpha(0) = 0$, the conclusion follows immediately. □

An obvious difficulty here is to determine in which function, u or v, the δ-distribution should be placed. Even if this issue were resolved, non-uniqueness would still be a problem. In fact, non-uniqueness can appear in the regular part (as is well known), or in the singular part. Concerning the regular part, non-uniqueness can be removed by the entropy inequalities. The singular part can be controlled by the principle of minimizing the possible singularities of a solution. More precisely, it can be required that the solution contain a minimal number of δ-distributions An example of how these ideas can be used to provide uniqueness will be given for a special class of solutions of the shallow-water system in Section 4. In particular, the minimization principle will be formalized in Definition 4.2.

The principle of minimizing the number of delta distributions can also be used to resolve the problem of interaction of delta distributions. Indeed, when two delta distributions interact, they can split into several new delta distributions, or they can merge into a single delta distributions. If the number of delta distributions is required to be minimal, they will always have to merge, since the number of delta distributions then will be a monotonically decreasing function of t. The merging of two δ shock waves into a single δ shock wave has been reviewed in [25]. Since it is known how to resolve interactions of delta distributions, it would be straightforward to extend Theorem 4.1 to the case where several initial jumps are present in the initial data. An obvious way to proceed further is to use the wave front tracking arguments [3] in order to obtain the singular solution for (1.1) endowed with arbitrary L^∞ or BV initial data. However, it seems that we need additional admissibility criterions in order to formalize the wave front tracking procedure. This will be subject of further investigations.

3 Weak asymptotics for the shallow-water system

In this section, we shall construct weak asymptotic solutions for the shallow-water system (1.2). According to Definition 1.2, we need to find families of distributions, such that

$$\left.\begin{array}{rcl}\partial_t u_\varepsilon + \partial_x \left(v_\varepsilon + \frac{u_\varepsilon^2}{2} \right) & = & o_{\mathcal{D}'}(1), \\ \partial_t v_\varepsilon + \partial_x \left(u_\varepsilon + u_\varepsilon v_\varepsilon \right) & = & o_{\mathcal{D}'}(1). \end{array}\right\} \tag{3.1}$$

In addition, initial data are to be satisfied, so that we need

$$u_\varepsilon(x,0) \rightharpoonup u(x,0) = 0 \text{ and } v_\varepsilon(x,0) \rightharpoonup v(x,0). \tag{3.2}$$

Here the weak convergence indicated above means convergence in the sense of distributions as ε tends to 0. This notation will be adhered to in the remainder of this article. In order to introduce the ansatz for a weak asymptotic solution , we introduce a smooth, nonnegative even function $\rho : \mathbb{R} \to \mathbb{R}$ such that

$$\int_{\mathbb{R}} \rho(z)\, dz = 1, \quad \text{supp } \rho \subset (-1,1).$$

Then, denote

$$R_\varepsilon(x) = \tfrac{1}{\sqrt{\varepsilon}} \rho \left(\tfrac{x-2\varepsilon}{\varepsilon} \right), \quad \delta_\varepsilon(x) = \tfrac{1}{\varepsilon} \rho \left(\tfrac{x+2\varepsilon}{\varepsilon} \right). \tag{3.3}$$

Notice that

$$R_\varepsilon(x)\delta_\varepsilon(x) = 0 \text{ for every } x \in \mathbb{R}. \tag{3.4}$$

Furthermore, defining $\rho_0 = \int_{\mathbb{R}} \rho^2(z)\, dz$, it is not difficult to check that

$$R_\varepsilon \rightharpoonup 0, \quad R_\varepsilon^2(x) \rightharpoonup \rho_0 \delta(x), \quad \text{and} \quad \delta_\varepsilon(x) \rightharpoonup \delta(x). \tag{3.5}$$

The first example of a weak asymptotic solution for the shallow-water equations is a stationary delta distribution in v, centered at the origin.

Proposition 3.1. *Define the constant $a_0 = \pm \frac{1}{2}\rho_0$. Let $p = 1$ and $q = 0$ when $a_0 < 0$, and let $p = 0$ and $q = 1$ when $a_0 > 0$. The pair of families of smooth functions given by*

$$\begin{aligned} u_\varepsilon(x,t) &= (p + iq) R_\varepsilon(x), \\ v_\varepsilon(x,t) &= a_0 \delta_\varepsilon(x), \end{aligned} \tag{3.6}$$

represents a weak asymptotic solution of (1.2).

Proof. Consider the first equation in (3.1). Note first that $\partial_t u_\varepsilon = 0$. Moreover, from (3.5) we have

$$\begin{aligned}
&\left\langle \partial_x v_\varepsilon, \phi \right\rangle \to -a_0 \phi'(0),\\
&\left\langle \partial_x \tfrac{u^2}{2}, \phi \right\rangle \to -\tfrac{1}{2}(p+iq)^2 \rho_0 \phi'(0) + o(1)
\end{aligned}$$

as ε tends to zero. Assume first that $a_0 > 0$, so that $p = 0$ and $q = 1$. From the last two relations, and the definitions of a_0 and ρ_0, it can be concluded that the first equation of (3.1) is satisfied in the sense of Definition 1.1.

The proof of the second relation is straightforward since $\partial_t v_\varepsilon = 0$ and $u_\varepsilon \rightharpoonup 0$ as $\varepsilon \to 0$. Moreover, $u_\varepsilon v_\varepsilon \equiv 0$ as observed in (3.4). Finally, it is also immediate that the initial data are satisfied in the sense required by (3.2). The case when $a_0 < 0$ is proved in exactly the same way. $\square$

While this simple example is actually a special case of the next example, we have chosen to include it as a separate result because it shows so clearly the utility of the complex-valued extension of the weak asymptotic method. Indeed, notice that if only real-valued distributions were allowed as weak asymptotic solutions in this example, then this would force the coefficient a_0 to be negative.

In the next example, we shall show that the weak asymptotic framework allows the construction of more general solutions than the one given by (3.6). In addition to the Dirac delta initial data in v, a jump discontinuity is specified in the second dependent variable u. Thus let initial data be given by

$$u(x,0) = \begin{cases} u_1, & x < 0, \\ u_2, & x > 0, \end{cases} \tag{3.7}$$

$$v(x,0) = a_0 \delta(x), \tag{3.8}$$

where we assume $u_2 < u_1$ for the sake of simplicity.

In order to construct the weak asymptotic solution, let $c = \frac{1}{2}\frac{u_1^2 - u_2^2}{u_1 - u_2}$ be given as usual from the Rankine-Hugoniot conditions, and take the functions R_ε and δ_ε as in the previous example. Next, introduce the function G_ε representing a regularized step function with values u_1 and u_2 by defining

$$G_\varepsilon(z) = \begin{cases} u_1, & z \le -5\varepsilon, \\ c, & -3\varepsilon \le z \le 3\varepsilon \\ u_2, & z \ge 5\varepsilon, \end{cases}$$

so that $u_1 \le G_\varepsilon \le u_2$.

Proposition 3.2. *Define*

$$u_\varepsilon(x,t) = G_\varepsilon(x-ct) + (p(t)+iq(t))R_\varepsilon(x-ct),$$
$$v_\varepsilon(x,t) = \alpha(t)\delta_\varepsilon(x-ct). \tag{3.9}$$

where $\alpha(t) = (u_1-u_2)t + a_0$, *and* p *and* q *are chosen such that* $\rho_0(p+iq)^2 = -2\alpha$. *Then* (u_ε) *and* (v_ε) *represent the weak asymptotic solution to the shallow-water system* (1.2) *with initial data* (3.7), (3.8).

Proof. To prove that (u_ε) and (v_ε) represent the weak asymptotic solution of the shallow-water system for an appropriate choice of p, q and α, we substitute the given ansatz for (u_ε) and (v_ε) into the equations (3.1). Since $R_\varepsilon \rightharpoonup 0$ and $R_\varepsilon G_\varepsilon \equiv cR_\varepsilon \rightharpoonup 0$, it needs to be shown that

$$\begin{aligned} \partial_t G_\varepsilon(x-ct) + \partial_x\Big(\alpha(t)\delta_\varepsilon(x-ct) + \tfrac{G_\varepsilon^2(x-ct)+(p+iq)^2R_\varepsilon^2(x-ct)}{2}\Big) &= o_{\mathcal{D}'}(1), \\ \partial_t\big(\alpha(t)\delta_\varepsilon(x-ct)\big) + \partial_x\big(G_\varepsilon(x-ct) + G_\varepsilon(x-ct)\alpha(t)\delta_\varepsilon(x-ct)\big) &= o_{\mathcal{D}'}(1). \end{aligned} \tag{3.10}$$

Focusing first on the first equation in (3.10), note the following limits:

$$\partial_x\left(\alpha(t)\delta_\varepsilon(x-ct)\right) \rightharpoonup -\alpha(t)\delta'(x-ct),$$

and

$$\tfrac{(p+iq)^2}{2}\partial_x R_\varepsilon^2(x-ct) \rightharpoonup -\tfrac{(p+iq)^2}{2}\rho_0\delta'(x-ct).$$

Thus these terms cancel in the limit if p and q are chosen such that

$$\rho_0(p+iq)^2 = -2\alpha. \tag{3.11}$$

Of course, α is not yet known, but will be determined from the second equation in (3.10). In any case, choosing p and q in this way, the first equation from (3.10) reduces to

$$\partial_t G_\varepsilon(x-ct) + \tfrac{1}{2}\partial_x G_\varepsilon^2(x-ct) = o_{\mathcal{D}'}(1). \tag{3.12}$$

Furthermore, notice that $G_\varepsilon \to u$ in L^1_{loc}, where

$$u(x,t) = \begin{cases} u_1, & x < ct, \\ u_2, & x > ct, \end{cases}$$

and that u represents a weak solution to the Burgers equation $\partial_t u + \partial_x u^2/2 = 0$ so long as c is chosen to be $\frac{u_1+u_2}{2}$. This immediately implies that (3.12) is satisfied, from which it follows that the first equation in (3.10) is satisfied.

In order to choose α so that the second equation in (3.10) is satisfied, let $\varepsilon \to 0$ there to obtain

$$(\alpha' + u_2 - u_1)\delta(x - ct) = 0.$$

Setting the coefficient to zero and taking initial data (3.8) into account yields

$$\alpha(t) = (u_1 - u_2)t + a_0, \tag{3.13}$$

which concludes the proof. ☐

Contemplating the expression for $\alpha(t)$ derived in the proof, it appears that for certain choices of a_0, u_1 and u_2, the coefficient $\alpha(t)$ multiplying the delta distribution will be equal to zero for some positive time $t_0 > 0$. At that moment, the δ-shock disappears and one may continue the solution for $t \geq t_0$ by solving the usual Riemann problem for the shallow-water system. On the other hand, the distributionals limits of (u_ε) and (v_ε) represent the weak solution along the entire temporal axis, and we may take them as global solutions to our problem. The resulting non-uniqueness will be resolved in the next section in Definition 4.2 by introducing a solution concept which calls for minimization of singularities of a solution. Finally, note that similar solutions exist for more complicated Riemann-type initial data, such as combinations of jumps and delta distributions in both u and v.

4 Generalized weak solutions of the shallow-water system

In this section, it will be shown that the weak asymptotic solutions constructed in the previous section represent solutions to the shallow-water system also in the sense of Definition 2.1. With the same provisos as in Section 2, Definition 2.1(a) can be written in the special case of (1.2) as follows.

Definition 4.1. *A graph* Γ *and a pair of distributions* (u, v) *where* v *is represented in the form*

$$v(x,t) = V(x,t) + \sum_{i \in I} \alpha_i(x,t)\delta(\gamma_i),$$

with $u, V \in L^\infty(\mathbb{R} \times \mathbb{R}_+)$*,* $\alpha_i \in C^1(\Gamma)$*,* $i \in I$*, is called a generalized* δ*-shock wave solution of system* (1.2) *with initial data* $u_0(x)$ *and* $V_0(x) + \sum_{I_0} \alpha_k(x_0^k, 0)\delta\big(x - x_k^0\big)$ *if* *the integral identities*

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}} \Big(u\partial_t\varphi + (\tfrac{u^2}{2} + V)\partial_x\varphi\Big)\, dxdt + \int_{\mathbb{R}} u_0(x)\varphi(x,0)\ dx = 0, \tag{4.1}$$

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}} (V\partial_t\varphi + (uV+u)\partial_x\varphi)\ dxdt \tag{4.2}$$

$$+\sum_{i\in I}\int_{\gamma_i} \alpha_i(x,t)\tfrac{\partial\varphi(x,t)}{\partial \mathbf{l}} + \int_{\mathbb{R}} V^0(x)\varphi(x,0)\ dx + \sum_{k\in I_0}\alpha_k(x_k^0,0)\varphi(x_k^0,0) = 0,$$

hold for all test functions $\varphi \in \mathcal{D}(\mathbb{R}\times\mathbb{R}_+)$.

An appropriate version of Definition 2.1(b) may also be given, but we focus here on the more traditional view that of solutions that are singular in the second variable, i.e. in v. The main reason for this is that we are able to obtain unqiueness for a certain class of these solutions by adding suitable requirements in Definition 4.2. Now concerning the example defined in Proposition 3.1, it is straightforward to verify that it represents a weak solution in the sense of Definition 4.1. Note first that if u_ε and v_ε are defined as in (3.6), then $u_\varepsilon \rightharpoonup 0$ and $v_\varepsilon \rightharpoonup a_0\delta$. Thus u and V are zero, and all terms in (4.1) are therefore zero. The relation (4.2) reduces to

$$a_0\int_{\mathbb{R}_+} \partial_t\varphi\ dt + a_0\varphi(0,0) = 0,$$

which is obviously true. Thus we have the following proposition.

Proposition 4.1. *The weak asymptotic solution to* (3.1) *defined in* (3.6) *converges to the pair of distributions*

$$u = 0 \quad \textit{and} \quad v = a_0\delta(x) \tag{4.3}$$

which is a solution of the shallow-water equations in the sense of Definition 4.1.

Next, consider the example given in Proposition 3.2. Here, it can be seen that $u_\varepsilon \rightharpoonup u_1 + (u_2-u_1)H(x-ct)$, and $v_\varepsilon \rightharpoonup \alpha(t)\delta(x-ct)$, where H is the Heaviside function and $\alpha(t)$ is defined in (3.13). Then we can prove the following proposition.

Proposition 4.2. *The weak asymptotic solution to* (3.1) *defined in* (3.9) *converges to the pair of distributions*

$$u = u_1 + (u_2-u_1)H(x-ct) \quad \textit{and} \quad v = \big((u_1-u_2)t + a_0\big)\delta(x-ct)$$

which is a solution of the shallow-water system in the sense of Definition 4.1 with initial data (3.7) *and* (3.8).

Proof. As already observed in the proof of Proposition 3.2, the Rankine-Hugoniot condition $c = \frac{u_1+u_2}{2}$ guarantees that (4.1) is satisfied since V is identically zero. Next, it can be seen that (4.2) reduces to

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}} u\ \partial_x\varphi\ dxdt\ +\ \int_{\{x=ct\}} \alpha(t)\tfrac{\partial\varphi}{\partial \mathbf{l}}\ +\ \alpha(0)\varphi(0,0) = 0.$$

This relation can be verified by elementary integration. ☐

As Definitions 2.1 and 4.1 appear to be a natural generalizations of the classical weak solution concept, it is no surprise that uniqueness does not hold if Definition 4.1 is used. However, for the shallow-water system, the problem of uniqueness of weak solutions can be resolved in some special cases. Following ideas from [24], an additional condition guaranteeing uniqueness will be put forward in Definition 4.2. First, however, to prove non-uniqueness in the present definition, we show that there is an infinite family of functions satisfying the equation (1.2) in the sense of Definition 4.1 with zero initial data.

Proposition 4.3. *Let $u_1 > 0$ be arbitrary, and let $a_0 = 2u_1$. The pair of functions*

$$\begin{aligned} u(x,t) &= \begin{cases} -u_1, & -u_1t \le x \le 0, \\ \ \ u_1, & 0 \le x \le u_1t, \\ \ \ 0, & \textit{otherwise}, \end{cases} \\ v(x,t) &= -a_0t\delta(x) + u_1t\delta(x+u_1t) + u_1t\delta(x-u_1t), \end{aligned} \tag{4.4}$$

represents a solution of the shallow-water equations (1.2) *in the sense of Definition 4.1 with zero initial data.*

Proof. First, notice that in the case at hand, the regular part V of the distribution v is equal to zero. Therefore, (4.1) reduces to

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}} \Big(u\partial_t\varphi + \tfrac{u^2}{2}\ \partial_x\varphi\Big)\ dxdt + \int_{\mathbb{R}} u_0(x)\varphi(x,0)\ dx = 0, \tag{4.5}$$

where the second integral is zero. The relation (4.5) is true since the function u satisfies the Rankine-Hugoniot conditions at its jumps and (4.5) is exactly the integral formulation of the Burgers equation.

Next, substituting u and v from (4.4) into the second equation in Definition 4.1, we see that we need to check

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}} u\partial_x\varphi\ dxdt + \int_{\{x=-u_1t\}} (u_1t)\tfrac{\partial\varphi(x,t)}{\partial \mathbf{l}} - \int_{\{x=0\}} (a_0t)\tfrac{\partial\varphi(x,t)}{\partial \mathbf{l}} + \int_{\{x=u_1t\}} (u_1t)\tfrac{\partial\varphi(x,t)}{\partial \mathbf{l}} = 0.$$

Noticing that

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}} u\partial_x\varphi \; dxdt = u_1 \int_{\mathbb{R}_+} \varphi(-u_1t,t) \; dt + u_1 \int_{\mathbb{R}_+} \varphi(u_1t,t)) \; dt - 2u_1 \int_{\mathbb{R}_+} \varphi(0,t)) \; dt,$$

$$\int_{\{x=-u_1t\}} (u_1t)\tfrac{\partial\varphi(x,t)}{\partial \mathbf{l}} = -u_1 \int_{\mathbb{R}_+} \varphi(-u_1t,t)dt,$$

$$\int_{\{x=u_1t\}} (u_1t)\tfrac{\partial\varphi(x,t)}{\partial \mathbf{l}} = -u_1 \int_{\mathbb{R}_+} \varphi(u_1t,t)dt,$$

$$\int_{\{x=0\}} (a_0t)\tfrac{\partial\varphi(x,t)}{\partial \mathbf{l}} = a_0 \int_{\mathbb{R}_+} \varphi(0,t)dt,$$

we conclude that (4.6) holds. □

Note that this weak solution could be constructed with the help of the weak asymptotics method, but here we have chosen to give the definition directly, without using the definition (3.2). Now to remove the non-uniqueness, we consider weak solutions of the shallow-water system (1.2) for which the regular part $V(x,t)$ of v is identically zero. For this special class of weak solutions, we may introduce the following admissibility concept.

Definition 4.2. *Consider a weak solution $u \in L^\infty(\mathbb{R}\times\mathbb{R}_+)$ and $v = \sum_{i\in I}\alpha_i(x,t)\delta(\gamma_i)$, to* (1.2) *in the sense of Definition 4.1. Then, we say that (u,v) is an admissible weak solution if the function u satisfies the Oleinik condition*

$$\frac{u(x_2,t)-u(x_1,t)}{x_2-x_1} \le \frac{1}{t}, \tag{4.6}$$

for almost every $x_1, x_2 \in \mathbb{R}$, and v contains a minimal number of delta distributions.

It is clear that the assumption $V \equiv 0$ and the integral relation (4.1) imply that we are dealing with the Burgers equation. In this case, one might expect that the Oleinik entropy condition would be sufficient to provide uniqueness of the solution. However, as illustrated in the following example, this is not the case.

Proposition 4.4. *Let u_1 and u_2 be arbitrary, and let $c = \frac{u_1+u_2}{2}$. Then for any three real numbers β_1, $\tilde{c}_1$ and $\tilde{c}_2$, the distributions*

$$u = u_1 + (u_2-u_1)H(x-ct) \quad \text{and}$$
$$v = \big((u_1-u_2)t + a_0\big)\delta(x-ct) \; + \; \beta_1\delta(x-\tilde{c}_1t) \; - \; \beta_1\delta(x-\tilde{c}_2t),$$

are solutions of the shallow-water system in the sense of Definition 4.1 with initial data (3.7) *and* (3.8).

The proof of this proposition is similar to the proof of Proposition 4.3, and is therefore omitted. To prohibit the kind of singular behavior shown in the example above, the admissability condition in Definition 4.2 demands that the number of delta distributions be minimized. In other words, (u, v) is admissible if $V \equiv 0$, and if given any other solution $\tilde{v}(x,t) = \sum_{i\in\tilde{I}} \tilde{\alpha}_i(x,t)\delta(\tilde{\gamma}_i)$ of (1.2) in the sense of Definition 4.1 the inequality

$$\mathrm{card}(I) \leq \mathrm{card}(\tilde{I}),$$

holds. Of course, here only those $i \in I$ and $\tilde{i} \in \tilde{I}$ for which $\alpha_i \not\equiv 0$ and $\tilde{\alpha}_{\tilde{i}} \not\equiv 0$ are counted. Now in the framework of definitions 4.1 and 4.2 it is possible to prove the following theorem.

Theorem 4.1. *The Cauchy problem for the shallow-water equations* (1.2) *with initial data* (3.7)*,* (3.8)*, where* $u_1 > u_2$ *has a unique admissible weak solution in the sense of definitions 4.1 and 4.2.*

If $u_1 < u_2$*, then there exists no* δ*-shock weak solution with zero regular part in* v *in the sense of Definition 4.1 which is admissible in the sense of Definition 4.2.*

Proof. Since the function u must satisfy (4.1) and the Oleinik admissibility conditions, the standard theory of conservation laws shows that

$$u = \begin{cases} u_1, & x < ct, \\ u_2, & x > ct, \end{cases}$$

where $c = (u_1 + u_2)/2$. Next, since we demand $V = 0$, the integral relation (4.2) implies that for every $\varphi \in C_0^1(\mathbb{R}_+ \times \mathbb{R})$:

$$\int_{\mathbb{R}_+} (u_1 - u_2)\varphi(ct,t)\, dt = -\sum_{i\in I} \int_{\gamma_i} \alpha_i(x,t)\frac{\partial\varphi(x,t)}{\partial \mathbf{l}} - a_0\varphi(0,0),$$

where γ_k, $k \in I$, are manifolds where δ-shocks forming the solution are supported. From the latter expression, we conclude that v must contain at least one delta distribution supported on the curve $\{(x,t):\ x = ct,\ t \in \mathbb{R}^+\}$. Having established this fact, the same computation as in the proof of Proposition 3.2 implies that the coefficient must be given by $\alpha(x,t) = (u_1 - u_2)t + a_0$, i.e. the solution can be constructed by the use of only one delta distribution which makes it unique (since any other solution must contain the term $\alpha(x,t)\delta(x - ct)$).

If $u_1 < u_2$, then the admissible solution to the Burgers equation is the rarefaction wave connecting u_1 and u_2. Therefore, the function u in Definition 4.1 must be the rarefaction wave, and hence $u_x \in L^\infty(\mathbb{R}_+ \times \mathbb{R})$. Thus integrating by parts in the third term of (4.2), and keeping in mind that $V=0$, we see that (4.2) reduces to

$$\int_{\mathbb{R}_+\times\mathbb{R}} u_x\varphi(x,t)\,dxdt = -\sum_{i\in I}\int_{\gamma_k} e_k(x,t)\frac{\partial\varphi(x,t)}{\partial \mathrm{l}} - \sum_{k\in I_0} e_k^0\varphi(x_k^0,0). \tag{4.7}$$

Furthermore, since $u_x \neq 0$ on an open set, a test function φ can be chosen which is equal to zero on γ_k, $k\in I$, but such that $\varphi u_x > 0$ on a set of positive measure which does not contain γ_k, $k\in I$. Thus, it can be concluded that the relation (4.7) cannot be satisfied for all test functions φ. □

5 Weak asymptotics and generalized weak solutions for the Brio system

In this section, we shall construct weak asymptotic solutions for the Riemann problem associated to the Brio system (1.3), and then show that the weak asymptotic solution converges to the generalized weak solution to the system in the sense of Definition 2.1. To construct weak asymptotic solutions we need to find families of distributions (u_ε), (v_ε), such that

$$\left.\begin{array}{rcl} \partial_t u_\varepsilon + \partial_x\left(\frac{u_\varepsilon^2+v_\varepsilon^2}{2}\right) &=& o_{\mathcal{D}'}(1), \\ \partial_t v_\varepsilon + \partial_x\left(v_\varepsilon(u_\varepsilon-1)\right) &=& o_{\mathcal{D}'}(1), \end{array}\right\} \tag{5.1}$$

$$u_\varepsilon \rightharpoonup u, \quad v_\varepsilon \rightharpoonup v \quad \text{as} \quad \varepsilon\to 0, \tag{5.2}$$

and such that $u(x,0)=U_0(x)$ and $v(x,0)=V_0(x)$, where

$$U_0(x) = \begin{cases} u_1, & x<0, \\ u_2, & x>0, \end{cases}$$

$$V_0(x) = \begin{cases} v_1, & x<0, \\ v_2, & x>0. \end{cases}$$

We shall prove the following theorem.

Theorem 5.1. **a)** *If $u_1 \neq u_2$ then there exist weak asymptotic solutions (u_ε), (v_ε) of the Brio system* (1.3)*, such that the families (u_ε) and (v_ε) have distributional limits*

$$u(x,t) = U_0(x-ct), \tag{5.3}$$

$$v(x,t) = V_0(x-ct) + \alpha(t)\delta(x-ct), \tag{5.4}$$

where

$$c=\frac{u_1^2+v_1^2-u_2^2-v_2^2}{u_1-u_2} \quad \text{and} \quad \alpha(t)=\left(c(v_2-v_1)+(v_1(u_1-1)-v_2(u_2-1))\right)t. \quad (5.5)$$

b) *If $v_1\neq v_2$ then there exist weak asymptotic solutions (u_ε), (v_ε) of the Brio system* (1.3)*, such that the families (u_ε) and (v_ε) have distributional limits*

$$\begin{aligned} u(x,t) &= U_0(x-ct)+\alpha(t)\delta(x-ct), && (5.6)\\ v(x,t) &= V_0(x-ct), && (5.7)\end{aligned}$$

where

$$c=\frac{v_1(u_1-1)-v_2(u_2-1)}{v_1-v_2} \quad \text{and} \quad \alpha(t)=\big(c(u_2-u_1)+(u_1^2+v_1^2-u_2^2-v_2^2)\big)\,t. \quad (5.8)$$

Proof. **a)** We use the same function $\rho\in C_c^\infty(\mathbb{R})$ as in Section 3, and define

$$\begin{aligned} R_\varepsilon(x,t)&=\tfrac{i}{\varepsilon}\rho((x-ct-2\varepsilon)/\varepsilon)-\tfrac{i}{\varepsilon}\rho((x-ct+2\varepsilon)/\varepsilon),\\ \delta_\varepsilon(x,t)&=\tfrac{1}{\varepsilon}\rho((x-ct-4\varepsilon)/\varepsilon)+\tfrac{1}{\varepsilon}\rho((x-ct+4\varepsilon)/\varepsilon).\end{aligned} \quad (5.9)$$

Next, define smooth functions U_ε and V_ε such that

$$U_\varepsilon(x,t)=\begin{cases} u_1, & x<ct-20\varepsilon,\\ c+1, & ct-10\varepsilon\le x\le ct+10\varepsilon,\\ u_2, & x\ge ct+20\varepsilon,\end{cases}$$

$$V_\varepsilon(x,t)=\begin{cases} v_1, & x<ct-20\varepsilon,\\ 0, & ct-10\varepsilon\le x\le ct+10\varepsilon,\\ v_2, & x\ge ct+20\varepsilon.\end{cases}$$

Notice that

$$R_\varepsilon\rightharpoonup 0,\quad U_\varepsilon R_\varepsilon\rightharpoonup 0,\quad \text{and}\quad U_\varepsilon\delta_\varepsilon\rightharpoonup (c+1)\delta(x-ct). \quad (5.10)$$

Moreover, we have

$$V_\varepsilon R_\varepsilon\equiv 0,\quad V_\varepsilon\delta_\varepsilon\equiv 0,\quad \text{and}\quad \delta_\varepsilon R_\varepsilon\equiv 0. \quad (5.11)$$

Now make the ansatz

$$\begin{aligned} u_\varepsilon(x,t)&=U_\varepsilon(x,t),\\ v_\varepsilon(x,t)&=V_\varepsilon(x,t)+\alpha(t)(\delta_\varepsilon(x,t)+R_\varepsilon(x,t)),\end{aligned}$$

and substitute it into equations (5.1). Notice first of all that

$$v_\varepsilon^2(x,t) = V_\varepsilon^2 + \alpha^2(t)(R_\varepsilon^2 + \delta_\varepsilon^2)$$

by invoking (5.11). Focussing on the expression $R_\varepsilon^2 + \delta_\varepsilon^2$, we take $\varphi \in C_c^\infty(\mathbb{R})$ and consider the integral

$$\begin{aligned}
&\int_{\mathbb{R}} (R_\varepsilon^2 + \delta_\varepsilon^2)\varphi \, dx \\
&= \int_{\mathbb{R}} \tfrac{1}{\varepsilon^2}\Big(-\rho^2((x-ct+2\varepsilon)/\varepsilon) - \rho^2((x-ct-2\varepsilon)/\varepsilon) \\
&\qquad\qquad + \rho^2((x-ct+4\varepsilon)/\varepsilon) + \rho((x-ct-4\varepsilon)/\varepsilon)\Big)\varphi \, dx = \mathcal{O}(\varepsilon).
\end{aligned}$$

In the above reasoning, use was made of the following computation.

$$\begin{aligned}
&\int_{\mathbb{R}} \tfrac{1}{\varepsilon^2}\left(\rho^2((x-ct+\alpha\varepsilon)/\varepsilon) + \rho^2((x-ct-\beta\varepsilon)/\varepsilon)\right)\varphi(x)\, dz \\
&= \int_{\mathbb{R}} \tfrac{1}{\varepsilon}\rho^2(z)\left(\varphi(ct+\varepsilon(z-\alpha)) + \varphi(ct+\varepsilon(z+\beta))\right)\, dz \\
&= \int_{\mathbb{R}} \tfrac{1}{\varepsilon}\rho^2(z)\left(2\varphi(ct) + \varepsilon\varphi'(ct)(\beta-\alpha)\right) dz + \mathcal{O}(\varepsilon), \qquad \text{for } \alpha,\beta \in \mathbb{R}.
\end{aligned}$$

The last relation was found by making the changes of variables $(x-ct+\alpha\varepsilon)/\varepsilon = z$ and $(x-ct-\beta\varepsilon)/\varepsilon = z$, and observing that $\int z\rho^2(z)dz = 0$ since ρ is an even function. In the case at hand, we use $\alpha = \beta = 2$ for the first integral, and $\alpha = \beta = 4$ in the second integral. Finally, it becomes plain that

$$v_\varepsilon^2 = V_\varepsilon^2 + o_{\mathcal{D}'}(1). \tag{5.12}$$

Therefore, taking into account Definition 1.1, from the first equation in (5.1), we conclude that we need to check whether

$$\partial_t U_\varepsilon + \partial_x(U_\varepsilon^2 + V_\varepsilon^2) = o_{\mathcal{D}'}(1). \tag{5.13}$$

But this is indeed satisfied thanks to the choice of the constant c which was found from the Rankine-Hugoniot condition for the first equation in (1.3).

Let us now consider the second equation in (5.1). First, notice that

$$\begin{aligned}
\partial_x(v_\varepsilon(u_\varepsilon - 1)) &= \partial_x\left(U_\varepsilon V_\varepsilon + (c+1)\alpha(t)\delta_\varepsilon - V_\varepsilon - \alpha(t)\delta_\varepsilon\right) + o_{\mathcal{D}'}(1) \\
&= (v_1(1-u_1) + v_2(u_2-1)\delta(x-ct) + c\alpha(t)\delta'(x-ct)) + o_{\mathcal{D}'}(1).
\end{aligned}$$

Next, note also that

$$\partial_t v_\varepsilon = -c(v_2 - v_1)\delta(x-ct) + \alpha'(t)\delta(x-ct) - c\alpha(t)\delta'(x-ct) + o_{\mathcal{D}'}(1).$$

Adding the latter two expressions, we obtain

$$\begin{aligned}
&\partial_t v_\varepsilon + \partial_x \left(v_\varepsilon(u_\varepsilon - 1)\right)\\
&= \left(-c(v_2 - v_1) + \alpha'(t) + (v_1(1-u_1) + v_2(u_2 - 1)\right)\delta(x - ct) + o_{\mathcal{D}'}(1).
\end{aligned}$$

From here, we conclude that choosing α as given in (5.5), the first equation in (5.1) is satisfied, as well. This concludes the proof of part (a).

b) In this case, an appropriate weak asymptotic solution is given by

$$\begin{aligned}
&u_\varepsilon(x,t) = U_\varepsilon(x,t) + \alpha(t)(\delta_\varepsilon(x,t) + R_{1\varepsilon}(x,t)) + \sqrt{\alpha(t)}R_{2\varepsilon}(x,t)),\\
&v_\varepsilon(x,t) = V_\varepsilon(x,t),
\end{aligned}$$

where

$$c = \frac{v_1 u_1 - v_2 v_2}{v_1 - v_2} - 1 \quad \text{and} \quad \alpha(t) = \left(c(u_1 - u_2) - (u_1^2 + v_1^2 - u_2^2 - v_2^2)\right)t,$$

and

$$U_\varepsilon(x,t) = \begin{cases} u_1, & x \leq x < ct - 20\varepsilon \\ 0, & ct - 10\varepsilon \leq x \leq ct + 10\varepsilon \\ u_2, & x \geq ct + 20\varepsilon \end{cases}$$

$$V_\varepsilon(x,t) = \begin{cases} v_1, & x \leq x < ct - 20\varepsilon \\ 0, & ct - 10\varepsilon \leq x \leq ct + 10\varepsilon\ ; \\ v_2, & x \geq ct + 20\varepsilon \end{cases}$$

$$\begin{aligned}
&R_{1\varepsilon}(x,t) = \tfrac{i}{\varepsilon}\rho((x - ct - 2\varepsilon)/\varepsilon) - \tfrac{i}{\varepsilon}\rho((x - ct + 2\varepsilon)/\varepsilon);\\
&R_{2\varepsilon}(x,t) = \tfrac{(p+iq)}{\sqrt{\varepsilon}}\rho((x - ct)/\varepsilon);\\
&\delta_\varepsilon(x,t) = \tfrac{1}{\varepsilon}\rho((x - ct - 4\varepsilon)/\varepsilon) + \tfrac{1}{\varepsilon}\rho((x - ct + 4\varepsilon)/\varepsilon),
\end{aligned}$$

where $p = 1$, $q = 0$ if $c \geq 0$ and $p = 0$, $q = 1$ if $c < 0$, and ρ is the same smooth non-negative even function as used in the previous examples. We omit the proof since it follow the proof of (a) verbatim. □

Next, we shall prove that the limit distributions u and v given in Theorem 5.1 represent generalized weak solutions to (1.3) with initial data $u(x,0) = U_0(x)$ and $v(x,0) = V_0(x)$. We start by adapting Definition 2.1 from [16] to the Brio system. With the notations from the previous section, we introduce the following definition.

Definition 5.1. a) *A graph* Γ *and a pair of distributions* (u, v) *where* v *is represented in the form*

$$v(x,t) = V(x,t) + \sum_{i \in I} \alpha_i(x,t)\delta(\gamma_i),$$

with $u, V \in L^\infty(\mathbb{R} \times \mathbb{R}_+)$, $\alpha_i \in C^1(\Gamma)$, $i \in I$, *is called a generalized* δ*-shock wave solution of system* (1.2) *with initial data* $u_0(x)$ *and* $V_0(x) + \sum_{I_0} \alpha_k(x_0^k, 0)\delta\big(x - x_k^0\big)$ *if the integral identities*

$$\begin{aligned}
&\int_{\mathbb{R}_+}\int_{\mathbb{R}} \Big(u\partial_t\varphi + \tfrac{u^2+V^2}{2}\partial_x\varphi\Big)\ dxdt + \int_{\mathbb{R}} u_0(x)\varphi(x,0)\ dx = 0,\\
&\int_{\mathbb{R}_+}\int_{\mathbb{R}} (V\partial_t\varphi + (V(u-1))\partial_x\varphi)\ dxdt\\
&+\sum_{i\in I}\int_{\gamma_i} \alpha_i(x,t)\tfrac{\partial\varphi(x,t)}{\partial \mathbf{l}} + \int_{\mathbb{R}} V_0(x)\varphi(x,0)\ dx = 0 + \sum_{k\in I_0} \alpha_k(x_k^0,0)\varphi(x_k^0,0) = 0,
\end{aligned}$$

hold for all test functions $\varphi \in \mathcal{D}(\mathbb{R} \times \mathbb{R}_+)$.

Definition 5.1. b) *A graph* Γ *and a pair of distributions* (u, v) *where* u *is represented in the form*

$$u(x,t) = U(x,t) + \sum_{i \in I} \alpha_i(x,t)\delta(\gamma_i),$$

with $U, v \in L^\infty(\mathbb{R} \times \mathbb{R}_+)$, $\alpha_i \in C^1(\Gamma)$, $i \in I$, *is called a generalized* δ*-shock wave solution of system* (1.2) *with initial data* $U_0(x) + \sum_{I_0} \alpha_k(x_0^k, 0)\delta\big(x - x_k^0\big)$ *and* $v_0(x)$ *if the integral identities*

$$\begin{aligned}
&\int_{\mathbb{R}_+}\int_{\mathbb{R}} \Big(U\partial_t\varphi + \tfrac{U^2+v^2}{2}\partial_x\varphi\Big)\ dxdt\\
&+\sum_{i\in I}\int_{\gamma_i} \alpha_i(x,t)\tfrac{\partial\varphi(x,t)}{\partial \mathbf{l}} + \int_{\mathbb{R}} u_0(x)\varphi(x,0)\ dx + \sum_{k\in I_0} \alpha_k(x_k^0,0)\varphi(x_k^0,0) = 0,\\
&\int_{\mathbb{R}_+}\int_{\mathbb{R}} (v\partial_t\varphi + (v(U-1))\partial_x\varphi)\ dxdt + \int_{\mathbb{R}} V_0(x)\varphi(x,0)\ dx = 0,
\end{aligned}$$

hold for all test functions $\varphi \in \mathcal{D}(\mathbb{R} \times \mathbb{R}_+)$.

Recall that in the case $v_1 < 0 < v_2$ there exists no Lax-admissible solution to the Riemann problem (1.3), with the initial data U_0 and V_0 (see [22]). If v_1 and v_2 do not satisfy this relation, we have the classical (and thus unique) Lax admissible solution to the appropriate Riemann problem consisting of the elementary waves, i.e. shock and rarefaction waves. As we shall see in the next theorem, we can also have the δ-shock wave solution. In this case, we would naturally choose the classical solution as this is also consistent with the minimization principle used in Definition 4.2. At the moment, we have no concept providing uniqueness of δ-shock wave solution for the Brio system, and we shall therefore confine ourselves to an existence statement.

Theorem 5.2. *a) The distributions defined by* (5.3), (5.4) *and* (5.5) *are a δ-shock wave solution of* (1.3) *with initial data U_0 and V_0 in the sense of Definition 5.1 (a).*

Theorem 5.2. *The distributions defined by* (5.6), (5.7) *and* (5.8) *are a δ-shock wave solution of* (1.3) *with initial data U_0 and V_0 in the sense of Definition 5.1 (b).*

The proof of this theorem follows along the lines of the general result in Theorem 2.1. Note that the distributions u and v given in the previous theorem are, as announced, the ones appearing in Theorem 5.1. Thus, in this way we might say that Theorem 5.1 represents a justification of the solution concept defined in Definition 5.1 and used in Theorem 5.2.

6 Conclusion

In the paper, the existence of δ-shock wave solutions for two important systems of hyperbolic conservation laws has been proved. The existence has been shown using the solution concept developed in [16], and the definition given in [16] has been extended to the case of nonlinear flux functions. This extension has been illustrated by exhibiting explicit examples using the method of weak asymptotics. The complex-valued corrections defined in this paper proved to be crucial in the construction of these examples. In particular for the Brio system (1.3), the question of possible existence of δ-shock wave solutions raised in [22] was answered positively.

Previous results on existence of δ-shock wave solutions for the Rieman problems included assumptions such as overcompressivity [14, 16, 17, 31, 46], demanding that the speed c of the δ-shock satisfy the relation

$$\lambda_k(u_2, v_2) \leq c \leq \lambda_k(u_1, v_1), \tag{6.1}$$

where λ_k, $k = 1, 2$ denote the characteristic speeds for the left and right states (u_1, v_1) and (u_2, v_2), respectively. This condition actually means that the characteristics from both sides of the δ-shock enter the shock trajectory.

In the case of the shallow-water system, as we have already noticed, every Riemann problem admits standard Lax-admissible solution. In the case of initial data (3.2), an admissible solution in the sense of Definition 4.2 satisfies (6.1) if only the regular parts are taken into account. Indeed, the regular parts are $(u_1, 0)$ for the left state, and $(u_2, 0)$ as the right state, and c is given by $c = \frac{u_1+u_2}{2}$. For such states, the characteristic speeds are equal to $\lambda_1(u_1,0)=\lambda_2(u_1,0)=u_1$ and $\lambda_1(u_2,0)=\lambda_2(u_2,0)=u_2$ (see [23]), and it is plain that

$$u_2 \le c = \frac{u_1+u_2}{2} \le u_1.$$

The characteristic speeds for the Brio system are (see [22])

$$\lambda_1(u,v) = u - 1/2 - \sqrt{1/4+v^2}, \quad \lambda_2(u,v) = u - 1/2 + \sqrt{1/4+v^2}.$$

In the most interesting case when $v_1 < 0 < v_2$, i.e. when the standard Lax solution of of the Riemann problem for (1.3) does not exist, it is clear that we can have either overcompressivity, compressivity, or undercompressivity, depending on the choice of the left and right states. In this sense, the suitability of the δ-shock wave solution to (1.3) remains unresolved.

It is also clear that additional requirements are needed in order to justify δ-shock wave solutions in the case of a general system. The first such requirement could be the existence of the weak asymptotic solution to the general Riemann problem for (1.1), and the convergence in the sense of distributions towards the δ-shock wave solution. The admissibility condition defined in Definition 4.2 was tailored to the shallow-water system. For a more general result, one would have to find different conditions when aiming for uniqueness. A possibility would be to require entropy conditions similar to those given in [40], but this will be left to future work.

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